Comments on the California Mathematics Framework

Written between June 26 and July 7

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These comments are not meant to be comprehensive. They fall into two main categories: specific edits; and mentions of inconsistencies in use of terms. For example, terms such as "fluency" are used inconsistently within the CMF. They are also used in ways that are inconsistent with references such as the National Research Council's *How People Learn* and *Adding It Up.*

Chapter 1

Big ideas, fluency, and learning progressions. The terms "big idea," "fluency," and "learning progression" are used in the CMF in ways that don't seem to acknowledge how they are related. Here is a brief summary of their relationships. I have put examples of the CMF's use of "fluency" in the appendix.

The terms "big idea" and "fluency" are used as descriptors in the study of expertise, in particular, studies of experts in chess, physics, and teaching (see *How People Learn*, 2000, pp. 32–37).

How People Learn (hereafter HPL I) says (p. 30):

> 1. Experts notice features and meaningful patterns of information that are not noticed by novices.
> 2. Experts have acquired a great deal of content knowledge that is organized in ways that reflect a deep understanding of their subject matter.
> 3. Experts' knowledge cannot be reduced to sets of isolated facts or propositions but, instead, reflects contexts of applicability: that is, the knowledge is "conditionalized" on a set of circumstances.
> 4. Experts are able to flexibly retrieve important aspects of their knowledge with little attentional effort.
> 5. Though experts know their disciplines thoroughly, this does not guarantee that they are able to teach others.
> 6. Experts have varying levels of flexibility in their approach to new situations.

In HPL I, item 2 above is described further in terms of "big ideas." Experts' knowledge is organized around "core concepts or 'big ideas' that guide their thinking about their domains" (HPL I, p. 36).

Item 4 above is also described as fluent retrieval of relevant knowledge (HPL I, p. 49).

From these findings about expertise, HPL I draws implications for instructional design:

> • using "important ideas" to organize curriculum (p. 42).
>
> • strik[ing] the appropriate balance between activities designed to promote understanding and those designed to promote the automaticity of skills necessary to function effectively without being overwhelmed by attentional requirements. Students for whom it is effortful to read, write, and calculate can encounter serious difficulties learning. (HPL I, p. 139)

Sections 1 and 2 discuss (respectively) big ideas and automaticity.

Section 1. How big ideas for learning appear in the Progressions for the CCSS. The *Progressions for the CCSS* uses "learning progression" to describe developmental sequences documented by researchers (see p. 2). For example,

> A learning trajectory has three parts: a specific mathematical goal, a developmental path along which children develop to reach that goal, and a set of instructional activities that help children move along that path. (Clements & Sarama, 2009, p. viii)

Douglas Clements (one of the CCSS and *Progressions* writers) and Julie Sarama describe how HPL I's "important ideas" might be "big ideas" of a learning trajectory:

> The first part of a learning trajectory is a math goal. Our goals include the "big ideas of math": clusters of concepts and skills that are mathematically central and coherent, consistent with children's thinking, and generative of future learning. These big ideas come from mathematicians, researchers, and teachers (CCSSO/NGA, 2010; Clements, 2004; NCTM, 2006; NMP, 2008). They include math content but *also research on students' thinking about and learning of math*. As an example, one big idea is that *counting can be used to find out how many in a collection*. (Clements & Sarama, 2021, p. 3, emphases theirs)

Along with other ideas, *counting can be used to find out how many in a collection* is described on page 11 of the *Progressions for the CCSS.*

From counting to counting on Students understand that the last number name said in counting tells the number of objects counted.[K.CC.4b] Prior to reaching this understanding, a student who is asked "How many kittens?" may regard the counting performance itself as the answer, instead of answering with the cardinality of the set. Experience with counting allows students to discuss and come to understand the second part of K.CC.4b—that the number of objects is the same regardless of their arrangement or the order in which they were counted. This connection will continue in Grade 1 with the more advanced counting-on methods in which a counting word represents a group of objects that are added or subtracted and addends become embedded within the total[1.OA.6] (see page 23). Being able to count forward, beginning from a given number within the known sequence,[K.CC.2] is a prerequisite for such counting on. Finally, understanding that each successive number name refers to a quantity that is one larger[K.CC.4c] is the conceptual start for Grade 1 counting on. Prior to reaching this understanding, a student might have to recount entirely a collection of known cardinality to which a single object has been added.

From spoken number words to written base-ten numerals to base-ten system understanding The Number and Operations in Base Ten Progression discusses the special role of 10 and the difficulties that English speakers face because the base-ten structure is not evident in all the English number words. See page 55.

K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality.

b Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

K.CC.2 Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality.

c Understand that each successive number name refers to a quantity that is one larger.

Source: *Progressions*, p. 11

Learning progressions documented by research do not exist for all of K–12 mathematics, thus the development of the CCSS drew on other research in education (such as studies of curriculum in other countries) and the structure of mathematics. These were also part of early drafts of the *Progressions for the CCSS.*

In describing the development of the CCSS, lead writer Bill MacCallum said, "state standards have direct policy and legal consequences. . . . they are flat lists of performance objectives of even grain size, designed to be delivered into the hands of assessment writers without further human intervention." Because of this constraint, the early *Progressions*

> were then sliced into grade level standards. From that point on the work focused on refining and revising the grade level standards, thus, the early drafts of the progressions documents do not correspond to the 2010 Standards.
>
> The Progressions for the Common Core State Standards are updated versions of those early progressions drafts, revised and edited to correspond with the Standards by members of the original Progressions work team, together with other mathematicians, statisticians, and education researchers not involved in the initial writing. They note key connections among standards, point out cognitive difficulties and pedagogical solutions, and give more detail on particularly knotty areas of the mathematics. (*Progressions for the CCSS*, p. iv)

Section 2. Automaticity. Automaticity is very important for experts.

> Automatic and fluent retrieval are important characteristics of expertise. . . within the overall process of problem solving there are a number of subprocesses that, for experts, vary from fluent to automatic. Fluency is important because effortless processing places

> fewer demands on conscious attention. Since the amount of information a person can attend to at any one time is limited (Miller, 1956), ease of processing some aspects of a task gives a person more capacity to attend to other aspects of the task. . . . Many instructional environments stop short of helping all students develop the fluency needed to successfully perform cognitive tasks. (HPL I, p. 44)

As with "big ideas," automaticity has been interpreted in the context of mathematics education. The National Research Council report *Adding It Up*, which focuses on mathematics from prekindergarten to grade 8, describes mathematical proficiency as having five intertwined components (NRC, 2001, p. 116). This includes automaticity:

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*—ability to formulate, represent, and solve mathematical problems
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Another well-known approach to analyzing mathematical behavior, Schoenfeld (2016, p. 19), also includes knowledge of procedures and the ability to use them appropriately, noting that:

> The limits on working memory also suggest that for "knowledge-rich" domains [such as chess and mathematics], there are severe limitations to the amount of "thinking things out" that one can do; the contents of the knowledge base are critically important. . . . [C]hess experts had "vocabularies" of chess positions, some 50,000 well-recognized configurations, which they recognized and to which they responded automatically. These vocabularies formed the base (but not the whole) of their competence."

Along with the authors of *Adding It Up*, Schoenfeld makes the case that mathematical knowledge is not *solely* a collection of procedures, but does include procedures. Chapters 1 and 3 of the CMF often seem to disagree with this view (see examples in appendix).

The table below uses examples from the CCSS to sketch how big ideas and automaticity are connected and how knowledge may change over the grades, using Schoenfeld's distinction between problem and exercise. An exercise is something for which one has easy access to a solution procedure; a problem is not. Thus, one person's problem may be another person's exercise.

Expert	Example of big idea	Examples of automaticity and non-automaticity
Mathematician		Solving quadratic equation by applying quadratic formula. Proving Fermat's last theorem.
Expert kindergarten teacher	Declarative knowledge: Counting can be used to find out how many in a collection	
Beginning first grader (expert user of kindergarten math)	Old idea: Counting can be used to find out how many in a collection. New idea: seeing addition and subtraction structure in problems such as "6 bunnies were sitting on the grass. Some more bunnies hopped there. Then there were 11 bunnies. How many bunnies hopped over to the first 6 bunnies?"	Fluency in saying count sequence Fluency in counting objects, pairing count words and objects Addition and subtraction within 20: Finding the value of $9 + 8$ is a problem to be solved, using strategies such as counting on, making 10, etc.
Beginning third grader (expert user of grade 2 math)	Old idea: sees addition and subtraction structure in addition and subtraction situations (Table 2, *Progressions*). New idea: see multiplication and division structure in multiplication and division situations (Table 3, *Progressions*).	Knows from memory sums within 20. Finding the value of $9 + 8$ is an exercise. Multiplication within 100: Finding the value of 9×8 is a problem to be solved, using strategies such as properties of operations.

References for sections 1 and 2.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. Routledge.

Clements, D. H., & Sarama, J. (2021). *Learning and teaching early math: The learning trajectories approach*. Routledge.

Common Core Standards Writing Team. (2022). *Progressions for the Common Core State Standards for Mathematics (May 26, 2022).* Tucson, AZ: Institute for Mathematics. https://mathematicalmusings.org/2023/05/24/version-of-progressions-with-revised-appendix/

McCallum, W. (2015). The U.S. Common Core State Standards in Mathematics. https://www.math.arizona.edu/~wmc/Research/2012_04_22_ICME.pdf. Published in *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 547–560). https://link.springer.com/book/10.1007/978-3-319-17187-6?noAccess=true#page=544)

National Research Council. (2000). *How people learn: Brain, mind, experience, and school: Expanded edition*. J. Bransford, A. Brown, and R. Cocking (Eds.). Washington, DC: National Academy Press.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education*, *196*(2), 1–38. https://journals.sagepub.com/doi/pdf/10.1177/002205741619600202

Lines 47–55: research is invoked twice. Even one mention of research seems to promise more than the CMF delivers. Why not be more straightforward and say that this is something of a vision document, as are many documents that guide curriculum and instruction?

Line 53: The meaning of "multidimensional" is not explained here. Also, the CMF does not explain how teaching in multidimensional ways meets learning needs.

Lines 188–190: "All mathematical ideas can be considered in different ways—visually; through touch or movement; through building, modeling, writing and words; through apps, games and other digital interfaces; or through numbers and algorithms." This is an interesting claim but how do we know it's true, even for K–12 mathematics?

Lines 205–206: "In many other countries, the standards guiding content in each grade are fewer, higher, and deeper, with greater coherence and integration." "Many" seems like an exaggeration. Also, what does it mean for standards to have integration? Suggested edit: "Studies of high-achieving countries, find that their standards guiding content are fewer and higher, with greater coherence (Schmidt, Houang, & Cogan, 2002)." Even this statement is somewhat exaggerated, Schmidt et al. only examined grades 1–8. Moreover, the U.S. seems to be the only country with standards rather than a course of study.

In particular, as mentioned above:

> Unlike the NCTM standards, state standards have direct policy and legal consequences, and are used as a basis for writing assessments. They are flat lists of performance objectives of even grain size, designed to be delivered into the hands of assessment writers without further human intervention. (McCallum, 2015)

This situation places constraints on the style and implementation of standards that seem to be quite different from the affordances of other countries, e.g., Japan and South Korea. One difference is that these other countries don't have annual state-wide or country-wide standardized testing.

For an overview of “standards” implementation in South Korea, see *Mathematics Curriculum, Teacher Professionalism, and Supporting Policies in Korea and the United States*.

References for this comment.

Japan course of study for elementary mathematics (23 pages long): https://www.mext.go.jp/component/english/__icsFiles/afieldfile/2011/03/17/1303755_004.pdf

Timeline for textbook production and approval in Japan: https://www.mext.go.jp/en/policy/education/elsec/title02/detail02/sdetail02/1383719.html

Mathematics Curriculum, Teacher Professionalism, and Supporting Policies in Korea and the United States: Summary of a Workshop, National Academy of Sciences. http://www.nap.edu/catalog/21753/mathematics-curriculum-teacher-professionalism-and-supporting-policies-in-korea-and-the-united-states.

Schmidt, W., Houang, R., & Cogan, L. (2002). A coherent curriculum. *American Educator*, 1–17.

Lines 206–209: “Topics are studied more deeply, with applications to real world problems. Mathematical practices include collaborative problem-solving strategies, heterogeneously grouped classrooms, and an integrated approach to mathematics from grade school through high school.” Delete “Mathematical”, this is about instructional practices.

Lines 222–225: “Moreover, the data lay bare a serious equity issue. There are significant racial and socioeconomic math achievement gaps; Black, American Indian or Alaska Native, and Latino students in particular are, on average, lower-achieving on state and national tests.” This mentions inequity in achievement but not possible contributing factors such as school funding via property taxes. It does not mention inequity in attention from policy-makers such as that described by Martin (2009).

Lines 262–264: “All students deserve powerful mathematics instruction. High-level mathematics achievement is not dependent on rare natural gifts, but rather can be cultivated (Leslie et al., 2015; Boaler, 2019a, b; Ellenberg, 2014).”

I suggest deleting “(Leslie et al., 2015; Boaler, 2019a, b; Ellenberg, 2014)” for the following reasons.

Leslie et al*.* Leslie et al. does not say anything about whether or not mathematics achievement is dependent on rare natural gifts. It is about some professors’ *beliefs* about whether achievement in mathematics (or another field) is dependent on rare natural gifts, as suggested by its title: “Expectations of Brilliance Underlie Gender Distributions Across Academic Disciplines.”

Boaler, 2019a, b. Boaler, 2019a, b are not listed in the references for chapter 1. In the previous draft of the framework they referred to

> Boaler, Jo. 2019a. “Developing Mathematical Mindsets: The Need to Interact with Numbers Flexibly and Conceptually.” *American Educator* 42(4): 28.
>
> Boaler, Jo. 2019b. *Limitless Mind. Learn, Lead and Live without Barriers*. New York: Harper Collins.

According to imab-cfird-jul23item01, Attachment 2, the neuroscientific discussions in the draft CMF have been revised “to avoid overgeneralizations and ensure that citations are correct and research findings are clearly and accurately captured.” Shouldn’t efforts also be made to avoid using references with overgeneralizations about neuroscience research? Both Boaler, 2019a and Boaler, 2019b have been criticized for inaccuracies about neuroscience research.

“Developing Mathematical Mindsets” (https://files.eric.ed.gov/fulltext/EJ1200568.pdf) is excerpted from *Mathematical Mindsets*. It says:

> We know that when we learn mathematics, we engage in a brain process called “compression.” When you learn a new area of mathematics that you know nothing about, it takes up a large space in your brain. . . . But the mathematics you have learned before and know well, such as addition, takes up a small, compact space in your brain.

In her review of *Mathematical Mindsets*, Victoria Simms says:

> This statement implies a physical change in the state of the brain. Importantly, Boaler suggests that compression cannot apply to rules and methods, but only to concepts. This assertion appears to create a new neuromyth.

See Victoria Simms (2016) Mathematical mindsets: unleashing students’ potential through creative math, inspiring messages and innovative teaching, *Research in Mathematics Education*, 18:3, 317-320, DOI: 10.1080/14794802.2016.1237374. Victoria Simms is a lecturer in psychology, Ulster University, Northern Ireland.

Limitless Mind (Boaler, 2019b) has been criticized for its interpretation of research in neuroscience, e.g.:

> Similarly, Boaler takes evidence that brain regions are connected to one another to suggest that people benefit from a “multidimensional approach” to teaching and learning. In teaching math, for instance, such an approach would focus not just on solving problems and applying formulas but also on building such skills as asking good questions, interpreting a problem in various ways, using logic and reasoning, and explaining concepts to others. The author states that “anyone can learn the content of any subject with a multidimensional approach.” Not only is this a very strong claim about the equipotentiality of learners, but its link to research on brain connectivity is nothing but tenuous.

See Ansari, D. (2020). The Case for Limitlessness Has Its Limits. *Education Next, 20*(2), 83-84. Daniel Ansari is a professor in the Department of Psychology and the Faculty of Education at Western University, Canada.

Ellenberg, 2014. Ellenberg, 2014 argues mathematics is an extension of common sense that but does not give evidence that mathematics achievement has occurred by using this perspective without special gifts.

Lines 272–274: "All students, regardless of background, language of origin, differences, or foundational knowledge are capable and deserving of depth of understanding and engagement in rich mathematics tasks." What is foundational knowledge? Prior learning?

Lines 291–294: "Fixed notions about student ability have led to considerable inequities in mathematics education. Particularly damaging is the idea of the "math brain" (Heyman, 2008)—that people are either born with a brain that is suited for math or not, in which case they should expect little success." This is formulated as causal but no evidence is given. The point that the notion of "math brain" isn't well supported could be made without making causal claims. Suggested edit:

Lines 322–326: "Stanford University psychologist Carol Dweck and her colleagues have conducted research studies in different subjects and fields for decades showing that people's beliefs about personal potential can change the ways their brains operate and influence what they achieve." I suggest deleting "change the ways their brains operate and" unless a Dweck article can be found that shows such changes.

Lines 326–328: "One of the important studies Dweck and her colleagues conducted took place in mathematics classes at Columbia University (Carr et al., 2012), where researchers found that young women received messaging that they did not belong in the discipline." The article (Carr et al.) cited is not about this study.

This is disappointing because the same sentence with the same reference appears in the previous version of Chapter 1—despite the statement in imab-cfird-jul23item01 a, Attachment 2 that "Citations to research and other sources have been reviewed and updated to ensure accuracy."

Lines 367–369: Delete "and connect mathematical concepts, such as number sense. Teachers teach these ideas in multidimensional ways that meet varied student learning needs." The end of the first sentence seems to say that number sense is a disconnected concept. Teaching in multidimensional ways is advocated several times earlier in the document so doesn't need to be here.

Lines 709–717: "In this framework, *rigor* refers to an integrated way in which conceptual understanding, strategies for problem-solving and computation, and applications are learned so that each supports the other. Using this definition, conceptual understanding cannot be considered rigorous if it cannot be *used* to analyze a novel situation encountered in a real-world application or within mathematics itself (for new examples and phenomena). Computational speed and accuracy cannot be called rigorous unless it is accompanied by conceptual understanding of the strategy being used, including why it is appropriate in a given situation. And a correct answer to an application problem is not rigorous if the solver cannot explain both the ideas of the model used and the methods of calculation."

Do the state assessments already measure rigorous conceptual understanding, computational speed and accuracy, and answers to application problems as defined above? How does the understanding specified by this definition differ from what is already in place?

Lines 737–740: "In other words, they experience reasoning that Ellenberg (2014, 48) describes as understanding "all the way down to the bottom." This is the basis of mathematical rigor, often expressed in terms of validity and soundness of arguments."

Like Ellenberg (and many others), I associate rigor with Cauchy. Because my understanding of rigor in mathematics is close to "the use of logical deductions from stated hypotheses to prove theorems," the first sentence seems to say the CMF meaning of rigor is mathematical rigor. (The "bottom" is the axiom system used as a foundation.)

But then the next sentence would then be interpreted as "mathematical rigor" is the basis of "mathematical rigor." Presumably this is not the intended meaning.

My confusion about this passage arises from the use of the Ellenberg quote. I suggest deleting the following two sentences (lines 738–739):

> Rigorous reasoning enables understanding "all the way down to the bottom" (Ellenberg, 2014, 48).
>
> In other words, they experience reasoning that Ellenberg (2014, 48) describes as understanding "all the way down to the bottom."

and deleting "("all the way to the bottom")" on line 762 in

> The Drivers of Investigation provide broad reasons to think rigorously ("all the way to the bottom") in ways that enable students to recognize, value, and internalize linkages between and through topics (Content Connections).

Other chapters seem to use "rigor" and "rigorous" with different meanings. Sometimes rigor is said to occur throughout K–12 and sometimes it is associated with Algebra 1 or with deductive proof.

> Chapter 6, line 2460
> When fifth grade students organize two-dimensional shapes in a hierarchical structure, they are demonstrating the informal deduction stage of growth. At higher grade levels, students move to formal deduction and rigor.
>
> Chapter 6, line 2460
> As noted earlier, the foundational mathematics content, or big ideas, across transitional kindergarten through grade twelve progresses in accordance with the CA CCSSM principles of focus, coherence, and rigor.
>
> Chapter 8, line 734
> By completing Algebra I and Geometry or Mathematics I and II, students will satisfy the requirements of California Assembly Bill 220 of the 2015 legislative session that requires students to complete two mathematics courses in order to receive a diploma of graduation from high school, with at least one course meeting the rigor of Algebra I.
>
> Chapter 8, line 658
> Researchers followed three cohorts in the earlier tracked sequence and three cohorts in the more rigorous untracked sequence.
>
> Chapter 12, line 1002
> Claim 3 refers to a recurring theme in the CA CCSSM content and practice standards: the ability to construct and present a clear, logical, and convincing argument. For older students this may take the form of a rigorous deductive proof based on clearly stated axioms. For younger students this will involve justifications that are less formal. Assessment tasks that address this claim typically present a claim and ask students to provide a justification or counterexample.

Lines 380–382: "Big ideas are central to the learning of mathematics, link numerous mathematics understandings into a coherent whole, and provide focal points for student investigations (Charles, 2005)." Charles says only the first two—nothing about focal points.

Lines 382–386: "Big ideas and the connections among them serve as a schema—a map of the intellectual territory—that supports conceptual understanding. Learning scientists find that people learn more effectively when they understand a map of the domain and how the big ideas fit together (National Research Council, 2000). Within that map, they can then locate facts and details and see how they, too, fit." I cannot find any assertion such as "Learning scientists find that people learn more effectively when they understand a map of the domain and how the big ideas fit together" in the book which is referenced. (Is this meant as a reference to the "conceptual framework" on p. 17?) I do, however, see assertions about big ideas with respect to experts and with respect to curriculum:

> We turn now to the question of how experts' knowledge is organized and how this affects their abilities to understand and represent problems. Their knowledge is not simply a list of facts and formulas that are relevant to their domain; instead, their knowledge is

> organized around core concepts or "big ideas" that guide their thinking about their domains. (p. 36)
>
> The fact that experts' knowledge is organized around important ideas or concepts suggests that curricula should also be organized in ways that lead to conceptual understanding. Many approaches to curriculum design make it difficult for students to organize knowledge meaningfully. Often there is only superficial coverage of facts before moving on to the next topic; there is little time to develop important, organizing ideas. History texts sometimes emphasize facts without providing support for understanding (e.g., Beck et al., 1989, 1991). (p. 42)

Line 412: delete "learning" in "learning progressions." "Learning progression" has a different meaning as described on p. 2 of *Progressions for the CCSS*:

> Within the United States, researchers who study children's learning have identified developmental sequences associated with constructs such as "teaching–learning paths," "learning progressions," or "learning trajectories."
>
> For example,
>
>> A learning trajectory has three parts: a specific mathematical goal, a developmental path along which children develop to reach that goal, and a set of instructional activities that help children move along that path. (Clements & Sarama, 2009, *Learning and Teaching Early Math: The Learning Trajectories Approach*, Routledge, p. viii)

Lines 436–437 of chapter 1 and 1654 of vignettes: what does it mean to organize data into number lines and coordinate planes? I had to look at the vignette to figure this out. Suggested edit: "Students use double number lines to determine how fast the swimmer swims, and record values of time and distance in a table. They plot the swimmer's rate vs time on the coordinate plane."

Line 457: Are the components of this figure going to turn into yet another thing that teachers are required to write on the board or in lesson plans? It seems just an overlay that will distract from standards.

Note its source: "Adapted from the California Digital Learning Integration and Standards Guidance, 2021." In particular the source appears to be Figure 5.1 on page 132. The caption on this figure says "The Drivers of Investigation, Content Connections, and Mathematical Practices from the Mathematics Framework."

The CDE web site says "The California Digital Learning Integration and Standards Guidance was adopted by the State Board of Education (SBE) at their meeting on May 12–13, 2021."

Lines 713–715: "Teachers who identify and discuss big ideas become attuned to the math that is most important and develop greater appreciation of the connections between tasks and ideas

(Boaler, Munson, and Williams, 2018).” Boaler et al. says something similar (“When teachers work on identifying and discussing big ideas, they become attuned to the mathematics that is most important and that they may see in tasks, they also develop a greater appreciation of the connections that run between tasks and ideas”) but does not provide evidence for this assertion. Suggested edit: delete lines 713–715 or find a reference that provides evidence for this assertion.

Lines 932–933: “Students learn best when they are actively engaged in making sense of the world around them.”

> Asking which teaching technique is best is analogous to asking which tool is best—a hammer, a screwdriver, a knife, or pliers. In teaching as in carpentry, the selection of tools depends on the task at hand and the materials one is working with. (HPL I, p. 22)

Are these two statements contradictory? They may be if only certain types of classroom activity are interpreted as “actively engaged.”

Chapter 2

Lines 69–80: Suggested edit: Delete lines 69–80.

Reason: the tables give no information about women’s CAASPP performance and graduation rates. Yet, the conclusion seems to be that women’s K–12 mathematics performance explains their underrepresentation in STEM. Note that for 2021 unrounded percentages for Percent Met or Exceeded (Levels 3 + 4) were: females 33.05%; males 34.46%.

For California, grade 8 NAEP averages: 270 for males, 269 for females.

Among California high school students:

- Females are *slightly overrepresented* in advanced and AP mathematics courses (Asim et al., 2019, Figures 3 and 5). The situation was similar in 1998 (Danenberg, 2001, Figures 2 and 3).

- Over 70% of computer science course-takers are male (Scott et al., 2019, Figures 8 and 9). The situation was similar in 1998 (Danenberg, 2001, Figures 2 and 3).

STEM includes mathematics but is not identical to mathematics. The fact that women undergraduates are better represented in mathematics than in engineering and computer science suggests that their mathematical K–12 preparation as measured by test scores is unlikely to be the only factor—or even a factor at all—in explaining their underrepresentation in engineering and computer science. In contrast, findings from studies such as *Talking About Leaving* and the K–12 gaps for non-white and non-Asian students described in lines 38–66 suggest that inadequate K–12 mathematical preparation disproportionately impedes their access to higher education and STEM careers.

References.

Minahil Asim, Michal Kurlaender, and Sherrie Reed, August 2019. *12th Grade Course-taking and the Distribution of Opportunity for College Readiness in Mathematics.* Stanford, CA: Policy Analysis for California Education. https://files.eric.ed.gov/fulltext/ED600439.pdf

Anne Danenberg, February 2001. Who's lagging now? Gender differences in secondary course enrollments. *California Counts: Population Trends and Profiles, 2*(3). Public Policy Institute of California. https://www.ppic.org/wp-content/uploads/content/pubs/cacounts/CC_201ADCC.pdf

Seymour, E. (2019). Then and Now: Summary and Implications. In *Talking about Leaving Revisited* (pp. 437-473). Springer, Cham. See also David Bressoud's *Notices of the American Mathematical Society* review of *Talking about Leaving Revisited*.

Alison Scott et al., June 2019. *Computer Science in California's Schools: An Analysis of Access, Enrollment, and Equity.* Kapor Center and CsforCA. https://www.kaporcenter.org/wp-content/uploads/2019/06/Computer-Science-in-California-Schools.pdf

Chapter 3

Lines 55–56: replace "seeing parallels between numbers and functions" by "seeing parallels between numbers and polynomials" Reason: "Polynomial," "polynomial function," and "function" cannot used interchangeably. A polynomial can be viewed as determining a function but isn't itself a function. Also, not all functions are polynomial functions.

Line 63: There is an incomplete sentence. "Chapter 5 describes progressions for." Chapter 5 is about mathematical foundations for data science.

Line 123: "Primary grade" is used to mean "early elementary" in the CMF but this meaning is not made explicit in the text. It should be because "primary" is sometimes used as a synonym for "elementary" in discussions of education.

Lines 156–159: "The eight California Common Core Standards for Mathematical Practice (SMP), implemented in tandem with the California Common Core Content Standards for Mathematics, offer a carefully constructed pathway that supports the gradual growth of number sense across grade levels." "Pathway" is misleading (see quotation below).

The CCSS document states:

> These Standards do not dictate curriculum or teaching methods. For example, just because topic A appears before topic B in the standards for a given grade, it does not necessarily mean that topic A must be taught before topic B. A teacher might

> prefer to teach topic B before topic A, or might choose to highlight connections by teaching topic A and topic B at the same time. Or, a teacher might prefer to teach a topic of his or her own choosing that leads, as a byproduct, to students reaching the standards for topics A and B.
>
> What students can learn at any particular grade level depends upon what they have learned before. Ideally then, each standard in this document might have been phrased in the form, "Students who already know ... should next come to learn ...." But at present this approach is unrealistic—not least because existing education research cannot specify all such learning pathways. (p. 5)

Line 178: replace "Seeing parallels between numbers and functions" by "Seeing parallels between numbers and polynomials" Reason: "Polynomial," "polynomial function," and "function" cannot used interchangeably. A polynomial can be viewed as determining a function but isn't itself a function. Also, not all functions are polynomial functions.

Line 180: replace "Seeing parallels between numbers and functions" by "Seeing parallels between numbers and polynomials"

Lines 210–212: "fluency has sometimes been equated with speed, which may account for the common, but counterproductive, use of timed tests for practicing facts (Henry and Brown, 2008)." Henry and Brown's article is about states that changed NCTM's recommendation that students develop recall of addition and subtraction facts by the end of grade 2 into "students should have memorized addition and subtraction facts by the end of grade 1." In particular, the authors suggest that in writing its 1998 standards, California misinterpreted the educational practices of high-performing countries as drill rather than consulting the work of Karen Fuson (part of the CCSS development group) and her colleagues who have studied educational practices in these countries (p. 157). There was evidence that the state-approved textbooks and professional development did not provide the California teachers information about the educational practices of high-performing countries—which, as adapted to the U.S. context, are described in the *Progressions for the CCSS.*

Line 245: "Counting and cardinality" should be above "Know number names and the count sequence" and in the same box.

Line 463: replace "solve" by "compute"—an expression is not a problem.

Line 1478: replace "whether a system of numbers or a system of functions" by "whether a system of numbers or a system of polynomials"

Line 1482–1483: replace "explore similar divisibility concepts in the new territories of polynomials and rational functions" by "explore similar divisibility concepts in the new territories of polynomials and rational expressions"

Line 1485: replace "solving" by "solve"

Line 1496: replace “with a fractions representation of rational numbers” by “with rational numbers represented in the form *a*/*b*”

Line 1500: “logarithms and exponential functions” has several possible intended meanings “logarithms and exponents” or “logarithmic and exponential functions” or “logarithmic and exponential expressions in one variable”

Lines 1525–1528: “Students’ conception of the complex number system and its itinerant properties grows further with adding, subtracting, and multiplying complex numbers together (N-CN.2), just as they have manipulated prior types of numbers, such as rational numbers, with these same operations.” “Itinerant” is an interesting word choice. What does it mean in this context?

Chapter 6

Lines 226–228: “Big ideas are central to the learning of mathematics, link numerous mathematics understandings into a coherent whole, and provide focal points for student investigations (Charles, 2005).” Charles says only the first two—nothing about focal points. This sentence also appears in Chapter 1.

Lines 247–249: “Investigations motivate students to learn focused, coherent, and rigorous mathematics. They also help teachers to focus instruction on the big ideas.” This seems to say that anything called an investigation has these properties.

Lines 268–269: “X axis” is written “*x*-axis”; “Y axis” is written “*y*-axis.” These are lines. The figure doesn’t show lines.

Line 312: replace “Practices” by “Practice”

Line 356: “In addition” is ambiguous. Does it mean “also” or does it refer to students’ work with the operation of addition. Suggested edit: delete “In addition,”

Lines 358–359, 369: Note that this text no longer appears in the Progressions.

Chapter 10

Line 337: replace “*Principles to Action”* by “*Principles to Actions”*

Line 780–781: delete “of learning”

Line 793: “using strategies that show connections between different mathematical ideas on various topics across grade levels.” Teaching strategies or solution strategies

Chapter 12

Line 63: replace “eliminated SAT tests from the admissions process” by “eliminated SAT scores from the admissions process”

Line 66: replace “students to reason and problem solve” by “students to reason and solve problems”

Line 73–78: In particular: “Math anxiety . . . timed tests are a major cause of this debilitating, often life-long condition (Boaler, 2014)” Please note disagreement and lack of evidence about this claim.

> A particularly damaging assessment practice to avoid is the use of timed tests to assess speed of mathematical fact retention, as such tests have been found to prompt mathematics anxiety. When anxious, the working memory—the part of the brain needed for reproducing mathematics facts—is compromised. Math anxiety has been recorded in students as young as five years old (Ramirez et al., 2013), and timed tests are a major cause of this debilitating, often life-long condition (Boaler, 2014). Alternative activities can be used that develop mathematics fact fluency through engaging, conceptual visual activities instead of anxiety-producing speed tests.

Lines 93–94: Suggested edit: Delete “instead of narrow procedures” in “Chapters 6, 7, and 8 set out an approach to mathematics teaching through big ideas that integrates mathematical content and practices instead of narrow procedures.” The phrase “narrow procedures” is confusing because it suggests the existence of a special kind of procedures called “narrow procedures.”

Lines 232–232: replace “The NCTM Principles to Action state” by “The NCTM’s *Principles to Actions* states.”

Lines 271–272: Suggested edit: Replace “identify what quality work is” by “identify what high-quality work is”

Line 304: Suggested edit: Replace “problem-solve” by “solve problems”

Line 522: Figure 12.15. Why do none of the teacher’s comments mention units?

Lines 875–876: Suggested edit: “You have a collection of objects and your friend gives you six more. How many do you have and how do you know?” The figure suggests that “five” should be inserted before “objects.”

Lines 888–889: Not clear what this is asking “What whole number is between?” Between the two points?

Line 892: Suggested edit: Delete “over”

Line 893: Suggested edit: Replace “showing the integer-ordered pairs” by “showing the ordered pairs for integer values of *x.”*

Line 893: Suggested edit: Replace “Write a story that might be modeled by this function. Explain how your story models the function” by “Write a story that might be modeled by this function. Explain how the function models your story.”

Line 1010: Claim 3 says “For older students this may take the form of a rigorous deductive proof based on clearly stated axioms. For younger students this will involve justifications that are less formal.” Does “rigorous” have the same meaning as in Chapter 1?

Line 1062: is there a difference between “directly connects” and “connects”? What is it supposed to signify?

Chapter 13

Lines 156, 357, 501: delete “learning” in “learning progressions.” Line 506: delete “of learning.” See p. 2 of *Progressions for the CCSS.*

Line 195: Consider inserting a paragraph like this:

> The *Progressions* note key connections among standards within and between grades, point out cognitive difficulties and pedagogical solutions, and give more detail on particularly knotty areas of the mathematics. For example, they note connections between K–5 Measurement and Data standards and standards for work with numbers (p. 74). They give a side-by-side comparison of the standards for measurement of area in Grades 3 and 5 and the measurement of volume in Grades 5 and 6 (p. 89). They display multiplication and division situations for equal groups, arrays, and comparisons (p. 32), and analogous measurement situations (p. 103).

Lines 528–534: delete. The content of these lines is repeated in lines 543–559.

Line 580–581: “Kilpatrick, Swafford, and Findell, 2001” refers to National Research Council. 2001. *Adding It Up: Helping Children Learn Mathematics*. Washington, DC: National Academy Press.

Comments on citations and references

Citations in text. Citations in text do not have a consistent order. Some are ordered by date, some by alphabetic order, and some seem to have no order. Examples are below.

Citations not ordered alphabetically or by date:

> Chapter 1, Lines 159–161: They understand that the intellectual authority of mathematics rests in mathematical reasoning itself—mathematics makes sense! (Nasir, 2002; Gresalfi et al., 2009; Martin, 2009; Boaler and Staples, 2008).

Citations ordered by date, not alphabetically:

> Chapter 1, Lines 168–172: When students engage in mathematical tasks, they are recruiting both domain-specific and domain-general brain systems, and the pattern of activation across these systems differs depending on the type of mathematical task the students are performing (Vogel and De Smedt, 2021; Sokolowski, Hawes, and Ansari, 2023).

Citations ordered alphabetically, not by date:

> Chapter 1, Lines 306–308: Further, when students change their mindsets, from fixed to growth, their mathematics achievement increases (Blackwell, Trzesniewski, and Dweck, 2007; Dweck, 2008; Yeager et al., 2019).

One book referenced by two different authors.

> Appendix, Lines 41–42: Bransford, John, Ann L. Brown, and Rodney R. Cocking. 2000. *How People Learn* (Vol. 11). Washington, DC: National Academy Press.

> Appendix, Lines 152–153: National Research Council. 2000. *How People Learn: Brain, Mind, Experience, and School: Expanded Edition*. Washington, DC: The National Academies Press.

One book referenced by two different names and three different groups of authors.

> Appendix, lines 455, 919: National Research Council. 2001. *Adding It Up: Helping Children Learn Mathematics*. Washington, DC: National Academy Press.

> Appendix, line 1630: National Research Council and Mathematics Learning Study Committee. 2001. Adding It Up: Helping Children Learn Mathematics. National Academies Press.

Appendix, line 1680: Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. 2001. *Adding It All Up: How Children Learn Mathematics*. Washington, DC: National Academy Press.

Two books with same author and date.

Appendix, line 709: National Academies of Sciences, Engineering, and Medicine. 2018. How people learn II: Learners, contexts, and cultures. Washington, DC: National Academies Press.

Appendix, line 791: National Academies of Sciences, Engineering, and Medicine. 2018. *Data Science for Undergraduates: Opportunities and Options.* Washington, DC: The National Academies Press.

Updated reference. There is a minor update of the reference below posted at https://mathematicalmusings.org/2023/05/24/version-of-progressions-with-revised-appendix/.

Appendix, Lines 72, 587, 767, 861, 1023, 1707: Common Core Standards Writing Team. 2022. Progressions for the Common Core State Standards for Mathematics (February 28, 2023). Tucson, AZ: Institute for Mathematics and Education, University of Arizona. https://mathematicalmusings.org/wp-content/uploads/2023/02/Progressions.pdf.

Wrong format. The Mathematical Education of Teachers II is in a book series, not a journal.

Appendix, Lines 1333–1335: Conference Board of the Mathematical Sciences. 2012. *The Mathematical Education of Teachers II* (Issues in Mathematics Education 17). Providence, RI: American Mathematical Society. https://www.cbmsweb.org/the-mathematical-education-of-teachers/

Appendix. Inconsistencies in the CMF's Use of "Fluency"

In the CMF, there are several descriptions of fluency, for example:

> Chapter 14 (glossary), lines 189–190: "Fluency. The ability to select and flexibly use appropriate strategies to explore and solve problems in mathematics."
>
> Chapter 3, lines 56–57: "number fluency—the ability to use strategies that are flexible, efficient, and accurate"
>
> Chapter 3, lines 188–195: "Fluency in mathematics . . . is a topic is of continuing importance across all grade levels. . . . fluency means that students use strategies that are flexible, efficient, and accurate to solve problems in mathematics.

These descriptions seem to combine aspects of procedural fluency and strategic competence as described in the National Research Council report *Adding It Up.* This report describes mathematical proficiency as having five components (NRC, 2001, p. 116). Two of the five are:

> • *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
>
> • *strategic competence*—ability to formulate, represent, and solve mathematical problems

"Procedure" is not defined in the CMF glossary. "Strategy" as defined in the glossary includes the usual meaning of procedure: "Mental or written method chosen for approaching or solving a problem; may be invented by a student."

However, several lines down from its definition, fluency is said to involve procedures:

> Chapter 3, lines 212–218: "Fluency involves more than speed, however, and requires knowing, efficiently retrieving, and appropriately using facts, procedures, and strategies, including from memory. Achieving fluency builds on a foundation of conceptual understanding, strategic reasoning, and problem solving (NGA Center and CCSSO, 2010; NCTM, 2000, 2014). To develop fluency, students need to have opportunities to explicitly connect their conceptual understanding with facts and procedures (including standard algorithms) in ways that make sense to them (Hiebert and Grouws, 2007)."

One motivation for the CMF definition of fluency may be an attempt to avoid timed tests of "math facts," possibly because such tests are claimed (without evidence) to create math anxiety (Chapter 12, lines 73–78) and claimed (citing one study) to be counterproductive (Chapter 3, lines 210–212). Yet, much better documented factors in math anxiety exist and receive no attention in the CMF. The cognitive psychologist Sian Beilock and her colleagues have studied math anxiety and published numerous articles on contributing factors, among them:

Beilock, S. L., Gunderson, E. A., Ramirez, G., & Levine, S. C. (2010). Female teachers' math anxiety affects girls' math achievement. *Proceedings of the National Academy of Sciences*, *107*(5), 1860-1863.

Maloney, E. A., & Beilock, S. L. (2012). Math anxiety: Who has it, why it develops, and how to guard against it. *Trends in Cognitive Sciences*, *16*(8), 404-406.

Maloney, E. A., Ramirez, G., Gunderson, E. A., Levine, S. C., & Beilock, S. L. (2015). Intergenerational effects of parents' math anxiety on children's math achievement and anxiety. *Psychological Science*, *26*(9), 1480-1488.

Another motive for the CMF's treatment of fluency may be concern about procedures. Chapters 1 and 3 seem to be avoiding the words "procedure" and "procedural." Some occurrences are: "memorized procedures" (Chapter 1, line 141), "narrow tests of procedural knowledge" (Chapter 1, line 784), "teachers can do much to help students learn 'real' mathematics as big ideas that are related to one another rather than a list of procedures and tricks that must be memorized" (Chapter 3, lines 136–139), "rote procedural skills" (Chapter 3, line 1351). This avoidance and the lack of distinction between procedures and strategies is awkward because Chapter 2 distinguishes between them (e.g., lines 340–341, 876).