Homework 01

This homework is due at 11 PM on January 30, 2026.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Course Setup**

   Please complete the following steps to get access to all course resources.

   (a) Visit the course website at http://eecs127.github.io/ and familiarize yourself with the syllabus.

   (b) Verify that you can access the class Ed site at https://edstem.org/us/courses/93760/discussion.

   (c) Verify that you can access the class Gradescope site at https://www.gradescope.com/courses/1228786.

2. What Prerequisites Have You Taken?

The prerequisites for this course are

- MATH 54 (Linear Algebra & Differential Equations),
- CS 70 (Discrete Mathematics & Probability Theory), and
- MATH 53 (Multivariable Calculus).

Please fill out the following Google form: https://forms.gle/gLgvN769ZxoDNzn59 to tell us which of these courses, or their equivalents, you have taken. If you are unsure of course material overlap, please refer to the MATH 54, CS 70 websites (http://www.sp22.eecs70.org/, https://lin-lin.github.io/MATH54/), and the MATH 53 textbook (*Multivariable Calculus* by James Stewart). **For the response to this question, write the secret word revealed at the end of the form.**

The course material this semester will rely on knowledge from these prerequisite courses. If you feel shaky on this material, please use the first week to reacquaint yourself with it. We expect you to handle this review on your own; we will not prioritize questions about prerequisite material in office hours.

3. Orthogonality

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ be two linearly independent unit-norm vectors; that is, $\|\vec{x}\|_2 = \|\vec{y}\|_2 = 1$.

(a) Show that the vectors $\vec{u} = \vec{x} - \vec{y}$ and $\vec{v} = \vec{x} + \vec{y}$ are orthogonal.

(b) Find an orthonormal basis for $\mathrm{span}(\vec{x}, \vec{y})$, the subspace spanned by $\vec{x}$ and $\vec{y}$.

4. Least Squares

The Michaelis-Menten model for enzyme kinetics relates the rate y of an enzymatic reaction to the concentration x of a substrate, as follows:

$$y = \frac{\beta_1 x}{\beta_2 + x}, \tag{1}$$

for constants $\beta_1, \beta_2 > 0$.

(a) Show that the model can be expressed as a linear relation between the values $1/y = y^{-1}$ and $1/x = x^{-1}$. Specifically, give an equation of the form $y^{-1} = w_1 + w_2 x^{-1}$, specifying the values of w_1 and w_2 in terms of β_1 and β_2.

(b) In general, reaction parameters β_1 and β_2 (and, thus, w_1 and w_2) are not known a priori and must be fit from data — for example, using least squares. Suppose you collect m measurements (x_i, y_i), $i = 1, \ldots, m$ over the course of a reaction. Formulate the least squares problem

$$\vec{w}^\star = \operatorname*{argmin}_{\vec{w}} \|X\vec{w} - \vec{y}\|_2^2, \tag{2}$$

where $\vec{w}^\star = \begin{bmatrix} w_1^\star & w_2^\star \end{bmatrix}^\top$, and you must specify $X \in \mathbb{R}^{m\times 2}$ and $\vec{y} \in \mathbb{R}^m$. Specifically, your solution should include explicit expressions for X and $\vec{y}$ as a function of (x_i, y_i) values and a final expression for $\vec{w}^\star$ in terms of X and $\vec{y}$, which should contain only matrix multiplications, transposes, and inverses.

Assume without loss of generality that $x_1 \neq x_2$.

(c) Assume that we have used the above procedure to calculate values for $w_1^\star$ and $w_2^\star$, and we now want to estimate $\widehat{\vec{\beta}} = \begin{bmatrix} \widehat{\beta}_1 & \widehat{\beta}_2 \end{bmatrix}^\top$. Write an expression for $\widehat{\vec{\beta}}$ in terms of $w_1^\star$ and $w_2^\star$.

NOTE: This problem was taken (with some edits) from the textbook *Optimization Models* by Calafiore and El Ghaoui.

5. Subspaces and Dimensions

Consider the set $\mathcal{S}$ of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0. \tag{3}$$

(a) Find a 2×3 matrix A for which $\mathcal{S}$ is exactly the null space of A.

(b) Determine the dimension of $\mathcal{S}$ and find a basis for it.

6. Vector Spaces and Rank

The *rank* of a $m \times n$ matrix A, $\mathrm{rank}(A)$, is the dimension of its *range*, also called span, and denoted $\mathcal{R}(A) := \{A\vec{x} \,:\, \vec{x} \in \mathbb{R}^n\}$.

(a) Assume that $A \in \mathbb{R}^{m\times n}$ takes the form $A = \vec{u}\vec{v}^\top$, with $\vec{u} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $\vec{u}, \vec{v} \neq \vec{0}$. (Note that a matrix of this form is known as a *dyad*.) Find the rank of A.

HINT: Consider the quantity $A\vec{x}$ for arbitrary $\vec{x}$, i.e., what happens when you multiply any vector by matrix A.

(b) Show that for arbitrary $A, B \in \mathbb{R}^{m\times n}$,

$$\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B), \tag{4}$$

i.e., the rank of the sum of two matrices is less than or equal to the sum of their ranks.

HINT: First, show that $\mathcal{R}(A + B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$, meaning that any vector in the range of $A + B$ can be expressed as the sum of two vectors, each in the range of A and B, respectively. Remember that for any matrix A, $\mathcal{R}(A)$ is a subspace, and for any two subspaces S_1 and S_2, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.[1] (Note that the sum of vector spaces $S_1 + S_2$ is not equivalent to $S_1 \cup S_2$, but is defined as $S_1 + S_2 := \{\vec{s_1} + \vec{s_2} | \vec{s_1} \in S_1, \vec{s_2} \in S_2\}$.)

(c) Consider an $m \times n$ matrix A that takes the form $A = UV^\top$, with $U \in \mathbb{R}^{m\times k}$, $V \in \mathbb{R}^{n\times k}$. Show that the rank of A is less than or equal to k. *HINT: Use parts 6(a) and 6(b), and remember that this decomposition can also be written as the* dyadic expansion

$$A = UV^\top = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_k^\top \end{bmatrix} = \sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top, \tag{5}$$

for $U = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix}$ and $V = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$.

[1]This fact can be proved by taking a basis of S_1 and extending it to a basis of S_2 (during which we can only add *at most* $\dim(S_2)$ basis vectors). This extended basis must now also be a basis of $S_1 + S_2$. Thus, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.

7. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.