Graph-Theoretic Analysis of Belief Systems Dynamics under Logic Constraints

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Abstract

Opinion formation cannot be described solely as an ideological deduction from a set of principles about the social world. Repeated social interactions and logic constraints on truth statements are consequential for the construction of beliefs systems as well. In spite of recent developments on models and algorithmic approaches for the study of such complex relations, their analysis remains an outstanding problem. Will the belief system converge? How long does it takes? What value will it converges to? Existing algebraic conditions for answering these questions cannot be easily tested on large-scale complex networks. Here we provide graph-theoretic answers for a model of opinion dynamics of a belief system with logic constraints. We show how the belief system properties depend on the social network where agents interact and the set of logic constraints that relate beliefs on different topics. Moreover, we provide explicit dependencies for a variety of commonly used large-scale network models.

The analysis and modeling of opinion dynamics has generated interest that spans several decades of interdisciplinary research[1–9]. Social interaction models assume the opinion dynamics are based on agents exchanging beliefs and then constructing new ones by some internal aggregating scheme[10,11]. Mathematically, these pooling operations are modeled as weighted averages where the weights show the relative importance assigned by an individual to others. However, convergence among other properties do not depend explicitly on the chosen interaction weights but the topology of the underlying graphs. Therefore, understanding the role of the networks involved in the structural features of a belief system is of critical importance and can have direct implications for better decision-making and policy design[12–16].

Most of the existing models assume social interaction happens repeatedly and beliefs on separate topics form independently. This simple characterization provided tools for analyzing the long-term behaviors using systems theory. Nevertheless, this has been shown insufficient to explain the existence of shared beliefs in a population[15]. Recently proposed generalizations of opinion dynamic models integrate functional interdependencies among issues that coherently bound ideas and attitudes[15,17]. Mainly, *logic constraints* in belief systems provide a successful model for the evolution of opinions in both large-scale populations and small groups. These *logic constraints* imply that believing a specific statement is true may depend on the belief that others are true as well. Nonetheless, existing algebraic tools can be too complicated to use when facing large-scale and complex networks[17].

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In this paper, we study belief systems with multi-dimensional opinion dynamics and logic constraints[15]. Furthermore, we analyze how the structural properties of the social network of agents and the set of logic constraints influence a belief system from a *graph-theoretic* point of view. We describe this influence for the convergence of beliefs; the expected convergence time and its stationary value. Informally, we answer the following three questions with graph-theoretic conditions that are easily accessible for a number of commonly used topologies in large-scale complex networks.

1. When does a belief system converges?
2. How long does it take converge?
3. What does it converge to?

Results

Belief System with Logic Constraints

Friedkin et al.[15,17] describe a belief system with logic constraints as a group of n agents that periodically exchange and update their opinions about a set of m different truth statements that have logical dependencies between them. After each effective social interaction, the agents use the shared opinions and its dependencies to update their beliefs. The agents exchange their opinions by interacting over a social network that indicates the personal relations between them. These relations are captured on a graph $\mathcal{G} = (V, E)$, where V is the set of agents and E is a set of edges and there is an edge between two nodes if the corresponding agents share their opinions. Analogously, a graph $\mathcal{T} = (W, D)$ models the set of logic constraint where there is an edge between two topics if the belief in a statement affects the belief in the other.

The generalized dynamics of the belief system is defined as follows. First, every agent aggregates its opinions on every topic according to the imposed logic constraints. Thus, opinions are modified to take into account the dependencies on other topics. Second, the aggregation follows the social network. Here opinions are again modified to take into account those coming from other agents. Finally, a new opinion is formed as a combination of the most recent aggregation and the initial opinion which models adversity to deviate from initial beliefs. Here, we model the opinion of an agent on a specific truth statement being true or false is a value between zero and one. A value of zero indicates an agent strongly believes a statement is false, whereas a value of one implies the agent thinks the statement is true. Similarly, a value of 0.5 shows maximal uncertainty about a topic. The aggregation steps consist of convex combinations of the available values where weights represent the relative influence. This model is described in the following equations (1) for an arbitrary agent i.

$$\hat{x}_k^i(u) = \sum_{v=1}^{m} C_{uv} x_k^i(v) \qquad \text{(Aggregation by logic constraints)} \qquad (1a)$$

$$\bar{x}_k^i(u) = \sum_{j=1}^{n} A_{ij} \hat{x}_k^j(u) \qquad \text{(Aggregation by social network)} \qquad (1b)$$

$$x_{k+1}^i(u) = \lambda^i \bar{x}_k^i(u) + (1-\lambda^i) x_0^i(u) \qquad \text{(Influence of inital beliefs)} \qquad (1c)$$

where $0 \leq x_k^i(u) \leq 1$ represents the opinion of an agent i at time k on a certain topic u and $\hat{x}_k^i(u)$ and $\bar{x}_k^i(u)$ are the intermediate aggregation steps. The parameters $0 \leq A_{ij} \leq 1$ represents the weights an agent i assigns to the information coming from its neighbor j, for example A_{13} is how agent 1 weights the opinions shared by agent 3. This parameters are compliant with the network $\mathcal{G}$ in the sense that if there is an edge in the graph, then the corresponding weight is non-zero. Similarly, the parameters $0 \leq C_{uv} \leq 1$ are compliant with the graph $\mathcal{T}$ and represents the strength of the logic constraints, that is, the weight an opinion on a statement has on the opinion on other statements. Moreover, the parameter $0 \leq \lambda^i \leq 1$ indicates how stubborn is an agent. If $\lambda^i = 1$, agent i is said to be *maximally open* or *oblivious* to new opinions. Conversely if $\lambda^i = 0$ the agent i is said to be *maximally closed* or *intransigent*. Finally, we can group the parameters $\{A_{ij}\}$ into a n-by-n matrix A, known as the *social influence structure*, and parameters $\{C_{uv}\}$ into a m-by-m matrix C known as the as the *multi-issues dependent structure*[17]. These matrices are nonnegative and we assume the set of weights makes them row stochastic. Therefore, the sum of weights assigned by an agent to its neighbors should add up to one or the weights a belief assigns to the beliefs related by logic constraints add up to one. For example, Fig. 1 shows an example of a belief system with 4-agents and 3-truth statement.

Figure 1(c) shows the belief system generated by the network of agents in Fig. 1(a) and set of logic constraints in Fig. 1(b). This new graph is much "larger" than the individual agents or statements separately, it effectively has $2nm$ nodes. The belief of each agent on each truth statement is a separate node. Also, initial beliefs are separate nodes. The model of this larger graph of the belief system can be compactly restated as

$$x_{k+1} = Px_k \tag{2}$$

where $x \in [0,1]^{2nm}$ is a state that stacks the beliefs of all agents on all topics along side the initial beliefs, with

$$x_k = \Big[\underbrace{x_k^1(1), \dots, x_k^1(m)}_{\text{Beliefs of Agent 1}}, \underbrace{x_k^2(1), \dots, x_k^2(m)}_{\text{Beliefs of Agent 2}}, \dots, \underbrace{x_k^n(1), \dots, x_k^n(m)}_{\text{Beliefs of Agent } n}, \\ \underbrace{x_0^1(1), \dots, x_0^1(m)}_{\text{Initial Beliefs of Agent 1}}, \underbrace{x_0^2(1), \dots, x_0^2(m)}_{\text{Initial Beliefs of Agent 2}}, \dots, \underbrace{x_0^n(1), \dots, x_0^n(m)}_{\text{Initial Beliefs of Agent } n} \Big]'$$

and

$$P = \left[\begin{array}{c|c} (\Lambda A) \otimes C & (\boldsymbol{I}_n - \Lambda) \otimes \boldsymbol{I}_m \\ \hline \mathbf{0}_{nm} & \boldsymbol{I}_{nm} \end{array} \right]$$

where $\mathbf{0}_{nm}$ is a zeros matrix of size $n \times m$, $\boldsymbol{I}_{nm}$ is an identity matrix of size $n \times m$, $\otimes$ indicates the Kronecker product (see Supplementary Definition 2) and Λ is a diagonal matrix with the i-th diagonal entry being λ^i. This allows for the definition of the belief system graph $\mathcal{P}$, compliant with the matrix P, by adding an edge from j to i if $P_{ij} > 0$. In other words, there is an edge going from j to i in $\mathcal{P}$ if $x_{k+1}(i)$ is influenced by $x_k(j)$. See Supplementary equation (1) for an example of a matrix P for the belief system in Fig.1(c) assuming that $\lambda^i = 0.5$ for all agents.

Figure 2 shows an example, where the network of 5 agents is a cycle graph, and the network of 4 logical constraints is a directed path. The product graph between the cycle graph and the path graph has a single closed component (shown as the green nodes in Fig.2(c)). Figure 2(d) shows dynamics of the belief vector as the number of social interactions increases. The opinion on all 4 topics convergences to a single value for all agents. Figure 2(d) shows the dynamics of the belief vector when no logic constraints are considered. In this case, the agents reach some agreement on the final value, but this consensus value is different for each of the statements. Particularly, one can see that when logic constraints are present, the final consensus value corresponds to the average of the values in the strongly connected component. See Supplementary Fig. 1 for an additional example of the influence of the logic constraints in the resulting belief system matrix and Supplementary Fig. 3 for the case of Fig. 2 when the network of agents is a complete graph.

Does it converge?

The convergence of the belief system can be stated as a question of the existence of a limit of the beliefs as more social interactions happen. That is, whether or not there exists a vector of opinions x_∞ such that

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} P^k x_0 = x_\infty$$

for any initial value x_0.

The belief system in equation (2) converges to equilibrium if and only if every closed strongly connected component of the graph $\mathcal{P}$ is aperiodic[4,18]. Recall that a strongly connected component is closed if it has no outgoing links in the condensation of $\mathcal{P}$, see Methods. For example, if $\lambda^i < 1$, then the nodes corresponding to the initial beliefs, that is, $x_0^i(\cdot)$ will be closed strongly connected components in the condensation of $\mathcal{P}$. Consequently, every node with $\lambda^i < 1$ will not be part of a closed strongly connected component since there will be an outgoing edge to the initial beliefs nodes. The set of oblivious nodes, for which $\lambda^i = 1$, are the only nodes able to form closed strongly connected components. Furthermore, in the case of large-complex networks, the Kosaraju's algorithm or the Tarjan algorithm[19] can compute the strongly connected components in linear time.

The convergence condition depends on the topology of the graph $\mathcal{P}$. Nevertheless, we ask how does the existence of the limit depend on the topology of the network of agents and the set of logic constraints (see the Supplementary Note 1). Mainly, we want to relate the convergence of the belief system with the strongly connected component of the factor graphs $\mathcal{G}$ and $\mathcal{T}$.

The matrix P has two diagonal blocks, one corresponding to the initial beliefs and one involving the product $\Lambda A \otimes C$. The initial belief nodes can be regarded as closed strongly connected components with a single node in them and therefore non-bipartite. Moreover, all agents with $\lambda^i \neq 1$ will necessarily be transient nodes and do not affect the convergence of the belief system. Thus, one can focus on the closed strongly connected components of $A_c \otimes C$, where A_c is the minor of P with the maximally open agents with associated subgraph $\mathcal{G}_c$.

The product $A_c \otimes C$ can be written in its block diagonal form, where each of the blocks is the product of one strongly connected component from $\mathcal{G}_c$ and of from $\mathcal{T}$ (see Supplementary Lemma 2). Moreover, if either of the strongly connected components is bipartite so is their tensor product (see Supplementary Theorem 1). An immediate conclusion of this fact is that the process in equation (2) converges to equilibrium if an only if every closed strongly component of the graph $\mathcal{T}$ is non-bipartite and every closed strongly component composed of only oblivious agents is non-bipartite.

In Fig.1, the network of agents has a single closed strongly connected component composed of the node 4. Similarly, the network of topics has a single closed strongly connected component, the node 3. Thus, we can say that even if all agents are oblivious the belief system will converge to a final set of beliefs. In Fig.2, the belief system has one closed strongly connected component, and it is a cycle graph formed by the beliefs on the statement green. This strongly connected component corresponds to the product of the cycle graph and the green node of the logic constraints. Moreover, the cycle graph will be non-bipartite if the number of nodes is odd. Thus, if one considers an even number of nodes in the cycle network of agents, the belief system will not converge.

How long does it take to converge?

We seek to characterize the time required by the process by equation (2) to be arbitrarily close to its limiting value regarding properties of the graphs $\mathcal{G}$ and $\mathcal{T}$. Specifically, the number of agents and truth statements, and the topology of the graphs. Formally, we will bound the number of iterations required for the beliefs to be at a distance of at most ϵ from their final value (assuming they converge). We will express this estimate in terms of the total variation distance (see Methods), for this we define the convergence time as

$$T(\epsilon) = \min_k \left\{ \frac{\|x_k - x_\infty\|_{TV}}{\|x_0 - x_\infty\|_{TV}} \leq \epsilon \right\}$$

where x^k evolves according to equation (2). Informally, the value $T(\epsilon)$ shows the minimum number of social interactions required for the belief system to reach a set of beliefs that are arbitrarily close to their final value regardless of their initial condition. Moreover, this number depends explicitly on the graphs $\mathcal{G}$ and $\mathcal{T}$ and the number of agents and number of truth statements.

The dynamics of the belief system in equation (2) are closely related to the dynamics of a Markov chain with transition matrix P. Convergence to a stationary distribution of a random walk with transition probability P on the graph $\mathcal{P}$ implies convergence of equation 2. Therefore, bounds on the convergence time based on the mixing properties of this Markov chain provide rates of convergence for the belief system. Notably, the convergence time is proportional to the maximum time required for the random walk to get *absorbed* into a closed strongly connected component plus the time needed for such component to *mix*. Figure 3 illustrates this by considering two random walks X and Y with the same transition matrix P.

If we call L the maximum expected mixing time among all closed strongly connected components, and H the maximum expected time to get absorbed into a closed component, then if the graph $\mathcal{P}$ has no bipartite closed strongly connected components, the belief system will be ϵ close to its limiting distribution after $O((L + H)\log(1/\epsilon))$ steps (see Supplementary Theorem 3 for a formal statement and proof of this result). Therefore, not only we have an estimate of the convergence time of the belief system in terms of the topology of the graph $\mathcal{P}$, but we know this convergence happens exponentially fast.

We have shown that each of the strongly connected components of the graph $\mathcal{P}$ is the product of one component from $\mathcal{G}$ and one component from $\mathcal{T}$. Moreover, the mixing and absorbing times for a random walk on a product graph is the maximum between the mixing and absorbing time of the individual factors (see Supplementary Lemma 4). Thus, we have an explicit characterization of the convergence time in terms of the components of the networks of agents and the network of logic constraints. For example, in Fig. 2, the expected absorbing time is of the order of the number of nodes in the path, that is m, and then the mixing time of cycle[20–22] graphs is of the order n^2. Thus, the expected convergence time for the belief system is $O(\max(n^2, m)\log(1/\epsilon))$. Figure 4 shows a numerical analysis on this bound and verifies it holds. Particularly, Fig. 4(a) shows how the convergence time changes when the number of nodes in the cycle graph increases, and Fig. 4(b) shows how the convergence time changes when the number of truth statements in the directed path graph increases. Moreover, Fig. 4(c) shows the convergence to the final beliefs happens exponentially fast.

Table 1 presents the estimates for the expected convergence time for belief systems composed of well-known classic graphs, see Fig. 2 for plots of some of these common graphs. We use existing results about the mixing time for these graphs (see Supplementary Table 1 for a detailed list of references on each of the studied graphs) to provide an estimate of the convergence time of the resulting belief system. Particularly, our method allows the direct estimation of the dynamics of a belief system when large-scale complex networks are involved. For example, we provide convergence time bounds for the case where networks follow random graph models, namely: the geometric random graphs, the Erdős-Rényi model, and the Newmann-Watts small-world networks. These graphs are usually considered for its ability to represent the behavior of complex networks encountered in a variety of fields[23–26] (see Supplementary Fig. 8).

Figure 5 shows experimental results for the convergence time of a belief system for a subset of the factor graphs in Table 1. For every pair of graphs, we show how the convergence time increases as the number of agents or the number of truth statements change. One can particularly observe the maximum type behavior on the convergence time as predicted by the theoretical bounds. See Supplementary Fig. 4 and Fig. 6 for additional numerical results on other combinations of graphs from Table 1 and Fig. 5 and Fig. 7 for their linear convergence rates.

What does it converge to?

So far we have shown the conditions for a belief system to convergence and the corresponding convergence time. Convergence implies the existence of a vector x_∞ where the set of beliefs settle as the number of interactions increases. Moreover, we are interested in a characterization of this limit vector that allows a rapid computation of its value. The Supplementary Lemma 2 shows that one can always group the nodes in the graph $\mathcal{P}$ into strongly connected components. These components are either open or closed. For example, assume there is a closed strongly connected component S and call P_S the minor of the matrix P taking into account only the nodes in S. Then, P_S corresponds to the transition matrix of an irreducible and aperiodic Markov chain with stationary distribution π_S, where $\pi'_S P_S = \pi'_S$. The vector π_S is effectively the left-eigenvector of the matrix P_S corresponding to the eigenvalue 1. Moreover, it follows[18,22] that

$$\lim_{k\to\infty} x_k^S = \pi'_S x_0^S \mathbf{1}_{|S|}$$

Additionally, recall that every strongly connected component of $\mathcal{P}$ is the product of two strongly connected components, one from the network of agents and one from the logic constraints network. Thus, $P_S = A_S \otimes C_S$ which implies that $\pi_S = \pi_S^A \otimes \pi_S^C$, for some A_S and C_S where π_S^A and π_S^C are the corresponding left eigenvalues of the factor components of P_S. Therefore, the final beliefs of those nodes in a closed strongly connected component is a weighted average of the initial beliefs, and the weights (sometimes called the social power) are determined by the product of the left-eigenvectors of the factors. Particularly, the value π_S indicates the limit distribution of the component, that is, it tells the probability of finding a random on a specific node after a long time.

On the other hand, the beliefs on those open strongly connected components might have outgoing edges to several other components, in this case the beliefs will converge to

$$\lim_{k\to\infty} x_k^S = \sum_S p_{iS} \pi'_S x_0^S$$

where p_{iS} is the probability of absorption of node i into the source S (see Supplementary Section 4). Therefore, the limiting values for the nodes corresponding to an open strongly connected components will be a convex combination of the limiting values of the sources it is connected.

Numerical Analysis

Next, we provide a numerical analysis of three Large-Scale Networks from the Stanford Network Analysis Project (SNAP)[27], see Fig. 6. Table 2 shows the description of the networks used. In the three cases, we select the largest strongly connected component of the graph and use it as a representative of the network structure and the mixing properties of the graph. Furthermore we assume the weights are uniform among all neighbors of an agent.

Figure 7 shows the convergence time of a belief system when the network of agents is the three large-scale complex networks described in Table 2. Results show that the predicted maximum type behavior holds. The convergence time remains constant and of the order of the convergence time of the network of agents, until the mixing time of the network formed by the logic constraints is larger. Then, then total convergence time increases based on the specific topology of the graphs.

Even though the random graph generating models, such at the Erdős-Rényi graphs, the Newmann-Watts graph, and the geometric random graphs, has been proposed to model the dynamics and properties of real large-scale complex networks, for example, rapid mixing or linear convergence of the beliefs. Existing approaches for the computation of such properties in real-world social networks are mainly simulation-based or require extensive computations for the approximation of the spectral properties of the graphs[28,29]. Our analysis based on the properties of the networks provide a structural explanation of why such behavior holds. Specifically, we can explain fast mixing or equivalently linear convergence of beliefs from the existence of highly influential cliques that drive the dynamics of the complete belief system. Figure 9 shows the cumulative influence of the nodes in each of the graphs. That is, the weight an ordered subset of the nodes has on the final value of the beliefs. In this case, since we are considering a strongly connected component, the weights are determined by the values of the left-eigenvalue of the weight matrix corresponding to the eigenvector 1.

Figure 9 shows the existence of highly influential nodes. Particularly, assume we want to study a group of M nodes that affect the 20% of the final opinion. Moreover, assume there is a number K such that after K social interactions the set of beliefs lower are bounded by $\frac{1}{5M}$. Then one can conclude that the graph will be rapid mixing with a mixing time of the order $O(MK\log(1/\epsilon))$ since any two random walks will intersect with probability $1/M$ every K steps. Table 3 shows the values for K and M of the graphs studied in this section.

Discussion

In a recent paper, Friedkin et al.[15] proposed a new model that integrates logic constraints into the evolution opinions of a group of agents in a belief system. Logic constraints among truth statements have a significant impact on the evolution of opinion dynamics. Such restrictions can be modeled as graphs that represent the positive or negative influence the beliefs on specific topics have on others. Starting from this context, we have here approached this model by constructing an extended representation of a belief system, where opinions of all agents on all topics as well as their corresponding initial values are nodes in a larger graph. This larger graph is composed of the Kronecker product of the graphs corresponding to the network of agents and the network of logical constraints respectively.

In this study, we have provided graph-theoretic arguments for the characterization of the convergence properties of such opinion dynamic models based on extensive existing knowledge of convergence and mixing time of random walks on graphs using the theory of Markov chains. We have shown that convergence occurs if every strongly connected component of the network of logic constraints is non-bipartite and every strongly connected component of oblivious agents is non-bipartite as well. Moreover, to be arbitrarily close to their limiting value we require $O((L + H)\log(1/\epsilon))$ time steps. The parameter L is the maximum coupling time for a random walk among the closed strongly connected components of the product graph, and H is the maximum time required for a random walk, that starts on a transient component, to get absorbed by a closed component. Our analysis applies to broad classes of networks of agents and logic constraints for which we have provided bounds regarding the number of nodes in the graphs. Finally, we show that the limiting opinion value is a convex combination of the nodes in the closed strongly connected components and this convergence happens exponentially fast.

Our framework offers analytical tools that deepen our abilities for modeling, control and synthesis of complex network systems, particularly man-made, and can inspire further research in domains where opinion formation and networks interact naturally, such as neuroscience and social sciences. Finally, extending this analysis to other opinion formation models that use different aggregating strategies may require further study of Markov processes and random walks.

Methods

Random Walks, Mixing and Markov chains

Consider a graph $\mathcal{G} = (V, E)$ composed of V nodes with a set of edges E and a compliant associated row-stochastic matrix A. A random walk on the graph $\mathcal{G}$ is the event of a token moving from one node to another according to some probability distribution. This dynamics is captured by a Markov chain $X = (X_k)_0^\infty$ with transition matrix A such that $A_{y,x} = \mathbb{P}\{X_{k+1} = y | X_k = x\}$. This Markov chain is called *ergodic* if it is irreducible and aperiodic. For an ergodic Markov chain, there exists a unique stationary distribution π and is the position of any random walk on the graph as the time goes to infinity, that is $\mathbb{P}\{X_k = j\} \to \pi_j$ as $k \to \infty$. The stationary distribution is invariant to the transition matrix, that is $\pi A = \pi$. It follows immediately that its convergence reduces to analyzing powers of P (Theorem 4.9 in[22]).

Now, define the distance to stationarity as

$$d(k) = \max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{TV}$$

where $\|\mu - \nu\|_{TV}$ is the *total variation distance* between two probability distributions μ and ν on a sigma algebra $\mathcal{F}$ on a sample space Ω defined as

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|.$$

Moreover, define the mixing time of the Markov chain as

$$t_{\text{mix}}(\epsilon) = \min_k \{k : d(k) \leq \epsilon\}.$$

and we say the Markov chain is rapid mixing if $t_{\text{mix}}(\epsilon) = \text{poly}(\log n, \log \frac{1}{\epsilon})$. Finally, it holds that

$$\frac{\lambda_2}{2(1-\lambda_2)} \log\left(\frac{1}{2\epsilon}\right) \leq t_{\text{mix}}(\epsilon) \leq \frac{\log n + \log(1/\epsilon)}{1 - \lambda_2} \tag{3}$$

where λ_2 is the second largest left-eigenvalue of the transition matrix A.

The Coupling Method

Consider two independent Markov chains $X = (X_k)_0^\infty$ and $Y = (Y_k)_0^\infty$, with the same transition matrix P. Then, define the *coupling time* K the smallest k such that $X_k = Y_k$, that is $K = \min_k\{X_k = Y_k\}$. Note, that K is a random variable and it depends on P as well as the initial distributions of the processes X_k and Y_k, usually for analysis purposes we pick $Y_n = \pi$. Finally, lets define the quantity L_P as the maximum expected coupling time of a Markov chain with transition matrix P over all possible initial distributions of the processes X_k and Y_k, then

$$L_P = \max_{u,v} \mathbb{E}[K] \qquad \text{where} \qquad X_0 = u \text{ and } Y_0 = v.$$

In words, this L_P is the maximum expected time it takes for two random walks, with the same transition matrix and arbitrary initial states, to intersect.

Condensation of a Graph

Consider a strongly connected directed graph $\mathcal{G}$. The directed acyclic graph resulting from the contraction of every strongly connected component into a single vertex is called a condensation of $\mathcal{G}$. For example, the condensation of a complete graph is a single vertex. Fig. 10 shows a graph composed of 12 nodes, with three strongly connected components and the resulting condensation of the graph.

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Authors Contributions

AN, AO, and CAU conceived the project, derived the analytical results and wrote the manuscript. CAU performed the numerical simulations and analyzed the data.

Additional information

The authors declare that they have no competing financial interests. Correspondence and requests for materials should be addressed to CAU (cauribe2@illinois.edu)

Table 1: Maximum Expected Convergence Time for the belief system with Logic Constraints for different networks of Agents with n nodes and networks of truth statements with m nodes. The approximated maximum expected convergence time identified as $\approx$ should be understood in terms of the order $O(\cdot)$, that is, an estimate up to constant terms. Additionally, all the estimates provided should be multiplied by the accuracy term $\log(1/\epsilon)$.

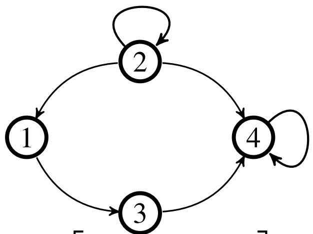
Network of Agents	**Logic Constraints**	**Maximum Expected Convergence Time $\approx$**
Complete	Directed Path	m
Cycle	Directed Path	$\max(n^2, m)$
Cycle	Path	$\max(n^2, m^2)$
Dumbbell Graph	Complete Binary Tree	$\max(n^2, m)$
k-d Cube with Loops	Complete Binary Tree	$\max((1-1/k), m$
k-d Hypercube $\{0,1\}^k$	Complete Binary Tree	$\max(k \log k, m)$
Lovasz Graph $\mathcal{C}_n^k$	Dumbbell	$\max(1 - 1/(kn^2), m^2)$
2-d Grid	Star	$\max(n \log n, m)$
3-d Grid	Two Joined Star	$\max(n^{2/3} \log n, m)$
k-d Grid	Star	$\max(k^2 n^{2/k} \log n, m)$
2-d Torus	2-d Grid	$\max(n^2, m \log m)$
3-d Torus	Star	$\max(n^2, m)$
k-d Torus	k-d Grid	$\max(n^2 k \log k, k^2 m^{2/k} \log m)$
Lollipop	Star	$\max(n^3, m)$
Barbell	Star	$\max(n^3, m)$
Eulerian: d-degree and expansion	Dumbbell	$\max(\lvert E\rvert^2, m^2)$
Eulerian: d-degree, max-degree weights	Dumbbell	$\max(n^2 d, m^2)$
Lazy Eulerian with degree d-degree	Dumbbell	$\max(n\lvert E\rvert, m^2)$
Lamplighter on k-Hypercube	Bolas	$\max(k2^k, m^3)$
Lamplighter on (k,n)-Torus	Bolas	$\max(kn^k, m^3)$
Geometric Random: $\mathcal{G}^d(n,r)$	Bolas	$\max(r^{-2} \log n, m^3)$
Geometric Random: $r = \Omega(\text{polylog}(n))$	Bolas	$\max(\text{polylog}(n), m^3)$
Erdős-Rényi: $\mathcal{G}(n, c/n)$, $c > 1$	Dumbbell	$\max(\log^2 n, m^2)$
Erdős-Rényi: $\mathcal{G}(n, c/n)$, $c > 1$	Newman-Watts	$\max(\log^2 n, \log^2 m)$
Erdős-Rényi: $\mathcal{G}(n, (1+\delta)/n)$, $\delta^3 n \to \infty$	Dumbbell	$\max((1/\delta^3) \log^2(\delta^3 n), m^2))$
Erdős-Rényi: $\mathcal{G}(n, 1/n)$	Dumbbell	$\max(n, m^2)$
Newman-Watts : $\mathcal{G}(n, k, c/n)$, $c > 0$	Path	$\max(\log^2 n, m^2)$
Expander	Undirected Path	m^2
Exponential Random: High temperature	Path	$\max(n^2 \log n, m^2)$
Exponential Random: Low temperature	Path	$\max(\exp(n), m^2)$
Connected Undirected with Metropolis Weights	Expander	n^2
Connected	Expander	$\lvert E\rvert \text{diam}(\mathcal{G})$

Table 2: Datasets of Large-Scale Networks. Description, number of nodes, number of edges, simulated mixing time and upper bound on mixing time of the three datasets used for on the numerical analysis. The upper bound on the mixing time is computed from the second largest eigenvalue bound in equation (3)

Graph	Nodes	Edges	Type	Upper Bound on Mixing Time	Description
`wiki-Vote`[30]	1300	103663	Directed	145	Wikipedia who-votes-on-whom network
`ca-GrQc`[31]	4158	13428	Undirected	12308	Collaboration network of arXiv General Relativity
`ego-Facebook`[32]	3927	88234	Undirected	53546	Social circles from Facebook

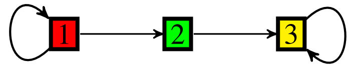
Table 3: F

Graph	K	% of Nodes	M
wiki-Vote[30]	121	10%	25
ca-GrQc[31]	365	11%	1700
ego-Facebook[32]	365	11%	1829



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)



$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

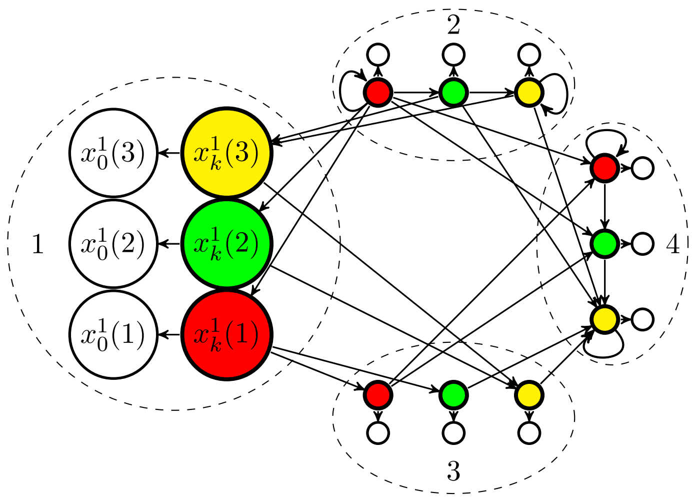


Figure 1: **A belief system with** 4 **agents and** 3 **truth statements**. Agents are represented as nodes/circles, numbered from 1 to 4, and the network of influences among them is shown as edges between nodes. The truth statements or topics are color-coded, that is, the truth statement 1 is represented as a red square. (a) Agent 2 is influenced by its own opinion and agents 4 and 1, agent 1 follows the opinion of agent 3 which in turn follows the opinion of agent 4, agent 4 follows its own opinion only. A possible matrix A for this social network is shown bellow the graph. This indicates that agent 2 assigns a higher weight of $\frac{1}{2}$ to the opinion of agent 1 than the weight it assigns to the opinion of communicated by agent 4. (b) The truth statement 1 is influenced by the belief that statement 2 is true, statement 2 directly follows the belief in statement 3. (c) The aggregated system of beliefs as described by equations 2 composed of agent's interaction graph in (a) and the truth statements in (b). A possible matrix C for this social network is shown below the graph. The belief that the truth statement 1 is true is influenced (with a weight of $\frac{1}{2}$) by the opinion that the truth statement 2 is true.

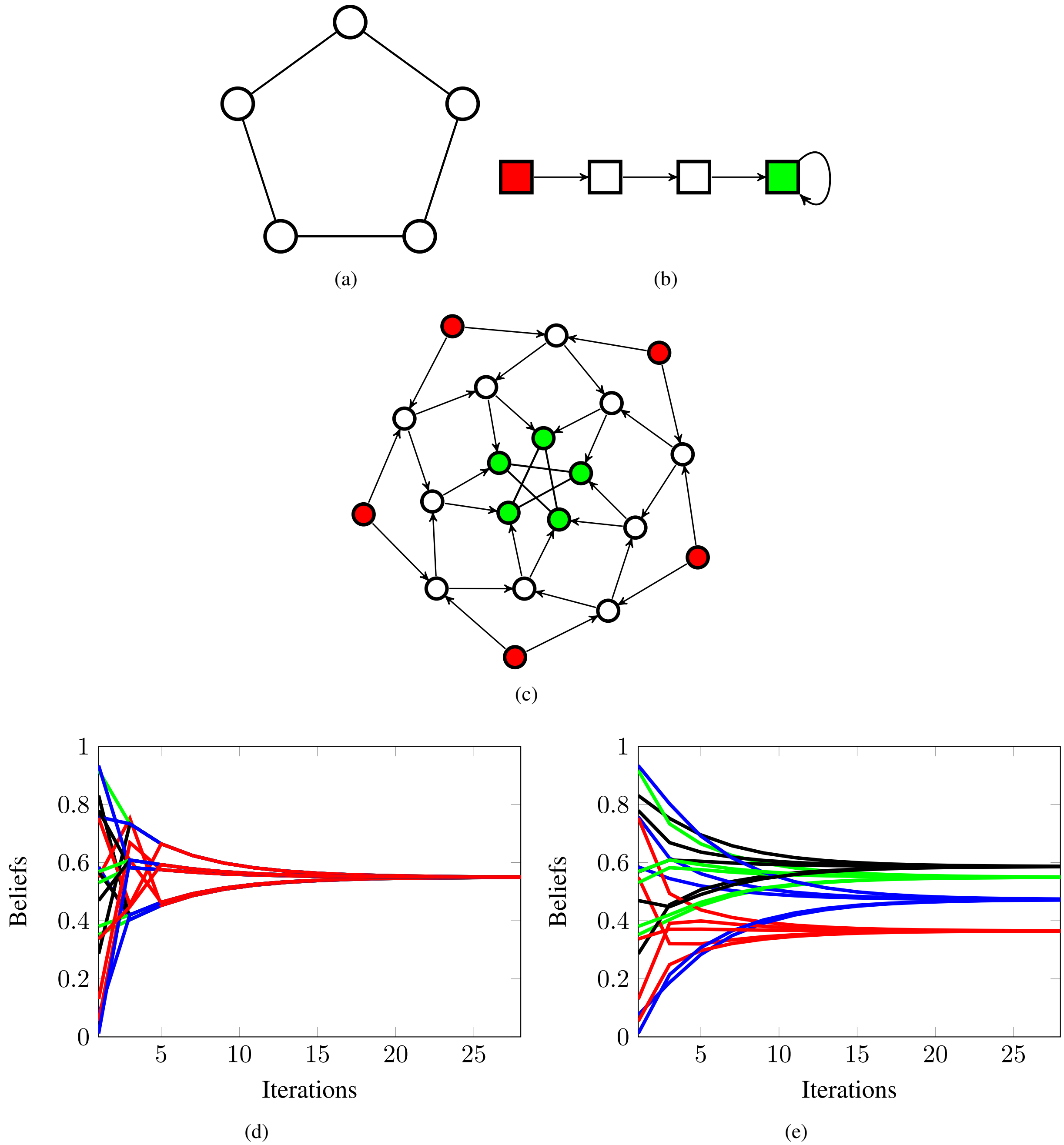


Figure 2: An belief system with agents on a cycle graph and logic constraints on a path graph. (a) A cycle graph $\mathcal{G}$ with 5 oblivious agents. (b) A path graph $\mathcal{T}$ with 4 nodes. (c) The aggregated belief system. that is graph $\mathcal{P}$. (d) The belief dynamics with logic constrains. (e) The belief dynamics with no logic constraints. The beliefs of all agents have been color coded per truth statement. This shows agents reach an agreement or a consensus value on each of the topics separately.

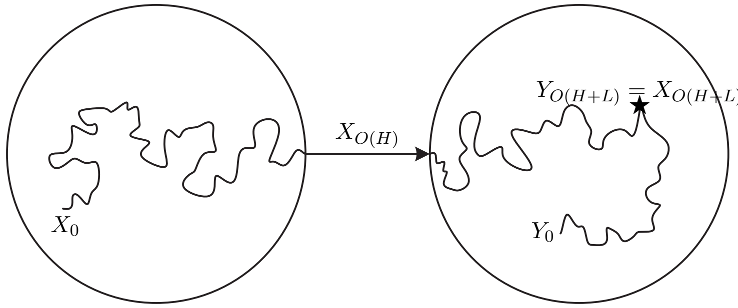


Figure 3: A random walk starts at X_0 in a transient state and evolves according to some transition matrix P, after $O(H)$ time steps (the absorbing time), it gets absorbed into a closed connected component. Then, after $O(L)$ time steps (the coupling time) it crosses path with another random walk Y_k starting at π the stationary distribution of P. Then after $O((L+H)\log 1/\epsilon)$ time steps, the random walk X_0 is arbitrarily close to its limit value.

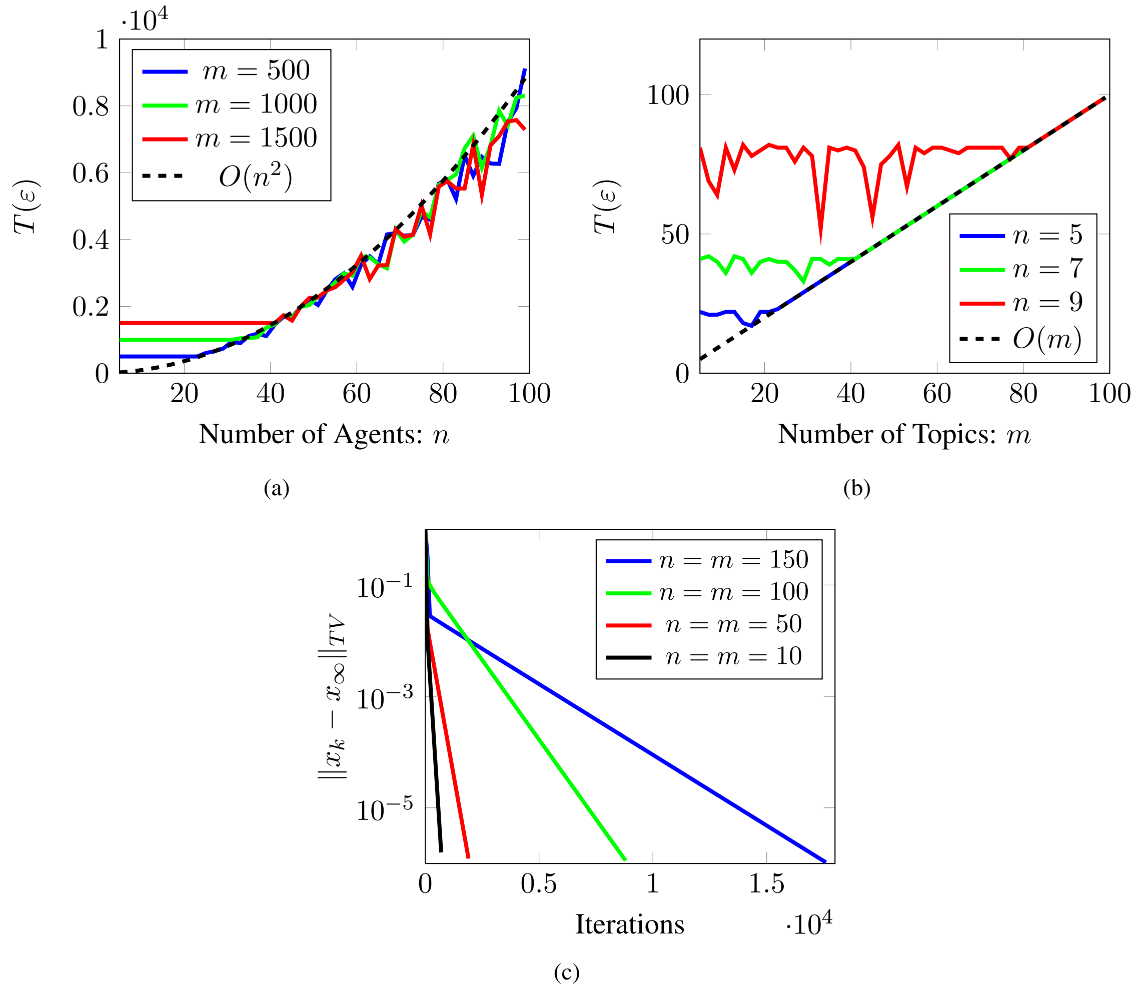


Figure 4: Convergence time for a belief system with a cycle as a social network and a path as logic constraints. Varying the number of agents for an undirected cycle, (b). Varying the number of truth statements for a directed path. (c) The exponential convergence rate of the belief system.

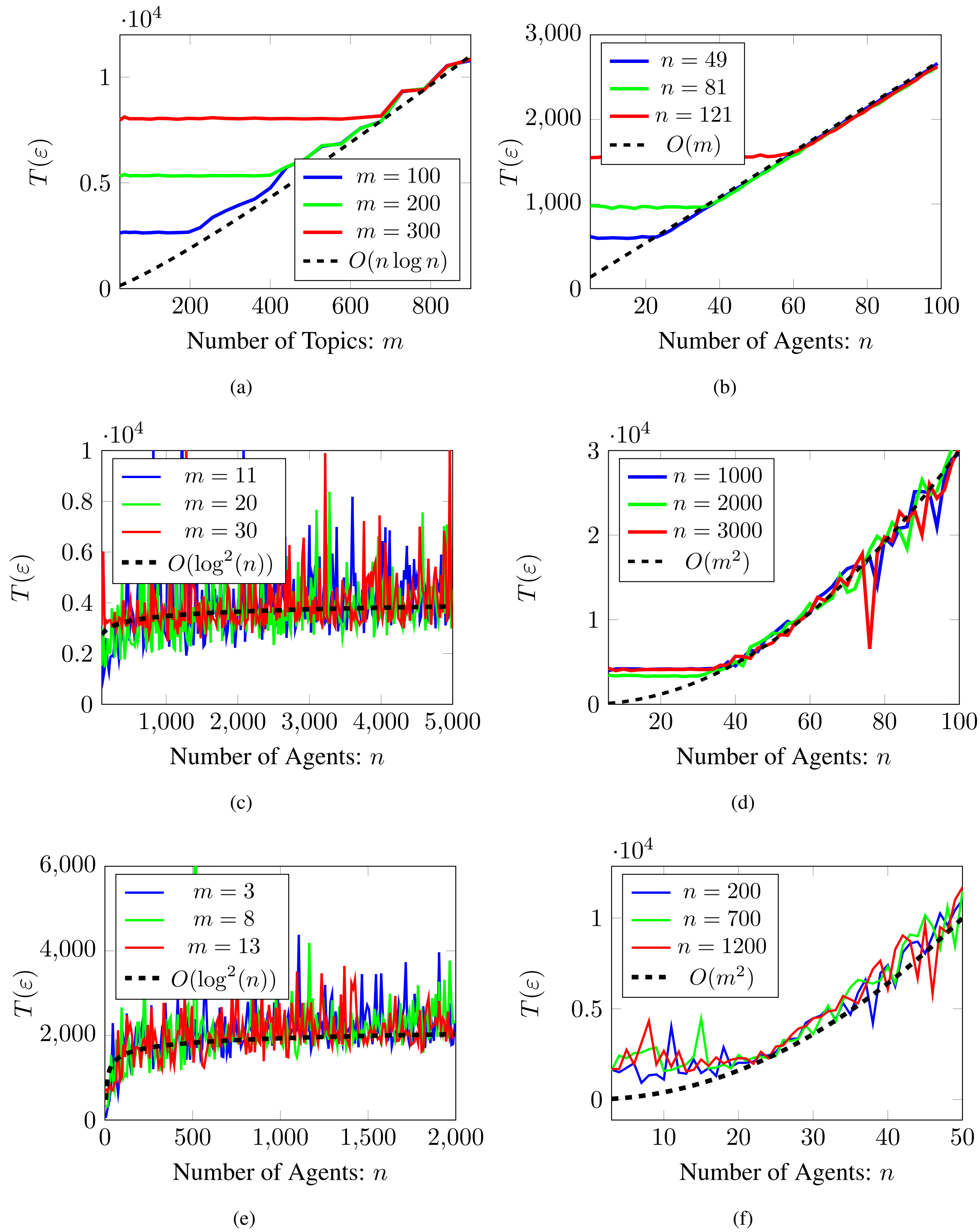
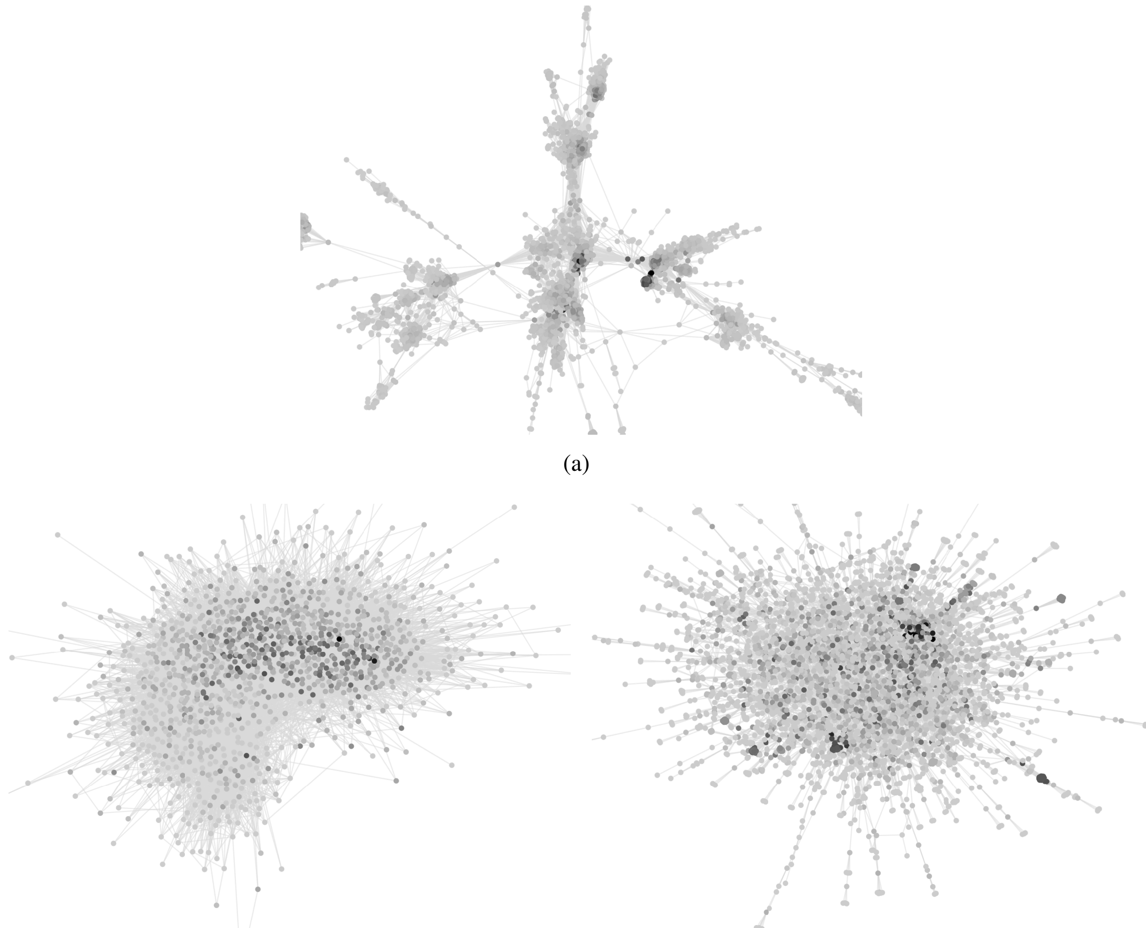


Figure 5: Convergence time or various belief systems. (a,b) Varying the number of agents on a 2d-Grid and the number of truth statements on a star graph. (c,d) Varying the number of agents on a Erdős-Rényi(ER) graph and the number of truth statements on a dumbbell graph. (c) Varying the number of agents on a Newmann-Watts(NW) small-world graph and the number of truth statements on a path graph.



(a)

(b) (c)

Figure 6: Large-Scale Complex Networks from the Stanford Network Analysis Project (SNAP). (a) The `wiki-Vote` graph, each node represents a Wikipedia administrator and an directed edge represents a vote used for promoting a user to admin status. (b) The `ego-Facebook`, nodes are anonymized users from Facebook and edges indicated friendship status between them. (c) The `ca-GrQc` graph is a collaboration network from arXiv authors with paper submitted to General Relativity and Quantum Cosmology category, edges indicated co-authorship of a manuscript. The gray scale in the node colors shows the relative social power according to the left-eigenvector corresponding to the eigenvalue 1.

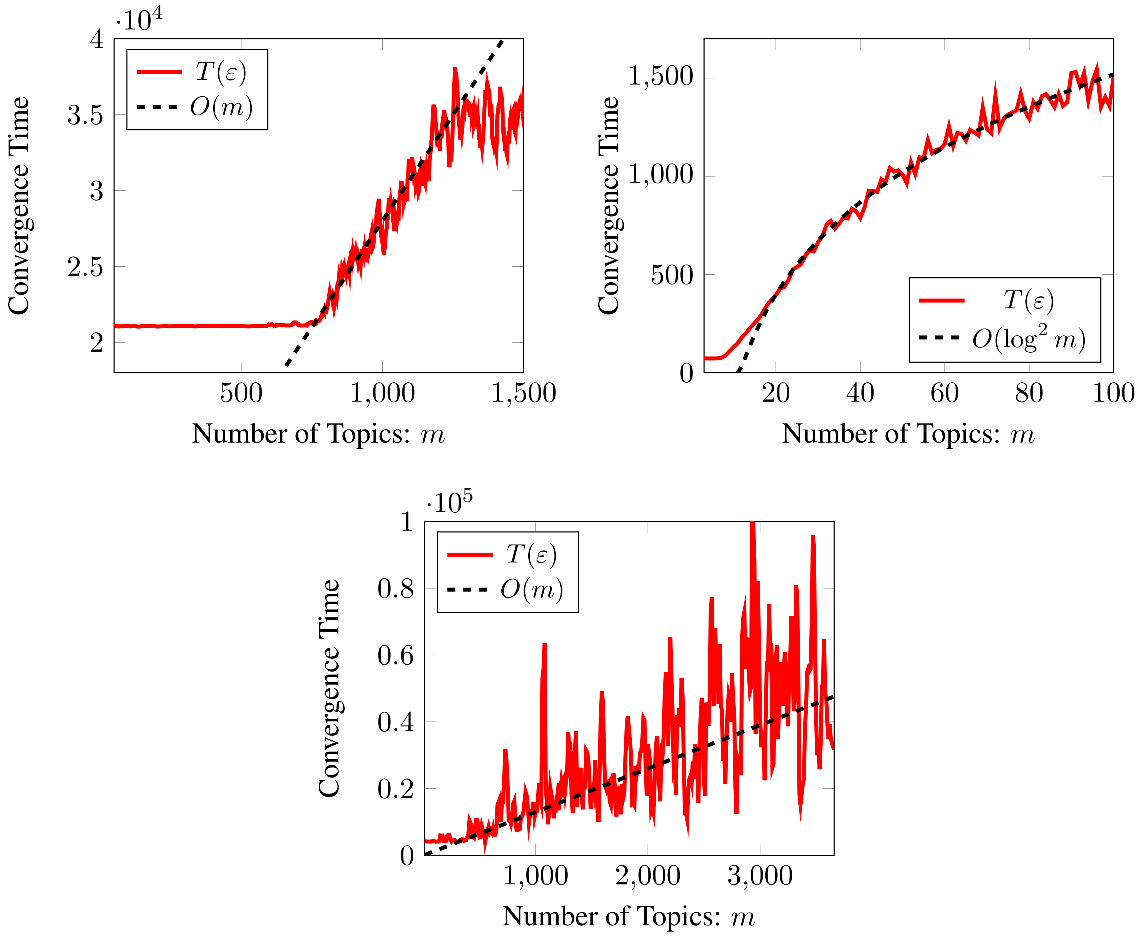


Figure 7: Convergence Time of Belief System with Large-Scale Complex Networks. (a) The social network is the `ego-Facebook` graph, and the logic constraints form a complete binary tree (CBT) with an increasing number of topics. (b) The social network is the `wiki-Vote` graph and the logic constraints form Newmann-Watts small-world graph with an increasing number of topics. (c) The social network is the `ca-GrQc` arXiv collaboration graph, and the logic constraints form an Erdős-Rényi graph with an increasing number of topics.

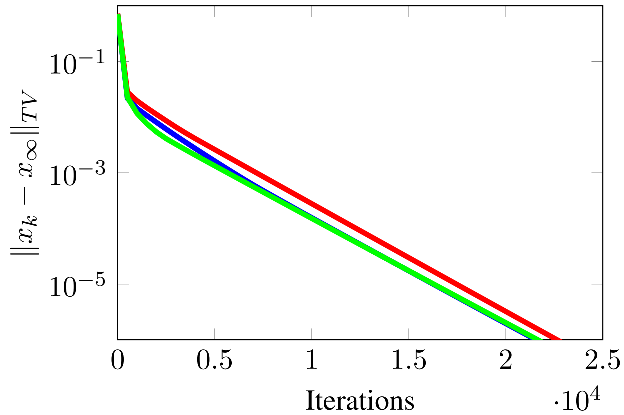


Figure 8: Total variation distance between the beliefs and its limiting value as the number of iteration increases. Results are shown for a particular subset of randomly selected agents.

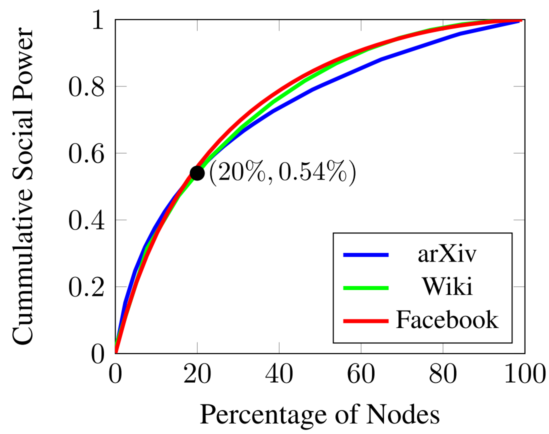


Figure 9: Cumulative Social Power of the agents. Each of the nodes in the graphs considered has some weight in the final value achieved by the belief system. In all three cases, the 20% most important nodes accounts for the 50% of the final value.

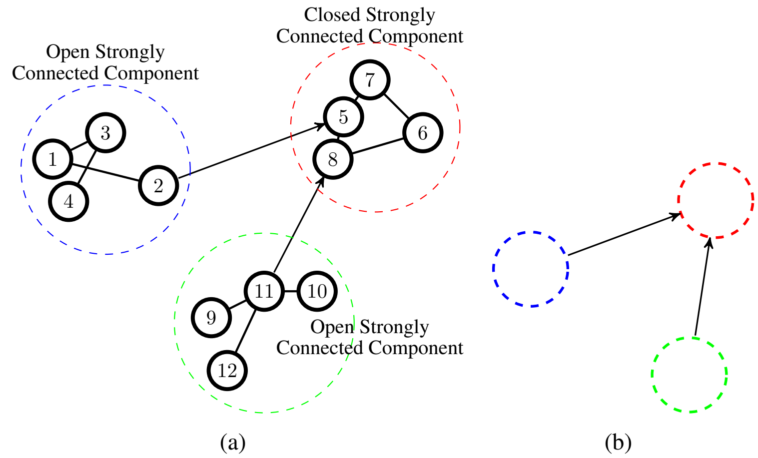
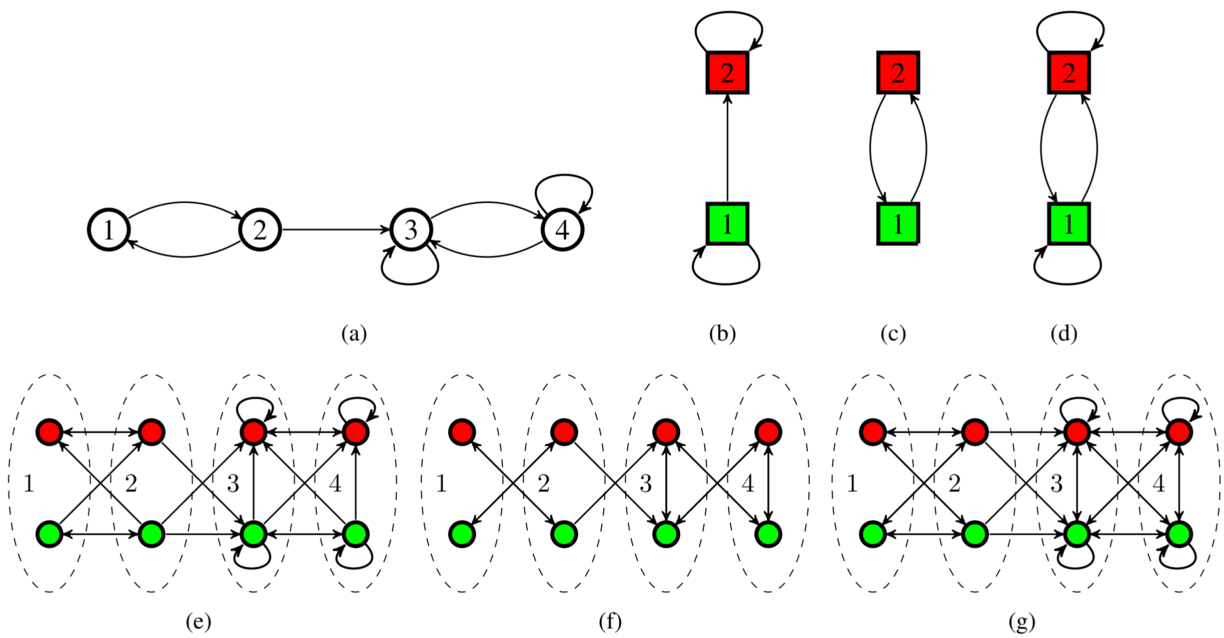


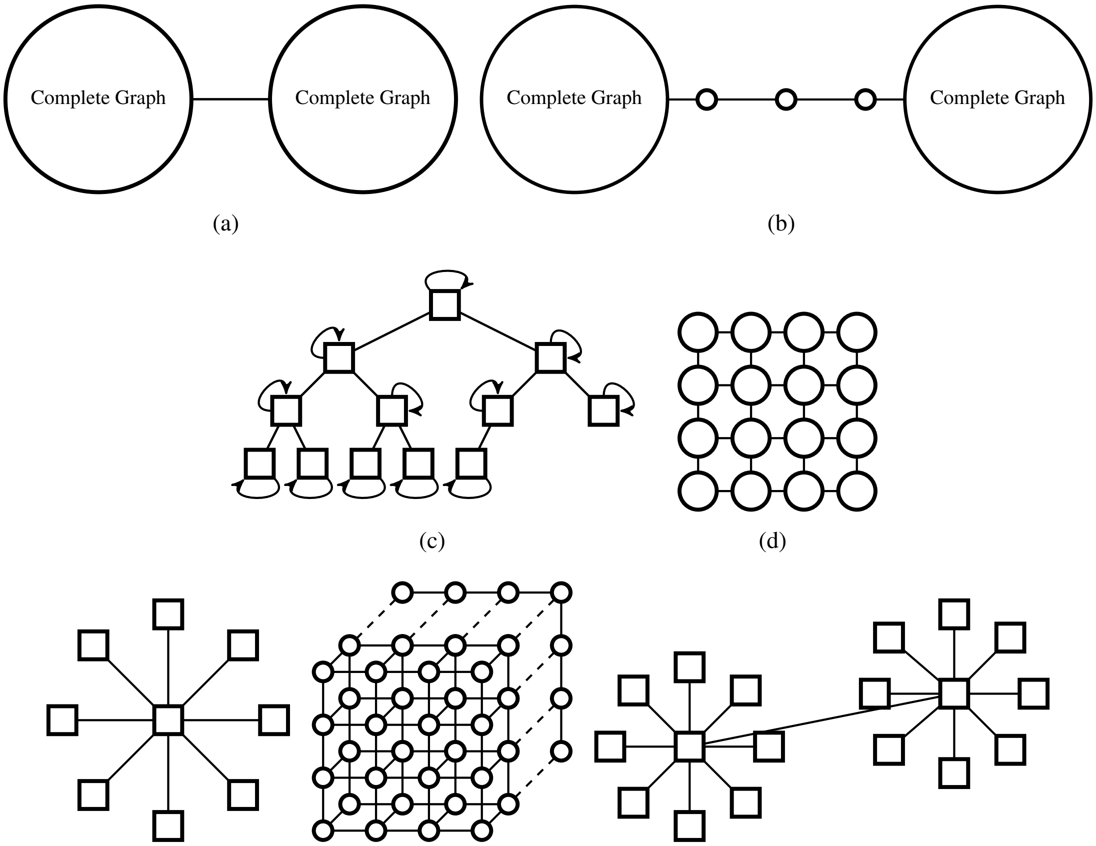
Figure 10: Strongly connected components and Condensation of a Graph. (a) A graph with 12 nodes and 3 strongly connected components. The strongly connected component composed by nodes 5, 6, 7 and 8 is closed since it only has incoming edges, one from node 2 and one from node 3. (b) The condensation of the graph in (a). Each strongly connected component is taken as a node in the condensation.

Supplementary Material: Graph-Theoretic Analysis of Belief Systems Dynamics under Logic Constraints

Angelia Nedić, Alex Olshevsky and César A. Uribe



Suppementary Figure 1: The influence of the logic constraints in the resulting aggregated belief system. (a) The network of agents, where agent 1 follows the opinion of agent 2, agent 2 is influenced by agent 1 and 3, agent 3 is influenced by its own opinion and the opinion of agent 4 and agent 4 is influenced by agent 3 as well as its own. (b) The opinion on statement 1 is influenced by the belief on statement 2. (c) The opinion on statements 2 and 1 follow each other. (d) The opinion on statements 2 and 1 influence each other (e-g) The belief systems with the network of agents in (a) and logic constraints in (b-d).

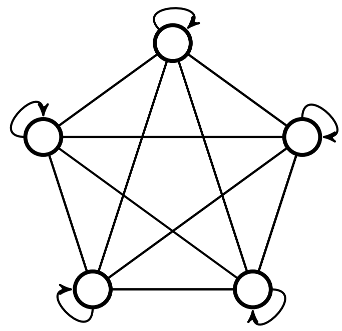


(a) (b)

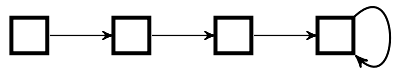
(c) (d)

(e) (f) (g)

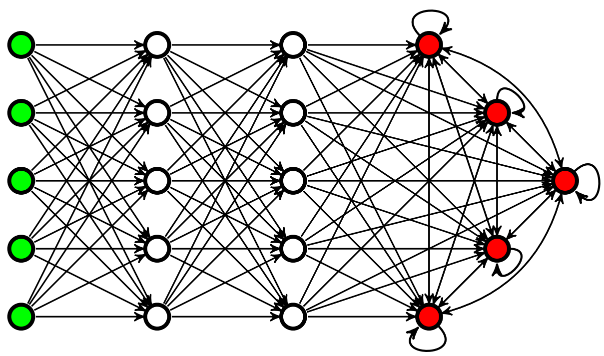
Suppementary Figure 2: Examples of common graph families. (a) Dumbbell graph, two complete graph connected by an edge. (b) Bolas graph, two complete graphs connected by a path. (c) Complete binary tree. (d) 2-d grid or lattice. (e) Star graph. (f) 3-d grid. (g) Two star graph connected on their centers.



(a)

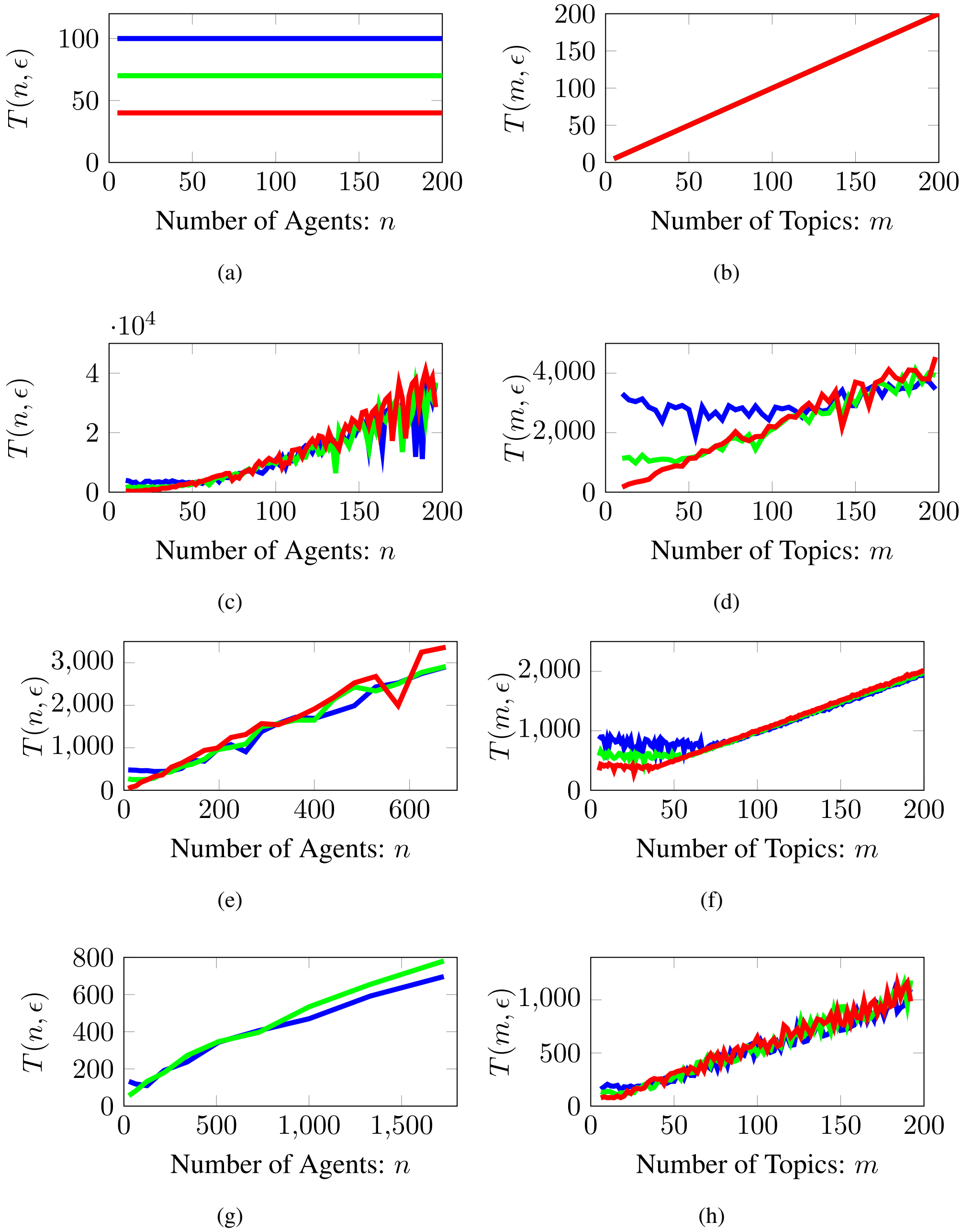


(b)

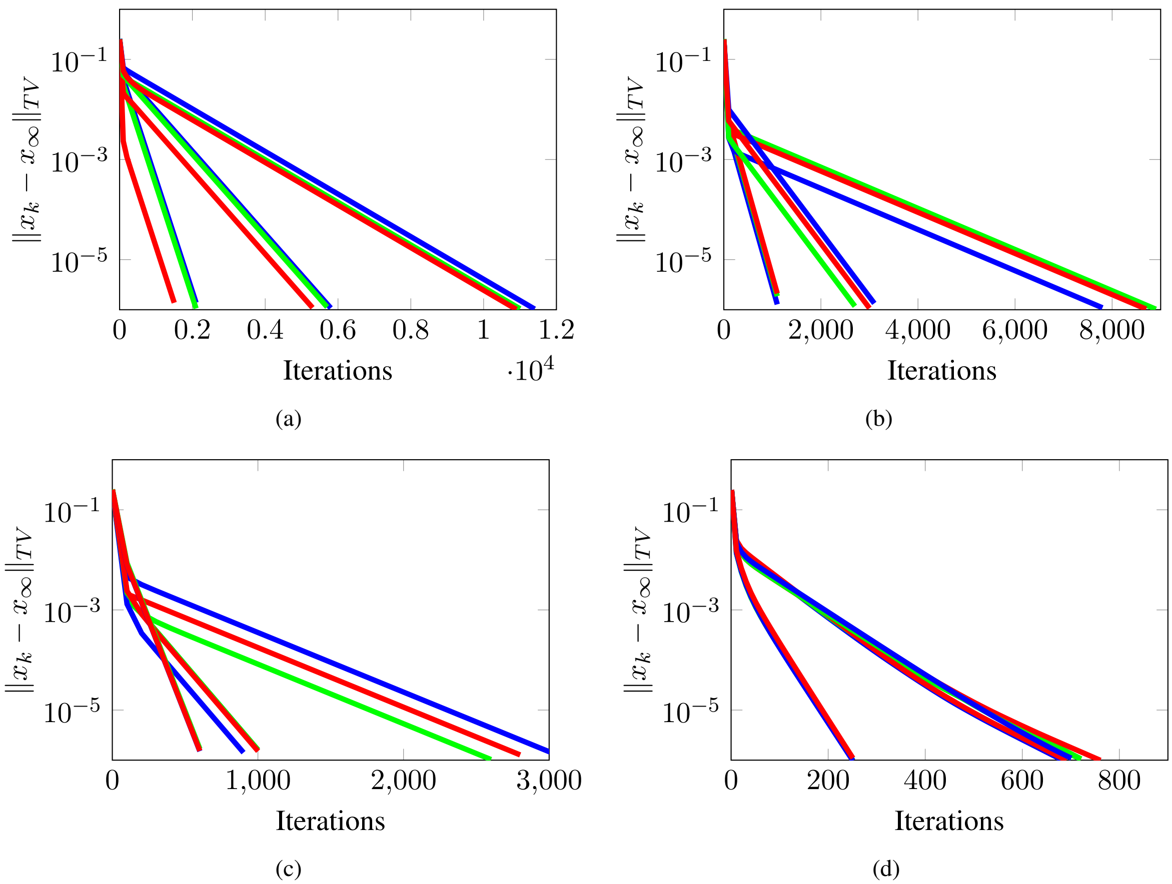


(c)

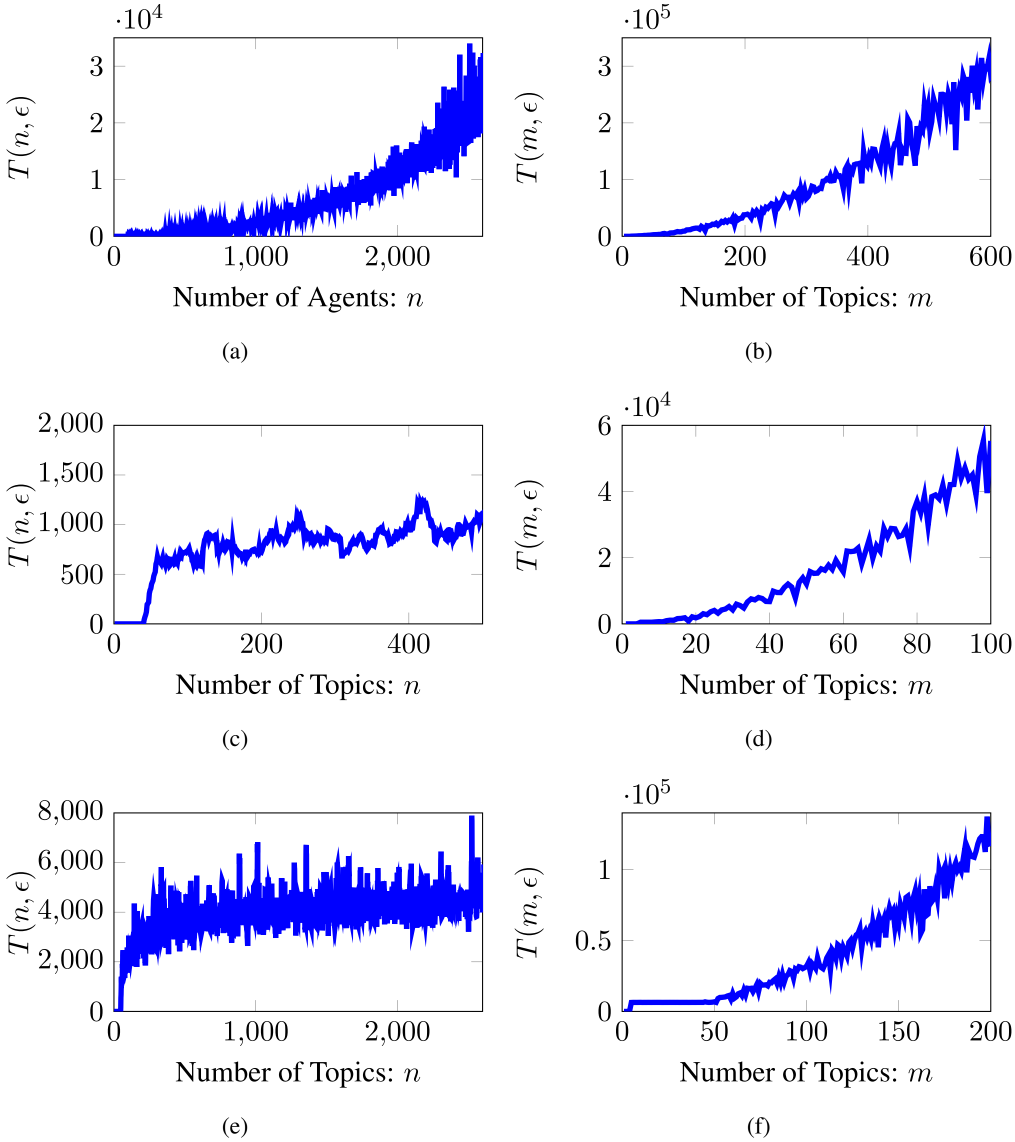
Suppementary Figure 3: Two examples of graph product between a complete graph/cycle graph with 5 nodes and a path graph of 4 logical belief constraints. (a) Shows a complete graph with 5 agents. (b) Shows a path graph with 5 nodes. (c) Shows a cycle graph with 5 agents. (d) The resulting product graph between the complete graph in (a) and the path graph in (b). (e) The The resulting product graph between the complete graph in (c) and the path graph in (b).



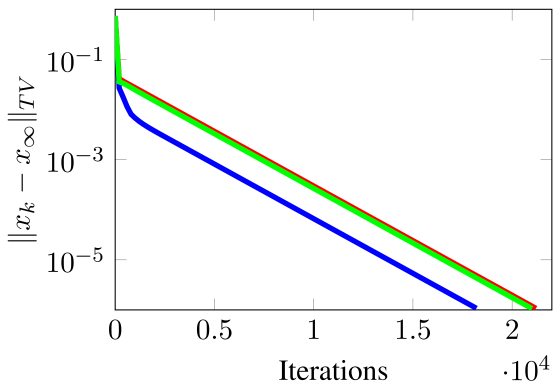
Suppementary Figure 4: Convergence time for different examples of networks of agents and network of truth statements in a belief system. Varying the number of agents for a: (a) complete graph, (c) undirected cycle, (c) dumbbell graph, (e) 2-d grid, (g) 3-d grid. Varying the number of truth statements for a: (b) directed path, (d) complete binary tree, (f) star graph, (h) two joined star graphs.



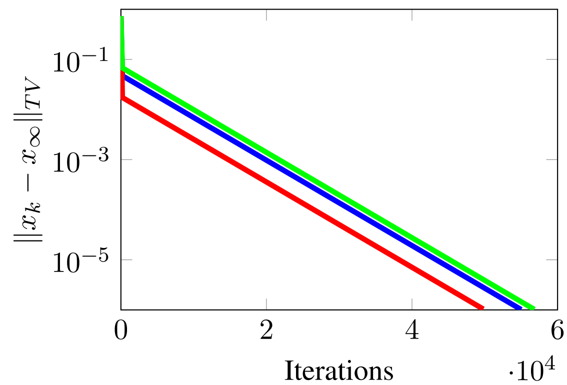
Suppementary Figure 5: Exponentially Fast Convergence of the Belief System. Distance to the final value of a belief system with: (a) a directed cycle network of agents and a directed path of truth statements, (b) a dumbbell network of agents and a complete binary tree of truth statements, (c) a 2-d grid of agents and a star network of truth statements, (d) a 3-d grid of agents and a two-joined star network of truth statements.



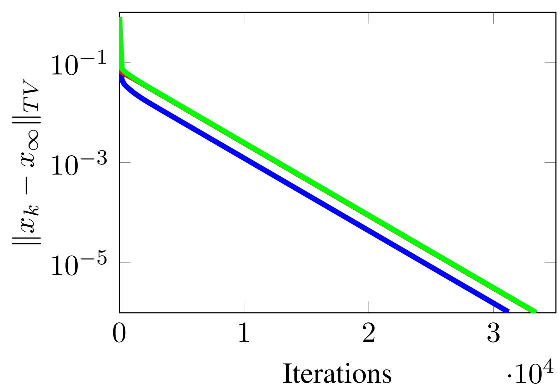
Suppementary Figure 6: Convergence time dependency for random graphs. (a,b) Network of Agents is a Geometric Random Graph and the network of truth statements is the Bolas graph, (c-d) The network of agents is an Erdős-Rényi random graph and the network of truth statements is a Dumbbells graph, (e-f) the network of agents is a Small-World graph and the network of agents is an undirected path.



(a)

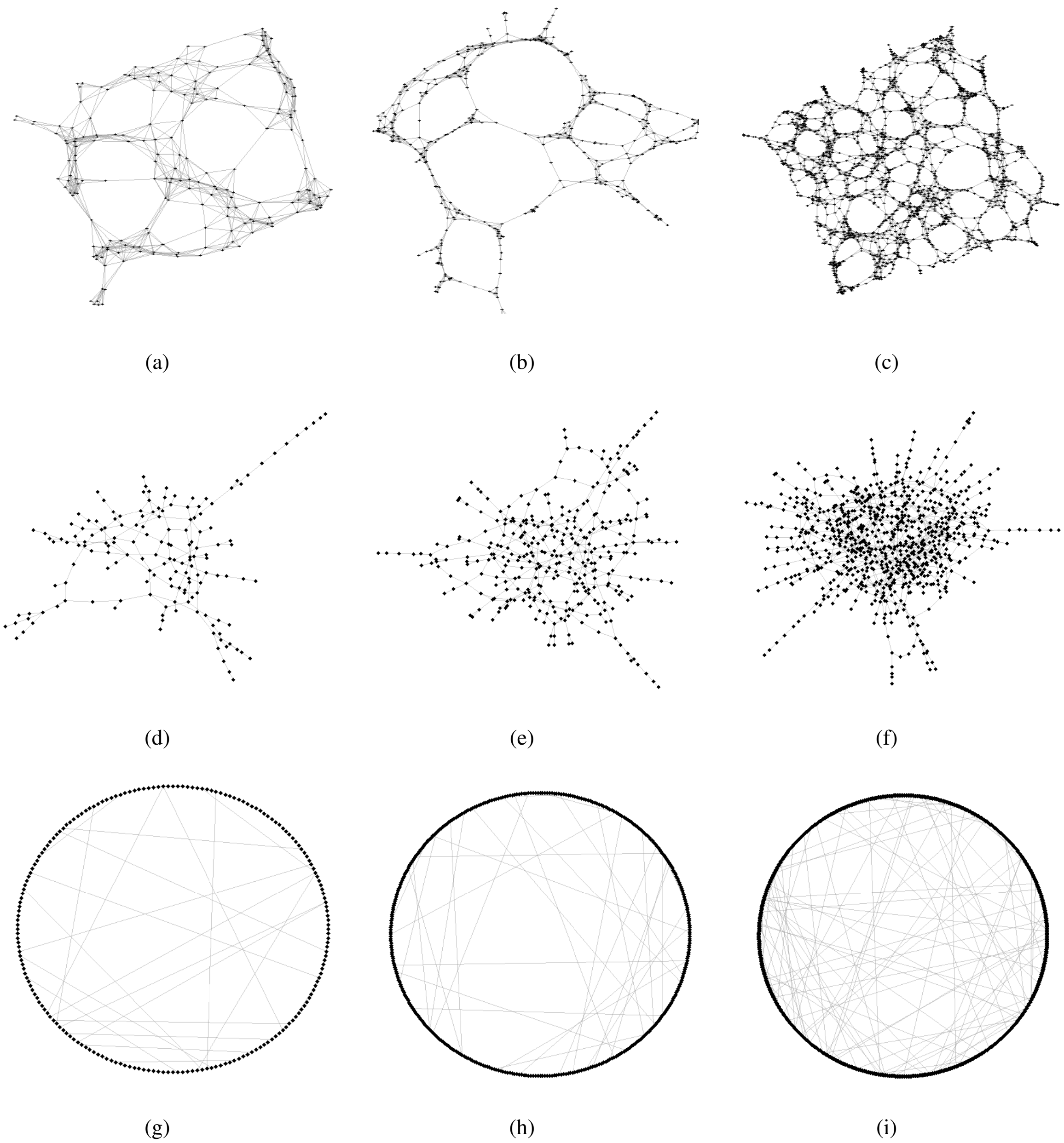


(b)



(c)

Suppementary Figure 7: Geometric convergence of the Believe system with random networks of agents. (a) Distance to the final value for a network of 200 agents modeled as a Geometric Random graph and a network of 150 truth statements modeled as a Bolas graph, (b) Distance to the final value for a network of 500 agents modeled as a Erdős-Rényi Random graph and a network of 100 truth statements modeled as a dumbbell graph, (c) Distance to the final value for a network of 500 agents modeled as a Small-World Random graph and a network of 100 truth statements modeled as a undirected path graph.



Suppementary Figure 8: Samples of Random graphs. (a-c) Geometric Random Graphs with 200, 400 and 2000 nodes respectively. A geometric random graph is the result of randomly placing n nodes in a metric space and adding an edge between two nodes if and only if their distance is smaller that certain radius r[1]. (d-f) Erdős-Rényi Random Graphs with 200, 400 and 1000 nodes respectively. An $\mathcal{G}_{n,p}$ Erdős-Rényi graph is the result of adding edges independently with probability p to a set of n nodes[2]. (g-i) Small-World Random Graphs with 200, 400 and 1000 nodes respectively. The Newman-Watts graph $H_{n,k,p}$ is the random graph obtained from a (n, k)-ring graph by independent adding edges with probability p[3].

Table 1: Maximum Expected Convergence Time for the belief system with Logic Constraints for different networks of Agents with n nodes and networks of truth statements with m nodes.

Network Topology	**Mixing Time**
Complete	$O(m \log 1/\epsilon)$
Cycle	$O(n^2 \log 1/\epsilon)$
Path [4]	$O(n^2 \log 1/\epsilon)$
Dumbbell Graph [5]	$O(n^2 \log 1/\epsilon)$
Complete Binary Tree [6–8]-Section 5.3.4	$O(m \log 1/\epsilon)$
k-d Cube with Loops [9]	$O((1-1/k) \log 1/\epsilon)$
k-d Hypercube $\{0,1\}^k$ [8]-Section 5.3.3	$O(k \log k \log 1/\epsilon)$
Lovasz Graph $\mathcal{C}_n^k$ [9]	$O((1-1/(kn^2)) \log 1/\epsilon)$
2-d Grid [10,11]	$O(n \log n \log 1/\epsilon)$
Star Graph [12]	$O(m \log 1/\epsilon)$
3-d Grid [10,11]	$O(n^{2/3} \log n \log 1/\epsilon)$
Two Joined Star Graphs	$O(m \log 1/\epsilon)$
k-d Grid [10,11]	$O(k^2 n^{2/k} \log n \log 1/\epsilon)$
2-d Torus [13]	$O(n^2 \log 1/\epsilon)$
3-d Torus [13]	$O(n^2 \log 1/\epsilon)$
k-d Torus [13]	$O(n^2 k \log k) \log 1/\epsilon)$
Lollipop [14]	$O(n^3 \log 1/\epsilon)$
Barbell [14]	$O(n^3 \log 1/\epsilon)$
Eulerian: d-degree and expansion [15]	$O(\|E\|^2 \log 1/\epsilon)$
Lazy Eulerian with degree d-degree [16]	$O(n\|E\| \log 1/\epsilon)$
Eulerian: d-degree, max-degree weights and expansion [15]	$O(n^2 d \log 1/\epsilon)$
Lamplighter on k-Hypercube [13]	$O(k2^k \log 1/\epsilon)$
Lamplighter on (k,n)-Torus [13]	$O(kn^k \log 1/\epsilon)$
Bolas Graph [17]	$O(m^3 \log 1/\epsilon)$
Geometric Random Graph: $\mathcal{G}^d(n,r)$ [18]	$O(r^{-2} \log n \log 1/\epsilon)$
Geometric Random Graph: $\mathcal{G}^2(n, \Omega(\text{polylog}(n)))$ [19]	$O(\text{polylog}(n) \log 1/\epsilon)$
Erdős-Rényi: $\mathcal{G}(n, c/n)$, $c > 1$ [20,21]	$O(\log^2 n \log 1/\epsilon)$
Erdős-Rényi: $\mathcal{G}(n, (1+\delta)/n)$, $\delta^3 n \to \infty$ [22,23]	$O((1/\delta^3) \log^2(\delta^3 n) \log 1/\epsilon)$
Erdős-Rényi: $\mathcal{G}(n, 1/n)$ [24]	$O(n \log 1/\epsilon)$
Newman-Watts (small-world) Graph [25]	$O(\log^2 n \log 1/\epsilon)$
Expander Graph [26]	$O(m^2 \log 1/\epsilon)$
Exponential Random Graph: High temperature [27]	$O(n^2 \log n \log 1/\epsilon)$
Exponential Random Graph: Low temperature [27]	$O(\exp(n) \log 1/\epsilon)$
Any Connected Undirected Graph [28]	$O(n^2 \log 1/\epsilon)$
Any Connected Graph	$O(\|E\| \text{diam}(\mathcal{G}) \log 1/\epsilon)$

Supplementary Equation

$$P = \left[\begin{array}{c|c}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 \\
0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix} & \frac{1}{2} \cdot \boldsymbol{I}_{12} \\
\hline
\mathbf{0}_{12} & \boldsymbol{I}_{12}
\end{array}\right]. \tag{1}$$

Supplementary Note 1: Previous results

A belief system with logic constraints will converge to equilibrium if and only if either $\lim_{k\to\infty}(\Lambda A)^k = 0$ or $\lim_{k\to\infty}(\Lambda A)^k \neq 0$ and $\lim_{k\to\infty} C^k$ exists[29]. Moreover the limiting distribution is characterized as a solution of

$$X_\infty = \Lambda A X_\infty C' + (I - \Lambda) X_0.$$

A more detailed approach takes into account the characterization of those agents that do not have a persistent influence of its initial opinions. Define the set of *oblivious* agents as those agents i such that $\lambda^i = 1$[30], i.e. maximally open to new opinions with no dependence on the initial beliefs, and is not influenced by other agent j whose $\lambda^j < 1$. For this definition, then we can represent the matrices A and Λ with a block structure as

$$A = \begin{bmatrix} A^{11} & A^{12} \\ 0 & A^{22} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \Lambda^{11} & 0 \\ 0 & I \end{bmatrix}$$

where A^{22} represent the subgraph of oblivious agents. Thus, the belief system is convergent if and only if $\lim_{k\to\infty} C^k$ exists and denoted as C^* and $\lim_{k\to\infty}(A^{22})^k =$ exists and denoted as A^{22}_*[30]. If the above conditions hold then

$$X_\infty = \begin{bmatrix} (I - \Lambda^{11} A^{11} \otimes C)^{-1} & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} (I - \Lambda^{11}) \otimes I_m & (\Lambda^{11} A^{12} A^{22}) \otimes C C_* \\ 0 & A^{22}_* \otimes C_* \end{bmatrix} X_0.$$

Supplementary Note 2: The Kronecker Product of Graphs

Definition 1 ([31]) *Let A be a $m \times n$ matrix, and C be a $p \times q$ matrix, the* ***Kronecker product*** *$A \otimes C$ is the $mp \times nq$ matrix defined as:*

$$A \otimes C = \begin{bmatrix} a_{11}C & \dots & a_{1n}C \\ \vdots & \ddots & \vdots \\ a_{m1}C & \dots & a_{mn}C \end{bmatrix}$$

or explicitly

$$A \otimes C = \begin{bmatrix} a_{11}\begin{bmatrix} c_{11} & \dots & c_{1q} \\ \vdots & \ddots & \vdots \\ c_{p1} & \dots & c_{pq} \end{bmatrix} & \dots & a_{1n}\begin{bmatrix} c_{11} & \dots & c_{1q} \\ \vdots & \ddots & \vdots \\ c_{p1} & \dots & c_{pq} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ a_{m1}\begin{bmatrix} c_{11} & \dots & c_{1q} \\ \vdots & \ddots & \vdots \\ c_{p1} & \dots & c_{pq} \end{bmatrix} & \dots & a_{mn}\begin{bmatrix} c_{11} & \dots & c_{1q} \\ \vdots & \ddots & \vdots \\ c_{p1} & \dots & c_{pq} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11} & \dots & a_{11}c_{1q} & \dots & a_{1n}c_{11} & \dots & a_{1n}c_{1q} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{11}c_{p1} & \dots & a_{11}c_{pq} & \dots & a_{1n}c_{p1} & \dots & a_{1n}c_{pq} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}c_{11} & \dots & a_{m1}c_{1q} & \dots & a_{mn}c_{11} & \dots & a_{mn}c_{1q} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1}c_{p1} & \dots & a_{m1}c_{pq} & \dots & a_{mn}c_{p1} & \dots & a_{mn}c_{pq} \end{bmatrix}$$

Next, we will enumerate some useful properties of the Kronecker product.

1. Bilinearity and associativity: for matrices A, B and C, and a scalar k, it holds:

$$\begin{aligned} A \otimes (B + C) &= A \times B + A \otimes C \\ (A + B) \times C &= A \otimes C + B \otimes C \\ (kA) \otimes C &= A \otimes (kB) = k(A \otimes B) \\ (A \otimes B) \otimes C &= A \otimes (B \otimes C) \end{aligned}$$

2. Non-Commutative: In general $A \times B \neq B \otimes A$. However, there exists commutation matrices P and Q such that:

$$A \otimes B = P(B \otimes A)Q$$

   and if A and B are square matrices then $P = Q'$

3. Mixed-product property: for matrices A, B, C and D:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

Now, we are ready to introduce the Kronecker product of graphs.

Definition 2 *The Kronecker (also known as categorical, direct, cardinal, relational, tensor, weak direct or conjunction) product $G_1 \otimes G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_1)$ is a graph $G = (V, E)$ where $V = V_1 \times V_2$ and $|V| = |V_1||V_2|$; and $(u, u') \to (v, v') \in E$ if and only if $u \to v \in E_1$ and $u' \to v' \in E_2$. Moreover, it is equivalent to the Kronecker product of the adjacency matrices of G_1 and G_2.*

Theorem 1 (Theorem 5.29 in[32]) *Let $\mathcal{G}$ and $\mathcal{H}$ be graphs with at least one edge. Then $\mathcal{G} \otimes \mathcal{H}$ is connected if and only if both $\mathcal{G}$ and $\mathcal{H}$ are connected, and at least one of them is nonbipartite. Furthermore, if both $\mathcal{G}$ and $\mathcal{H}$ are connected and bipartite then $\mathcal{G} \otimes \mathcal{H}$ has exactly two connected components.*

Supplementary Note 3: Main Technical Results

We start with a technical lemma about the strongly connected components of a product of two graphs.

Lemma 2 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two graphs with n_1 and n_2 strongly connected components respectively. Every strongly connected component of the tensor product $\mathcal{G}_1 \otimes \mathcal{G}_2$ is either: the tensor product of a strongly connected component of $\mathcal{G}_1$ and a strongly connected component of $\mathcal{G}_2$ where at most one of them is bipartite; or a component resulting from the tensor product if both of the factors are bipartite.*

Proof. Let A_1 and A_2 denote the weight matrices for graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ restrictively. Initially, note that there exists two permutation matrices P_1 and P_2 such that we can rearrange both matrices A_1 and A_2 into block diagonal matrices where each of the blocks represents a strongly connected component. Then

$$P_1' A_1 P_1 = \begin{bmatrix} A_1^1 & * & * & * \\ 0 & A_1^2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & A_1^{n_1} \end{bmatrix} \quad \text{and} \quad P_2' A_2 P_2 = \begin{bmatrix} A_2^1 & * & * & * \\ 0 & A_2^2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & A_2^{n_2} \end{bmatrix}.$$

Moreover, define $P = P_1 \otimes P_2$ and by properties of the Kronecker product, c.f. definition 1 it follows that

$$(P_1' A_1 P_1) \otimes (P_2' A_2 P_2) = P'(A_1 \otimes A_2)P$$

where P is also a permutation matrix and

$$P'(A_1 \otimes A_2)P = \begin{bmatrix} A_1^1 \otimes A_2 & * & * \\ 0 & \ddots & * \\ 0 & \cdots & A_1^{n_1} \otimes A_2 \end{bmatrix}$$

Finally, by property 4 in definition 1 there exists a permutation matrix Q such that

$$Q'(P'(A_1 \otimes A_2)P)Q = \begin{bmatrix} A_2 \otimes A_1^1 & * & * \\ 0 & \ddots & * \\ 0 & \cdots & A_2 \otimes A_1^{n_1} \end{bmatrix}$$

$$= \begin{bmatrix} A_2^1 \otimes A_1^1 & * & * & * & * & * & * \\ 0 & \ddots & * & * & * & * & * \\ 0 & \cdots & A_2^{n_2} \otimes A_1^1 & * & * & * & * \\ 0 & \cdots & 0 & \ddots & * & * & \\ 0 & \cdots & \cdots & 0 & A_2^1 \otimes A_1^{n_1} & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \ddots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & A_2^{n_2} \otimes A_1^{n_1} \end{bmatrix}$$

Therefore, every block in the diagonal is the product of two strongly connected components, one from each graph. If both factors are bipartite, then the resulting block will not be connected. □

We are now ready to state our main technical result regarding the expected mixing time of Markov Chains whose transition probability matrix is a Kronecker product of two stochastic matrices.

Theorem 3 *Suppose $\mathcal{G}(P)$ has no bipartite closed strongly connected component. Let L be the maximum expected coupling time of a random walk in a closed strongly connected component. Moreover, let H be maximum expected time for a random walk on an open strongly connected component to get absorbed into a closed one. Then, the belief system described in equation* (2) *is ϵ close to its limiting value after $O((L+H)\log 1/\epsilon)$ steps.*

Proof. We use the coupling method to bound the convergence time of the belief system[33]. Initially, we show that all opinions x_k^i, such that i lies in a closed strongly connected component, will converge to some stationary point.

Let i belong to a closed strongly connected component S and let P_S be the matrix obtained by looking at the minor of P corresponding to entries in S. S is closed and P_S is row-stochastic, then the Perron-Frobenius theory tells us there exists some vector π_S such that

$$\pi_S' P_S = \pi_S'.$$

Now, define two independent random walks $X = (X_k)_0^\infty$ and $Y = (Y_n)_0^\infty$ with the same transition matrix P_S. X starts from a distribution π_S, and Y from some other arbitrary stochastic vector v. Moreover, *couple* the processes Y and X by defining a new process W such that

$$W_k = \begin{cases} Y_k, & \text{if } k < K \\ X_k, & \text{if } k \geq K \end{cases}$$

where $K = \min\{k : Y_k = X_k\}$ is called the *coupling time*. Each walk moves according to P_S, so if we correlate them by moving them together after they intersect, we have not changed the fact that, individually, they move according to P_S. With this construction of the coupling we have that

$$\|v' P_S^k - \pi_S\|_1 \leq \max_{u,v} \mathbb{P}_{u,v}\{K > k\}$$

and by the Markov inequality

$$\|v' P_S^k - \pi_S\|_1 \leq \frac{\max_{u,v} \mathbb{E}[K]}{k}.$$

We have that $L = \max_{u,v} \mathbb{E}[K]$ is the maximum expected time it takes for two random walks in the source S to intersect starting from two distinct nodes u and v. Thus, after $O(L \log 1/\epsilon)$ steps, $\|v^T P^T - \pi\|_1 < \epsilon$, for any v and u.

We have shown that x_k^i for i in a closed strongly connected component S converges to $\pi_S' x_0^S$ at a geometric rate. Here x_0^S stacks those x_0^i that belong to S.

Now, consider the case where i belongs to an open strongly connected component. Let M be the set of states in such connected component. Stacking up x_k^i over i in M into the vector x_k^M, observe that

$$x_{k+1}^M = Z x_k^M + R y_k$$

where Z is strongly connected and substochastic, meaning some rows add up to less than 1. The entries of y_k come from nodes in other strongly connected components. Let us assume for now that they belong to closed strongly connected nodes and the matrix R represents how they influence the nodes in M. We have already shown that y_k converges; call its limit y_∞.

Now consider a random walk that moves around M according to Z; the moment it steps out of M to a closed strongly connected component we say it hits an absorbing state.

Let q_k^i be the probability the walk is at state i in M at time k. Then

$$q_{k+1}^{i}{}' = q_k^{i}{}' Z$$

and let H_i be the expected time to get absorbed starting from node i and let

$$H = \max_i H_i$$

Then, by Markov's inequality, regardless of where the random walk starts, the probability that it takes more than $2H$ time to get absorbed is at most $1/2$. Thus, after $O(H \log 1/\epsilon)$ steps we have that $\|q_i(t)\|_1 < \epsilon$.

Now, let z_∞ be the vector that satisfies

$$z_\infty = Z z_\infty + R y_\infty \tag{2}$$

which we know exists since every eigenvalue of Z must be strictly less than 1 (since $Z^k \to 0$). Then, if we define

$$\Delta_k = x_k^M - z_\infty$$

then subtracting the updates of x_M and z_∞,

$$\Delta_{k+1} = Z\Delta_k + R(y_k - y_\infty). \tag{3}$$

It follows that Δ_k goes to zero since we have already shown that $y_k \to y_\infty$, and $Z^k \to 0$.

Moreover, this argument shows that after $O((L + H) \log 1/\epsilon)$ steps every node is within ϵ of the limiting value. $\square$

Lemma 4 *A random walk with transition probability $A_1 \otimes A_2$ has a coupling time $L = \max\{L_1, L_2\}$, where L_1 and L_2 are the coupling times for random walks with transition probabilities A_1 and A_2 respectively. Similarly, the absorbing time is $H = \max\{H_1, H_2\}$.*

Proof. Consider a random walk $X = (X_k^1, X_k^2)_0^\infty$ on the graph product whose transition matrix is $A_1 \otimes A_2$. Each of the coordinates is also a random walk with the corresponding transition matrix. Coupling between X and a random walk starting at the stationary distribution occurs when each coordinate couples. Thus, the coupling time of X is $L = \max\{L_1, L_2\}$, where L_1 and L_2 are the coupling times for random walks each of the coordinates respectively. Similarly, the absorbing time of a random walk on a transient component defined by a product graph requires both coordinates are absorbed individually, thus $H = \max\{H_1, H_2\}$. $\square$

Supplementary Note 4: Computation of Limiting Distribution

In order to compute the absorption probabilities consider a random walk moving on the graph until it is absorbed into a closed strongly connected component. One can create a graph where closed strongly connected components are taken as single nodes, then for each transient strongly connected M component one can construct a matrix

$$P_M = \begin{bmatrix} Z & R \\ 0 & I \end{bmatrix}$$

where Z is the set of weights assigned to nodes inside the transient component and R is the set of weights assigned to each of the sources. It follows that

$$\lim_{k\to\infty} P_M^k = \begin{bmatrix} 0 & NR \\ 0 & I \end{bmatrix}$$

where $N = (I + Z + Z^2 + \cdots)$. The matrix NR is the absorbing probability matrix, where $[NR]_{iS} \triangleq p_{iS}$ is the probability of being absorbed by a source S starting from node i. Then

$$\lim_k x_k^i = \sum_S p_{iS} \pi_S' x_0^S$$

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