APM466H1 Assignment 1

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Fundamental Questions - 25 points

1. (a) If a country's currency issuing institutions issue more money than necessary without a corresponding increase in productivity, that is, the production of goods does not increase accordingly, then due to supply and demand, more money will have to be used to buy goods, and the currency will depreciate in the international view.

   (b) In long term part, if forward rate equals to spot rate, – i.e spot rate does not change in long term, that is $f_{t-i,\ t},\ s_t\ and\ s_{t-i}$ in $(1+\frac{f_{t-i,\ t}}{2})^{t-(t-i)} = \frac{(1+s_t)^t}{(1+s_{t-i})^{t-i}}$ are equal, then the yield curve will flatten.

   (c) Quantitative easing is similar to printing money indirectly, because when zero or near-zero interest rates are in place, the central bank buys long-term bonds, such as Treasury bonds, to flood the market with liquidity and encourage spending and borrowing. The (US) Fed provides credit support to households, small businesses and major employers, and allows unlimited purchases of TREASURIES and agency mortgage-backed securities (MBS) on demand.

2. I choose "CAN 0.5 Feb.28.22", "CAN 2.75 May.31.22", "CAN 1.75 Feb.28.23", "CAN 1.5 May.31.23", "CAN 2.25 Feb.29.24", "CAN 1.5 Aug.31.24", "CAN 1.25 Feb.28.25", "CAN 0.5 Aug.31.25", "CAN 0.25 Feb.28.26", "CAN 1 Aug.31.26" and "CAN 1.25 Feb.28.27" (refers to the Canadian Government bond with a maturity in Month day, year and a coupon of c). First I choose to use the bonds with maturity 3-10 years, then I collected the data in 2022.01.10-2022.01.14 and 2022.01.17-2022.01.21. I noticed that most of the bonds have maturity 02.28-29 and 08.30-31 with 5-year term. Since we want to construct a "0-5 year" yield and spot curves with 10-11 bonds, the ideal interval for years until maturity of each bond should be: 0-0.5 year, 0.5-1, 1-1.5, . . . , 4.5-5 year and 5+ year. So I choose all the bonds with 5-year term and find that there are two missing 5-year bond in 0.5-1 and 1-1.5 year. Thus we use two 10-year bond whose maturity date are in 0-1 and 1-2 year respectively instead.

3. First of all, PCA is if we have an n-dimensional data set with m data, and 2e want to reduce the dimension from the n down to k dimension. Ideally output is the data set with k dimension and m data to represent the original data set as much as possible. Let A be an n × n covariance matrix, and we have $Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$ where λ is the eigenvalue and x is the eigenvector. The eigenvalues and the eigenvectors are the solution for the function above. $x \rightarrow y = Ax$ implies: x can be transformed to y by A. That is the eigenvector corresponding to the eigenvalue is the ideal coordinate axis, and the eigenvalue is equal to the variance of the corresponding dimension of the data in the rotated coordinates. The value of eigenvalues represents the contribution of corresponding eigenvectors to the whole matrix after orthogonalization. When we solve the eigenvalues and eigenvectors we sort the eigenvalues first. The larger the eigenvalues are, the more relevant the corresponding eigenvectors are in the data set.

Table 1: Bonds' Yield (YTM) from the selected 11 Bonds

	2022/01/10	2022/01/11	2022/01/12	2022/01/13	2022/01/14	2022/01/17	2022/01/18	2022/01/19	2022/01/20	2022/01/21
0-0.5	0.0036	0.0046	0.0042	0.0039	0.0047	0.0041	0.0060	0.0057	0.0054	0.0041
0.5-1	0.0052	0.0053	0.0053	0.0059	0.0059	0.0064	0.0067	0.0064	0.0067	0.0061
1-1.5	0.0096	0.0096	0.0097	0.0099	0.0100	0.0108	0.0111	0.0111	0.0110	0.0106
1.5-2	0.0102	0.0102	0.0104	0.0106	0.0108	0.0115	0.0118	0.0118	0.0118	0.0114
2-2.5	0.0117	0.0117	0.0119	0.0120	0.0121	0.0129	0.0133	0.0133	0.0133	0.0128
2.5-3	0.0129	0.0126	0.0126	0.0128	0.0130	0.0138	0.0141	0.0143	0.0142	0.0137
3-3.5	0.0138	0.0136	0.0138	0.0138	0.0142	0.0149	0.0153	0.0153	0.0153	0.0147
3.5-4	0.0145	0.0143	0.0141	0.0141	0.0144	0.0151	0.0155	0.0158	0.0157	0.0153
4-4.5	0.0147	0.0146	0.0147	0.0146	0.0149	0.0156	0.0161	0.0162	0.0162	0.0157
4.5-5	0.0152	0.0151	0.0152	0.0151	0.0153	0.0161	0.0166	0.0167	0.0166	0.0160
5+	0.0157	0.0156	0.0157	0.0156	0.0158	0.0165	0.0171	0.0171	0.0171	0.0165

Table 2: Summary of 2022.01.10 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00589	0.00081	7.25240	5e-05
years_until_maturity_0110	0.00221	0.00027	8.30286	2e-05

Empirical Questions - 75 points

4.

(a) This part shows the steps for YTM calculation.

Recall:

$$P_n = cash\ flow_1 \times (1+\frac{r_1}{2})^{-2t_1} + cash\ flow_2 \times (1+\frac{r_2}{2})^{-2t_2} + \cdots$$

where P is the dirty price, p_i is the cash flow in period t_i, $r(t_i)$ is the yield to maturity we want and t_i is the remaining maturity from now (counted by year).

Based on the bonds we selected, the ideal interval for years until maturity of each bond should be: 0-0.5 year, 0.5-1 year, 1-1.5 year, 1.5-2 year, 2-2.5 year, 2.5-3 year, 3-3.5 year, 3.5-4 year, 4-4.5 year, 4.5-5 year and 5+ year. However since we had two missing 5-year bond, and we use two 10-year bond instead of the two missing 5-year bond. So we do not have years until maturity in: 0.5-1 year and 1.5-2 year in this case. We want to keep our thought, them we assume that they has the position in 0.5-1 year and 1.5-2 year. Nevertheless when we do calculations we need to pay attention to their real years until maturity to find their cash flow, ytm, ect.

Recall: $DirtyPrice : P_i = accrued\ interest + Clean\ Price = n/360 * annual\ coupon\ payment$ where n is number of days since last payment between "today" and coupon payment date.

$$coupon\ payment = \frac{annual\ coupon\ rate \times face\ value}{number\ of\ coupon\ payments\ per\ year}$$

$$Cash\ Flow_i = \begin{cases} coupon\ payment, & i < years\ until\ maturity \\ coupon\ payment + face\ value, & i = years\ until\ maturity \end{cases}$$

Thus we have:

With the ytm in table 1 above, we can plot the yield curve now (by concatenating YTM with different years until maturity on the same date):

The x axis is year and y axis is yield to maturity. There are 10 curves in figure 1 and each stands for one date with identified color. Each line has the similar tendency – may growing with decreasing growth rate with the shape of the curve seems to convex upward. (we couldn't know for sure, and it need more analysis.)

The last part is the interpolation for the obtained ytm above. Since we need to use exactly 1-year yield in Q5 and Q6 and we only have ytm for approximately years until maturity: 1.2, 2.2, 3.2, 4.2 and 5.2, thus I

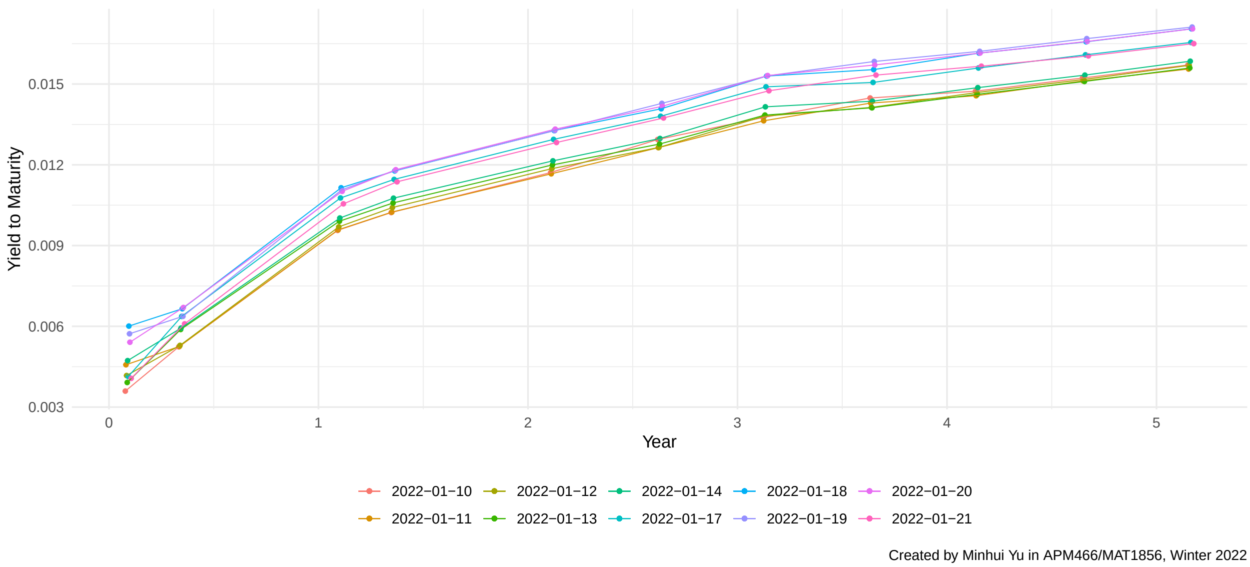

Figure 1: 5-Year Yield Curve for the Selected 11 Bonds

Table 3: Summary of 2022.01.11 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00619	0.00071	8.75088	1e-05
years_until_maturity_0111	0.00209	0.00023	9.01469	1e-05

Table 4: Summary of 2022.01.12 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00614	0.00076	8.05009	2e-05
years_until_maturity_0112	0.00212	0.00025	8.50808	1e-05

Table 5: Summary of 2022.01.13 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00635	0.00080	7.92608	2e-05
years_until_maturity_0113	0.00206	0.00026	7.87747	3e-05

Table 6: Summary of 2022.01.14

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00666	0.00074	9.02707	1e-05
years_until_maturity_0114	0.00204	0.00024	8.47741	1e-05

Table 7: Summary of 2022.01.17 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00691	0.00091	7.63257	3e-05
years_until_maturity_0117	0.00217	0.00030	7.34853	4e-05

Table 8: Summary of 2022.01.18 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00765	0.00074	10.38631	0e+00
years_until_maturity_0118	0.00209	0.00024	8.69330	1e-05

Table 9: Summary of 2022.01.19 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00746	0.00079	9.47304	1e-05
years_until_maturity_0119	0.00216	0.00026	8.43659	1e-05

choose to fit linear regression model for collected date on each date. And we will have 11 linear regression model in total. (Note: Linear regression model may not be the best fitted model for the data, however, since we do not have too much data (11 data in one model), the estimated ytms could be closed to the real ones.) From the summary table (which are shown in rmd) of each model, we observed that p-value for both β_0 and β_1 are smaller than 0.05. Thus there exists significant difference, so we need to reject $H_0 : \beta_0 = 0$ and $H_0 : \beta_1 = 0$. And using the linear regression model, we predicted ytm with years until maturity: 1, 2, 3, 4, and 5 in each date, which is shown in table 2.

(b)

$$P_i = \sum_{i=1}^{n} cash\ flow_i \times (1 + \frac{r(t_i)}{2})^{(-t_i \times 2)}$$

Recall: $P_n = \sum_{i=1}^{n} cash\ flow_i \times (1 + \frac{r_i}{2})^{-2t_i}$, where $n \in [1, 11]$, where r_i is the spot rate, P_i represents the dirty price and t_i is the years until maturity. The difference between calculating YTM and spot rate is YTM has same r for one equation while spot rate has different r for one equation.

The pseudo-code for calculating spot rate:

Recall time_interval is a vector with elements 0.5-1 year, 1-1.5 year, 1.5-2 year, 2-2.5 year, 2.5-3 year, 3-3.5 year, 3.5-4 year, 4-4.5 year, 4.5-5 year and 5+ year; dates is a vector with elements 2022/01/10, 2022/01/11, 2022/01/12, 2022/01/13, 2022/01/14, 2022/01/17, 2022/01/18, 2022/01/19, 2022/01/20, and 2022/01/21; years_until_maturity is a 11 $\times$ 10 matrix, and years_until_maturity[i,j] represents the years until maturity for i^{th} date and j^{th} bond in data set; cash_flow is a 11 $\times$ 11 matrix and cash_flow[i,j] represents the cash flow for i^{th} bond in j^{th} year interval.

Table 10: Summary of 2022.01.20 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00746	0.00080	9.37245	1e-05
years_until_maturity_0120	0.00214	0.00026	8.27593	2e-05

Table 11: Summary of 2022.01.21 Model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.00669	0.00091	7.39072	4e-05
years_until_maturity_0121	0.00222	0.00029	7.53234	4e-05

Table 12: Bonds' Yield (YTM) with Maturity 1-5

2022/01/10	2022/01/11	2022/01/12	2022/01/13	2022/01/14	2022/01/17	2022/01/18	2022/01/19	2022/01/20	2022/01/21	years until maturity
0.0081	0.0083	0.0083	0.0084	0.0087	0.0091	0.0097	0.0096	0.0096	0.0089	1
0.0103	0.0104	0.0104	0.0105	0.0107	0.0112	0.0118	0.0118	0.0118	0.0111	2
0.0125	0.0124	0.0125	0.0125	0.0128	0.0134	0.0139	0.0139	0.0139	0.0133	3
0.0147	0.0145	0.0146	0.0146	0.0148	0.0156	0.0160	0.0161	0.0160	0.0156	4
0.0169	0.0166	0.0167	0.0167	0.0169	0.0178	0.0181	0.0183	0.0182	0.0178	5

$let\ spot\_rate\ be\ a\ new\ 11 \times 10\ matrix\ with\ all\ original\ elements\ are\ 0$
$let\ the\ row\ name\ of\ spot\_rate\ be\ time\_interval$
$let\ the\ column\ name\ of\ spot\_rate\ be\ dates$
$for\ i\ in\ the\ range\ of\ \#\ of\ row$
$\quad for\ j\ in\ the\ range\ of\ \#\ of\ column$
$\quad\quad there\ are\ 11\ cases\ in\ total$
$\quad\quad if\ i == 1\ and\ i == 2$
$\quad\quad\quad //\ since\ years\_until\_maturity[2, j] < 0.5$
$\quad\quad\quad //\ so\ spot\_rate[1, j]\ and\ spot\_rate[2, j] will\ be\ calculated\ by\ the\ samemethod$
$\quad\quad\quad spot\_rate[i, j] \leftarrow ytm[i, j]$
$\quad\quad else\ if\ i == 3$
$\quad\quad\quad let\ f(r) = cash\_flow[i, 1] * (1 + spot\_rate[1, j]/2)^{-years\_until\_maturity[1,j]*2}$
$\quad\quad\quad\quad - cash\_flow[i, 2] * (1 + spot\_rate[2, j]/2)^{-years\_until\_maturity[2,j]*2}$
$\quad\quad\quad\quad - cash\_flow[i, 3] * (1 + r/2)^{-years\_until\_maturity[3,j]*2}$
$\quad\quad\quad spot\_rate[i, j] \leftarrow uniroot(f(r), c(0, 1))\$root$
$\quad\quad else\ if\ i == 4$
$\quad\quad\quad //\ since\ years\_until\_maturity[4, j] < 1.5$
$\quad\quad\quad //\ so\ spot\_rate[3, j]\ and\ spot\_rate[4, j] will\ be\ calculated\ by\ the\ samemethod$
$\quad\quad\quad let\ f(r) = cash\_flow[i, 1] * (1 + spot\_rate[1, j]/2)^{-years\_until\_maturity[1,j]*2}$
$\quad\quad\quad\quad - cash\_flow[i, 2] * (1 + spot\_rate[2, j]/2)^{-years\_until\_maturity[2,j]*2}$
$\quad\quad\quad\quad - cash\_flow[i, 3] * (1 + r/2)^{-years\_until\_maturity[4,j]*2}$
$\quad\quad\quad spot\_rate[i, j] \leftarrow uniroot(f(r), c(0, 1))\$root$
$\quad\quad //\ r\ in\ the\ remaining\ cases\ is\ consistent\ with\ the\ calculation\ method\ of\ r\ in\ case\ i == 3$
$\quad\quad else\ if\ i == 5$
$\quad\quad \vdots$
$\quad\quad \vdots$
$\quad\quad else\ if\ i == 11$
$\quad\quad\quad \ldots$
$//\ now\ we\ are\ done\ for\ calculated\ spot\ rate$

And here is the plot for spot rate:

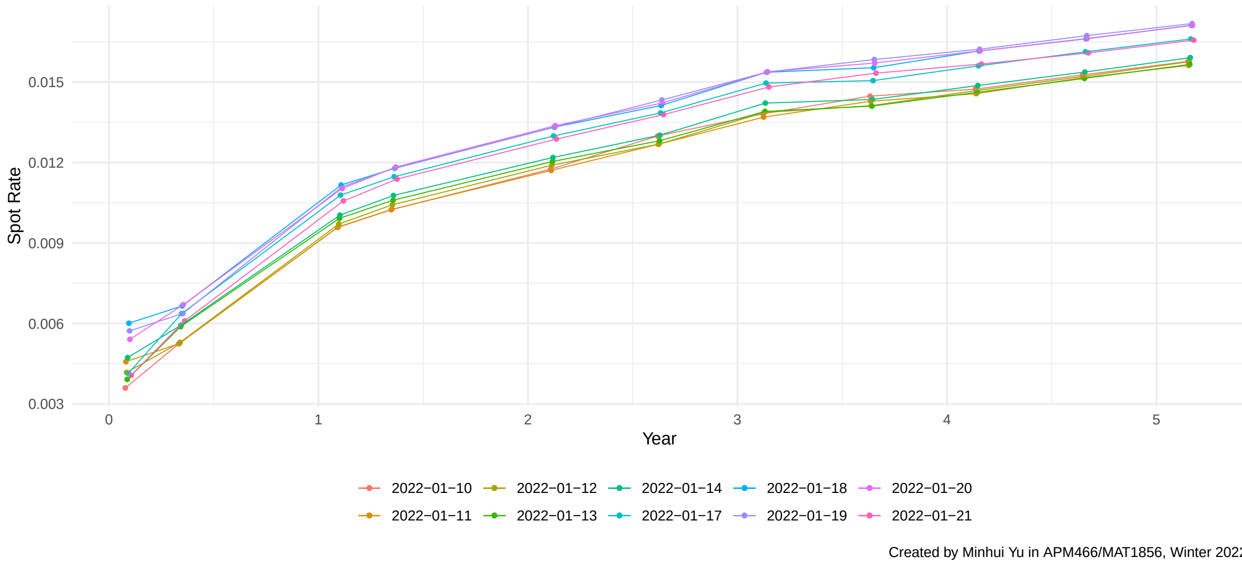

Figure 2: 5-Year Spot Rate Curve for the Selected 11 Bonds

The spot rate curves is very similar to the bond's yield curves. They both have years on the x axis, and the y axis of the spot rate curve is the spot rate. The reason is that spot rate is also known as the zero interest rate which means we find it by discount the cash flow and discount rate is what we need.

(c) For forward rate we have the equation below, where t represents the years until maturity and s_t represents the spot rate with years until maturity t, $f_{1yr,\ iyr}$ represents the forward rate starting in one year and going i more years where i $\in \{1, 2, 3, 4\}$. The reason why I choose spot_rate[3,j] is the years until maturity of spot_rate[2,j] is about 0.333 while the years until maturity of spot_rate[3,j] is about 1.1 which is more closer to 1 year.

$$(1+\frac{f_{t-i,\ t}}{2})^{t-(t-i)} = \frac{(1+s_t)^t}{(1+s_{t-i})^{t-i}} \Rightarrow f_{t-i,\ t} = 2\times\left[\sqrt[i]{\frac{(1+s_t)^t}{(1+s_{t-i})^{t-i}}}-1\right]$$

$$(focus\ on\ the\ data\ in\ one\ date) \Rightarrow f_{1yr-iyr} = f_{3,\ 2i+1} = 2\times\left[\sqrt[2i-2]{\frac{(1+s_{2i+1})^{2i+1}}{(1+s_3)^3}}-1\right]$$

The pseudo-code for calculating forward rate:

$$let\ forward\_rate\ be\ a\ new\ 4 \times 10\ matrix\ with\ all\ original\ elements\ are\ 0$$
$$let\ f\_time\_interval\ be\ a\ vector\ with\ given\ year\ interval\ inside$$
$$//\ given\ year\ interval\ are\ 1yr-1yr,\ 1yr-2yr,\ 1yr-3yr,\ and\ 1yr-4yr$$
$$let\ the\ row\ name\ of\ forward\_rate\ be\ f\_time\_interval$$
$$let\ the\ column\ name\ of\ forward\_rate\ be\ dates$$
$$for\ i\ in\ the\ range\ of\ \#\ of\ row$$
$$for\ j\ in\ the\ range\ of\ \#\ of\ column$$
$$let\ forward\_rate[i,j] \leftarrow 2 * (((1 + spot\_rate[2(i+1)+1,j]/2)^{years\_until\_maturity[(i+1)*2+1,j]} / (1 + spot\_rate[3,j]/2)^{years\_until\_maturity[3,j]})^{(1/(years\_until\_maturity[2(i+1)+1,j] - years\_until\_maturity[3,j]))} - 1)$$
$$//\ now\ we\ are\ done\ for\ calculated\ forward\ rate$$

And here is the plot for forward rate:

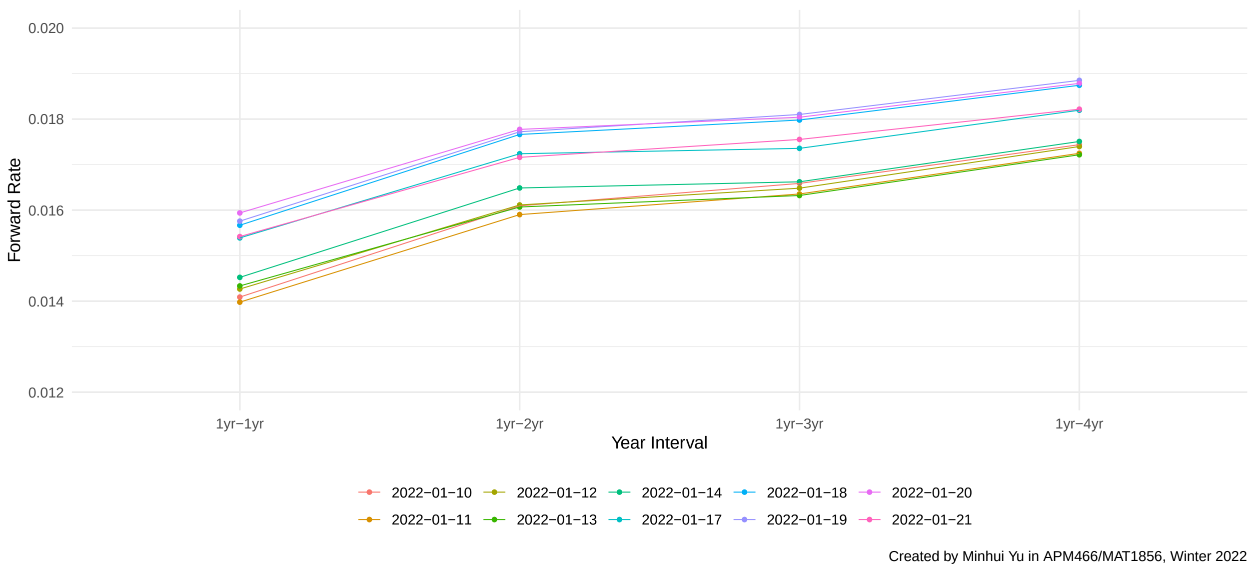

Figure 3: 1-Year Forward Rate Curve with Terms Ranging from 2-5 Years for the Selected 11 Bonds

The x axis is time interval 1yr-1yr, 1yr-2yr, 1yr-3yr, and 1yr-4yr and the y axis is forward rate. There are 10 curves in figure 1 and each stands for one date with identified color. Most curve shows that the forward rate will increase for long time interval. Note that the forward rate here from derivation is implicit and does not represent the market's expectation of forward rate

Table 13: Covariance Matrix for Dailylog-Returns of Yield

	year_1	year_2	year_3	year_4	year_5
year_1	0.00169	0.00125	0.00095	0.00073	0.00057
year_2	0.00125	0.00097	0.00078	0.00065	0.00054
year_3	0.00095	0.00078	0.00067	0.00059	0.00052
year_4	0.00073	0.00065	0.00059	0.00054	0.00051
year_5	0.00057	0.00054	0.00052	0.00051	0.00050

Table 14: Covariance Matrix for Dailylog-Returns of Forward Rate

	1yr-1yr	1yr-2yr	1yr-3yr	1yr-4yr
1yr-1yr	0.00059	0.00054	0.00049	0.00047
1yr-2yr	0.00054	0.00055	0.00052	0.00049
1yr-3yr	0.00049	0.00052	0.00053	0.00049
1yr-4yr	0.00047	0.00049	0.00049	0.00046

5.

Let C_1 = Cov(log-return of yield) and C_2 = Cov(log-return of forward rate). Since log-return of yield is calculated by yield in table 2 above which has 5 year's yield, then log-return of yield is a 9×5 matrix where 10 represents there are 9 day_i-day_{i+1} differences with 10 date in original and 5 represents 1-year, 2-year, ..., 5-year time series. Similarly, log-return of forward rate is a 9×4 matrix with 4 year interval as the same in Q4.(c). Since we focus on times series for both daily return-log yield and forward rate, then when we calculate the covariance of them respectively based on the formula: $cov\left(\sum_{i=1}^{n} X_i,\ \sum_{i=1}^{n} X_i\right) = var\left(\sum_{i=1}^{n} X_i\right)$ where X_i has a time series $X_{i,\ j} = \log(r_{i,\ j+1}/r_{i,\ j})$. And based on the discussion above, n = 5 for C_1 and n = 4 for C_2, j $\in \{1, 2, \ldots, 9\}$. The time series $X_{i,\ j}$ is essentially the rate of return on the term as it compounds to infinity and logarithms are additive. So $X_{i,\ j}$ represents the log difference between rate $r_{i,\ j+1}$ and $r_{i,\ j}$ and it can also eliminate part of the linear influence (degree of independent) between $r_{i,\ j+1}$ and $r_{i,\ j}$ – the covariance became smaller.

6.

Here are the eigenvalues and the eigenvectors of the covariance matrix for dailylog-returns of yield.

$$\begin{pmatrix} \lambda_{y_1} \\ \lambda_{y_2} \\ \lambda_{y_3} \\ \lambda_{y_4} \\ \lambda_{y_5} \end{pmatrix} = \begin{pmatrix} 3.9616 \times 10^{-3} \\ 4.1833 \times 10^{-4} \\ 3.2897 \times 10^{-8} \\ 1.0236 \times 10^{-12} \\ 6.7514 \times 10^{-18} \end{pmatrix}, \quad \begin{pmatrix} \xi_{y_1}, & \xi_{y_2}, & \xi_{y_3}, & \xi_{y_4}, & \xi_{y_5} \end{pmatrix} = \begin{pmatrix} 0.625 & 0.590 & -0.480 & -0.173 & -0.033 \\ 0.495 & 0.108 & 0.534 & 0.628 & 0.252 \\ 0.405 & -0.216 & 0.454 & -0.409 & -0.645 \\ 0.340 & -0.450 & 0.015 & -0.475 & 0.676 \\ 0.291 & -0.626 & -0.527 & 0.428 & -0.250 \end{pmatrix}$$

Here are the eigenvalues and the eigenvectors of the covariance matrix for dailylog-returns of forward rate

$$\begin{pmatrix} \lambda_{f_1} \\ \lambda_{f_2} \\ \lambda_{f_3} \\ \lambda_{f_4} \end{pmatrix} = \begin{pmatrix} 2.0363 \times 10^{-3} \\ 8.1254 \times 10^{-5} \\ 1.3659 \times 10^{-5} \\ 1.7300 \times 10^{-6} \end{pmatrix}, \quad \begin{pmatrix} \xi_{f_1}, & \xi_{f_2}, & \xi_{f_3}, & \xi_{f_4} \end{pmatrix} = \begin{pmatrix} -0.514 & 0.782 & 0.352 & 0.006 \\ -0.513 & 0.042 & -0.841 & -0.166 \\ -0.500 & -0.502 & 0.395 & -0.585 \\ -0.472 & -0.366 & 0.113 & 0.794 \end{pmatrix}$$

λ_{y_1} is corresponding to ξ_{y_1}. λ_{y_1} is the variance of the corresponding dimension of the data after transformation and implies the contribution of ξ_{y_1} to the covariance matrix of log-return yield with percentage $\lambda_{y_1} \sum_{i=1}^{5} \lambda_{y_i}$. ξ_{y_1} implies the direction of change in yield., which is in the wave.

References and GitHub Link to Code

[1] Oeis.org. 2019. List of LaTeX mathematical symbols - OeisWiki. [online] Available at: https://oeis.org/wiki/List_of_LaTeX_mathematical_symbols [Accessed 14 February 2022].

[2] Latex mathematical Formulas. “Latex mathematical Formulas - detailed tutorial” Published October 2020. https://blog.csdn.net/NSJim/article/details/109045914

[3] Rpubs.com. 2022. RPubs - Bond Valuation and Analysis in R. Accessed 14 February 2022. https://rpubs.com/Sergio_Garcia/bond_valuation_analysis_r

Github Link: https://github.com/minhui-yu-helen/APM466H1-Assignment-1.git