UNIVERSITY OF CALIFORNIA, BERKELEY
DEPARTMENT OF STATISTICS

Homework #1

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Course: *EECS 227AT* – Professor: *Gireeja Ranade*
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Problem 1. Course Setup

Please complete the following steps to get access to all course resources.

(a) Visit the course website at `http://eecs127.github.io/` and familiarize yourself with the syllabus.

(b) Verify that you can access the class Ed site at `https://edstem.org/us/courses/93760/discussion`.

(c) Verify that you can access the class Gradescope site.

Answer.

Problem 2. Prerequisites

The prerequisites for this course are:

- MATH 54 (Linear Algebra & Differential Equations),
- CS 70 (Discrete Mathematics & Probability Theory),
- MATH 53 (Multivariable Calculus).

Please fill out the Google form provided on the homework PDF. For your written response, write the *secret word* revealed at the end of the form.

Answer. PinkpantheR

Problem 3. Orthogonality

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ be two linearly independent unit-norm vectors, i.e.,

$$\|\vec{x}\|_2 = \|\vec{y}\|_2 = 1.$$

(a) Show that the vectors $\vec{u} = \vec{x} - \vec{y}$ and $\vec{v} = \vec{x} + \vec{y}$ are orthogonal.

(b) Find an orthonormal basis for $\text{span}(\vec{x}, \vec{y})$.

Answer.

(a) To show that $\vec{u}$ and $\vec{v}$ are orthogonal, we need to show that their dot product is zero:

$$\vec{u} \cdot \vec{v} = (\vec{x} - \vec{y}) \cdot (\vec{x} + \vec{y}).$$

Expanding the dot product, we have:

$$\vec{u} \cdot \vec{v} = \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{y} - \vec{y} \cdot \vec{x} - \vec{y} \cdot \vec{y}.$$

Since the dot product is commutative, $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$. Also, given that $\|\vec{x}\|_2 = \|\vec{y}\|_2 = 1$, we have $\vec{x} \cdot \vec{x} = 1$ and $\vec{y} \cdot \vec{y} = 1$. Therefore,

$$\vec{u} \cdot \vec{v} = 1 + \vec{x} \cdot \vec{y} - \vec{x} \cdot \vec{y} - 1 = 0. \Rightarrow \vec{u} \perp \vec{v}$$

(b) We use the Gram-Schmidt process. Starting with the vectors $\vec{x}$ and $\vec{y}$. First, we take $\vec{u}_1 = \vec{x}$. Next, we project $\vec{y}$ onto $\vec{u}_1$:

$$\vec{p}_{\vec{y}} = \frac{\vec{y} \cdot \vec{u}_1}{\|\vec{u}_1\|_2^2} \vec{u}_1 = (\vec{y} \cdot \vec{x})\vec{x}.$$

Now, we subtract this projection from $\vec{y}$ to get $\vec{u}_2$:

$$\vec{u}_2 = \vec{y} - \vec{p}_{\vec{y}} = \vec{y} - (\vec{y} \cdot \vec{x})\vec{x}.$$

Next, we normalize $\vec{u}_1$ and $\vec{u}_2$ to get the orthonormal basis vectors:

$$\vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|_2} = \vec{x},$$

$$\vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|_2} = \frac{\vec{y} - (\vec{y} \cdot \vec{x})\vec{x}}{\|\vec{y} - (\vec{y} \cdot \vec{x})\vec{x}\|_2}.$$

Thus, the orthonormal basis for $\text{span}(\vec{x}, \vec{y})$ is given by the vectors $\vec{e}_1$ and $\vec{e}_2$:

$$\left\{ \vec{x}, \frac{\vec{y} - (\vec{y} \cdot \vec{x})\vec{x}}{\|\vec{y} - (\vec{y} \cdot \vec{x})\vec{x}\|_2} \right\}.$$

Problem 4. Least Squares

The Michaelis–Menten model for enzyme kinetics relates the rate y of an enzymatic reaction to the concentration x of a substrate, as follows:

$$y = \frac{\beta_1 x}{\beta_2 + x}, \tag{1}$$

for constants $\beta_1, \beta_2 > 0$.

(a) Show that the model can be expressed as a linear relation between the values $1/y = y^{-1}$ and $1/x = x^{-1}$. Specifically, give an equation of the form

$$y^{-1} = w_1 + w_2 x^{-1},$$

specifying the values of w_1 and w_2 in terms of β_1 and β_2.

(b) In general, reaction parameters β_1 and β_2 (and, thus, w_1 and w_2) are not known a priori and must be fit from data, for example using least squares. Suppose you collect m measurements (x_i, y_i), $i = 1, \ldots, m$, over the course of a reaction.

Formulate the least squares problem

$$\vec{w}^\star = \arg\min_{\vec{w}} \|X\vec{w} - \vec{y}\|_2^2, \tag{2}$$

where

$$\vec{w}^\star = \begin{bmatrix} w_1^\star \\ w_2^\star \end{bmatrix},$$

and you must specify $X \in \mathbb{R}^{m\times 2}$ and $\vec{y} \in \mathbb{R}^m$. Specifically, your solution should include explicit expressions for X and $\vec{y}$ as a function of (x_i, y_i) values and a final expression for $\vec{w}^\star$ in terms of X and $\vec{y}$, which should contain only matrix multiplications, transposes, and inverses.

Assume without loss of generality that $x_1 \neq x_2$.

(c) Assume that we have used the above procedure to calculate values for $w_1^\star$ and $w_2^\star$, and we now want to estimate

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}.$$

Write an expression for $\hat{\boldsymbol{\beta}}$ in terms of $w_1^\star$ and $w_2^\star$.

NOTE: This problem was taken (with some edits) from the textbook *Optimization Models* by Calafiore and El Ghaoui.

Answer.

(a) Taking the reciprocal of both sides of equation (1), we have:

$$y^{-1} = \frac{\beta_2 + x}{\beta_1 x} = \frac{\beta_2}{\beta_1 x} + \frac{x}{\beta_1 x} = \frac{\beta_2}{\beta_1} x^{-1} + \frac{1}{\beta_1}.$$

Thus, we can express the model as a linear relation between y^{-1} and x^{-1}:

$$y^{-1} = \frac{1}{\beta_1} + \frac{\beta_2}{\beta_1} x^{-1}.$$

Therefore, we have:

$$w_1 = \frac{1}{\beta_1}, \quad w_2 = \frac{\beta_2}{\beta_1}.$$

(b) To formulate the least squares problem, we first need to express the measurements in terms of y^{-1} and x^{-1}. For each measurement (x_i, y_i), we have:

$$y_i^{-1} = w_1 + w_2 x_i^{-1}.$$

We can rewrite this in matrix form as:

$$\begin{bmatrix} y_1^{-1} \\ y_2^{-1} \\ \vdots \\ y_m^{-1} \end{bmatrix} = \begin{bmatrix} 1 & x_1^{-1} \\ 1 & x_2^{-1} \\ \vdots & \vdots \\ 1 & x_m^{-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Thus, we can define:

$$X = \begin{bmatrix} 1 & x_1^{-1} \\ 1 & x_2^{-1} \\ \vdots & \vdots \\ 1 & x_m^{-1} \end{bmatrix} \in \mathbb{R}^{m \times 2}, \quad \vec{y} = \begin{bmatrix} y_1^{-1} \\ y_2^{-1} \\ \vdots \\ y_m^{-1} \end{bmatrix} \in \mathbb{R}^m.$$

The least squares problem can then be expressed as:

$$\vec{w}^\star = \arg\min_{\vec{w}} \|X\vec{w} - \vec{y}\|_2^2.$$

The solution to this least squares problem is given by the normal equation:

$$\vec{w}^\star = (X^\top X)^{-1} X^\top \vec{y}.$$

(c) From part (a),

$$w_1 = \frac{1}{\beta_1} \quad \Rightarrow \quad \beta_1 = \frac{1}{w_1}, \qquad w_2 = \frac{\beta_2}{\beta_1} \quad \Rightarrow \quad \beta_2 = \beta_1 w_2 = \frac{w_2}{w_1}.$$

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{w_1^\star} \\ \frac{w_2^\star}{w_1^\star} \end{bmatrix}.$$

Problem 5. Subspaces and Dimensions

Consider the set S of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0. \tag{3}$$

(a) Find a 2×3 matrix A for which S is exactly the null space of A.

(b) Determine the dimension of S and find a basis for it.

Answer.

(a) We can express the given equations in matrix form as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}.$$

Thus, the matrix A is given by:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

(b) To determine the dimension of S and find a basis for it, we first need to find the null space of the matrix A. We can perform row reduction on the matrix A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{bmatrix} \xrightarrow{R_2 \leftarrow -\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$$

Now, we can express the system of equations represented by the row-reduced matrix:

$$x_1 + 2x_2 + 3x_3 = 0,$$
$$x_2 + 2x_3 = 0.$$

From the second equation, we have:

$$x_2 = -2x_3.$$

Substituting this into the first equation, we get:

$$x_1 + 2(-2x_3) + 3x_3 = 0 \implies x_1 - 4x_3 + 3x_3 = 0 \implies x_1 - x_3 = 0 \implies x_1 = x_3.$$

Letting $x_3 = t$, we can express the solution as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, the null space of A is spanned by the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and the dimension of S is 1.

Therefore, a basis for S is given by:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Problem 6. Vector Spaces and Rank

The rank of a $m \times n$ matrix A, $\text{rank}(A)$, is the dimension of its range, also called span, and denoted

$$\mathcal{R}(A) := \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}.$$

(a) Assume that $A \in \mathbb{R}^{m\times n}$ takes the form

$$A = \vec{u}\vec{v}^\top,$$

with $\vec{u} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $\vec{u}, \vec{v} \neq \vec{0}$. (Note that a matrix of this form is known as a dyad.) Find the rank of A.

HINT: Consider the quantity $A\vec{x}$ for arbitrary $\vec{x}$.

(b) Show that for arbitrary $A, B \in \mathbb{R}^{m\times n}$,

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B). \tag{4}$$

HINT: First, show that $\mathcal{R}(A + B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$. Remember that for any matrix A, $\mathcal{R}(A)$ is a subspace, and for any two subspaces S_1 and S_2, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$. The sum of vector spaces is defined as

$$S_1 + S_2 := \{\vec{s}_1 + \vec{s}_2 \mid \vec{s}_1 \in S_1, \vec{s}_2 \in S_2\}.$$

(c) Consider an $m \times n$ matrix A that takes the form

$$A = UV^\top,$$

with $U \in \mathbb{R}^{m\times k}$ and $V \in \mathbb{R}^{n\times k}$. Show that the rank of A is less than or equal to k.

HINT: Use parts (a) and (b), and remember that this decomposition can also be written as the dyadic expansion

$$A = UV^\top = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_k^\top \end{bmatrix} = \sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top. \tag{5}$$

Answer.

(a)

$$A\vec{x} = \vec{u}\vec{v}^\top\vec{x} = \vec{u}(\vec{v}^\top\vec{x}).$$

The term $\vec{v}^\top\vec{x}$ is a scalar, so we can denote it as $c = \vec{v}^\top\vec{x}$. Therefore,

$$A\vec{x} = c\vec{u}.$$

This shows that the image of any vector $\vec{x}$ under the transformation defined by A is a scalar multiple of the vector $\vec{u}$. Hence, the range of A is spanned by the single vector $\vec{u}$. Since $\vec{u} \neq \vec{0}$, the range of A is one-dimensional. Therefore, the rank of A is:

$$\text{rank}(A) = 1.$$

(b) To show that $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$, we first need to show that $\mathcal{R}(A+B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$. Let $\vec{y} \in \mathcal{R}(A+B)$. Then, by definition of the range, there exists a vector $\vec{x} \in \mathbb{R}^n$ such that:

$$\vec{y} = (A+B)\vec{x} = A\vec{x} + B\vec{x}.$$

Since $A\vec{x} \in \mathcal{R}(A)$ and $B\vec{x} \in \mathcal{R}(B)$, thus: $\mathcal{R}(A+B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$.
Now, using the property of dimensions of subspaces, we have:

$$\dim(\mathcal{R}(A+B)) \le \dim(\mathcal{R}(A) + \mathcal{R}(B)) \le \dim(\mathcal{R}(A)) + \dim(\mathcal{R}(B)).$$

Therefore,

$$\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B).$$

(c) Using the dyadic expansion of A from equation (5), we can express A as a sum of k dyads:

$$A = \sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top.$$

From part (a), we know that each dyad $\vec{u}_i \vec{v}_i^\top$ has rank 1. Therefore, we can apply the result from part (b) to find the rank of A:

$$\text{rank}(A) = \text{rank}\left(\sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top\right) \le \sum_{i=1}^{k} \text{rank}(\vec{u}_i \vec{v}_i^\top) = \sum_{i=1}^{k} 1 = k.$$

Thus, we have shown that the rank of A is less than or equal to k:

$$\text{rank}(A) \le k.$$

Problem 7. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.
NOTE: If you didnt work with anyone, you can put "none" as your answer.

Answer. none