

Rotation in the Space*

(Com S 477/577 Notes)

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The position of a point after some rotation about the origin can simply be obtained by multiplying its coordinates with a matrix. One reason for introducing homogeneous coordinates is to be able to describe translation by a matrix so that multiple transformations, whether each is a rotation or a translation, can be concatenated into one described by the product of their respective matrices. However, in some applications (such as spaceship tracking), we need only be concerned with rotations of an object, or at least independently from other transformations. In such a situation, we often need to extract the rotation axis and angle from a matrix which represents the concatenation of multiple rotations. The homogeneous transformation matrix, however, is not well suited for the purpose.

1 Euler Angles

A rigid body in the space has a coordinate frame attached to itself and located often at the center of mass. This frame is referred to as the *body frame* or *local frame*. The position, orientation, and motion of the body can be described using the body frame relative to a fixed reference frame, called the *world frame*.

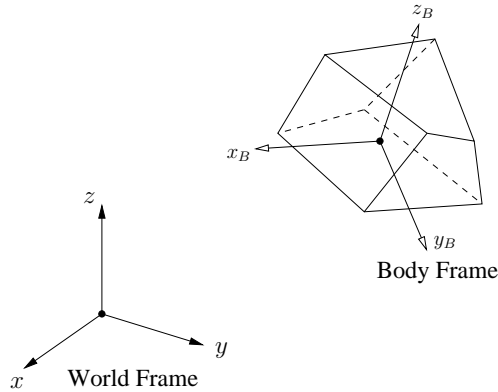


Figure 1: Body frame (local) vs. world frame (global).

The rigid body has six degrees of freedom: its position given by the x , y , z coordinates of its center of mass (i.e., the origin of the local frame) in the world frame, and its orientation described

*The material is partially based on Chapters 3-6 of the book [2]

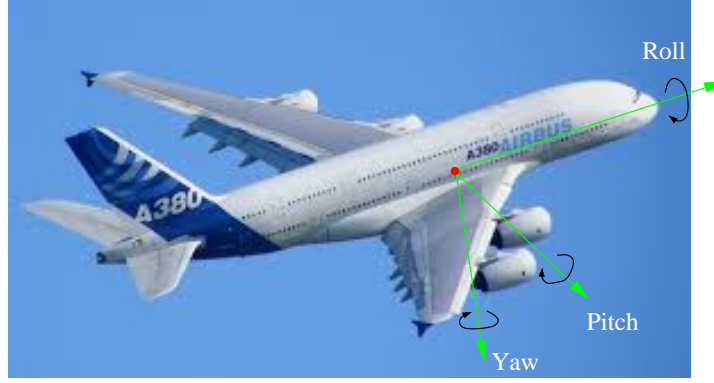


Figure 2: Roll, pitch, yaw of an aircraft.

by three angles of rotation of its body frame from the world frame. To describe this rotation, one choice is to use three angle about the x -, y -, and z -axes. A more convenient choice, from the perspective of the rotating body, is to use three angles about the axes of the body frame. The latter three angles are called *Euler angles*.

For example, an aircraft in flight can perform three independent rotations: *roll*, about an axis from nose to tail; *pitch*, nose up or down about an axis from wing to wing; and *yaw*, nose left or right about an axis from top to bottom.

There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. Here we introduce the Z - Y - X Euler angles. The body frame starts in the same orientation as the world frame. To achieve its final orientation, the first rotation is by an angle ϕ , about the body z -axis, the second rotation by an angle $\theta \in [0, \pi]$ about the body y -axis, and the third rotation by an angle ψ about the body x -axis. Here, ϕ , θ , and ψ correspond to as yaw, pitch, and roll, respectively. The three rotations are illustrated in Figure 3, where x'_B - y'_B - z'_B and x''_B - y''_B - z''_B are the two intermediate configurations of the body frame x_B - y_B - z_B .

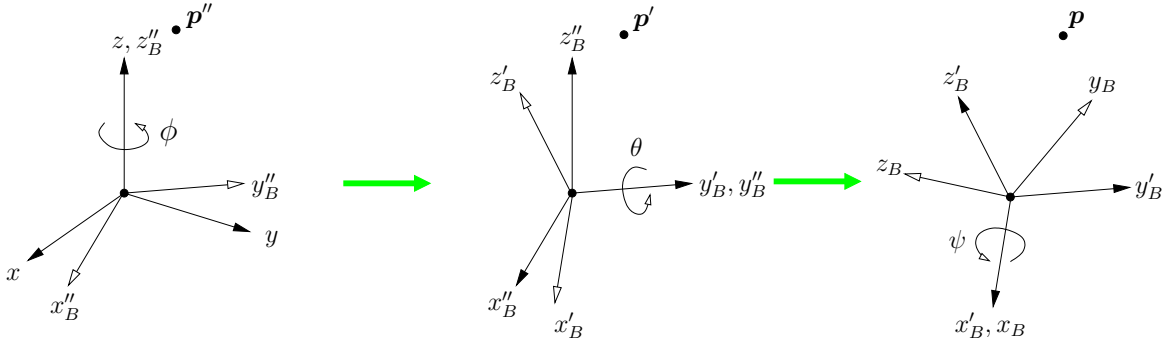


Figure 3: Z - Y - X Euler angles ϕ (yaw), θ (pitch), ψ (roll). Here, \mathbf{p} , \mathbf{p}' , and \mathbf{p}'' are the same point of coordinates in the frames x_B - y_B - z_B , x'_B - y'_B - z'_B , and x''_B - y''_B - z''_B , respectively.

Now we derive the homogeneous transformation matrix. Consider a point \mathbf{p} in the body frame x_B - y_B - z_B . Let us determine its coordinates in the original frame x - y - z by considering the four frames backward: x_B - y_B - z_B , x'_B - y'_B - z'_B , x''_B - y''_B - z''_B , and x - y - z . Consider a point \mathbf{p} with its coordinates given in the body frame. Since the frame x_B - y_B - z_B is obtained from the frame x'_B - y'_B - z'_B by a rotation

about the x'_B -axis through an angle ψ , the coordinates of \mathbf{p} in the latter frame is

$$\mathbf{p}' = \text{Rot}_x(\psi)\mathbf{p}.$$

Next, the frame $x'_B-y'_B-z'_B$ is obtained from the frame $x''_B-y''_B-z''_B$ after a rotation about the y''_B -axis through an angle θ . So the same point has coordinates

$$\mathbf{p}'' = \text{Rot}_y(\theta)\mathbf{p}' = \text{Rot}_y(\theta)\text{Rot}_x(\psi)\mathbf{p}$$

in the frame $x''_B-y''_B-z''_B$. Similarly, the coordinates of the point in the frame $x-y-z$ is

$$\text{Rot}_z(\phi)\mathbf{p}'' = \text{Rot}_z(\phi)\text{Rot}_y(\theta)\text{Rot}_x(\psi)\mathbf{p}.$$

Thus the transformation matrix associated with the Z - Y - Z Euler angles are

$$\begin{aligned} \text{Rot}_{zyx}(\phi, \theta, \psi) &= \text{Rot}_z(\phi) \cdot \text{Rot}_y(\theta) \cdot \text{Rot}_x(\psi) \\ &= \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{pmatrix} \\ &= \begin{pmatrix} \cos \phi \cos \theta & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \\ \sin \phi \cos \theta & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi \\ -\sin \theta & \cos \theta \sin \psi & \cos \theta \cos \psi \end{pmatrix} \quad (1) \end{aligned}$$

Euler angles are defined in terms of three moving axes. Singularities happen when two of these axes coincide. Consider the z - x - z angles α, β, γ with $\beta = 0$. Here, the first and the third rotations are about the same axis. All the Euler angles $(\alpha, 0, \gamma)$ with the same $\alpha + \gamma$ value thus describe one rotation. Namely, one degree of freedom is lost. Linear interpolation from one orientation to another is not well-behaved. Imagine, when the latitude and longitude values are interpolated. What will happen when longitude goes towards 90 degrees to reach the north pole? All latitude values there make no difference as they end up describing the same point! This is a phenomenon referred to as “gimbal lock”, drawing its name from certain orientation with three nested moving gimbals in which two of the three axes become collinear — restricting the available rotations to only two axes. (Watch a nice video tutorial on gimbal lock at <https://www.youtube.com/watch?v=zc8b2Jo7mno>).

The above singularity issue with Euler angles is because they form a 3D box $[0, 2\pi]^3$, while the rotations constitute a 3D projective space. The mapping between the two spaces cannot be “continuous both ways” (or, strictly speaking, a homeomorphism¹).

2 Arbitrary Rotation Axis

Let \mathbf{l} be a vector that is undergoing a rotation of the amount θ about some axis through the origin with an arbitrary orientation. This can be viewed as a rotation about a line that was treated in Appendix A in the notes titled “Transformations in Homogeneous Coordinates”. The resulting vector can be found by first rotating the axis to the z -axis, performing the rotation, and rotating the axis back to its original orientation. Now let us present a simpler approach with direct geometric meaning.

¹a bijective and continuous function whose inverse is also continuous

Let $\hat{\mathbf{k}} = (k_x, k_y, k_z)^T$ be the unit vector along the axis. To obtain the resulting \mathbf{l}' from the rotation, we decompose \mathbf{l} into two parts: one along $\hat{\mathbf{k}}$ and the other in the plane Π containing the origin and perpendicular to $\hat{\mathbf{k}}$:

$$\mathbf{l}_{\parallel} = (\mathbf{l} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}, \quad (2)$$

$$\begin{aligned} \mathbf{l}_{\perp} &= \mathbf{l} - \mathbf{l}_{\parallel} \\ &= \mathbf{l} - (\mathbf{l} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}. \end{aligned} \quad (3)$$

As shown in Figure 4, the component \mathbf{l}_{\parallel} is not affected by rotation. We need only determine the vector \mathbf{l}'_{\perp} that results from applying the same rotation to \mathbf{l}_{\perp} .

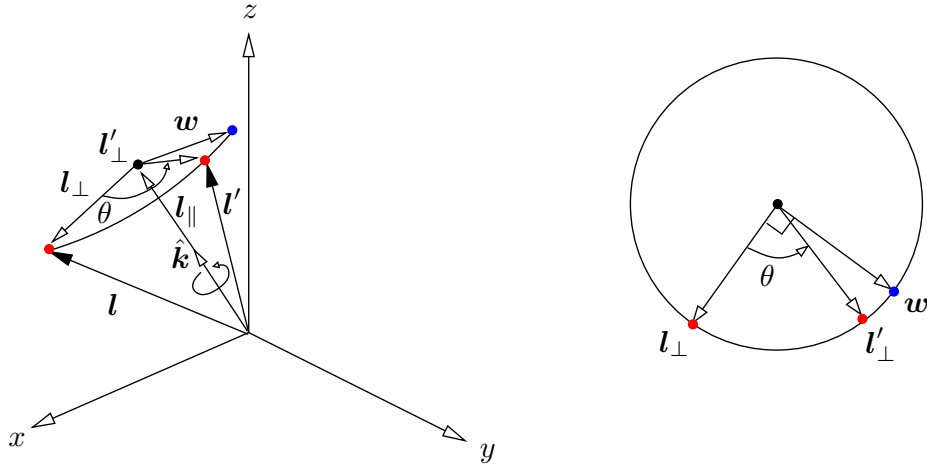


Figure 4: The vector \mathbf{l} is rotated to \mathbf{l}' about the axis $\hat{\mathbf{k}}$.

The plane Π is spanned by two orthogonal vectors: \mathbf{l}_{\perp} and

$$\mathbf{w} = \hat{\mathbf{k}} \times \mathbf{l}_{\perp} = \hat{\mathbf{k}} \times \mathbf{l}. \quad (4)$$

It is easy to see that $\mathbf{l}_{\perp}, \mathbf{w}, \hat{\mathbf{k}}$ are the axes of a right-handed system. Since $\hat{\mathbf{k}}$ is orthogonal to \mathbf{l}_{\perp} , \mathbf{l}_{\perp} and \mathbf{w} have the same length. The vector \mathbf{l}'_{\perp} lies in the plane Π , and has the form

$$\mathbf{l}'_{\perp} = \mathbf{l}_{\perp} \cos \theta + \mathbf{w} \sin \theta. \quad (5)$$

Finally, we have the rotated vector from \mathbf{l} :

$$\begin{aligned} \mathbf{l}' &= \mathbf{l}_{\parallel} + \mathbf{l}'_{\perp} \\ &= (\mathbf{l} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} + \mathbf{l}_{\perp} \cos \theta + \mathbf{w} \sin \theta && \text{(by (2) and (5))} \\ &= (\mathbf{l} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} + (\mathbf{l} - (\mathbf{l} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}) \cos \theta + (\hat{\mathbf{k}} \times \mathbf{l}) \sin \theta && \text{(by (3) and (4))} \\ &= \mathbf{l} \cos \theta + (\hat{\mathbf{k}} \times \mathbf{l}) \sin \theta + \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{l})(1 - \cos \theta). \end{aligned} \quad (6)$$

Equation (6) is called *Rodrigues' rotation formula*.

Note that

$$\begin{aligned}
\mathbf{l} \cos \theta &= \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & \cos \theta \end{pmatrix} \mathbf{l}; \\
\hat{\mathbf{k}} \times \mathbf{l} &= \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \mathbf{l}; \\
\hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \mathbf{l}) &= (\hat{\mathbf{k}}\hat{\mathbf{k}}^T) \mathbf{l} \\
&= \begin{pmatrix} k_x^2 & k_x k_y & k_x k_z \\ k_y k_x & k_y^2 & k_y k_z \\ k_z k_x & k_z k_y & k_z^2 \end{pmatrix} \mathbf{l}.
\end{aligned}$$

Substituting the above into (6), we express \mathbf{l}' as the product of the following 3×3 rotation matrix with \mathbf{l} :

$$\text{Rot}_{\hat{\mathbf{k}}}(\theta) = \begin{pmatrix} k_x k_x (1 - \cos \theta) + \cos \theta & k_x k_y (1 - \cos \theta) - k_z \sin \theta & k_x k_z (1 - \cos \theta) + k_y \sin \theta \\ k_x k_y (1 - \cos \theta) + k_z \sin \theta & k_y k_y (1 - \cos \theta) + \cos \theta & k_y k_z (1 - \cos \theta) - k_x \sin \theta \\ k_x k_z (1 - \cos \theta) - k_y \sin \theta & k_y k_z (1 - \cos \theta) + k_x \sin \theta & k_z k_z (1 - \cos \theta) + \cos \theta \end{pmatrix}. \quad (7)$$

3 Rotation Matrix

We have seen the use of a matrix to represent a rotation. Such a matrix is referred to as a *rotation matrix*. In this section we look at the properties of rotation matrix. Below let us first review some concepts from linear algebra.

3.1 Eigenvalues

An $n \times n$ matrix A is *orthogonal* if its columns are unit vectors and orthogonal to each other, namely, if $A^T A = I_n$, where I_n is the $n \times n$ identity matrix. An orthogonal matrix has determinant $\det(A) = \pm 1$.

A complex number λ is called the *eigenvalue* of A if there exists a vector $\mathbf{x} \neq 0$ such that $A\mathbf{x} = \lambda\mathbf{x}$. It tells whether the vector \mathbf{x} is stretched or shrunk or reversed or left unchanged — when it is multiplied by A . The vector \mathbf{x} is called an *eigenvector* of A associated with the eigenvalue λ . The set of all eigenvalues is called the *spectrum* of A .

The matrix A has exactly n eigenvalues (multiplicities included). They are the roots of the n th degree polynomial $\det(A - \lambda I_n)$, called the *characteristic polynomial*. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of this polynomial, then

$$\det(A - \lambda I_n) = (-1)^n (\lambda - \lambda_1)^{\delta_1} \cdots (\lambda - \lambda_k)^{\delta_k}, \quad \text{for some } \delta_1, \dots, \delta_k > 0.$$

Here δ_i , $1 \leq i \leq k$, is the *algebraic multiplicity* of λ_i .

The *geometric multiplicity* of λ_i is $n - \text{rank}(A - \lambda_i I_n)$. It specifies the maximum number of linearly independent eigenvectors associated with λ_i .

EXAMPLE 1. The diagonal matrix aI_n has the characteristic polynomial $(a - \lambda)^n$. So a is the only eigenvalue while every vector $\mathbf{x} \in C^n$, $\mathbf{x} \neq 0$ is an eigenvector. Both algebraic and geometric multiplicities of a are n .

EXAMPLE 2. The n th order matrix

$$A = \begin{pmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & a & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}$$

also has the characteristic polynomial $(a - \lambda)^n$ and a as its only eigenvalue. The algebraic multiplicity of a is n . However, the rank of $A - aI_n$ is $n - 1$; hence the geometric multiplicity of a is $n - (n - 1) = 1$.

The product of the n eigenvalues equals the determinant of A . The sum of the n eigenvalues equals the sum of the n diagonal entries; this number is called the *trace* of A and denoted $\text{Tr}(A)$.

The matrix A is *invertible* if and only if it does not have a zero eigenvalue. A symmetric matrix has real eigenvalues. A symmetric and positive definite matrix has positive eigenvalues.

3.2 Matrices Describing Rotations

All rotation matrices applying to points in the space are 3×3 .

Theorem 1 *A 3×3 matrix describes a rotation about some axis through the origin if and only if it is orthogonal and has determinant 1.*

Proof (\Rightarrow) We first show that any rotation matrix R is orthogonal and $\det(R) = 1$. Let \mathbf{l} be the corresponding rotation axis and θ the angle of rotation about the axis. The inner product of two vectors, say, \mathbf{l}_1 and \mathbf{l}_2 , must be preserved under the rotation; namely,

$$(R\mathbf{l}_1)^T(R\mathbf{l}_2) = \mathbf{l}_1^T\mathbf{l}_2.$$

The above implies that

$$\mathbf{l}_1^T(R^T R - I_3)\mathbf{l}_2 = 0.$$

Because \mathbf{l}_1 and \mathbf{l}_2 are arbitrarily chosen, it must hold that

$$R^T R - I_3 = 0 \quad \text{and thus} \quad R^T R = I_3.$$

Therefore, R is orthogonal. Further more, from the equation

$$\det(R^T R) = \det(R)^T \det(R) = (\det(R))^2 = 1$$

we infer that $\det(R)$ must be 1 or -1 . But a rotation preserves the right-handedness of a coordinate frame, so we cannot have $\det(R) = -1$. For example, the diagonal matrix with diagonal entries 1, 1, and -1 would describe a reflection about the x - y plane instead of a rotation. We have thus shown that $\det(R) = 1$.

(\Leftarrow) Suppose R is a 3×3 orthogonal matrix with determinant 1. We want to show that it represents a rotation. First, we show that there exists a vector \mathbf{u} such that $R\mathbf{u} = \mathbf{u}$. This vector

will be along the rotation axis. It suffices to establish that the matrix R has an eigenvalue of 1, which is true if and only if $\det(R - I_3) = 0$. We have

$$\begin{aligned}
\det(R - I_3) &= \det(R^T) \det(R - I_3) \\
&= \det(R^T R - R^T) \\
&= \det(I_3 - R^T) && (A \text{ orthogonal}) \\
&= \det((I_3 - R)^T) \\
&= \det(I_3 - R) \\
&= -\det(R - I_3).
\end{aligned}$$

Thus $\det(R - I_3) = 0$. Next, we show that the plane containing the origin and orthogonal to \mathbf{u} maps to itself under the rotation. Consider an arbitrary vector \mathbf{l} in this plane, i.e., $\mathbf{l}^T \mathbf{u} = 0$. The vector $R\mathbf{l}$ is orthogonal to \mathbf{u} because

$$\begin{aligned}
(R\mathbf{l})^T \mathbf{u} &= \mathbf{l}^T R^T R \mathbf{u} \\
&= \mathbf{l}^T I \mathbf{u} \\
&= \mathbf{l}^T \mathbf{u} \\
&= 0.
\end{aligned}$$

Finally, the inner product of two vectors \mathbf{l}_1 and \mathbf{l}_2 is invariant under the transformation because R is orthogonal. Therefore, the lengths of \mathbf{l}_1 and \mathbf{l}_2 and the angles between them are preserved under R . Hence the transformation represented by R is a rotation. \square

A rotation about an axis not through the origin cannot be simply represented by a 3×3 orthogonal matrix with determinant 1. Below we reason this by contradiction. First, let \mathbf{l} be the direction of the axis, and \mathbf{b} be a point on the axis. Since the origin is not on the axis, $\mathbf{b} \neq \mathbf{0}$. Every point $\mathbf{p} \in \mathbb{R}^3$ is transformed into $R(\mathbf{p} - \mathbf{b}) + \mathbf{b}$, where $R \neq I_3$ is a rotation matrix. Assume that the transformation is equivalent to some rotation about the origin. Then there exists a rotation matrix R' such that

$$R(\mathbf{p} - \mathbf{b}) + \mathbf{b} = R'\mathbf{p}. \quad (8)$$

For any $\mathbf{q} \neq \mathbf{p}$, it also holds

$$R(\mathbf{q} - \mathbf{b}) + \mathbf{b} = R'\mathbf{q}. \quad (9)$$

Now, subtract (9) from (8):

$$R(\mathbf{p} - \mathbf{q}) = R'(\mathbf{p} - \mathbf{q}).$$

Since \mathbf{p} and \mathbf{q} are arbitrary points, the vector $\mathbf{p} - \mathbf{q}$ is arbitrary. For the above equation to always hold, $R = R'$ must be true. Equation (8) now reduces to

$$R\mathbf{b} = \mathbf{b},$$

which implies that the vector \mathbf{b} (from the origin) is aligned with the rotation axis \mathbf{l} , in other words, \mathbf{l} passes through the origin. Thus, a contradiction.

Two successive rotations about the same axis in \mathbb{R}^3 is equivalent to one rotation about this axis through an angle which is the sum of the two rotation angles. Even if they are not about the same axis through the origin, their composition is still a rotation through some angle about some axis. The reason is that the product of their rotation matrices, each orthogonal with determinant 1, is still orthogonal with determinant 1. Thus, by Theorem 1 the product matrix describes a rotation. To state this formally, *the composition of any two rotations is equivalent to a rotation.*

3.3 Recovery of Rotation Axis and Angle

Consider the following orthogonal matrix with determinant 1:

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}.$$

Theorem 1 says that it represents a rotation about some axis \mathbf{l} in the space. We can extract the Euler angles ϕ , θ , and ψ based on the form (1). How to find the axis of rotation? It follows from the proof of Theorem 1 that \mathbf{l} must be an eigenvector of R which corresponds to the eigenvalue 1. Namely,

$$(R - I_3)\mathbf{l} = 0,$$

from which we can obtain a solution vector by setting one of the non-zero components in \mathbf{l} to 1.

Note that \mathbf{l} is also an eigenvector of R^T for

$$\begin{aligned} R^T \mathbf{l} &= R^T R \mathbf{l} \\ &= I_3 \mathbf{l} \\ &= \mathbf{l}. \end{aligned}$$

We have

$$\begin{aligned} \mathbf{0} &= (R - R^T)\mathbf{l} \\ &= \begin{pmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \end{aligned} \tag{10}$$

where

$$\begin{aligned} q_1 &= r_{23} - r_{32}, \\ q_2 &= r_{31} - r_{13}, \\ q_3 &= r_{12} - r_{21}. \end{aligned}$$

The last equation above implies that the rotation axis \mathbf{l} must be in the same direction as $\mathbf{q} = (q_1, q_2, q_3)^T$. We simply let

$$\mathbf{l} = \mathbf{q} = \begin{pmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{pmatrix}. \tag{11}$$

Having determined the rotation axis \mathbf{l} , we look at how to recover the rotation angle. If the rotation happens to be about the z -axis through an angle θ , then we have the rotation matrix

$$R = \text{Rot}_z(\theta) \equiv \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And the rotation angle θ satisfies $\cos \theta = r_{11}$. We can also utilize the trace

$$\text{Tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2 \cos \theta$$

and obtain

$$\cos \theta = \frac{\text{Tr}(R) - 1}{2}, \quad (12)$$

$$\begin{aligned} \sin \theta &= \pm \sqrt{1 - \cos^2 \theta} \\ &= \pm \sqrt{\frac{3 - \text{Tr}(R)^2 + 2\text{Tr}(R)}{4}}. \end{aligned} \quad (13)$$

The sign of $\sin \theta$ must agree with that of r_{21} .

In fact, the solution of θ from (12)–(13) generalizes to the situation with an arbitrary rotation axis. To see this, we use the old trick of applying a sequence of rotations, represented by a matrix Q , to transform the axis \mathbf{l} into the z -axis. Then we carry out the rotation about the z -axis through the angle θ . Finally, we apply a sequence of reverse rotations Q^T to transform the z -axis back to \mathbf{l} . The rotation matrix is represented as

$$R = Q^T \text{Rot}_z(\theta) Q. \quad (14)$$

The trace function is commutative, that is, $\text{Tr}(CD) = \text{Tr}(DC)$ for any two square matrices of the same dimensions.² This property allows us to derive the following from (14):

$$\begin{aligned} \text{Tr}(R) &= \text{Tr}(Q^T \text{Rot}_z(\theta) Q) \\ &= \text{Tr}\left(\left(Q^T \text{Rot}_z(\theta)\right) Q\right) \\ &= \text{Tr}\left(Q \left(Q^T \text{Rot}_z(\theta)\right)\right) \\ &= \text{Tr}\left(\left(Q Q^T\right) \text{Rot}_z(\theta)\right) \\ &= \text{Tr}\left(I_3 \text{Rot}_z(\theta)\right) \\ &= \text{Tr}\left(\text{Rot}_z(\theta)\right) \\ &= 1 + 2 \cos \theta. \end{aligned}$$

Hence the rotation angle θ still satisfies equation (12) when the rotation axis is arbitrary. How do we determine the sign in (13)? First, we normalize the rotation axis \mathbf{l} from (11) and obtain a unit vector $\hat{\mathbf{l}} = (l_x, l_y, l_z)$. The rotation about $\hat{\mathbf{l}}$ through an angle θ is described by the following rotation matrix according to (7):

$$\text{Rot}_{\hat{\mathbf{l}}}(\theta) = \begin{pmatrix} l_x l_x (1 - \cos \theta) + \cos \theta & l_x l_y (1 - \cos \theta) - l_z \sin \theta & l_x l_z (1 - \cos \theta) + l_y \sin \theta \\ l_x l_y (1 - \cos \theta) + l_z \sin \theta & l_y l_y (1 - \cos \theta) + \cos \theta & l_y l_z (1 - \cos \theta) - l_x \sin \theta \\ l_x l_z (1 - \cos \theta) - l_y \sin \theta & l_y l_z (1 - \cos \theta) + l_x \sin \theta & l_z l_z (1 - \cos \theta) + \cos \theta \end{pmatrix}.$$

²The proof is directly based on the definition of trace.

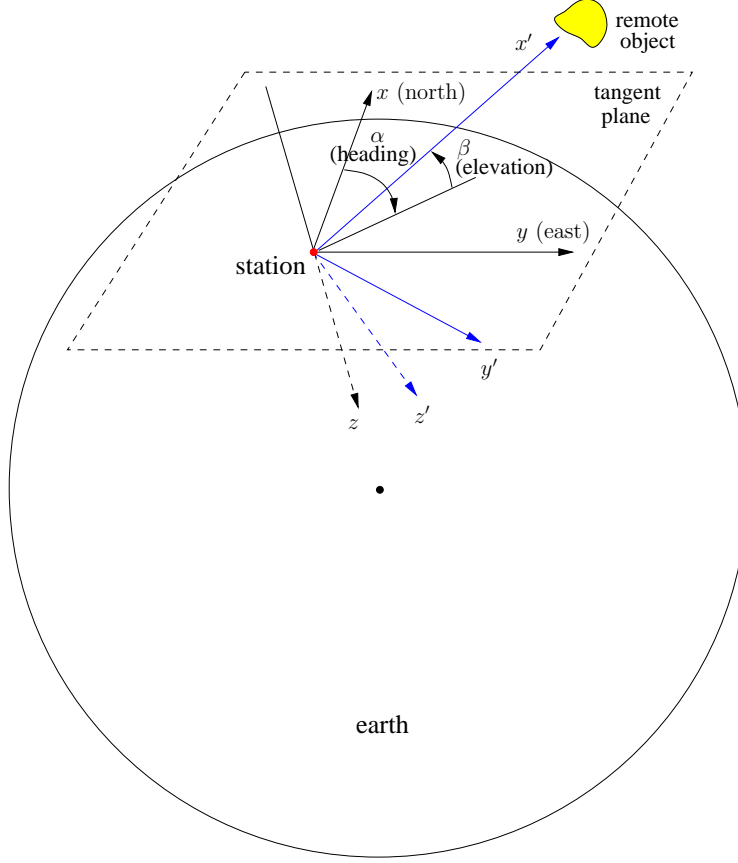


Figure 5: Tracking an aircraft from a ground station.

Comparing an off-diagonal element in the matrix R , say r_{21} , with its corresponding entry in $\text{Rot}_{\hat{\mathbf{i}}}(\theta)$ gives us

$$\begin{aligned} \sin \theta &= \frac{r_{21} - l_x l_y (1 - \cos \theta)}{l_z} \\ &= \frac{2r_{21} - l_x l_y (3 - \text{Tr}(R))}{2l_z}. \end{aligned} \tag{15}$$

The angle $\theta \in [0, 2\pi)$ of rotation is thus completely determined from (12) and (15).

EXAMPLE 3. Here we describe an application of rotations in tracking. Consider a remote object, such as an aircraft, which is being tracked from a station on the earth. We define a local coordinate system at the station with the xy -plane being the tangent plane to the earth. The x - and y -axes point in the directions of the North and East, respectively, and the z -axis points toward the center of the earth. The coordinate system is drawn in Figure 5.

The *heading* α is the angle in the tangent plane between the North and the projection of the direction to the remote object being tracked. The *elevation* is the angle between the tangent plane and the direction to the object. The tracking transformation is a rotation about the z -axis through the angle α , followed by a rotation about the new y -axis through the angle β . After these two rotations, the x -axis is pointing toward the object. The tracking coordinate system is made up of the new x' -, y' -, and z' -axes after the two rotations.

Let R be the 3×3 matrix to represent the composite rotation; it relates the tracking coordinate system to the station's coordinate system. Namely, a point $\mathbf{p} = (p_x, p_y, p_z)^T$ in the (re-oriented) tracking frame is $R\mathbf{p}$ in the station's frame. Then we have

$$\begin{aligned} R &= \text{Rot}_z(\alpha) \cdot \text{Rot}_y(\beta) \\ &= \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \beta & -\sin \alpha & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta & \cos \alpha & \sin \alpha \sin \beta \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}. \end{aligned}$$

This composite tracking transformation can be represented as an equivalent rotation through some angle θ about some axis \mathbf{l} . By (11) we know that the rotation axis is given by

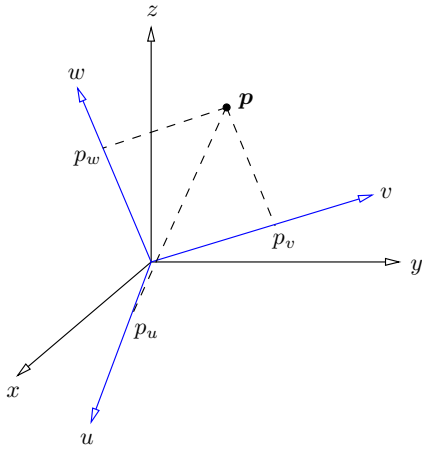
$$\mathbf{l} = \begin{pmatrix} \sin \alpha \sin \beta \\ -\cos \alpha \sin \beta - \sin \beta \\ -\sin \alpha \cos \beta - \sin \alpha \end{pmatrix} \equiv \begin{pmatrix} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \\ -\cos \frac{\alpha}{2} \sin \frac{\beta}{2} \\ -\sin \frac{\alpha}{2} \cos \frac{\beta}{2} \end{pmatrix}. \quad (16)$$

Let $\hat{\mathbf{l}} = (l_x, l_y, l_z)$ be the normalization of \mathbf{l} . By (12) and (15) the angle of rotation satisfies the following equations:

$$\begin{aligned} \cos \theta &= \frac{\cos \alpha \cos \beta + \cos \alpha + \cos \beta - 1}{2}, \\ \sin \theta &= \frac{2 \sin \alpha \cos \beta - l_x l_y (3 - \text{Tr}(A))}{2l_z}. \end{aligned}$$

The domain for the tracking angles is $\alpha \in (-\pi, \pi]$ and $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. As β increases from $\frac{\pi}{2} - \epsilon$ to $\frac{\pi}{2} + \epsilon$, for small ϵ , α instantaneously jumps to $\alpha + \pi$. For this reason, $(\alpha, \frac{\pi}{2})$ is a singular point, and the phenomenon is gimbal lock.

4 Coordinates in a New Coordinate Frame



We are in the x - y - z frame with its three axes defined by the following unit vectors:

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consider another (right-handed) frame u - v - w with the same origin and three axes defined by the unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ (all expressed in the x - y - z frame). Take an arbitrary point, say $\mathbf{p} = (p_x, p_y, p_z)^T$, in the x - y - z frame. What are the new coordinates $\mathbf{p}' = (p_u, p_v, p_w)^T$ of \mathbf{p} when expressed in the u - v - w frame?

The answer is simpler than you might think. Finding the new coordinates $(p_u, p_v, p_w)^T$ is equivalent to rewriting \mathbf{p} in

the following form:

$$\begin{aligned}\mathbf{p} &= p_u \hat{\mathbf{u}} + p_v \hat{\mathbf{v}} + p_w \hat{\mathbf{w}} \\ &= (\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) \begin{pmatrix} p_u \\ p_v \\ p_w \end{pmatrix}.\end{aligned}\tag{17}$$

Since

$$\mathbf{p} = p_x \hat{\mathbf{x}} + p_y \hat{\mathbf{y}} + p_z \hat{\mathbf{z}},\tag{18}$$

we want to describe the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in terms of the unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$. Simply decompose each vector from the first set along the three vectors from the second set, namely,

$$\begin{aligned}\hat{\mathbf{x}} &= (\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}) \hat{\mathbf{u}} + (\hat{\mathbf{x}} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} + (\hat{\mathbf{x}} \cdot \hat{\mathbf{w}}) \hat{\mathbf{w}}, \\ \hat{\mathbf{y}} &= (\hat{\mathbf{y}} \cdot \hat{\mathbf{u}}) \hat{\mathbf{u}} + (\hat{\mathbf{y}} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} + (\hat{\mathbf{y}} \cdot \hat{\mathbf{w}}) \hat{\mathbf{w}}, \\ \hat{\mathbf{z}} &= (\hat{\mathbf{z}} \cdot \hat{\mathbf{u}}) \hat{\mathbf{u}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{w}}) \hat{\mathbf{w}}.\end{aligned}$$

Write the above equations compactly as a matrix product:

$$(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) = (\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) \begin{pmatrix} \hat{\mathbf{x}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{u}} \\ \hat{\mathbf{x}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{v}} \\ \hat{\mathbf{x}} \cdot \hat{\mathbf{w}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{w}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{w}} \end{pmatrix},$$

and substitute the above into (18):

$$\begin{aligned}\mathbf{p} &= (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \\ &= (\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) \begin{pmatrix} \hat{\mathbf{x}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{u}} \\ \hat{\mathbf{x}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{v}} \\ \hat{\mathbf{x}} \cdot \hat{\mathbf{w}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{w}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{w}} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}.\end{aligned}$$

Comparing the above equation with (17), from nonsingularity of the matrix $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}})$ we infer that

$$\begin{pmatrix} p_u \\ p_v \\ p_w \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{u}} \\ \hat{\mathbf{x}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{v}} \\ \hat{\mathbf{x}} \cdot \hat{\mathbf{w}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{w}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{w}} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}.\tag{19}$$

What is the matrix R describing the rotation from the x - y - z frame to the u - v - w frame? In (19), $\mathbf{p}' = R' \mathbf{p}$, where R' is a rotation matrix. Then R is simply the inverse of R' , that is, $R = R'^{-1} = R'^T$ given as

$$R = \begin{pmatrix} \hat{\mathbf{x}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{x}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{x}} \cdot \hat{\mathbf{w}} \\ \hat{\mathbf{y}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{y}} \cdot \hat{\mathbf{w}} \\ \hat{\mathbf{z}} \cdot \hat{\mathbf{u}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{v}} & \hat{\mathbf{z}} \cdot \hat{\mathbf{w}} \end{pmatrix}.$$

5 Rigid Body Transformation in Three Dimensions

A point under a rotation about some axis through a point \mathbf{q} is transformed to $R(\mathbf{p} - \mathbf{q}) + \mathbf{q}$, where R is the rotation matrix. Rewriting the image point as $R\mathbf{p} + (I_3 - R)\mathbf{q}$, we see that the same rotation can be represented as one about an axis through the origin in the same orientation and through the same angle, followed by a translation. Meanwhile, a transformation in which a translation \mathbf{b} precedes a rotation R is equivalent to one in which the same rotation precedes a new translation $R\mathbf{b}$, because

$$R(\mathbf{p} + \mathbf{b}) = R\mathbf{p} + R\mathbf{b}.$$

Any sequence of translations and rotations can be reordered into a sequence of rotations followed by a sequence of translations. The rotation sequence is equivalent to a single rotation, while the translation sequence is equivalent to a single translation.

A rigid body transformation consists of translations and rotations only. Based on the above reasoning, it is equivalent to a rotation followed by a translation. Namely, it is described as a mapping $T : \mathbf{p} \mapsto R\mathbf{p} + \mathbf{b}$, where $R = (r_{ij})$ is a rotation matrix, and \mathbf{b} is a translation vector.

A rigid body transformation in three dimensions is often referred to as a *spatial displacement*. Unless the transformation is a pure rotation, there does not exist a fixed point under such transformation.

Theorem 2 (Chasles' Theorem) *Every spatial displacement $T(\mathbf{p}) = R\mathbf{p} + \mathbf{b}$ is the composition of a rotation about some axis, and a translation along the same axis.*

Instead of proving the theorem, below we describe how to determine the rotation axis ℓ and the translation [3, pp. 19–20]. The axis ℓ is a line in the space called the *screw axis*. Any point on the line is constrained to move along the line under the transformation T .

Following the derivation from Section 3, the direction \mathbf{l} of the axis is given in (11). Essentially, $\mathbf{l} \times \mathbf{u} = (R - R^T)\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^3$. We decompose the translation vector \mathbf{b} into a part \mathbf{b}_{\parallel} along the axis ℓ and a part \mathbf{b}_{\perp} orthogonal to it:

$$\mathbf{b} = \mathbf{b}_{\parallel} + \mathbf{b}_{\perp},$$

where

$$\begin{aligned} \mathbf{b}_{\parallel} &= \left(\mathbf{b} \cdot \frac{\mathbf{l}}{\|\mathbf{l}\|} \right) \frac{\mathbf{l}}{\|\mathbf{l}\|} \\ &= \frac{1}{\|\mathbf{l}\|^2} (\mathbf{b} \cdot \mathbf{l}) \mathbf{l}, \\ \mathbf{b}_{\perp} &= \mathbf{b} - \frac{1}{\|\mathbf{l}\|^2} (\mathbf{b} \cdot \mathbf{l}) \mathbf{l}. \end{aligned}$$

As shown in Figure 6, \mathbf{c} is the perpendicular from the origin to ℓ . It is transformed to the point

$$R\mathbf{c} + \mathbf{b} = R\mathbf{c} + \mathbf{b}_{\perp} + \mathbf{b}_{\parallel}.$$

Since $\mathbf{c} \perp \ell$, its rotation $R\mathbf{c}$ about an axis parallel to ℓ and through the origin must be perpendicular to ℓ as well. The point $R\mathbf{c} + \mathbf{b}_{\perp}$ must be in the plane containing \mathbf{c} and the perpendicular to ℓ . Meanwhile, the same point must also lie on the axis ℓ in order for $R\mathbf{c} + \mathbf{b}$, its sum with \mathbf{b}_{\parallel} , to lie on ℓ . We thus infer

$$\mathbf{c} = R\mathbf{c} + \mathbf{b}_{\perp},$$

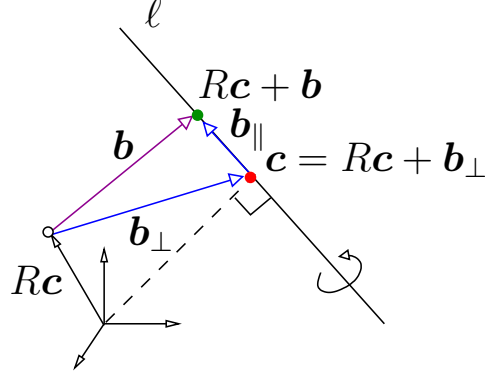


Figure 6: Any rigid body transformation is equivalent to a rotation about some axis, followed by a translation along the same axis.

from which we obtain

$$\mathbf{c} = (I_3 - R)^{-1} \mathbf{b}_\perp,$$

where the inverse $(I_3 - R)^{-1}$ exists due to the uniqueness of \mathbf{c} . The moment \mathbf{m} of ℓ is

$$\begin{aligned} \mathbf{l} \times \mathbf{c} &= (R - R^T) \mathbf{c} \\ &= (R - R^T)(I_3 - R)^{-1} \mathbf{b}_\perp. \end{aligned} \tag{20}$$

Since

$$\begin{aligned} (R^T + I_3)(I_3 - R) &= R^T + I_3 - I_3 - R \\ &= R^T - R, \end{aligned}$$

we have

$$-(R^T + I_3) = (R - R^T)(I_3 - R)^{-1}.$$

Substitute the above into (20):

$$\mathbf{m} = \mathbf{l} \times \mathbf{c} = -(R^T + I_3) \mathbf{b}_\perp. \tag{21}$$

The screw axis is thus given the Plücker coordinates (\mathbf{l}, \mathbf{m}) .

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