Tensor Field Reconstruction Based on Eigenvector and Eigenvalue Interpolation

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Summary. Interpolation is an essential step in the visualization process. While most data from simulations or experiments are discrete many visualization methods are based on smooth, continuous data approximation or interpolation methods. We introduce a new interpolation method for symmetrical tensor fields given on a triangulated domain. Differently from standard tensor field interpolation, which is based on the tensor components, we use tensor invariants, eigenvectors and eigenvalues, for the interpolation. This interpolation minimizes the number of eigenvectors and eigenvalues computations by restricting it to mesh vertices and makes an exact integration of the tensor lines possible. The tensor field topology is qualitatively the same as for the component wise-interpolation. Since the interpolation decouples the "shape" and "direction" interpolation it is shape-preserving, what is especially important for tracing fibers in diffusion MRI data.

1 Introduction

Typically visualization deals with discrete data; nevertheless, many visualization methods are designed to generate a continuous data representation. We need an appropriate interpolation or approximation method, which must be chosen from a large number of possibilities, which can influence the visualization result significantly. This choice is often guided by two - sometimes conflicting - goals: (i) the interpolation should be "simple," meaning it should simplify our computations; and (ii)the interpolation should be "natural," meaning that it should represent the real data as well as possible without introducing too many artifacts. Often losses due to simple interpolation schemes are accepted to simplify the computations.

In many cases a method accomplishing simplicity well uses a linear interpolation schema on a triangulated or tetrahedrized domain. For scalar fields, a unique piecewise linear interpolation is defined by a given triangulation. For vector fields, there are two obvious linear interpolations schemas: one based on the interpolation of the vector field components, and the other one based on or alternatively the direction and length of the vectors. In most cases, both approaches lead to similar results. Considering tensor fields, there are even more ways to interpolate linearly. The most common approach is a linear interpolation of the tensor components [4, 9]. But since the entities we are mostly interested in are not the tensor components but tensor invariants, which are in general not linear, the interpolation is not really simple. There are other problems related to this interpolation, e.g. the sign of the determinate is not preserved.

There has been some work done in the area of tensor field interpolation based on the tensor components. Besides the linear approaches more advanced interpolation methods based on components have been developed with goals of noise reduction or feature preservation [1, 6, 8].

There are also some papers using direction interpolation. in context of diffusion MRI data with the goal of tracing anatomical fibers [5, 2]. Most of these approaches are specific to diffusion MRI data with the goal of tracing anatomical fibers. Such approaches often focus on regions with high anisotropy where no degenerated points exist and the issue of direction assignment is not so important. These interpolation lead to linear direction field, but are not consistent in regions containing degenerated points.

We introduce a linear interpolation schema for symmetrical tensor fields that combines the advantages of linear interpolation of components, which delivers a consistent field, with the advantages of eigenvector and eigenvalue-based interpolation generating a simple direction field and being shape preserving. It is based on eigenvectors and eigenvalues guided by the behavior of eigenvectors for the component-wise interpolation. It is "shape preserving" and minimizes the number of locations where we have to compute the eigendirections. We discuss its properties and compare it to the standard interpolation, i.e., linear interpolation of tensor components.

2 Outline - Idea

Our goal is the interpolation of the tensor field based on the linear interpolation of eigenvalues and eigenvectors. Since there are two eigenvalues and two eigendirections with each two possible orientations, there exists a variety of linear interpolations based on eigenvectors and eigenvalues, see Figure 1. To specify the interpolation uniquely we first have to assign the eigenvalues to each other and then assign orientations to the eigenvectors. When using vectors for the interpolation, we have to consider the fact that not all structures occurring in tensor fields can be simulated by global vector fields, e.g., winding numbers of half integers. This means that an assignment of directions is not possible globally, but it is possible for simply connected regions, that do not contain degenerate points. The situation is complicated by the fact that for discrete data the existence of a degenerate point inside a cell depends on the chosen interpolation.

We discuss one possible eigenvector eigenvalue interpolation schema which simulates the topology of the component-wise interpolation but decouples the interpolation of "shape," represented by the eigenvalues, and direction. Our orientation assignment for eigenvectors is guided by the behavior of the eigenvectors in case of a component-wise interpolation such that the resulting field topology is qualitatively the same as for the component-wise interpolation. This means that we obtain the same number and types of degenerate points in each triangle. The exact position of the degenerate points and separatrices varies slightly.

In the following sections, we describe first the criterion for the assignment of eigenvalues to each other. Then we discuss the behavior of the eigenvectors for the linear interpolation of tensor components, which is essential for the orientation assignment to the eigendirections. Since the existence of degenerate points inside the triangle influences the interpolation, the next step is to define degenerate points on the basis of eigenvectors at the vertices. The results from these discussions are finally used to define the interpolation of eigenvalues and eigenvectors.

3 Basics and Notation

We use the term tensor field for symmetric 2D tensors of second order defined on a triangulated two-dimensional domain. Using a fixed coordinate basis, each tensor can be expressed as a 2×2 matrix, given by four independent scalars. A tensor $\mathbf{T}$ is called symmetric if, for any coordinate basis, the corresponding array of scalars is symmetric. The symmetric part of the tensor is defined by three independent scalars and is represented by a symmetric matrix. We use the following notation:

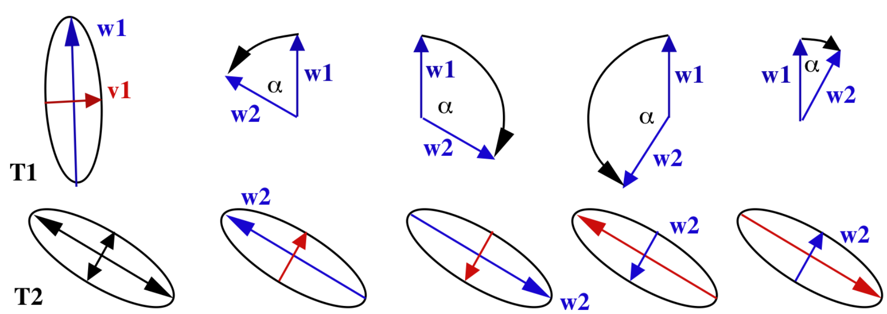

Fig. 1. Symmetric tensors can uniquely be represented by their eigendirections and eigenvalues, here represented by the ellipses. Using the eigenvalues and eigendirections for interpolation we have several possibilities for grouping them and assigning orientations to eigendirections. If we restrict rotation angles to values smaller than π, there are four different assignments resulting in four different rotation angles.

Point	Tensor	Eigenvalues	Eigenvectors
P_1	T_1	λ_1	$\pm\mathbf{v}_1 = (v_{11}, v_{12})$
		μ_1	$\pm\mathbf{w}_1 = (w_{11}, w_{12})$
P_2	T_2	λ_2	$\pm\mathbf{v}_2 = (v_{21}, v_{22})$
		μ_2	$\pm\mathbf{w}_2 = (w_{21}, w_{22})$
P_3	T_3	λ_3	$\pm\mathbf{v}_3 = (v_{31}, v_{32})$
		μ_3	$\pm\mathbf{w}_3 = (w_{31}, w_{32})$
$P(\beta_1, \beta_2, \beta_3)$	$T(\beta_1, \beta_2, \beta_3)$	$\lambda(\beta_1, \beta_2, \beta_3)$	$\pm\mathbf{v}(\beta_1, \beta_2, \beta_3) = (v_1, v_2)$
		$\mu(\beta_1, \beta_2, \beta_3)$	$\pm\mathbf{w}(\beta_1, \beta_2, \beta_3) = (w_1, w_2)$

Table 1. Notation for eigenvectors and eigenvalues in triangle P_1, P_2, and P_3. The variables λ and μ are chosen such that $\lambda \geq \mu$. The functions $\lambda(t)$ and $\mu(t)$ are continuous but not necessarily differentiable everywhere. The functions $\mathbf{v}(t)$ and $\mathbf{w}(t)$ might not be continuous. The "$\pm$" in front of the eigenvectors allude to the fact that the eigenvectors are bidirectional and have no orientation.

$$\mathbf{T} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} d + \Delta & F \\ F & d - \Delta \end{pmatrix}, \tag{1}$$

where $\Delta = \frac{E-G}{2}$ and $d = \frac{E+G}{2}$.

A tensor $\mathbf{T}$ is characterized by its *eigenvalues* λ and μ and corresponding *eigenvectors* $\pm\mathbf{v}$ and $\pm\mathbf{w}$. For symmetric tensors, the eigenvalues are always real and the eigenvectors mutually orthogonal. We call the eigenvector with the larger eigenvalue major eigenvector and the smaller eigenvalue minor eigenvector. Further, $\Delta = (\lambda + \mu)/2$. In most points the two eigenvectors of a tensor are defined uniquely, each assigned to one eigenvalue. This is no longer the case for points where both eigenvalues are the same, i.e., $\lambda = \mu$, the so-called *degenerate points*. In tensor field topology the degenerate points play a similar role as zeros (critical points) in vector fields [3, 7]. Independently of the eigenvalues, an isolated degenerate point can be defined by the number of windings an eigenvector performs when moving on a closed line around the degenerate point. The undirected eigenvector field allows winding-numbers to be multiples of one half. The rotational direction provides us with additional information about the characteristics of the degenerate point.

To describe the points inside a triangle with vertices P_1, P_2, and P_3 we use barycentric coordinates β_1, β_2 and β_3, $0 \leq \beta_i \leq 1$, where

$$P(\beta_1, \beta_2, \beta_3) = \beta_1 P_1 + \beta_2 P_2 + \beta_3 P_3, \qquad \sum_{i=1}^{3} \beta_i = 1. \tag{2}$$

The triangle edges are named e_i, $i = 1, 2, 3$, where the index i is chosen according to its opposite vertex P_i. Table 1 summarizes the notation we use for the eigenvalues and eigenvectors at the vertices P_i, $i = 1, 2, 3$, and in the interior of the triangle. We always use "$\pm\mathbf{v}$" and "$\pm\mathbf{w}$" when referring to

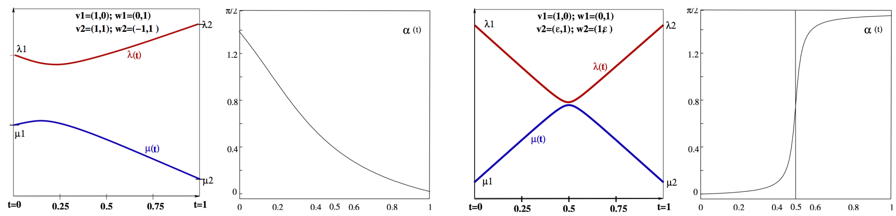

Fig. 2. Two examples of the behavior of the eigenvalues and eigenvectors for component-wise linear interpolation when moving along the edge. The images on the right show an example where the rotation angle is almost $\pi/2$. The rotation takes place in a very small region where the eigenvalues reach their extrema.

eigenvectors to allude to the fact that the eigenvectors are bidirectional and have no orientation. We use $\mathbf{v}$ and $\mathbf{w}$ when referring to vectors representing the eigenvectors with an arbitrary but fixed direction for $\mathbf{v}$, e.g., considering the way they were generated by a numerical computation. The direction of $\mathbf{w}$ is defined in a way such that $\mathbf{v}$ and $\mathbf{w}$ is a right-handed system. The assignment of the names λ and μ to the eigenvalues is critical for the interpolation based on eigenvalues and eigenvectors. For component-wise interpolation it is given implicitly. The requirement for the assignment to be continuous resolves this ambiguity: assigning the same name to all major eigenvalues and to all minor eigenvalues. Thus, we can define the variable names in a way that $\lambda_i \geq \mu_i$ for all i, without loss of generality. If there is a degenerate point inside a cell the eigenvalue might not be differentiable there.

4 Linear Interpolation of Tensor Components

The most commonly used interpolation scheme for tensors is linear interpolation of tensor components. It is a consistent approach and produces a globally continuous tensor field approximation. We use this field to guide the orientation assignment to the vector field. To be able to do so we start with a detailed analysis of the eigenvector behavior for a component-wise tensor interpolation.

For linear interpolations, the value at a point P is already uniquely specified by the values in two points P_1 and P_2 whose connection passes trough P. Therefore, many properties can already be observed when considering the linear interpolation in-between two points:

$$P(t) = (1-t)\,P_1 + t\,P_2, \;\; T(t) = (1-t)\,T(P_1) + t\,T(P_2), \;\; t \in [0,1]. \tag{3}$$

The eigenvalues at $P(t)$ are given by

$$\lambda(t) = d(t) + \sqrt{F^2(t) + \Delta^2(t)} \text{ and } \mu(t) = d(t) - \sqrt{F^2(t) + \Delta^2(t)}\,. \tag{4}$$

There exists a degenerate point in $P(t_0)$ when both eigenvalues $\lambda(t_0)$ and $\mu(t_0)$ are the same. This is equivalent to $F(t) = \Delta(t) = 0$.

Observation 1
For linear interpolation of tensor components, there exists only a degenerate point on the connection of two points if the following three conditions are satisfied:

$$\begin{array}{lr} (a) & F_1 \cdot F_2 \leq 0 \\ (b) & \Delta_1 \cdot \Delta_2 \leq 0 \\ (c) & F_1 \Delta_2 - \Delta_1 F_2 = 0 \end{array} \tag{5}$$

If $F_1 = F_2 = 0 = \Delta_1 = \Delta_2$, the entire connection consists of degenerate points. In all other cases, there exists an isolated degenerate point located in $P(t_0)$, where

$$t_0 = \frac{F_1}{F_1 - F_2} = \frac{\Delta_1}{\Delta_1 - \Delta_2}. \tag{6}$$

If the denominators are equal to zero, the entire edge is degenerate. The eigenvalue in the degenerate point is $E(t_0) = G(t_0)$.

If we assume that P_1 is not a degenerate point and use the eigenvectors of T_1 as coordinate basis, then $F_1 = 0$ and $\Delta_1 \neq 0$. Thus, the third condition (5c) reduces to $F_2 = 0$, meaning that the eigenvectors of T_1 are eigenvectors of T_2 as well. The second condition (5c) implies that the corresponding eigenvectors are rotated about $\pi/2$.

Observation 2
For linear interpolation of tensor components, there exists a degenerate point on the connection of two not-degenerate points P_1 and P_2 if and only if

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 = \mathbf{w}_1 \cdot \mathbf{w}_2. \tag{7}$$

Thus, the existence of degenerate points on an edge is independent from the eigenvalues on the vertices.

We now consider the behavior of eigenvectors when moving along an edge without degenerate points. The eigenvectors are-well defined everywhere and

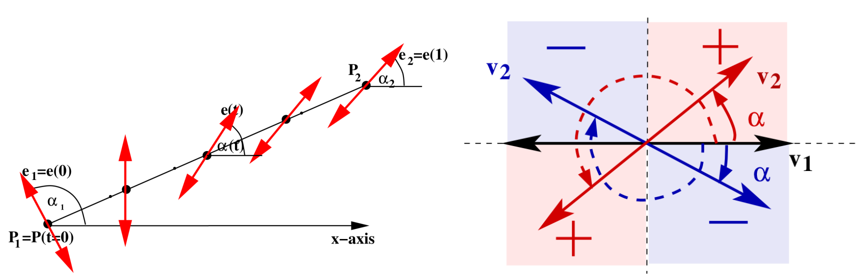

Fig. 3. Change of eigenvector along the connection of two points. The rotation direction of the eigenvectors depends on the quadrant in which the second eigenvector lies. Only one possible rotation angle exists with absolute value smaller than $\pi/2$.

change continuously. The change can be expressed by the angle $\alpha(t)$ formed by the eigenvector $\mathbf{v}(t)$ and the x-axis, see Figure 3. The special choice of the angle does not influence the result for the change of the angle. Geometrical considerations using the tensor components to express the eigenvector allow us to obtain after simplification:

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = \frac{1}{2}\frac{\Delta(t)\dot{F}(t) - F(t)\dot{\Delta}(t)}{\Delta^2(t) + F^2(t)}. \tag{8}$$

For linear interpolation of tensor components, this results in

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = \frac{1}{2}\frac{F_2\Delta_1 - F_1\Delta_2}{\Delta^2(t) + 4F^2(t)}. \tag{9}$$

Since the denominator is always greater than zero the sign, and thus the rotational direction, is determined by the numerator. Integrating Equation 9 shows that the absolute rotation angle is smaller than $\frac{3}{4}\pi$.

If there exists a degenerate point $D = P(t_0)$ on the edge, it can easily be seen that $\mathbf{v}(t) = \mathbf{v}_1$, $\mathbf{w}(t) = \mathbf{w}_1$ for $0 \leq t < t_0$, and $\mathbf{v}(t) = \mathbf{v}_2$, $\mathbf{w}(t) = \mathbf{w}_2$ for $t_0 < t \leq 1$.

Observation 3
When moving along an edge the rotation angle is limited to an absolute value of $\pi/2$. The direction of the rotation is given by

$$F_2\Delta_1 - F_1\Delta_2.$$

If this expression is smaller than zero, the eigenvector is rotated clockwise; if it is larger it is rotated counter-clockwise; if it is equal to zero, then either both points have the same eigenvectors or there exists a degenerate point on the edge, the rotation is $\pi/2$ and the rotation direction is undetermined.

5 Interpolation Based on Eigenvectors and Eigenvalues

We use the observations from the last sections to define a tensor interpolation inside a triangle based on an eigenvector and eigenvalue interpolation. We use the notations as defined in Table 1.

There are some issues we have to take care in order to use a vector interpolation applied to a tensors. First, we have to define a local criterion for the assignment of an orientation to the eigenvectors $\pm\mathbf{v}_1$ and $\pm\mathbf{w}_1$. Second, we have to show that the rotation angle is smaller than π such that it can be represented by a vector interpolation. Third, we have to show that the interpolated vectors are orthogonal everywhere and thus are valid eigenvectors. The second point was already shown to be satisfied in the last section. The third point can be shown to hold by performing a simple scalar product calculation. The most critical point, the first one, is discussed in the next sections.

Among all possible assignments, we chose an assignment that reflects the continuous change of the eigenvectors defined by the component-wise interpolation. It is important to differentiate between triangles containing a degenerate point and those that do not, since the eigenvector behavior of triangles with a winding number of half integers cannot be simulated by a simple vector interpolation.

5.1 Edge Labeling

Since a consistent global orientation assignment is not possible, we first define arbitrarily directed eigenvectors $\mathbf{v}_i$ and $\mathbf{w}_i$. Instead of changing directions we label the edges according to the behavior of the eigenvectors when moving along the edge e_i:

$$l(e_i) = \begin{cases} 0 & \text{if there exists a degenerate point on the edge, meaning } \mathbf{v}_j \cdot \mathbf{v}_k = 0 \\ \text{-1} & \text{if the directions of } \mathbf{v}_j \text{ and } \mathbf{v}_k \text{ do not match the direction propagation, meaning } \mathbf{v}_j \cdot \mathbf{v}_k < 0 \\ 1 & \text{if the directions of } \mathbf{v}_j \text{ and } \mathbf{v}_k \text{ match the direction propagation, meaning } \mathbf{v}_j \cdot \mathbf{v}_k > 0 \end{cases} \tag{10}$$

where $i, j, k \in \{1, 2, 3\}$ are cyclic indices.

5.2 Interpolation in Triangles without Degenerate Points

The tensor inside the triangle is defined by its eigenvectors $\mathbf{v}$ and $\mathbf{w}$ and eigenvalues λ and μ. We use the edge labelling defined in the last paragraph to define the interpolation of the eigenvectors, see Figure 4,

$$\begin{aligned} \mathbf{v}(\beta_1, \beta_2, \beta_3) &= \beta_1 \mathbf{v}_1 + \beta_2 l(e_3) \mathbf{v}_2 + \beta_3 l(e_2) \mathbf{v}_3, \\ \mathbf{w}(\beta_1, \beta_2, \beta_3) &= \beta_1 \mathbf{w}_1 + \beta_2 l(e_3) \mathbf{w}_2 + \beta_3 l(e_2) \mathbf{w}_3, \end{aligned} \tag{11}$$

and eigenvalues λ and μ,

$$\lambda(\beta_1, \beta_2, \beta_3) = \sum_{i=1}^{3} \beta_i \lambda_i, \qquad \mu(\beta_1, \beta_2, \beta_3) = \sum_{i=1}^{3} \beta_i \mu_i. \tag{12}$$

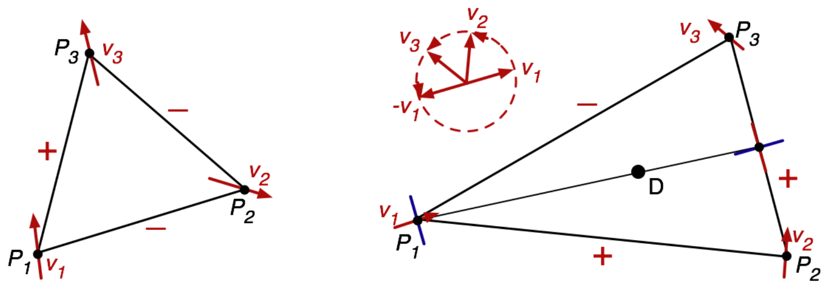

Fig. 4. The left figure shows an example of a triangle without a degenerate point. The right figure shows an example of a triangle containing a degenerate point. The edges of the triangle are labelled with a "+" if the direction definition of the two adjacent eigendirections matches the assignment defined by the interpolation of tensor components. They are labelled with "-" if one of the directions must be reversed to obtain a consistent assignment.

5.3 Existence of Degenerate points

We distinguish three cases: (i) isolated degenerate points; (ii) degenerate lines; and (ii) degenerate triangles. The interesting case is the isolated degenerate point. For linear interpolation, there only two types of isolated degenerate points exist, *wedge points* with a winding number of $1/2$ and *trisector points* with a winding number of $-1/2$. Using our edge labelling convention we obtain a criterion for the existence of a degenerate point inside the triangle.

Observation 4
Let $e_i, i = 1, 2, 3$ be the edges of the current triangle, and let $l(e_i)$ be the edge labelling as defined by Equation 10. The product of the labels of the triangle edges take on the following values:

$$\prod_{i=1,2,3} l(e_i) = \begin{cases} 0 & \text{if there exists a degenerate point on at least one of the edges. If there exist two degenerate edges, we have a degenerate line. If there exist three degenerate edges, the entire triangle is degenerate.} \\ 1 & \text{if there is no degenerate point inside the triangle.} \\ \text{-1} & \text{if there exists an isolated degenerate point inside the triangle.} \end{cases}$$

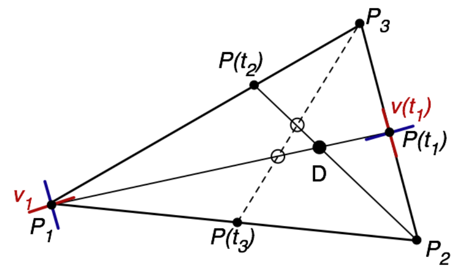

Fig. 5. To show that the location of a degenerate point is well-defined we have to show that the three lines connecting the vertices and their opposite points intersect in one point.

5.4 Location of Degenerate Points

Since degenerate points at vertices can be detected easily, we restrict our considerations, in this section, to triangles without degenerate behavior at the vertices. Initially, we also assume that there is no degenerate point along the edges, thus the product of the edge labels is -1. Figure 4 shows an example of a triangle containing a degenerate point. We know that the existence of a degenerate point on a line connecting any two points A and B is equivalent to the eigendirections $\mathbf{v}_A$ and $\mathbf{v}_B$ being perpendicular, see Section 4, thus $\mathbf{v}_A \cdot \mathbf{v}_B = 0$. We use this fact to determine the location of the degenerate point in a triangle. First, we define an eigenvector field on the triangle boundary by linear interpolation. For each edge e_i, we define

$$\mathbf{v}(t) = (1-t)\cdot \mathbf{v}_j + t \cdot l(e_i) \cdot \mathbf{v}_k, \qquad t \in]0,1[, \tag{13}$$

where $i, j, k \in \{1,2,3\}$ are cyclic indices. Even though the vector field $\mathbf{v}$ on the boundary is not continuous at all vertices, the corresponding un-oriented direction field $\pm\mathbf{v}$, is continuous defining a continuous rotation angle varying from zero to π or $-\pi$.

The mean value theorem implies that there exist three parameters $t_i \in]0,1[, i = 1,2,3$, for each vertex one, such that $\mathbf{v}_i \cdot \mathbf{v}(t_i) = 0$. Thus, for every vertex there exists a point on the opposite edge with rotation angle $\pi/2$. We call this point the opposite point of the vertex, see Figure 5. The following equation defines the parameters t_i:

$$\mathbf{v}_i \cdot (\ (1-t_i)\mathbf{v}_j + l(e_i)t_i\mathbf{v}_k\ ) = 0, \tag{14}$$

where $i, j, k \in \{1,2,3\}$ are cyclic indices. Using these definitions we finally define the location of the degenerate point in the following way:

Definition
The location of the degenerate point is defined as the intersection of the connections of the triangle vertices and their opposite points. Using barycentric coordinates, the location of the degenerate point D is defined by :

$$\beta_i = \left| \frac{(P(t_j) - P_j) \times (P_j - P_i)}{(P(t_j) - P_j) \times (P(t_i) - P_i)} \right|, \tag{15}$$

with cyclic indices $i, j, k \in \{1, 2, 3\}$ where

$$P(t_i) = (1 - t_i)P_j + t_i P_k \qquad \text{and} \qquad t_i = \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{\mathbf{v}_i \cdot (\mathbf{v}_j - l_i \mathbf{v}_k)}.$$

To prove that the point D is well-defined we have to show that the resulting point does not depend on the choice of i and j. From the definition of t_i, $i = 1, 2, 3$, Equation 14, it follows that

$$t_1 t_2 t_3 = (1 - t_1)(1 - t_2)(1 - t_3). \tag{16}$$

This is exactly the condition that three lines through the vertices and points on the opposite edge defined by parameters t_i intersect in one point. Since all other eigenvectors result from linear interpolation of the vectors at the vertices it can be seen that also all other connectors of points with their opposite point intersect in the same point. Thus, the point D is well-defined and can be use to define the location of the degenerate point. The location of the degenerate point depends only the eigendirection at the triangle vertices, it is independent of the eigenvalues.

If the degenerate point lies on an edge, we cannot use this definition since the three connecting lines degenerate to one line. In this case we use the eigenvalues at the vertices to determine the degenerate point.

5.5 Eigenvalue Definition in the Degenerate Point

A linear interpolation of the eigenvalues at the tree vortices would not lead to multiple eigenvalues. Instead, we interpolate the mean eigenvalue $d = 1/2(\lambda_i + \mu_i)$ and set the deviator $\Delta = 1/2(\lambda_i - \mu_i)$ to zero. If β_i, $i = 1, 2, 3$, are the coordinates of the degenerate point inside the triangle, its eigenvalue is defined by

$$\nu = \frac{1}{2} \sum_{i=1}^{3} \beta_i (\lambda_i + \mu_i) \tag{17}$$

5.6 Interpolation of Triangles with Degenerate Point

For the interpolation of triangles containing degenerate points, we subdivide them by inserting an additional vertex D in the degenerate point. It is connected to the three triangle vertices. The tensor in the new point is defined as

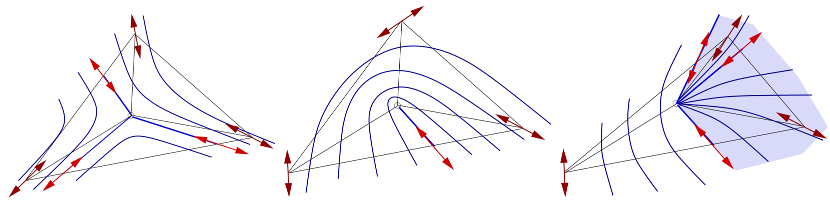

Fig. 6. The left figure is an example of a trisector point for one eigenvector field. The middle and right figure are two examples showing wedge points, with one and three radial tensor lines, respectively.

the degenerate tensor with eigenvalue ν as defined by Equation 17. Each new triangle is interpolated separately. The eigenvectors, which are not defined in the degenerate point, are set to zero, in correspondence to vector field singularities. The final interpolation of the eigenvectors is performed in the new triangle with vertices P_i, P_j, and D. Let $P = P(\beta_1, \beta_2, \beta_3) := \beta_i P_i + \beta_j P_j + \beta_k D$, using cyclic indices i, j, k. Then the eigenvectors in P are defined by

$$\begin{aligned} \mathbf{v}(\beta_i, \beta_j, \beta_k) &= \beta_i \mathbf{v}_i + \beta_j l(e_k) \mathbf{v}_k \text{ and} \\ \mathbf{w}(\beta_i, \beta_j, \beta_k) &= \beta_i \mathbf{w}_i + \beta_j l(e_k) \mathbf{w}_k, \end{aligned} \tag{18}$$

Thus, the eigenvectors are independent from the coordinate β_k. The eigenvalues are defined in the non-degenerated case as described in Section 5.2.

5.7 Classification of the Degenerate Point

The neighborhood of the degenerate point is characterized by segments separated by radial tensor lines. In our case, the radial tensor lines are straight lines and are defined by their intersection $P(t_r)$ with the triangle boundary:

$$\mathbf{v}(t) \times (P(t_v) - D) = 0, \qquad \mathbf{w}(t) \times (P(t_w) - D) = 0, \qquad t_v, t_w \in [0, 1]. \tag{19}$$

These are quadratic equations for each edge with a maximum of two solutions per edge and eigenvector field. For the entire triangle, one or three solutions per eigenvector field are possible.

Trisector Point

The trisector point has a winding number of $-1/2$. It is characterized by three separatrices and three hyperbolic sectors for each eigenvector field, see Figure 6.

Wedge Point

A wedge point has a winding number of $1/2$. It is characterized by one to three radial tensor lines. These radial lines define either one hyperbolic sector or one hyperbolic and two parabolic sectors, see Figure 6.

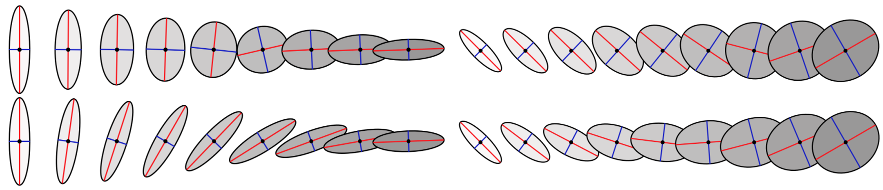

Fig. 7. Two examples comparing the results of linear component-wise tensor interpolation (top row) and linear interpolation of eigenvectors and eigenvalues (bottom row). The second interpolation is much more shape-preserving, and the change of directions is much more uniform.

6 Results

We provide some examples to illustrate the basic differences of the two interpolation methods. In the figures we use ellipses to represent tensors. The half axes are aligned to the eigenvector field, and the radii represent eigenvalues. The half axes are defined as

$$r_1 = 1 + c\lambda \qquad \text{and} \qquad r_2 = 1 + c\mu, \tag{20}$$

where c is a normalization factor. This approach allows us to represent non positive definite tensors as well as ellipses.

The first example shows interpolations on a line, see Figure 7. It illustrates, that the decoupling of eigenvalue and eigenvector interpolation preserves shape. The change of the direction is much more continuous than for component-wise interpolation. Similar observations can be made for the interpolation inside a triangle without degenerate point, see Figure 8. Figure 9 shows two examples of triangles with degenerate points, one with wedge point and one with trisector point. Since interpolation of the eigenvectors is only piece wise linear, due to the subdivision, it allows a more general structure when compared with component-wise interpolation. The qualitative structure of the mesh generated by integrating the eigenvector fields are the same. The position of the degenerate point varies slightly, but the type of the degenerate point is always the same. It can happen that, for eigenvector interpolation the wedge point has two more radial tensor lines, resulting in an additional parabolic, sector, see 9(b).

7 Acknowledgements

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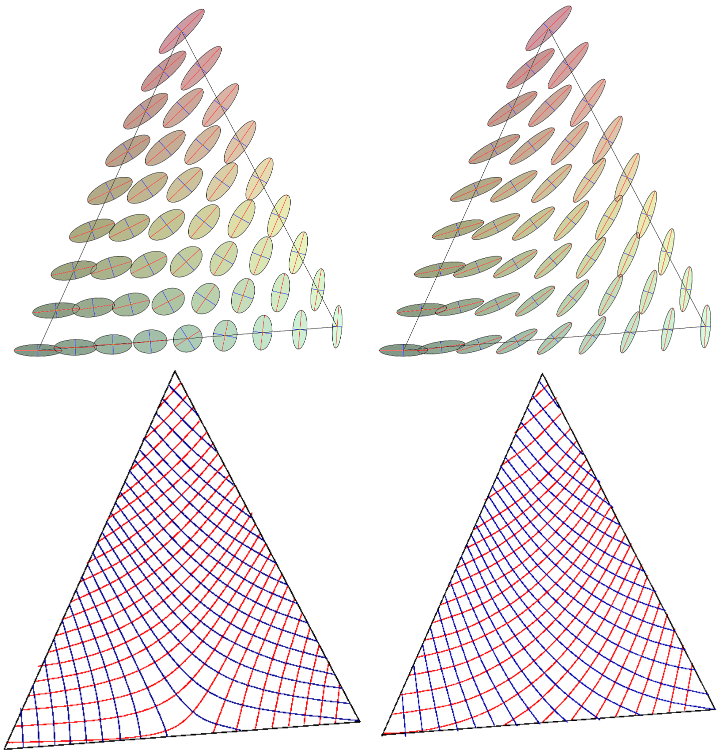

Fig. 8. Comparison of component-wise (left) and eigenvalue, eigenvector-based (right) tensor interpolation inside a triangle without degenerate point. The upper row shows the tensors represented as ellipses, and the second row shows the mesh resulting form integrating the eigenvector fields.

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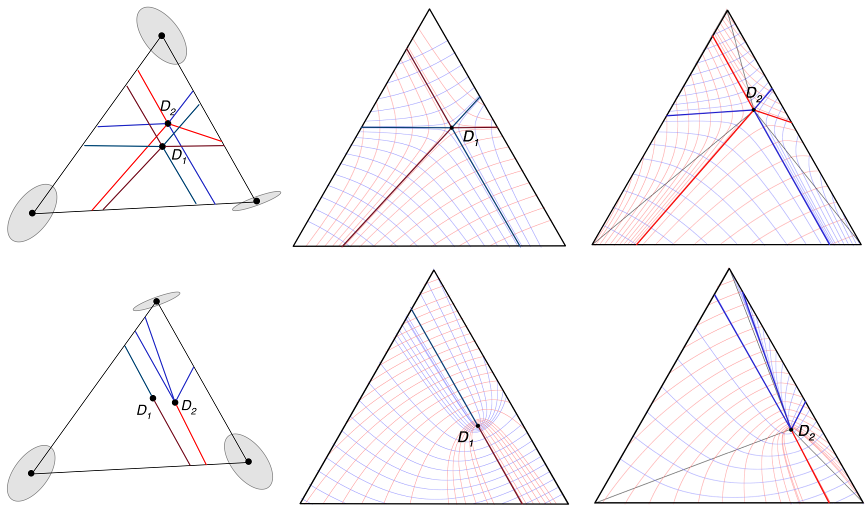

Fig. 9. Comparison of component-wise (middle) and eigenvalue, eigenvector-based (right) tensor interpolation inside a triangle with degenerate point. The upper row shows a triangle with trisector point, and the bottom row a triangle with wedge point. The triangles on the left compare the separatrices for both interpolations. For the wedge point case, there exist two more radial lines with two additional parabolic sectors for the eigenvector interpolation.

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