svd_transformation

February 17, 2026

1 Readme

Places where solutions are required are marked with **#TODO**
You will **NOT** need to modify any section not marked as #TODO to answer this question.
Make sure the helper file. svd_transformation_helper.py is in the same folder as this .ipynb
Make sure you have numpy, matplotlib and itertools packages installed for python

1.0.1 In this notebook:

Part (b) has 3 subparts i, ii, and iii
Part (c) has 4 subparts i, ii, iii and iv
Part (d) has 2 subparts i,ii
Part (e) has only 1 subpart

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
!gdown 1cV2RxKzE-02nGVMXi-092dytf6-Cg-3D -O svd_transformation_helper.py
from svd_transformation_helper import visualize_function
from svd_transformation_helper import matrix_equals, is_orthonormal
```

```
Downloading…
From (original):
https://drive.google.com/uc?id=1cV2RxKzE-02nGVMXi-092dytf6-Cg-3D
From (redirected): https://drive.google.com/uc?id=1cV2RxKzE-02nGVMXi-092dytf6-
Cg-3D&confirm=t&uuid=aa75dbad-20db-4848-ab58-e7a663de25a3
To: /content/svd_transformation_helper.py
   0% 0.00/4.63k [00:00<?, ?B/s] 100% 4.63k/4.63k [00:00<00:00, 13.0MB/s]
```

[ ]:
```
DISABLE_CHECKS = False #Set this to True only if you get Value Errors about
 ↪inputs even
#when you are sure that what you are inputting is correct.
#WARNING: Setting this to True and entering wrong inputs can lead to all kinds
 ↪of crazy results/errors


def visualize(U = np.identity(2), D = np.ones(2), VT = np.identity(2),
 ↪num_grid_points_per_dim = 200,\
```

```
    disable_checks = DISABLE_CHECKS, show_original = True, show_VT = True,␣
↪show_DVT = True, show_UDVT = True):
    '''
    Inputs:
    A has singular value decomposition A = U np.diag(D) VT
    U: 2 x 2 orthonormal matrix represented as a np.array of shape (2,2)
    D: Diagonal entries corresponding to the diagonal matrix in SVD represented␣
↪as a np.array of shape (2,)
    VT: 2 x 2 orthonormal matrix represented as a np.array of shape (2,2)
    num_grid_points_per_dim: Spacing of points used to represent circle␣
↪(Decrease this if plotting is slow)
    disable_checks: If False then have checks in  place to make sure dimensions␣
↪of VT, U are correct, etc.
    show_original: If True plots original unit circle and basis vectors
    show_VT: If True plots transformation by VT
    show_DVT: If True plots transformation by DVT
    show_UDVT: If True plots transformation by UDVT
    '''

    visualize_function(U=U, D=D, VT=VT,␣
↪num_grid_points_per_dim=num_grid_points_per_dim,␣
↪disable_checks=disable_checks,\
                       show_original=show_original, show_VT=show_VT,␣
↪show_DVT=show_DVT, show_UDVT=show_UDVT)
```

2 Effect of the linear transformation by an orthonormal matrix V^T

A 2 x 2 orthonormal matrix can be viewed as a linear transformation that performs some combination of rotations and reflections. Note that both rotation and reflection are operations that preserve the length of vectors and the angle between them.

2.1 V^T as a rotation matrix

First we set V^T as a counter-clockwise rotation matrix.

2.1.1 (b) i: Fill in the function "get_RCC(theta)" to return a 2 x 2 matrix that, when applied to a vector x, rotates it by theta radians counter clockwise.

Example: If $V^T = RCC\left(\frac{\pi}{4}\right)$ and x $= \begin{bmatrix}1\\0\end{bmatrix}$, then,

$$V^T \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

[ ]:
```
def get_RCC(theta):
    '''
```

```
    Returns a 2 x 2 orthonormal matrix that rotates x by theta radians␣
↪counter-clockwise
    '''

    RCC = np.array([[np.cos(theta), -np.sin(theta)], [np.sin(theta), np.
↪cos(theta)]])


  ␣
↪####################################################################################
    #Some assertions (WARNING: Do not modify below code)
    if DISABLE_CHECKS is False:
        if not isinstance(RCC, np.ndarray) or isinstance(RCC, np.matrix):
                raise ValueError('RCC must be a np.ndarray')
        if len(RCC.shape) != 2 or (RCC.shape != np.array([2,2])).any():
                raise ValueError('RCC must have shape [2,2]')
    return RCC
```

2.1.2 get_RCC(theta) function test

If the function get_RCC(theta) is defined correctly then you should not get any ERROR statement here.

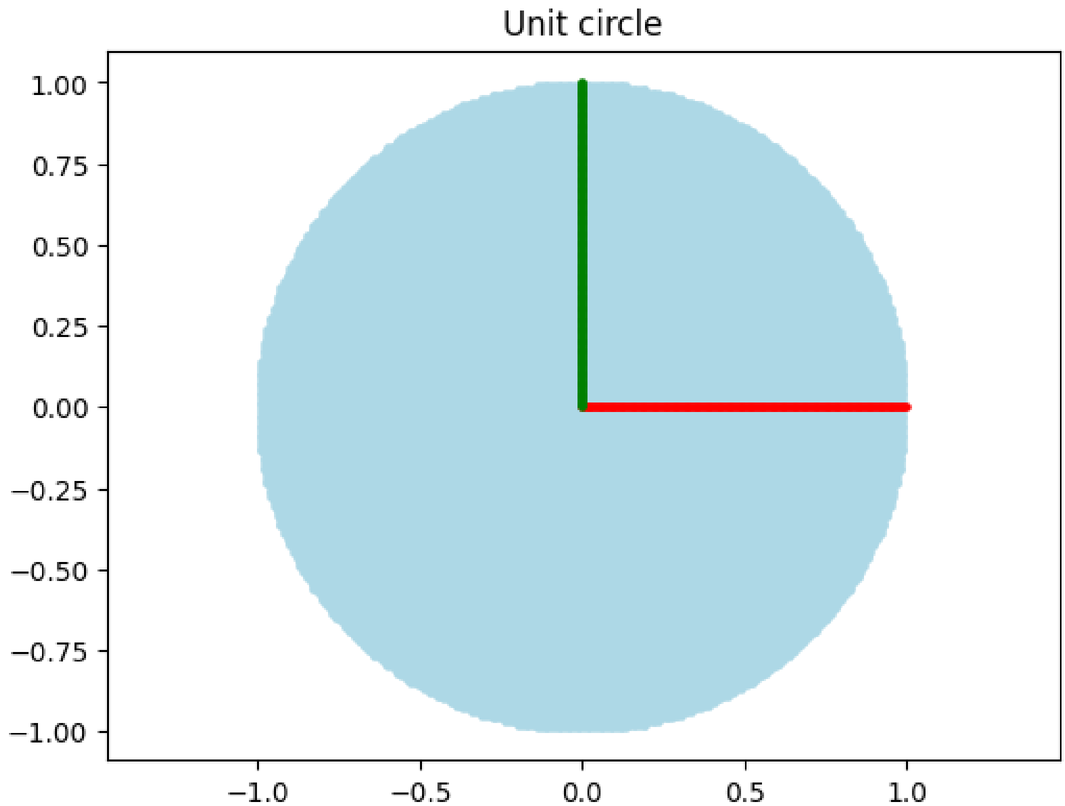
```
[ ]: x = np.array([[1,0]]).T
     V_test = get_RCC(np.pi/4)
     y = V_test @ x
     expected_y = np.array([[1/np.sqrt(2), 1/np.sqrt(2)]]).T
     print("y:")
     print(y)
     print("Expected y:")
     print(expected_y)
     if not matrix_equals(y, expected_y):
         print("ERROR: y does not match expected_y. Check if function get_RCC(theta)␣
      ↪is completed correctly")
     else:
         print("MATCHED: y matches expected_y!")
```

```
y:
[[0.70710678]
 [0.70710678]]
Expected y:
[[0.70710678]
 [0.70710678]]
MATCHED: y matches expected_y!
```

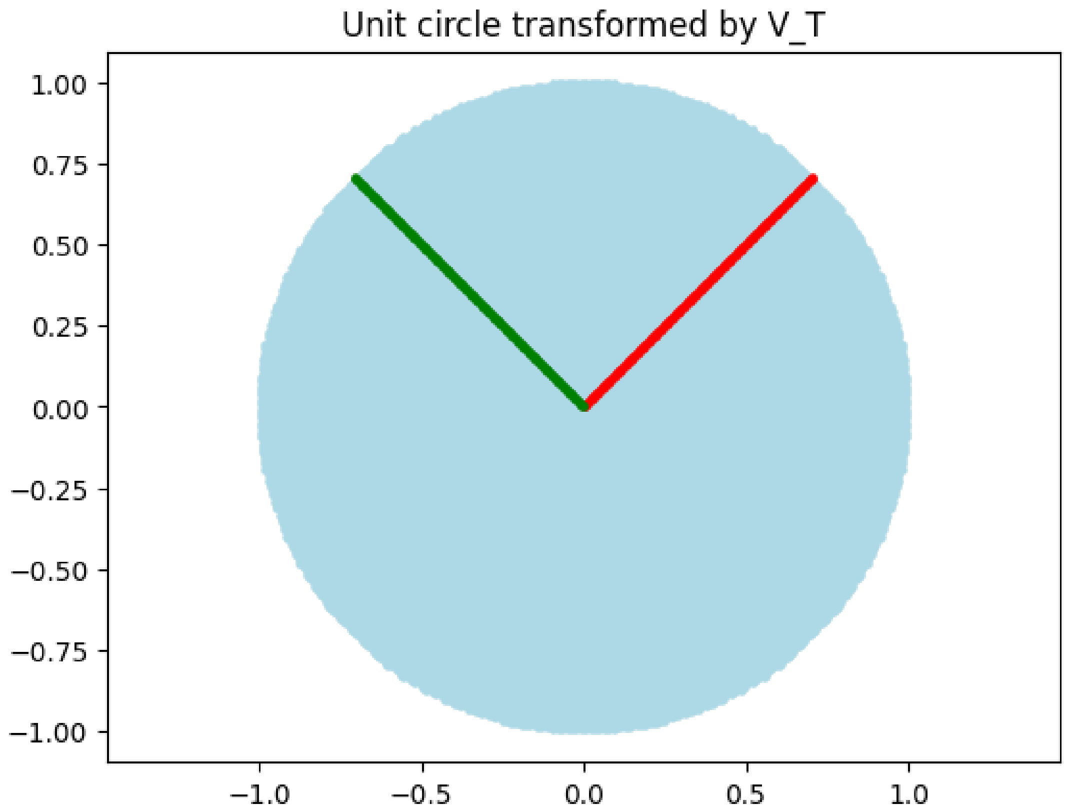
Next we observe how V^T transforms the unit circle and unit basis vectors when:

1) $V^T = RCC\left(\frac{\pi}{4}\right)$

[ ]:
```
VT_1 = get_RCC(np.pi/4)
visualize(VT = VT_1, show_DVT=False, show_UDVT=False)
```

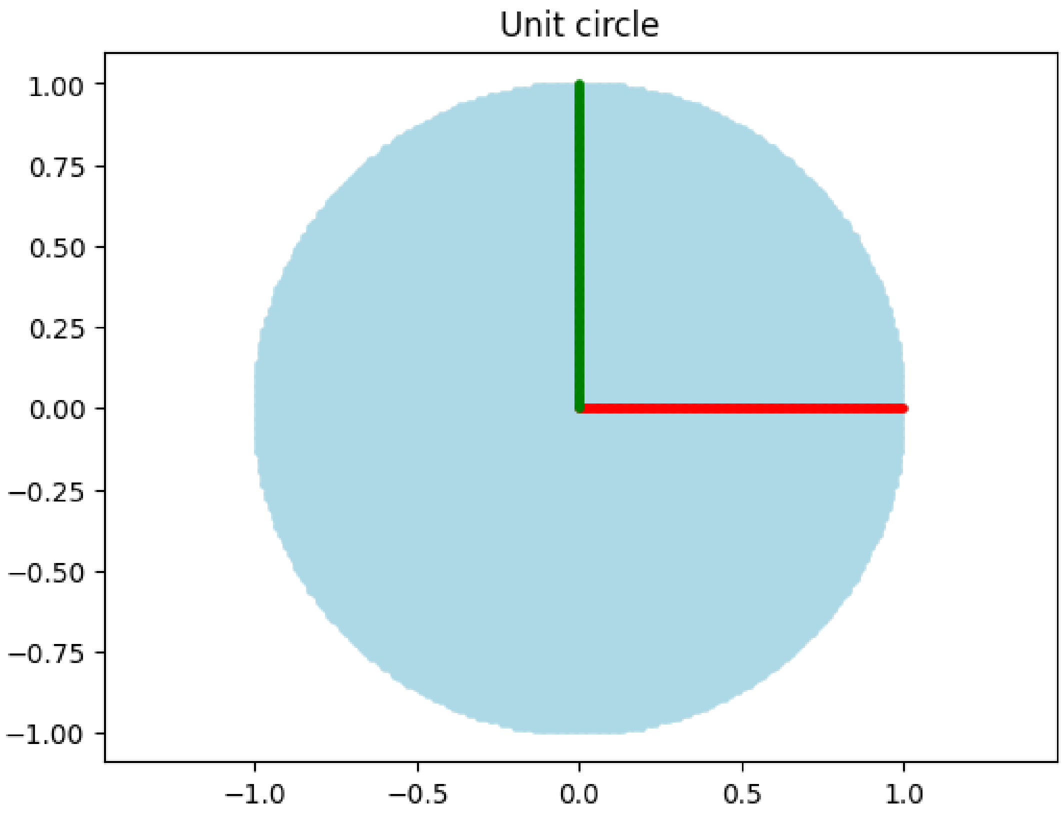



```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```

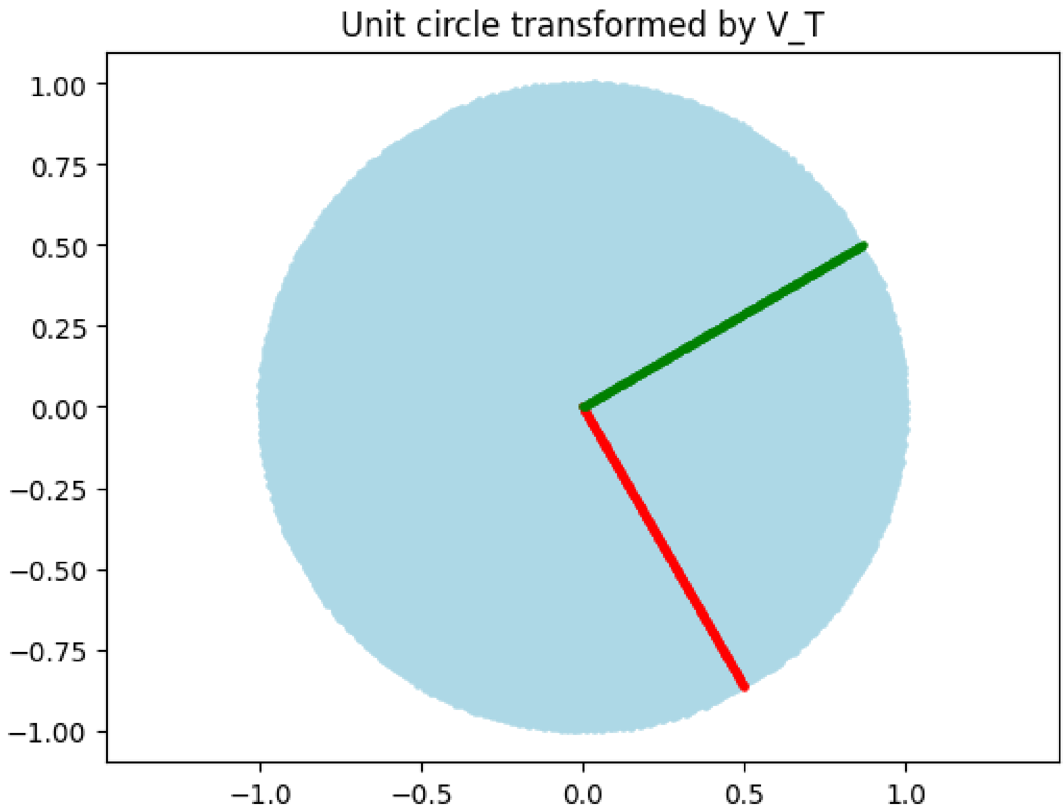


2) $V^T = RCC\left(\frac{-\pi}{3}\right)$

[ ]:
```
VT_2 = get_RCC(-np.pi/3)
visualize(VT = VT_2, show_DVT=False, show_UDVT=False)
```




```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```



Next we consider the case where V^T is a reflection matrix.

2.2 V^T as a reflection matrix

A reflection matrix is another type of orthonormal matrix.

2.2.1 (b) ii: Fill in the function "get_RFx()" to return a 2 x 2 matrix that when applied to a vector x reflects it about the x-axis.

Example: If $V^T = RFx()$ and $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then,

$$V^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

[ ]:
```
def get_RFx():
    '''
    Returns a 2 x 2 orthonormal matrix that reflects about x-axis
    '''



    RFx = np.array([[1,0], [0,-1]])
```

```
␣
↪##################################################################################
    #Some assertions (WARNING: Do not modify below code)
    if DISABLE_CHECKS is False:
        if not isinstance(RFx, np.ndarray) or isinstance(RFx, np.matrix):
                raise ValueError('RFx must be a np.ndarray')
        if len(RFx.shape) != 2 or (RFx.shape != np.array([2,2])).any():
                raise ValueError('RFx must have shape [2,2]')
    return RFx
```

2.2.2 get_RFx() function test

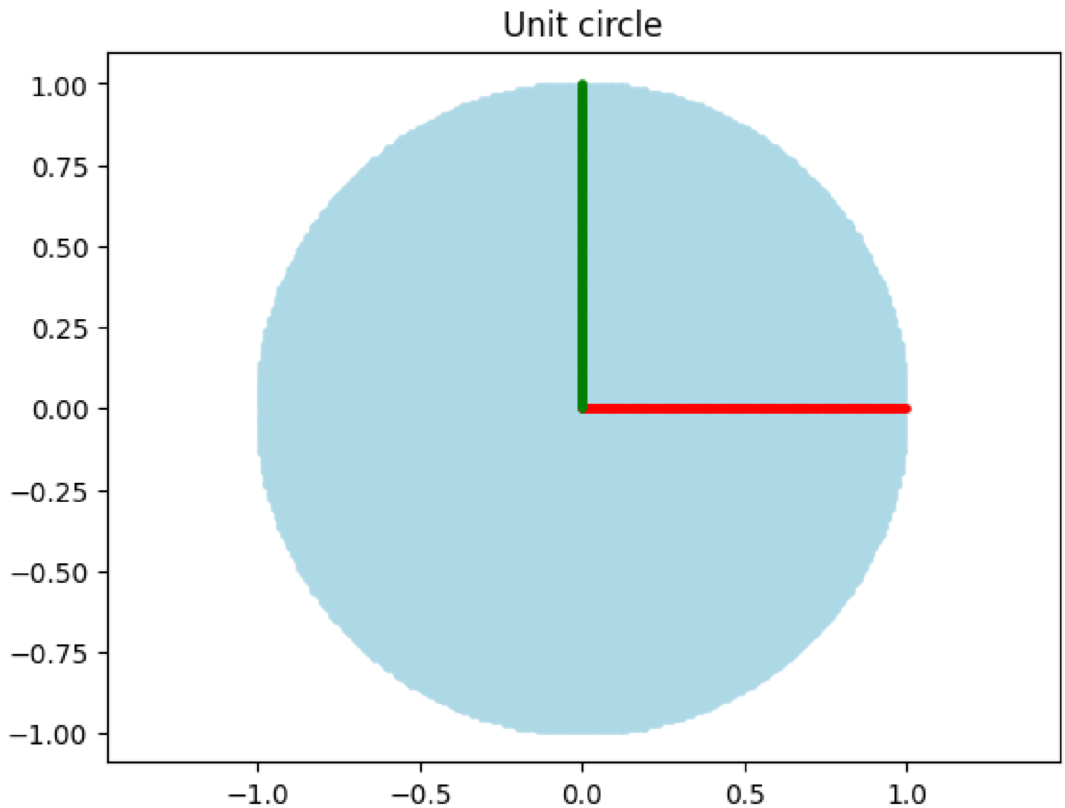
If the function get_RFx() is defined correctly then you should see a MATCHED statement here.

[ ]:
```
x = np.array([[1,1]]).T
V_test = get_RFx()
y = V_test @ x
expected_y = np.array([[1, -1]]).T
print("y:")
print(y)
print("Expected y:")
print(expected_y)
if not matrix_equals(y, expected_y):
    print("ERROR: y does not match expected_y. Check if function get_RFx() is␣
 ↪completed correctly")
else:
    print("MATCHED: y matches expected_y!")
```

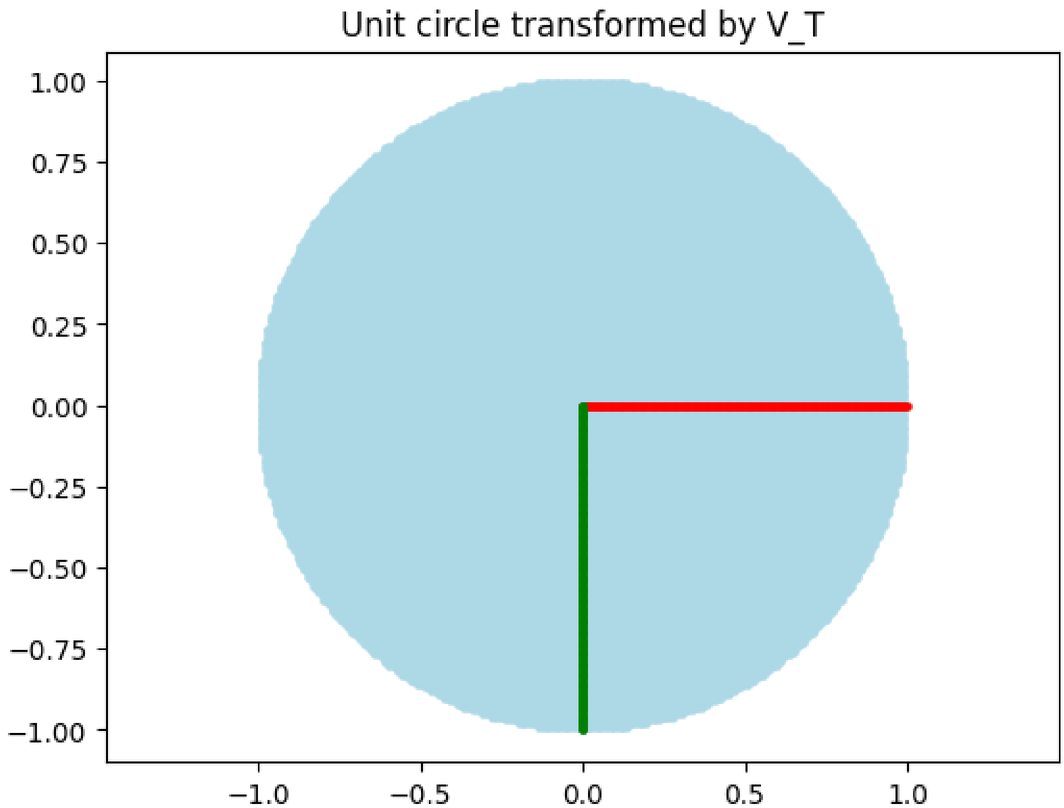
```
y:
[[ 1]
 [-1]]
Expected y:
[[ 1]
 [-1]]
MATCHED: y matches expected_y!
```

$V^T = RFx()$

[ ]:
```
VT_3 = get_RFx()
visualize(VT = VT_3, show_DVT=False, show_UDVT=False)
```




```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```



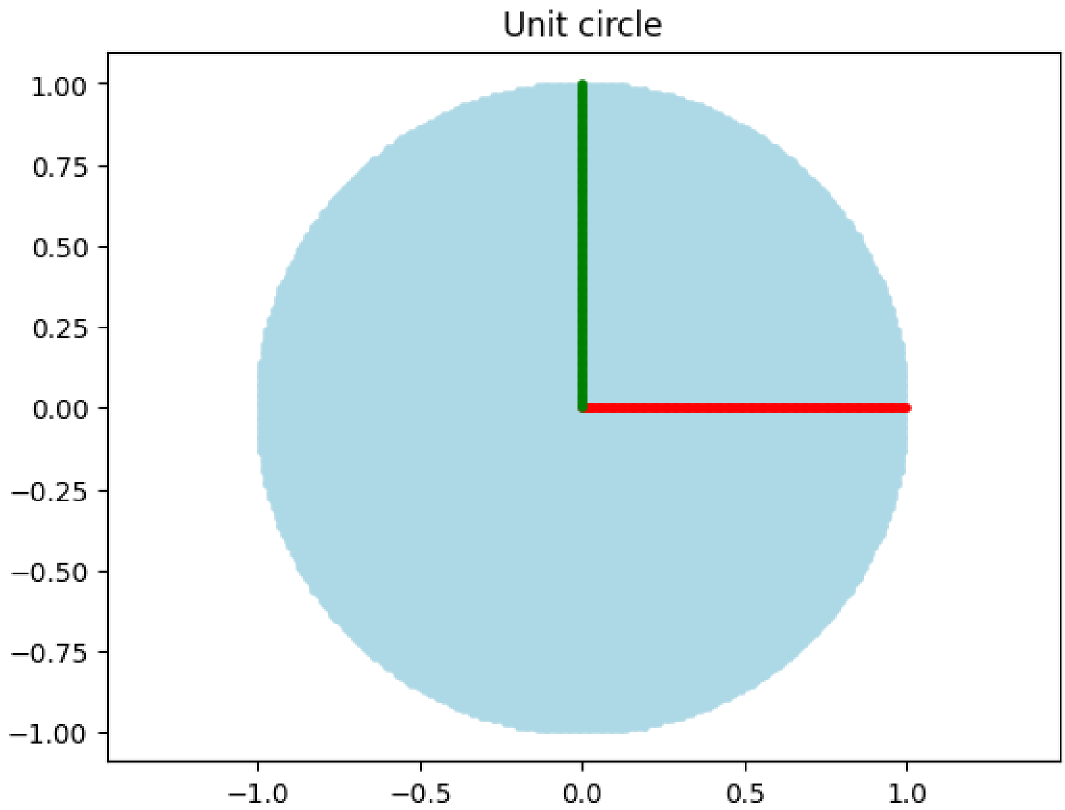
2.3 V^T as a composition of reflection and rotation matrix

In general an orthonormal transformation can be viewed as compositions of rotation and reflection operators. Next we observe the effect of setting

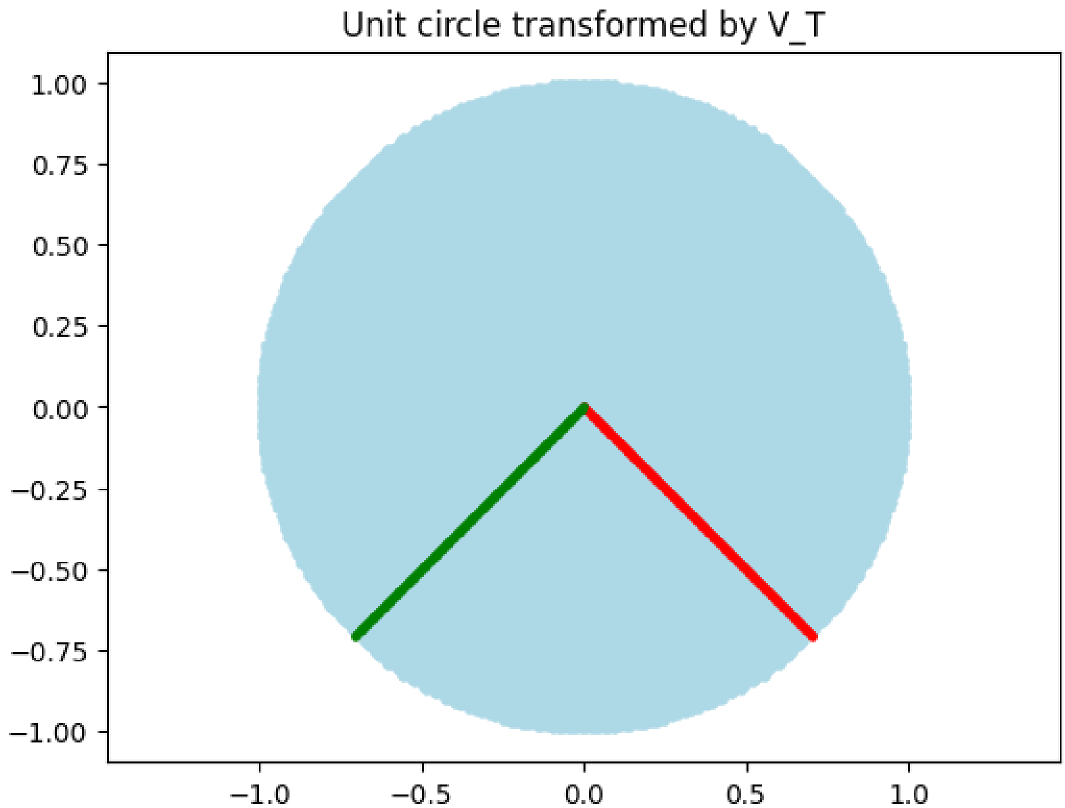
$V^T = RFx() \cdot RCC\left(\frac{\pi}{4}\right)$

[ ]:
```
VT_4 = VT_3 @ VT_1
#Check that VT_4 is still orthonormal
print("VT_4 is orthonormal?: ", is_orthonormal(VT_4))
visualize(VT = VT_4, show_DVT=False, show_UDVT=False)
```

```
VT_4 is orthonormal?:  True
```

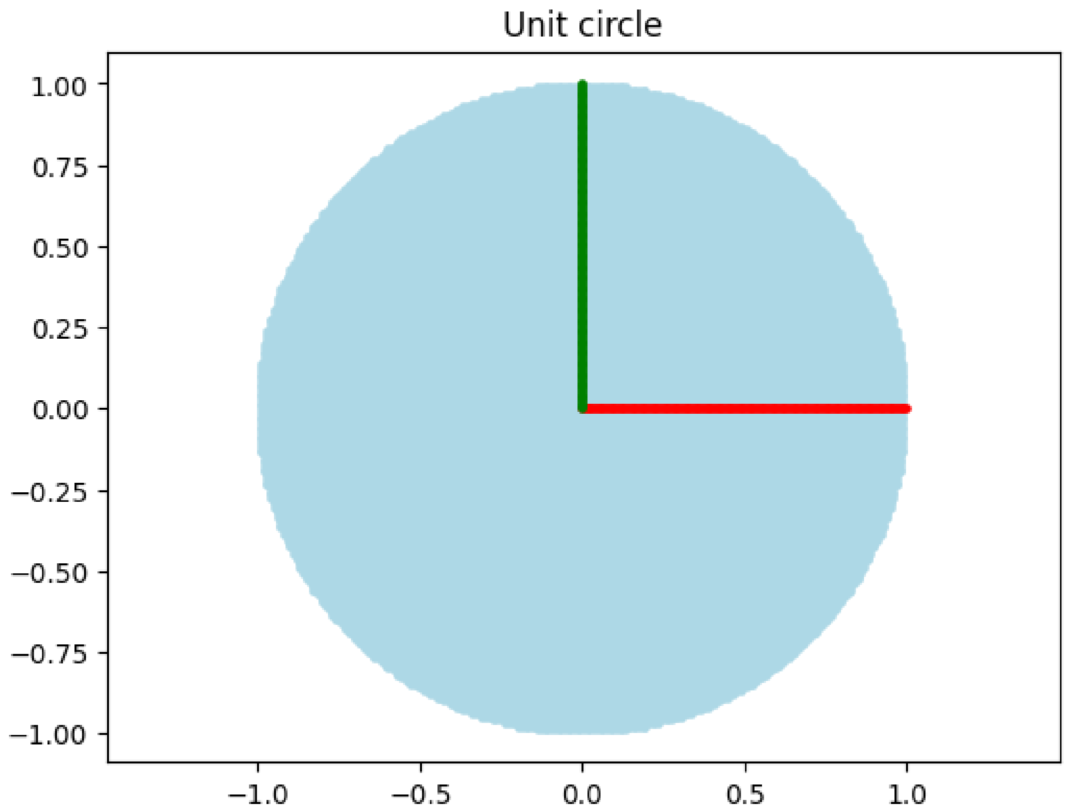



```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```

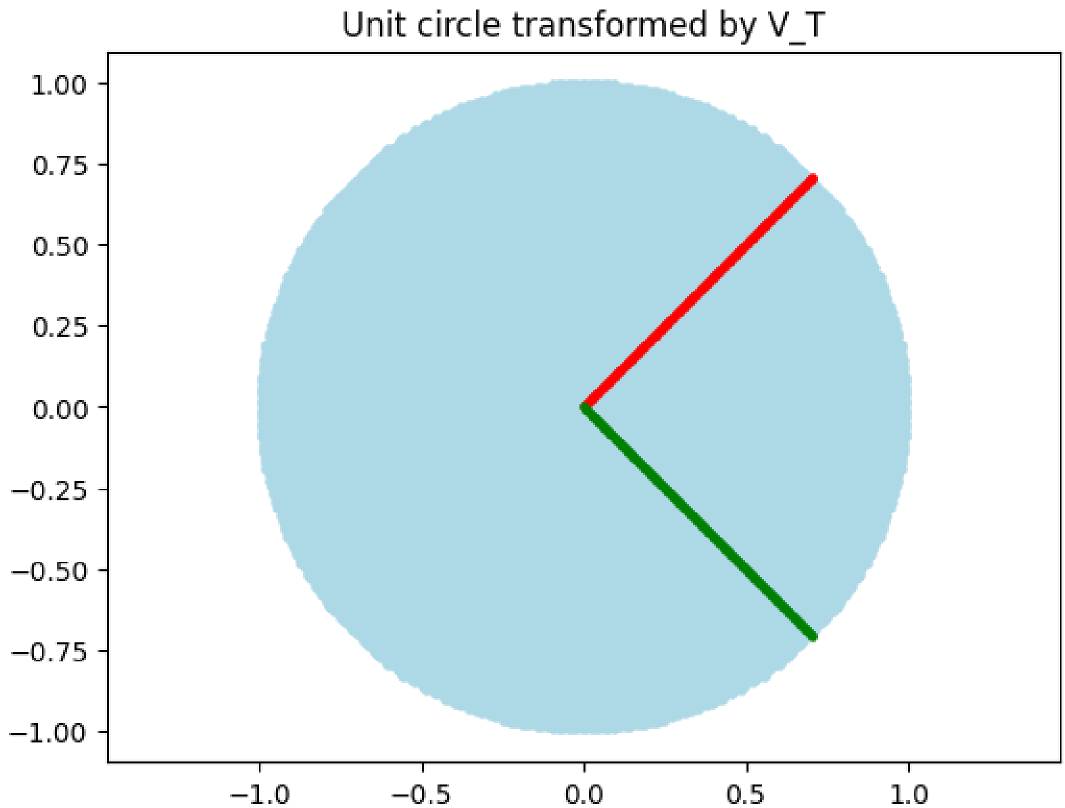


2.3.1 (b) iii: Comment on the effect of $V^T = RCC\left(\frac{\pi}{4}\right) \cdot RFx()$. Is it same as the case when $V^T = RFx() \cdot RCC\left(\frac{\pi}{4}\right)$?

[ ]:
```
VT_5 = VT_1 @ VT_3
visualize(VT = VT_5, show_DVT=False, show_UDVT=False)
```




```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```



No, they are not commutative.

3 Effect of linear transformation by diagonal matrix D

The diagonal matrix D with entries σ_1 and σ_2, transforms the unit circle into an ellipse with x direction scaled by σ_1 and y direction scaled by σ_2.

If $\sigma_1 > \sigma_2$, then the major axis of the ellipse will be along the x-axis.

If $\sigma_1 < \sigma_2$, then the major axis of the ellipse will be along the y-axis.

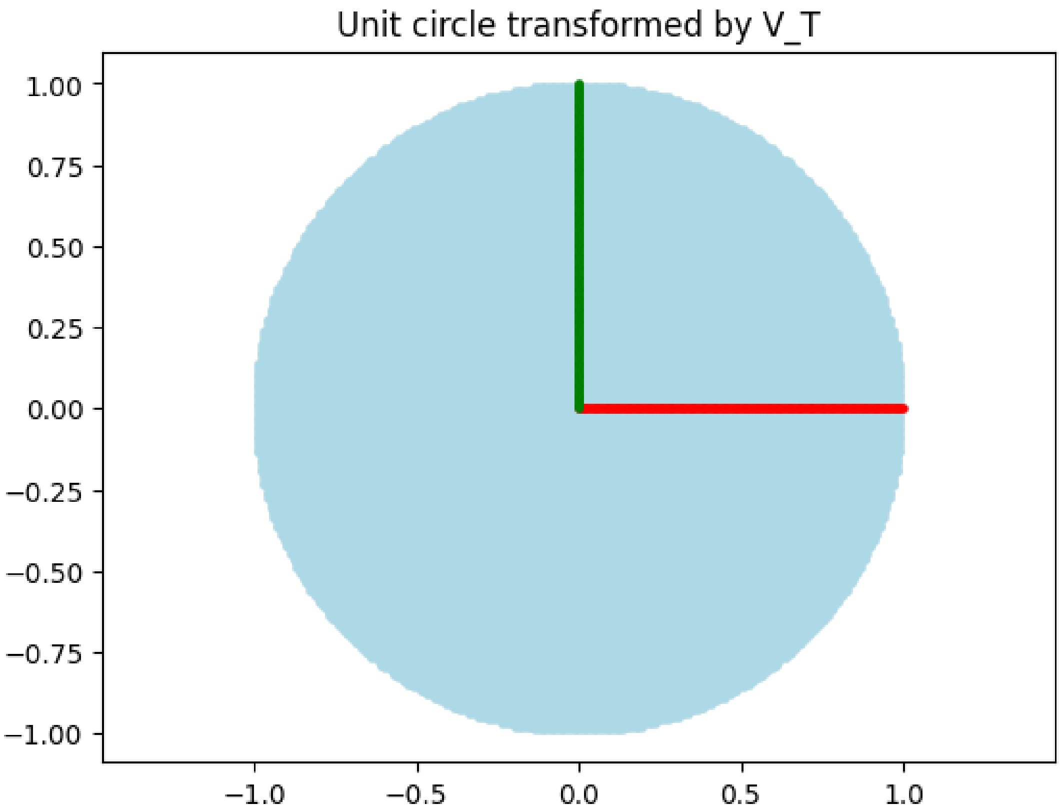
If $\sigma_1 = \sigma_2$, then the ellipse will have both axis equal (i.e it is a circle).

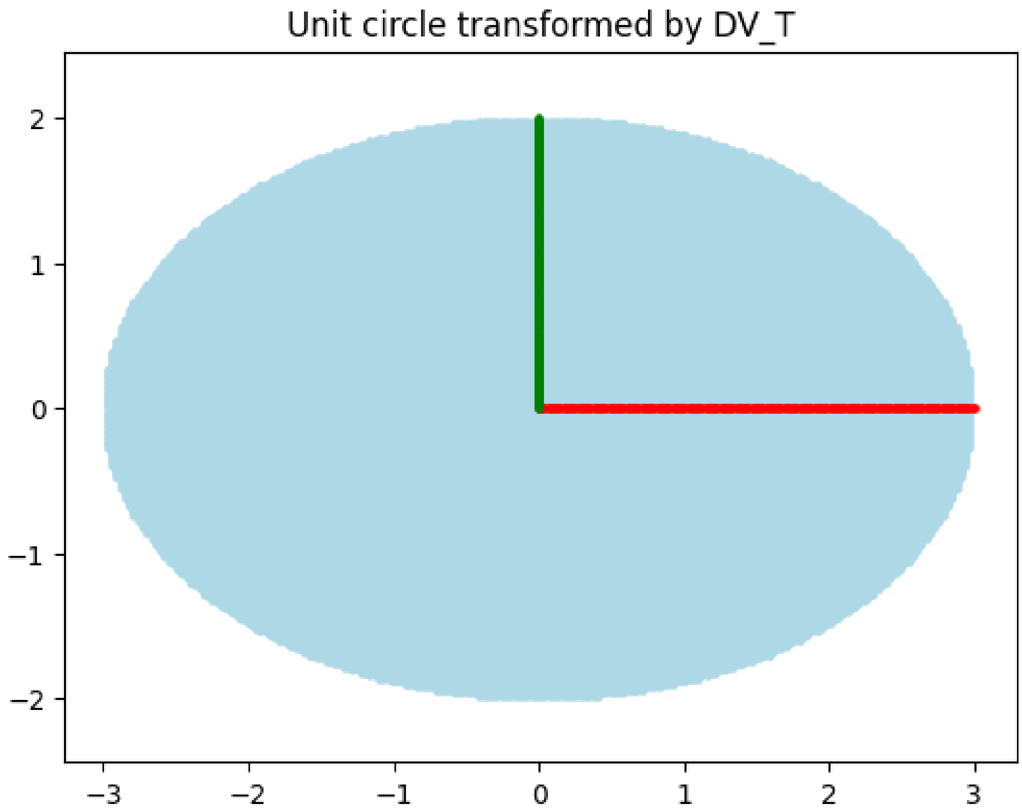
Note that multiplying by D, does not rotate or reflect points in any way. It is a purely scaling operation where different directions get scaled by different values based on entries of D.

3.0.1 (c) i: Comment on the length of major and minor axis of the ellipse and their orientation with respect to X and Y axis when D has entries [3, 2]. Here V is identity.

[ ]:
```
D_1 = np.array([3, 2])
visualize( D = D_1, show_original=False, show_UDVT=False)
```

```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```



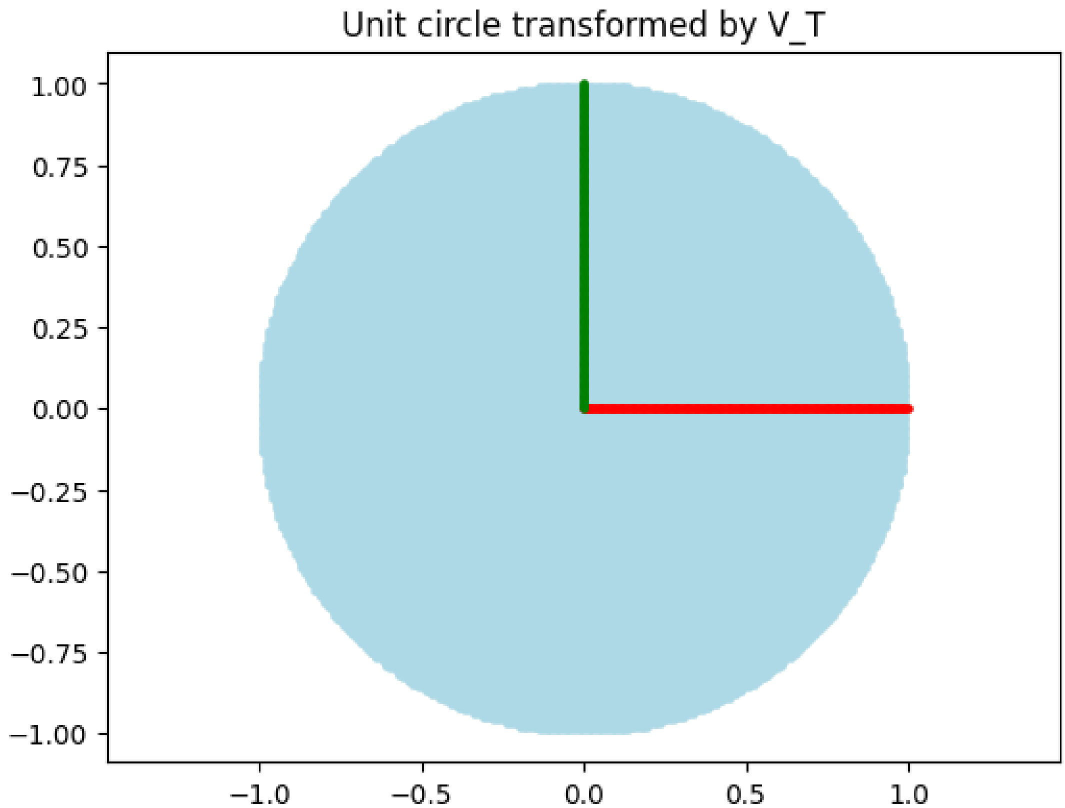


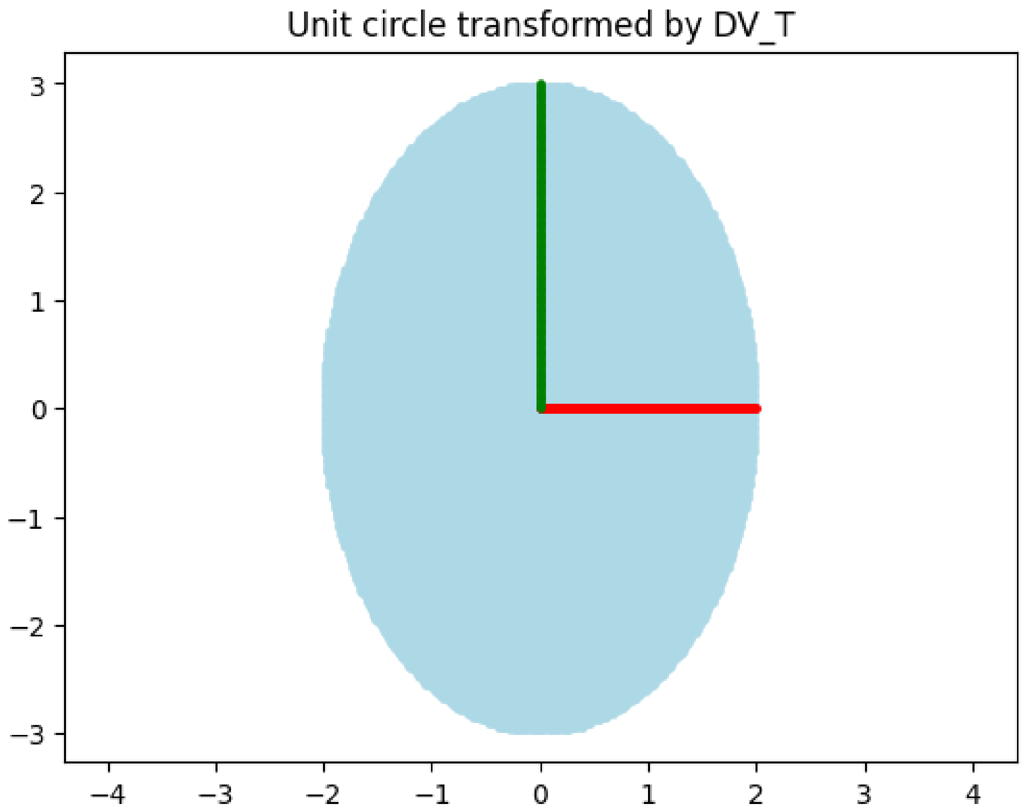
The orientation of the axis is the same, but the lengths have been changed proportionally to the diagonal elements in D. Larger diagonal element correspond longer axis.

3.0.2 (c) ii: Comment on the length of major and minor axis of the ellipse and their orientation with respect to X and Y axis when D has entries [2, 3]. Here V is the identity matrix.

[ ]:
```
D_2 = np.array([2, 3])
visualize( D = D_2,  show_original=False, show_UDVT=False)
```

```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```


Unit circle transformed by V_T
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.0
−0.5
0.0
0.5
1.0

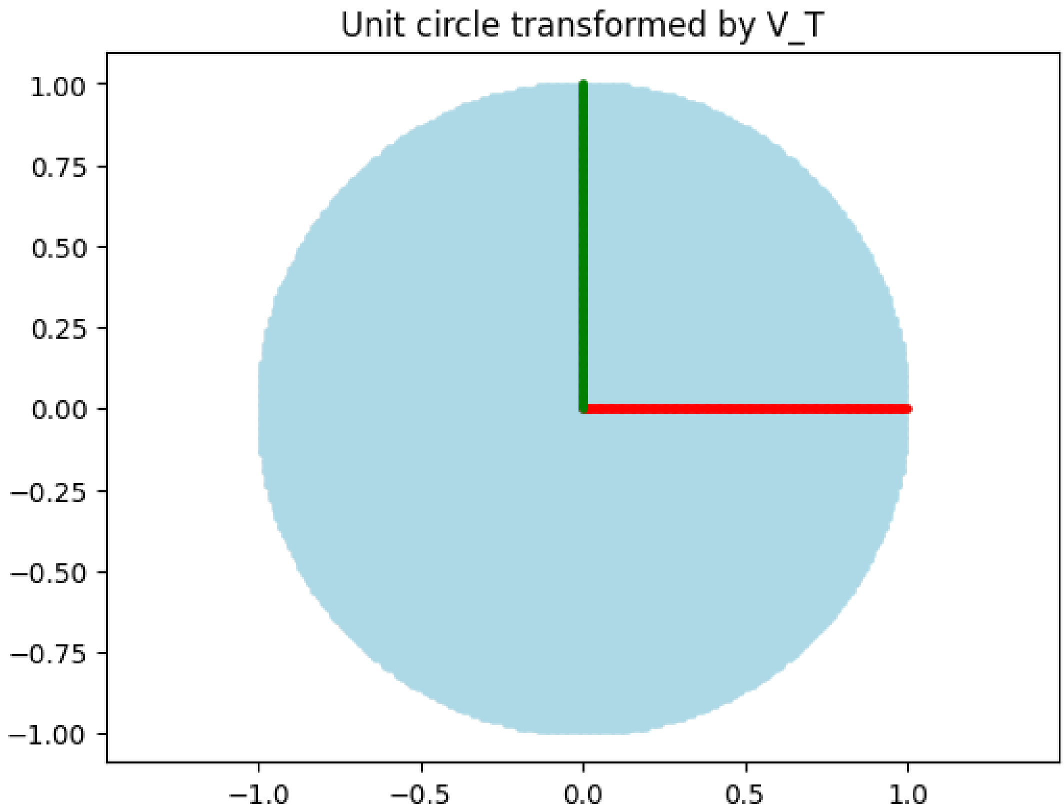


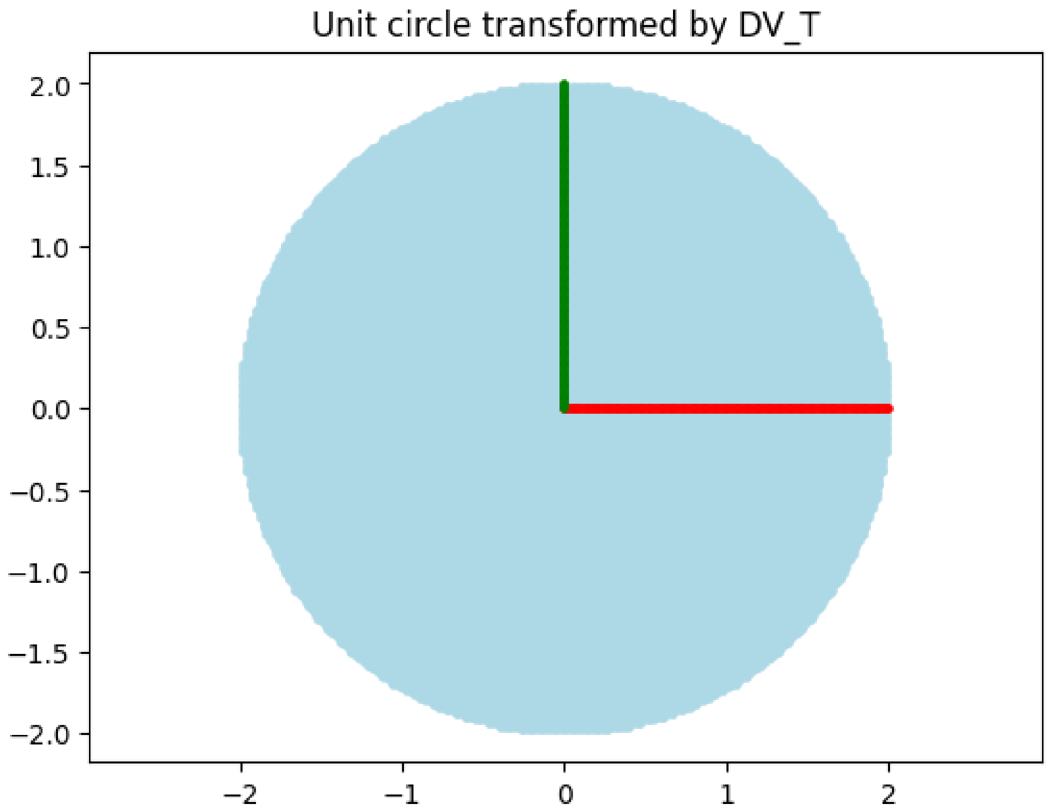
Still, the green and red axis stay the same. The major axis is the green line.

3.0.3 (c) iii: What can you say about the ellipse when D has entries [2, 2]? Here V is the identity matrix.

[ ]:
```
D_3 = np.array([2, 2])
visualize( D = D_3, show_original=False, show_UDVT=False)
```

```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```


Unit circle transformed by V_T
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.0
−0.5
0.0
0.5
1.0

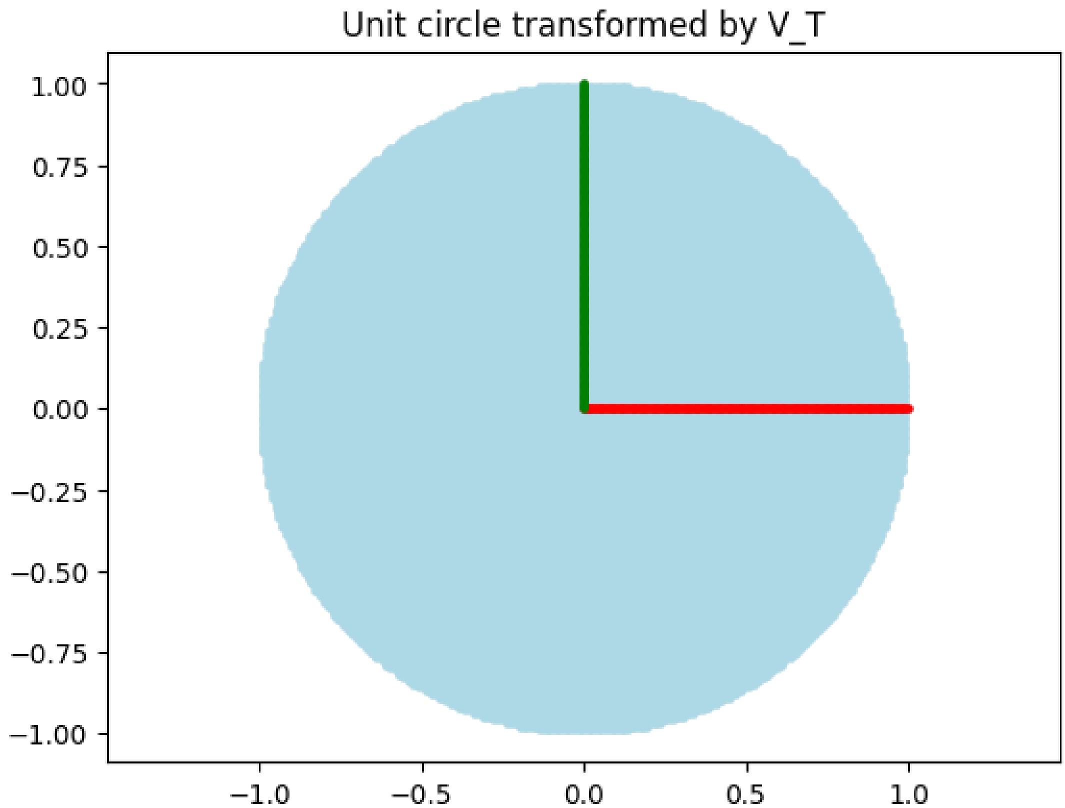


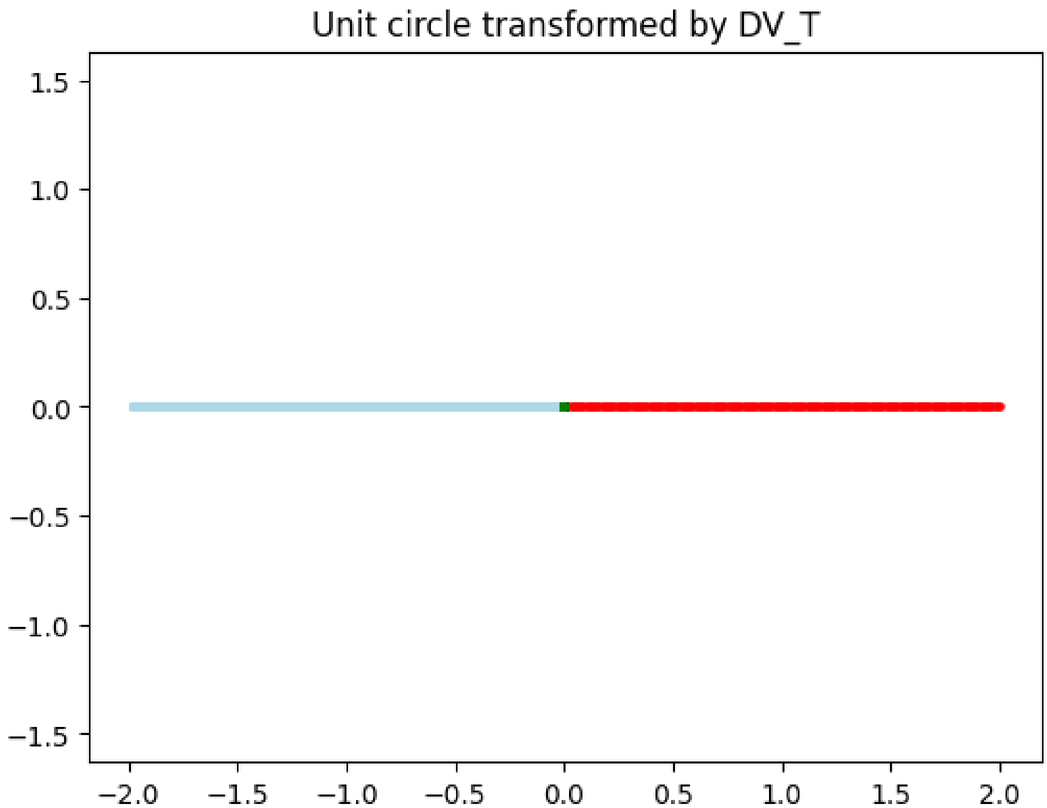
The whole unit circle is magnified by 2, with orientation the same.

3.0.4 (c) iv: What can you say about the ellipse when D has entries [2, 0]? Here V is the identity matrix.

[ ]:
```
D_4 = np.array([2, 0])
visualize( D = D_4, show_original=False, show_UDVT=False)
```

```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```


Unit circle transformed by V_T
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.0
−0.5
0.0
0.5
1.0



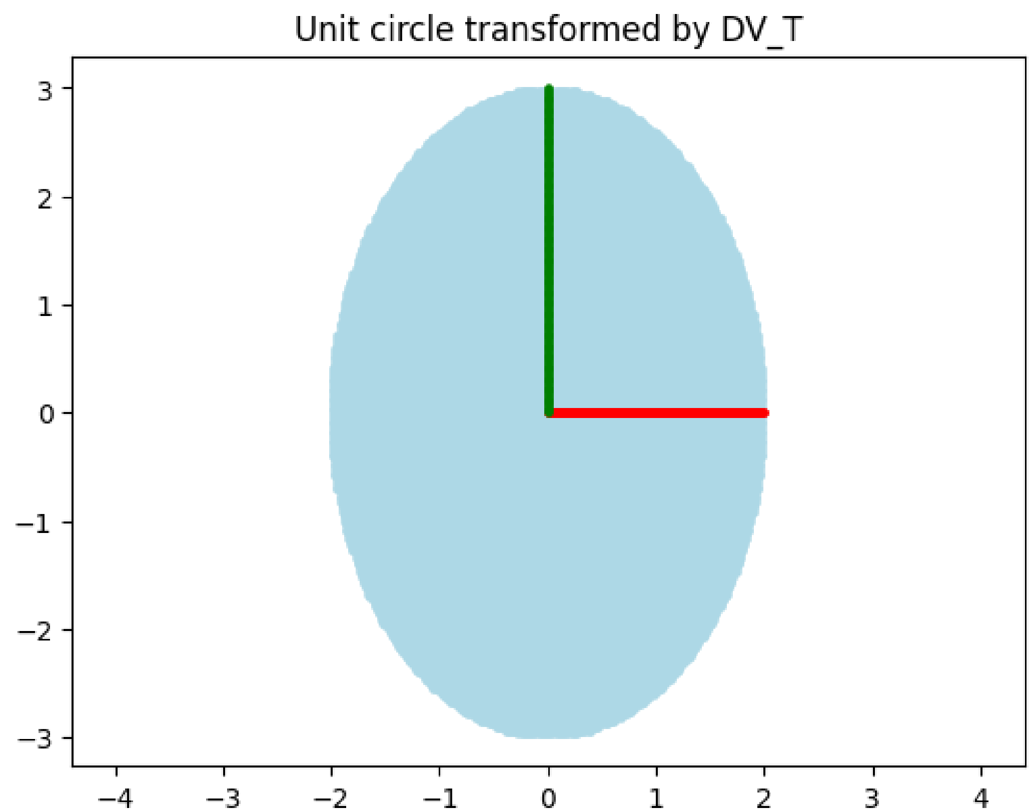
The dimension is reduced to 1. We project the ellipse to a line on x-axis.

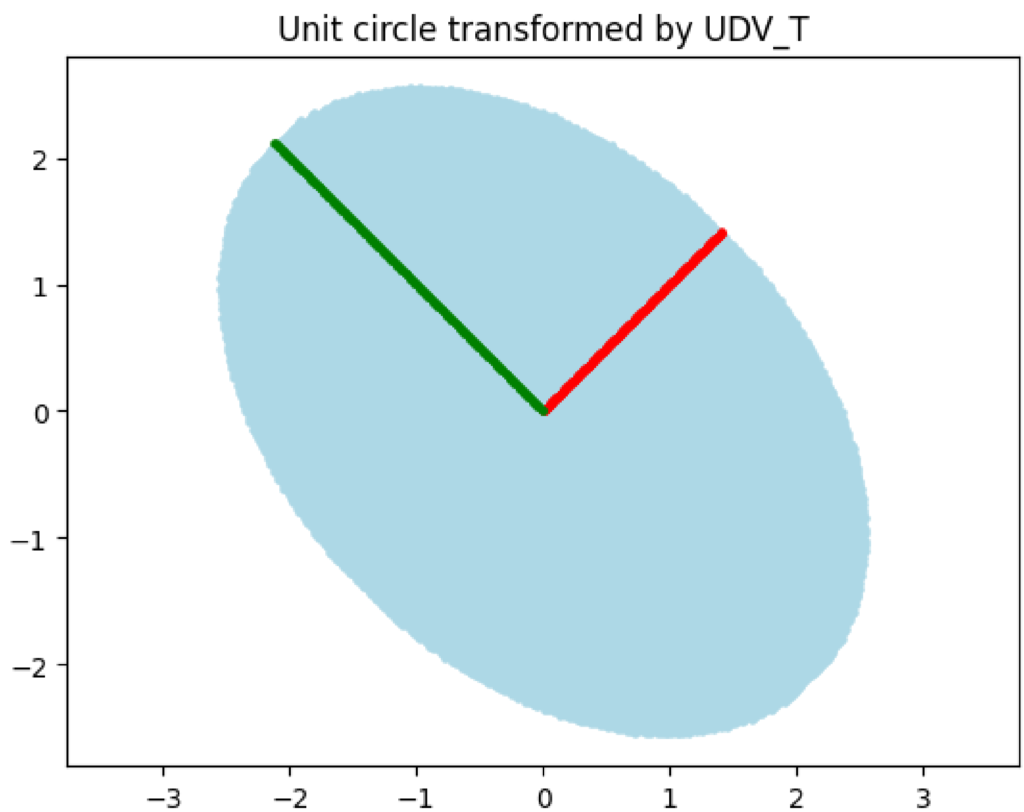
4 Effect of the linear transformation by an orthonormal matrix U

As we saw before for V^T, a 2 x 2 orthonormal matrix can be viewed as a linear transformation that performs some combination of rotations and reflections.

4.0.1 (d) i: Comment on the effect of $U = RCC\left(\frac{\pi}{4}\right)$ as in cell below. The value of D is in the code below and V is the identity matrix. What happened to the ellipse? Did the length of the major and minor axis change?

[ ]:
```
U_1 = get_RCC(np.pi/4)
visualize( U = U_1, D =np.array([2,3]), show_original=False, show_VT=False)
```

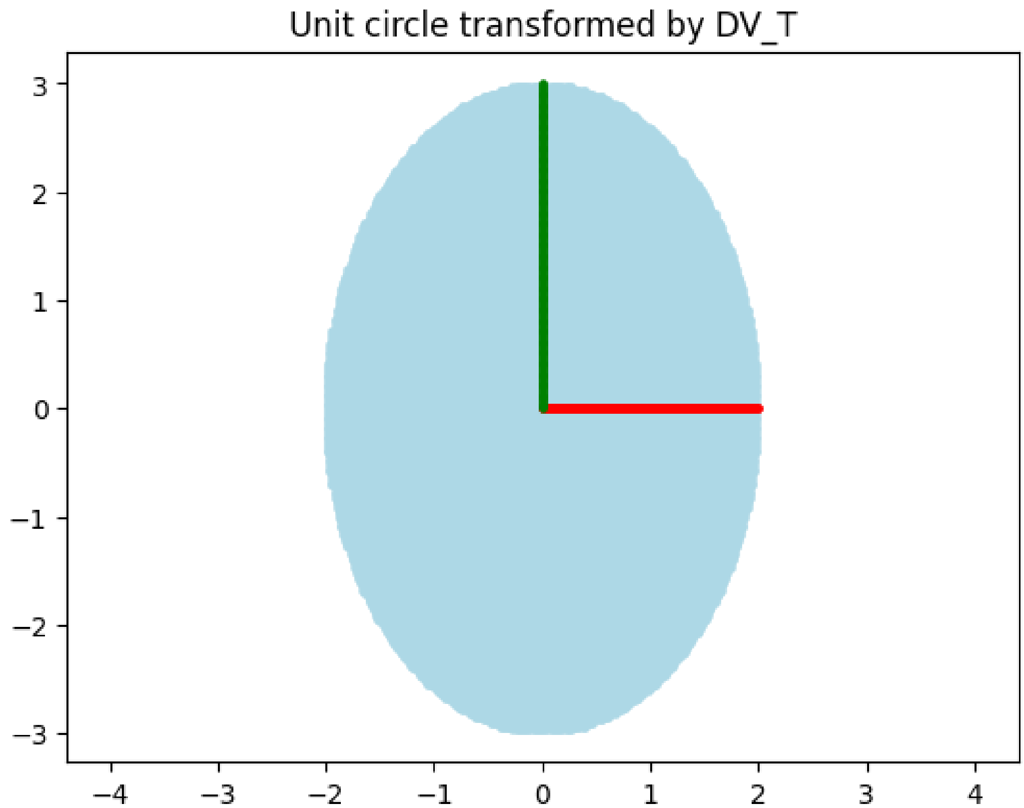

Unit circle transformed by DV_T
3
2
1
0
−1
−2
−3
−4
−3
−2
−1
0
1
2
3
4



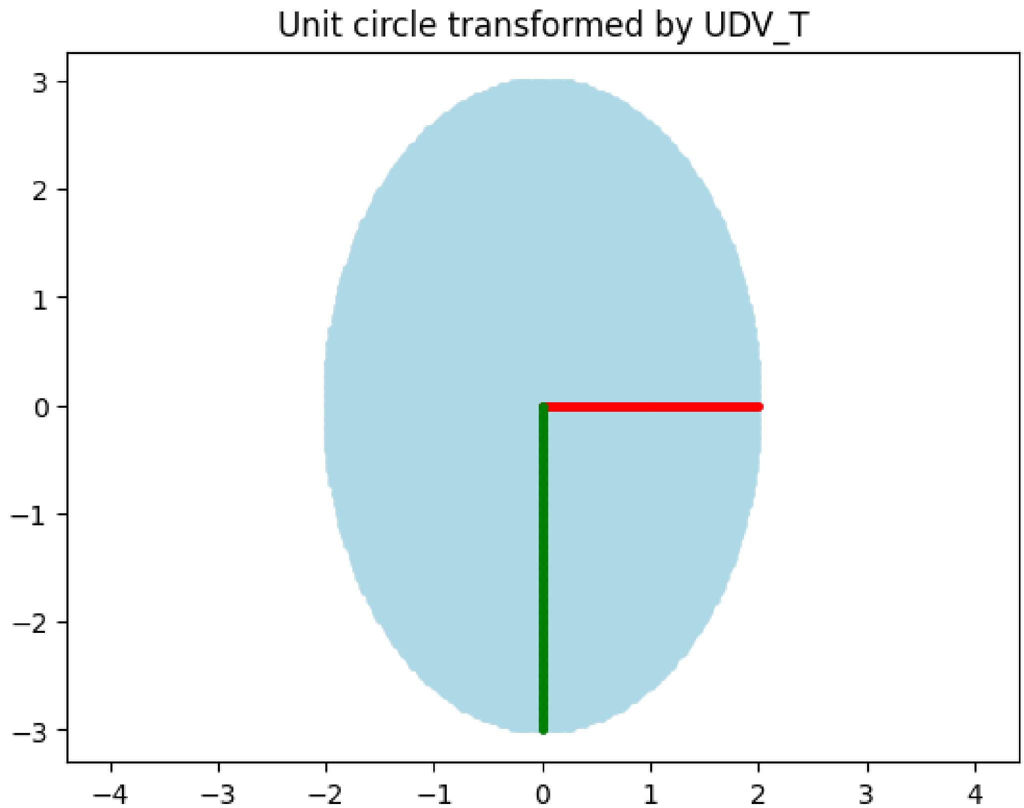
The ellipse is rotated by \$ / 4 \$. No, the magnitude of it stays the same.

4.0.2 (d) ii: Comment on the effect of $U = RFx()$ as in cell below. The value of D is in the code below and V is the identity matrix. What happened to the ellipse? Did length of major and minor axis change?

[ ]:
```
U_2 = get_RFx()
visualize( U = U_2, D =np.array([2,3]), show_original=False, show_VT=False)
```



Unit circle transformed by DV_T
3
2
1
0
−1
−2
−3
−4
−3
−2
−1
0
1
2
3
4

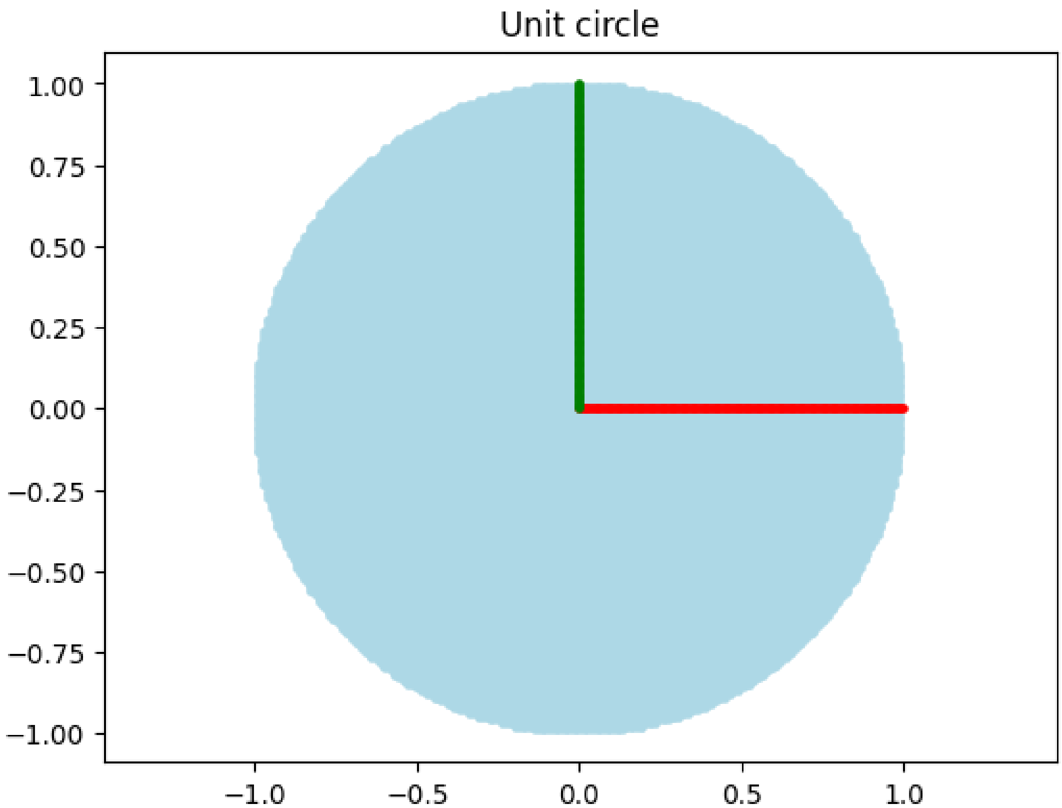


It is reflected by x-axis. No, the magnitude is the same.

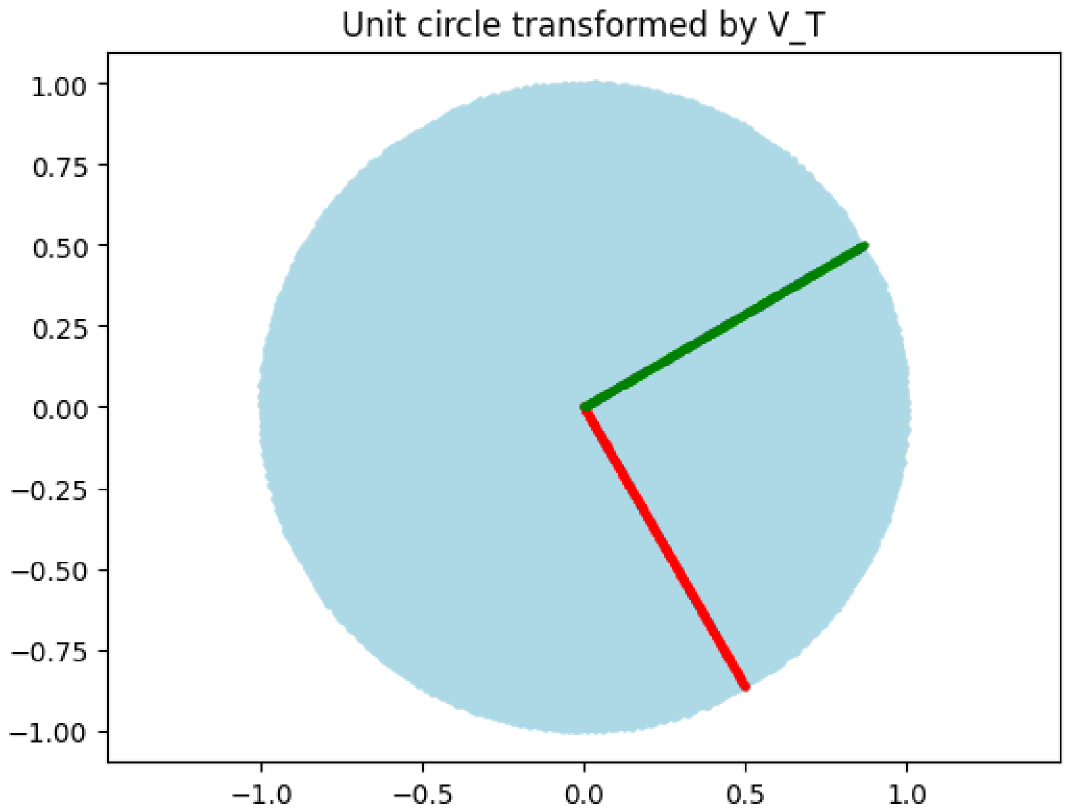
5 Putting everything together. Effect of linear transformation by UDV^T

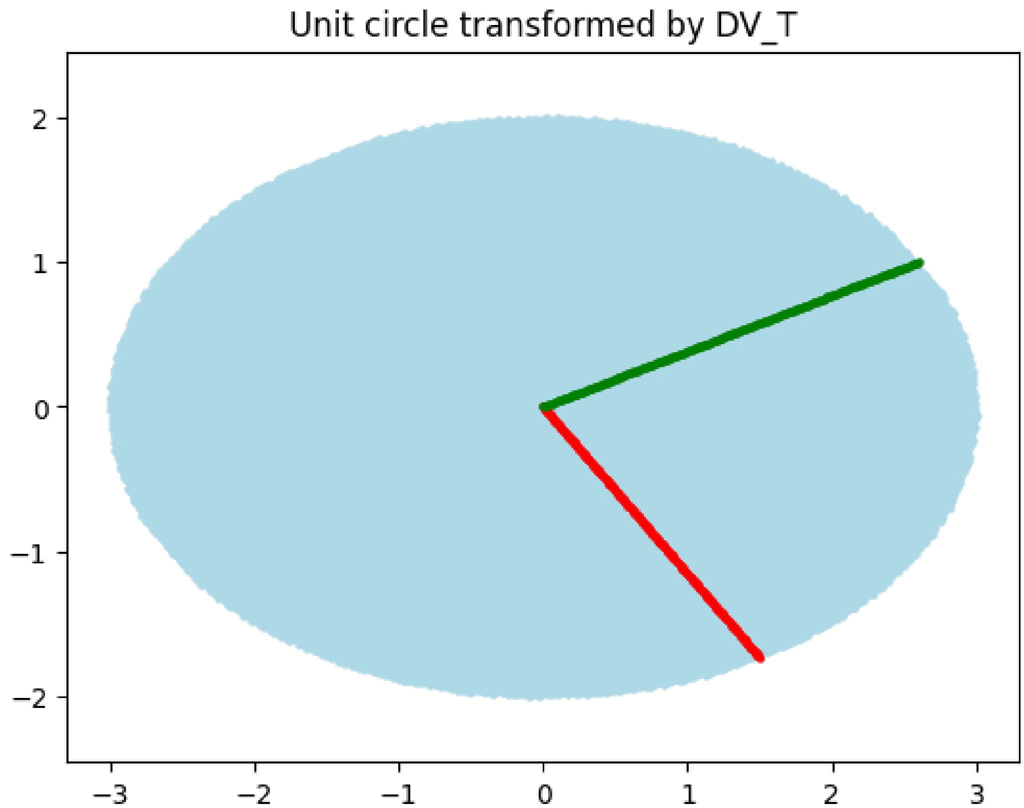
5.0.1 Case I

[ ]:
```
U = get_RCC(np.pi/4)
VT = get_RCC(-np.pi/3)
D = np.array([3,2])
visualize(U = U, VT= VT, D=D)
```

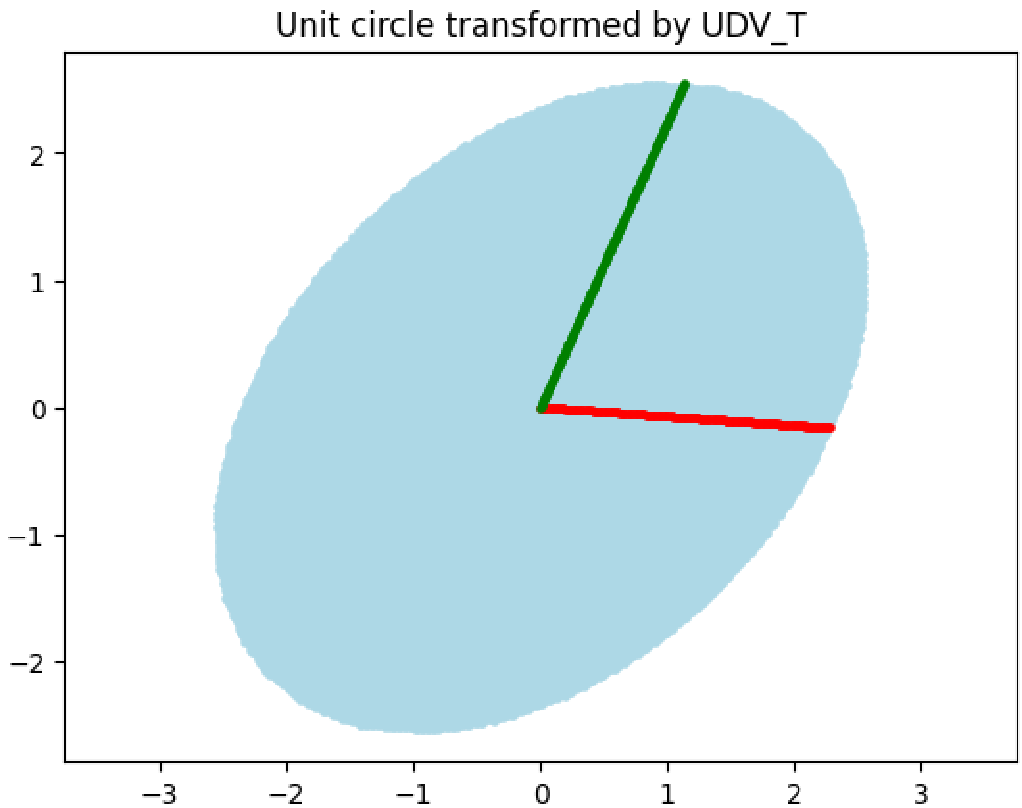



```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```


Unit circle transformed by V_T
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.0
−0.5
0.0
0.5
1.0



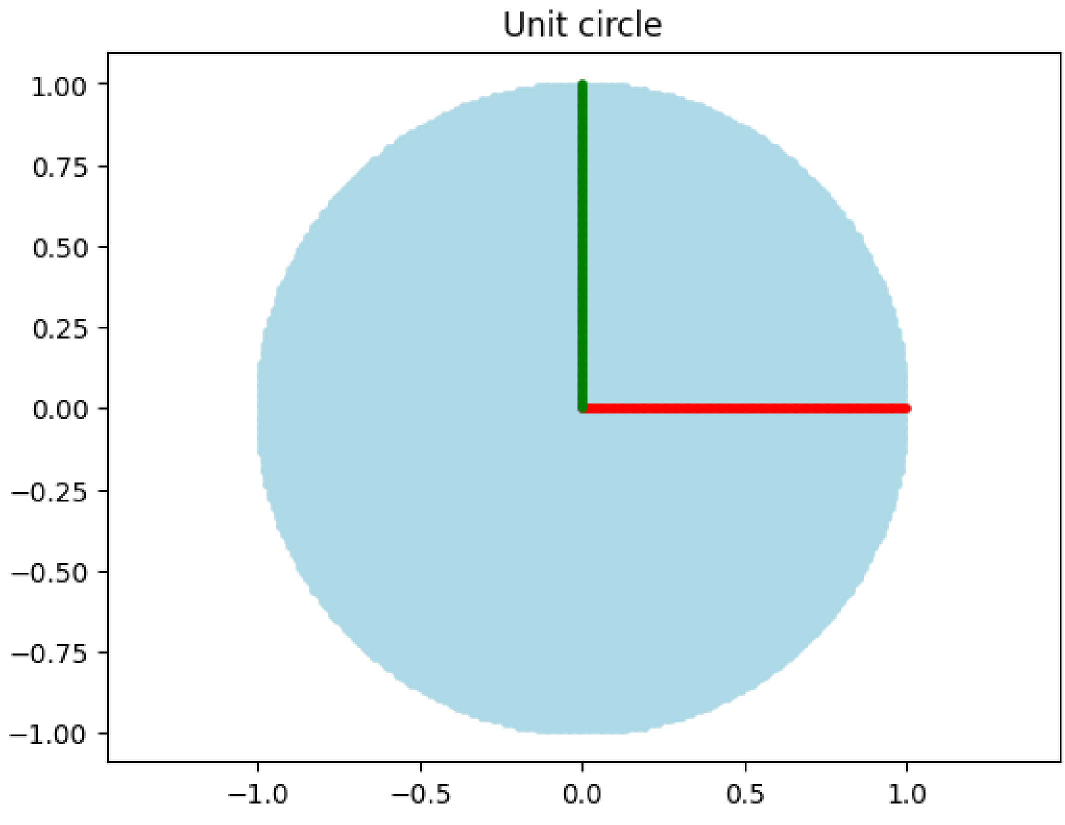
Unit circle transformed by DV_T
2
1
0
−1
−2
−3
−2
−1
0
1
2
3



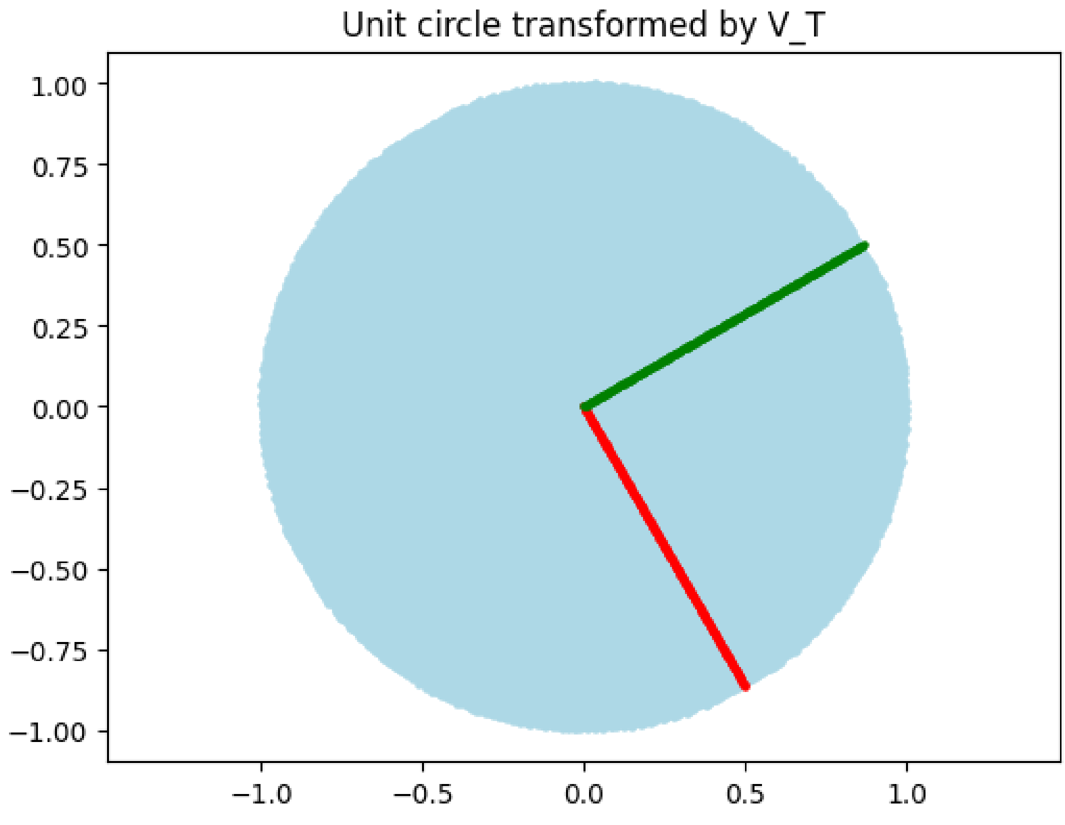
The above figures show the transformation after each step.

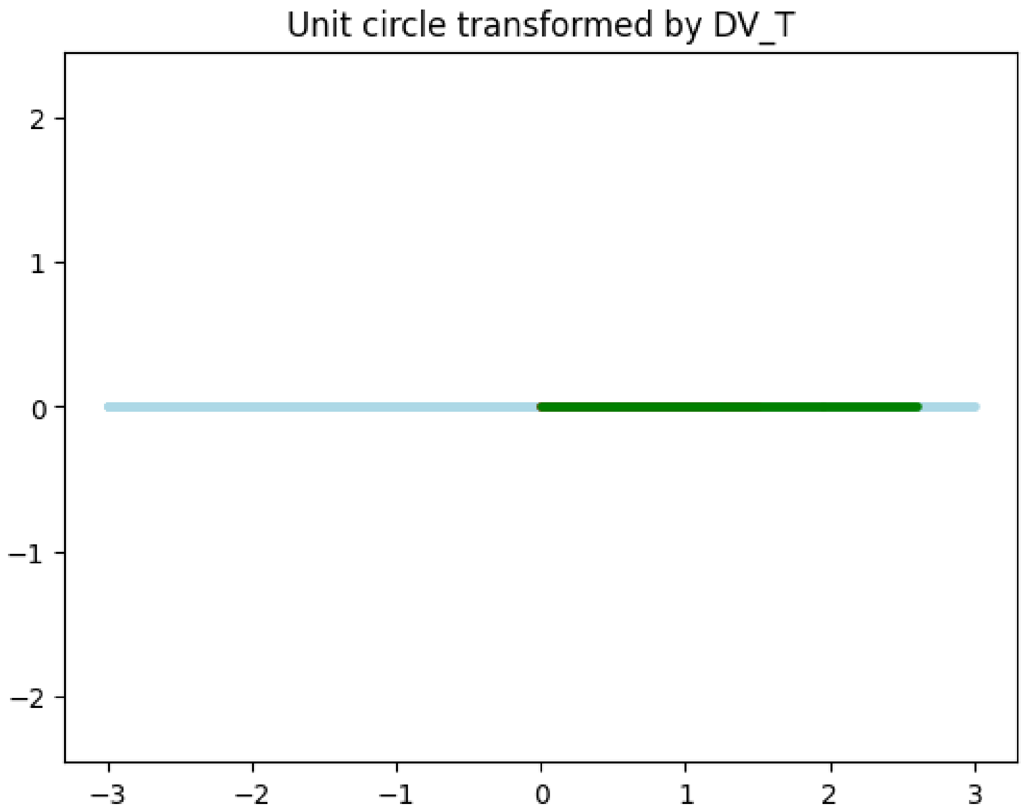
5.0.2 Case II

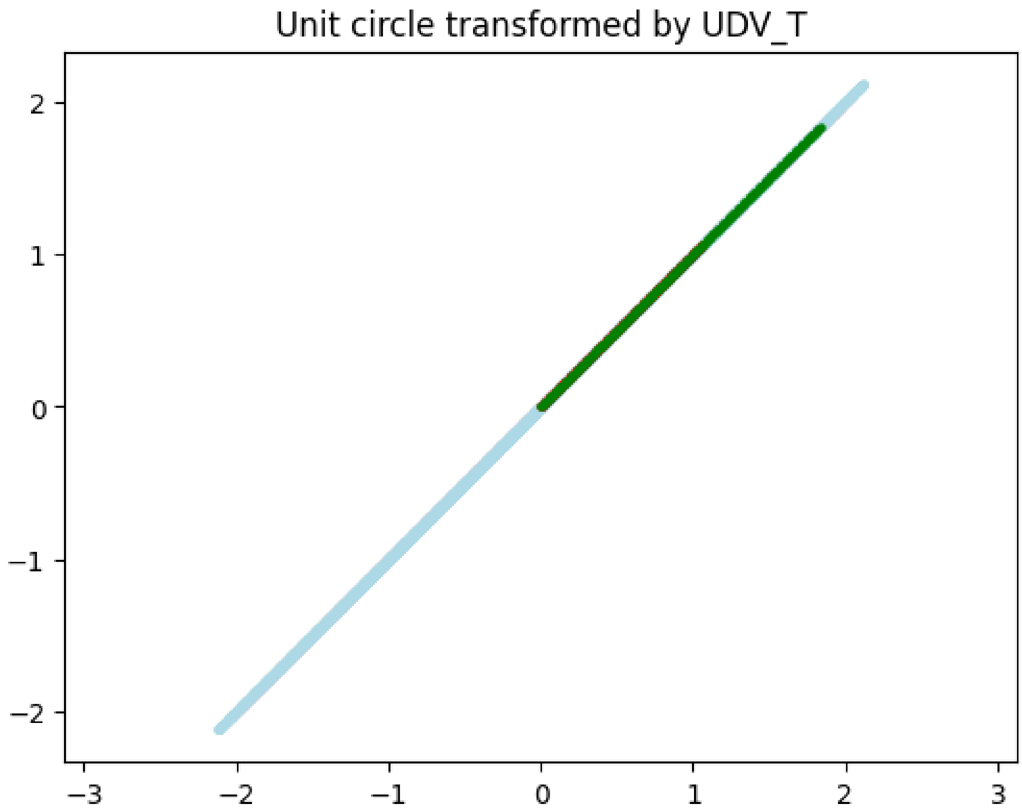
[ ]:
```
U = get_RCC(np.pi/4)
VT = get_RCC(-np.pi/3)
D = np.array([3,0])
visualize(U = U, VT= VT, D=D)
```




```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```


Unit circle transformed by V_T
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.0
−0.5
0.0
0.5
1.0

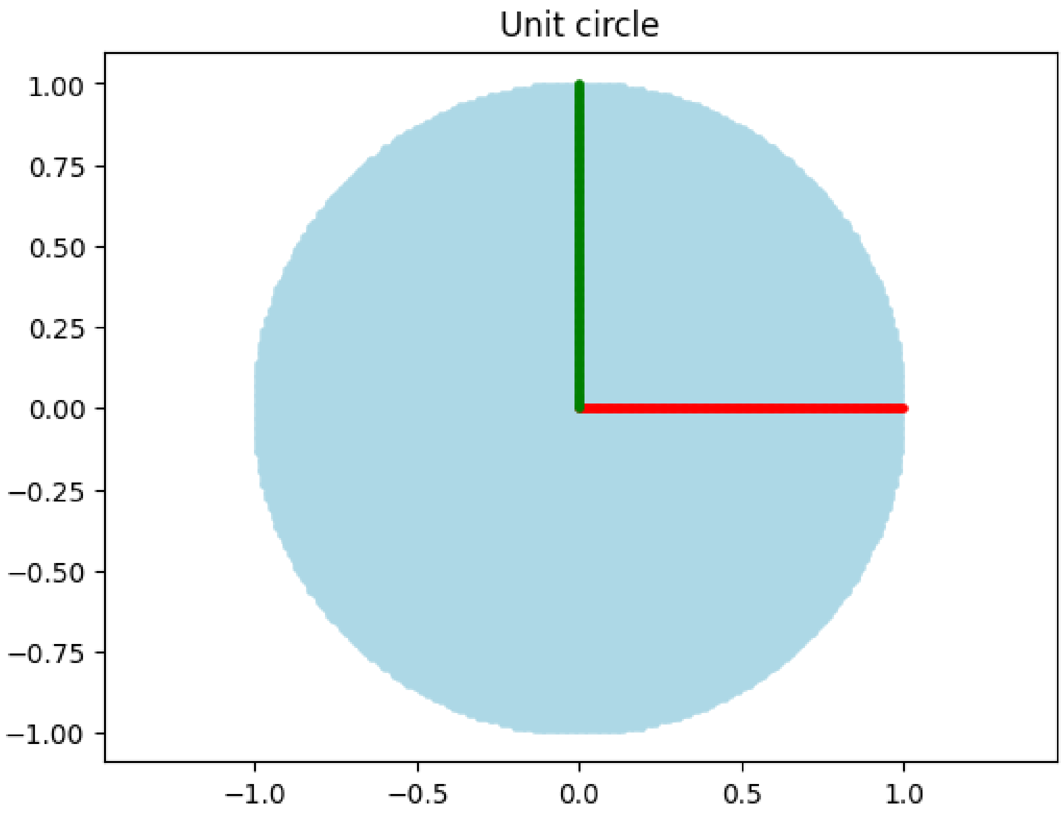

Unit circle transformed by DV_T
2
1
0
−1
−2
−3
−2
−1
0
1
2
3



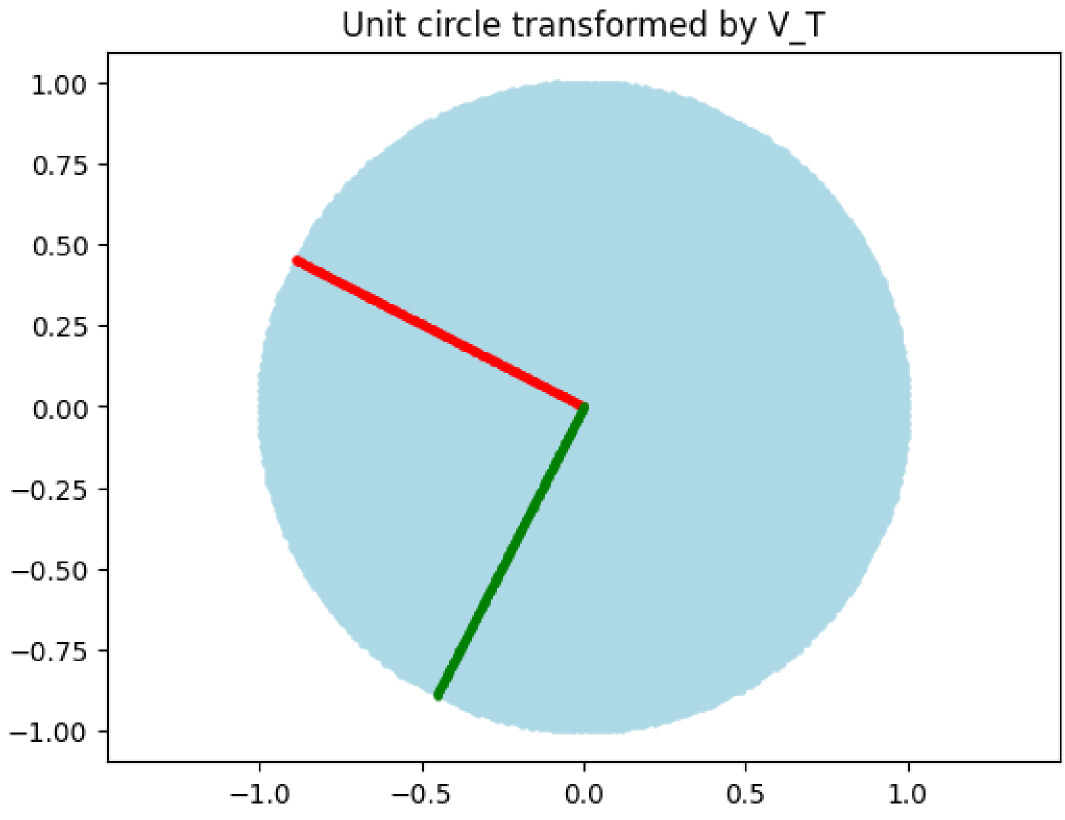
The above figures show the transformation after each step.

5.0.3 Case III

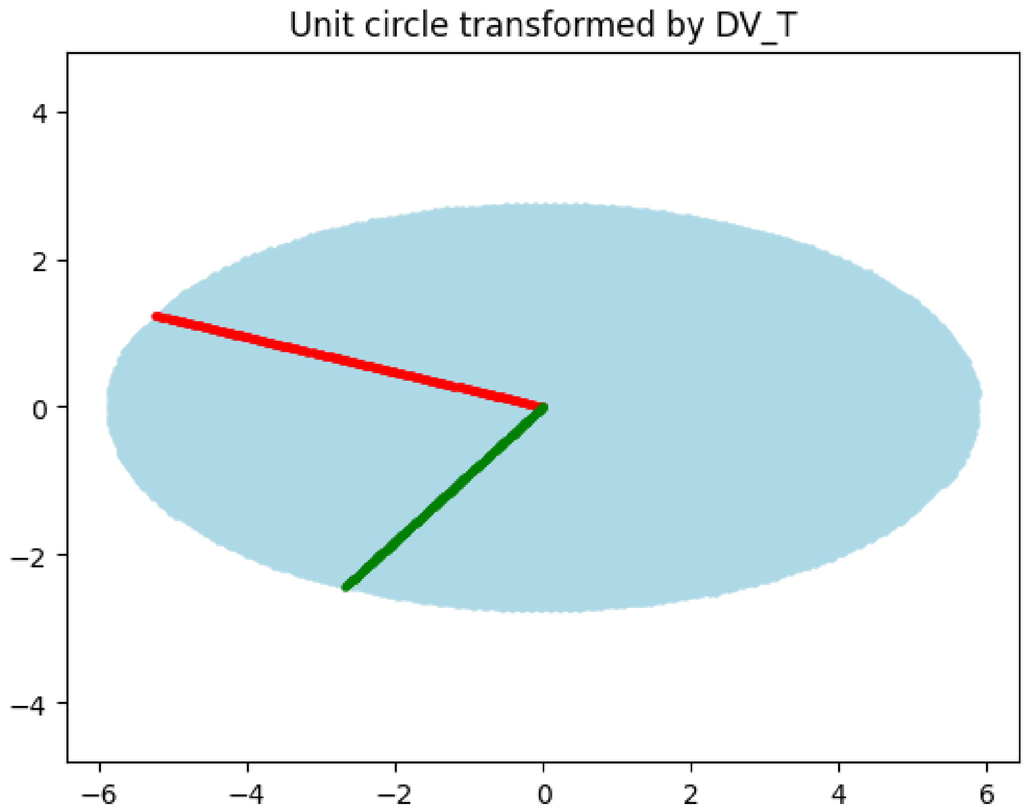
[ ]:
```
A = np.array([[5, 3], [2, -2]])
U,D,VT = np.linalg.svd(A)
visualize(U = U, D=D, VT=VT)
```

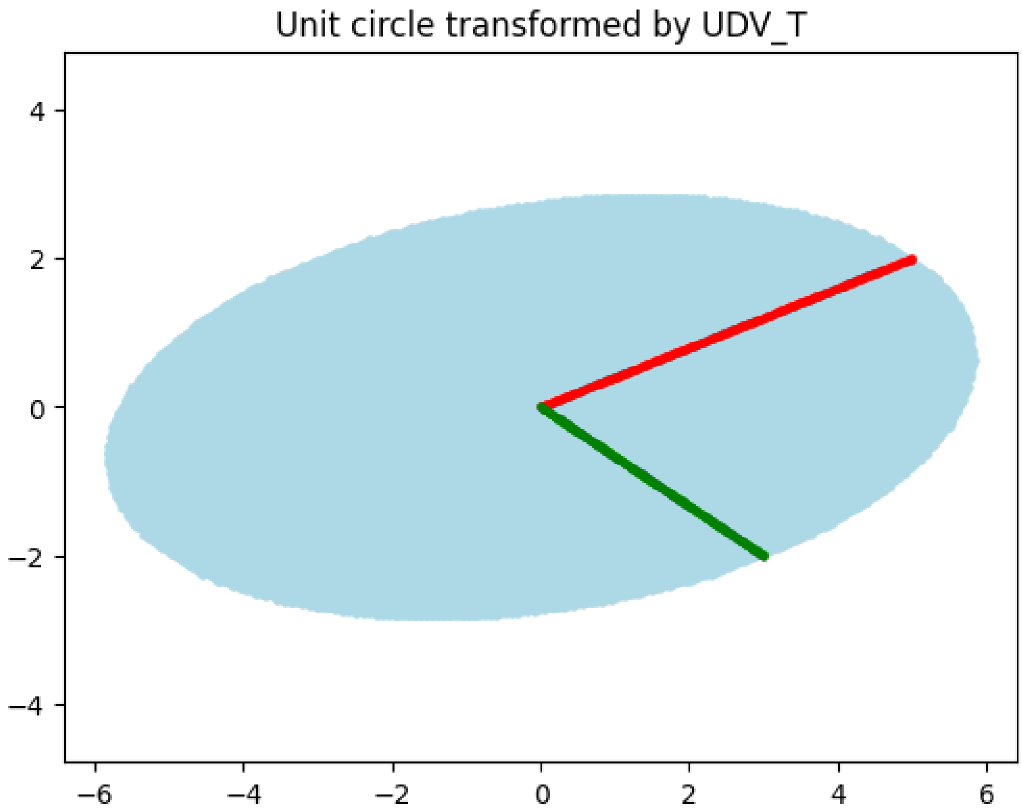



```
WARNING:matplotlib.axes._base:Ignoring fixed x limits to fulfill fixed data
aspect with adjustable data limits.
```



Unit circle transformed by V_T
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.0
−0.5
0.0
0.5
1.0


Unit circle transformed by DV_T
4
2
0
−2
−4
−6
−4
−2
0
2
4
6



5.0.4 (e): For case III, based on the figures obtained by running the cell, answer the following questions:

1) Is V^T a pure rotation, pure reflection or combination of both?
2) Let σ_1 and σ_2 denote the entries of the diagonal matrix in SVD of A, with $\sigma_1 > \sigma_2$. What is an approximate value of $\frac{\sigma_1}{\sigma_2}$?
3) Is U a pure rotation, pure reflection or combination of both?

1) $V^\top$ is pure rotation
2) We can approximate it by the ratio of the lengths of the long axis and short axis of the ellipse after the transformation.
3) A combination of both.

6 Exploration Area (Not part of homework question)

You are free to visualize the effect of the SVD transformation on the unit circle for whatever matrix you desire

[ ]:
```python
# #Sample format 1
# U = get_RCC(np.pi/4)
# VT = get_RCC(-np.pi/3)
# D = np.array([3,2])
# visualize(U = U, VT= VT, D=D)
```

[ ]:
```python
# #Sample format 2
# A = np.array([[5, 3], [2, -2]])
# U,D,VT = np.linalg.svd(A)
# visualize(U = U, D=D, VT=VT)
```

[ ]:
```python
from google.colab import drive
drive.mount('/content/drive')
```

```
Mounted at /content/drive
```

[ ]:
```python

```