UNIVERSITY OF CALIFORNIA, BERKELEY
DEPARTMENT OF STATISTICS

Homework #2

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Problem 1. Proof of the Fundamental Theorem of Linear Algebra

In this question, we will prove the fundamental theorem of linear algebra. For any $A \in \mathbb{R}^{m\times n}$, let $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\text{rank}(A)$ denote the null space, range, and rank of A respectively. For any subspace S with dimension $\dim(S)$, let $S^\perp$ denote its orthogonal subspace.
The fundamental theorem of linear algebra states that

$$\mathcal{N}(A) \oplus \mathcal{R}(A^\top) = \mathbb{R}^n. \tag{1}$$

The proof technique we employ will first show that

$$\mathcal{N}(A) = \mathcal{R}(A^\top)^\perp. \tag{2}$$

Then we will prove that we can find orthonormal vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ such that

$$\mathcal{N}(A) = \text{span}(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_\ell)$$

and

$$\mathcal{R}(A^\top) = \text{span}(\vec{e}_{\ell+1}, \vec{e}_{\ell+2}, \dots, \vec{e}_n).$$

As a corollary, we get the ranknullity theorem:

$$\dim(\mathcal{N}(A)) + \text{rank}(A) = n. \tag{3}$$

(a) First, show that $\mathcal{N}(A) \subseteq \mathcal{R}(A^\top)^\perp$.

HINT: Consider $\vec{u} \in \mathcal{N}(A)$, $\vec{v} \in \mathcal{R}(A^\top)$ and show that $\vec{u}^\top \vec{v} = 0$.

(b) Now show that $\mathcal{R}(A^\top)^\perp \subseteq \mathcal{N}(A)$.

HINT: Show that any vector $\vec{v}$ that is orthogonal to all vectors in the range of $A^\top$ satisfies $A\vec{v} = 0$. To do this, consider $\vec{v} \in \mathcal{R}(A^\top)^\perp$ and what it implies for $\vec{v}^\top A^\top$.

Answer.

(a) Let $\vec{u} \in \mathcal{N}(A)$. Then we have, $A\vec{u} = \vec{0}$.

$\forall \vec{v} \in \mathcal{R}(A^\top), \quad \exists \vec{w}, \quad s.t. \vec{v} = A^\top \vec{w}.$

$$\vec{u}^\top \vec{v} = \vec{u}^\top A^\top \vec{w} = (A\vec{u})^\top \vec{w} = \vec{0}^\top \vec{w} = 0.$$

Thus, $\vec{u}$ is orthogonal to every vector in $\mathcal{R}(A^\top)$, which implies

$$\vec{u} \in \mathcal{R}(A^\top)^\perp.$$

Hence,

$$\mathcal{N}(A) \subseteq \mathcal{R}(A^\top)^\perp.$$

(b) Let $\vec{v} \in \mathcal{R}(A^\top)^\perp$. Then $\forall \vec{w}$,

$$\vec{v}^\top A^\top \vec{w} = 0.$$

This implies

$$(A\vec{v})^\top \vec{w} = 0 \quad \text{for all } \vec{w}.$$

Take $\vec{w} = A\vec{v}$, we have

$$(A\vec{v})^\top (A\vec{v}) = \|A\vec{v}\|_2^2 = 0.$$

Therefore,

$$A\vec{v} = \vec{0},$$
$$\mathcal{R}(A^\top)^\perp \subseteq \mathcal{N}(A).$$

Combining parts (a) and (b), we have

$$\mathcal{N}(A) = \mathcal{R}(A^\top)^\perp.$$

Problem 1. cont'd

(c) Note that we could apply the orthogonal decomposition theorem (Theorem 19 in the course reader) at this point to complete the proof. However, instead well work through how to re-derive that result directly.

Let $\dim(\mathcal{N}(A)) = \ell$ and let $\vec{e}_1, \ldots, \vec{e}_\ell$ be an orthonormal basis for $\mathcal{N}(A)$. Consider an extension of the basis to an orthonormal basis $\vec{e}_1, \ldots, \vec{e}_n$ for $\mathbb{R}^n$.

We will prove that $\vec{e}_{\ell+1}, \ldots, \vec{e}_n$ form a basis for $\mathcal{R}(A^\top)$ and as a consequence, the dimension of $\mathcal{R}(A^\top)$ is $n - \ell$.

i. Show that $\mathcal{R}(A^\top)$ lies in the span of $\vec{e}_{\ell+1}, \ldots, \vec{e}_n$.
HINT: Express any vector $\vec{u} \in \mathcal{R}(A^\top)$ as $\vec{u} = \sum_{i=1}^{n} \alpha_i \vec{e}_i$. What are the values of α_i?

ii. From part (i) we know that $\mathcal{R}(A^\top) \subseteq \text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$, but we want something stronger. Show that in fact

$$\mathcal{R}(A^\top) = \text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n).$$

HINT: First, prove

$$\dim(\mathcal{R}(A^\top)) = \dim(\text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)) = n - \ell$$

by contradiction. Assume $\dim(\mathcal{R}(A^\top)) = k < n - \ell$.
Show that a vector $\vec{u} \in \text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$ and $\vec{u} \notin \mathcal{R}(A^\top)$ cannot exist.
Specifically, let $\vec{f}_1, \ldots, \vec{f}_k$ be an orthonormal basis for $\mathcal{R}(A^\top)$. We can find a non-zero vector

$$\vec{u}^\perp = \vec{u} - \sum_{i=1}^{k} (\vec{f}_i^\top \vec{u}) \vec{f}_i$$

that is orthogonal to $\mathcal{R}(A^\top)$.
Does $\vec{u}^\perp$ lie in $\mathcal{N}(A)$? Does $\vec{u}^\perp$ also lie in $\text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$? Does this lead to a contradiction?
Think of $n - \ell = 3$ and $k = 2$ for visualization.
HINT: Second, you can use without proof the fact that for two subspaces $S_1 \subseteq S_2$, if $\dim(S_1) = \dim(S_2)$ then $S_1 = S_2$.

(d) Using part (c), argue why

$$\mathcal{N}(A) \oplus \mathcal{R}(A^\top) = \mathbb{R}^n$$

and why the ranknullity theorem holds.

Answer.

(c) $\{\vec{e}_1, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$.

(i) $\forall \vec{u} \in \mathcal{R}(A^\top)$. Since $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$,

$$\vec{u} = \sum_{i=1}^{n} \alpha_i \vec{e}_i.$$

We multiply both sides by $\vec{e}_i^\top$, since $\vec{e}_i$ are orthonormal and $\vec{e}_i \in \mathcal{R}(A^\top)^\perp$,

$$\alpha_i = \vec{e}_i^\top \vec{u} = 0 \quad \text{for } i = 1, \ldots, \ell.$$

Therefore,

$$\vec{u} = \sum_{i=\ell+1}^{n} \alpha_i \vec{e}_i,$$

$$\mathcal{R}(A^\top) \subseteq \text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n).$$

(ii) The dimension of $\text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$ is $n - \ell$.
Suppose for contradiction that

$$\dim(\mathcal{R}(A^\top)) = k < n - \ell.$$

Let $\{\vec{f}_1, \ldots, \vec{f}_k\}$ be an orthonormal basis for $\mathcal{R}(A^\top)$. Choose a nonzero vector

$$\vec{u} \in \text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n) \quad \text{with} \quad \vec{u} \notin \mathcal{R}(A^\top).$$

Define

$$\vec{u}^\perp = \vec{u} - \sum_{i=1}^{k} (\vec{f}_i^\top \vec{u}) \vec{f}_i.$$

Then $\vec{u}^\perp \neq \vec{0}$ and $\vec{u}^\perp \perp \mathcal{R}(A^\top)$, so

$$\vec{u}^\perp \in \mathcal{R}(A^\top)^\perp = \mathcal{N}(A).$$

At the same time, $\vec{u}^\perp$ lies in $\text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$ by construction. This contradicts the orthogonality between $\mathcal{N}(A) = \text{span}(\vec{e}_1, \ldots, \vec{e}_\ell)$ and $\text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$.
Hence,

$$\dim(\mathcal{R}(A^\top)) = n - \ell,$$

and therefore,

$$\mathcal{R}(A^\top) = \text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n).$$

(d) From part (c), we have an orthonormal basis of $\mathbb{R}^n$ such that

$$\mathbb{R}^n = \text{span}(\vec{e}_1, \ldots, \vec{e}_\ell) \oplus \text{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n) = \mathcal{N}(A) \oplus \mathcal{R}(A^\top).$$

This proves the fundamental theorem of linear algebra.
Since $\dim(\mathcal{N}(A)) = \ell$ and $\dim(\mathcal{R}(A^\top)) = \text{rank}(A)$,

$$\dim(\mathcal{N}(A)) + \text{rank}(A) = n.$$

Problem 2. Eigenvalues of Symmetric Matrices

Let $A \in \mathbb{S}^n$ (i.e., the set of $n \times n$ real symmetric matrices) with eigenvalues λ_i. Prove that all of the eigenvalues of A are real, i.e., $\lambda_i \in \mathbb{R}$ for each i.
HINT: Consider the quantity $(Av)^*v$ for eigenvector v, where * denotes the conjugate transpose. Note that this is the Hermitian inner product between Av and v.
NOTE: This exercise is part of the proof of the spectral theorem.

Answer. Let λ be an eigenvalue of A with (possibly complex) eigenvector $v \neq 0$, so

$$Av = \lambda v.$$

Consider

$$(Av)^*v = (\lambda v)^*v = \overline{\lambda}\, v^*v.$$

On the other hand, since A is real symmetric, we have $A^* = A$, and therefore

$$(Av)^*v = v^*A^*v = v^*Av = \lambda\, v^*v.$$

Thus,

$$\overline{\lambda}\, v^*v = \lambda\, v^*v.$$

Since $v \neq 0$, we have $v^*v > 0$, so

$$\overline{\lambda} = \lambda,$$

which implies $\lambda \in \mathbb{R}$.

Problem 3. Distinct Eigenvalues, Orthogonal Eigenspaces

Let $A \in \mathbb{S}^n$ (i.e., the set of $n \times n$ real symmetric matrices) and $(\lambda_1, \vec{u}_1)$, $(\lambda_2, \vec{u}_2)$ with $\lambda_1 \neq \lambda_2$ be distinct eigen-pairs of A.
Show that $\vec{u}_1^\top \vec{u}_2 = 0$, i.e., eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.
HINT: First try to prove that

$$\lambda_1 \vec{u}_1^\top \vec{u}_2 = \lambda_2 \vec{u}_1^\top \vec{u}_2,$$

then show that this implies $\vec{u}_1^\top \vec{u}_2 = 0$.
NOTE: This exercise is part of the proof of the spectral theorem.

Answer. Let

$$A\vec{u}_1 = \lambda_1 \vec{u}_1, \qquad A\vec{u}_2 = \lambda_2 \vec{u}_2.$$

Consider the scalar $\vec{u}_1^\top A \vec{u}_2$. On one hand,

$$\vec{u}_1^\top A \vec{u}_2 = \vec{u}_1^\top (\lambda_2 \vec{u}_2) = \lambda_2 \vec{u}_1^\top \vec{u}_2.$$

On the other hand, since A is symmetric, $A^\top = A$, so

$$\vec{u}_1^\top A \vec{u}_2 = \vec{u}_1^\top A^\top \vec{u}_2 = \lambda_1 \vec{u}_1^\top \vec{u}_2.$$

Therefore,

$$\lambda_2 \vec{u}_1^\top \vec{u}_2 = \lambda_1 \vec{u}_1^\top \vec{u}_2 \quad \Longrightarrow \quad (\lambda_2 - \lambda_1)\, \vec{u}_1^\top \vec{u}_2 = 0.$$

$$\lambda_1 \neq \lambda_2 \Longrightarrow \vec{u}_1^\top \vec{u}_2 = 0.$$

Problem 4. GramSchmidt

Any set of n linearly independent vectors in $\mathbb{R}^n$ could be used as a basis for $\mathbb{R}^n$. However, certain bases could be more suitable for certain operations than others. For example, an orthonormal basis could facilitate solving linear equations.

(a) Given a matrix $A \in \mathbb{R}^{n\times n}$, it could be represented as

$$A = QR, \tag{4}$$

where $Q \in \mathbb{R}^{n\times n}$ is an orthonormal matrix and $R \in \mathbb{R}^{n\times n}$ is an upper-triangular matrix.

For the matrix A, describe how the GramSchmidt process could be used to find the Q and R matrices, and apply this to

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 4 & -4 & -7 \\ 0 & 3 & 3 \end{bmatrix}. \tag{5}$$

(b) Given an invertible matrix $A \in \mathbb{R}^{n\times n}$ and an observation vector $\vec{b} \in \mathbb{R}^n$, the solution to

$$A\vec{x} = \vec{b} \tag{6}$$

is given by $\vec{x} = A^{-1}\vec{b}$.

For the matrix $A = QR$ from part (a), assume that we want to solve

$$A\vec{x} = \begin{bmatrix} 8 \\ -6 \\ 3 \end{bmatrix}. \tag{7}$$

By using the fact that Q is orthonormal, find $\vec{v}$ such that

$$R\vec{x} = \vec{v}. \tag{8}$$

Then, given the upper-triangular matrix R and $\vec{v}$, find the elements of $\vec{x}$ sequentially.

(c) Given an invertible matrix $B \in \mathbb{R}^{n\times n}$ and an observation vector $\vec{c} \in \mathbb{R}^n$, find the computational cost of finding the solution $\vec{z}$ to

$$B\vec{z} = \vec{c}$$

using the QR decomposition of B.

Assume that Q and R are available, and adding, multiplying, and dividing scalars take one unit of computation.

As examples: Computing an inner product $\vec{a}^\top\vec{b}$ is $O(n)$; Matrixvector multiplication is $O(n^2)$; Matrix inversion is $O(n^3)$.

This is why $A^{-1}\vec{b}$ is usually not computed directly.

Answer.

(a) For

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 4 & -4 & -7 \\ 0 & 3 & 3 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} -3 \\ -4 \\ 3 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ -7 \\ 3 \end{bmatrix},$$

we compute $\|a_1\| = 5$, so $q_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$.

Next,

$$r_{12} = q_1^\top a_2 = \frac{1}{5}(3, -4, 0) \cdot (-3, -4, 3) = -5, \quad u_2 = a_2 - r_{12}q_1 = a_2 + 5q_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix},$$

so $\|u_2\| = 3$ and $q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For the third vector,

$$r_{13} = q_1^\top a_3 = \frac{1}{5}(3, 4, 0) \cdot (1, -7, 3) = -5, \quad r_{23} = q_2^\top a_3 = (0, 0, 1) \cdot (1, -7, 3) = 3,$$

$$u_3 = a_3 - r_{13}q_1 - r_{23}q_2 = a_3 + 5q_1 - 3q_2 = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \quad \|u_3\| = 5, \quad q_3 = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}.$$

Thus one valid QR decomposition is

$$Q = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & 0 \end{bmatrix}, \qquad R = Q^\top A = \begin{bmatrix} 5 & -5 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}.$$

(b)

$$Q^\top A\vec{x} = Q^\top\vec{b} \quad \Longrightarrow \quad (Q^\top Q)R\vec{x} = Q^\top\vec{b} \quad \Longrightarrow \quad R\vec{x} = \vec{v},$$

where

$$\vec{v} = Q^\top\vec{b}.$$

Plug in the values:

$$\vec{v} = Q^\top\vec{b} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & -\frac{3}{5} & 0 \end{bmatrix}\begin{bmatrix} 8 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}.$$

Solve $R\vec{x} = \vec{v}$ by backward substitution:

$$\begin{bmatrix} 5 & -5 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}.$$

From the last row: $5x_3 = 10 \Rightarrow x_3 = 2$.
From the second row: $3x_2 + 3x_3 = 3 \Rightarrow x_2 = -1$.
From the first row: $5x_1 - 5x_2 - 5x_3 = 0 \Rightarrow x_1 = 1$.

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(c) Given $B = QR$ and $\vec{c}$, to solve $B\vec{z} = \vec{c}$ we compute

$$QR\vec{z} = \vec{c} \implies R\vec{z} = Q^\top \vec{c}.$$

- $\vec{y} = Q^\top \vec{c}$: this is n inner products of length n. Each inner product costs n multiplications and $(n-1)$ additions. Total cost is

$$n^2 \text{ multiplications } + n(n-1) \text{ additions } = 2n^2 - n.$$

- Solve the upper-triangular system backward. For row i, forming $\sum_{j=i+1}^{n} r_{ij} z_j$ uses $(n-i)$ multiplications and $(n-i)$ additions/subtractions, and 1 division. Summing over $i = 1, \dots, n$ gives n^2.

Therefore, assuming Q and R are already available, solving $B\vec{z} = \vec{c}$ via QR costs $O(n^2)$ operations. $(3n^2 - n)$

Problem 5. Determinants

Consider a unit box B in $\mathbb{R}^2$, i.e., the square with corners

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}.$$

Define $A(B)$ as the parallelogram generated by applying matrix A to every point in B.

(a) For

$$A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix},$$

calculate the location of each corner of $A(B)$.

(b) Write the area of $A(B)$ as a function of $\det(A)$.

HINT: How are the basis vectors $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$ transformed by matrix multiplication?

(c) Calculate the area of $A(B)$ for each of the following:

i. $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

ii. $\begin{bmatrix}2 & 1\\4 & 3\end{bmatrix}$

iii. $\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

iv. $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

Answer.

(a) The corners of B are

$$\vec{p}_1 = \begin{bmatrix}0\\0\end{bmatrix}, \vec{p}_2 = \begin{bmatrix}0\\1\end{bmatrix}, \vec{p}_3 = \begin{bmatrix}1\\0\end{bmatrix}, \vec{p}_4 = \begin{bmatrix}1\\1\end{bmatrix}.$$

With $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, their images are:

$$A\vec{p}_1 = \begin{bmatrix}0\\0\end{bmatrix}, \quad A\vec{p}_2 = \begin{bmatrix}2\\4\end{bmatrix}, \quad A\vec{p}_3 = \begin{bmatrix}1\\3\end{bmatrix}, \quad A\vec{p}_4 = \begin{bmatrix}3\\7\end{bmatrix}.$$

So the corners of $A(B)$ are $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}3\\7\end{bmatrix}$.

(b) The unit square B is spanned by the standard basis vectors $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$. Under A, these become Ae_1 and Ae_2, which form adjacent sides of the parallelogram $A(B)$.

Thus, the area of $A(B)$ is given by the magnitude of the cross product of Ae_1 and Ae_2:

$$\text{Area}(A(B)) = \|Ae_1 \times Ae_2\|.$$

which equals the absolute value of the determinant of the matrix formed by Ae_1 and Ae_2 as columns:

$$\text{Area}(A(B)) = \left|\det\left[Ae_1 \quad Ae_2\right]\right|.$$

Since

$$\left[Ae_1 \quad Ae_2\right] = A\left[e_1 \quad e_2\right] = AI = A,$$

we have

$$\text{Area}(A(B)) = |\det(A)| \cdot \text{Area}(B) = |\det(A)| \cdot 1 = |\det(A)|.$$

(c) Compute $\det(A)$ and take absolute value.

i. $\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$, so Area $= |-2| = 2$.

ii. $\det\begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix} = 2 \cdot 3 - 1 \cdot 4 = 2$, so Area $= |2| = 2$.

iii. $\det\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} = 1 \cdot 4 - 2 \cdot 2 = 0$, so Area $= 0$.

iv. $\det\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = 0 \cdot 0 - (-1) \cdot 1 = 1$, so Area $= 1$.

Problem 6. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.
NOTE: If you didnt work with anyone, you can put "none" as your answer.

Answer. none