UC Berkeley, Department of Statistics
Statistics 201B Final Exam - Fall 2018

NAME:________________________

SID:____________________________

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Problem 7	
Total	

Note: Some distribution formulas are available at the end of the exam book.

1. (24pts) Let $X_1, \ldots, X_n \overset{iid}{\sim} Gamma(\alpha, \beta)$.

   (a) (6 pts) Find a sufficient statistic for the parameter β, assuming α is known. Is the statistic you found above minimally sufficient? Justify your answer.

   (b) (6 pts) Find the MLE of β assuming α is known.

   (c) (6 pts) Find the Fisher information and construct an approximate 95% normal-based confidence interval for β.

   (d) (6 pts) Find the asymptotic distribution for the MLE of β assuming α is known.

Cont'd with Problem 1.

2. (12 pts) Let X and Y be two random variables with joint distribution F. Suppose we observe pairs $(x_1, y_1), \cdots, (x_n, y_n)$, a random sample from F .

   (a) (6 pts) Without making any assumptions about F, form a statistic for testing $H_0 : P(X > Y) = 0.5$.

   (b) (6 pts) How would you calculate the p-value?

3. (6 pts) Let $X_1, \ldots, X_n \overset{iid}{\sim} Normal(u, 1)$ and let $Y_1, \ldots, Y_n \overset{iid}{\sim} Normal(v, 4)$. $X_i's$ and $Y_i's$ are independent of each other. Find the Fisher information matrix for the parameter (u, v).

4. (12 pts) (Cont'd with problem 3.) Now we consider a Bayesian setting.

Let $X_1, \ldots, X_n \overset{iid}{\sim} Normal(u, 1)$ and let $Y_1, \ldots, Y_n \overset{iid}{\sim} Normal(v, 4)$. $X_i's$ and $Y_i's$ are independent of each other.

(a) (6pts) Assume the conjugate prior distribution for u is $Normal(a, b)$. Find the posterior distribution for u, conditioning on $X_1, \ldots, X_n$. It is fine to write the distribution family and specify the parameters; you do not need to write out the PDF or CDF.

(b) (6pts) Suppose we use squared error loss, and let $\hat{u}_n$ be the Bayes estimator based on observing $X_1, ..., X_n$. Find $\hat{u}_n$.

5. (24 pts + 10 pts (extra)) Consider the regression model $Y_i = r(X_i) + \epsilon_i$ for $i = 1, \ldots, n$, with $\epsilon_1, \ldots, \epsilon_n$ *iid* and $\epsilon_i \sim N(0, \sigma^2)$.

   (a) (6 pts) Consider $X_1, \ldots, X_n$ as given. Find a Kernel estimator for $r(x)$. Specify what kernel function you use.

   (b) (10 pts) (Optional; for extra credit) Cont'd with (a). Is the above estimator unbiased at a given x? What is the variance of the above estimator at a given x? Explain the tradeoff between bias and variance.

   (c) (6 pts) Now consider $X_1, \ldots, X_n$ as given and $r(X_i) = X_i$. Find the MLE for σ.

   (d) (6 pts) Cont'd with (c). What is the MLE's asymptotic distribution? Approximate asymptotic variance?

   (e) (6 pts) Cont'd with (c). Now we assume that $n = 2m$, $\epsilon_i \sim N(0, \sigma_1^2)$ when $i = 1, ..., m$, and $\epsilon_i \sim N(0, \sigma_2^2)$ when $i = m+1, ..., 2m$. Carry out a test for testing $H_0 : \sigma_1 = \sigma_2$ vs. $H_1 : \sigma_1 \neq \sigma_2$.

Cont'd with Problem 5.

6. (12 pts) Suppose $X_1, ..., X_n$ are i.i.d. Poisson variables with mean λ and we are interested in estimating $p = P(X_i = 0) = e^{-\lambda}$.

   (a) (6 pts) One estimator for p is the proportion of zeros in the sample, $\tilde{p} = \#\{i \leq n : X_i = 0\}/n$. Find the limiting distribution for $\sqrt{n}(\tilde{p} - p)$.

   (b) (6 pts) Another estimator would be the maximum likelihood estimator $\hat{p}$. Give a formula for $\hat{p}$ and determine the limiting distribution for $\sqrt{n}(\hat{p} - p)$.

7. (10 pts) Consider a decision problem with possible states of nature θ_1 and θ_2 . Let X be a random variable with probability function $p(x|\theta)$:

$$P(X=0|\theta_1)=0.2, P(X=1|\theta_1)=0.8; P(X=0|\theta_2)=0.4, P(X=1|\theta_2)=0.6.$$

Two non-randomized actions a_1 and a_2 are considered with the following loss function:

$$L(\theta_1, a_1(0))=1, L(\theta_1, a_1(1))=2, L(\theta_1, a_2(0))=4, L(\theta_1, a_2(1))=0;$$

$$L(\theta_2, a_1(0))=3, L(\theta_2, a_1(1))=1, L(\theta_2, a_2(0))=1, L(\theta_2, a_2(1))=4.$$

(a) (5 pts) Suppose θ has the prior distribution $\Lambda(\theta)$ defined by $P(\theta=\theta_1)=0.9, P(\theta=\theta_2)=0.1$. What is the Bayes rule with respect to $\Lambda(\theta)$?

(b) (5pts) Find the minimax rule(s).

Distributions

- $X \sim N(\mu, \sigma^2)$

$$f(x;\mu,\sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

- $X \sim Gamma(\alpha, \beta)$ with $\alpha > 0$ and $\beta > 0$,

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

- $X = (X_1, \ldots, X_k) \sim Multinomial(n, p)$ where $p = (p_1, \ldots, p_k)$

$$f(x;p) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

- $X \sim Beta(\alpha, \beta)$

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$$E[X] = \frac{\alpha}{\alpha+\beta}$$

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$$

- $X \sim Poisson(\lambda)$

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E[X] = Var[X] = \lambda$$