

Vector and Tensor Field Topology Simplification, Tracking, and Visualization

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Vorwort

Diese Dissertation entstand im Umfeld von Prof. Dr. Hans Hagen, der mir die Möglichkeit gab, in seiner Arbeitsgruppe in Kaiserslautern zu arbeiten. Er weckte mein Interesse für die Arbeit von Thomas Schreiber, was zu meiner ersten Veröffentlichung führte und den Grundstein für mein Schaffen in den letzten 3 Jahren legte. Daß ich an so vielen internationalen Konferenzen aktiv teilnehmen konnte wäre ohne seine Hilfe nicht denkbar gewesen. Für das alles möchte ich mich hier bedanken. Das Thema dieser Dissertation verdanke ich Dr. Gerik Scheuermann, der mir sein Wissen im Bereich der Vektorfeldtopologie zur Verfügung stellte. Seine enge Zusammenarbeit und ständige Unterstützung waren mir eine wertvolle Hilfe. Meine Arbeit war nicht selten abhängig vom Wohlwollen der Technik, die des öfteren gegen mich arbeitete. Nicht nur in diesen Augenblicken war ich froh, so hilfsbereite und kompetente Kollegen wie Holger Burbach und Thomas Wischgoll zu haben, die mir unermüdlich zur Seite standen. In Hinblick auf die theoretischen Hintergründe brachte mich der Hinweis von Ingrid Hotz auf die Differentialgeometrie sehr voran. Dazu haben mich die Mitarbeiter des Projekts FAnToM, welches auch den Rahmen für meine Implementierungen darstellte, sowohl in der Programmierung als auch in der Diskussion verschiedener Problematiken sehr unterstützt. Dabei ist vor allem Christoph Garth für seine unverzichtbare Hilfe bei der NURBS-Visualisierung zu danken. Auch Stefan Clauss leistete einen nicht unerheblichen Beitrag. Erwähnt werden sollen auch die Kolleginnen und Kollegen aus der AG Hagen, welche in für mich anstrengenden Phasen geduldig meine Laune ertragen haben. Dank geht an Inga, Roger und Gunther für ihre Freundschaft und Hilfe. Spezieller Dank gilt Mady, die mir in mütterlicher Fürsorge unter die Arme gegriffen hat und in meiner Anfangszeit ein Zuhause gab. Und, last but not least, führe ich hier Tom Bobach und Jan Frey an, die sich einfach in keine spezielle Danksagungskategorie stecken lassen.

Abstract

Among existing techniques for the visualization of vector and tensor fields, topology-based methods offer a synthetic and accurate way to depict the structure of the associated flows. This approach was introduced quite recently in Scientific Visualization but its roots and inspiration are to be found in the genius work of Poincaré at the end of the 19th century. Practically, the considered domain is partitioned by a graph into subregions of homogeneous qualitative behavior. Extracting and visualizing this graph permits to convey the most meaningful properties of large vector or tensor datasets. This has motivated the design of many topology visualization schemes that aim at fitting the requirements of today's applications. Yet, there are deficiencies that still inconvenience their use in typical practical cases. This thesis attacks two of them.

First, turbulent flows exhibit very complicated structures. Hence, their topologies result in a visual clutter that is of little use for interpretation. To overcome this limitation, two new methods are presented to simplify the topology while ensuring structural consistency with the original data. The first one is based on a scaling strategy and enables the merging of close singularities by means of local grid deformations. It applies to vector and tensor fields. The second one works out successive removals of pairs of critical points and is designed for vector fields defined over triangulations. It permits to retain only the most significant features, according to any prescribed criterion. Both methods act on the field along with its topology and clarify the visualization results.

Second, existing techniques for time-dependent vector and tensor fields provide no way to depict the continuous topology evolution over time. They miss essential features called bifurcations that correspond to dramatic structural changes. A new technique is proposed that accurately detects and characterizes bifurcations in vector and tensor fields. It produces three-dimensional pictures where the whole topological graph is precisely tracked over time, describing surfaces that provide the structure of the time-dependent topology.

Zusammenfassung

Unter den heutigen Methoden für Vektor- und Tensorfeldvisualisierung bieten topologiebasierte Verfahren die Möglichkeit, die Struktur einer Strömung exakt darzustellen. Obwohl topologiebasierte Methoden erst seit kurzem in der Visualisierung wissenschaftlicher Daten eingesetzt werden, können ihre Wurzeln bereits in der bahnbrechenden Arbeit von Poincaré am Ende des neunzehnten Jahrhunderts gefunden werden. Im Prinzip wird das Definitionsgebiet des betrachteten Feldes durch einen Graphen in Regionen gleichartigen qualitativen Verhaltens unterteilt. Durch Extraktion und Visualisierung dieses Graphen kann man die bedeutsamsten Eigenschaften eines großen Vektor- oder Tensorfelds hervorheben. Diese Möglichkeit motivierte die Entwicklung vieler topologiebasierter Visualisierungsverfahren, die darauf abzielen, die Anforderungen heutiger Anwendungen zu erfüllen. Dennoch haben sie immer noch Schwächen, die ihren Einsatz in der Praxis erschweren. Diese Dissertation entwickelt Lösungsansätze für zwei dieser Schwächen.

Zum einen weisen turbulente Strömungen sehr komplizierte Strukturen auf. Folglich wirkt die Visualisierung ihrer Topologie häufig unübersichtlich. Um diesen Nachteil zu überwinden, werden zwei Verfahren vorgestellt, die die Topologie vereinfachen, aber dennoch ihre Struktur konsistent zu den Originaldaten halten. Das erste basiert auf einer Skalierungsstrategie und verschmilzt nahe beieinanderliegende Singularitäten durch lokale Deformationen des Gitters. Das zweite arbeitet auf Vektorfeldern, die über Triangulierungen definiert sind, und entfernt schrittweise Paare von kritischen Punkten. Es ermöglicht, in Abhängigkeit eines vorgegebenen Kriteriums nur die wichtigsten Eigenschaften beizubehalten. Beide Methoden vereinfachen das Vektorfeld zusammen mit dessen Topologie und resultieren in einer übersichtlicheren Darstellung.

Zum anderen bieten existierende Techniken für zeitabhängige Vektor- und Tensorfelder keine Möglichkeit, die stetige Veränderung der Topologie über die Zeit darzustellen. Sie übergehen Bifurkationen, die drastischen Änderungen in der Struktur eines Felds entsprechen. Diese Arbeit stellt eine neuartige Technik vor, die automatisch Bifurkationen in Vektor- und Tensorfeldern findet und charakterisiert. Sie erzeugt dreidimensionale Bilder, die den vollständigen topologischen Graphen über die Zeit hinweg präzise verfolgen und Flächen angeben, die die Struktur der zeitabhängigen Topologie vermitteln.

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Chapter 1

Introduction

Computers have become indissociable from our daily life. Their capability to process continually increasing amounts of data has found various fields of application. They ease innumerable tasks and open new horizons by pushing back the limits of our understanding. Scientific Visualization in particular is a very active discipline of Computer Science, in charge of producing images that convey meaningful aspects of numerical datasets. The appeal of these geometric depictions is due to the fantastic ability of human beings to comprehend and interpret visual information.

Vector and tensor fields are traditionally objects of major interest for visualization. The significance of these symbolic objects is due to their key role in the description of many notions in physics and, by extension, in engineering sciences. This is especially true in fundamental physics, optics, solid mechanics or fluid dynamics on one hand but also civil engineering, aeronautics, turbomachinery or climate prediction on the other hand, to mention just a few. From a theoretical viewpoint, vector and tensor fields have received much attention from mathematicians, leading to a precise and rigorous framework that greatly facilitates their practical study. In particular, Poincaré's work [Poi75, Poi99] laid the foundations of a genius geometric interpretation of vector fields associated to dynamical systems. This provides an aesthetic way to apprehend the signification hidden behind the language of multivariate data. Nowadays, following this very rich theoretical inheritance, analysts typically center their attention on the topology of large vector and tensor datasets provided by Computational Fluid Dynamics (CFD) or Finite Elemente Methods (FEM). Informally, the topology is the structure of a field: Practically, this is a graph where so-called singularities play the role of vertices and are connected by special curves that constitute the edges. It partitions the domain of interest into subdomains of equivalent qualitative nature. Therefore, extracting and studying this structure permits to focus the analysis on essential

properties. For visualization purposes, the depiction of the topology results in synthetic representations that transcribe the fundamental characteristics of the data. These ideas are at the basis of the topological approach that has gained an increasing interest in vector and tensor field visualization during the last decade. First introduced for planar vector fields by Helman and Hesselink in [HH89a], this technique has experienced many contributions and extensions that aimed at better fitting the actual requirements of practical applications. A crucial milestone on this way was the PhD thesis of Delmarcelle [Del94] that transposed the vector formalism to symmetric, second-order tensor fields. Indeed, a tensor field can be interpreted as the set of its eigenvector fields which induces a deep similarity to vector fields.

At present, some remaining challenges must still be addressed to broaden the scope of topology-based vector and tensor field visualization. One of them is the excessive complexity of structural graphs associated with turbulent flows. As a matter of fact, the presence of a large number of close singularities leads to cluttered images that confuse interpretation since meaningful information cannot be distinguished from local, small-scale details. So far, existing techniques to solve this problem are limited to the graph representation and provide no field description consistent with the simplified topology. As a consequence, no alternative visualization technique like e.g. drawing of integral curves can be applied afterward. Another deficiency is the lack of specific technique to properly visualize the topology of parameter-dependent vector or tensor fields. In fact, any attempt in this direction must face the fundamental issue of bifurcations. Indeed, an additional parameter enables the occurrence of structural changes that leads to dramatic modifications of the topology. This topic has thus raised a wide interest in the mathematics community leading to numerous theoretical advances. It follows that an accurate and precise visualization of such fields must permit both detection and identification of bifurcations to provide insights into the continuous structural evolution. Up to now, the proposed methods content themselves with displaying persistent features and observing that topological consistency may have been lost between successive discrete parameter values.

In this context, the contributions of this thesis are the following: First, the mathematical connections between vector and tensor fields are precisely exposed and commented (chapter 3). This is, at least to our knowledge, the first time that a rigorous theoretical framework is explicitly formulated for the topological visualization of two-dimensional symmetric tensor fields. In particular, concepts taken from differential geometry are used to properly define the notion of tensor lines and prove their existence in the vicinity of singular points. Second, the study of low-order interpolation schemes from the topological viewpoint enables us to detect and model singularities with arbitrary

structures in piecewise linear vector and tensor fields (chapter 4). Next, the problem of visual clutter encountered by topology-based schemes with turbulent flows is attacked and solved for both vector and tensor fields defined over structured or arbitrary grids (chapter 6). Our method is based on a scaling approach that simulates the merging of close singularities thanks to local grid deformations. This permits to replace a complex local structure by a new, simpler one, that presents the same aspect in the large and preserves consistency. This scheme is completed by a second one, designed for planar vector fields defined over piecewise linearly interpolated triangulations (chapter 8). In this case, simplification takes place via the successive pairwise pruning of singularities, monitored by arbitrary qualitative or quantitative criteria. At last, a topology-based visualization technique for time-dependent planar vector and tensor fields is proposed (chapter 7). In accordance with what precedes, while tracking the topology over time, we detect and characterize the bifurcations that affect the local or global structure. This leads to a three-dimensional visualization of these continuous transitions, where time is embedded in a 3D space-time grid.

Structure of the text The mathematical foundations of two-dimensional vector field topology are first introduced in chapter 2. All useful notions are precisely defined and explained. Following the ideas of Poincaré and of his many successors, we emphasize a geometric approach. Special attention is paid here to the theory of bifurcations and structural stability to lay the theoretical basis required for the methods presented later on.

This framework is next extended to symmetric, second-order two-dimensional tensor fields in chapter 3. The theory originally proposed by Delmarcelle is rigorously investigated and completed to fit the needs of our applications. In particular, the existing parallel between vector and tensor fields is underlined by means of covering spaces. The theory of bifurcations is sketched in this context, keeping the same geometric approach as before.

After this mathematical introduction, common grid structures and associated interpolation schemes are presented in chapter 4. The related topological properties are discussed for the two-dimensional case. The chapter ends with a method for the extraction and modeling of singularities with arbitrarily complex structure in piecewise linear fields. This completes the description of all the concepts required for our visualization purposes.

The state of the art in topology-based visualization of vector and tensor fields is described in chapter 5. The original methods for vectors and tensors are presented along with the improvements and extensions achieved since their introduction. In particular, existing methods for topology simplification and time-dependent topology visualization are considered.

Chapter 6 provides all details about our topology scaling method of vector and tensor fields defined over structured and unstructured grids. Results are shown on both artificial and practical turbulent datasets. All demonstrate the ability of our technique to clarify the depiction while preserving all meaningful aspects of the original data.

Our theoretical knowledge about bifurcation theory is used in chapter 7 to accurately visualize the topological evolution of parameter-dependent vector and tensor fields. Pictures generated by our method are proposed that illustrate various types of bifurcations. The tracking of singularities together with associated structures is processed on datasets chosen for their interesting topological properties. We show the resulting three-dimensional images.

Chapter 8 describes a new method for the simplification of piecewise linear planar vector fields. It permits to remove insignificant singularities pairwise with respect to any user-prescribed criteria without altering the global nature of the associated flow. An interpretation in terms of bifurcations underlines the continuous, natural flavor of the algorithm. The application focuses on a turbulent CFD dataset: The results confirm that a very high simplification rate can be achieved although consistency is preserved.

Finally, a few comments on the numerical issues related to the presented methods are given in appendix A.

Datasets The new methods presented in this thesis were tested with practical CFD datasets provided by Wolfgang Kollmann, from the Mechanical and Aeronautical Engineering Department of the University of California at Davis. These vector and tensor datasets correspond to numerical simulations of a swirling jet with inflow into a steady medium. They are of great interest in the context of this work because they are turbulent and exhibit very complicated structures. Further details can be found in [SHJ00]. Prof. Kollmann provided also many helpful comments on the role of topology in fluid dynamics. I would like to take the opportunity to thank him for this contribution.

Chapter 2

Vector Fields and Dynamical Systems

This chapter is devoted to the theoretical framework of vector field topology as it is defined and used in Scientific Visualization. The essential idea is the notion of phase portrait of a dynamical system that provides a vector field with a geometric interpretation. Special features of major interest, called critical points and closed orbits, play a key role. Therefore, a precise qualitative analysis has been designed to study their properties. Furthermore, when considering parameter-dependent vector fields, the fundamental concepts of structural stability and bifurcation naturally arise to describe and understand the qualitative changes that may be encountered. An overview of all these topics is given.

2.1 Basic Notions and Fundamental Theorems

In this section, the notions of dynamical system, flow and phase portrait related to a vector field are defined and the fundamental theorems ensuring the existence and uniqueness of the solution to the Cauchy problem are given.

2.1.1 First Definitions

Definition 1 *The differential equation associated with a vector field f is a system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

where \mathbf{x} is a n -dimensional function of an independent real variable t (say time) and $\mathbf{f} : (I \subseteq \mathbb{R}) \times (U \subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is a smooth function.

Remark 1 If \mathbf{f} does not depend on time, the equation is called **autonomous**. Otherwise the system is said **non-autonomous**.

The addition of an initial condition to the differential equation leads to the Cauchy problem:

Definition 2 The Cauchy problem is defined as the system

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

The equation $\mathbf{x}(t_0) = \mathbf{x}_0$ is called **initial condition** of the problem.

Remark 2 In practice, one usually takes $\mathbf{x}(0) = \mathbf{x}_0$ as initial condition, that is $t_0 = 0$. Note that in the case of an autonomous system, the problem is unchanged by any translation of time, i.e. is independent of the choice of t_0 .

When dealing with a differential equation, one typically studies the properties of its associated flow which is defined as follows.

Definition 3 The vector field \mathbf{f} generates a **flow** $\phi_t : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\phi_t := \phi(\mathbf{x}, t)$ is a smooth function defined for $(\mathbf{x}, t) \in U \times (I \subseteq \mathbb{R})$ satisfying

$$\left. \frac{d}{dt} \phi(\mathbf{x}, t) \right|_{t=\tau} = \mathbf{f}(\tau, \phi(\mathbf{x}, \tau))$$

for all $(\mathbf{x}, \tau) \in U \times I$.

Property 1 With the definition above, ϕ_t satisfies the group properties:

1. $\phi_0 = id$
2. $\phi_t \circ \phi_s = \phi_{t+s}$ and therefore $\phi_t \circ \phi_{-t} = id$.

Remark 3 For a given $t \in \mathbb{R}$, the point $\phi_t(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$, can be seen as the new position reached after time t by a particle located at \mathbf{x} at $t = 0$, submitted to the flow ϕ .

Remark 4 With the definition above, the function $\phi(\mathbf{x}_0, .) : t \mapsto \phi(\mathbf{x}_0, t)$ is a solution of the Cauchy problem with initial condition $\mathbf{x}(0) = \mathbf{x}_0$. This solution will be referred to as **integral curve**, **trajectory** or **orbit** in the following.

If one considers all integral curves as a whole, one forms the phase portrait.

Definition 4 The family of all integral curves as subsets of $\mathbb{R}^2 \times \mathbb{R}$ is called **phase portrait** of the dynamical system.

2.1.2 Existence and Uniqueness

The results cited here are taken from [ZDH92]. Before stating the fundamental theorem on local existence and uniqueness of a solution to the Cauchy problem, the Lipschitz condition for a function must be introduced.

Definition 5 *The function $f : V \subseteq E \rightarrow E$, E normed vector space, is said to be **Lipschitz** on V if there exists a positive constant $K > 0$ such that*

$$\forall x, y \in V, \|f(x) - f(y)\|_E \leq K\|x - y\|_E.$$

K is called Lipschitz constant of f.

Definition 6 *The function $f : V \subseteq E \rightarrow E$, E normed vector space, is said to be **locally Lipschitz** on V , if for any point $x_0 \in V$, there exists a constant $b > 0$, such that*

$$\forall y, z \in \{x \in V, \|x - x_0\|_E < b\} \subset V, \|f(y) - f(z)\|_E \leq K_{x_0}\|y - z\|_E.$$

Here, K_{x_0} is a constant depending on x_0 .

One can now state the fundamental theorem on the local existence and uniqueness of a solution of the Cauchy problem.

Theorem 1 *Let $U \subseteq \mathbb{R}^n$ be an open subset of real euclidean space, let $I \subseteq \mathbb{R}$, $\mathbf{f} : I \times U \rightarrow \mathbb{R}^n$ be a continuous, Lipschitz function with respect to $\mathbf{x} \in U$ and $\mathbf{x}_0 \in U$. Then there is some $a > 0$ and a unique solution*

$$x : (-a, a) \rightarrow U$$

to the Cauchy problem with initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

Proof: see [ZDH92, pp. 1-4].

This theorem is only local because it deals with the existence and uniqueness of a solution of the Cauchy problem in a neighborhood of the initial point \mathbf{x}_0 . Fortunately, the solution can be continued to a larger (open) interval as described in the next theorem.

Theorem 2 *Suppose that $\mathbf{f}(t, \mathbf{x})$ is continuous for (t, \mathbf{x}) in the region $G = (I \subseteq \mathbb{R}) \times (U \subseteq \mathbb{R}^n)$, and satisfies a local Lipschitz condition with respect to \mathbf{x} . Then the solution of the Cauchy problem can be continued to the boundary of G (possibly ∞).*

Proof: The proof of this theorem results from the uniqueness of the solution to the Cauchy problem (c.f. theorem 1) and from the continuity of the flow (see [ZDH92, p. 13]).

In the special case of vector fields defined over a compact region, which typically occurs in practice, the previous theorem ensures the global existence of solutions.

Remark 5 *If $\mathbf{f}(t, \mathbf{x})$ is continuous for (t, \mathbf{x}) in the region $\mathbb{R} \times M \subseteq \mathbb{R}^n$, with M compact, and satisfies there a local Lipschitz condition with respect to \mathbf{x} , then either the solutions to the Cauchy problem $\mathbf{x} = \phi(t)$ are unbounded (and therefore leave M through its boundary) or they exist in the interval $(-\infty, +\infty)$.*

The practical relevancy of integral curves is due to their continuity with respect to initial conditions. This is formalized in the next theorem.

Theorem 3 *Consider the Cauchy problem*

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \\ \mathbf{x}(0) = \boldsymbol{\eta}, \end{cases}$$

where $\mathbf{f}(t, \mathbf{x})$ is continuous for (t, \mathbf{x}) in the region $G = (I \subseteq \mathbb{R}) \times (U \subseteq \mathbb{R}^n)$ and satisfies a Lipschitz condition with respect to \mathbf{x} , then there exists a constant $a' > 0$ and a unique solution

$$x : (-a', a') \longrightarrow U.$$

Moreover, $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\eta})$ is a continuous function of $(t, \boldsymbol{\eta})$.

Proof: see [ZDH92, p. 10].

The fundamental meaning of the previous theorem is that if the initial condition of the Cauchy problem is only approximated and therefore provided with a small error, then the corresponding solution will be related in a “nice” way to the one of the original problem.

2.2 Critical Points

We deal in the following with critical points of vector fields that only depend on the space variable \mathbf{x} , i.e. that are autonomous. The main focus is on two-dimensional vector fields where a complete classification of critical points has been provided by Andronov et al. [ALG73].

Definition 7 A critical point $\mathbf{x}_0 \in U \subseteq \mathbb{R}^n$ of a vector field $\mathbf{f} : U \rightarrow \mathbb{R}^n$ is characterized by

$$\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$$

Remark 6 Critical points are also known as **singular points** or simply **singularities**. Since the constant function $\mathbf{x}(t) \equiv \mathbf{x}_0$ is a solution of the differential equation related to \mathbf{f} , they may also be called **fixed** or **equilibrium points**. A point that is not singular is called **regular**.

Remark 7 Because of the local uniqueness of the solutions of the Cauchy problem, integral curves generally cannot meet. Now, the fundamental particularity of a critical point, as opposed to a regular point, is that integral curves can possibly meet asymptotically at its position as t approaches $\pm\infty$.

When studying a critical point, one is typically interested in its stability. The precise definitions are given next. Note that the following notions are not classical (although they constitute straightforward generalizations of the traditional definitions) but conform to the formalism introduced in [Sch00]. The reason for this choice is that they better fit typical qualitative considerations in Scientific Visualization.

Definition 8 A critical point $\underline{\mathbf{x}}$ is said to be ω - (resp. α -) **stable** if for every neighborhood $V \subset U$ of $\underline{\mathbf{x}}$ there exists a neighborhood $W \subset V$ such that every solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0 \in W$ is defined and lies in V for all $t > 0$ (resp. $t < 0$). Furthermore, if W can be chosen so that $\mathbf{x}(t) \rightarrow \underline{\mathbf{x}}$ when $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$), then $\underline{\mathbf{x}}$ is said to be ω - (resp. α -) **asymptotically stable**.

We now consider two-dimensional vector fields. A fundamental special case is provided by the critical points of linear vector fields.

2.2.1 Critical Points of Planar Linear Vector Fields

The following results are taken from [HS74, pp. 82-96] that offers a complete treatment of this subject.

Definition 9 A vector field $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is (affine) **linear** if there exists a matrix $A \in \mathbb{R}(n, n)$ and a vector $\mathbf{b} \in \mathbb{R}^n$ such that

$$\forall \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

If furthermore, $\mathbf{b} = \mathbf{0}$, \mathbf{f} is **homogeneous linear** and the coordinate origin O is a critical point.

Remark 8 If A is invertible (i.e. $\det A \neq 0$), setting $\mathbf{x}_0 = A^{-1}\mathbf{b}$, one gets $\mathbf{g}(x) \equiv \mathbf{f}(\mathbf{x} - \mathbf{x}_0) = A\mathbf{x}$. That is, with a convenient translation of the coordinate system, one obtains a homogeneous linear vector field. Note that in this case \mathbf{x}_0 is the only critical point of the vector field.

The generic classification of critical points of a homogeneous linear vector field is based on the following property.

Property 2 The critical points of homogeneous vector fields are characterized by the eigenvalues of their matrix A .

In the two-dimensional case, the matrix A has two eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$. The possible cases are enumerated next.

- **Case 1.** A has real eigenvalues with opposite signs. The zero is called a **saddle point**.

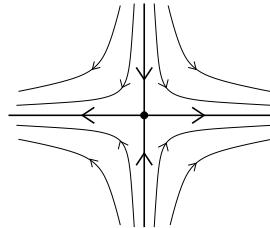


Figure 2.1: Saddle point

- **Case 2.** Both eigenvalues have negative real parts. The zero is called a **sink**, because any integral curve tends toward O for $t \rightarrow \infty$.

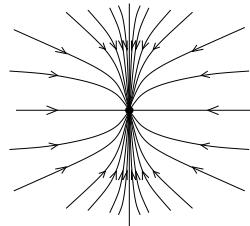


Figure 2.2: Node sink

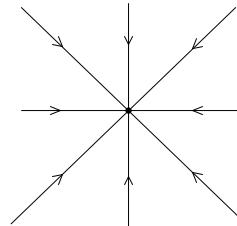


Figure 2.3: Focus sink

- **Case 2a.** A is diagonalizable and its eigenvalues are different. The zero is called a **node sink**. The eigenvector related to the eigenvalue with largest (resp. smallest) modulus corresponds to the direction of “fast” (resp. “slow”) convergence. The special case where the eigenvalues are equal is called a **focus sink**.

- **Case 2b.** A is not diagonalizable but has one real negative eigenvalue. The zero is called an **improper node sink**.
- **Case 2c.** A has two complex conjugate eigenvalues with negative real parts. The zero is called a **spiral sink**.

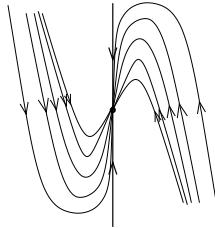


Figure 2.4: Improper node sink

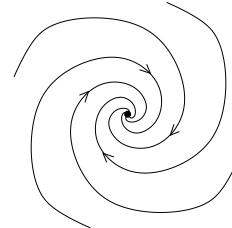


Figure 2.5: Spiral sink

- **Case 3.** Both eigenvalues have positive real parts. The zero is called a **source**, because any integral curve tends toward it for $t \rightarrow -\infty$.
 - **Case 3a.** A is diagonalizable and its eigenvalues are different. The zero is a **node source**. If both eigenvalues are equal, the zero is a **focus source**.

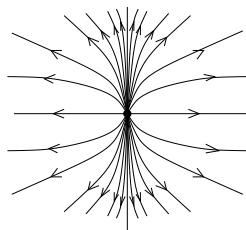


Figure 2.6: Node source

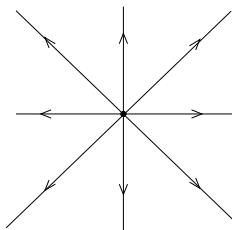


Figure 2.7: Focus source

- **Case 3b.** A is not diagonalizable but has one real positive eigenvalue. The zero is called an **improper node source**.

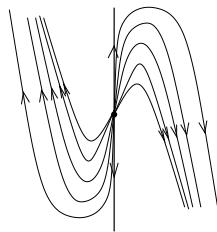


Figure 2.8: Improper node source

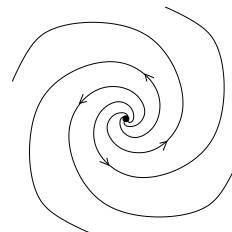


Figure 2.9: Spiral source

- **Case 3c.** A has two complex conjugate eigenvalues with positive real parts. The zero is called a **spiral source**.
- **Case 4.** A has pure imaginary (conjugate) eigenvalues. The zero is called a **center**.

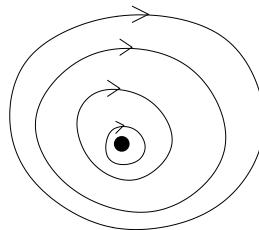


Figure 2.10: Center

2.2.2 Non-Linear Case

If one now focuses on general, non-linear vector fields, the study of a critical point concentrates on the local behavior of the phase portrait in its vicinity. More precisely, the vector field is linearized at the considered critical point. The approximating linear vector field is thus defined as

$$\frac{d\mathbf{y}}{dt} = D\mathbf{f}(\mathbf{x}_0)\mathbf{y},$$

where $D\mathbf{f}$ is the Jacobian matrix of the vector field \mathbf{f} , that is $(\frac{\partial f_i}{\partial x_j})$. Note that this local approximation is only valid if the Jacobian matrix has full rank at the critical point.

Definition 10 *If a critical point \mathbf{x}_0 of a vector field \mathbf{f} is such that the Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$ has full rank (i.e. $\det D\mathbf{f}(\mathbf{x}_0) \neq 0$), then the critical point is said to be of **first order**. Otherwise, it is said to be of **higher order** or **non-linear**.*

Hyperbolic critical points are a subclass of first order critical points.

Definition 11 *A critical point \mathbf{x}_0 of a vector field \mathbf{f} is said to be **hyperbolic** if the Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$ has no eigenvalue with zero real part.*

Among all possible types of critical points, special attention is paid to sinks and sources. Their definition is based on the property of the Jacobian matrix at their location. They generalize the linear sinks and sources encountered previously.

Definition 12 If all eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$ have negative real parts, then the critical point \mathbf{x}_0 is called a **sink**. Conversely, if all eigenvalues have positive real parts, \mathbf{x}_0 is called a **source**.

The intuitive meaning of this classification is stated more precisely by the following theorem.

Theorem 4 Let \mathbf{x}_0 be a sink of the vector field \mathbf{f} with corresponding flow ϕ . Suppose every eigenvalue of $D\mathbf{f}(\mathbf{x}_0)$ has real part less than $-c$, $c > 0$. Then there is a neighborhood U of \mathbf{x}_0 such that there is an euclidean norm satisfying

$$|\phi_t(\mathbf{x}) - \mathbf{x}_0| \leq \exp^{-tc} |\mathbf{x} - \mathbf{x}_0|$$

for all $\mathbf{x} \in U$, $t \geq 0$.

Proof: see [HS74, pp. 181-182].

Remark 9 With the definitions above, a sink (resp. source) is a ω - (resp. α -) stable critical point. If furthermore $D\mathbf{f}(\mathbf{x}_0)$ has no eigenvalue with zero real part, it is ω - (resp. α -) asymptotically stable.

The fundamental relation between the local aspect of the integral curves of a non-linear vector field in the vicinity of a critical point and its linear approximation is given by the next theorem.

Theorem 5 (Hartman-Grobman) If a critical point \mathbf{x}_0 of a non-linear vector field \mathbf{f} is hyperbolic then there is a homeomorphism h defined on some neighborhood U of \mathbf{x}_0 locally taking integral curves of the non-linear flow ϕ_t related to \mathbf{f} to those of the corresponding linear flow. The homeomorphism preserves the sense of integral curves and can also be chosen to preserve parametrization.

Proof: see [Har64, Theorem 7.1, p. 244].

Consequently, for hyperbolic critical points, the aspect of the integral curves in their neighborhood will be similar to one of the ten cases enumerated previously. Fig. 2.11 gives an idea of this correspondence.

The general classification of non-linear critical points in the plane distinguishes two types of critical points: The center and the non-center types.

Definition 13 A critical point that is approached by no integral curve is said to be of **center type**. If on the contrary, at least one integral curve converges to it, it is of **non-center type**.

Remark 10 In the non-center case, there are actually two integral curves at least that converge toward the critical point.

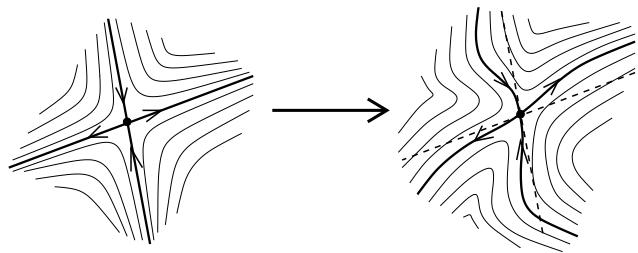


Figure 2.11: Relation between a linear and a non-linear saddle point

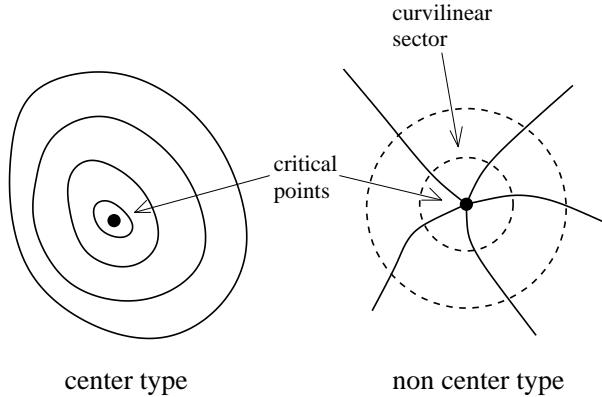


Figure 2.12: Center and non-center types

In the non-center case, the integral curves converging to the critical point determine so-called **curvilinear sectors**. These notions are illustrated in Fig. 2.12. The possible natures of a curvilinear sector are introduced next.

Definition 14 *The curvilinear sectors of a critical point can be of three different types.*

- *Case 1. If one (bounding) converging integral curve tends toward 0 for $t \rightarrow \infty$, and if the other tends toward 0 for $t \rightarrow -\infty$, and if every integral curve passing through the sector leaves it for both $t \rightarrow \infty$ and $t \rightarrow -\infty$, the sector will be called a **hyperbolic** or **saddle sector**. In this case, both bounding converging integral curves are called **separatrices of the singular point O**.*
- *Case 2. If both bounding converging integral curves tend to O for $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$) and if every integral curve through the sector tends toward 0 for $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$) without leaving it and leaves the sector for $t \rightarrow -\infty$ (resp. $t \rightarrow \infty$), the sector is called a **parabolic sector**.*

- *Case 3. If both bounding converging integral curves are actually the same and if all integral curves through a point inside this loop form nested loops tending to O for both $t \rightarrow \infty$ and $t \rightarrow -\infty$, the sector is called an elliptic sector.*

An illustration of the different sector types is proposed in Fig. 2.13.

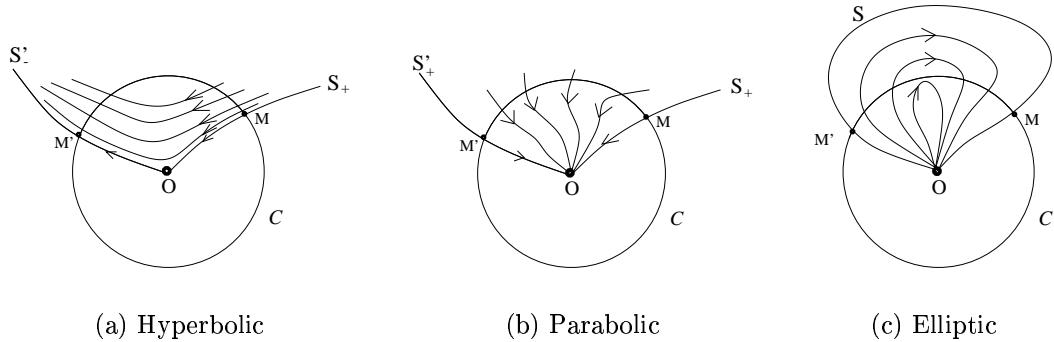


Figure 2.13: Sector types

Remark 11 By considering a neighborhood with a small enough radius, one can always get a decomposition in the sector types defined above (analytical case).

Remark 12 The boundaries of an elliptic sector cannot be determined locally. If one restricts the study of an elliptic sector to a neighborhood of the critical point, this sector will always be bounded on both sides by parabolic sectors.

Remark 13 An integral curve bounding a hyperbolic sector is called separatrix because it separates two sets of integral curves that diverge from another as $t \rightarrow \infty$ or $t \rightarrow -\infty$, as illustrated in Fig. 2.14.

Remark 14 In the special case of linear vector fields, the only critical point presenting hyperbolic sectors and thus separatrices is the saddle point: It has four separatrices that are the integral curves converging toward it along the direction of its eigenvectors.

Consequently, any singular point may be characterized by the type, angular location, and number of its curvilinear sectors. The precise meaning of this characterization is the following.

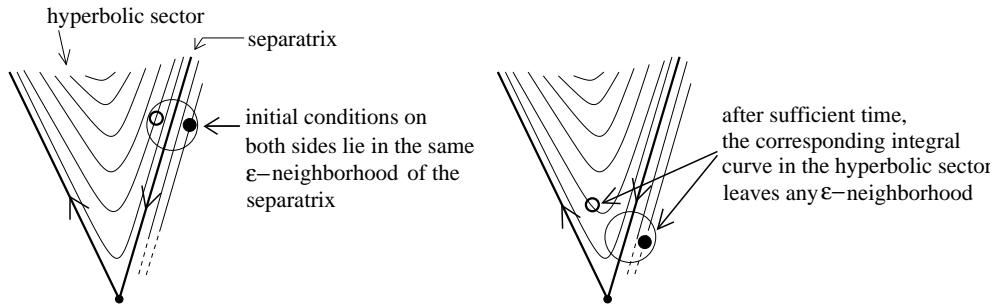


Figure 2.14: A separatrix

Theorem 6 *If the structures of two singular points are related through a one-to-one correspondence between their respective separatrices converging for $t \rightarrow \infty$, separatrices converging for $t \rightarrow -\infty$ and elliptic regions then there exists a curve-preserving topological mapping of a neighborhood of the first onto a neighborhood of the second preserving orientation and direction of t .*

Proof: see [ALG73, ch. 8].

Moving away from the vicinity of a critical point to consider its influence on the global structure of the phase portrait, one introduces the notion of basin.

Definition 15 *The union of all integral curves that tend toward a critical point x_0 as $t \rightarrow \infty$ is called the ω -basin of x_0 . The union of all integral curves that tend toward a critical point x_0 for $t \rightarrow -\infty$ is called the α -basin of x_0 .*

Remark 15 *The ω -basin of a source is reduced to the source itself whereas the α -basin of a sink is reduced to the sink itself.*

2.3 Closed Orbits

Definition 16 *A closed orbit is a periodic solution to the Cauchy problem, that is, if γ denotes the trajectory of an closed orbit ϕ_t with initial condition $x \in \gamma$, there exists a $t_1 \neq 0$ such that $\phi_{t_1} = x$.*

Corollary 1 *With the notations above and by uniqueness of the solution of the Cauchy problem, it follows that $\phi_{nt_1} = x$, for all $n \in \mathbb{Z}$.*

As for critical points, one defines the asymptotic stability of closed orbits.

Definition 17 Let γ be a closed orbit of a dynamical system with flow ϕ . γ is said to be **ω -asymptotically stable** if for every open set $V \supset \gamma$, there exists an open set $W \supset V \supset \gamma$ such that $\phi_t(W) \subset V$ for all $t > 0$ and

$$\lim_{t \rightarrow \infty} d(\phi_t(x), \gamma) = 0,$$

where $d(x, \gamma)$ is the minimal distance from x to a point on γ . Replacing t by $-t$ in the definition above, one defines similarly an **α -asymptotically stable closed orbit**.

Fig. 2.15 shows an asymptotically stable closed orbit.

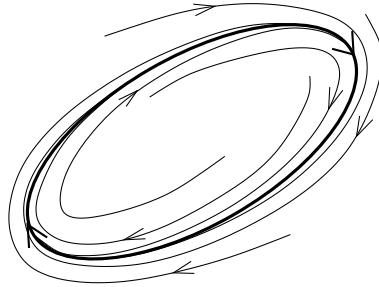


Figure 2.15: Asymptotically stable closed orbit

The study of a closed orbit can be done by the use of the Poincaré map.

Definition 18 Let $\Sigma \subset \mathbb{R}^n$ be a local cross section of a closed orbit γ of a flow ϕ_t in \mathbb{R}^n . The section is chosen such that the flow is everywhere transverse to it (that is not tangent). Let $U \subset \Sigma$ be a neighborhood of the unique point p where the closed orbit intersects Σ . The **Poincaré map** is then defined as the map $P : U \rightarrow \Sigma$ such that

$$\forall q \in U, P(q) = \phi_\tau(q),$$

where τ is the time taken for the orbit $\phi_t(q)$ based at q to first return to Σ . (See Fig. 2.16.)

Remark 16 With the notations above, p is a fixed point for the map P .

Note that P actually defines a discrete map

$$q_{n+1} = P(q_n), n \in \mathbb{Z}$$

The stability of p for the map P corresponds to the stability of the closed orbit γ for the flow ϕ_t . Classification is based, as in the vector case, on the eigenvalues of the linearization of the Poincaré map at p . In the case of two-dimensional vector fields, this map is one-dimensional and there is a single (real) eigenvalue. The definitions are as follows.

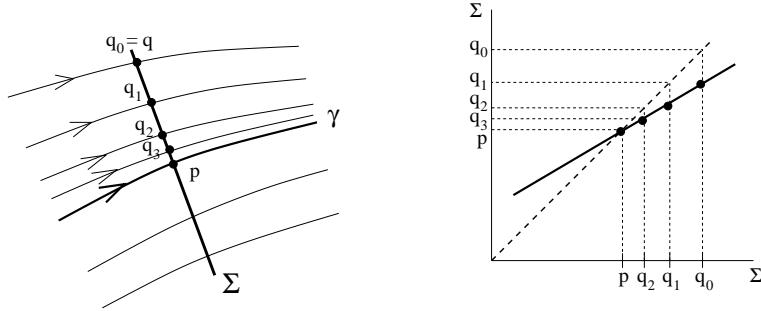


Figure 2.16: Poincaré map

Definition 19 Let ϕ_t be a two-dimensional flow on the plane \mathbb{R}^2 . Let γ be a closed orbit of ϕ_t and P the corresponding one-dimensional Poincaré map defined on some cross section Σ of γ . If the linearization of P at its fixed point $p \in \gamma$ has an eigenvalue $\lambda < 1$, p is said to be a **sink** for P . If $\lambda > 1$, p is said to be a **source** for P .

Remark 17 With the notations above, the value of λ is necessarily a positive value in the planar case because a negative value would correspond to integral curves that alternatively meet the cross section on each side of p (i.e. of the closed orbit), which is impossible because integral curves cannot cross the closed orbit.

The link between the stability of p for P and the stability of γ is formalized, in the general case, by the next theorem.

Theorem 7 With the notations above, if p is a sink for the Poincaré map P , i.e.

$$\lim_{n \rightarrow \infty} P^n(q) = p, \text{ for all } q \text{ in } U,$$

then the closed orbit γ is ω -asymptotically stable. Similarly, if p is a source for the Poincaré map P , i.e.

$$\lim_{n \rightarrow -\infty} P^n(q) = p, \text{ for all } q \text{ in } U,$$

then the closed orbit γ is α -asymptotically stable.

Proof: see [HS74, pp. 282-283].

As for critical points, one can define the hyperbolicity of a closed orbit, which completes the classification introduced previously between sinks and sources in two dimensions.

Definition 20 *If the linearization of the Poincaré map P at the fixed point p (lying on the closed orbit) has no eigenvalue with unit modulus, then p is a hyperbolic fixed point for the discrete map P and the closed orbit is said to be hyperbolic as well.*

Remark 18 *In the two-dimensional case, a closed orbit is either hyperbolic and then α - or ω -asymptotically stable (i.e. attracting or repelling) or non-hyperbolic.*

Two non-hyperbolic closed orbits are shown in Fig. 2.17.



Figure 2.17: Non-hyperbolic closed orbits

2.4 Topological Graph

The concepts introduced so far are the constituent parts of the structure of the phase portrait of a dynamical system, also called topological graph or simply topology. By extension, one calls it topology of the corresponding vector field. The definition used in the following is given next.

Definition 21 *The topology of a planar vector field f is built up of all critical points, separatrices and closed orbits of f .*

Note that the notion of separatrix, as defined previously, is closely related to the local structure of the phase portrait in the vicinity of a critical point. This definition is actually too restrictive. As a matter of fact, the essential property of a separatrix is to separate groups of integral curves that have

different asymptotic behaviors. In other words, a separatrix locally divides the domain of definition of a vector field into two subdomains inside which all integral curves converge to the same critical point (or closed orbit) for $t \rightarrow \infty$ and all converge to the same critical point (or closed orbit) for $t \rightarrow -\infty$. An equivalent definition consists in considering a separatrix as the intersection of the closure of two basins.

An interesting problem arises when dealing with vector fields defined over closed bounded domains (compacts). In this case, the asymptotic behavior of all integral curves is not only determined by the critical points and the closed orbits located inside the domain but also by the restriction of the vector field to the boundary of the domain: This boundary can locally act as source (where the vector field is directed inwards), sink (where the vector field is directed outwards) or saddle (separating the former two). Therefore, the generalized notion of separatrix also includes the integral curves starting at the boundary saddles. A presentation of this topic from the visualization viewpoint can be found in [SHJ00]. The previous notions are illustrated in Fig. 2.18.

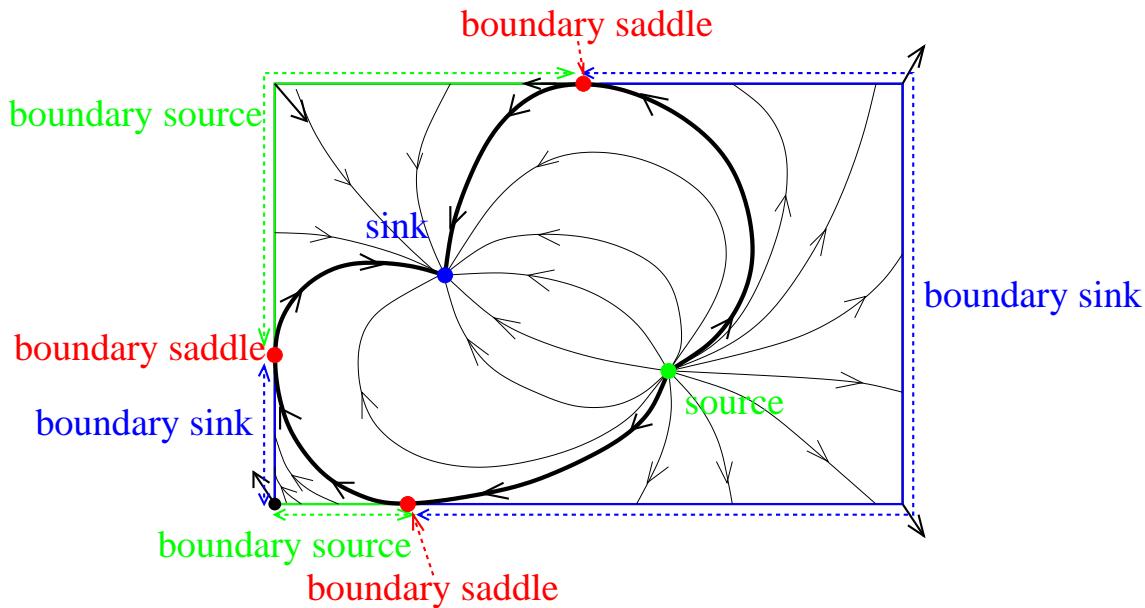


Figure 2.18: Topology of a vector field over a bounded domain

Remark that this topology contains no separatrix emanating from critical points.

2.5 Poincaré Index

The index is a fundamental concept that has been introduced by Poincaré in the qualitative theory of dynamical systems. This notion has many theoretical and practical applications. It is based on the rotation of a planar vector field along a simple closed curve. The definitions are given next as well as fundamental theorems that will prove extremely useful in the following.

At a position where a two-dimensional vector field $\mathbf{f} = (f_X, f_Y)^T$ is not vanishing (i.e. away from every critical point), one can define the angle between the positive x -axis and the direction of \mathbf{f} as the scalar θ satisfying

$$\cos \theta = \frac{f_X}{\sqrt{f_X^2 + f_Y^2}}, \quad \sin \theta = \frac{f_Y}{\sqrt{f_X^2 + f_Y^2}}.$$

Furthermore, if \mathbf{f} is continuous, the angle θ is uniquely determined modulo 2π and is a continuous function of the position. The index of a simple (non self-intersecting) closed curve is then defined as follows.

Definition 22 *The index of a simple closed curve Γ of the plane relative to a continuous vector field \mathbf{f} is the number of positive field rotations while traveling once along Γ in positive direction, that is (with the notations above)*

$$I(\Gamma, \mathbf{f}) = \frac{1}{2\pi} \oint_{\Gamma^+} d\theta,$$

where the curve Γ contains no critical point of the vector field \mathbf{f} .

Remark 19 *The index is an integer by continuity of θ along the closed curve.*

The index of a simple closed curve around a saddle point is illustrated in Fig. 2.19.

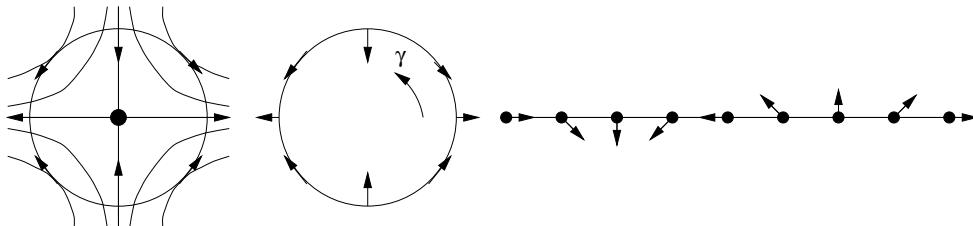


Figure 2.19: Simple closed curve of index -1

One can now state the first fundamental theorem about the index.

Theorem 8 *Let Γ be a simple closed curve in the domain of definition U of a continuous vector field \mathbf{f} and let $\bar{\Gamma}$ denote the closure of the domain bounded by Γ . If all points of $\bar{\Gamma}$ are in U and the interior of Γ contains no critical point, then the index of Γ relative to \mathbf{f} is zero: $I(\Gamma) = 0$.*

Proof: See [ALG73, p. 194].

This theorem generalizes as follows.

Theorem 9 *With the hypotheses above, if the interior domain of Γ contains simple closed curves $\Gamma_1, \dots, \Gamma_n$, then one has the relation*

$$I(\Gamma) = I(\Gamma_1) + \dots + I(\Gamma_n)$$

Proof: See [ALG73, p. 194].

We now come to the important notion of index of an isolated critical point.

Definition 23 *The index of an isolated critical point \mathbf{x}_0 of a continuous vector field \mathbf{f} is defined as the index of any simple closed curve Γ containing \mathbf{x}_0 in its interior and enclosing no other critical point, either in or on Γ .*

One gets immediately a corollary of theorem 9:

Corollary 2 *With the hypotheses above, let Γ be a simple closed curve enclosing n critical points $\mathbf{x}_1, \dots, \mathbf{x}_n$ in its interior, then the index of Γ is given by*

$$I(\Gamma) = I(\mathbf{x}_1) + \dots + I(\mathbf{x}_n)$$

Getting back to first-order critical points, one gets the following property.

Theorem 10 *The index of a critical point of first order is either +1 (sink, source and center) or -1 (saddle point).*

Proof: See [ALG73, p. 199].

Considering closed orbits, one obtains:

Theorem 11 *The index of a closed orbit of a dynamical system related to a continuous vector field \mathbf{f} is +1.*

Proof: See [ALG73, p. 195].

Theorem 11 together with theorem 9 lead to the following corollary.

Corollary 3 *A closed orbit that contains only first order critical points contains at least one critical point of index +1. Furthermore, if it contains several first order critical points, then this number is odd, say $2n + 1$, and there are n saddle points and $n + 1$ critical points of index +1 (sinks, sources or centers).*

2.6 Structural Stability

The previous sections focused on autonomous dynamical systems. Now, if the considered vector field depends on an additional parameter, the structure of the phase portrait may change as the value of this parameter evolves. Therefore, the analysis of non-autonomous dynamical systems is concerned with the essential question of structural stability of the phase portrait, i.e. the ability of a given topology to maintain its qualitative nature under small changes of the parameter value. The present section introduces the notions required to precisely define structural stability, and states the fundamental theorem of Peixoto. The first definition precises the nature of the “small changes” mentioned above.

Definition 24 Let F be a function of class C^r in \mathbb{R}^n , $r \geq 1$ (that is r times continuously differentiable) and $\epsilon > 0$. Then G is a C^1 ϵ -perturbation if there exists a compact $K \subset \mathbb{R}^n$ such that $F = G$ on $\mathbb{R}^n \setminus K$ and for all $i \in \{0, \dots, n-1\}$, one has

$$\left\| \frac{\partial}{\partial x_i} (F - G) \right\| < \epsilon$$

The preservation of the qualitative nature of a dynamical system is intimately related to the notion of equivalence as defined next.

Definition 25 Two C^r vector fields \mathbf{f} and \mathbf{g} are said to be C^k equivalent ($k \leq r$) if there exists a C^k diffeomorphism (all derivatives are invertible and their inverse are continuous) h which takes orbits $\phi_t^{\mathbf{f}}(\mathbf{x})$ of \mathbf{f} to orbits $\phi_t^{\mathbf{g}}(\mathbf{x})$ of \mathbf{g} , preserving sense but not necessarily parametrization by time. If furthermore h does preserve parametrization by time, then h is called **conjugacy**.

Remark 20 This definition means that for any \mathbf{x} and t_1 , there is a t_2 such that

$$h(\phi_{t_1}^{\mathbf{f}}(\mathbf{x})) = \phi_{t_2}^{\mathbf{g}}(h(\mathbf{x}))$$

The structural stability is now defined as follows.

Definition 26 A C^r vector field \mathbf{f} is **structurally stable** if there is an $\epsilon > 0$ such that all C^1 ϵ -perturbations of \mathbf{f} are topologically (i.e. C^0) equivalent to \mathbf{f} .

The focus is again on planar vector fields. One first needs to introduce special cases of separatrices of first order critical points.

Definition 27 A separatrix connecting two saddle points is called a **heteroclinic connection**. A closed separatrix connecting a saddle point with itself is called a **homoclinic connection**.

Saddle connections are shown in Fig. 2.20.

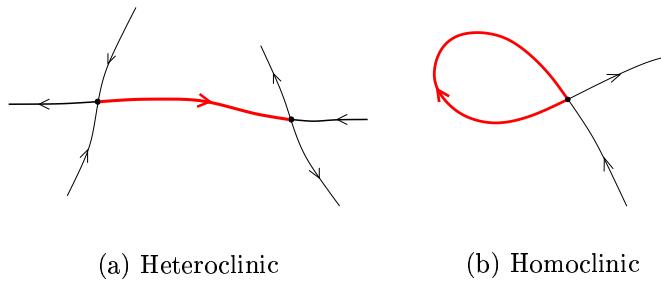


Figure 2.20: Saddle connections

We now have all the definitions required to state the fundamental Peixoto theorem [Pei62] on structural stability for two-dimensional flows defined over compact domains. For convenience and to keep in the scope of this thesis, we restrict it to the euclidean case.

Theorem 12 (Peixoto) *A C^r vector field on a two-dimensional compact planar domain of \mathbb{R}^2 is structurally stable if and only if:*

1. *the number of fixed points and closed orbits is finite and each is hyperbolic;*
2. *there are no orbits connecting saddle points (heteroclinic or homoclinic).*

Remark 21 *The actual theorem deals with two-dimensional manifolds and is concerned with non-wandering sets and orientability, concepts that go beyond the theoretical needs of the present overview.*

Practically, Peixoto's theorem implies that a planar vector field typically presents saddle points, sinks and sources as well as attracting or repelling closed orbits. Furthermore, it asserts that non-hyperbolic critical points or closed orbits are unstable because small perturbations can make them hyperbolic. Saddle connections, as far as they are concerned, can be broken by small perturbations as well. The next section is concerned with such structural transitions.

2.7 Bifurcations

The term bifurcation was originally used in the literature to describe the splitting of equilibrium points in a parameter-dependent dynamical system, as the value of this parameter comes to change over its domain of definition: If one depicts the curve describing the successive positions of the equilibrium over

the space embedding euclidean and parameter space, one notices for a particular parameter value the presence of a fork that leads to several alternative equilibria. This basic idea is illustrated in Fig. 2.21 for the simplest case of one-dimensional euclidean space (variable x) and one-dimensional parameter space (variable μ). The structure associated with the parameter value where the fork

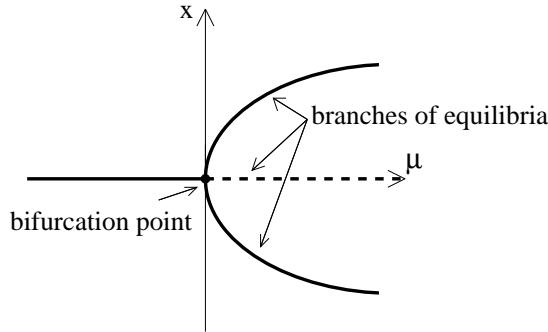


Figure 2.21: Bifurcation diagram

occurs is thus unstable for slight changes of this value can lead to another different structure (non equivalent in the sense of definition 25). Therefore, one gets the following definition.

Definition 28 *A value μ_0 of the parameter μ for which the flow is not structurally stable is called a **bifurcation value** of μ .*

Remark 22 *In the definition above, the notion of bifurcation is restricted to a one-dimensional parameter space. This is motivated by the fact that the approach of structural stability developed in this thesis is limited to time-dependent vector fields where time is the only (considered) parameter. However, bifurcation problems go far beyond this restriction and actually apply to any n -dimensional parameter space.*

The theory of bifurcations is thus concerned with the structural transitions that occur at the bifurcation values of the parameter. These transitions may be very complicated and an exhaustive classification is impossible, even in the simple case treated here. Yet, two categories exist: On one hand, some bifurcations only affect the nature of a critical point or a closed orbit, and the corresponding new stable state (reached after transition) is to be found in a neighborhood. These bifurcations are called *local bifurcations*. On the other hand, bifurcations that change the global structure of the flow and cannot be deduced from local information are called *global bifurcations*. Both types are considered next.

2.7.1 Local Bifurcations

In this section, one focuses on local bifurcations that involve two-dimensional time-dependent vector fields (where time can be seen as any one-dimensional parameter). For simplicity, we shall consider here only the simplest local bifurcations that occur in this case, which suffices for the visualization methods to come.

As said previously, local bifurcations correspond to structural changes that take place in the vicinity of a critical point or a closed orbit. Now, from the viewpoint of critical points or closed orbits, structural instability is due to non-hyperbolicity, as defined previously. One considers in the following the most common cases, in the sense of practical applications, which is actually deeply related to considerations inherited from the mathematical theory of transversality, as explained in [GH83].

Bifurcations of Critical Points

Recalling a previous definition, the non-hyperbolicity of a critical point is due to the fact that the Jacobian matrix at the corresponding position has an eigenvalue with zero real part. For two-dimensional vector fields, two situations may thus be encountered: One eigenvalue is zero or both are conjugate and purely imaginary.

Saddle-Node Bifurcations This type of bifurcation is also called **static fold** by some authors [AS82]. This is the most usual case of local bifurcation related to a non-hyperbolic critical point with zero eigenvalue. Considering the corresponding dimension, the graph of the one-dimensional bifurcation is as shown in Fig. 2.22. In the situation depicted here, no critical point is

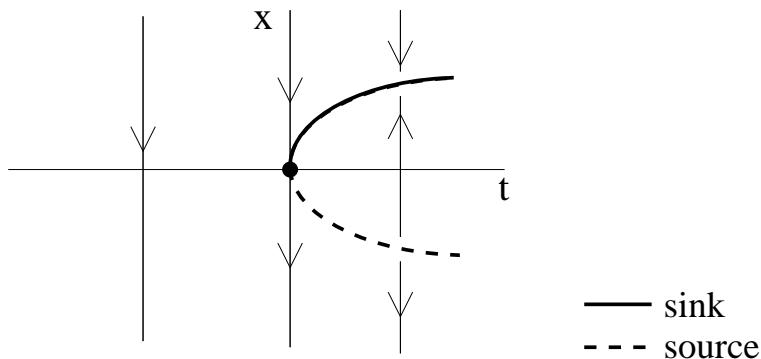


Figure 2.22: Saddle-node bifurcation (1D)

present at the beginning. At the bifurcation point, a critical point appears and the associated eigenvalue is zero. The non-hyperbolicity of this singularity is characterized by both an attracting and a repelling nature at the same time. This critical point is then replaced by a source and a sink, moving away from another. For this reason, this bifurcation is also called **pairwise creation**. Getting back to a two-dimensional vector field, the successive local aspects of the topology are shown in Fig. 2.23. Note that this two-dimensional

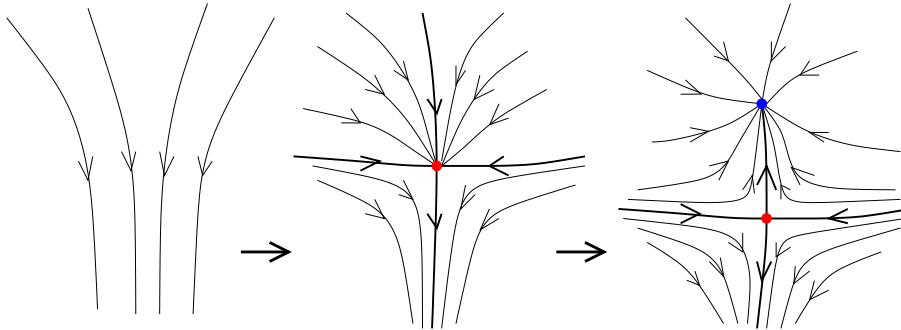


Figure 2.23: Saddle-node bifurcation (2D)

picture has been obtained by adding a stable (i.e. attracting) eigenspace to the previous (one-dimensional) critical points. The presence of a saddle point and a sink after transition justifies the name chosen for this bifurcation. Obviously, the same kind of bifurcation is obtained by adding an unstable (repelling) eigenspace to both one-dimensional critical points: The sink is replaced by a source.

If one inverts the direction of time, one gets additional bifurcations: Starting with a saddle point and a sink (resp. source), close to another, both critical points come to merge together to form a non-hyperbolic critical point that disappears to let place to a local structure where no critical point is present. This reverse bifurcation is therefore called **pairwise annihilation**.

Pitchfork Bifurcations This second type of bifurcation is also related to the presence of a zero eigenvalue in the Jacobian. Its major difference to the previous one is that a critical point maintains through the bifurcation point. Considering the dimension associated with the zero eigenvalue, one obtains the one-dimensional situation proposed in Fig. 2.24.

In this case, a stable sink is present originally. At the bifurcation point this critical point becomes non-hyperbolic but keeps its attracting nature. After bifurcation, two stable sinks move away from another, separated by a stable source between them. If one gets back to a two-dimensional problem, e.g. by

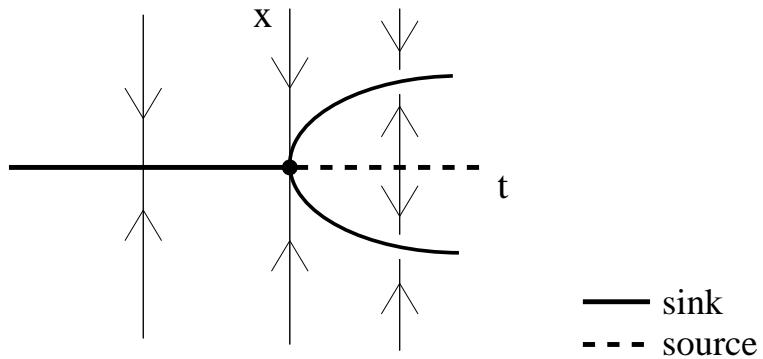


Figure 2.24: Pitchfork bifurcation

adding an attracting (stable) dimension to this picture, one gets the bifurcation shown in Fig. 2.25.

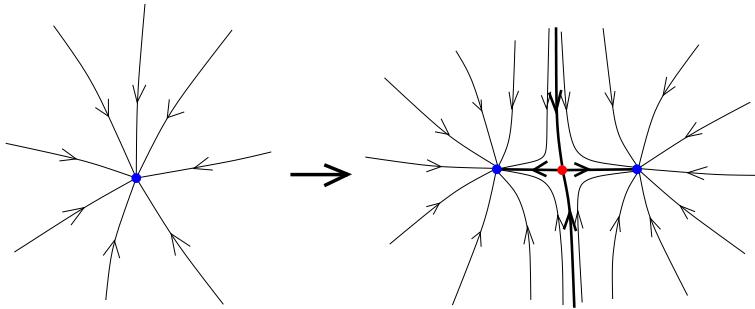


Figure 2.25: Pitchfork bifurcation (2D)

If one adds a repelling dimension to the one-dimensional bifurcation, one obtains a final configuration with two saddle points connected through a sink. Similar pictures are obtained by replacing the original sink by a source. Additional cases correspond to a reversed time direction.

Hopf Bifurcations This is the bifurcation that typically occurs when the Jacobian has two purely imaginary conjugate eigenvalues. Recalling what was said previously about linear vector fields, in the first order case such a critical point is a center and is surrounded by infinitely many closed orbits. This bifurcation is fundamentally two dimensional, as opposed to the saddle-node presented previously. The situation is illustrated in Fig. 2.26.

The evolution is as follows: Before the bifurcation point, a spiral sink is present. The attracting character of this sink weakens until it vanishes: No

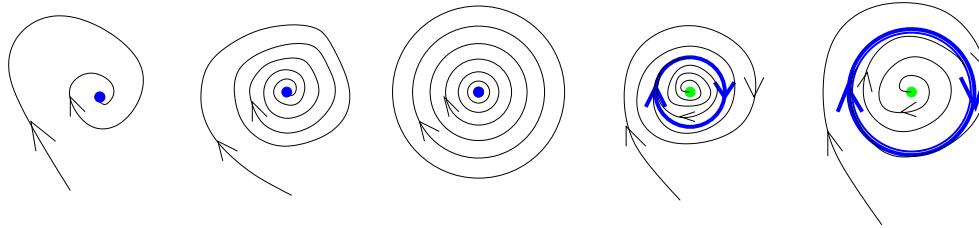


Figure 2.26: Hopf bifurcation

integral curve converges toward the critical point that becomes a center point. This non-hyperbolic singularity gives raise to a spiral source (i.e., the attracting nature has mutated in a repelling one through a center nature) surrounded by an attracting closed orbit, that ensures global consistency of the new local structure with the rest of the topology (the whole structure acts as attracting in the large, like before).

Once again, one obtains additional similar bifurcations by reversing the direction of the flow (the sink at the beginning becomes a source and the source after bifurcation becomes a sink while the surrounding closed orbit has now a repelling nature). Inverting the direction of time provides at last two more bifurcations where a closed orbit surrounding a critical point disappears to let place to another critical point with opposite (attracting/repelling) nature.

Bifurcations of Closed Orbits

The non-hyperbolicity of a closed orbit corresponds to an eigenvalue of the Poincaré map with unit modulus. In the case of two-dimensional vector fields, this map is one-dimensional and the only possible cases are eigenvalues equal to 1 or -1. Nevertheless, as shown in section 2.3, negative values are impossible in the planar two-dimensional case. Therefore, only the case of an eigenvalue equal to 1 is treated.

Periodic Fold This kind of bifurcation is the equivalent of the (one-dimensional) saddle-node bifurcations mentioned previously for the Poincaré map. The most common situation is illustrated in Fig. 2.27.

Before the bifurcation point, an attracting spiral is present and no closed orbit can be found in its vicinity. This corresponds to a Poincaré map that has originally no fixed point on any cross section of the flow in this vicinity. A fixed point appears then with associated eigenvalue 1. At this instant, a closed orbit surrounds the spiral source that acts as attracting from outside and as repelling from inside. This is an unstable structure that is then replaced by two stable

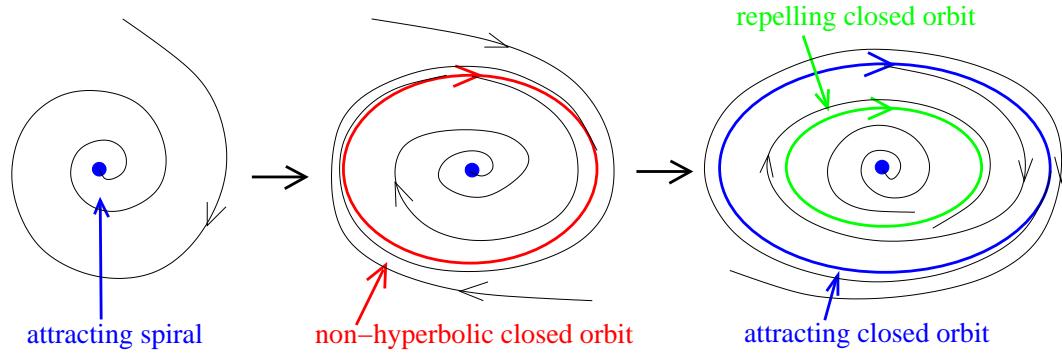


Figure 2.27: Periodic fold

closed orbits, inside one another and moving away from another: The external closed orbit has an attracting nature (which ensures global consistency with the rest of the topology) while the internal closed orbit has repelling nature. Both closed orbits have now Poincaré maps with hyperbolic fixed points.

2.7.2 Global Bifurcations

As said previously, global bifurcations correspond to structural changes that involve global aspects of the flow. As a consequence, their analysis cannot be reduced to the neighborhood of a critical point or a closed orbit. Furthermore they may be very complicated and a complete, systematic classification is impossible. For this reason, this presentation only deals with some simple cases. Their common characteristic is to involve saddle connections. These connections are unstable in the sense of Peixoto's theorem. They can be of two types: Heteroclinic or homoclinic, as defined previously.

Basin Bifurcation This first global bifurcation is based on a heteroclinic saddle connection. Only two saddle points are involved. It entails radical changes in the basins in the vicinity of these critical points. The evolution is depicted in Fig. 2.28. As one can see, before the bifurcation point, a separatrix tending toward a saddle point comes closer to a second separatrix emanating from the other saddle point. At the bifurcation point, both separatrices merge: The saddle points are connected through a heteroclinic connection. This unstable state is replaced by a new configuration where the respective position of the separatrices is inverted, with respect to the original situation: These separatrices have been swapped.

The following bifurcations involve homoclinic connections.

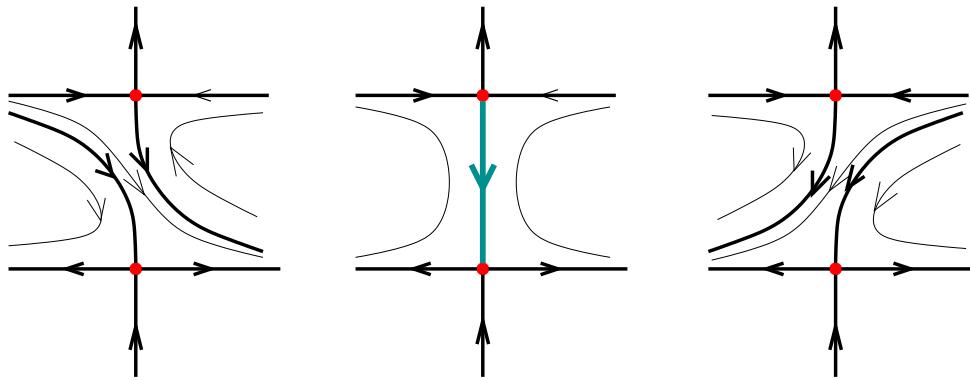


Figure 2.28: Basin bifurcation

Homoclinic Bifurcation with Center Point This bifurcation is shown in

Fig. 2.29 (see also [GH83, pp. 291-292]). Originally, a saddle point is con-

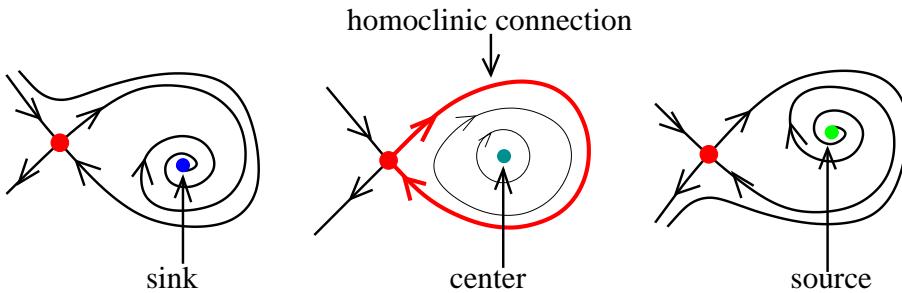


Figure 2.29: Homoclinic bifurcation with center point

nected to a sink by one of its separatrices. The attracting nature of the sink weakens until it becomes a center: This is the bifurcation point. The center nature of the second critical point constrains the separatrix that reached it so far to become homoclinic, that is to return to the saddle point. This unstable state is immediately replaced by a configuration where the center has become a source and two separatrices have been swapped with respect to the original situation. The whole process, from the viewpoint of the sink, is analogous to a Hopf bifurcation. Nevertheless, as a saddle point is involved too, the bifurcation becomes global and no closed orbit remains after transition. Inverting the direction of time (which corresponds here to replacing the original sink by a source) provides an additional bifurcation.

A similar bifurcation exists that results in the creation of a closed orbit.

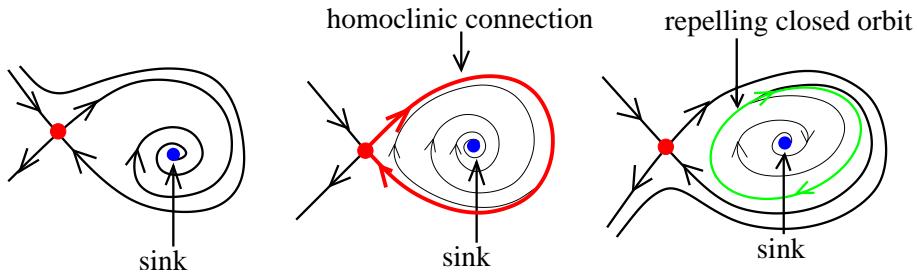
Homoclinic Bifurcation with Closed Orbit Consider Fig. 2.30.

Figure 2.30: Homoclinic bifurcation with closed orbit

Like in the previous case, one has at the beginning a saddle point connected to a sink. Yet, as opposed to the bifurcation above, the present one has no incidence on the nature of the sink. Here, the separatrix connecting the saddle point to the sink becomes homoclinic at the bifurcation point. The sink maintains at this moment so the homoclinic closed orbit acts as a source (repelling nature) in its interior. After the bifurcation point, a repelling closed orbit is emitted from the homoclinic connection which induces a new stable situation where the original separatrix now converges towards a closed orbit surrounding the sink. An analogous situation is obtained when replacing the sink by a source. Inverting the direction of time, one obtains the progressive disappearance of a closed orbit through a homoclinic connection of a saddle point located in its vicinity.

The last example presented in this brief overview of global bifurcations is called Ω -explosion in [Zee82].

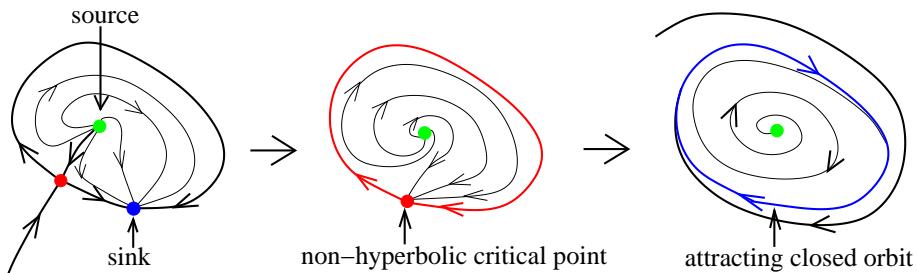
Explosion Bifurcation It involves three critical points as shown in Fig. 2.31.

Figure 2.31: Explosion bifurcation

Before the bifurcation point, a saddle point is connected to a source by one separatrix and to a sink by both of its repelling separatrices. The saddle point and the sink get closer and merge at the bifurcation point. This gives raise to a non-hyperbolic critical point, lying on a closed integral curve. This configuration is then replaced by an attracting closed orbit surrounding the unchanged source, i.e. the non-hyperbolic critical point has disappeared. The name *explosion* of this kind of bifurcation is due to the fact that the attractor of the source, originally reduced to a single point (the sink) “explodes” at the bifurcation point to become a whole closed orbit.

Analogous bifurcations exist with inverted roles of sink and source and by inverting the time direction.

Chapter 3

Topology of Symmetric, Second-Order, Planar Tensor Fields

In this chapter, we are concerned with symmetric, second-order, planar tensor fields. The topological approach was recently introduced in Scientific Visualization by the PhD thesis of Delmarcelle [Del94]. Yet its theoretical framework is inherited from differential geometry. The main idea consists in studying the structure of the eigenvector fields associated with a real symmetric tensor field. As we will see, this leads to the qualitative analysis of line fields (or fields of directions). Such fields lack orientation. Therefore they exhibit topological features that are unknown in the standard (oriented) vector case. Nevertheless, using the profound connections between both, we shall extend many results of the previous chapter obtained in the treatment of dynamical systems. Practically, the presentation is structured as follows. After a brief definition of convenient notations, the notion of tensor line is introduced: Results on existence and uniqueness are given. This naturally leads to the definition of singular points. Analogous to vector fields, singularities are the unique locations where tensor lines can meet. Hence they locally determine the topological structure of the field. The geometric study of singularities yields local configurations that may be classified in a way similar to the sector decomposition presented previously in section 2.2.2. After a treatment of the autonomous (or steady) case, we finally turn to time-dependent tensor fields and study the question of structural stability. To do so, we make use of the knowledge provided by the analysis of dynamical systems (section 2.6). The focus is on basic local bifurcations as well as on simple global structural transitions. This eventually provides the whole mathematical material required for the visualization methods to come.

3.1 Definitions

This preliminary section is only intended to give basic notions of tensor analysis. For a detailed treatment of this subject, see e.g. [Tho65] for a mathematical presentation or [You78] for an overview from the applications' viewpoint. First, one defines the notion of tensor.

Definition 29 *A tensor of order p is a geometric invariant that corresponds to a linear transformation of an euclidean space of dimension n into an euclidean space of dimension n^{p-1} . Considered in a particular cartesian coordinate system, it is thus described by n^p scalar components. Practically, a tensor of order 0 is equivalent to a **scalar** (by convention), a tensor of order 1 is equivalent to a **vector** and a tensor of order 2 is equivalent to a **matrix**, all with appropriate behavior under changes of the parametrization.*

The definition above is too general for our purpose. In fact, we restrict our considerations to second-order tensor fields.

Definition 30 *A second-order tensor field \mathbf{T} defined over a subset U of a n -dimensional euclidean space E is a map that associates every point $\mathbf{x} \in U$ with a second-order tensor, that is a linear transformation $\mathbf{T}(\mathbf{x})$ of E into itself. Focusing on a particular cartesian coordinate system, $\mathbf{T}(\mathbf{x})$ is described by a $n \times n$ matrix, i.e. characterized by n^2 real components.*

Remark 23 *Tensors fields of higher order are possible, too. For instance, the gradient field of a second-order tensor field is of third order, see section 3.4.2.*

In the following, the considered tensor fields are always supposed to be Lipschitz continuous and are considered in a cartesian coordinate system (such tensors are called *cartesian tensors*). Furthermore, the basis is supposed to be orthonormal. We are interested in symmetric, second-order tensor fields.

Definition 31 *A symmetric, second-order tensor field on a subset U of an euclidean n -dimensional space E is a second-order tensor field that maps every $\mathbf{x} \in U$ to a self-adjoint operator. Thus, it is everywhere associated with a symmetric matrix $\mathbf{T}(\mathbf{x}) = \mathbf{T}(\mathbf{x})^t$ with respect to any particular orthonormal basis.*

In the symmetric case, a tensor field can thus be seen as a multivariate function determined by $\frac{1}{2}n(n+1)$ independent scalar functions.

Remark 24 *An arbitrary second-order tensor field can always be decomposed into its symmetric and anti-symmetric parts*

$$\mathbf{T} = \mathbf{S} + \mathbf{A},$$

where

$$\mathbf{S} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) \quad \text{is the symmetric part}$$

and

$$\mathbf{A} = \frac{1}{2}(\mathbf{T} - \mathbf{T}^T) \quad \text{is the anti-symmetric part.}$$

We now turn to the focus of the current presentation: Symmetric second-order tensor fields of dimension 2. For convenience, they are often simply called *tensor fields* in the following. Their matrix representation in a cartesian coordinate system is of the form

$$\begin{aligned} \mathbf{T} : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto \mathbf{T}(x, y) = \begin{pmatrix} T_{11}(x, y) & T_{12}(x, y) \\ T_{12}(x, y) & T_{22}(x, y) \end{pmatrix}. \end{aligned}$$

From the structural point of view, a symmetric tensor field is fully characterized by its deviator part. The definitions are as follows.

Definition 32 An isotropic n -dimensional tensor corresponds to a diagonal matrix of the form $\mu \mathbf{I}_n$, where μ is a real scalar and \mathbf{I}_n denotes the identity matrix in \mathbb{R}^n .

This suggests the definition of the isotropic part of a symmetric tensor.

Definition 33 Let T be a n -dimensional symmetric tensor. Its isotropic part is defined as the matrix $\mu \mathbf{I}_n$, where μ is the mean of the (real) eigenvalues of T , i.e.

$$\mu = \frac{1}{n} \operatorname{tr} T$$

with $\operatorname{tr} T$ denoting the trace of T .

When one subtracts the isotropic part of a symmetric matrix, the result is a deviator matrix:

Definition 34 One calls deviator a trace free tensor. Hence the deviator part \mathbf{D} of a symmetric tensor \mathbf{T} is obtained by the relation

$$\mathbf{T} = \mu \mathbf{I}_n + \mathbf{D},$$

where $\mu \mathbf{I}_n$ is the isotropic part of \mathbf{T} , as defined above.

Remark 25 It follows from this definition that a deviator part in two-dimensions is of the form

$$\mathbf{D} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{R}$.

We restrict our considerations to deviator tensor fields and adopt these notations for further investigation, in particular in the solving of the eigensystem (see next).

As mentioned in the introduction, the structural analysis of tensor field is based on the topology of so-called eigenvector fields. These notions are introduced next.

3.2 Eigenvector Fields

Basically the tensor fields that we consider are matrix-valued functions. The traditional analysis of matrix systems in linear algebra focuses on the properties of the eigensystem. It determines stable and unstable subspaces of the euclidean space with respect to the considered linear map. The expression of this system is detailed next.

We first need to underline the characteristic of eigenvectors: By definition the eigensystem of a linear system with matrix \mathbf{M} can be written

$$\mathbf{M} \mathbf{e} = \lambda \mathbf{e}$$

where the eigenvector \mathbf{e} is non-zero and the eigenvalue λ is a complex number in general. Now, let \mathbf{e} be an eigenvector of \mathbf{M} with respect to the eigenvalue λ , it holds

$$\forall \nu \in \mathbb{R}, \nu \neq 0, \mathbf{M}(\nu \mathbf{e}) = \nu \mathbf{M}\mathbf{e} = \nu \lambda \mathbf{e} = \lambda(\nu \mathbf{e})$$

i.e. $\nu \mathbf{e}$ is an eigenvector associated with λ too. In other words, an eigenvector is determined modulo a non-zero scalar coefficient. Practically, this means that this kind of vector has neither norm nor orientation. This characteristic plays a fundamental role in the following.

Remark 26 In the equations above, the eigenvalues are not explicitly mentioned: This underlines the fact that the topology of tensor fields, as we are about to define it, only focuses on the structural aspects of a field and consequently neglects the quantitative information associated with eigenvalues, as it is typically interpreted in physics. We will get back to this remark during our presentation of the corresponding visualization methods in chapter 5.

Now, as far as symmetric real matrices are concerned, they possess a very useful property that greatly facilitates their study.

Property 3 *A symmetric real linear system of dimension $n \in \mathbb{N}$ has an eigenspace spanned by n orthogonal eigenvectors. Furthermore, the associated eigenvalues are real.*

Proof: see [HS74, pp. 207-208].

If we turn back to deviator matrices and reformulate the eigensystem in the following way (eliminating the eigenvalue)

$$\mathbf{D}\mathbf{e} \times \mathbf{e} = \mathbf{0}$$

(\times denotes cross product), we obtain an equation in $\mathbf{e} = (e_1, e_2)^T$

$$2\alpha e_1 e_2 + \beta(e_2^2 - e_1^2) = 0. \quad (3.1)$$

Straightforward calculus leads to the following expression for an eigenvector

$$\mathbf{e} = \begin{pmatrix} \beta \\ -\alpha \pm \sqrt{\alpha^2 + \beta^2} \end{pmatrix} \quad (3.2)$$

but for convenience, we set for some angle θ

$$\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

i.e. we consider normalized eigenvectors. Thus, Equation 3.1 can be rewritten

$$\alpha \sin 2\theta - \beta \cos 2\theta = 0 \quad (3.3)$$

The special case $\alpha = \beta = 0$ which corresponds to a zero deviator matrix \mathbf{D} leads to a singular eigensystem. In this case, every nonzero vector is an eigenvector. The case $\alpha = 0, \beta \neq 0$ leads to $\theta \equiv \frac{\pi}{4}[\frac{\pi}{2}]$ (this notation stands for $\frac{\pi}{4}$ modulo $\frac{\pi}{2}$). Otherwise, we may write

$$\tan 2\theta = \frac{\beta}{\alpha} \quad (3.4)$$

which is equivalent to

$$\theta \equiv \frac{1}{2} \arctan \left(\frac{\beta}{\alpha} \right) [\frac{\pi}{2}]. \quad (3.5)$$

A geometric interpretation is proposed in Fig. 3.1. Note that the angle solution θ is determined modulo $\frac{\pi}{2}$ which is due to the orthogonality of both

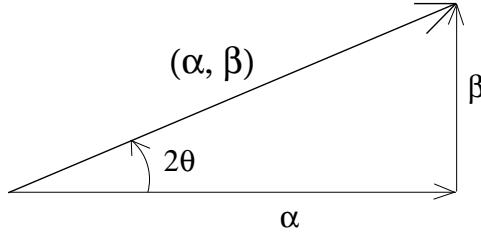


Figure 3.1: Signification of equation 3.4

eigenvectors in the symmetric real case on one hand and to the arbitrary orientation of an eigenvector in general on the other hand. Therefore, if e is an eigenvector (e.g. a normalized one), then every vector obtained by rotation of e by an angle $k\frac{\pi}{2}$ is an eigenvector too. Moreover, the special case $\alpha = 0$ corresponds to the limit case $\theta \equiv \frac{\pi}{4}[\frac{\pi}{2}]$ as $\tan 2\theta$ tends toward $\pm\infty$ in Equation 3.4.

Through its corresponding eigensystem, any symmetric real tensor field can now be associated a set of two orthogonal eigenvector fields. For convenience, the notations are as follows in the remaining of this presentation.

Definition 35 Let $\lambda_1 > \lambda_2$ be the two real eigenvalues of the symmetric tensor field \mathbf{T} (i.e. λ_1 and λ_2 are both scalar fields as functions of the coordinate vector (x, y)). The corresponding eigenvector fields e_1 and e_2 are respectively called **major** and **minor eigenvector field**.

In other words, solving the eigensystem for each position (x, y) of the considered planar domain yields two eigenvectors that are classified according to their eigenvalue. This finally defines two orthogonal (eigen-)vector fields over the domain. Once again, these eigenvector fields have neither norm nor orientation. We now resume computation and determine the eigenvalues.

$$\det(\mathbf{D} - \lambda I_2) = 0 \iff \lambda^2 - (\alpha^2 + \beta^2) = 0$$

Hence

$$\lambda_{1,2} = \pm\sqrt{\alpha^2 + \beta^2}$$

where λ_1 is the major and λ_2 the minor eigenvalue. Equation 3.4 can now be formulated in an equivalent way

$$\beta \tan^2 \theta + 2\alpha \tan \theta - \beta = 0$$

where

$$\tan \theta = \frac{+\sqrt{\alpha^2 + \beta^2} - \alpha}{\beta}$$

is the solution associated to the major eigenvector and

$$\tan \theta = \frac{-\sqrt{\alpha^2 + \beta^2} - \alpha}{\beta}$$

the one associated to the minor eigenvector.

At this stage, one should mention that the distinction between minor and major eigenvector fields is only possible if both eigenvalues are not equal. This is almost always the case but there exist singularities where this distinction is no longer possible. We should get back to this later on.

We now come to tensor lines that will be the objects of our structural analysis.

3.3 Tensor Lines

These curves are defined as follows.

Definition 36 *A tensor line computed in a Lipschitz continuous eigenvector field, is a curve that is everywhere tangent to the direction of the field.*

Because of the lack of both norm and orientation, the tangency is expressed at each position in the domain in terms of lines. For this reason, an eigenvector field can be thought of as a **line field** for our purpose. Consequently, the theorems introduced in the context of dynamical systems cannot be applied directly to ensure existence and uniqueness of such curves. Notice furthermore that the definition implies that tensor lines are only defined over regions where the scalar eigenvalue fields are not equal (see section 3.4).

These preliminary considerations allow us to deal with tensor lines defined over regions without degenerate point. Let the open U be such a region. The major and minor eigenvector fields can thus be determined at any position over U . If one focuses on a particular eigenvector field, one can find a normalized vector field \mathbf{v} that is everywhere tangent to the associated line field (see Fig. 3.2). (In fact, there are two possibilities for this vector field, depending on the arbitrary choice of the orientation.) Practically, this consists in determining a continuous angular function θ^* defined modulo 2π that is everywhere equal to the angular coordinate θ of the line field, modulo π .

$$\forall (x, y) \in U^2, \quad \theta^*(x, y) \equiv \theta [\pi]$$

Note that inconsistency in the local determination of a tangent vector field only occurs in the neighborhood of a degenerate point as we will show. But for

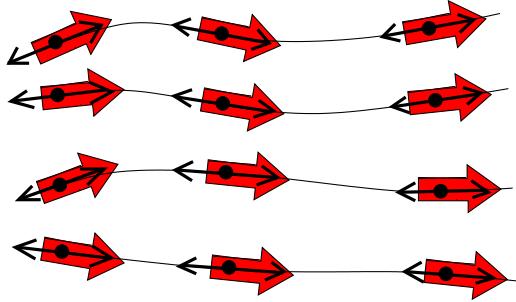


Figure 3.2: Line field and an associated normalized vector field

now, we have a well-defined vector field and the tensor lines locally agree with the integral curves of this vector field. Note that these integral curves exist and are unique (modulo the arbitrary choice of orientation for the vector field) by Theorem 1! By treating in this way every neighborhood along the path of a tensor line, we are clearly able to proceed “integration” until one either leaves the considered domain (in the bounded case) or one enters the neighborhood of a degenerate point. The argumentation here is actually the same as the one used to continue the local solutions of the Cauchy problem in Theorem 2 (i.e., one makes use of the uniqueness of the integral curves in every neighborhood and identifies the different curves in the intersections). The study of the tensor lines in the vicinity of a degenerate point is considered next.

3.4 Degenerate Points

We first give the definition suggested previously.

Definition 37 *A degenerate point of a two-dimensional second-order, symmetric tensor field is a location (x_0, y_0) where the field is isotropic: At this position, every non-zero vector is eigenvector.*

In the general case of two-dimensional symmetric real matrices, this corresponds to a diagonal matrix of the form

$$\mathbf{T}(x_0, y_0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \mathbf{I}_2$$

i.e. an isotropic tensor. In the case of a deviator matrix, this is a zero matrix by definition.

Remark 27 Like the vector case, the singularities of tensor fields have several names in practice. Indeed these points may be called **singular** or **umbilic**. The latter is taken from the formalism of differential geometry. In the following however, we adopt the terminology introduced in [Del94] and call them **degenerate points**.

The fact that every vector is an eigenvector explains why tensor lines can meet at a degenerate point. Following the same logic as in our presentation of vector fields, we start our presentation of degenerate points by considering the simplest, linear case.

3.4.1 Degenerate Points in Planar Linear Fields

If the tensor field is linear, we can interpret the deviator field

$$\mathbf{D} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

as a matrix-valued function with coefficients α and β being linear scalar functions of the coordinate (x, y) . As we saw, the position of a singularity is characterized by a zero value of the deviator. This means $\alpha(x_0, y_0) = \beta(x_0, y_0) = 0$. Hence this yields a linear system in (x_0, y_0) . If this system is non-singular, its solution is then the position of **the only singularity of the linear tensor field**. For convenience, we translate the coordinate system by taking this degenerate point as new origin. Since \mathbf{D} is zero at the origin, we obtain the following formula:

$$\mathbf{D}(x, y) = \begin{pmatrix} \alpha_1 x + \alpha_2 y & \beta_1 x + \beta_2 y \\ \beta_1 x + \beta_2 y & -\alpha_1 x - \alpha_2 y \end{pmatrix}$$

Adopting notations in polar coordinates $(x, y) = \rho(\cos \theta, \sin \theta)$, it comes

$$\mathbf{D}(x, y) = \rho \mathbf{D}\mathbf{e}_\theta$$

with $\mathbf{e}_\theta = (\cos \theta, \sin \theta)^T$. This means that the eigensystem is independent of the distance ρ to the origin. We now determine the positions where the coordinate vector \mathbf{u}_θ is parallel to the corresponding eigenvector:

$$\mathbf{D} \mathbf{e}_\theta \times \mathbf{e}_\theta = 0$$

With Equation (3.4) p. 44, this yields

$$\tan 2\theta = \frac{\beta_1 \cos \theta + \beta_2 \sin \theta}{\alpha_1 \cos \theta + \alpha_2 \sin \theta}$$

Since

$$\tan 2\theta(1 - \tan^2 \theta) = 2 \tan \theta,$$

we set $u = \tan \theta$ and obtain

$$2u(\alpha_1 + \alpha_2 u) = (1 - u^2)(\beta_1 + \beta_2 u).$$

This finally leads to the following third-order polynomial equation in $u \equiv \tan \theta$

$$\beta_2 u^3 + (\beta_1 + 2\alpha_2)u^2 + (2\alpha_1 - \beta_2)u - \beta_1 = 0. \quad (3.6)$$

The last equation has either 1 or 3 real roots that all correspond to angles along which the tensor lines asymptotically reach the origin (“asymptotically” refers to the construction used to “integrate” the tensor lines in the previous section). These angles are defined modulo π so we actually obtain 6 possible angle solutions for radial eigenvectors. Since we limit our discussion to a single (minor / major) eigenvector field, we are finally concerned with up to 3 radial eigenvectors.

The special importance of these radial tensor lines is actually explained by their interpretation as separatrices (a formal definition is given in section 3.4.2). In fact, the linear case presents two major types of linear degenerate points as shown in Fig. 3.3. The geometry of the tensor lines in the vicinity of a **trisec-**

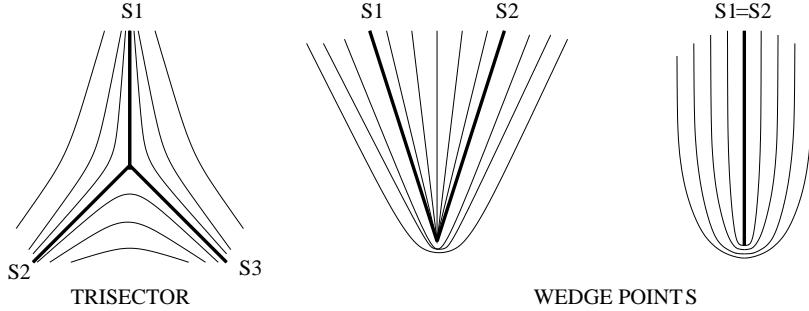


Figure 3.3: Linear Degenerate Points

tor exhibits three hyperbolic sectors (for consistency with the terminologies introduced in section 2.2.2, see also section 3.4.2). These sectors are bounded by 3 separatrices: They correspond to the three radial tensor lines determined by the polynomial equation above. As far as **wedge points** are concerned, there exist two types: The first one has two separatrices that bound a hyperbolic and a parabolic sector. The second one has a hyperbolic sector that is bounded by a single separatrix. This separatrix actually represents a parabolic sector reduced to a single tensor line. A wedge point with two separatrices is related to a situation where 3 roots exist to the polynomial in Eq. 3.6: Two

correspond to actual separatrices and the third lies within the parabolic sector. A wedge with one single separatrix occurs when the polynomial has a unique real root.

Remark 28 *The depiction of a wedge point with two separatrices in Fig. 3.3 could suggest that infinitely many radial lines exist within the parabolic sector. Yet this is impossible in the linear case since these radial directions are determined by a cubic polynomial. As a matter of fact, such a polynomial can only have infinitely many roots if all coefficients are zero. In our case, this leads to $\alpha_1 = \alpha_2 = \beta_1 = \beta_2$: This is a degenerate case where linear approximation is no longer valid, as we explain next. Hence, actual tensor lines in a parabolic sector looks here like those shown in Fig. 3.4.*

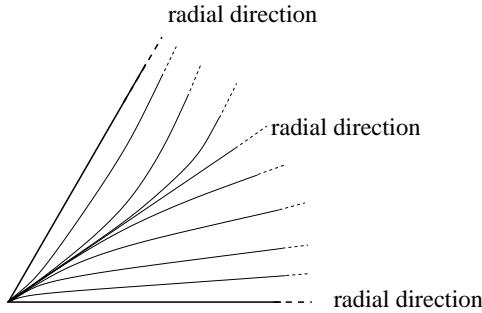


Figure 3.4: Tensor lines in the parabolic sector of a wedge point

3.4.2 Nonlinear Degenerate Points

The configurations encountered so far are the simplest examples of degenerate points that exist in tensor fields. They are valid when the linear element in the Taylor expansion of the tensor is nowhere isotropic in the vicinity of the singularity. In other words, let $\nabla\mathbf{D}(O)$ be the third-order differential of the deviator \mathbf{D} at the origin O , that coincides with the position of the degenerate point. The Taylor expansion of \mathbf{D} in the vicinity of the degenerate point is thus

$$\mathbf{D}(dX) = \mathbf{D}(O) + \nabla\mathbf{D}(O)dX + o(dX).$$

Since by hypothesis $\mathbf{D}(O) = 0$, the eigensystem at any point dX can be written

$$(\nabla\mathbf{D}(O)dX) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \times \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + o(dX) = 0.$$

Hence, the structure of the eigenvector field can be linearly approximated in the vicinity of O if and only if

$$\forall dX \in U, (\nabla \mathbf{D}(O)dX) \neq \mathbf{0}$$

(where U is a vicinity of the origin O) since we are dealing with deviators (a deviator is isotropic if and only if it is zero). With our notations we have

$$\nabla \mathbf{D}(O)dX = \begin{pmatrix} \frac{\partial \alpha}{\partial x}dx + \frac{\partial \alpha}{\partial y}dy & \frac{\partial \beta}{\partial x}dx + \frac{\partial \beta}{\partial y}dy \\ \frac{\partial \beta}{\partial x}dx + \frac{\partial \beta}{\partial y}dy & -\frac{\partial \alpha}{\partial x}dx - \frac{\partial \alpha}{\partial y}dy \end{pmatrix}.$$

Hence this is non-zero for any $dX \neq \mathbf{0}$ if and only if

$$\begin{vmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{vmatrix} \neq 0 \quad (3.7)$$

(i.e. if the corresponding matrix is non-singular). A zero value of the determinant above thus characterizes nonlinear degenerate points.

Remark 29 *In the linear case, equation 3.7 is equivalent to*

$$\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0.$$

Therefore, every point on the line through the degenerate point of equation $\alpha_1x + \alpha_2y = 0$ is a degenerate point (degenerate case).

Remark 30 *The determinant above is called δ in [Del94] and is used to distinguish between what Delmarcelle calls simple and multiple degenerate points.*

To study the geometric properties of the tensor lines in the vicinity of a nonlinear degenerate point, we now involve new theoretical material. This permits to overcome the difficulty induced by the lack of orientation around a singularity. Practically, the goal is to relate the line field to a vector field that characterizes its geometrical structure. The ideas described next are taken from [Spi79, pp. 324-332].

As a preliminary, some notions of algebraic topology are required. For a general introduction to this subject, refer to [Jän96] or [Gra75]. In the following we consider topological spaces. Their formal definition is given next.

Definition 38 *A topological space is a pair (X, O) where X is a set and O is a set of subsets of X called open sets such that*

- (i) *Every union of open sets is open.*

- (ii) *The intersection of two open sets is open.*
- (iii) \emptyset and X are open.

We are interested in the relation between topological spaces.

Definition 39 *Let X be a topological space. A **topological space over X** is defined as a pair (Y, π) , composed of a topological space Y and a continuous surjective map $\pi : Y \rightarrow X$. This pair is simply denoted Y for convenience. Moreover $\pi^{-1}(x)$ is denoted Y_x and is called the **fiber** over x . The restriction for an open U ($\pi^{-1}(U), \pi|_{\pi^{-1}(U)}$) is written $Y|U$.*

We now consider a special type of topological space.

Definition 40 *A topological space Y over X is **trivial** if there exists a topological space F such that Y is isomorphic to the canonical projection of $X \times F$ onto X .*

If one relaxes this condition, one defines local triviality.

Definition 41 *A topological space Y over X is called a **locally trivial bundle** with fiber if for each $x \in X$ there is a neighborhood U of x such that $Y|U$ is trivial.*

The case where the fibers are discrete is of particular interest to us.

Definition 42 *A locally trivial bundle with fiber is called **covering space** if all fibers are discrete. For simply connected spaces X , the cardinality of the fibers is constant. If every point of X is covered n times, the covering space is called **n -fold covering**.*

An illustration of a covering space is proposed in Fig. 3.5. This basic type of covering space can actually be extended to include coverings without local triviality. We proceed by defining branched covering spaces.

Definition 43 *If a topological space Y over X is everywhere locally trivial with discrete fiber except at some points p_i where it is not a homeomorphism, it is called a **branched covering space** and the points p_i are its **branch points**.*

A simple example of branched covering space is provided by the map $z \mapsto z^2$ that takes $\{z \in \mathbb{C}, 0 \leq |z| < 1\}$ onto itself. This induces a 2-fold branched covering space where 0 is the branch point. Next we have the following fundamental **path lifting property**.

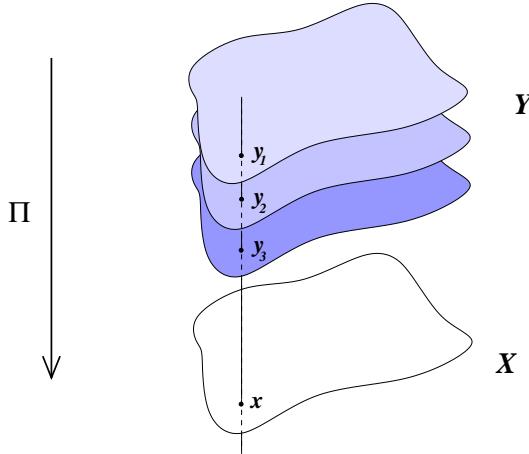


Figure 3.5: A 3-fold covering space

Property 4 Let (Y, π) be a covering space of X . Let $c : I \subset \mathbb{R} \rightarrow X$ be a curve on X and $a \in Y$ such that $\pi(a) = c(0)$. Then there is a unique curve $c' : I \rightarrow Y$ such that $\pi c' = c$ and $c'(0) = a$.

Proof: See [Gra75, pp. 35-36].

With these new notions we can now turn back to the geometric structure of line fields in the vicinity of a degenerate point. We consider the vicinity U of an arbitrary degenerate point p . The line field Δ is then defined over $U - \{p\}$ and associates every point q with a one-dimensional subspace (except zero):

$$\begin{aligned} \Delta : U - \{p\} &\longrightarrow \mathbb{R}^2 \\ q &\longmapsto \{\mu v, \mu \in \mathbb{R} - \{0\}, |v| = 1\}. \end{aligned}$$

We now construct a 2-fold covering space ω where π^{-1} associates every point q in $U - \{p\}$ with the two unit vectors in $\Delta(q)$. This map is locally equivalent to the map $\pi : z \mapsto z^2$ mentioned previously and p is the branch point. It maps a vector field defined over $\pi^{-1}(U - \{p\})$ onto the line field around p . Hence if $\theta : u \in [0, 2\pi[\rightarrow \mathbb{R}$ is the angular coordinate of this vector field along some arbitrary closed curve around p , the angle coordinate ϕ of the line field obtained by projection π satisfies

$$\phi(2u) = \theta(u) + 2u.$$

This means that we wrap the phase portrait around the origin. See also Property 8. These definitions are better understood when looking at Fig. 3.6.

Note that this construction can be extended to the whole definition space M of a line field Δ except at the locations $\{p_i\}$ of the singularities. If one

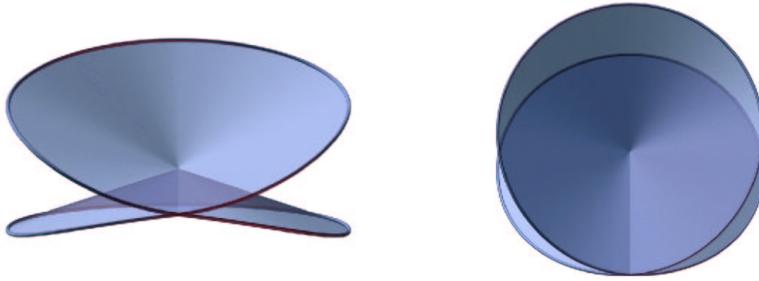


Figure 3.6: 2-fold branched covering space over the vicinity of a degenerate point

denotes by M' the space $\pi^{-1}(M - \{p_1, \dots, p_k\})$, it can be completed by adding new points $\{p'_1, \dots, p'_k\}$. For each p'_i , the neighborhood is defined as the set $p'_i \cup \pi^{-1}(A_i - \{p_i\})$ with A a neighborhood of p_i in M . If one deals with a line field defined on a manifold M , then the set obtained by adding all the new points to M' is a manifold too, see [Spi79]. At this stage the path lifting property cited above permits to relate the geometry of the tensor lines in the vicinity of a degenerate point to the phase portrait of the corresponding critical point obtained in the covering space. Two simple examples related to the linear case are given next. Fig. 3.7 shows the vector field corresponding to a wedge point that is defined in the covering space. Fig. 3.8 explicits this relation for a trisector point.

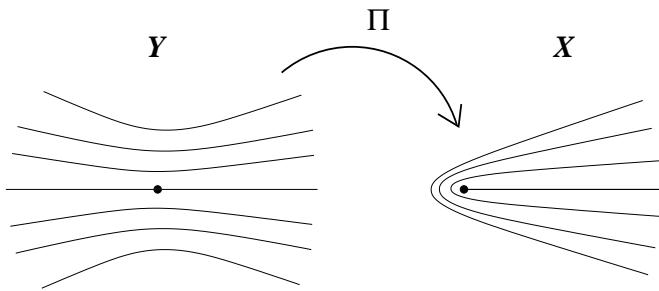


Figure 3.7: Covering vector field for a wedge point

Practically, the characterization of the local structure in terms of curvilinear sectors (corresponding to different asymptotic behaviors of the curves in the neighborhood of a critical point) can be used in this context to determine the type of a degenerate point. Hence, the possible natures of a sector are *parabolic*, *hyperbolic* or *elliptic* and the neighborhood of any degenerate point is formed by the union of such sectors, see Fig. 3.9. This relation also suggests the definition of the separatrix of a degenerate point.

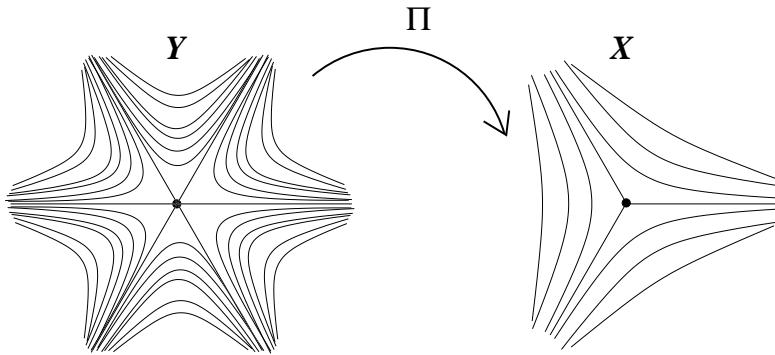


Figure 3.8: Covering vector field for a trisector point

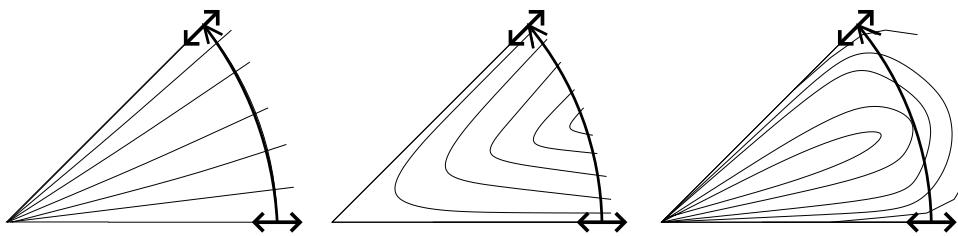


Figure 3.9: Sector types

Definition 44 *The boundary curve of a hyperbolic sector in the vicinity of a degenerate point is called **separatrix**. The set of all degenerate points together with their associated separatrices is called **topology of the symmetric tensor field** for further considerations.*

In fact, a complete definition of the topology involves **closed tensor lines** too. They are the direct equivalent of the closed orbits observed in vector fields. However they are very rare in practice and will not be considered by the visualization methods to come. Further details on that topic can be found in [Del94, pp. 139-146].

We need to extend an additional fundamental notion of dynamical systems that will prove similarly useful in the context of tensor fields.

3.5 Tensor Index

From the study of vector fields we know that the index of a critical point permits to characterize its action in the global structure of the field by counting the number of field rotations induced in its vicinity. We introduce next an

equivalent notion for tensor fields [Spi79].

In the following, one supposes for convenience that the degenerate point is located at the origin O . \mathbb{P}^1 denotes the projective line corresponding to the set of all lines through $0 \in \mathbb{R}^2$. S^1 is the sphere with unit radius around 0 in \mathbb{R}^2 and i the map on S^1 defined by $i(x) = \epsilon x$ for some $\epsilon > 0$. f_Δ is the map that associates every point on ϵS^1 with the corresponding line direction of the line field Δ in \mathbb{P}^1 . Furthermore, one denotes by α the homeomorphism that identifies \mathbb{P}^1 and S^1 as shown in Fig. 3.10. We can then define the index of a

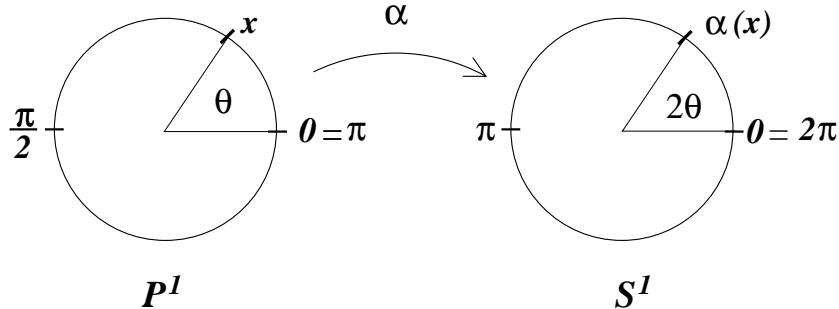


Figure 3.10: Homeomorphism between \mathbb{P}^1 and S^1

degenerate point as follows.

Definition 45 *The index of Δ at p_0 is defined as*

$$\frac{1}{2} \text{ degree } \alpha \circ f_\Delta \circ i$$

where the degree of a map β of S^1 onto itself at a non-singular point p is the quantity

$$\text{degree } (\beta, p \in S^1) = \sum_{q \in \beta^{-1}(p)} \text{sign } d\beta_q$$

which we denote degree β since this is constant over a simply connected neighborhood of p_0 . The sign of the differential $d\beta_q$ is 1 if it is orientation-preserving, -1 otherwise.

Remark 31 *The definition above is quite abstract. In fact, the degree of a map is the constant number of coverings of any non-singular point p with respect to β with sign depending on the orientation. Further details on that subject can be found in [Mil65].*

If the line field is spanned by a vector field v then the definition above agrees with the one given in the previous chapter, section 2.5. To see this,

let f_v be the map that associates every point q in a neighborhood U of the degenerate point p with the unit vector $\frac{v(q)}{|v(q)|} \in S^1$. Let π be the natural projection of S^1 onto \mathbb{P}^1 . Then one has the diffeomorphism

$$\pi \circ f_v \circ i \simeq f_\Delta \circ i$$

where both maps are defined from S^1 onto \mathbb{P}^1 . Since the map $\alpha \circ \pi$ has degree 2, one gets

$$\begin{aligned} \text{index of } \Delta \text{ at } p &= \frac{1}{2} \text{ degree } \alpha \circ f_\Delta \circ i \\ &= \frac{1}{2} \text{ degree } \alpha \circ \pi \circ f_v \circ i \\ &= \text{degree } f_v \circ i \\ &= \text{index of } v \text{ at } p \end{aligned}$$

The geometric interpretation of the index is again the number of field rotations along a closed curve enclosing the degenerate point, arbitrary close to it and traveled in positive direction. The illustration is proposed in Fig. 3.11. Note that the lack of orientation entails the existence of half-integer values. From

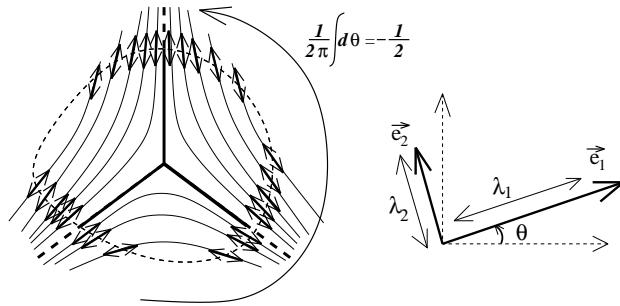


Figure 3.11: Tensor index

this observation, we come to the notion of index of a region enclosed by a curve just as for vectors.

Definition 46 *Let Δ be a Lipschitz continuous line field defined over an open $U \subset \mathbb{R}^2$. Let L be a simple closed curve in U such that no degenerate point of Δ is on L . Then the **index of the curve** L relative to Δ is defined as the number of positive field rotations obtained while rotating once along L in positive direction.*

This quantity is related to the following fundamental property.

Property 5 *Let p_1, \dots, p_k be the degenerate points of a Lipschitz continuous line field Δ contained in some region U of \mathbb{R}^2 . Then the index I_L of a closed curve*

L enclosing the $\{p_i\}$ relative to Δ depends on the indices $\{I_\Delta(p_i)\}$ of the $\{p_i\}$ as follows.

$$I_L = \sum_{j=1}^k I_\Delta(p_j)$$

Proof: See [Del94, pp. 106-107].

With this property, it is clear that the notion of index enables the description of the qualitative nature of a whole region: It “averages” the individual local contributions of all degenerate points in it. This is particularly useful in the study of structural stability, see section 3.6.1.

If one focuses on the linear case, one has the following useful property.

Property 6 *Let D be a linear (deviator) tensor field. The angle variation of D along any segment $[A, B]$, $A, B \in \mathbb{R}^2$, is smaller than $\frac{\pi}{2}$.*

Proof: according to equation 3.4, the angle θ (defined modulo π) satisfies

$$\tan 2\theta = \frac{\beta}{\alpha}$$

which entails the following differential equation

$$d\theta = \frac{1}{2} \frac{\alpha d\beta - \beta d\alpha}{\alpha^2 + \beta^2}.$$

By hypothesis D is linear and so are the scalar functions α and β . Hence, if we consider segment $[A, B]$, we can define a parametrization $t \in [0, 1] \rightarrow [A, B]$ such that α and β are linear functions of t along $[A, B]$, say $\alpha(t) = \alpha_0 + \alpha_1 t$, idem for β . Thus we get for the angle variation along $[A, B]$

$$\int_A^B d\theta = \frac{\alpha_0 \beta_1 - \alpha_1 \beta_0}{2} \int_0^1 \frac{dt}{at^2 + bt + c}$$

where a, b and c are functions of $\alpha_{0,1}$ and $\beta_{0,1}$. Furthermore, the discriminant $\Delta = b^2 - 4ac$ is negative. Therefore it follows (after calculus)

$$\int_A^B d\theta = \frac{\text{sign}(\alpha_0 \beta_1 - \alpha_1 \beta_0)}{2} (\text{atan}\mu_1 - \text{atan}\mu_0)$$

where μ_0 and μ_1 are two real scalars that depend on $\alpha_{0,1}$ and $\beta_{0,1}$. Since the function atan maps \mathbb{R} onto the open set $(-\frac{\pi}{2}, \frac{\pi}{2})$, we finally obtain

$$\left| \int_A^B d\theta \right| < \frac{1}{2}\pi$$

which completes the proof.

With this property, we can determine the index of a linear degenerate point.

Property 7 A linear degenerate point has either index $\frac{1}{2}$ or $-\frac{1}{2}$.

Proof: According to property 5, the index of a degenerate point can be computed on any closed curve that encloses no other degenerate point. In the linear case we can thus consider a closed curve composed of three segments (i.e. a triangle surrounding the degenerate point). It follows from the previous property that the angle variation along each segment is strictly smaller than $\frac{\pi}{2}$ (in absolute value). Consequently, the global angle change around the linear degenerate point is smaller than $\frac{3\pi}{2}$ and the corresponding index strictly smaller than $\frac{3}{4}$. The geometric interpretation of the index tells us that the actual value must be a multiple of a half-integer (by continuity of the eigenvector field along the curve). Since a linear degenerate point is present the index is nonzero and the possible values are finally $\frac{1}{2}$ and $-\frac{1}{2}$. QED

Practically a trisector has index $-\frac{1}{2}$ while a wedge point has index $+\frac{1}{2}$ (compare with Fig. 3.3). If we return to the presentation above, the tensor lines are interpreted as the projection of the phase portrait of a vector field defined in a 2-fold branched covering space. The respective indices are related by the following

Property 8 Let $\pi : Y \rightarrow X$ be a 2-fold branched covering space and let p be a branch point. Let Δ be a line field defined on a neighborhood of p except at p itself and let Δ' be the associated distribution defined on a neighborhood of $\pi^{-1}(p)$, except at $\pi^{-1}(p)$ itself satisfying $\pi\Delta' = \Delta$. Then the index i' of Δ' at $\pi^{-1}(p)$ is related to the index i of Δ at p by

$$i' = 2i - 1.$$

Proof: The demonstration of this property is based on the analogy with the map $z \mapsto z^2$. See [Spi79, pp. 328-329].

Remark 32 Compare with Fig. 3.7 and Fig. 3.8 shown previously: The critical point associated with a wedge point ($i = \frac{1}{2}$) has index $i' = 0$ whereas the critical point corresponding to a trisector point ($i = -\frac{1}{2}$) has index $i' = -2$ (monkey saddle).

Notice that this relation permits further to compute the Euler characteristic $\chi(M')$ of the manifold associated with a covering space [Spi79].

3.6 Parameter-Dependent Topology

By analogy with the presentation of dynamical systems, the natural question that arises at this stage is the structural stability of topology and degenerate

points as defined so far. Indeed, if an additional parameter (say time) is involved in the definition of the considered tensor field, the structures identified in previous considerations only correspond to instantaneous states of the evolving topology. Hence the fundamental question is the persistence of degenerate points under small perturbations of the parameter. This defines local stability. In this short presentation, we restrict our considerations to the simplest cases of local and global bifurcations to remain in the scope of the methods to come. A general theoretical treatment of these questions would require to properly extend in this context the knowledge gained about structural stability in the vector case. A very appealing way is to study the topological properties of the manifold associated with the covering space (see previous section).

3.6.1 Structural Stability

The observations proposed next are all based on geometric considerations. Yet a formal approach could explore structural stability from the viewpoint of its relation with a vector field defined in a proper covering space.

Degenerate Points

Following the ideas suggested by Peixoto's theorem (section 2.6) we first characterize stable degenerate points. As for critical points, the singularities obtained in the linear, non-singular case are the only stable ones. As a matter of fact, when considering the tensor lines in the neighborhood of a degenerate point, one can approximate their asymptotic behavior by the third-order differential $\nabla\mathbf{D}(O)$ except at locations where this differential is zero at some points in the vicinity of the degenerate point. It was shown in section 3.4.2 that this occurs if and only if

$$\begin{vmatrix} \frac{\partial\alpha}{\partial x} & \frac{\partial\alpha}{\partial y} \\ \frac{\partial\beta}{\partial x} & \frac{\partial\beta}{\partial y} \end{vmatrix} = 0.$$

This is by essence an unstable property since it corresponds to a degenerate case and thus arbitrary small perturbations in the coefficients of the differential $\nabla\mathbf{D}$ leads to one of both linear configurations. The stability of the linear degenerate points themselves is also related to $\nabla\mathbf{D}$. Namely as shown in [Del94], the value of the determinant above characterizes the type (trisector / wedge) of a degenerate point: A negative value corresponds to a trisector and a positive value corresponds to a wedge. Thus only a crossing of this determinant through zero can enforce the transition from a trisector point to a wedge which is again, a degenerate case. Therefore trisector and wedge are

both stable types. If we further investigate the wedge points, we may be interested in the possible instability of one of both types (with one single separatrix or two separatrices and an associated parabolic sector). We saw in 3.4 that the angle coordinates of the separatrices are solutions of a third order polynomial (Equation 3.6). The wedge point with two separatrices is thus associated with the case where three real roots exist and the wedge point with one separatrix occurs when the polynomial equation has one single real root. Transition between both takes place when a perturbation entails the change of this number of real roots. This is clearly an extraordinary fact and it separates both types from another in a stable way. Note that these ideas are used again in chapter 7.

Sepatrices

In the vector case we learned from Peixoto's theorem that separatrices connecting saddle points with another are unstable. The same holds in the tensor case for the separatrices that lie on the boundary of hyperbolic sectors at both ends. Examples are shown in Fig. 3.12. They show the heteroclinic case by analogy with previous terminologies. (Homoclinic connections are possible too.) The

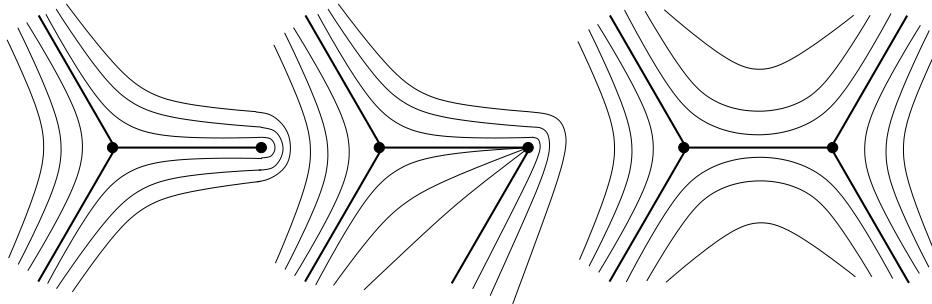


Figure 3.12: Heteroclinic connections in the tensor case

justification of this assertion is of geometric nature. It is based on the fact that adding an arbitrary small angle perturbation to the line field around any point along such a separatrix results in breaking the heteroclinic (resp. homoclinic) connection. See [GH83, pp. 61-62].

These brief explanations allows us to discuss our actual focus, namely simple local and global bifurcations in tensor fields. We start with the former ones.

3.6.2 Local Bifurcations

We consider here the bifurcations associated with the instability of degenerate points as defined previously. It was shown that such bifurcations occur when the differential ∇D becomes zero or when a special condition is fulfilled with respect to the number of roots to the polynomial equation 3.6. First, we present the bifurcation corresponding to the saddle-node bifurcation for vectors.

Pairwise Creation and Annihilation

A wedge and a trisector have opposite indices. Therefore a closed curve enclosing a trisector and a wedge has index 0. This simple fact is the basic idea behind pairwise creations or annihilations. As a matter of fact, the index zero computed along this closed curve shows that the combination of both degenerate points is structurally equivalent to a uniform flow. This local transition from a uniform flow to a line field with a wedge and a trisector is a **pairwise creation**. The reverse bifurcation (by inversion of the parameter) is thus a **pairwise annihilation** and entails the disappearance of both a wedge and a trisector. It can occur in several forms. Illustrations are proposed in Fig. 3.13 and Fig. 3.14. The former case corresponds to the disappearance of a separatrix of the trisector in the parabolic sector of the wedge. It results in the disappearance of two hyperbolic sectors and one parabolic one. The latter

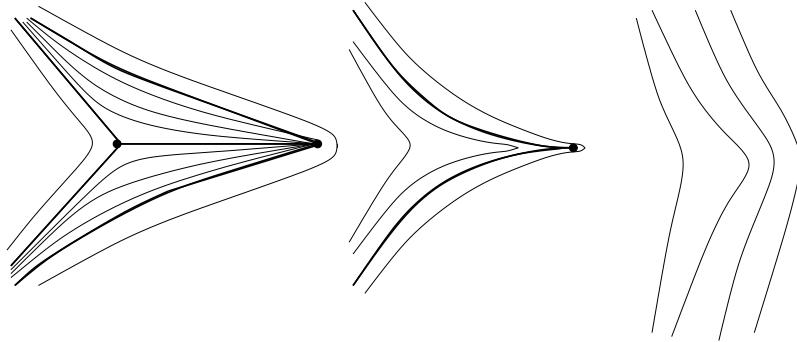


Figure 3.13: Pairwise annihilation by disappearance of the wedge's parabolic sector

case occurs when a hyperbolic sector of the wedge merges with a hyperbolic sector of the trisector. Observe the presence of a nonlinear singularity at both bifurcation points.

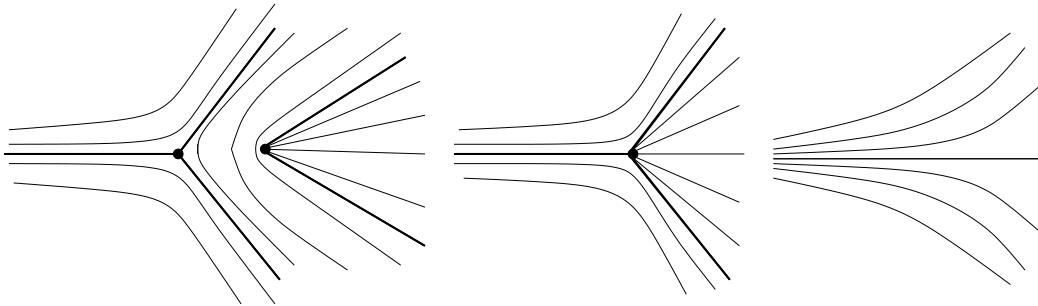


Figure 3.14: Pairwise annihilation by disappearance of the wedge's hyperbolic sector

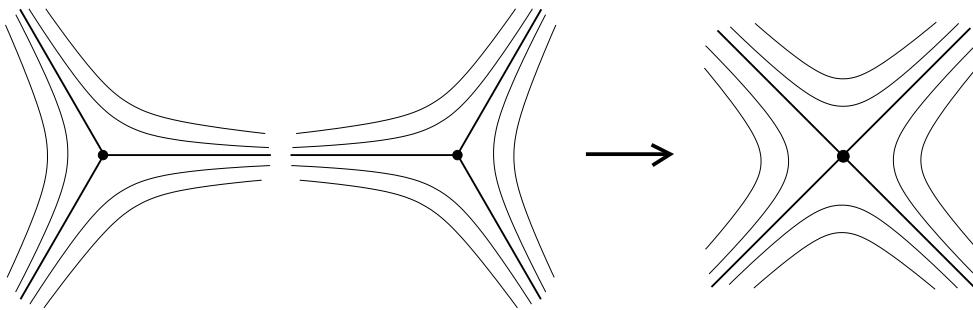


Figure 3.15: Merging of two trisectors

Homogeneous Mergings

We mention here bifurcations corresponding to the pairwise merging of degenerate points of same type. Additional details can be found in [Del94, pp. 128-131]. We know from what precedes that linear degenerate points of tensor fields have half-integer indices. Hence, when they come to merge with another singularity of the same type, the corresponding global index in the sense of Property 8 is an integer and one can show that this corresponds to the local phase portrait of a critical point, see [Spi79, pp. 326-327]. This becomes clear when looking at the merging of two trisectors as shown in Fig. 3.15. The fusion results in a singularity similar to a saddle point. This singularity has index $-\frac{1}{2} - \frac{1}{2} = -1$ as expected. According to previous remarks on structural stability, this new degenerate point is unstable. Consequently this bifurcation must be seen as the creation from two trisectors of a saddle that instantaneously vanishes to restore the two trisectors: It is a walk through.

The other kind of merging that we consider concerns two wedges. It is illustrated in Fig. 3.16. The result of this merging is an unstable singularity

that is similar to a focus (a node in the general case). It has index $\frac{1}{2} + \frac{1}{2} = +1$. Here this instantaneous configuration is replaced by two wedges after bifurcation.

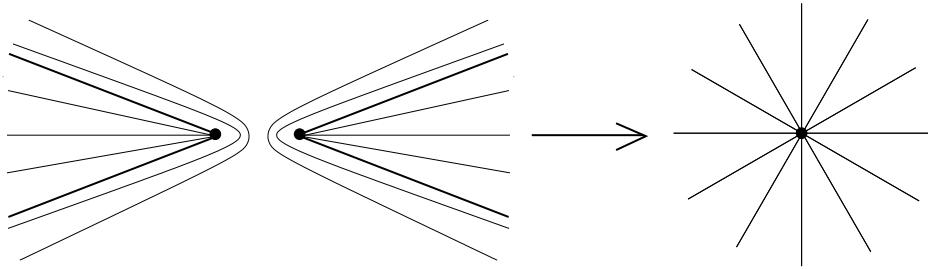


Figure 3.16: Merging of two trisectors

The last local bifurcation to be presented is not associated with a merging.

Wedge Bifurcation

This type of bifurcation was suggested by the remarks on the structural stability of wedge points: Each type of wedge corresponds to a specific number of real roots of a cubic polynomial, either 1 or 3. Now the transition from one type to another, say from a wedge with one separatrix to a wedge with two separatrices, means the appearance of a parabolic sector. From a structural viewpoint, this is a major change because from now on curves reach the wedge point that were sweeping past it so far. Refer to Fig. 3.17.

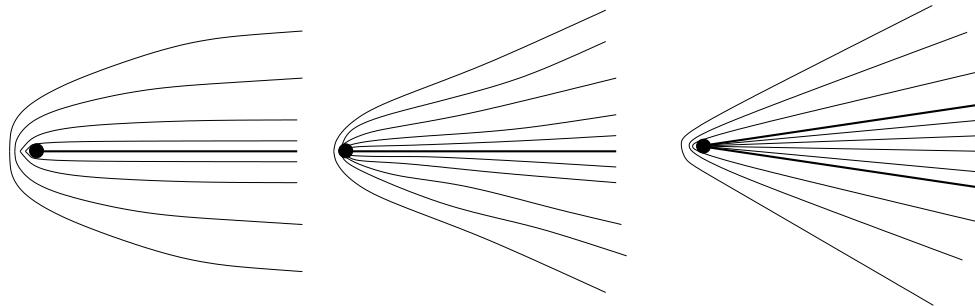


Figure 3.17: Creation of a parabolic sector in the vicinity of a wedge point

3.6.3 Global Bifurcations

To finish this presentation of structural transitions in line fields, we briefly consider the simplest types of global bifurcations. They are very similar to

the basin bifurcations encountered previously. They are intimately related to the unstable separatrices defined above: They occur when two separatrices emanating from two degenerate points come closer together to merge and then split. At the instant of merging, an unstable heteroclinic connection exists. As it breaks, it forces the swap of both separatrices. This modifies the asymptotic behavior of most curves in the concerned region. An example is proposed in Fig. 3.18 that involves 2 trisectors. Another one is given in Fig. 3.19 that is based on a heteroclinic connection between a trisector and a wedge. Compare to Fig. 2.28, p. 37.

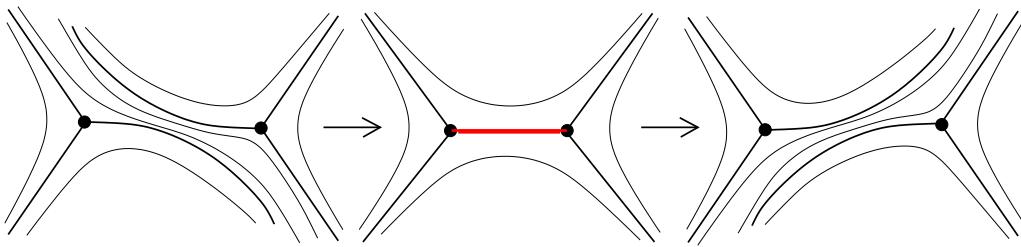


Figure 3.18: Global bifurcation with heteroclinic connection

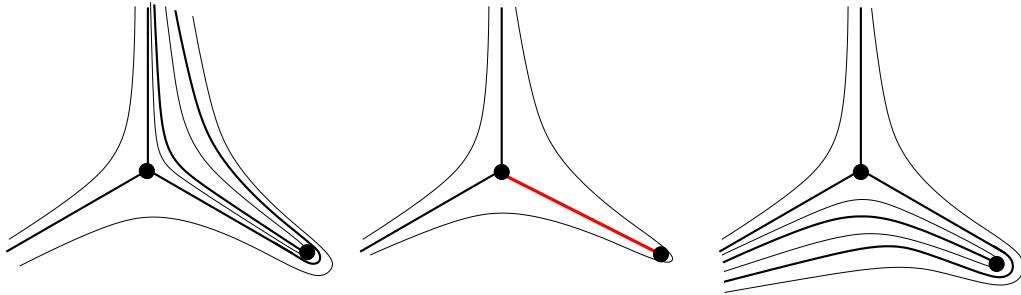


Figure 3.19: Global bifurcation with heteroclinic connection (cont.)

Chapter 4

Piecewise Interpolation and Singularities

Grids are at the basis of most methods in Scientific Visualization. They provide discrete numerical data (resulting from measurements or numerical simulations) with a geometric structure which enables a piecewise analytic interpolation. In this chapter, one offers a brief description of the grid structures typically used in practice. Furthermore, the associated interpolation schemes are presented and their computation is detailed. This material is finally used to consider piecewise interpolation from the point of view of critical and degenerate points. In particular, it is shown that singularities with arbitrary complex structure may be encountered and modeled in piecewise linear fields.

4.1 Grid Types

A grid represents the partition of a bounded domain of the euclidean space into subdomains called cells, based upon a point distribution in the domain. Basic definitions are given next. For an overview of this subject, see further e.g. [Nie97].

A **cell** in the euclidean space \mathbb{R}^n is defined by a set of points, called **vertices**, that determine its geometry. More precisely, a cell is built of a union of **simplices** of its vertices, where a simplex of $n + 1$ points P_i in \mathbb{R}^n is the set

$$S = \left\{ P \in \mathbb{R}^n, P = \sum_{i=0}^n \lambda_i P_i, \text{ with } \forall i \lambda_i \geq 0 \text{ and } \sum_{i=0}^n \lambda_i = 1 \right\},$$

i.e. the set of all affine convex combinations of the P_i . Note that the P_i must ensure non-zero volume for the corresponding set S . In practice, one

uses mostly the few cell types enumerated below. Note that these types are typical in Scientific Visualization. Yet cells with more complicated geometries are common in finite elements methods, see e.g. [Hue75].

1. Two-dimensional case:

- A **triangle** has 3 vertices: This is a simplex in \mathbb{R}^2 .
- A **quadrilateral** has 4 vertices and is made of two triangles. If all inner angles at the vertices of a quadrilateral are right angles, the cell is called **rectangle**.

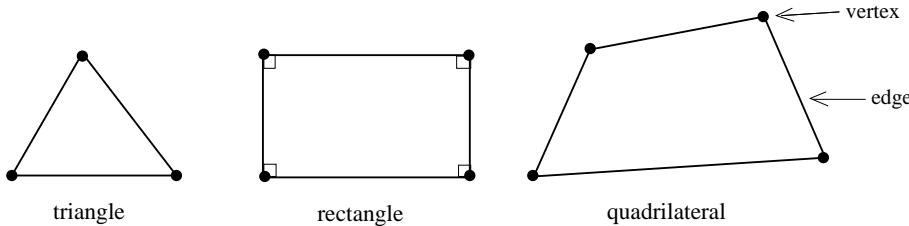


Figure 4.1: Cells in two dimensions

2. Three-dimensional case:

- A **tetrahedron** has 4 vertices: This is a simplex in \mathbb{R}^3 .
- A **prism** has 6 vertices and is made of 3 tetrahedrons.
- A **hexahedron** has 8 vertices and can be decomposed in 5 or more tetrahedrons. If the boundary of a hexahedron consists of rectangles, the cell is called a **voxel**.

One calls **face** the two-dimensional intersection of two neighbor cells. The one-dimensional intersection of two neighboring cells is called an **edge**. These notions are illustrated in Fig. 4.1 and Fig. 4.2.

A **grid** is defined as a finite set of cells. The grid G associated with a set of points must then fulfil the following requirements:

- (i) Every point is a vertex of a cell of G .
- (ii) The interiors of two cells do not intersect.
- (iii) The union of the cells of G is the whole domain.

Among all possible grid types, the simplest one is the rectilinear grid: A **rectilinear grid** is a grid where all cells are rectangles (in two dimensions) or voxels (in three dimensions): see Fig. 4.3.

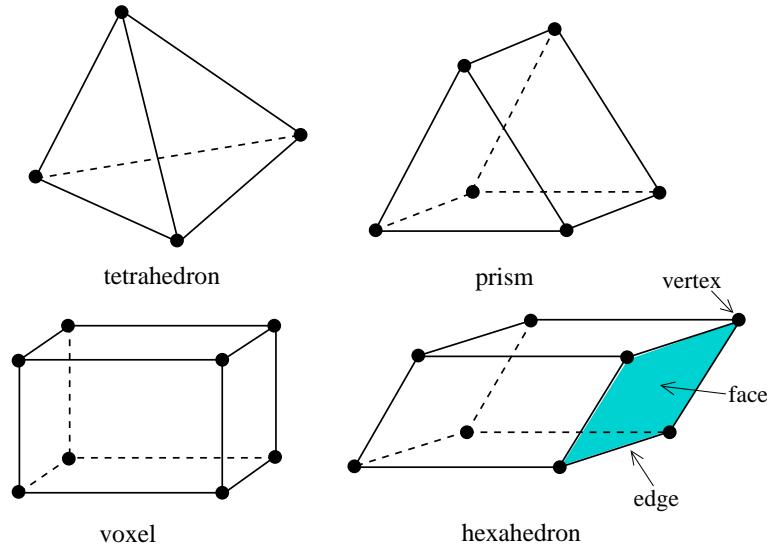


Figure 4.2: Cells in three dimensions

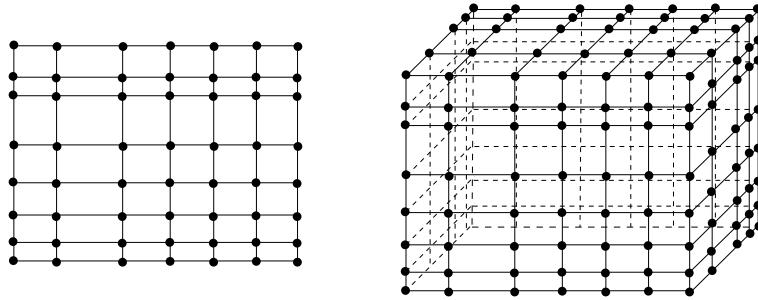


Figure 4.3: Rectilinear grids

A **curvilinear grid** is the generalization of a rectilinear grid: It consists of quadrilateral cells in two dimensions or hexahedra in three dimensions (see Fig. 4.4) and there exists a diffeomorphism ϕ that transforms it in a rectilinear grid. One distinguishes then between **physical space**, the euclidean space where a grid is curvilinear, and the **computational space** where the grid is rectilinear (see Fig. 4.5).

The diffeomorphism ϕ can be defined cell-wise as the function that maps the actual quadrilateral (resp. hexahedral) cell onto the canonic voxel $[0, 1]^2$ (resp. $[0, 1]^3$). Furthermore, to preserve the shape of the cell, $\psi = \phi^{-1}$ is required to be linear along the edges of the canonic voxel which ensures that edges are mapped onto edges. If one considers a quadrilateral $ABCD$ in the

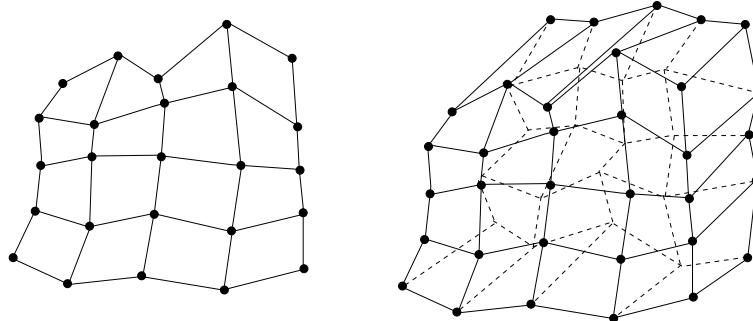


Figure 4.4: Curvilinear grids

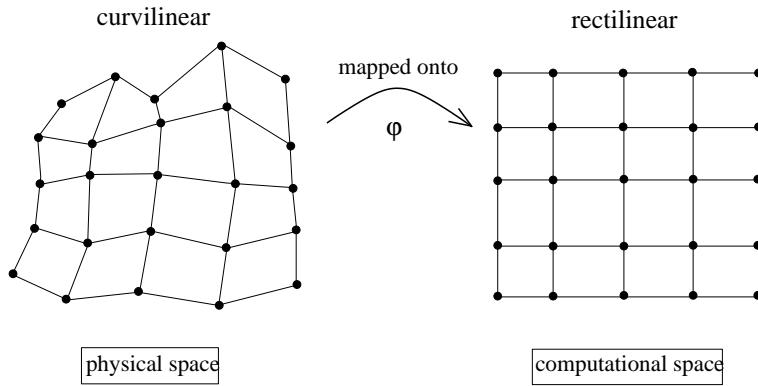


Figure 4.5: Physical and computational spaces

two-dimensional case, one obtains the following formula.

$$\begin{aligned} \psi : \quad \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (r, s) &\mapsto A + (B - A)r + (D - A)s + (A - B + C - D)rs \end{aligned}$$

In the three-dimensional case, with a hexahedron $ABCDEFGH$, the results are analogous:

$$\begin{aligned} \psi : \quad \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ (r, s, t) &\mapsto A + (B - A)r + (D - A)s + (E - A)t + (A - B + C - D)rs \\ &\quad + (A - B + F - E)rt + (A - D + H - E)st \\ &\quad + (-A + B - C + D + E - F + G - H)rst \end{aligned}$$

The function ϕ is then computed as the inverse of ψ : In the two-dimensional case, this problem is quadratic and can be solved analytically. The problem is indeed equivalent to the search for a critical point in a rectangle cell, as described in the following. In three dimensions, the problem is more complicated and a numerical search must be carried out.

If a grid cannot be transformed into a rectilinear grid, it is said to be **unstructured**. Such a grid has no global regular structure and can contain cells of different types. In general, it is related to a set of scattered points, i.e. without underlying point structure. A typical example is provided by a triangulation in two dimensions, resp. a tetrahedrization in three dimensions: These grids are exclusively made of simplices, see Fig. 4.6 for the planar case.

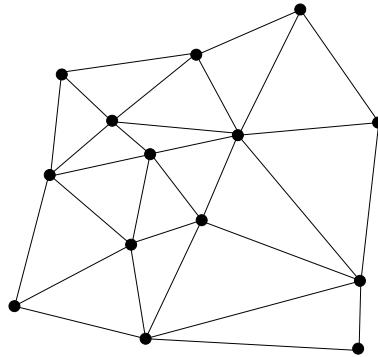


Figure 4.6: Unstructured grid: A triangulation

4.2 Interpolation Schemes

Once a geometric structure has been provided, an interpolation scheme must embed the original discrete information in a continuous function, defined over the whole grid. This continuous data is eventually processed for visualization. More precisely, the interpolation is processed cell-wise and the interpolant over the whole grid is thus defined as the union of these local cell interpolants. Therefore a good knowledge of the properties of an interpolation scheme is required for the design of visualization methods that apply to its resulting function. For this reason, special attention is paid in the following to the singularities in the planar case. The usual interpolation schemes associated with the introduced cell types are considered now.

4.2.1 Linear Interpolation over a Triangle

For each scalar component of the considered field (vector or tensor), a linear interpolant over a triangle T with vertices P_0, P_1, P_2 and associated values $\alpha_0, \alpha_1, \alpha_2$ is defined as the real function

$$\forall P(x, y) \in T, f(P) = a + bx + cy,$$

where a, b and c are the real coefficients of the interpolant. They are chosen to satisfy the following relations.

$$\forall i \in 0, \dots, 2, f(P_i) = \alpha_i \text{ that is } \begin{pmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}$$

An alternative formulation, based on **barycentric coordinates**, gives a better insight into the nature of this interpolant: For every point P in \mathbb{R}^2 , its barycentric coordinates with respect to the non-collinear points P_0, P_1, P_2 , are defined as the unique real triple (b_0, b_1, b_2) satisfying

$$P = \sum_{i=0}^2 b_i P_i \text{ with } \sum_{i=0}^2 b_i = 1,$$

see Fig. 4.7.

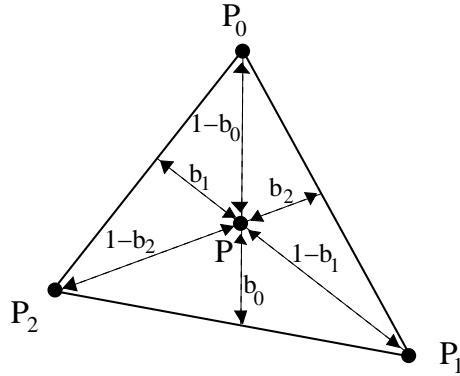


Figure 4.7: Barycentric coordinates in two-dimensions

If furthermore, $\forall i b_i \geq 0$, the point P lies in the triangle (P_0, P_1, P_2) . In this case, the interpolant F at P can be expressed as follows.

$$F(P) = \sum_{i=0}^2 b_i \alpha_i$$

where F is a multivariate function interpolating the multivariate values α_i s defined at the P_i s. In other words, the function value at a given point P is obtained by a weighted combination of the vertices' values, where the weights correspond to the proximity of P to the vertices.

Vector Case

In the two-dimensional vector case, a critical point is a location Q satisfying the following linear equation in its barycentric coordinates b_i

$$F(Q) = \sum_{i=0}^2 b_i \boldsymbol{\alpha}_i = \mathbf{0},$$

where the $\boldsymbol{\alpha}_i$ are the two-dimensional vectors defined at the vertices P_i . If the system is non-degenerate, the solution is unique and the considered linear vector field has a **unique critical point**. Furthermore, if the condition $\forall i, b_i \geq 0$ is satisfied, then the position found lies inside the triangle.

If one considers a single linearly interpolated edge, one has the following important property: The angle rotation of the vector field is always smaller than π . This is illustrated in Fig. 4.8. Hence, one has the following result.

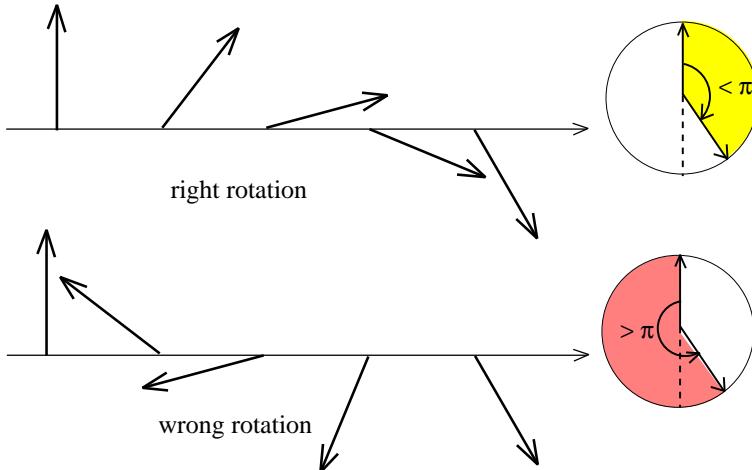


Figure 4.8: Angle rotation along a linear edge

Property 9 *The possible index of a linear vector field along the edges of a triangle is either 0, -1 or +1.*

Proof: The index is an integer value. Now, by definition, this value is given by

$$\text{index} = \frac{1}{2\pi} \sum_{i=0}^2 \Delta\theta_i,$$

where $\Delta\theta_i$ is the angle variation of the vector field along the i -th edge. Since $\forall i, |\Delta\theta_i| < \pi$, one has $|\text{index}| < 2$ which completes the proof.

Tensor Case

In the symmetric, two-dimensional tensor case, when considering only the deviator part, the linear interpolant is defined by

$$F(P) = \sum_{i=0}^2 b_i D_i,$$

where the D_i is the 2x2 deviator part of the symmetric matrix defined at P_i :

$$D_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & -\alpha_i \end{pmatrix}.$$

Now, a degenerate point Q is characterized by a zero matrix. This is equivalent to

$$\begin{cases} \sum_{i=0}^2 b_i \alpha_i = 0 \\ \sum_{i=0}^2 b_i \beta_i = 0, \end{cases}$$

which leads to a linear system analogous to the vector case. Once again, in non-degenerate cases, the solution is unique and the considered linear, symmetric tensor field has thus a **unique degenerate point**. It lies in the triangle if and only if $\forall i, b_i \geq 0$. As far as the tensor index is concerned, property 9 can be extended as follows.

Property 10 *The possible index of a linear tensor field along the edges of a triangle is either $-\frac{1}{2}$, 0 or $+\frac{1}{2}$.*

Proof: This is a direct corollary of property 7, p. 59.

4.2.2 Bilinear Interpolation over a Quadrilateral

The interpolant is first considered in computational space where the quadrilateral is mapped onto a rectangle. For each scalar component of the considered field (vector or tensor), a bilinear interpolant over a rectangle with vertices P_0, P_1, P_2, P_3 and associated values $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ is defined as the real function

$$\forall P(r, s), f(P) = a + br + cs + drs,$$

where a, b, c and d are the real coefficients of the interpolant and (r, s) are the local coordinates of P with respect to P_0 . Remark that the restriction of this interpolant to each edge of the rectangle is a linear (one-dimensional) function. The interpolation conditions lead to the following linear system.

$$\forall i \in 0, \dots, 3, f(P_i) = \alpha_i \text{ that is } \begin{pmatrix} 1 & r_0 & s_0 & r_0 s_0 \\ 1 & r_1 & s_1 & r_1 s_1 \\ 1 & r_2 & s_2 & r_2 s_2 \\ 1 & r_3 & s_3 & r_3 s_3 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix},$$

where a point P_i has local coordinates (r_i, s_i) .

Note that a direct combination of the values of the vertices is possible too. With the notations of Fig. 4.9, one obtains for each point $P(r, s)$ with $0 \leq r, s \leq 1$, the formula

$$f(P) = (1 - r)(1 - s)\alpha_0 + r(1 - s)\alpha_1 + rs\alpha_2 + (1 - r)s\alpha_3.$$

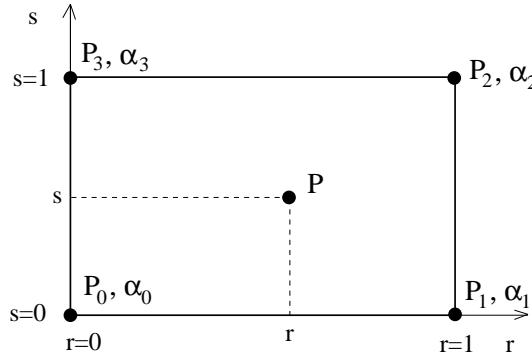


Figure 4.9: Local coordinates in a rectangle

Back in physical space, if ϕ denotes the map that transforms a quadrilateral cell in its rectangle equivalent in computational space, the interpolation at the position Q is then defined as follows.

$$g(Q) = f(\phi(Q))$$

This corresponds to first mapping the position Q onto its equivalent in computational space (analytically or numerically) and then evaluating the function there by bilinear interpolation. Remember that the coordinates' transformation from physical to computational space is an expansive task. For this reason, one usually carries out the whole computation in computational space, before converting the results back in physical space.

Vector Case

In the two-dimensional vector case, the location $Q(x, y)$ of a critical point in computational space satisfies the following non-linear system.

$$f(Q) = \mathbf{a} + \mathbf{b}r + \mathbf{c}s + \mathbf{d}rs = \mathbf{0},$$

where \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} are two-dimensional vectors. This leads to a quadratic equation in one of the two variables. Eventually, this system may have zero,

one or two solutions. Each of them must then be tested to lie in the quadrilateral. Now, the edges of a bilinearly interpolated quadrilateral are linearly interpolated. As said previously, the angle rotation along a linear edge is always smaller than π . For this reason, the angle variation of a vector field along the edges of a quadrilateral cell is the sum of four angles α_i , with $\forall i \in 0, \dots, 3, -\pi < \alpha_i < \pi$. Consequently, the angle variation is strictly comprised between -4π and 4π . Hence, like the triangle case, we have the following result.

Property 11 *The possible index of a linear vector field along the edges of a quadrilateral is either -1, 0 or +1.*

Practically, it means that if two critical points are located in the same quadrilateral, then they must have opposite indices (saddle point and source or sink).

Tensor Case

In the symmetric, two-dimensional tensor case, the interpolant has following expression in computational space:

$$F(P(r, s)) = D_0 + rD_1 + sD_2 + rsD_3,$$

where the D_i s are 2×2 deviator matrices of the form

$$D_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & -\alpha_i \end{pmatrix}.$$

Finding the location of a degenerate point consists in solving the following non-linear system:

$$\begin{cases} \alpha_0 + \alpha_1r + \alpha_2s + \alpha_3rs = 0 \\ \beta_0 + \beta_1r + \beta_2s + \beta_3rs = 0 \end{cases}$$

which is equivalent to the vector case. Furthermore, the index satisfies in this context the following property.

Property 12 *The possible index of a bilinear tensor field computed along the edges of a quadrilateral is either $-\frac{1}{2}$, 0 or $+\frac{1}{2}$.*

Proof: The reasoning used in the proof of property 11 can be also applied for tensors by recalling that the linear angle variation of a tensor field along an edge is (in absolute value) smaller than $\frac{\pi}{2}$ (property 6, p. 58).

This means that if a quadrilateral cell contains two degenerate points, they have opposite indices (a trisector and a wedge).

4.2.3 Linear Interpolation over a Tetrahedron

The linear interpolation over a tetrahedron constitutes the extension to the three-dimensional case of the linear interpolation over a triangle.

For each scalar component of the considered field (vector or tensor), the linear interpolant over a tetrahedron T with vertices P_0, P_1, P_2, P_3 and associated values $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ is defined as the real function

$$\forall P(x, y, z) \in T, f(P) = a + bx + cy + dz,$$

where a, b, c and d are the real coefficients of the interpolant. They are chosen to satisfy the following relation.

$$\forall i \in 0, \dots, 3, f(P_i) = \alpha_i \text{ that is } \begin{pmatrix} 1 & x_0 & y_0 & z_0 \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Once again, one can reformulate the problem in barycentric coordinates: For every point P in \mathbb{R}^3 , its barycentric coordinates with respect to non coplanar points P_0, P_1, P_2, P_3 are defined as the unique real triple (b_0, b_1, b_2, b_3) satisfying

$$P = \sum_{i=0}^3 b_i P_i \text{ with } \sum_{i=0}^3 b_i = 1.$$

If furthermore, $\forall i, b_i \geq 0$, the point P lies in the tetrahedron and the interpolant F at P can be expressed as

$$F(P) = \sum_{i=0}^3 b_i \alpha_i.$$

4.2.4 Trilinear Interpolation over a Hexahedron

The trilinear interpolant extends the bilinear interpolant defined over a quadrilateral. For each scalar component of the field, a trilinear interpolant over a hexahedron with vertices P_i and associated values $\alpha_i, i \in 0, \dots, 7$ is defined in computational space as the real function:

$$\forall P(r, s, t), f(P) = a + br + cs + dt + ers + fst + grt + hrst,$$

where $a..h$ are the real coefficients of the interpolant. They are chosen to satisfy the following relation. $\forall i \in 0, \dots, 4$,

$$f(P_i) = \alpha_i \text{ that is}$$

$$\begin{pmatrix} 1 & r_0 & s_0 & t_0 & r_0 s_0 & s_0 t_0 & r_0 t_0 & r_0 s_0 t_0 \\ \vdots & & & & & & & \vdots \\ 1 & r_7 & s_7 & t_7 & r_7 s_7 & s_7 t_7 & r_7 t_7 & r_7 s_7 t_7 \end{pmatrix} \times \begin{pmatrix} a \\ \vdots \\ h \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_7 \end{pmatrix}$$

As for rectangle cells, a direct combination of the values of the vertices is possible (adding a local third coordinate t to the ones introduced in the rectangle case): For each point $P(r, s, t)$ with $0 \leq r, s, t \leq 1$,

$$\begin{aligned} f(P) = & (1-r)(1-s)(1-t)\alpha_0 + r(1-s)(1-t)\alpha_1 + rs(1-t)\alpha_2 + (1-r)s(1-t)\alpha_3 + \\ & (1-r)(1-s)t\alpha_4 + r(1-s)t\alpha_5 + r st\alpha_6 + (1-r)st\alpha_7. \end{aligned}$$

In physical space, the interpolant g at a given position Q is then defined as follows (ϕ denotes the transformation map):

$$g(Q) = f(\phi(Q)).$$

As for the bilinear interpolation over quadrilateral cells, since the coordinates' transformation is expansive, one usually prefers to process the whole calculus in computational space before mapping back the results in physical space.

4.3 Singularities in Piecewise Linear Fields

We consider in this section the special case of piecewise linear, planar vector and tensor fields defined over a triangulation. As a matter of fact, the piecewise linearity of such fields extends the range of topological features that may be encountered in the linear case to singularities of higher order. Further, the low order of the piecewise linear interpolant enables a precise detection of the structure of such non-linear singular points.

Inside each triangle, the field is linear. So, if a singularity lies in its interior, it has the topological structure of one of the linear cases enumerated previously (see sections 2.2.1 and 3.4.1). Thus, the novelty with piecewise linear fields is due to singular points lying on the boundary of a triangle cell (edge or vertex). In this case, there is no neighborhood of the singular point completely lying in the definition domain of a single linear field (i.e. in a single triangle). Hence, the singularity is generally non-linear and local linear approximations of its neighborhood (Jacobian matrix in the vector case, third-order differential of the deviator in the tensor case) are unable to completely characterize its structure. Actually, as one shows next, a singularity lying on a grid vertex can have an arbitrary complex structure, depending on the connectivity of the vertex in the grid. This observation permits the modeling as well as the identification of any singularity in piecewise linear fields. Note that this special

case of singular point can be seen as the merging (at the bifurcation point) of several singularities lying in neighbor cells and getting infinitely close together. This perspective will be developed further in chapter 7. One first considers the case of singularities lying on the edge of a triangle before studying singularities located at a vertex.

4.3.1 Singularities on an Edge

For convenience, one distinguishes vector and tensor fields.

Critical Points

The possible situations are illustrated in Fig. 4.10 and Fig. 4.11.

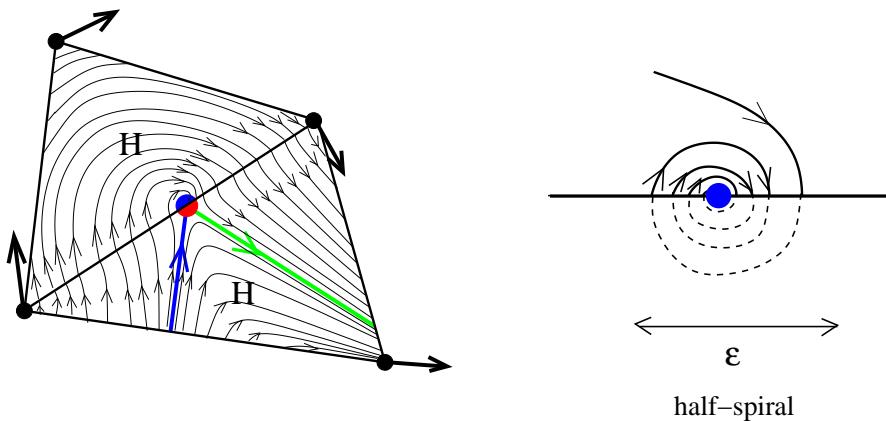


Figure 4.10: Saddle - spiral merging

As mentioned previously in the presentation of the bilinear interpolant, the index of a closed curve built of four linearly interpolated segments is either 0, -1 or +1. So, if two critical points are located in two cells sharing a common edge, they must have opposite indices. In particular, when a critical point lies on the common edge of two neighbor cells with global index 0, its phase portrait looks like the merging of a saddle point located in one cell with a sink or source lying in the other cell. Each of these two natures is determined by the value of the Jacobian matrix in the respective cell. Note that the Jacobian is constant over each cell in a piecewise linear interpolation. To determine the local structure of this critical point (considered as a single one), one needs to locate its separatrices, i.e. the boundary curves of its hyperbolic sectors, if any. Now, (exactly) two separatrices of the saddle point, considered in the

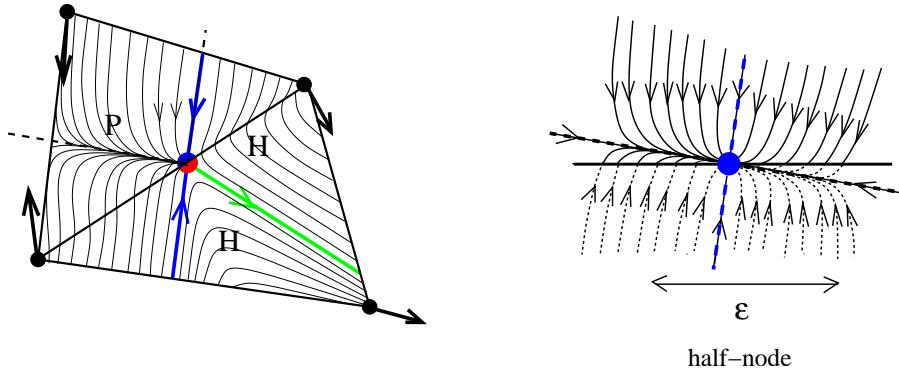


Figure 4.11: Saddle - node merging

corresponding cell, bound a hyperbolic sector entirely contained in this cell: By definition, they are separatrices of the non-linear critical point, too. If one considers the critical point from the other cell, it may be of two possible types: Either a spiral or a node.

The spiral case is shown in Fig. 4.10. Since the convergence toward a spiral occurs along rotating curves that get infinitely close to the singularity, the cut of a half of a spiral's neighborhood prevents any convergence to take place (see enlargements). For this reason, The two separatrices identified so far are the only ones and the singularity has two hyperbolic sectors.

The node case is shown in Fig. 4.11. Here the situation is different because there are infinite many integral curves converging to the half-node in the corresponding cell. The boundary between converging and non-converging integral curves is actually delimited by the eigenvector of the node related to the eigenvalue with maximal modulus, i.e. the direction of “fastest” convergence (see enlargements). This additional separatrix bounds a parabolic and a hyperbolic sector. Finally, the singularity has two hyperbolic sectors and one parabolic sector.

Degenerate Points

The possible cases are shown in Fig. 4.12.

As in the vector case, a degenerate point lying on an edge with index 0 can be seen as the merging of two simple degenerate points of the two linear tensor fields defined on both sides of the edge. Furthermore, these two degenerate points must have different natures: This is a direct corollary of property 6 p. 58, analogous to the vector case. Therefore, one is only concerned with the

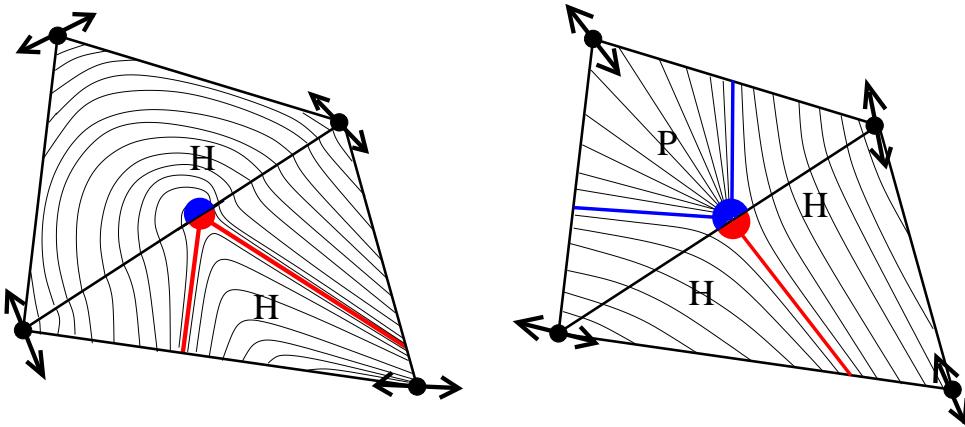


Figure 4.12: Degenerate Point on an Edge

merging of a trisector and a wedge point. The continuity of the tensor field across the edge prohibits other cases than those depicted in Fig. 4.12. The first case corresponds to the degenerate point obtained at the bifurcation point in the pairwise annihilation depicted in Fig. 3.13, p. 62. It corresponds to the situation where the parabolic sector of the wedge point faces a separatrix of the trisector. In the trisector's triangle, a complete hyperbolic sector is bounded by two separatrices of the half trisector. In the other triangle, no tensor line converges to the wedge point. Consequently there is no separatrix in this triangle and the singular point has two hyperbolic sectors. The second case corresponds to the degenerate point at the bifurcation point in the pairwise annihilation shown in Fig. 3.14, p. 63. It occurs when the hyperbolic sector of the wedge point disappears by merging with a hyperbolic sector of the trisector. It follows that this non-linear degenerate point has two hyperbolic sectors and that its separatrices are the remaining ones of the trisector and the separatrix (resp. both separatrices) of the wedge point in their respective triangles.

4.3.2 Singularities on a Vertex

The special case of a singular point lying on a vertex offers much more possibilities for the local structure of the field in its vicinity. This is easily explained by the fact that any neighborhood of such a point goes through every triangle incident to the considered vertex. Now, depending on the connectivity of this vertex, the topological complexity of the singularity arbitrarily increases as the number of these triangles does. In the following, one first focuses on the

identification of the local topological structure of a given singularity lying at a vertex. With these ideas in mind, one shows then how to simply model critical points with arbitrary complex structure in piecewise linear vector fields.

Local Topology Detection

Locating and analyzing the different topological sectors of a singular point lying on a vertex consists in seeking the boundary curves of its hyperbolic sectors and the different sets of nested loop curves tending to the singular point in both directions (elliptic behavior). The basic idea behind the search for separatrices is that they constitute a subset of the curves that converge toward the considered singularity along straight lines. Vector and tensor cases are distinguished next.

Vector Case One first proves a simple property that is used in the following.

Property 13 *In the neighborhood of a critical point lying on a vertex of a piecewise linear interpolated triangulation, the angle coordinate of the vector field does not depend on the distance to the critical point.*

Proof: Consider the situation shown in Fig. 4.13.

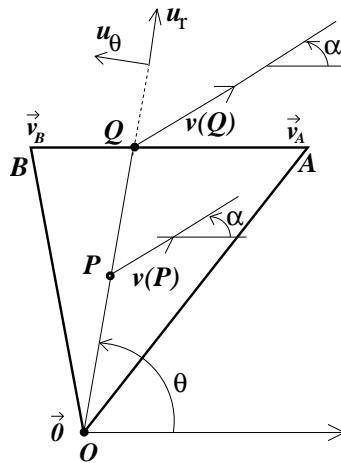


Figure 4.13: Angular coordinate in the vector case

In the triangle OAB , the vector value at P is linearly interpolated between O and Q . One gets

$$\begin{aligned} \mathbf{v}(P) &= u \mathbf{v}(Q) + (1 - u) \mathbf{v}(O) \\ &= u \mathbf{v}(Q) \end{aligned}$$

That is, $\mathbf{v}(P)$ is collinear to $\mathbf{v}(Q)$ and thus, taking O as coordinate origin, both vectors have the same angle coordinate. Q.E.D.

Using this property, one restricts the search for separatrices to the boundary of the cell stencil made of all cells incident to the “singular” vertex: A position on this boundary where the vector field is parallel to the coordinate vector is the initial condition of an integral curve that converges toward the critical point for $t \rightarrow \pm\infty$. One is thus concerned with the restriction of the vector field to a closed curve constituted by linearly interpolated line segments (see Fig. 4.14).

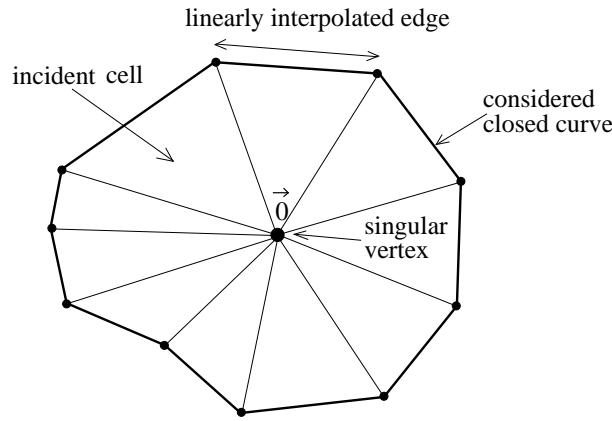


Figure 4.14: Stencil of a singular vertex

This restriction to each edge $[AB]$ is linear in one parameter (say t):

$$\mathbf{v}(t) \equiv \mathbf{v}((1-t)A + tB) = (1-t)\mathbf{v}_A + t\mathbf{v}_B$$

Therefore, one looks on each edge of the stencil boundary for parameter values where the vector field is collinear to the coordinate vector:

$$\mathbf{v}(t) \times \mathbf{u}_\theta(t) = \mathbf{0} \text{ (cross product), with } \mathbf{u}_\theta(t) \equiv \frac{\mathbf{OQ}(t)}{\|\mathbf{OQ}(t)\|}.$$

These positions correspond to a separatrix if the straight line joining the position to the singular vertex bounds a hyperbolic sector. Furthermore, one looks for positions where the vector field is orthogonal to the coordinate vector:

$$\mathbf{v}(t) \cdot \mathbf{u}_\theta(t) = 0 \text{ (scalar product).}$$

Both equations are quadratic and can be easily solved. This enables the distinction between hyperbolic and elliptic sectors and permits the complete

characterization of the critical point. To simplify the results, one adopts the following notations: at the positions where the vector field is orthogonal to the coordinate vector, one distinguishes angles where the cross-product $\mathbf{u}_\theta \times \mathbf{v}(t)$ is positive (called **orthogonal+**) from those where it is negative (called **orthogonal-**). At the positions where the vector field is parallel to the coordinate vector, one distinguishes angles where the scalar product $\mathbf{u}_\theta \cdot \mathbf{v}(t)$ is positive (called **parallel+**) from those where it is negative (called **parallel-**). These definitions are illustrated in Fig. 4.15.

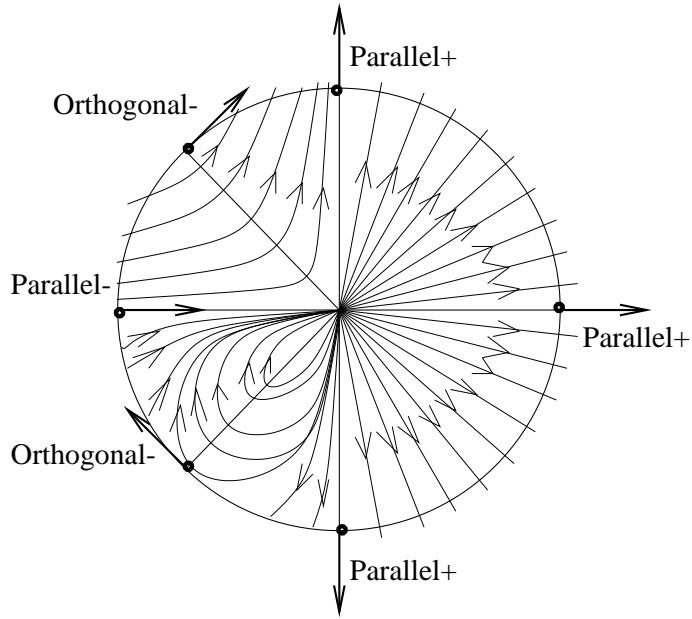


Figure 4.15: Notations

One obtains then the graph shown in Fig. 4.16 for the determination of a sector type.

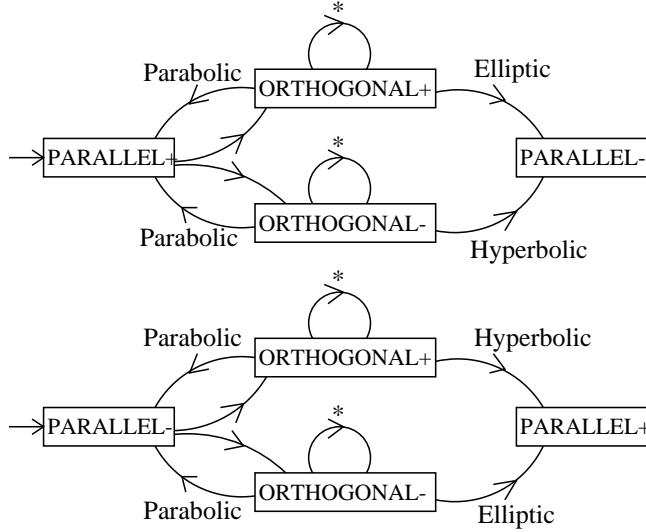


Figure 4.16: Sector type determination graph

Tensor Case Property 13 extends to the tensor case as follows.

Property 14 *In the neighborhood of a degenerate point lying on a vertex of a piecewise linear interpolated triangulation, the angle coordinate of both eigenvector fields does not depend on the distance to the degenerate point.*

Proof: It follows directly from the considerations of section 3.4.1.

Like the vector case, one uses this property to restrict the localization of separatrices to the piecewise linear boundary of the stencil. Yet, the corresponding eigenvector fields are not linear along each edge. From equation 3.2, one knows that the eigenvectors are given in the deviator case by following expression:

$$\mathbf{e} = \left(\beta(t), -\alpha(t) \pm \sqrt{\alpha^2(t) + \beta^2(t)} \right),$$

where α and β are linear functions in t along each edge. The *parallel* condition is here equivalent to the non-linear system

$$\begin{cases} \beta(t) = 0 \\ \text{or} \\ (x^2(t) - y^2(t))\beta(t) - 2x(t)y(t)\alpha(t) = 0, \end{cases}$$

where x and y are the linear functions describing the coordinates of a point $Q(t)$ along the edge. Hence, this equation is cubic and can be solved analytically.

Because of the lack of orientation of the eigenvector fields, the sector discrimination cannot be based on the distinction between **parallel+** and **parallel-** or **orthogonal+** and **orthogonal-**. An alternative approach, inspired by the definition of the tensor index, consists in computing the angle variation $\Delta\alpha$ of the eigenvector fields between two consecutive *parallel* positions. As a matter of fact, this value suffices to characterize each sector type:

- $\Delta\alpha = \theta$ in the *parabolic* case,
- $\Delta\alpha = \theta - \pi$ in the *hyperbolic* case,
- $\Delta\alpha = \theta + \pi$ in the *elliptic* case.

This is shown in Fig. 4.17.

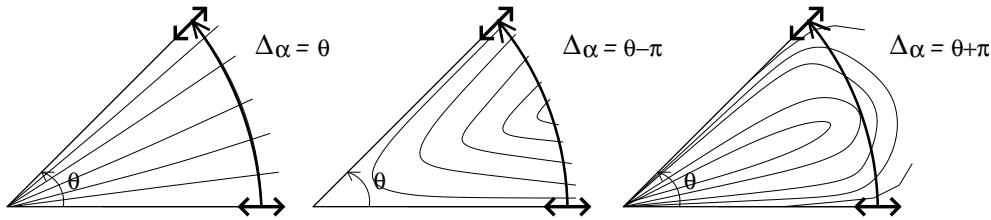


Figure 4.17: Angle variation in the parabolic, hyperbolic and elliptic case

Modeling of Critical Points

In this section, one deals with the following problem (see theorem 6):

Given a list of angles $(\omega_i)_{i=1,\dots,n_1}$ of ω -separatrices (i.e. separatrices that converge toward the critical point for $t \rightarrow +\infty$), a list of angles $(\alpha_i)_{i=1,\dots,n_2}$ of α -separatrices (i.e. separatrices that converge for $t \rightarrow -\infty$), and a list $(g_i)_{i=1,\dots,n_3}$ of elliptic sectors, build a piecewise linear vector field that exhibits an equivalent critical point.

Of course, the problem should be subdivided into modeling a single curvilinear sector starting and stopping at prespecified angular positions, given by the sorted values of the angles introduced above. Consequently, one treats the three possible sector type cases.

Remark: In the following, the neighborhood of the singular point is actually the set of triangles that are incident to the considered “singular” vertex.

Parabolic Sector According to the definitions of section 2.2.2, a parabolic sector is bounded by two separatrices of the same kind (both ω - or both α -separatrices). Consider two ω -separatrices located at $\theta = \omega_1$ and $\theta = \omega_2$ respectively. Building a triangle as shown in Fig. 4.18

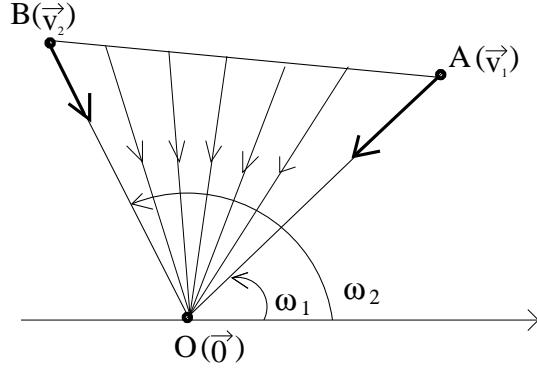


Figure 4.18: Piecewise linear parabolic sector

and setting $\mathbf{v}_1 = -\mathbf{u}_{\omega_1}$ and $\mathbf{v}_2 = -\mathbf{u}_{\omega_2}$, one gets the expected parabolic behavior for all integral curves starting inside the triangle ABC .

Hyperbolic Sector A hyperbolic sector is bounded by two separatrices of opposite kind (an ω - and an α -separatrix). Consider an ω -separatrix located at $\theta = \omega$ and an α -separatrix located at $\theta = \alpha$. Building a triangle as shown in Fig. 4.19,

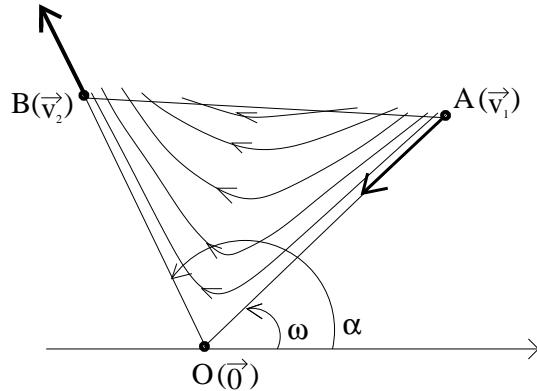


Figure 4.19: Piecewise linear hyperbolic sector

and setting $\mathbf{v}_1 = -\mathbf{u}_\omega$ and $\mathbf{v}_2 = +\mathbf{u}_\alpha$, one gets the expected hyperbolic behavior for all integral curves starting inside the triangle ABC .

Elliptic Sector In this case, one is not given two angle coordinates of two bounding separatrices but a curvilinear sector bounded by a loop integral curve that tends to O for both $t \rightarrow \infty$ and $t \rightarrow -\infty$. The modeling of such a sector by a piecewise linear vector field requires the curve itself to be described in terms of its tangential directions for $t \rightarrow \infty$ and $t \rightarrow -\infty$. Consider the picture in Fig. 4.20.

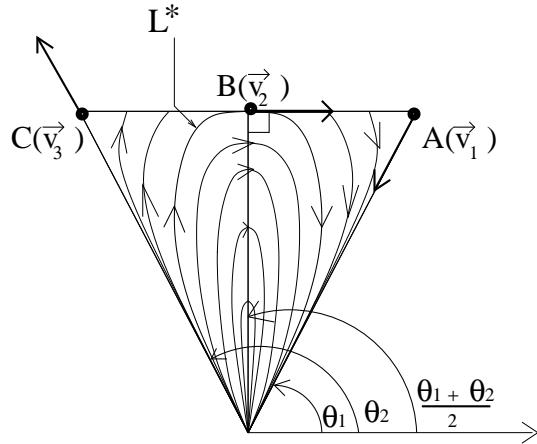


Figure 4.20: Piecewise linear elliptic sector

The angle coordinates θ_1 and θ_2 are the tangential direction of the loop curve L^* that bounds the modeled elliptic sector. The vector values at A and C are set as in the hyperbolic case. From the former case, one knows that the linear sector defined by A , B and O is hyperbolic. This means that an elliptic sector is not linear. To build such a sector, one has to split the triangle into two subtriangles. By setting the vector value at the corresponding additional point B as shown, one gets the expected elliptic behavior for all integral curves through points located inside the loop L^* . Finally, Fig. 4.21 illustrates a possible result of this modeling process.

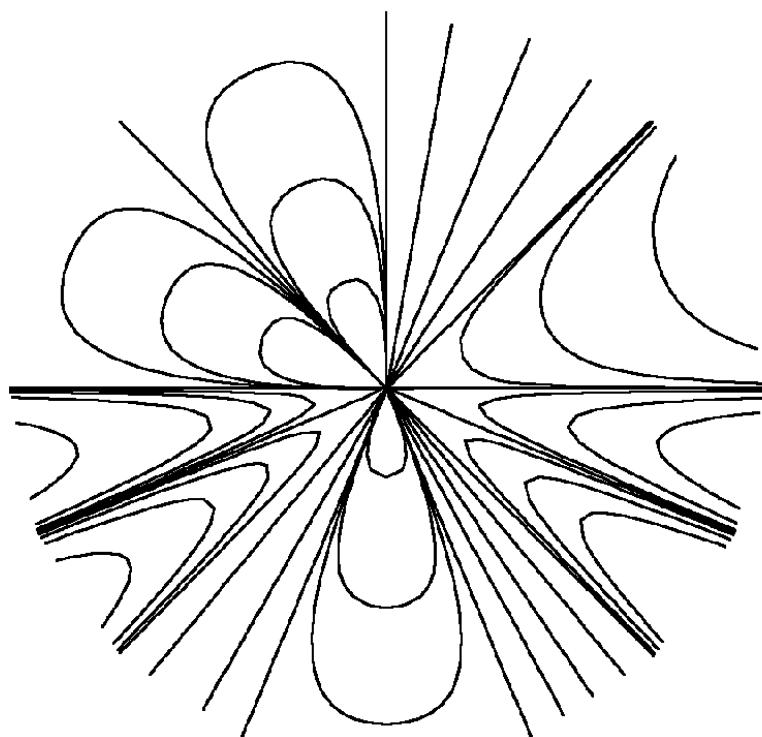


Figure 4.21: Modeled critical point

Chapter 5

Topology-Based Vector and Tensor Field Visualization: The State of the Art

An overview of the existing topology-based vector and tensor visualization techniques is proposed in this chapter. Note that this is only a (small) part of the flow visualization which is traditionally a major field of research in Scientific Visualization. Yet topological methods have received an increasing interest in the last decade and experienced many improvements and extensions since their introduction by Helman and Hesselink. Their motivation is the ability to tremendously reduce the volume of information required to analyze the data while conveying all important qualitative properties of the considered vector or tensor field. As we saw in previous chapters, the topology is the qualitative structure of a field and its depiction thus concentrates on the most meaningful aspects of the associated flow.

In fact, the origins of this topological approach can be found in numerous works carried out by fluid dynamicists from the 60's. Lighthill [Lig63], Tobak [TP82], Perry [PF74, Per84, PC87], Dallmann [Dal83], Chong [CPC90] and others applied the theoretical framework initiated by Poincaré [Poi75, Poi99] at the end of the 19th century to the study of three-dimensional flow fields. In particular, they showed how the concepts from the critical point theory apply to the characterization of flow patterns. First restricted to the visualization of two-dimensional vector fields, these ideas were next tested for three-dimensional flows. The appeal for these schematic depictions motivated their extension to the tensor case, following original work by Delmarcelle.

5.1 Vector Fields

In this section, we first introduce the basic topological technique for the visualization of vector fields in the two-dimensional case and then present the extensions that have been designed to improve the original method in order to fulfill the requirements of practical applications. We conclude the presentation by a brief description of additional visualization schemes that are also related to topology in some way.

5.1.1 Two-Dimensional Basics

The first application of topological techniques for the visualization of vector fields was proposed by Helman and Hesselink [HH89a, HH89b, HH90, HH91]. The primary concern of these authors was to design a software method for the automatic extraction and visualization of two-dimensional vector field topology. Their classification of critical points is restricted to a linear precision, i.e. is based on the eigenvalues of the Jacobian, as detailed previously in 2.2.1 p. 15. The major types are here saddles, nodes and spirals (called *foci* by the authors). Hence the edges of the topological graph can be defined in this context as the set of curves integrated from the saddle points along the direction of the eigenvectors. Additional points are also being considered that lie on the boundary of possible obstacles for the flow. As a matter of fact, the velocity of the associated flow is usually constrained to be zero on an obstacle's body which induces a linear decreasing of the tangential velocity with respect to the distance to the body. Yet there exist locations where the tangential velocity presents a singularity. This results in streamlines that start or end (in an asymptotic way) on the body. Consequently these curves along with their associated *attachment* or *detachment nodes* are drawn to complete the *topological skeleton* of the flow over the bounded domain. The topology is associated with a connectivity graph where the links between *originators* and *terminators* (saddles, attachment or detachment nodes) and the sinks or sources (nodes or spirals) are identified. Illustrations of these definitions are proposed in Fig. 5.1.

The same authors implemented an application of this basic technique to unsteady (parameter-dependent) two-dimension vector fields. The idea is to handle the one-dimensional parameter space as a third dimension and to connect curves lying in adjacent slices along the parameter line. To ensure consistency, this connection is only done between separatrices for which both start and end points can be connected. In this case, a ribbon is displayed that depicts the surface spanned by the motion of the separatrix over the parameter space. However if a topological transition has occurred between consecutive

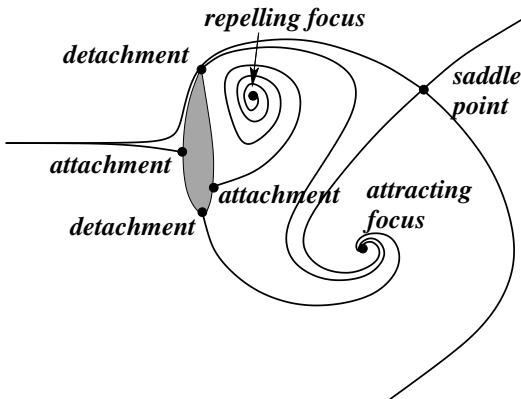


Figure 5.1: Topology Representation in 2D (from [HH89a])

slices, no strip is drawn. An example is shown in Fig. 5.2.

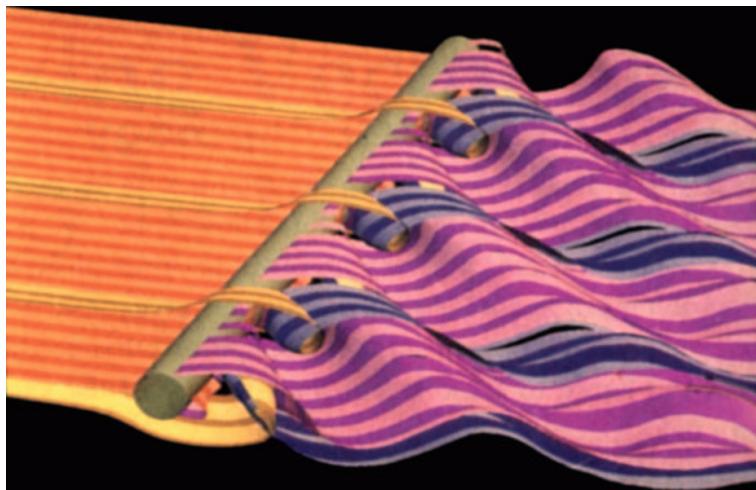


Figure 5.2: Time-dependent topology of a 2D flow (from [HH91])

In their work, Helman and Hesselink neglected the importance of closed orbits. Yet, from chapter 2 we know that they play a role similar to sinks and sources in the topology. Furthermore they are very common in two-dimensional datasets cut off from three-dimensional flows (this is illustrated in chapter 6). This lapse in the original topology depiction technique of 2D vector fields has been corrected recently by Wischgoll and Scheuermann [WS01]. Separatrix integration is conducted in a way that permits to detect closed regions where streamlines remain trapped. If no critical point is present in the region, the Poincaré-Bendixson theorem ensures the existence of a closed orbit (refer

to [GH83], p. 44). Its precise position is provided by the fixed point of the Poincaré map (see definition 18, p. 23).

Furthermore, the characterization of the topology of a two-dimensional vector field defined over a bounded region must take the boundary into account. This was proven by Scheuermann, Hamann and others in [SHJ00] who proposed a scheme to extract additional separatrices emanating from so-called boundary saddles. See section 2.4, p. 25 for further details.

5.1.2 Three-Dimensional Vector Field Topology

The early advances of topology-based vector field visualization in 2D have motivated the extension of the same principle to 3D flows. In their paper [GLL91], Globus, Levit and Lasinsky presented a software system called *TOPO* that extracts and visualize some topological aspects of three-dimensional vector fields. Here, the critical points are characterized by the three eigenvalues of the Jacobian, resulting in nodes, saddles and spiral saddles. Some examples are proposed in Fig. 5.3. The numerical issues like singularities' search or degeneracies are discussed. As far as the depiction is concerned, critical points are displayed as glyphs, conveying the properties of the Jacobian, and are connected by streamlines started from the saddles and the spiral saddles along the direction of the eigenvectors associated to the real eigenvalues. An analogous technique was also suggested by Helman and Hesslink in [HH91].

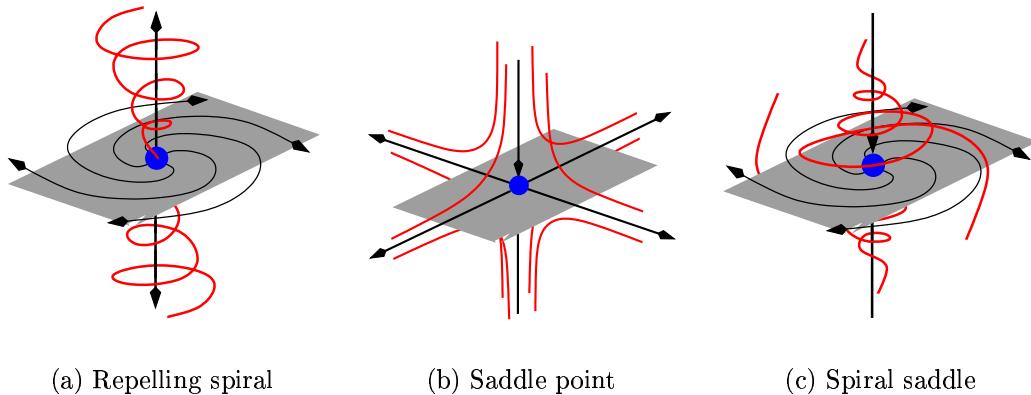


Figure 5.3: Examples of 3D critical points (from [wwwb])

Remark that these implementations do not provide a real depiction of three-dimensional topology. As a matter of fact, the partition of the domain into subregions where the flow is qualitatively uniform (as separatrices do in the

two-dimensional case) requires the computation and display of stream surfaces. A stream surface is the surface spanned by a streamline as its seed moves along a curve. Few algorithms exist for the construction of these surfaces [Hul92, SBH01] and this remains a challenging issue.

5.1.3 Surface Topology

A major concern in the study of three-dimensional vector fields is the qualitative behavior on (two-dimensional) surfaces of bodies in the flow. For this purpose, several topological techniques have been designed to extract and visualize the structure of curves integrated along solid surfaces: Supposing that the velocity field vanishes smoothly on the surface (noslip boundary), one considers the two-dimensional vector field obtained by projection of the three-dimensional velocity on the tangent plane of the body. Computing the topology of this field gives insight into the behavior of the flow in the vicinity of the flow. This has been done by Helman and Hesselink in [HH90, HH91] and by Globus, Levit and Lasinski in [GLL91]. An illustration of this principle is proposed in Fig. 5.4.

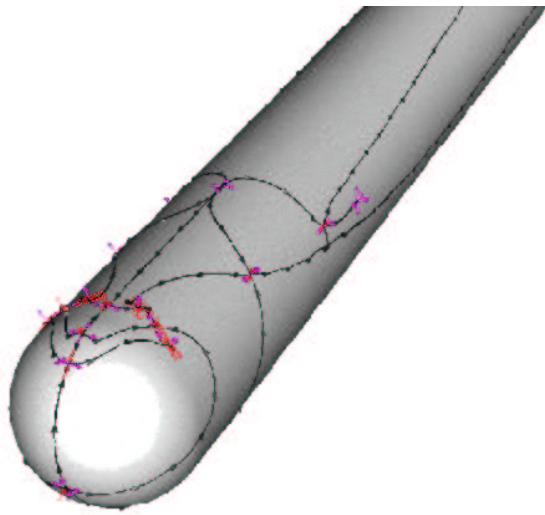


Figure 5.4: Surface topology on a hemisphere cylinder (from FAST [[www.a](#)])

The extracted structure can also serve as basis for the computation of surfaces of separation [HH90]. They can be seen as the three-dimensional equivalent of the curves emanating from the attachment or detachment nodes in the 2D case. These surfaces are connected to the body along attachment or

separation lines that correspond to the separatrices from the saddle points of the tangential topology, classified with respect to the sign of the eigenvalues. However Kenwright has shown that this method misses open attachment and separation lines (that do not start or terminate at critical points on the surface) and proposed new schemes to detect them completely [Ken98, KHL98].

5.1.4 Higher Order Critical Points

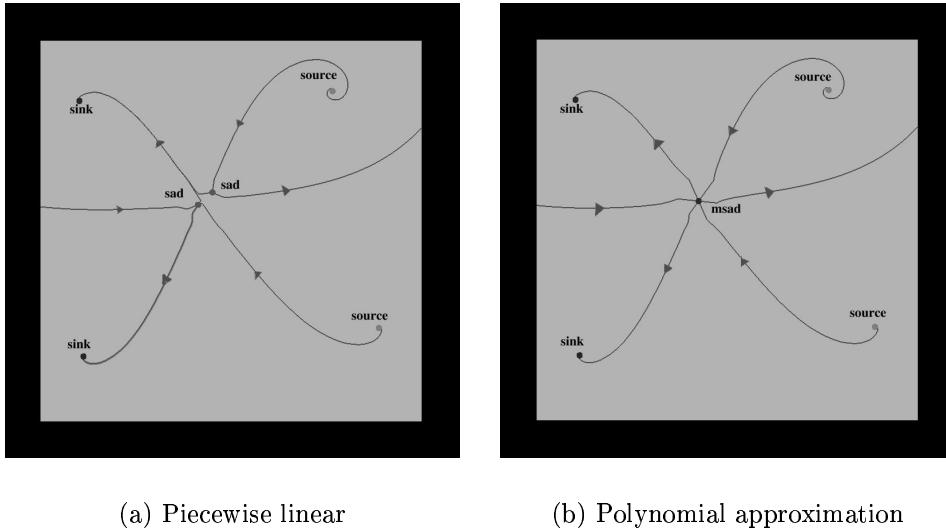


Figure 5.5: Extraction and visualization of a monkey saddle (from [SHK97])

The methods introduced so far are limited to linear precision in the characterization of critical points. This means that only hyperbolic singularities (see definition 11, p. 18) are considered. We saw previously that non-hyperbolic critical points are unstable (in the sense of structural stability) but when imposed constraints exist (e.g. symmetry or incompressibility of the flow), they can be encountered. To attack this deficiency, Scheuermann, Hagen, Krüger and others proposed a scheme for the extraction and visualization of higher-order critical points over piecewise linear two-dimensional vector fields [SHK97]. The basic idea is to identify regions where the absolute value of the index is bigger than 1 (by Theorem 9). In such regions, the original piecewise linear interpolant is replaced by a polynomial approximation function. The polynomial is designed in Clifford algebra, based on theoretical results presented by the same authors in [SHK98a, SKM98, SHK98b]. This permits to infer the actual underlying presence of a critical point with arbitrary index

that is next modeled and visualized as illustrated in Fig. 5.5. Note that an alternative technique, based on grid deformations, is proposed in chapter 6. The impossibility to model higher-order critical points with piecewise linear interpolation also motivated the use by Scheuermann, Tricoche and Hagen of Nielson's C^1 scattered data interpolation scheme [Nie79, Nie80, Nie83] for the extraction of vector field topology [STH99]. Getting back to the original ideas of Poincaré in his study of dynamical systems, Trott, Kenwright and Haimes proposed in [TKH00] a method to extract and visualize the non-linear topology of a “point at infinity” when the considered vector field is defined over an unbounded domain.

5.1.5 Topology Comparison

Using the topology to characterize the properties of vector fields induces an equivalence relation between fields that exhibit similar topological features. However, the corresponding depictions by means of classical visualization methods can conceal this similarity. This observation has motivated the design of feature comparison techniques that precisely evaluate the resemblance of vector fields. Lavin, Batra and Hessellink first introduced in [LBH98] quantities derived from the eigenvalues of the Jacobian that characterizes the types of two-dimensional critical points. This serves to define a convenient global metric over the vector field that permits the computation of a distance, used to measure the topological proximity of two vector fields. An extension to three-dimensional vector fields was presented by Batra and Hessellink in [BH99].

5.1.6 Topology Simplification

In general, the topology of two-dimensional vector fields provides a simple synthetic depiction that conveys the structural information while being restricted to an easily understandable graph. Unfortunately, in some cases, the structure becomes very complex and topology-based visualizations result in visual clutter. This is typically the case if the considered flow is turbulent (see chapter 6): A large number of close critical points can be observed along with associated separatrices that confuse the interpretation. In [LL99a], de Leeuw and van Liere proposed a method to remove critical points from the topological graph while preserving structural consistency of the simplified topology with the original one. Interconnected critical points of opposite indices are pruned pairwise when their distance is under a given threshold (see Fig. 5.6(a)). This globally preserves the structure of the flow since every pair is characterized by an index zero that is equivalent to a uniform flow, i.e. without critical point.

This original method was improved by its authors in [LL99b]. Flow regions

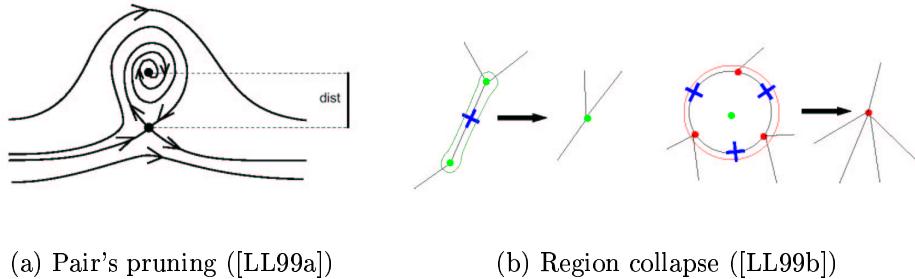


Figure 5.6: Local simplification

are defined as basins of sources or sinks, handling the boundary inflow (resp. outflow) parts as generalized sources (resp. sinks). When the global index of such a region is 1 (meaning a global sink or source nature for the region), the corresponding local topology is collapsed onto a linear sink or source which preserves the index and ensures consistency (see Fig. 5.6(b)). The restriction on the index of the region is made to avoid the creation of higher-order critical points by collapsing. The simplification is monitored by the area of the region to collapse. This criterion is namely considered a better metric by the authors to evaluate the influence of a critical point on the flow structure. Remark that both methods act directly on the graph without providing a corresponding vector field description.

5.1.7 Topology-Related Methods in Scientific Visualization

Piecewise linear interpolation of two-dimensional vector fields defined over triangulations may inconvenience the use of topology-based methods because it produces topological artifacts. This problem motivated the design of a data-dependent triangulation scheme by Scheuermann and Hagen in [SH98] that avoids artificial topological complexity of the associated vector field. For the purpose of data compression of planar vector fields Nielson, Jung and Sung presented a method based on the use of wavelets over curvilinear grids [NJS98]. They compared the consistency of the resulting vector field with the original one by means of their respective topologies. Yet the method enables no precise control on the topology deformation. On the contrary, Lodha, Renteria and Roskin created a topology-preserving compression method for planar vector fields defined over structured grids [LRR00]. Topology-preserving smoothing of vector fields was addressed by Westermann, Johnson and Ertl in [WJE01].

5.2 Tensor Fields

Compared to the vector case, the topology-based visualization of tensor fields has not profited from much research since its introduction by Delmarcelle. Therefore, the presentation is limited to the basic technique for two-dimensional tensor fields and the first attempts to extend it to the three-dimensional case.

5.2.1 Planar Topology

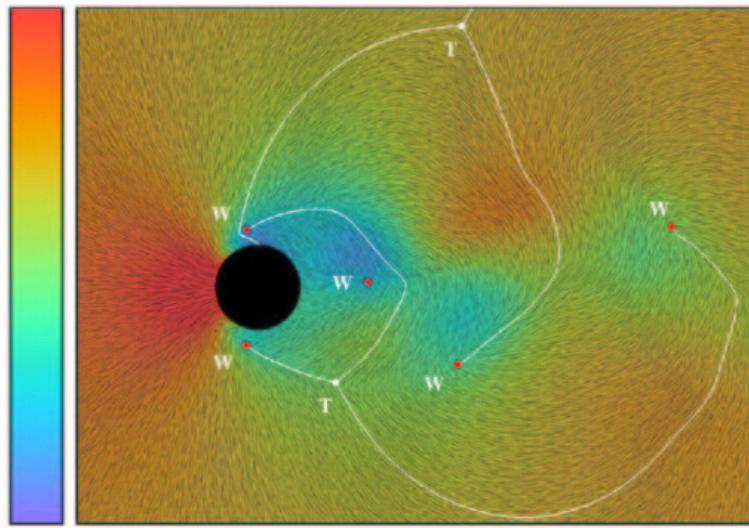


Figure 5.7: Topological depiction of a 2D symmetric tensor field (from [DH94])

Following the theoretical framework that he achieved in his PhD thesis [Del94], Delmarcelle first proposed a topology-based visualization method for two-dimensional, symmetric, second-order tensor fields [DH94]. This method extends to symmetric tensor fields the original scheme of Helman and Hesselink for planar vector fields. Here, topology extraction is based on the characterization of the degenerate points with linear precision and on the integration of the associated separatrices over the domain. Corresponding definitions were given in section 3, p. 40. Since the resulting graph only focuses on the directional information of the tensor field, no insight into quantitative information is conveyed in that way. Therefore Delmarcelle suggested to add a color mapping to display e.g. one eigenvalue or the difference between both. An illustration is proposed in Fig. 5.7. The texture shown in the background is obtained by Line Integral Convolution, a very popular scheme for flow visualization [CL93, SH97].

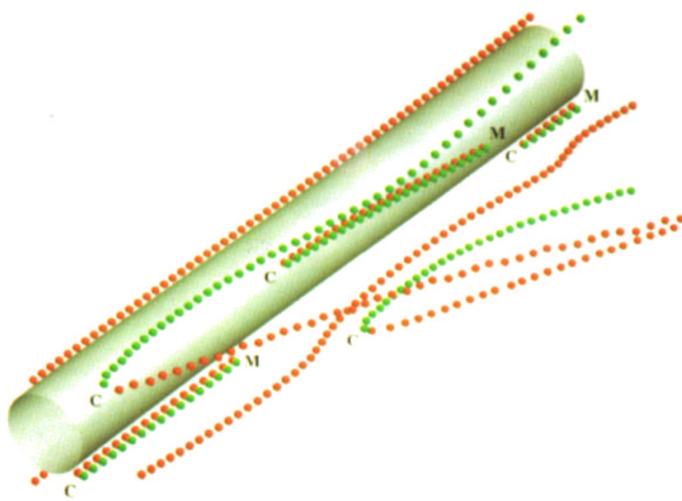


Figure 5.8: Discrete tensor topology tracking (from [DH94])

An extension to parameter-dependent 2D tensor fields can be obtained by tracking degenerate points over time in a graphical way. The 1D parameter space adds a third dimension to the considered 2D domain. The depiction shows the successive positions of the singularities along the discretized parameter line along with the inferred positions of pairwise mergings and creations, revealed by structural changes. This principle is illustrated in Fig. 5.8.

5.2.2 Three-Dimensional Topology

In the symmetric three-dimensional case, degenerate points correspond to locations where at least two of the three real eigenvalues are equal. The geometry of the tensor lines in their vicinity is first addressed by Hesselink, Lavin and Levy in [HLL97, LLH96]. However a complete characterization of 3D degenerate points is not achieved since the precise location of the separating surfaces (that generalize to 3D the separating curves computed in 2D) cannot be done analytically. Furthermore many numerical problems must be solved to properly locate the singularities in this case. Therefore this technique can still be considered “under construction”.

Chapter 6

Topology Scaling

Turbulent flows are generally associated with vector and tensor topologies characterized by the presence of many structures of very small scale. Their proximity and interconnection in the global depiction result in visual clutter with classical methods. Moreover, this deficiency is emphasized by low-order interpolation schemes, typically used in practice (like linear or bilinear interpolation), because they lack the local flexibility required to precisely reproduce close topological features. Consequently, they confuse the results by introducing artifacts. For these reasons, topology-based methods produce in this context pictures that inconvenience analysis by engineers or physicists because meaningful features cannot be distinguished from local details or numerical noise.

These local structures are in fact groups of first-order singularities (sinks, sources and saddle points in the vector case, trisector and wedge points in the tensor case) connected by separatrices, that are part of bigger structures. This interpretation induces a hierarchical approach of the topology, where the whole graph may be recursively decomposed into interconnected subgraphs. Closed orbits play here a special role because they isolate the part of the graph located in their interior from the rest of the graph, in their exterior. Yet, in all cases, the locality of a part of the topology can be determined by the proximity of the involved singular points.

The remarks above imply that a post-processing step is needed to reduce the complexity of the extracted topology in order to enable the interpretation of the visualized results. Moreover, they suggest the following strategy for scaling the resolution of a topology and therefore control the complexity of the graph:

1. Get the locations of all singularities contained in the domain.
2. Determine groups of “close” singularities according to a given measure.

3. For each group satisfying a prescribed proximity, replace the corresponding local structure by a consistent, simpler one.

A consistent structure, as mentioned in the last step, is in fact a new topological feature that can be embedded into the original topology instead of the original structure without modifying the direction and orientation (vector case) of the integral curves in their vicinity. According to the remarks above, the new structure will be considered simpler if it contains less singularities and less associated separatrices that emanate from them. The approach developed here is actually based on a fundamental property of singular points: Several close singular points of first-order are, in the large, equivalent to a single one of higher-order. The signification of this equivalence is better understood in the light of the notions introduced previously in chapters 2 and 3 about bifurcations: One can interpret this higher-order singularity as the structure obtained at the bifurcation point during the merging of close first-order singular points. Note that this generalizes the pairwise annihilations presented previously to a situation where arbitrary many singular points become closer to another to result in a single, mostly non stable, singular point at the instant of bifurcation. For this reason, the new, simpler local structure shall correspond to the local phase portrait of the singularity obtained by merging the previous close singular points. In fact, this merging process can be seen as the result of a graphical back zooming applied locally to the original complicated topological feature: The singularities appear closer as one moves away, until they cannot be distinguished from the large and look like a single one.

This provides the basic ideas of the two methods presented in this chapter. The first one is cell-based and handles vector and tensor fields defined over rectilinear grids with piecewise bilinear interpolation. It can be extended to general curvilinear grids processed in computational space. The second method works independently from the underlying cells and associated interpolation scheme and is thus suited for arbitrary grid structures. Both methods are detailed next. We conclude the presentation by a discussion that suggests further applications of this technique and points out possible improvements.

6.1 Scaling on Rectilinear Grids

This section is organized as follows. We first precise what type of input data can be handled by the method. Next, the clustering strategy is detailed that we use to determine the groups of close singularities to be merged. This enables the local deformation of the topology to simulate the fusion of all involved singular points, as we show. The last step consists in identifying the structure

of the new feature created in that way. Some examples are shown that illustrate the results of the method and the scaling effect achieved in the vector and tensor case.

6.1.1 Input Data

We deal here with two-dimensional vector or tensor fields defined over a planar rectilinear grid. The choice of this structured grid is motivated by the cell-based nature of the method. It can also be applied to an arbitrary curvilinear grid mapped onto its rectilinear equivalent in computational space. The interpolation scheme is bilinear, inducing the linearity of the interpolant along the edges of the grid. A preprocessing step consists in computing the position and type of the singular points that are provided together with a reference to the containing cell.

6.1.2 Cell-based Clustering

In this section, we are concerned with the problem of grouping singularities together, that are distributed over the domain spanned by the grid. The grouping criterion is the proximity of the singularities contained in the same group. This is a typical clustering problem, i.e. the decomposition of a set of data into homogeneous subsets (or clusters), by minimizing the variance of some characteristic measure in each cluster. Here, one requires the clustering process to handle the underlying cells, enabling afterward local deformations that best preserve the original vector or tensor field. The ideas developed here are based on a clustering scheme designed by T. Schreiber [Sch91] for computational geometry. The original scheme deals with scattered weighted points without grid. Thus, it is modified to take the cell structure into account. Practically, a cluster is a group of connected cells and corresponding singularities.

The whole process works independently from the nature (vector/tensor) of the underlying field since the clustering part only focuses on (singular) points located in the cells of the grid. First, we introduce some convenient notations. We denote by P_1, \dots, P_m the positions of the m singularities lying inside a particular cluster. We want to minimize the approximation error of these m singularities by a single point, where this point (also called cluster *center*) Q is chosen to be the best approximation (for a given norm) of the singularities' mean point by a grid vertex (see Fig. 6.1). The corresponding error is given by

$$S = \frac{\sum_{j=1}^m \omega_j \|P_j - Q\|}{\sum_{j=1}^m \omega_j}$$

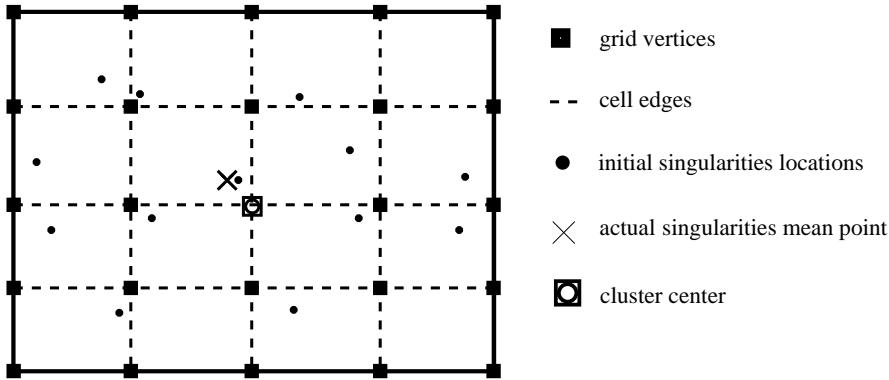


Figure 6.1: Cluster singularities and cluster center

where ω_i is the weight (set equal to 1 in our case, see discussion about possible other choices at the end of this chapter) associated with the i -th singularity. Here, the distance between the P_j and Q can be evaluated according to any norm (e.g. euclidean or infinity): The quantity S is used to measure the proximity of all singularities contained in a cluster and thus permits a numerical evaluation of its quality.

Hence, the aim of the clustering process is to partition the whole domain spanned by the structured grid into clusters that all have an associated error value smaller than a prescribed threshold τ (which is the only parameter of the method) and that contain all the cells.

We start with the whole grid considered as initial cluster. For the recursive subdivision, we proceed as follows: If a cluster does not satisfy the given error criterion, we split it into two subclusters. For this, we first compute the projected variances associated with a given cluster:

$$V_i = \sum_{j=1}^m \omega_j (P_j^i - Q^i)$$

where $i \in 0, 1$ is the considered coordinate axis ($P_j = (P_j^0, P_j^1)$). Now, the successive steps of the method are as follows.

For each cluster:

1. Take the best vertex approximation of all cluster's singular points as cluster center.
2. Compute the approximation error S .
If ($S > \tau$) go to step 3.
Otherwise stop.

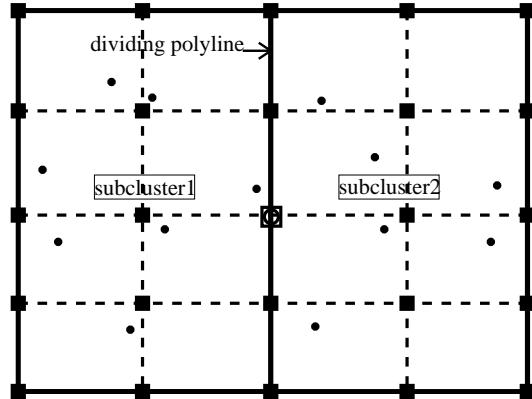


Figure 6.2: Cluster splitting

3. Compute the coordinate axis with largest projected variance (i.e. $\max(V_0, V_1)$).
4. Create 2 subclusters by splitting the cluster along an edge polyline through Q perpendicular to the selected coordinate axis.
Go to step 1.

The last step justifies the need for a cluster center to be a grid vertex. As a matter of fact, when splitting a cluster, one keeps processing cell groups in the form of Fig. 6.2. It also explains why the grid must be rectilinear: The subsplitting of the cluster must correspond to a minimization of the largest projected variance which is defined by projection on one of both coordinate axes. Note that this projection is the simplest and thus the fastest one which motivates this choice.

To ensure the termination of the algorithm, one requires, at each step, the existence of a best singularity mean point approximation by a grid vertex that does not lie on the cluster boundary. Otherwise, the recursive splitting of a cluster is stopped. An additional - and obvious - criterion is the presence of at least two singular points in the cluster.

In the end, the grid has been partitioned into cell groups that either contain singular points which are close to another, in the sense of the approximation error introduced previously, or that cannot be split further with respect to the termination criterion. These cell groups are (at least in computational space) rectangles. In each valid cluster (satisfying the proximity threshold), we can now process the merging of all contained singular points as described next.

6.1.3 Local Topology Deformation

Once a cluster has been isolated, its corresponding local topological structure must be simplified. In our case, this simplification consists in replacing all the singularities by a single one that presents the same aspect in the large while preserving consistency. To achieve it, we choose to simply let the field unchanged on the cluster boundary (piecewise linearly interpolated) and moreover to preserve continuity of the field across the boundary. In fact this even ensures the locality of the induced deformation.

From chapters 2 and 3, we know that the singular point resulting from the merging of several singularities typically presents a non-linear structure. Therefore, the singular point to be artificially created must be non-linear and the complexity of its structure depends on the number of singularities involved in the merging. As a consequence, it cannot be a singularity located in the interior of a rectangle cell contained in a final cluster (which is typically of linear nature). At this stage we make use of the property introduced in chapter 4 concerning the singular points lying on vertices of a piecewise linear triangulation. As shown previously, such a singularity may have arbitrary complexity. Practically, we process as follows: First, we remove all the rectangular cells contained in the cluster. Next, the resulting hole in the grid is filled up by a triangle stencil centered at the cluster center (defined after processing as the mean point of the contained singular points according to the chosen norm). The new internal cell structure is illustrated in Fig. 6.3: The triangles connect the cluster center with the grid vertices lying on the cluster boundary.

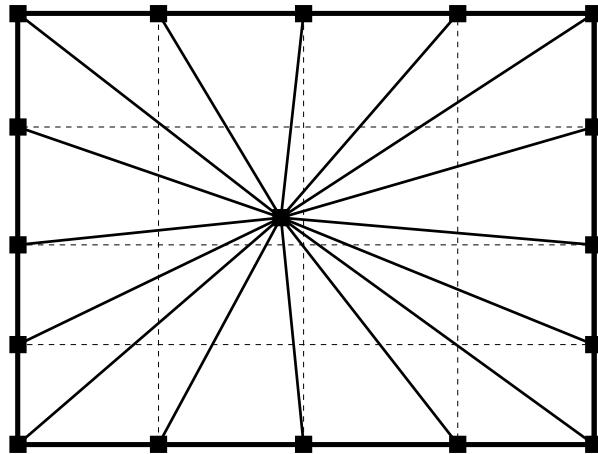


Figure 6.3: New cell structure in the cluster

Now, according to the ideas developed previously, we set a singular value to the internal vertex. In the vector case, it is a zero vector. In the symmetric

tensor case, it might be any isotropic matrix. Yet, as we choose to restrict our considerations to deviator tensor fields, the only possible value is a zero matrix. As far as the interpolation scheme is concerned, we apply a piecewise linear interpolation in the triangles. Remark that this choice is consistent with the piecewise linear nature of the restriction of the piecewise bilinear interpolant (defined elsewhere over the grid) to the edges of the cluster boundary. The singular point defined at the internal vertex is by construction the only one present in the cluster after modification. This is easily proven by recalling that an affine linear vector (resp. tensor) field has, at most, one single singular point (the degenerate case where the system to solve is singular and therefore infinitely many singularities are present would mean that a singular point is present on the cluster boundary, which is rejected by the method).

As a consequence, we have eventually created a singular point with arbitrary complex structure (the complexity depends on the number of triangles in the stencil) while preserving the piecewise linear value of the “exterior” field on the cluster’s boundary. Moreover this guarantees consistency of the local structure with the rest of the topology. In particular, the index of the cluster has been trivially maintained.

6.1.4 Structure Identification

Once this singularity has been created, its structure must be characterized to enable the drawing of its separatrices and thus depict the new simplified topology. The technique used here was presented in chapter 4, p. 81. Recall that it is based on the search, on each edge of the cluster’s boundary, for positions where the vector (resp. eigenvector) field is parallel to the vector emanating from the internal singular point. This corresponds to a quadratic (resp. cubic) polynomial equation. In the vector case, this search is completed by the location of positions where the vector field is orthogonal to the coordinate vector and the use of the sector discrimination graph shown in Fig. 4.16, p. 84. In the tensor case, an evaluation of the eigenvector’s angle variation between two consecutive *parallel* positions provides the missing nature and position of the different sectors, see chapter 4.

6.1.5 Results

This section presents the results of this method applied to a vector and a tensor dataset obtained by numerical simulations from Computational Fluid Dynamics (CFD). They both correspond to the simulation of a swirling jet with inflow into a steady medium. This results in a vortex breakdown, the turbulent nature of which makes these datasets very interesting for our purpose.

Vector Case

This dataset has been cut off from a three-dimensional vorticity vector field. The grid is structured and has 251×159 cells ranging from 0 to 15 along the x -coordinate, resp. -1.9 to 1.9 along the y -coordinate. The initial topology has a very complicated structure and contains 337 singular points and 624 associated separatrices, see Fig. 6.4. Note the presence of isolated critical points on the upper and lower left side of the picture: They are located in areas where the magnitude of the field is tiny and thus no meaningful streamline integration can be achieved numerically.

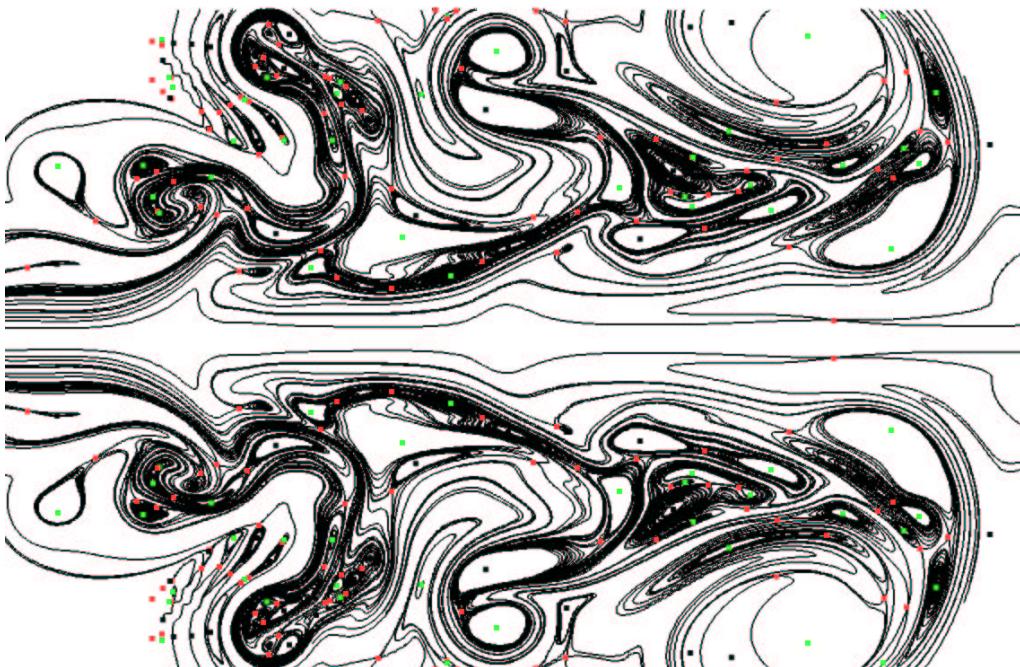


Figure 6.4: Original topology

We first apply the method with a threshold of 0.2. We use the euclidean norm to measure the proximity of the critical points. The resulting topology presents 140 critical points (88 of which are higher-order ones, i.e. artificially created by merging) and 359 associated separatrices: It is shown in Fig. 6.5. If one increases the threshold to a value of 0.4, the topology gets simplified as follows. There are 88 critical points remaining (6 of which are original ones), see Fig. 6.6.

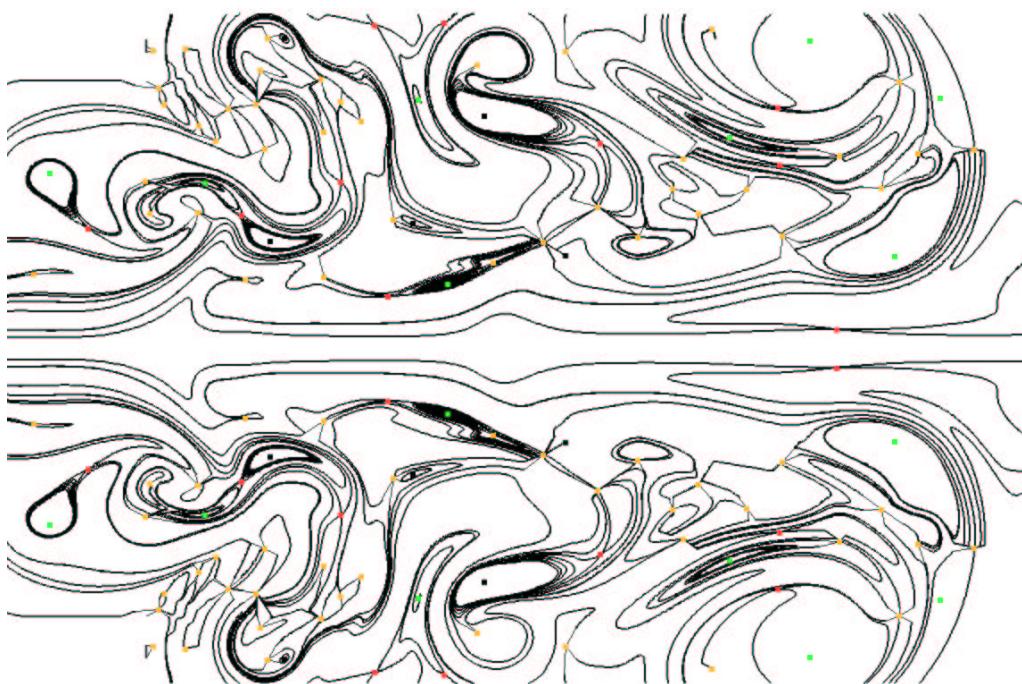


Figure 6.5: Scaled topology: Threshold = 0.2

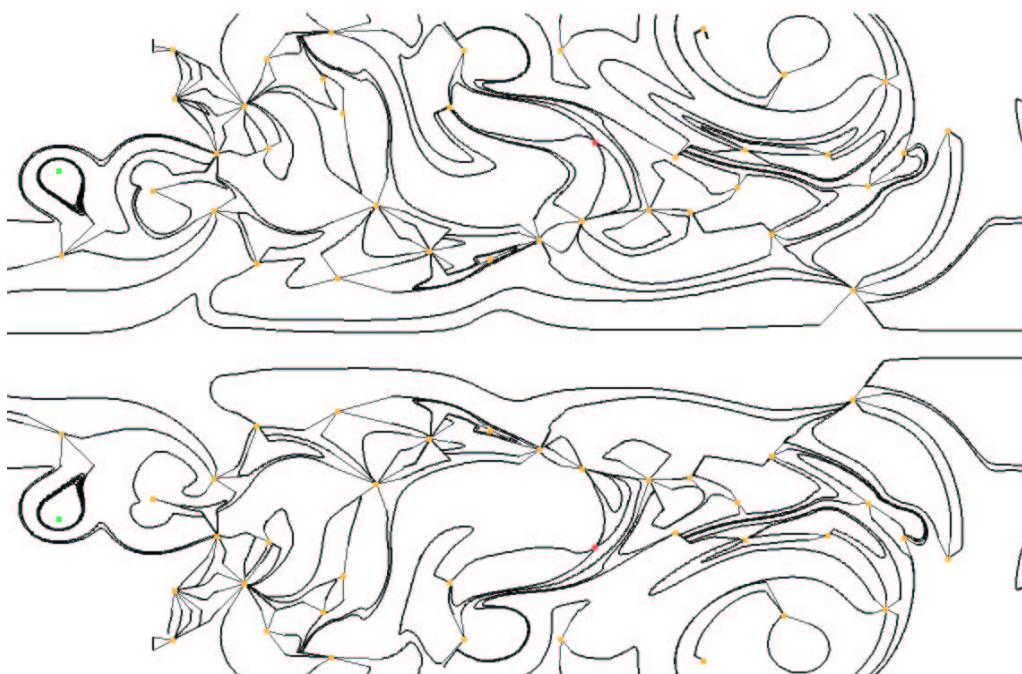


Figure 6.6: Scaled topology: Threshold = 0.4

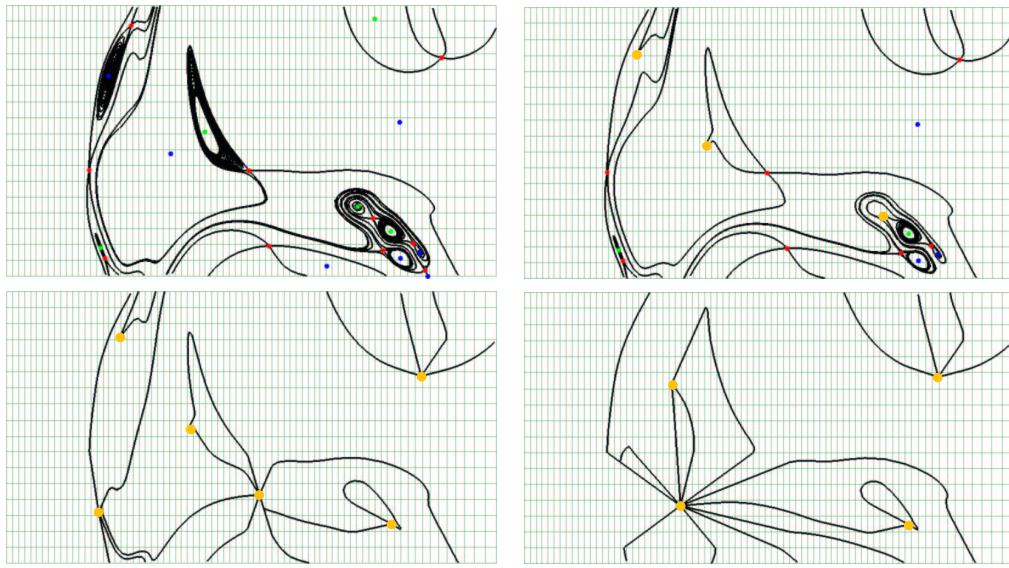


Figure 6.7: Local topology scaling: initial graph and simplifications with 0.1, 0.2 and 0.4 as thresholds

Note that in both cases, the global symmetry of the original graph has been well preserved after scaling, which is inherent to the scheme used to cluster the critical points. To demonstrate the local topological deformation induced by the method, Fig. 6.7 shows the successive topological aspects of a small part of the grid corresponding to increasing values of the threshold. Remark that the scaling effect applied to the critical points also induces the simplification of small closed orbits.

Tensor Case

This dataset is a two-dimensional cut of a three-dimensional rate of strain dataset corresponding to the same swirling jet simulation as for the vector case. The grid is rectilinear and has 124 x 101 cells ranging from 0 to 9.87 in x, resp. -3.85 to 3.85 in y. The original topology presents 61 degenerate points and 131 separatrices. It is shown in Fig. 6.8. Remark that this topology contains no cycle, as usual for tensor fields. One starts scaling with $threshold = 0.2$. The resulting topology contains 44 singularities (13 of which being non-linear) and 101 associated separatrices, see Fig. 6.9. (Non-linear singularities are depicted by big dots.) Increasing the threshold ($threshold = 0.4$), one simplifies the topology further. There are now 31 degenerate points (17 being non-linear) and 76 separatrices in the graph, see Fig. 6.10. The last stage of the scaling process is obtained with $threshold = 0.8$. This topology has 15 degenerate points (only 2 of them are original ones) and 42 separatrices. The result is shown in Fig. 6.11.

To observe the local effect of the method on close singularities, samples of a close-up corresponding to increasing thresholds are proposed in Fig. 6.12.

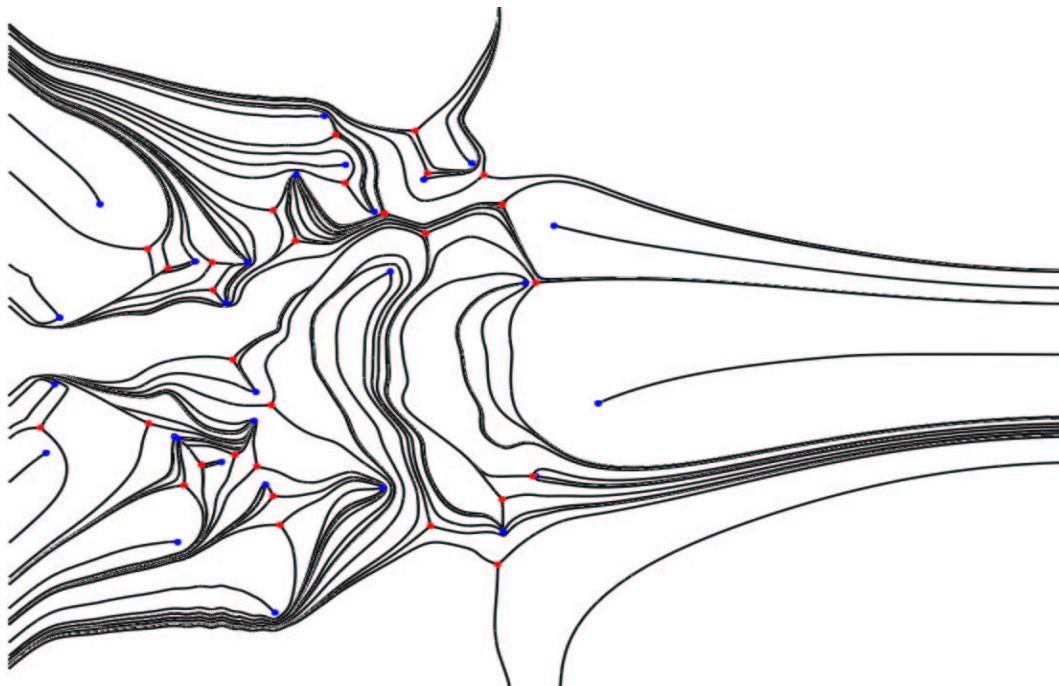
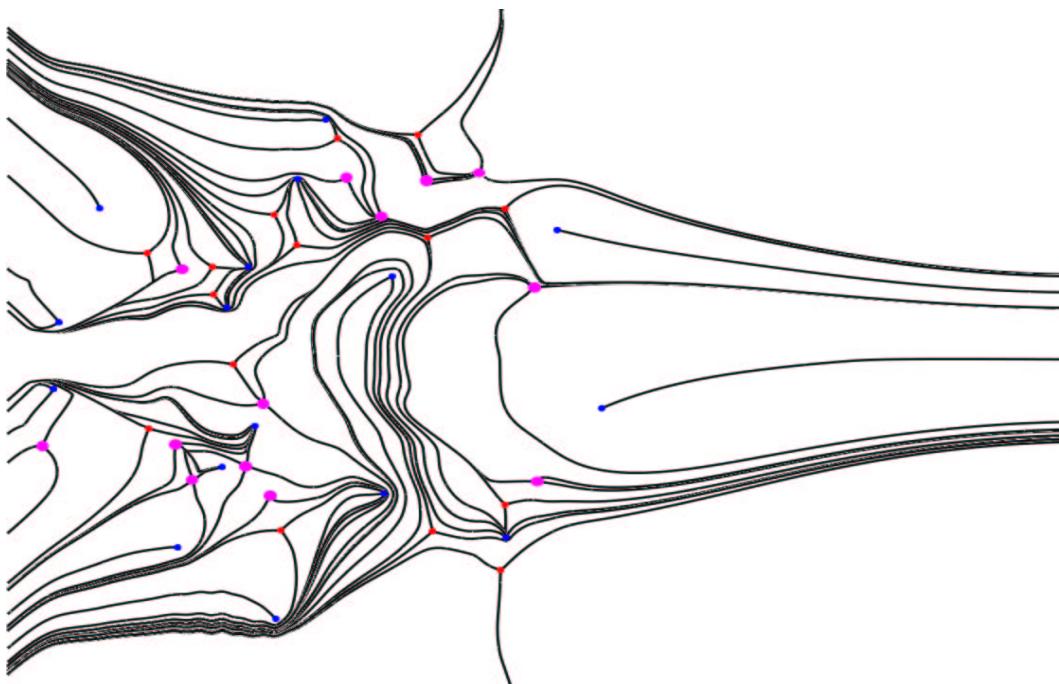
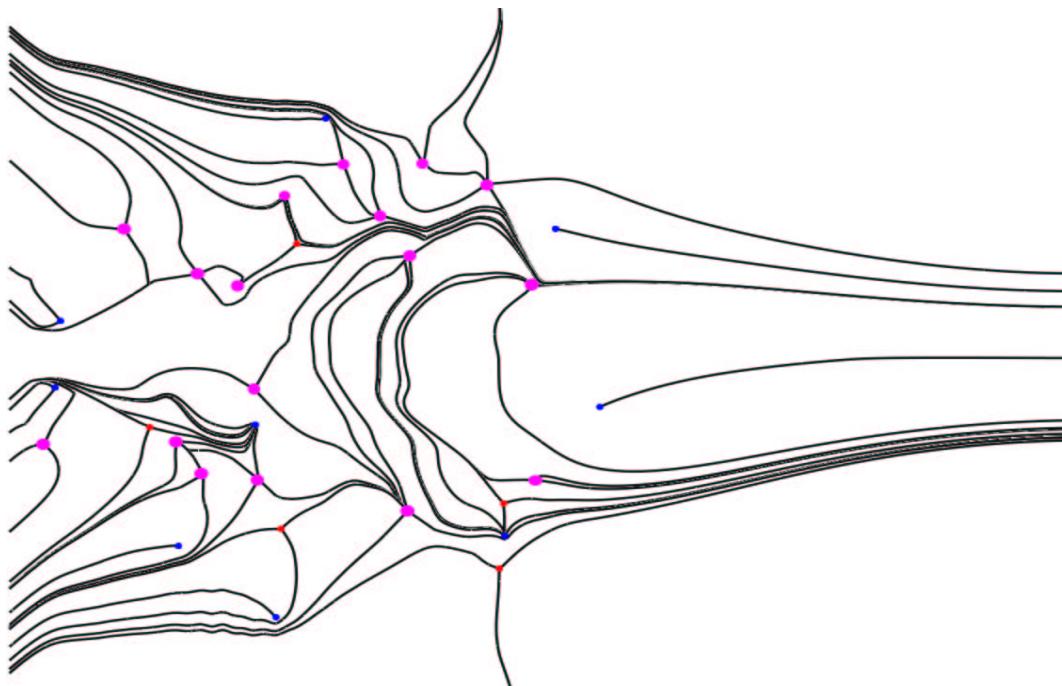
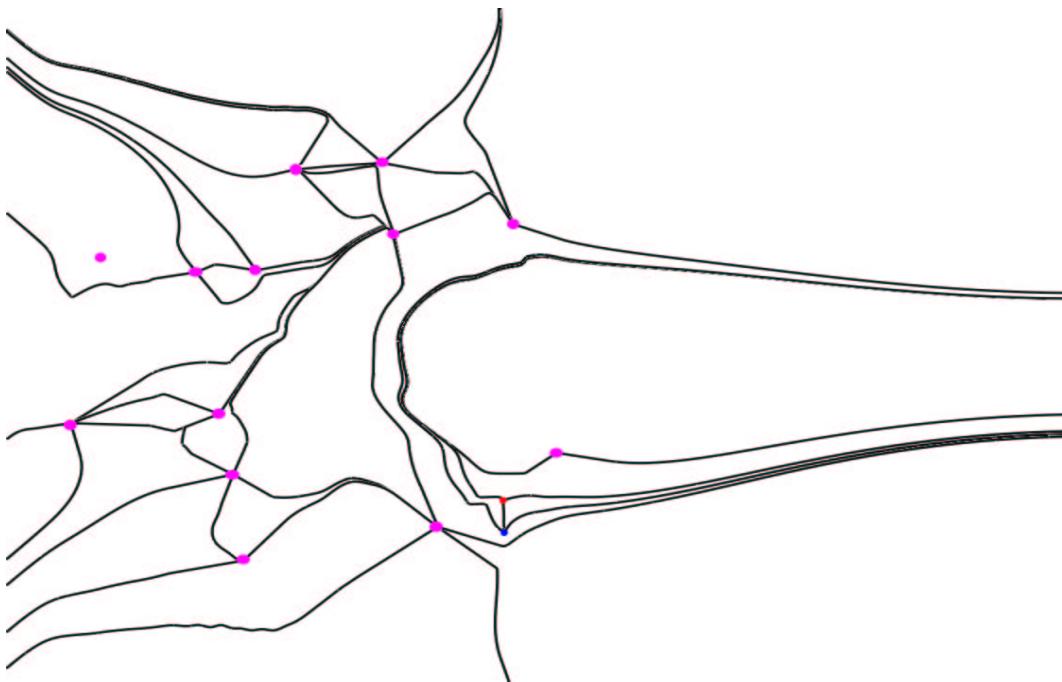


Figure 6.8: Original topology

Figure 6.9: Scaled topology ($threshold = 0.2$)

Figure 6.10: Scaled topology ($threshold = 0.4$)Figure 6.11: Scaled topology ($threshold = 0.8$)

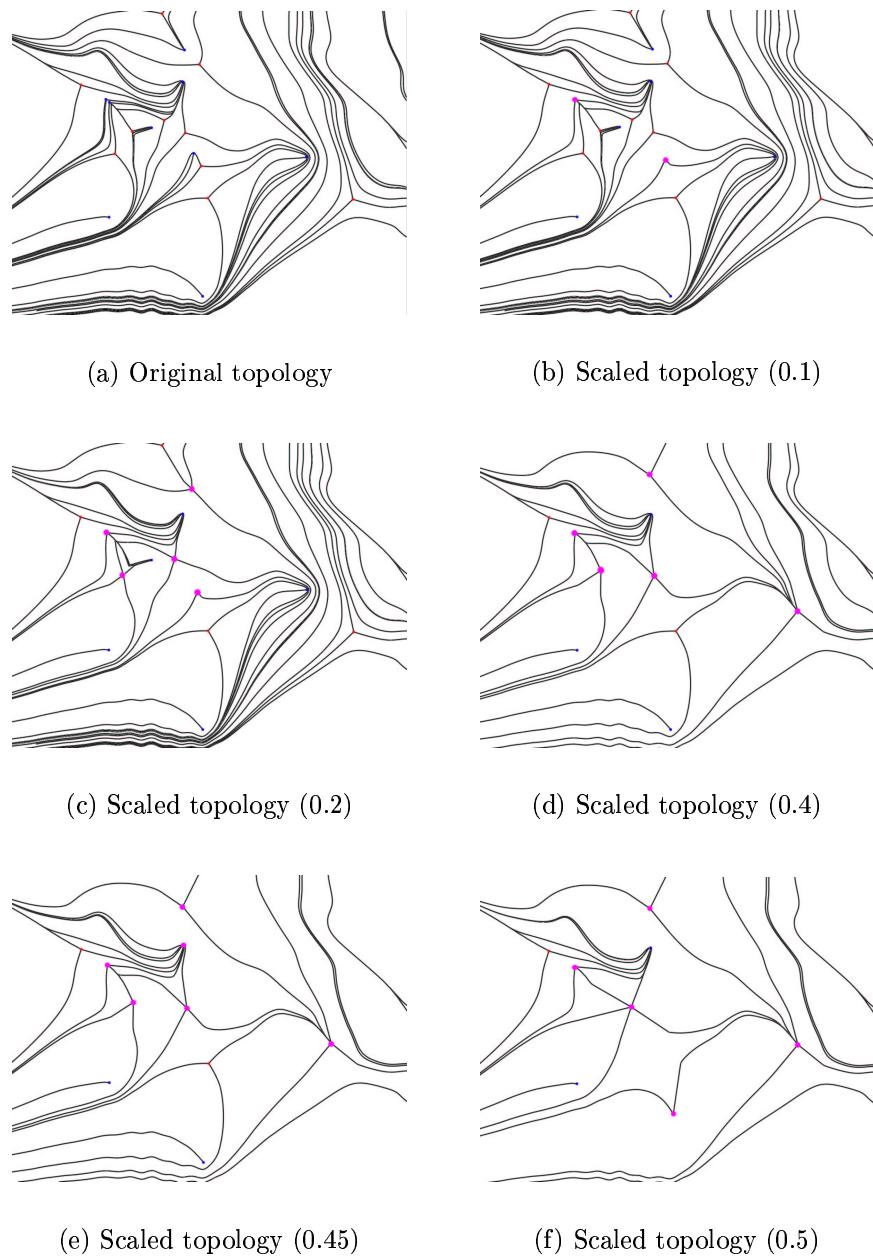


Figure 6.12: Samples of local topology scaling

6.2 Scaling on Arbitrary Grid Structure

As opposed to the previous method, no assumption is made here about the grid related to the considered vector or tensor field. In particular, no structure is expected and the cells can be arbitrarily interpolated. We just require the field to be continuous over the grid. Consequently, an important difference resides in the clustering scheme. It works independently from the cells' geometry which increases the flexibility of the method. This implies that the local topological deformation must be adapted as well to ensure continuity of the field through the cluster boundary. This section is structured as follows. The new, cell-independent, clustering scheme is presented first. The merging of the singularities contained in a cluster follows the same basic principle as in the previous method but is based on another interpolation scheme as explained next. The technique used to identify the simplified topological structure generalizes the one presented previously and is briefly explained then. At last, results are shown.

6.2.1 Cell-independent Clustering

The clustering scheme used in the unstructured case is in fact a simplified version of the scheme designed for rectilinear grids. The whole processing only focuses on the singularities distributed over the definition domain. The clusters are in this context, rectangles (or bounding boxes) that contain close singular points and that are informed about the cells that intersect their interior: See Fig. 6.13.

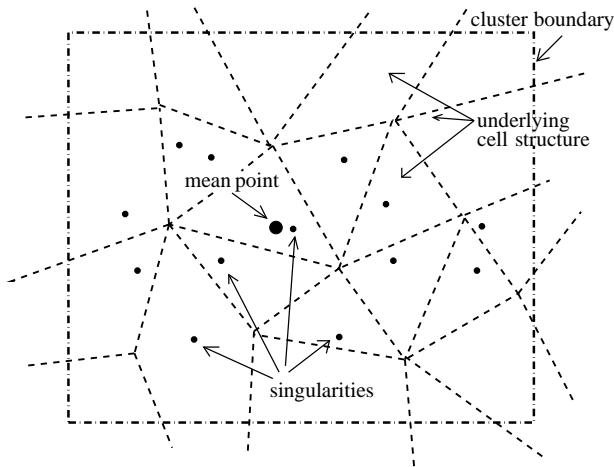


Figure 6.13: Cluster singularities and cluster center

As far as cluster subdivision is concerned, the strategy is the same as

previously: The approximation error

$$S = \frac{\sum_{j=1}^m \omega_j \|P_j - Q\|}{\sum_{j=1}^m \omega_j}$$

is computed, according to a particular norm, and if the value found is larger than the prescribed threshold τ , the projected variances are evaluated for both coordinate axes:

$$V_i = \sum_{j=1}^m \omega_j (P_j^i - Q^i).$$

The cluster is then split by a straight line through the singularities' mean point, orthogonal to the axis with largest projected variance. The original cluster enclosing the whole grid is recursively subdivided in that way until every subcluster fulfills the threshold. Compared to the structured case, the clustering is here processed independently from the underlying cell structure since we somehow apply a convenient artificial axisparallel grid. Remark that the splitting strategy used here is not the only possible one: A cluster can be subdivided in an arbitrary way if the convexity of the sub-clusters is preserved. Nevertheless, our choice leads to clusters with edges parallel to the coordinate axes which enables fast processing.

6.2.2 Local Topology Deformation

After the clustering step, we are left with a set of clusters that contain close singularities and know what cells are contained in them. We compute next, for each final cluster and for each contained cell, the possible intersections of the cell's edges with the cluster boundary. This can be done very efficiently because the cluster edges are parallel to the coordinate axes. Adding the 4 cluster corner points to the intersection positions found, we get a list of positions that we sort in a counterclockwise order. Now we isolate the interior domain of the cluster from the rest of the grid. This is done by removing all cells entirely contained in the cluster (without intersection with cluster boundary) and cutting away the part of every cell intersecting the cluster interior domain: This corresponds to superimposing locally a new small grid on the initial one (see Fig. 6.14).

The cut cells have now a modified geometry but keep their interpolation scheme to ensure continuity and consistency with the original field. In particular, the field value along the cluster boundary remains unchanged. Inside each cluster, we want to get a continuous piecewise analytic field description after modification that ensures the presence of a unique singularity located at the singularities' mean point. Furthermore, we want this description to preserve

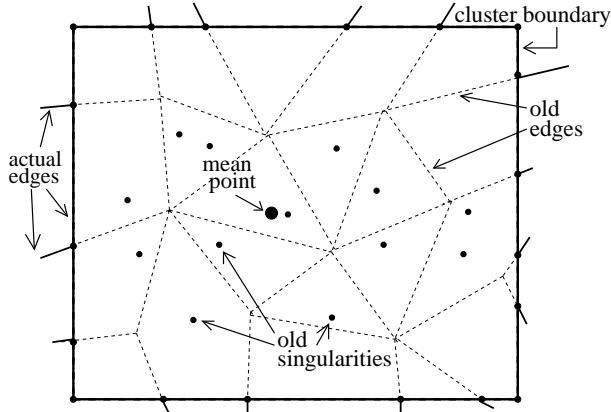


Figure 6.14: New grid structure around a final cluster

the field value on the cluster boundary. Consequently, we cover the cluster interior domain as follows: Inserting an additional vertex at the mean point position, we build a triangle stencil connecting this point with every position on the cluster boundary. Furthermore, we associate the new vertex with a singular value: See Fig. 6.15.

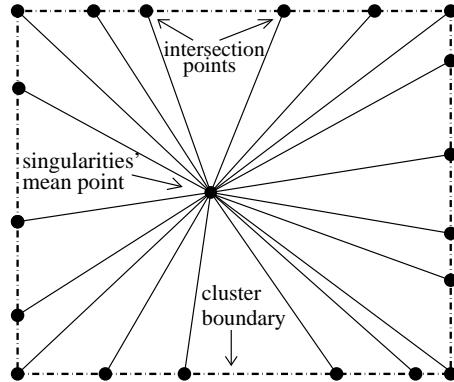


Figure 6.15: Triangle stencil inside a final cluster

In each of these triangles, we need an interpolation scheme that interpolates the field value on the cluster boundary. This is achieved by using a simple side-vertex interpolation scheme: The position of every point inside such a triangle is determined as shown in Fig. 6.16, so we get

$$Q(t) = (1 - t)A + tB \text{ and } P(t, u) = (1 - u)\Omega + uQ(t).$$

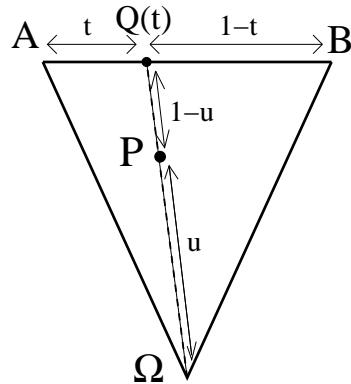


Figure 6.16: Side vertex interpolation

The interpolated value is (with \mathbf{f} denoting the considered field)

$$\mathbf{f}(P)(t, u) = u \mathbf{f}(Q(t)), \text{ since } \mathbf{f}(\Omega) = 0$$

where $\mathbf{f}(Q(t))$ is the original value on the cluster boundary (this is the same configuration as in Fig. 4.13, p. 81).

This ensures that the field on the boundary is preserved which guarantees continuity for the new piecewise analytic description. We can also claim that the new artificial singularity is the only one contained in the cluster after modification (otherwise, we would have, for some t , $Q(t) = 0$, and we would have a singular value on the boundary which cannot occur because such a case is rejected during the clustering process).

6.2.3 Structure Identification

As in the structured case, the identification of the structure corresponding to the deformed topology is based on the identification of the positions on the cluster boundary where the vector (resp. eigenvector) field is parallel to the coordinate vector from the cluster center. We saw that it leads to a quadratic polynomial equation in the vector case and a cubic equation in the tensor case, when the field is piecewise linear along the boundary. Now, in the general case, one has the following result: If the field on the boundary is a piecewise polynomial of degree n , then the system to solve for the “parallel” positions is of degree $n+1$ in the vector case and $n+2$ in the tensor case. Thus, if the field is polynomial of degree 3 or greater, an analytic solution of such a system will be typically impossible and therefore a numerical search is required. Note that this numerical search can be conducted without precise knowledge of the interpolant. A classical scheme for this purpose is Newton-Raphson (see [PTV92]). Once these positions have been detected, the identification is completed in the

vector case by the search for positions where the vector field is orthogonal to the coordinate vector and the use of the sector discrimination graph presented in Fig. 4.16, and in the tensor case by the (numerical) computation of the angle rotation of the field between two consecutive “parallel” positions as explained previously.

6.2.4 Results

We present here the results of the method applied to two artificial datasets, both defined over a Delaunay triangulation of scattered points. The vector and tensor values have been computed at random to result in turbulent topologies that require to be scaled. The interpolation scheme is in both cases piecewise linear.

Vector Case

The first dataset is a 2D vector dataset. The grid has 400 vertices, ranging from -5 to +5 in x and y . The original topology contains 189 critical points and 380 separatrices (see Fig. 6.17).

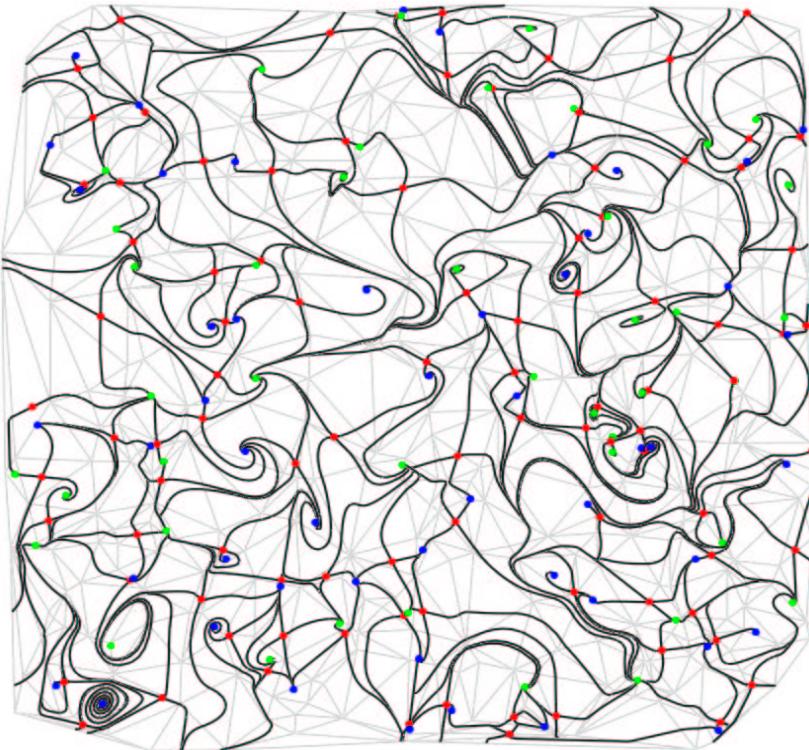


Figure 6.17: 1st example: Vector case

We first simplify this complicated topology with a clustering threshold of 0.5. The graph has now 114 critical points and 286 separatrices (see Fig. 6.18). To ease the interpretation, higher order singularities are depicted as big dots. Using a threshold of 1.5, there are 81 critical points and 188 separatrices remaining (Fig. 6.19).

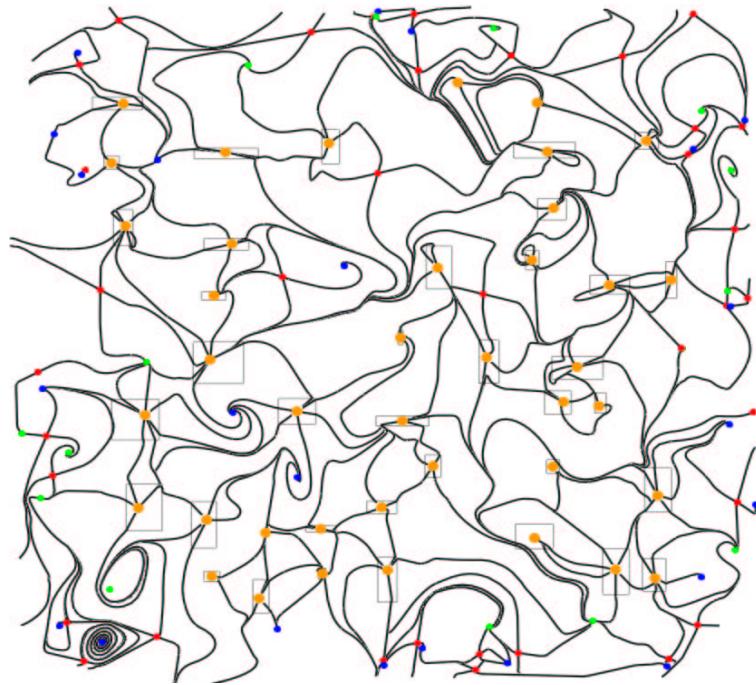


Figure 6.18: 1st example: threshold = 0.5

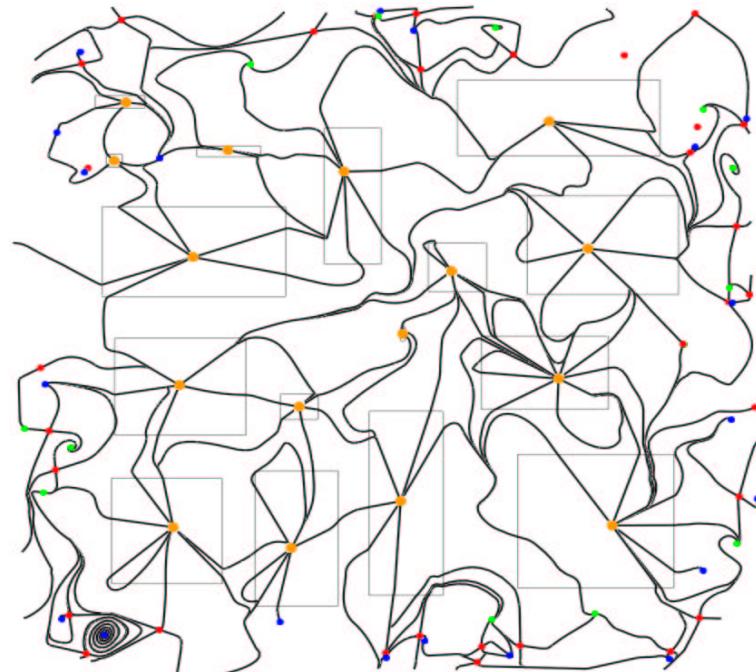


Figure 6.19: 1st example: threshold = 1.5

Note that the simplification process does not affect the topology close to the grid boundary (which explains the presence of many singularities) to preserve consistency to the original data.

Tensor Case

The second dataset is a planar symmetric, second-order tensor field. The grid has 300 vertices, ranging from -3 to +3 in x and from -2 to +2 in y . The original topology contains 114 degenerate points and 242 associated separatrices: See Fig. 6.20.

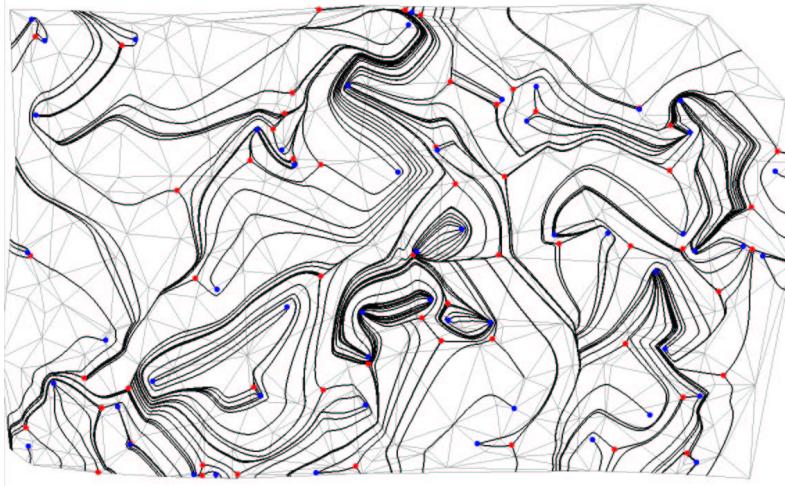


Figure 6.20: 2nd example: Tensor case

The first scaled topology is obtained with a threshold of 0.5. We use the infinite norm to determine the proximity of the singularities. The final clusters are depicted as rectangles. The higher-order singularities created artificially are shown as bigger points. The topology presents 69 degenerate points and 162 separatrices (see Fig. 6.21). With a threshold equal to 0.75, one obtains the topology shown in Fig. 6.22: There are 59 degenerate points remaining and 134 separatrices. Here again, the merging of the singular points take only place in the interior of the domain, preserving the features located close to the grid's border. The last result shown corresponds to a threshold of 1.5: See Fig. 6.23. The topology is now characterized by 49 degenerate points and 110 separatrices.

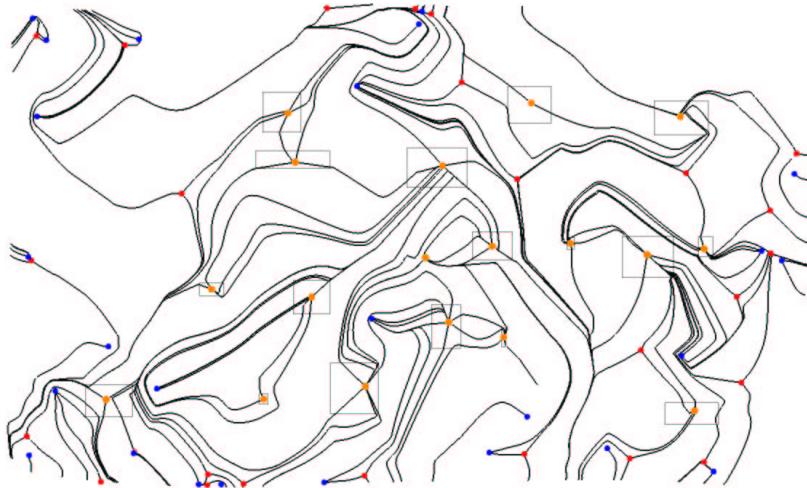


Figure 6.21: 2nd example: threshold = 0.5

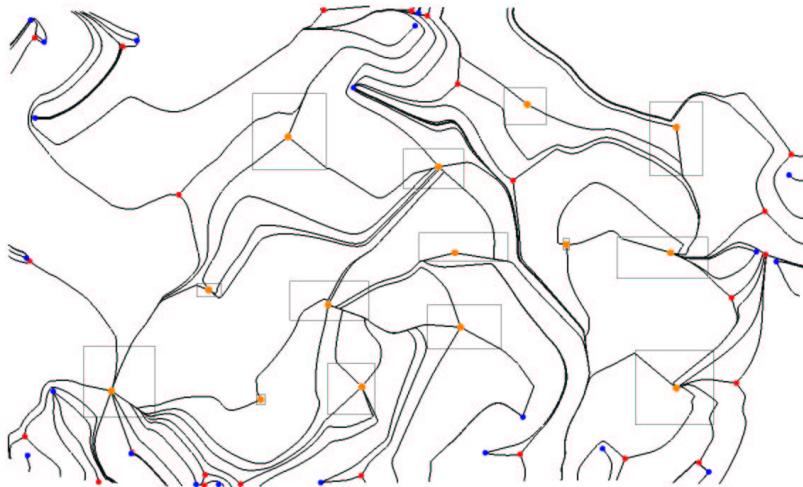


Figure 6.22: 2nd example: threshold = 0.75

6.3 Discussion

The two methods presented above are designed to attack the problem of visual clutter encountered with turbulent vector or tensor fields by topology-based schemes. The technique used here is based on a scaling approach of the structural complexity: Singularities that lie sufficiently close together should be merged because they cannot be properly distinguished in the global depiction on one hand, and because they are likely to correspond to interpolation artifacts on the other hand. As shown previously, the fusion of close singularities can be achieved on structured and unstructured grids. In both cases, the

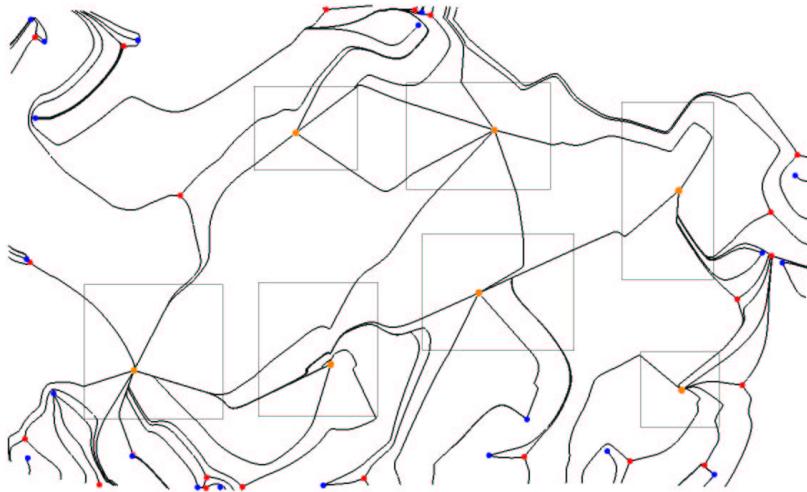


Figure 6.23: 2nd example: threshold = 1.5

scaling is obtained by local deformations of the grid. The resulting topology appears clarified since the number of singularities has been reduced along with that of associated separatrices. Remark that this technique can be applied as well when the objective is to recover higher-order singularities, inferred to be present in the underlying field. This constitutes an alternative to the scheme proposed in [SHK97]: Singular points of arbitrary indices can be obtained in that way while preserving a low-order interpolation scheme and ensuring continuity over the whole grid.

Now this approach is somehow purely geometrical. As a matter of fact the clustering scheme used to put singularities together only considers their relative position. In fact this property permitted to handle vector and tensor fields in a very similar way. Consequently qualitative aspects, like e.g. the types of the singularities, their interconnectivity or their structural influence, were ignored. In some cases, this might be insufficient. For instance, if one aims at filtering numerical noise, one expects to get undesirable singularities in regions where the norm of the field (euclidean norm for vectors, L^2 for deviators) is very small. The norm should therefore be taken into account when the merging of a given singular point with others is considered. Further, the interpretation of topology depictions in the context of fluid dynamics evaluates the relevancy of the singularities with respect to some additional quantities like e.g. vorticity or enstrophy (depending on the case, they can be either derived from the considered field or must be provided by another one). Here again, such criteria should then be evaluated and tested before modifying the topology. Actually these considerations are the motivation of an alternative simplification approach presented in chapter 8. However there

is a possibility to involve qualitative insight in the present method. As said in the presentation of the clustering scheme in section 6.1.2, our first strategy for weighting the singularities was a uniform law. This is a simple (and efficient) choice but it ignores any particular property of singular points. Now, we can replace these values by weights that reflect the importance of every singular points with respect to some relevancy function that depends on the considered interpretation. Note that we are able in that way to enforce the merging of several singular points to take place close to those with largest values and, by definition, with greatest importance.

Chapter 7

Topology Tracking

Topological methods, as defined and presented previously, offer an efficient and appealing way to visualize steady or instantaneous planar vector or tensor fields. Their structure is characterized by the fixed positions of the singularities, that constitute the nodes of a graph, the edges of which are special curves computed over the two-dimensional domain. But typical applications involve time as an additional parameter. Therefore, the topologies visualized with classical methods must be interpreted as instantaneous samples of a structure that evolves over time. Consequently, the visualization of the actual topology requires to reconnect these discrete samples to propose a continuous depiction of the evolution. A straightforward approach consists in doing this graphically: The successive topologies are considered as graphs that must be joined together (in an animation or a three-dimensional depiction where time is visualized as third dimension) in a consistent way. The major problem of this technique is the lack of qualitative continuity: There is no underlying continuous model and thus no ability to locate and identify the structural evolutions that may occur and thus lead to dramatic changes between two consecutive discrete time steps. These evolutions are known as bifurcations and are presented in chapters 2 and 3. They correspond to the transition from a stable state of the topology to another stable one, through an unstable, instantaneous intermediate state called bifurcation point. Practically, these stable states are all we can observe on discrete samples of a time-dependent topology. This prevents a proper visualization of bifurcations whereas they are key events in the temporal evolution of vector and tensor fields.

In the present chapter, a new method is proposed to track the topology of time-dependent vector and tensor fields. According to the remarks above, its basic principle is to build a three-dimensional space/time continuum thanks to a convenient interpolation scheme that enables an accurate detection of

topological changes. More precisely, keeping the same approach as original topological visualization methods, our focus is on singularities and on the local bifurcations affecting them (position, nature and existence). This means that global bifurcations, the study of which is essentially more complicated, are first let apart by the method to be possibly found after processing. The result can then be visualized as a three-dimensional picture where the bifurcations are easily identified as particular positions in the space/time domain. The singularities move along paths that are represented by curves and the separatrix curves span surfaces that partition the domain over time.

The chapter is structured as follows. The first section is devoted to a description of the grid structure designed to offer the required three-dimensional space/time continuum: A three-dimensional grid is used to reconnect the two-dimensional discrete samples over time. The input data is discussed too. The interpolation scheme used in the grid is presented next. The properties of the interpolant reduce the possible local bifurcations in the vector and tensor case to a few ones, as we show. Then, we explain the tracking of a singularity over a single cell: Trajectory, type and bifurcations on the way are considered. As next step, a global tracking is processed, based on the local information collected in each cell, that reconstitutes the path of a given singularity over the whole grid. Here again, attention is paid to possible additional bifurcations. The last task consists in integrating and displaying the surfaces spanned by the separatrix curves during their motion over time. This results in pictures similar to those proposed by Abraham and Shaw in [AS82]. We show the application of the method to a vector and a tensor dataset. We conclude the presentation with a discussion that points out further developments of this new visualization framework. In particular, the processing a CFD dataset permits us to comment the application of our technique to turbulent vector or tensor fields.

7.1 Grid Structure

The visualization of time dependent vector (or tensor) data has to deal with a higher dimensional mathematical space where time constitutes an additional dimension. This space must be handled continuously to enable the detection and depiction of bifurcations. In our two-dimensional case, time is handled as third coordinate axis and the whole data embedded in a three-dimensional scene.

Practically, we process a two-dimensional vector (resp. tensor) field lying on a triangulation with constant vertices over time: We dispose the several instantaneous states of the field parallel to another (each of them is called

time plane in the following), corresponding to their successive positions along the time line. A 3D grid evolves by connecting these planar grids together with 3D cells as shown in Fig. 7.1. There are several possible cell structures

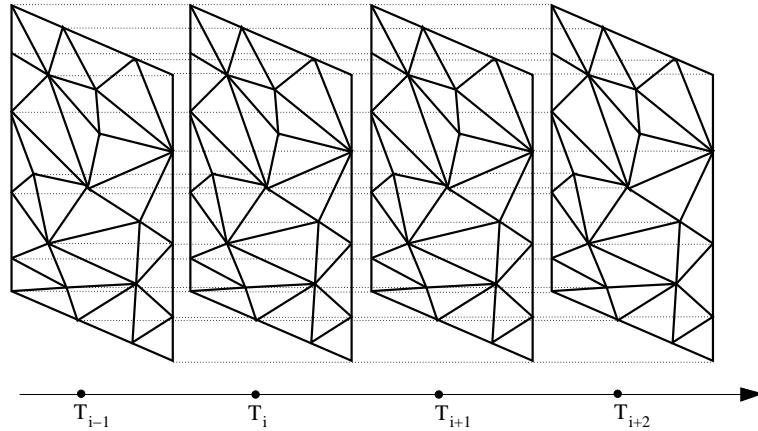


Figure 7.1: Grid structure

for connecting the successive time planes. Yet, a natural choice consists in joining together the corresponding triangles over time (they are obtained by translation along the time axis). This leads to prism cells, as in Fig. 7.2. Note

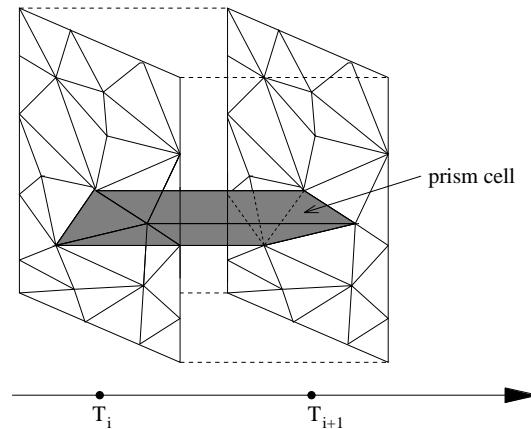


Figure 7.2: Prism cell

that this cell structure is preferred to a simple tetrahedrization of the point set because it preserves the topological continuity over time in each triangle.

7.2 Interpolation

The input data are two-dimensional vectors (resp. tensors). In each time plane, the field is originally piecewise linearly interpolated over the triangulation. The task consists in interpolating these successive planar fields over the three-dimensional grid presented previously. More precisely, in each prism cell one needs to interpolate between two linearly interpolated triangular faces. For simplicity in further processing, this temporal interpolation is linear too. Let \mathbf{f} be the multivariate (vector or tensor) interpolating function for a given prism cell and G be the closed interior domain of the cell, we get

$$\begin{aligned} \mathbf{f} : \mathbb{R}^3 \supset G &\longrightarrow \mathbf{T} \text{ (n-dimensional tensor space)} \\ (x, y, t) &\mapsto \boldsymbol{\alpha}(t) + \boldsymbol{\beta}(t)x + \boldsymbol{\gamma}(t)y =: (f_1(x, y, t), \dots, f_n(x, y, t)), \end{aligned}$$

(affine linear in x and y) with $f_i(x, y, t), i \in \{0, n - 1\}$ linearly interpolated in $t \in [t_j, t_{j+1}]$:

$$f_i(x, y, t) = \alpha_i(t) + \beta_i(t)x + \gamma_i(t)y,$$

and

$$\alpha_i(t) = \frac{t_{j+1} - t}{t_{j+1} - t_j} \alpha_i^j + \frac{t - t_j}{t_{j+1} - t_j} \alpha_i^{j+1} \text{ (idem for } \beta_i \text{ and } \gamma_i\text{).}$$

That means that the coefficients of the linear spatial interpolation in each triangular face are linearly interpolated over time between $t = t_j$ and $t = t_{j+1}$ and induce a linear spatial interpolation for the field in every plane orthogonal to the time axis (i.e. where time is constant).

7.3 Local Singularity Tracking

Tracking a singularity through a prism cell consists of two tasks: The first one is to compute the trajectory of a singular point through the cell and to determine its entry and exit positions. The second one is to identify the type of the singularity and to detect possible local bifurcations affecting it. Both aspects are described next, for the vector and the tensor case.

7.3.1 Vector Case

Path Equation

The affine linear nature of the restriction of the vector field to any time plane leads to the following singularity coordinates $(x(t), y(t))$ as function of the time

parameter t .

$$x(t) = \begin{vmatrix} -\alpha^0(t) & \gamma^0(t) \\ -\alpha^1(t) & \gamma^1(t) \\ \beta^0(t) & \gamma^0(t) \\ \beta^1(t) & \gamma^1(t) \end{vmatrix} \quad y(t) = \begin{vmatrix} \beta^0(t) & \alpha^0(t) \\ \beta^1(t) & \alpha^1(t) \\ \beta^0(t) & \gamma^0(t) \\ \beta^1(t) & \gamma^1(t) \end{vmatrix}$$

If t moves from t_j to t_{j+1} , the singularity position describes a 3D curve. Yet, we are only interested in the curve sections that intersect the interior domain of the considered prism cell (where the considered interpolant is defined). A simple way to determine them is to consider the singularities lying on the side faces of the prism: Two triangular faces lying in $t = t_j$ and $t = t_{j+1}$ and 3 quadrilaterals connecting them. We already showed that the interpolant is affine linear in both triangles (for they lie in time planes). As far as the quadrilateral faces are concerned, one can easily show that the restriction of the interpolant to these faces is bilinear. (This is because the vector normal to a quadrilateral face is always orthogonal to the time axis, by construction). Consequently, finding the position of a singularity in each prism face requires the solution of a simple linear/quadratic system: We sort the found (3D) positions in ascending time order (i.e. with respect to their third coordinate) and associate them pairwise. As a matter of fact, since at most one critical point can be present in a prism cell at a given time t (every instantaneous planar vector fields is affine linearly interpolated according to what precedes), we know that a critical point must first leave the cell before a singularity reenters it later. So, for each pair, we identify an **entry** and an **exit** position (see Fig. 7.3).

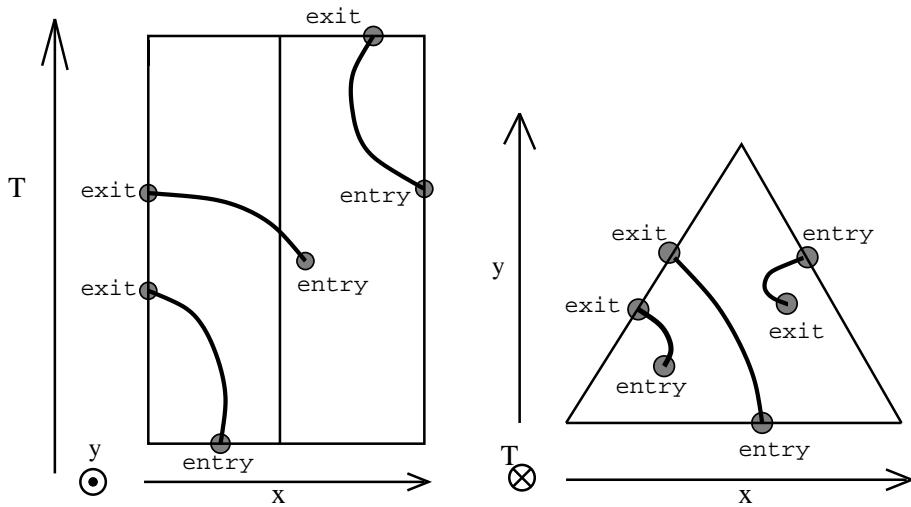


Figure 7.3: Entry and exit points of a path through a prism

Remark that the rational functions that describe the trajectory of a critical point through a prism are not defined at the instants t where the quantity $|\beta(t) \gamma(t)|$ is zero. Now, this is precisely the determinant of the Jacobian matrix of the vector field. This means that the trajectory diverges (the critical point leaves any bounded domain and the cell interior in particular) when t approaches a value where the Jacobian matrix is singular. These “singular” values of t correspond in fact to the roots of the quadratic equation above (β and γ being linear vector functions of t). Yet, these time values do not intervene in our computation because they always correspond to positions that lie outside the considered cell and are thus automatically neglected.

Path Type

In the following, only the generic types of linear critical points are distinguished since no finer characterization will be useful for our purpose. The possible types are *source*, *sink* or *saddle* (letting apart the *center* case, that corresponds to a transition between source and sink).

The possible transition from one type to another constitutes a local bifurcation as defined in chapter 2, section 2.7. We are interested here in the qualitative behavior of a critical point during its motion through the interior domain of a prism cell. Now, with our choice of the interpolation scheme, there is at a given time at most one single critical point present. So according to the presentation of the major types of local bifurcations, it comes out that only Hopf bifurcations are relevant to us. As a matter of fact, any other transition (e.g. fold or pitchfork bifurcation) would involve simultaneously two or more critical points present in the cell, either before or after the bifurcation point, which is impossible. Practically, such bifurcations can only be encountered on the common boundary of two neighboring cells, as detailed in the following. Actually, the kind of bifurcations we are interested in correspond here to the transition of a critical point from a type to another: The spontaneous disappearance (or creation) of a critical point inside a cell is also impossible, since one would move from a situation where the local index is ± 1 (a singularity is present) to a situation where this index equals 0 (no critical point in the cell), which would locally break consistency. Hence, with the generic types introduced previously the possible bifurcations concern the transition from a sink to a source and from a saddle to a sink or source (and vice versa). The first one is called Hopf bifurcation and was described in p. 34. The second one supposes again the local modification of the index (from -1 to +1) which is structurally inconsistent. Nevertheless, the following special case must be considered to understand what occurs in degenerate cases.

When the Jacobian has determinant zero, this matrix has one zero eigen-

value (and the other one is real and generally non zero). In this case, the variation of the time parameter implies a crossing of one eigenvalue from a negative value to a positive one through zero (or vice versa). So, one should observe the transition from a saddle point (one positive and one negative eigenvalue) to a source (the negative eigenvalue becomes positive) or a sink (the positive eigenvalue becomes negative) through an intermediate step where the Jacobian is singular. Nevertheless, the affine linear vector field defined in the prism has in general no critical point at this intermediate singular stage. As a matter of fact, the singular system describing the position of a singularity

$$\begin{pmatrix} \beta^0(t_0) & \gamma^0(t_0) \\ \beta^1(t_0) & \gamma^1(t_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} \alpha^0(t_0) \\ \alpha^1(t_0) \end{pmatrix}$$

is very unlikely to have solutions. This is due to the affine component

$$(\alpha^0(t_0), \alpha^1(t_0))^T$$

that varies independently from the Jacobian matrix. In fact, this system will have solutions if and only if this affine component is collinear to the eigenvector of the Jacobian matrix associated with its non-zero eigenvalue. In this (very) special case, the singular system has a whole line of zeros. Finally, in most cases, the singularity with index +1 will diverge (in the 2D space) until the critical time value is reached: There is no singularity at this time. After this, the singularity “returns” but its index is now +1! Remark that the local persistence of the index (see p. 28) has not been violated throughout this evolution: This is the global index of the vector field that has changed from -1 to +1.

Consequently, we only have to detect a possible Hopf bifurcation occurring between two consecutive entry and exit points of a critical point, and we assume otherwise that the singularity type remains constant along this section. Practically, the determination of a singularity type is based on the computation of the eigenvalues of the Jacobian matrix. In our case, decomposing the vectors β and γ in the canonical basis ($\beta = \beta^0 e_0 + \beta^1 e_1$, idem for γ), this matrix is

$$J(t) = \begin{pmatrix} \beta^0(t) & \beta^1(t) \\ \gamma^0(t) & \gamma^1(t) \end{pmatrix}.$$

Thus, we compute the Jacobian matrix and its associated eigenvalues at each entry and exit point and check if the types are the same. If this is the case, we can assert that no Hopf bifurcation has occurred since otherwise, at least two bifurcations would have taken place, which is impossible with our interpolant that varies linearly over time. If the type has changed, a Hopf bifurcation must be found on the way. Since a Hopf bifurcation corresponds to a transition

from a repelling to an attracting nature, the associated instantaneous nature of the singularity is a center, characterized by two conjugate pure imaginary eigenvalues. In particular, the trace of the Jacobian matrix is zero at this point, which is our criterion to determine the exact time location of the bifurcation. With the expression above, we obtain:

$$\text{tr}(J(t^*)) = \beta^0(t^*) + \gamma^1(t^*) = 0.$$

After straightforward calculus, we finally get

$$t^* = \frac{t_i(\beta_{i+1}^0 + \gamma_{i+1}^1) - t_{i+1}(\beta_i^0 + \gamma_i^1)}{\beta_{i+1}^0 - \beta_i^0 + \gamma_{i+1}^1 - \gamma_i^1}.$$

The corresponding position $(x(t^*), y(t^*), t^*)$ is the location of a Hopf bifurcation in the prism cell.

7.3.2 Tensor Case

Path Equation

By definition, locating a degenerate point is equivalent to finding a position in the plane where the tensor field is isotropic, i.e. of the form λI_2 . Now, we are dealing for simplicity with the deviator part of the symmetric tensor field: In this case, we seek a zero matrix ($\lambda = 0$).

$$\begin{pmatrix} a(t) & b(t) \\ b(t) & -a(t) \end{pmatrix} = \mathbf{O}, \text{ that is } a(t) = b(t) = 0.$$

As we deal, for each (fixed) value of t , with a piecewise linear tensor field, this equation system is equivalent to the following linear system in x and y (with analogous notations).

$$\begin{cases} \alpha_a(t) + \beta_a(t)x + \gamma_a(t)y = 0 \\ \alpha_b(t) + \beta_b(t)x + \gamma_b(t)y = 0 \end{cases}$$

Solving this system for x and y yields biquadratic rational functions in t , like the vector case.

$$x(t) = \frac{\begin{vmatrix} -\alpha_a(t) & \gamma_a(t) \\ -\alpha_b(t) & \gamma_b(t) \end{vmatrix}}{\begin{vmatrix} \beta_a(t) & \gamma_a(t) \\ \beta_b(t) & \gamma_b(t) \end{vmatrix}} \quad y(t) = \frac{\begin{vmatrix} \beta_a(t) & \alpha_a(t) \\ \beta_b(t) & \alpha_b(t) \end{vmatrix}}{\begin{vmatrix} \beta_a(t) & \gamma_a(t) \\ \beta_b(t) & \gamma_b(t) \end{vmatrix}}$$

These equations are only valid inside the cell so we are interested in the curve sections that intersect the interior domain of the cell. Like in the vector case,

we only consider the singularities lying on the side faces of the prism cell for at most one single degenerate point can be encountered in a planar, affine linear symmetric tensor field. The positions found on the cell boundary are then associated pairwise to describe successive `entry` and `exit` positions of the path through the prism.

Path Type

We now consider the corresponding successive natures of this degenerate point to get a complete picture of its evolution through the cell. From section 3.4, we know that the degenerate points are of two major types (trisector and wedge). An additional distinction is required in the following between the wedge points with two separatrices and the wedge points with a single separatrix. The major difference resides in the lack of a parabolic sector in the vicinity of the latter. Furthermore, we know that trisectors and wedges have opposite indices. Now, the tensor index of a closed curve is a local invariant in continuous tensor fields, as is the Poincaré index for vector fields. This means that if one takes the cell boundary (triangular in the two-dimensional physical space) as closed curve to compute the tensor index of the (single) contained degenerate point, the value will be constant as long as no degenerate point crosses the boundary, i.e. as long as no degenerate point enters or leaves the prism cell. In our case, it entails that a wedge point remains a wedge point (index $+\frac{1}{2}$) and that a trisector point remains a trisector point (index $-\frac{1}{2}$). That is the reason why, no type swap can occur in the context of our method inside a cell (the special case introduced in the vector case to give insight into the qualitative behavior of singularities through a degenerate Jacobian matrix can be extended here in a straightforward manner). The same result holds if the singularity leaves the cell to enter one of its neighbors by taking the boundary curve of both cells to compute the (constant) index. Furthermore, as a linear tensor field can have at most one degenerate point, several singularities can not meet or split in the interior of a cell but only at the common boundary of two cells. Like in the vector case, this restricts the possible location of a pairwise creation/annihilation to the boundary of a prism cell.

Consequently, the only bifurcation that can be observed in the interior of a prism cell is the *wedge bifurcation* (see section 3.6.2). The exact location of such a bifurcation is a difficult algebraic problem: According to section 3.4.1, the angular positions of the separatrices of a degenerate point are solutions of a cubic polynomial equation. Such an equation has 0, 1 or 3 real solutions. Here we deal with wedge points: A wedge point with a parabolic sector and two separatrices implies a polynomial equation with three roots (one of them corresponds to an angle located within the parabolic sector) and a wedge point

with no parabolic sector and a single separatrix is always associated with an equation with a single solution (there is no parabolic sector in this case, hence a single curve converges toward the degenerate point and its angle is the only root of the polynomial). Thus, we must find a parameter value t_0 such that the cubic polynomial equation has exactly one real root for $t > t_0$ and 3 real roots for $t < t_0$ (or vice versa), which is in our case a polynomial equation of degree 6. Practically, this is solved numerically: We compute successive positions of the degenerate point along the curve described by the formula in section 7.3.2 and check, in the case of a wedge point, if the number of its separatrices has changed. At this point, we start a binary search to approximate the exact bifurcation position with prescribed precision.

7.4 Global Singularity Tracking

At this stage, every cell knows the successive entry and exit positions of the paths of singular points that cross its interior domain, as well as the possible presence of bifurcations (Hopf in the vector case, wedge in the tensor case) on the way. Yet, this information is scattered and must be put together to offer a global view of the topology evolution over time. A fundamental aspect of this task is to detect and identify the bifurcations that may take place on the faces and involve several singularities: As mentioned previously, bifurcations that involve several (typically two) singularities must take place on the cells' boundary. They are detected during the reconnection of path sections lying in neighbor cells as we show next. The scheme is similar in both the vector and the tensor case. Solely the nature of the related bifurcations will be distinguished.

The current tracking scheme processes only two consecutive time planes at once, i.e. a single “time slice” made of prism cells joining two discrete time steps. This avoids the whole three-dimensional grid (and thus all the discrete time steps) to be present in memory for computation, which is of great interest in the context of this method. The tracking consists of two steps: First, we track every singularity located in the first time step until it leaves the slice. Second, we track backwards every singularity located in the next time step and that has not been treated so far until it leaves the slice. For this purpose, we dispose of a boolean array that indicates, for each prism cell, if it should be investigated or not: A cell should be investigated if a singularity's path was found in this cell during the local tracking step. At the beginning, every cell is marked. An additional information required for the efficient identification of bifurcations concerns the current temporal orientation of the tracking (i.e. either **forward** or **backward**), as explained in the following. We start in time plane $\{t = t_j\}$. The tracking direction is **forward**. For each cell marked in the

boolean array, we check the information collected during the cell analysis step: We seek a path section starting at an entry point located on the considered front triangular face. This provides us with the corresponding exit point at which the tracking will be proceeded. The exit point can either lie on a side face (quadrilateral) of the prism, that is the singularity has left the cell within the current time slice, or on the triangular face lying in $\{t = t_{j+1}\}$ (back face), which signifies that the singularity stays in the cell (triangular, in the two-dimensional physical space) between the instants t_j and t_{j+1} . In the latter case, we indicate in the boolean array of the next time plane $\{t = t_{j+1}\}$ that this triangle must be further inspected. If the path left the cell at $t^* < t_{j+1}$ (side face), then it entered a neighboring cell at the same time. Back to a two-dimensional representation, we get in time plane $\{t = t^*\}$ a singular point lying on the common edge of both neighboring triangle cells. The piecewise linear nature of the interpolant over this time plane is responsible for the discontinuity of the Jacobian matrix through this common edge. Therefore, this singularity may have another type when considered from the other side. This simple argument explains the possible existence of a bifurcation in such cases. Two situations may actually occur.

- There is a simple crossing of a singular point from one cell to another cell where no singularity was present so far. In this case, the type may change but the index computed along a curve enclosing both neighbor cells remains constant. Hence, in the vector case, a saddle remains a saddle (index -1) whereas a sink can become a source and vice versa (both index +1). The latter case is a Hopf bifurcation. In the tensor case, a trisector remains a trisector (index $-\frac{1}{2}$) while a transition from a wedge point with one separatrix to a wedge point with two separatrices can be observed (both index $+\frac{1}{2}$): This transition is then a wedge bifurcation. When tracking a sink/source (resp. a wedge), we easily detect the presence of a bifurcation by checking if the type is the same on the other side of the common face.
- The second situation corresponds to a merging of two singular points, coexisting in neighboring cells, on their common face (common edge in 2D). Now, an important property of the piecewise affine linear interpolant over a triangulation is that two neighboring triangles cannot contain two singularities with same index: In the vector case, an index ± 2 cannot be obtained on a closed polyline made of four linear interpolated edges, see reasoning in section 4.3.1. In the tensor case, a similar property exists (section 4.3.1) which prevents the existence of two degeneracies of same index in two neighbor cells. Consequently, only 2 singularities with opposite indices can merge in that way: A saddle and a source

(or a saddle and a sink) in the vector case and a trisector and a wedge point in the tensor case. This type of bifurcation has been considered in chapters 2 and 3: It is a pairwise annihilation or its inverse a pairwise creation.

When we leave the current cell through a side face, and a possible bifurcation has been detected, we identify this position (in particular we save its time coordinate). We proceed with tracking the path by asking the neighbor cell for the next “exit” position: Actually, we get a position corresponding to the other extremity of the path section starting on the current side face. This position can either be “earlier” ($t_{\text{exit}} < t^*$, in this case we have found an entry position at t_{exit} and an exit one at t^*) or “later” ($t_{\text{exit}} > t^*$ and the position found at t^* is indeed an entry one). We get three possible values for the time coordinate t_{exit} of the “exit” position.

1. The exit position lies in the time plane $\{t = t_{j+1}\}$: We have reached the next time plane. The corresponding triangle face is marked **true** in the boolean array corresponding to the next time plane. If the tracking direction was **backward** so far, then one has encountered a pairwise creation on the side face. In this case, the tracking direction is switched to **forward**.
2. The “exit” position lies in the time plane $\{t = t_j\}$: If we were tracking the current path **forward** so far, we have found a *pairwise annihilation* and the tracking direction is switched to **backward**. In this case, the corresponding triangle is set to **false** in the boolean array of $\{t = t_j\}$, indicating that this cell has been processed now.
3. The exit position lies on a side face at $t = t_{\text{exit}}$. If $t_{\text{exit}} < t^*$ and we were tracking the current path **forward** so far, we have found a *pairwise annihilation* and we switch the tracking direction to **backward**. If $t_{\text{exit}} > t^*$ and we were tracking the current path **backward** so far, we have found a *pairwise creation* and we switch the tracking direction to **forward**. In this case, we are back to the current situation: we enter the next cell through a side face.

The possible cases at an edge crossing are illustrated in Fig. 7.4. A sketch of the complete tracking scheme is proposed in Fig. 7.5.

Once the tracking has been conducted from the front side of the time slice, an analogous tracking is started from the back side, in backward direction. The cells that will be investigated are those that have not been marked in the next time step during forward tracking, which means that their back face has not been reached by a singularity coming from the front side of the slice. If a

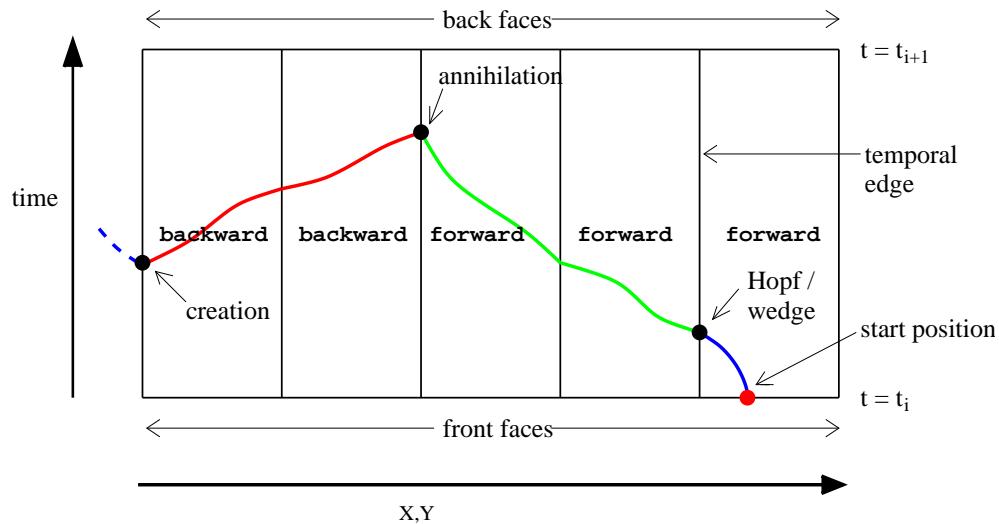


Figure 7.4: Possible cases when leaving a prism cell through a rectangular face

singularity is found at this stage in the back face, the cell is marked for further forward tracking in the next slice.

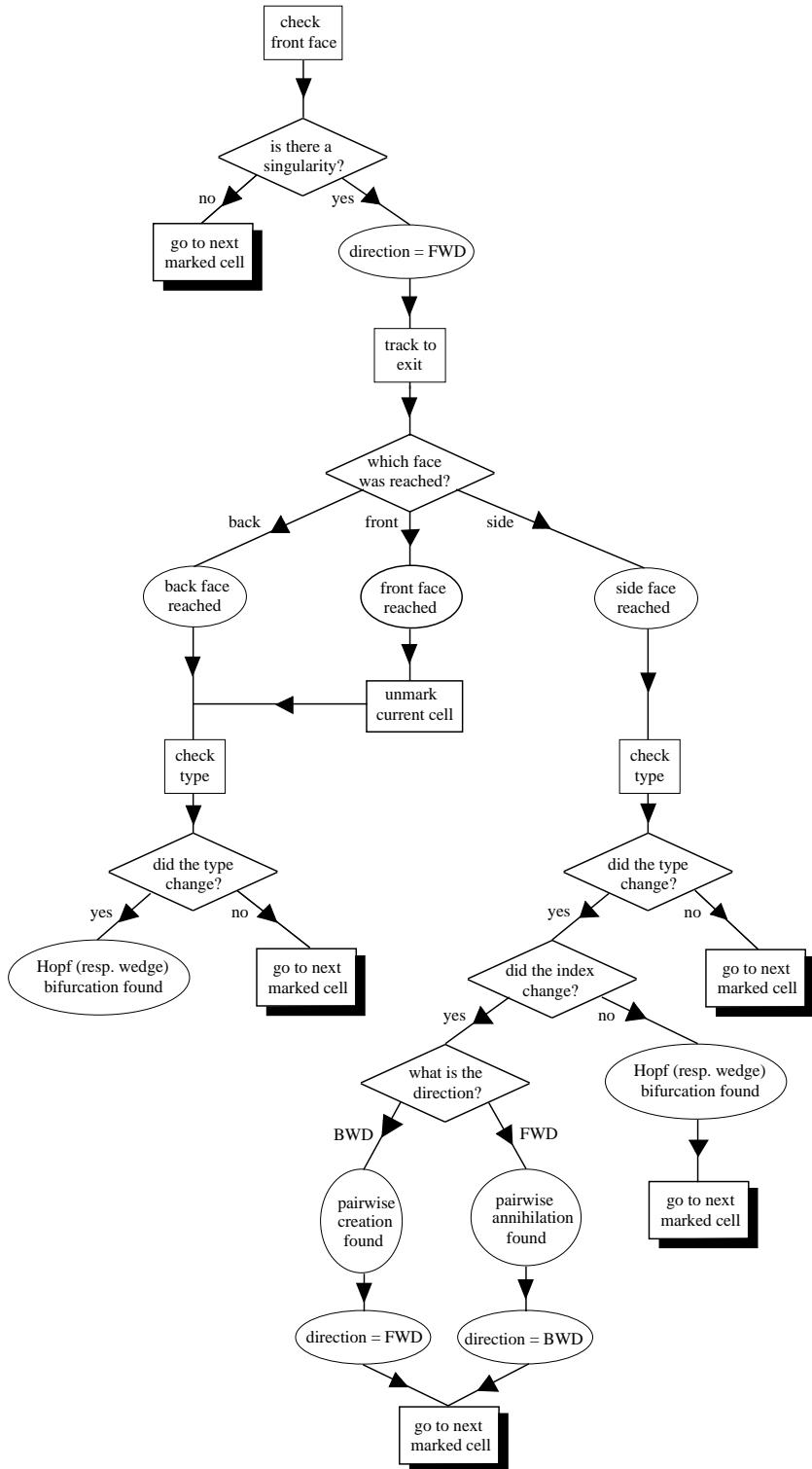


Figure 7.5: Global tracking scheme

7.5 Topology Depiction

Now, as singularities are only part of the topology in the two-dimensional steady case, the paths tracked over time must be associated with the corresponding separatrices to depict the whole topology of the field and visualize the structural evolution in the unsteady case. Practically, the separatrix curves change over time and thus span separatrix surfaces that bound volumes of the time-dependent topology. Recall that these separatrices are integrated from the saddle points in the vector case and from both the trisectors and wedge points in the tensor case.

7.5.1 Structures Tracking

Drawing these surfaces requires the previous determination of each singular path and, for each discrete position along the piecewise linear approximation of the path, the computation of the start directions of the separatrices. In the vector case, we consider only the saddle paths and compute at each position the eigenvalues and corresponding eigenvectors. This provides four start vectors (the associated directions are determined according to the sign of the eigenvalues). In the tensor case, we determine at each position along each path the angle of the separatrices. A start direction, necessary for the further correlation of the integration steps, is obtained by choosing always the direction moving away from the singular path. The depiction of the separatrix surfaces is then based on the integration of separatrix curves along these directions on one hand, and on the correlation of each separatrix curve with its predecessor, starting at the previous position along the path on the other hand. To ensure consistency, two aspects must be considered when joining separatrix curves together: First, they must start along eigenvectors that have approximately the same angle and the same start direction. In the vector case for instance, for a saddle path, there are at each position four starting curves. If one denotes $\mathbf{u}_i(t)$, $i \in 0, \dots, 3$ their starting (normal) vectors as a function of time, and with no assumption on the indices, we connect $\mathbf{u}_i(t)$ and $\mathbf{u}_j(t + \epsilon_1)$ if and only if

$$\begin{cases} \mathbf{u}_i(t) \cdot \mathbf{u}_j(t + \epsilon_1) > 0 \text{ (scalar product, same direction) and} \\ |\mathbf{u}_i(t) \times \mathbf{u}_j(t + \epsilon_1)| < \epsilon_2 \text{ (cross product, small angle between the two).} \end{cases}$$

The same principle applies to trisectors or wedges with two separatrices in the tensor case (for wedges with a single separatrix, this is trivial). Second, we must ensure that two related separatrices behave similarly in an asymptotic sense: we check if both corresponding separatrices reach the same singular path, the same cycle or close positions on the grid boundary. In this case, we add the new separatrix to the surface spanned by the old one (we add a new

“ribbon” to this surface). Otherwise, we check if the path reached previously has ended at a bifurcation point. In this case, we end the previous surface at the bifurcation by integrating a separatrix at the exact time position of the bifurcation and start a new one at the current separatrix. If no found bifurcation has occurred but the connectivity has changed, then we face a bifurcation that was not (and could not be) detected so far, as explained previously. As a matter of fact, when determining the topological correlation, we are finally able to detect unexpected changes that break local structural consistency and thus correspond to global bifurcations. In this case, we simply end the surface at the previous separatrix and start a new surface at the current separatrix. Doing this for each discrete position along the concerned paths, we are eventually able to depict all separating surfaces in the domain.

7.5.2 Surface Drawing

Once separatrices have been associated over time, one gets a set of curves that span a surface. The construction of this surface is done in two steps. Originally, the curves that result from numerical integration are piecewise linear. Therefore, they are first replaced by interpolating NURBS curves of degree 2 to increase smoothness. In the second step, these NURBS curves are embedded in a NURBS surface, after uniformization of their parametrizations. The algorithms used are taken from [PT97]. It results in smooth surfaces that better picture the continuous evolution and transformation of the structure.

7.6 Results

In the following, the topology tracking method is applied to a vector and a tensor dataset. The successive processing steps are illustrated. Examples of bifurcations are proposed to explicit their role in the qualitative evolution of the topology.

7.6.1 Vector Case

To test the method in the vector case, we first create an analytic vector field containing four critical points. The positions are functions of time, describing closed curves in the plane. Note that the mathematical description of such a vector field is done in Clifford algebra, according to the results presented in [SHK98a]. This field is sampled on a rectilinear point set for several values of the time parameter. The rotation of the critical points entails many structural changes for the topology, which is very interesting since many bifurcations are present and can be visualized. We show first the results of the singularity

tracking step: The path of each critical point through time has been tracked as well as all the local bifurcations that occur (indicated by small balls). The coordinate directions are displayed to give an impression (see Fig. 7.6). Color coding is as follows: Saddle paths are depicted in red, sinks in blue and sources in green; Hopf bifurcations are shown in yellow, annihilations in pink and creations in light blue. If one focuses on a particular bifurcation, one observes

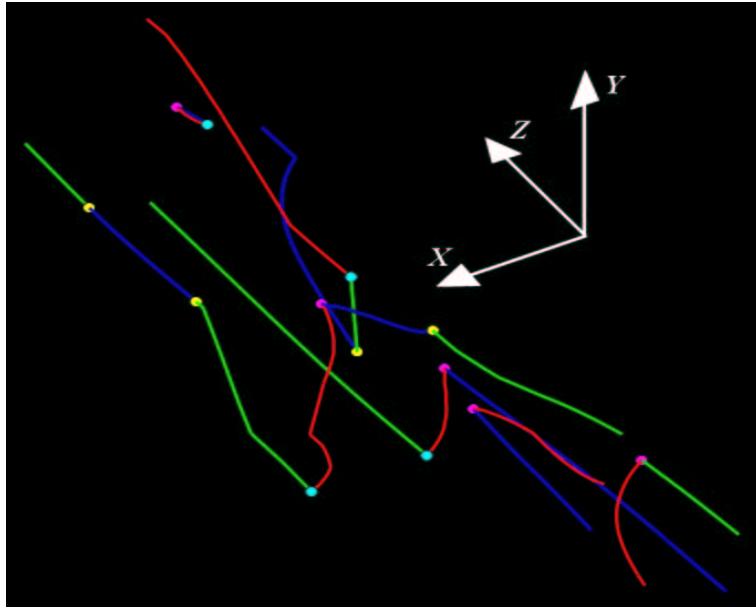


Figure 7.6: Singularities' path through time and associated bifurcations

how the separatrices evolve through the bifurcation point. In the case of a Hopf bifurcation for instance, we consider the picture without surfaces (see Fig. 7.7) and with separatrix surfaces (see Fig. 7.8). The creation of a closed

orbit can be easily seen. As far as the global topology and its evolution are concerned we first get the picture shown in Fig. 7.9: This corresponds to the depiction obtained after the integration of the separatrices from the saddle paths and before the computation of the separatrix surfaces that embed them (c.f. section 7.5). Note that the perspective used here is the same as in Fig. 7.6. When the separatrix surfaces are added to the structure, we get finally the picture presented in Fig. 7.10. The breaks that can be observed on the surfaces correspond to structural transitions associated with (global) bifurcations. The two colors used for surface depiction refer to the stable and unstable eigenspaces of the saddle points, respectively.

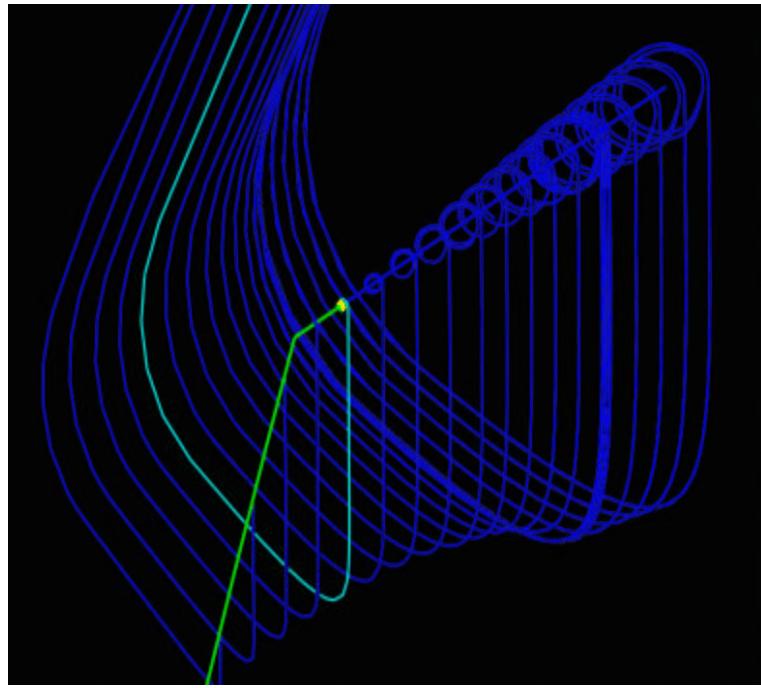


Figure 7.7: Hopf bifurcation: Separatrices

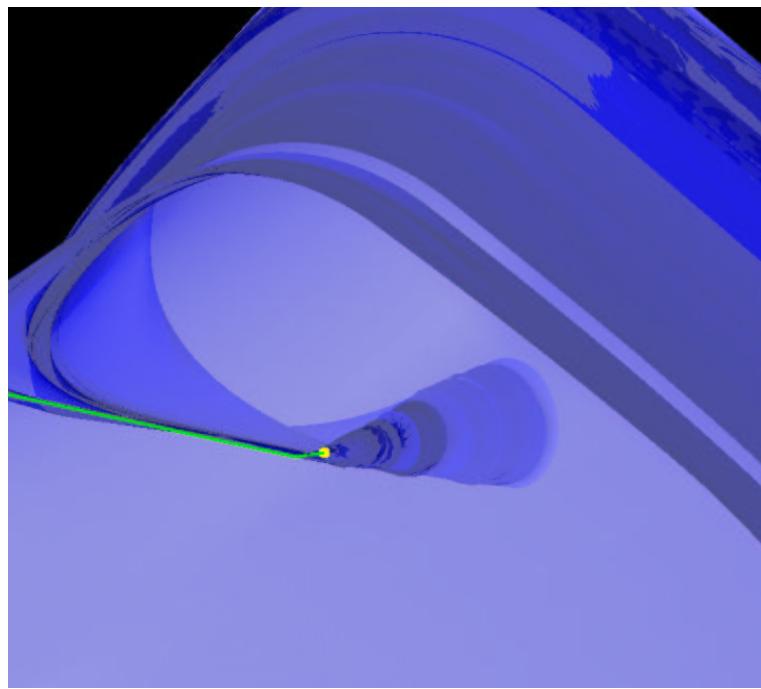


Figure 7.8: Hopf bifurcation: Separating surface

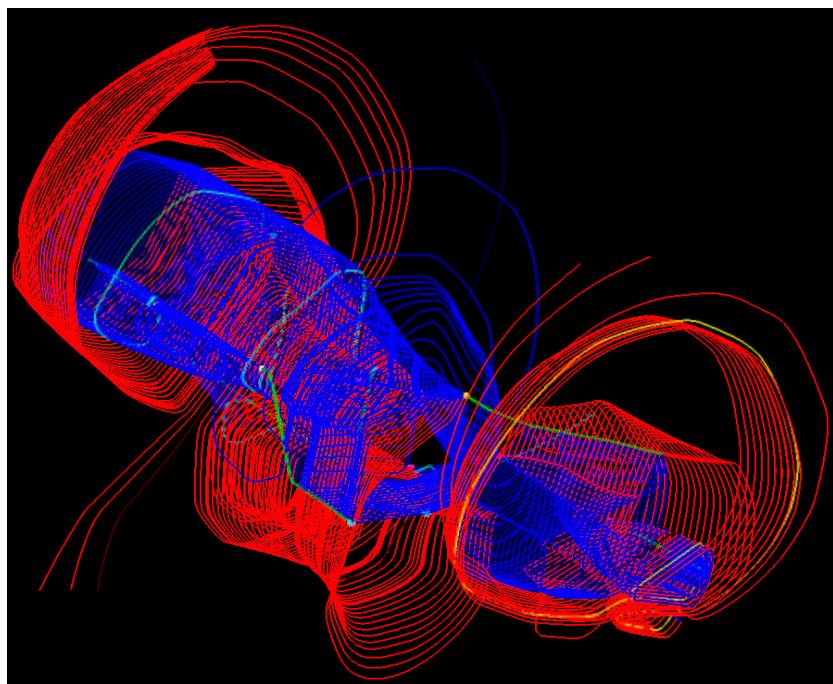


Figure 7.9: Overview of the topology evolution (curves)

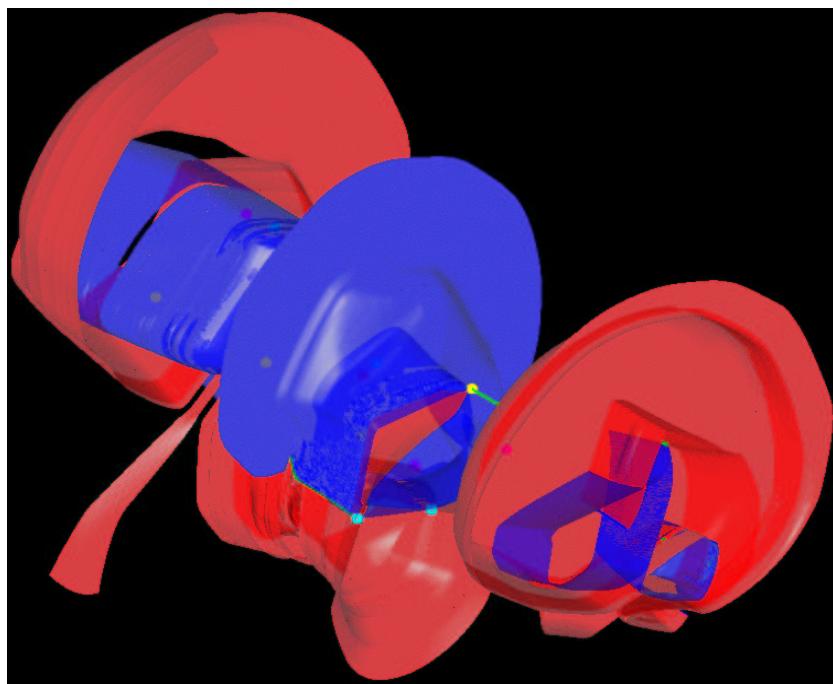


Figure 7.10: Overview of the topology evolution (surfaces)

7.6.2 Tensor Case

We present here the results of the method applied to an artificial symmetric tensor dataset. It is generated on a small rectilinear point set of 25 vertices that we triangulate to get a grid with 32 cells. We start with a topology containing 3 trisector points and 4 wedge points (3 with a single separatrix and 1 with two separatrices) as shown in Fig. 7.11. We process 11 consecutive time steps. Tracking the degeneracies over time, we get the picture shown in Fig. 7.12. Color coding is as follows: Trisectors are depicted in red, wedge points with two separatrices in dark blue and wedges with a single separatrix are shown in light blue. Bifurcations are marked as small balls: green ones indicate pairwise creation, red ones illustrate pairwise annihilation and yellow balls designate wedge swap bifurcations. Five bifurcations have been detected: two pairwise creations, two pairwise annihilations and one wedge swap.

Adding the separatrix surfaces to the degeneracies' paths, we complete the topology depiction and obtain the structures presented in Fig. 7.13 (separatrix surfaces emanating from a trisector point are colored in red while those coming from a wedge point are displayed in blue).

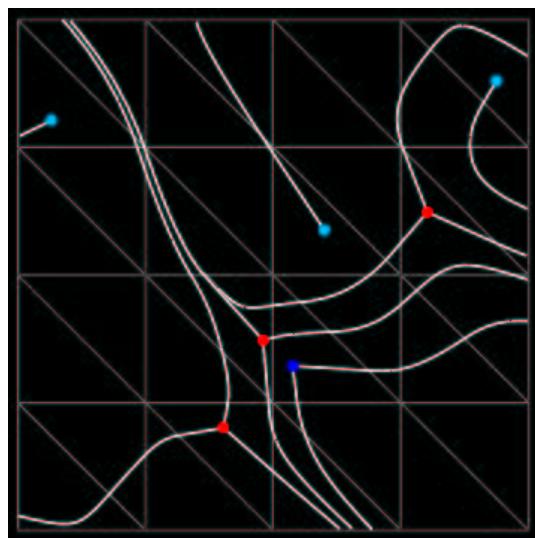


Figure 7.11: Start Topology

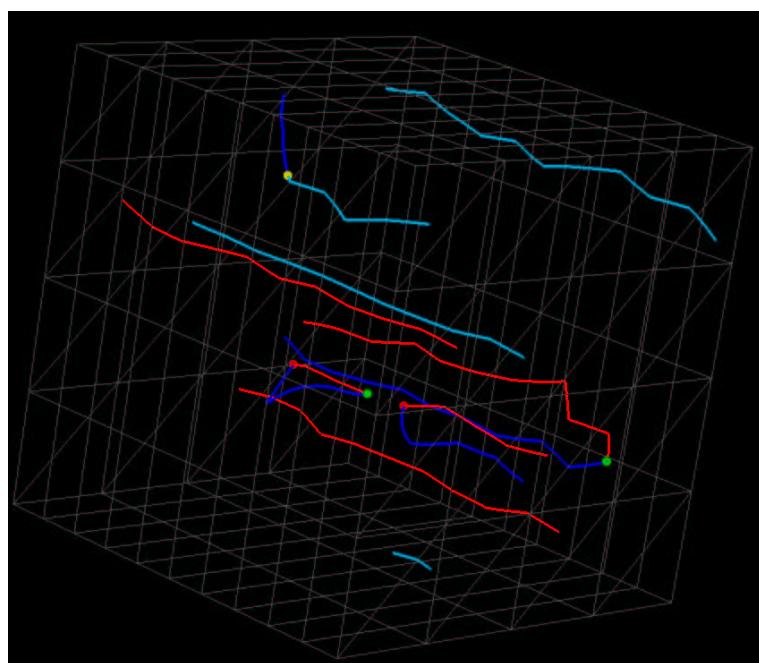


Figure 7.12: Degeneracies' paths with grid boundary

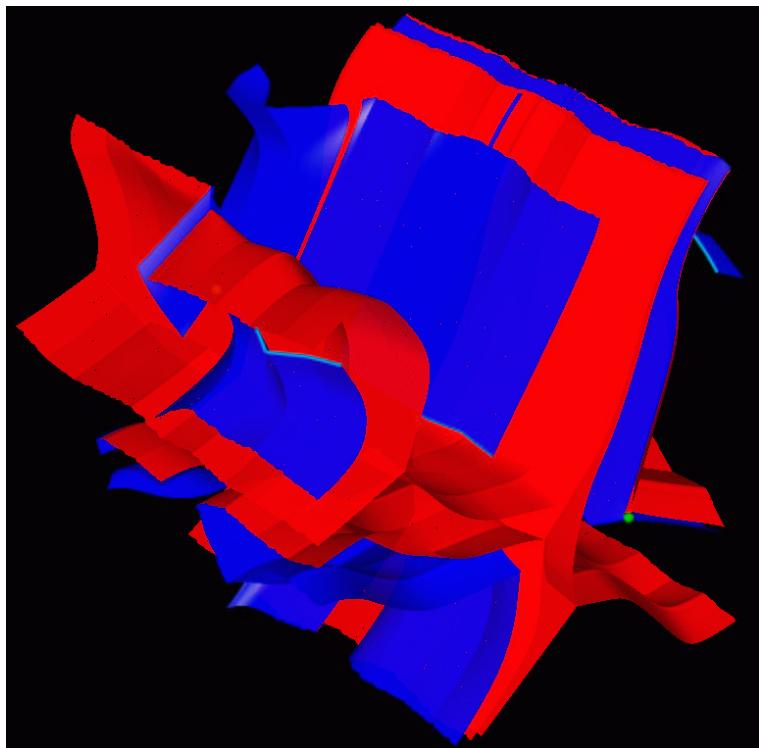


Figure 7.13: Picture of the complete topology evolution

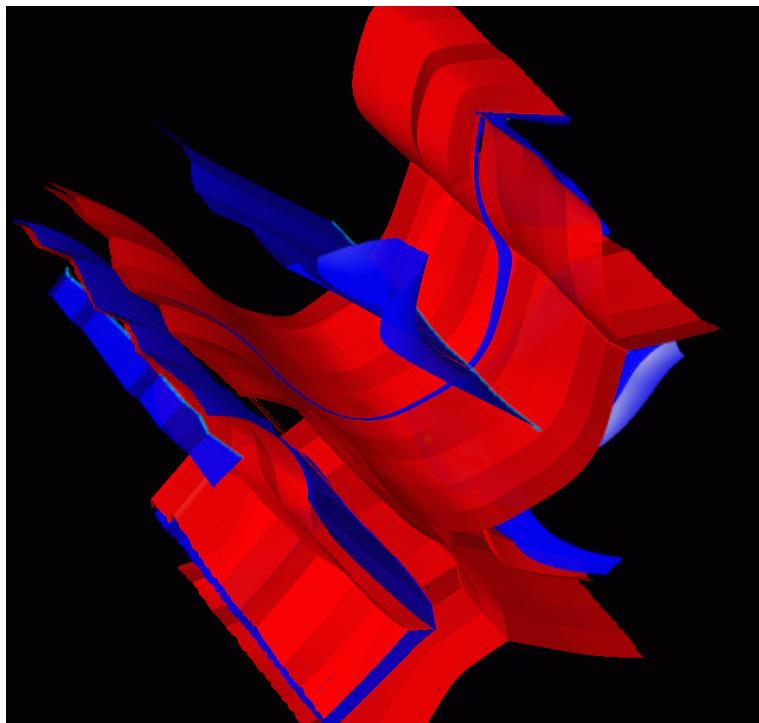


Figure 7.14: Complete topology in another perspective than Fig. 7.13

7.7 Discussion

Both examples considered in the previous section showed the ability of the method to accurately detect and identify the bifurcations that occur within a parameter-dependent vector or tensor field. This permits for the first time to visualize the continuous evolution of the topology since bifurcations are the key features that determine structural transformations. The choice of convenient space and time interpolation schemes restricts the range of possible bifurcations which enables both a precise and efficient tracking of critical points and associated separatrices. Yet, the datasets considered so far are artificial and their topologies are not too complex. Therefore, we now turn to a CFD tensor dataset to discuss possible improvements in the application of the method to turbulent, practical datasets.

This dataset is the rate of deformation tensor field of a swirling jet CFD simulation. The complete structured grid has 250×244 cells. We consider the symmetric part of the original data. For convenience, we process only a cell group containing 45×60 cells as shown on Fig. 7.15. Processing this set of

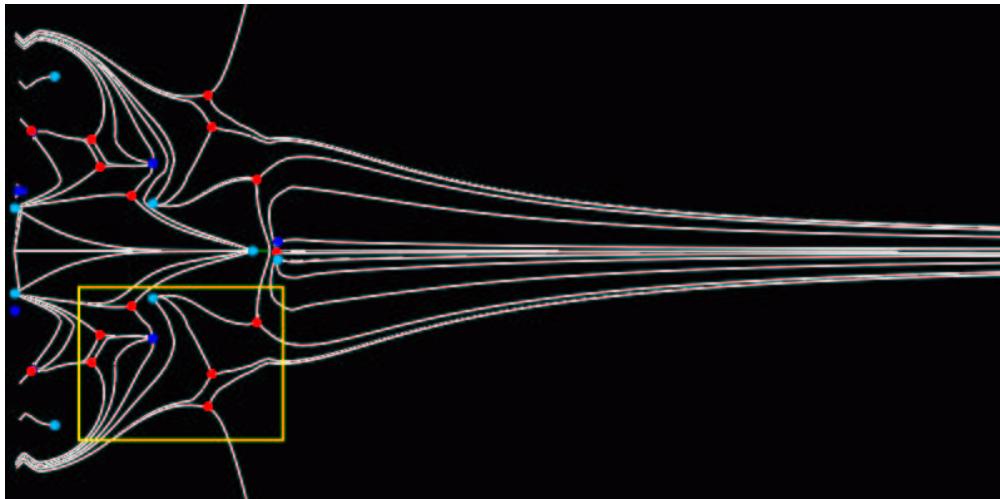


Figure 7.15: Start topology and considered part

cells over 25 time steps (see Fig. 7.16 to get an impression of the 3D structure), we obtain the degeneracies' paths shown in Fig. 7.17. The size of the balls indicating bifurcations has been reduced here to avoid a confused depiction. As a matter of fact, as one can see on this picture, tracking the topology of this turbulent tensor field results in a large number of encountered bifurcations. This effect is related in some extent to the linear scheme used for time interpolation along with the fact that the discretization steps along the time line are too big to track the topology very reliably: Many modifications affect the

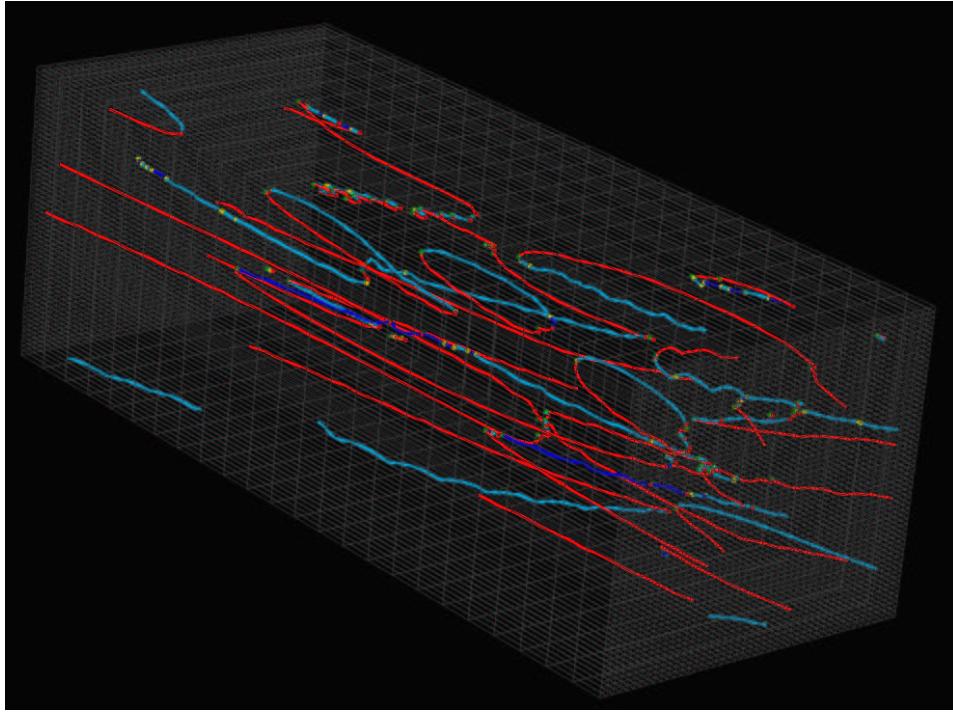


Figure 7.16: Degeneracies' paths and 3D grid

singularities between consecutive time planes and our linear time interpolant is limited to convey this evolution. Therefore, it would be convenient to increase the time interpolation continuity (like C^1 or C^2 for instance) to better reflect the nature of the underlying flow in such cases. Furthermore the linear spatial interpolation (in every time plane) induces the discontinuity of the Jacobian matrix (piecewise constant over the “instantaneous” triangulation in this case), which causes the occurrence of many local bifurcations at the locations where singularities leave a prism cell through a side face. If we zoom into the data to look at small topological features, we observe structures like those shown in Fig. 7.18. To attack this problem, one could think at increasing the space interpolation continuity too, like e.g. in [STH99]. However, this might cause trouble since the singularities’ locations become far more complicated in this case, which inconveniences the tracking step. Moreover, the nice property of the linear interpolant that exhibits at most one singularity would be lost and many additional bifurcation types should be taken into account. In fact, an appealing and efficient approach to overcome these problems seems to be a post-processing step that prunes features of small time or space scale out of the topology (inspired by the technique proposed in the next chapter for instance) to retain features of larger space scale and temporal persistence.

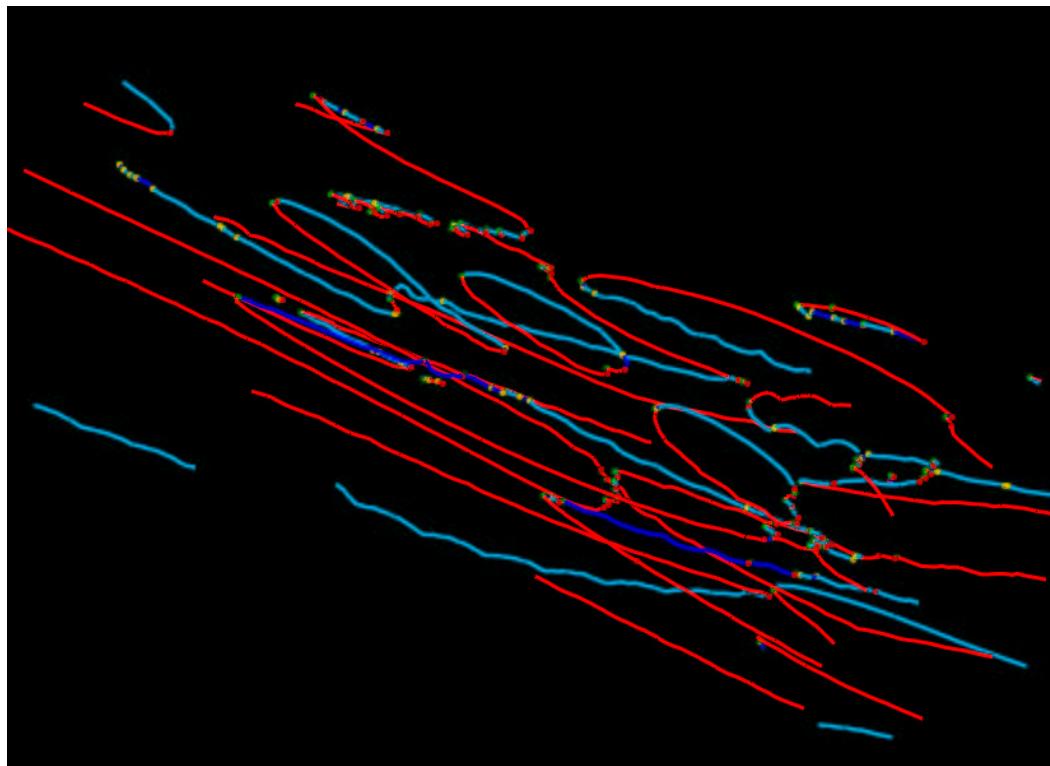


Figure 7.17: Degeneracies' paths

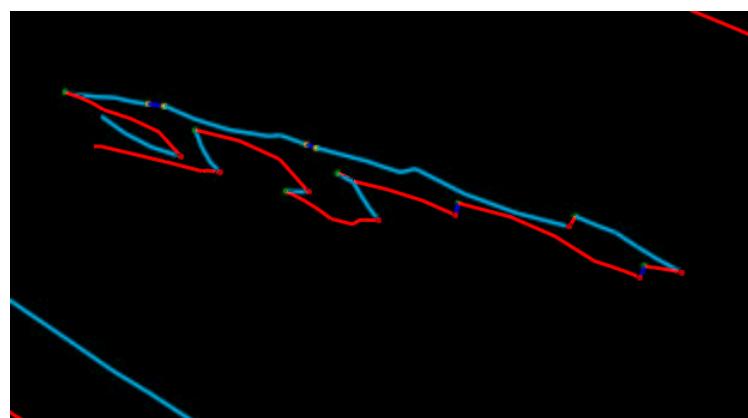


Figure 7.18: Local topological structures

Chapter 8

Continuous Topology Simplification

In chapter 6, a scaling approach was introduced to attack the problem of visual clutter encountered by topological methods in the processing of turbulent flows. The idea was to replace close singularities by an equivalent one of non-linear nature that locally approximates the phase portrait of the original field to clarify the depiction. Practically, the fusion of close singularities was simulated to result in a single one, with consistent structure in the large. With this technique close singularities are concentrated at a single point but their complete disappearance from the graph cannot be achieved since an artificial singularity is always created afterward. Moreover, scaling several critical points can lead to the removal of meaningful flow features because only spatial criteria are taken into account. The method presented in this chapter has been designed to overcome these drawbacks and to offer a continuous way to simplify the visualized topology in the context of vector fields. It is based on the following properties: Turbulent vector fields present a large number of interconnected pairs of first-order critical points with opposite indices, i.e. saddle points linked to sources or sinks. Furthermore we saw in chapter 2 that such pairs of singularities can annihilate: In a parameter-dependent evolution both critical points come closer together to merge and disappear which lets place to a uniform flow without remaining singularity in the considered neighborhood. One can thus remove pairs of critical points from the dataset by forcing pairwise annihilations locally that successively withdraw two singularities from the topology and let the others unchanged. This clearly induces a progressive simplification of the topological graph. In contrast to the scaling technique, no grid changes are necessary since the whole scheme uses small local changes of the vector values defining the vector field.

Practically, we first compute the topological graph and associate every edge

with numerical measures that evaluate its relevancy in the global structure. This is to enable a hierarchical simplification of the graph according to qualitative criteria that fit the considered application. Next, we sort the pairs of critical points according to these criteria and retain those with values over prespecified thresholds. Then we process all pairs sequentially: For each of them, we first determine a cell pad enclosing both critical points. In this pad, we slightly modify the vector values such that both critical points disappear. This deformation is controlled by angular constraints on the new vector values imposed by those kept constant on the frame of the pad. When every pair has been processed, we redraw the simplified topology. These successive steps are described next. We end with an application of the method to a CFD dataset.

8.1 Topology Computation

As a preprocessing step, the method requires the computation of the topological graph. This computation must be conducted in a way that provides all the information needed for pairing critical points as explained in the following. In the present method, we deal with a triangulation of vertices lying in the plane associated with 2D vector values. The interpolation scheme is piecewise linear. Therefore, we only consider topological features of first order: In this case, topology is defined as the graph built up of all saddle points, sinks, sources, closed orbits and the separatrices emanating from saddle points, see chapter 2.

Consequently, we process as follows: We start with the computation of all critical points in the grid. From each saddle point, we integrate the four related separatrices. For each separatrix, we check if it leaves the grid or if it reaches a critical point or a closed orbit. An accurate and effective detection of closed orbits can be achieved thanks to a scheme described in [WS01]. If a critical point is reached (sink or source, depending on the integration direction from the saddle point), we identify it among the set of all critical points and save this information for the current separatrix. Furthermore, we mark this sink or source as **connected**. If a closed orbit is reached, we must await the end of the complete topology computation to process this separatrix further. As a matter of fact, once all separatrices have been integrated, we look over all singularities for sinks or sources that are not **connected** and associate them with the separatrix surrounding the cycle that contains them, if any. This supposes that a single critical point (with index +1, c.f. index of a closed orbit) is present inside each limit cycle. Actually, any topological structure of index +1 may be encountered inside a closed orbit even if a single sink or source is most likely to occur in practice. At last, the separatrix gets as length the euclidean distance

between saddle point and isolated critical point. The reason for this choice is explained in the next section. Possible cases are illustrated in Fig. 8.1. This completes the topological information required for further processing.

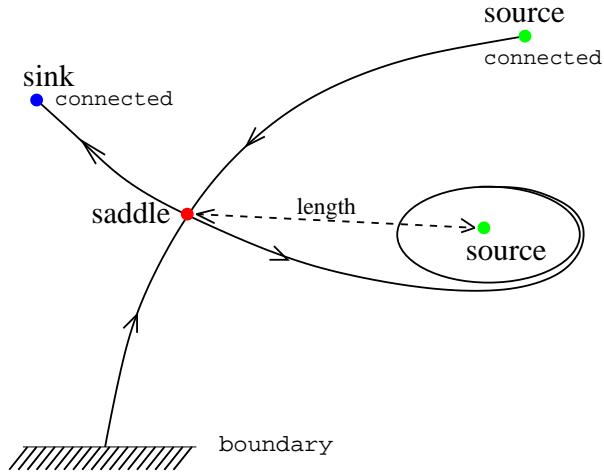


Figure 8.1: Topological connections of a saddle point

8.2 Pairing Strategy

The basic idea behind our simplification technique is to use the topological equivalence (in the sense of the index invariant) of a region containing several critical points with index sum 0 with the same region without critical point (see theorem 8, p. 28). More precisely, we aim at removing pairs of first order critical points of opposite indices (that is a saddle point, index -1, and a source or sink, index +1) to reduce the number of singularities present in the field and thus simplify the topology while keeping consistency with the original structure. As said previously, a local deformation of the vector field associated with the removal of a pair corresponds to a pairwise annihilation of two critical points with opposite indices. This type of bifurcation has been presented in chapter 2, p. 32. As shown previously in the topology tracking method (see chapter 7), the continuity of this transition can be illustrated by linearly interpolating, for each modified vertex, its value between the original vector value and the one obtained after modification. Depicting each intermediate aspect of the topology in the vicinity of both critical points shows how they become closer to merge and finally disappear. Considering time as third dimension, one gets the picture proposed in Fig. 8.2.

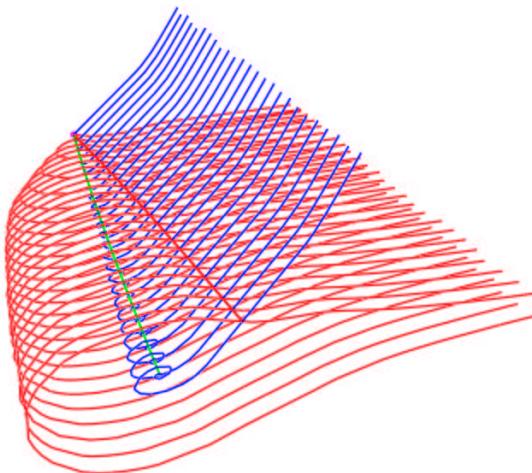


Figure 8.2: Continuous topological transition

Before starting the simplification we first need to determine which pairs of critical points may be removed in that way and to classify them according to the significance of the singularities in the field structure.

8.2.1 Connectivity

We require the singular points of a pair to be linked by a separatrix in the topological graph. This ensures that the topological transition associated with the disappearance of both singularities corresponds to a pairwise annihilation (see section 2.7.1). Yet, this criterion must be relaxed to handle isolated singularities lying in the interior domain of a closed orbit. This explains why we decided previously to connect a saddle point with an isolated critical point across the limit cycles enclosing it.

8.2.2 Additional Criteria

The importance of critical points mainly depends on the interpretation of the visualized vector field. For this reason one can make use of different measures to classify the relevancy of critical points and possibly consider a weighted combination of several of them to fit the domain of application.

Relevancy measures are for instance the euclidean distance between critical points, or the length of the edge (separatrix) connecting them (both measures apply to a pair of critical points of opposite indices), or the degree of a critical point of index +1 (sink or source), that is the number of saddle points it is connected to. Furthermore, in [LL99b], the authors suggest to use the area of

a source or sink's basin to evaluate the importance of critical points of index +1. Yet this basin-based method implies a computational effort that makes it unsuited for our method. Another interesting quantity based on fluid dynamics considerations is the absolute value of the vorticity of a sink or source. Spatial variation of vorticity in the vicinity of a critical point of index +1 also gives insight into the action of a source or sink on the field structure. Nevertheless, an accurate computation of such quantities is a tricky task, especially in the quite common case of planar vector fields cut off from 3D datasets: Higher order terms are involved and, when dealing with simulations, the underlying numerical schemes must be taken into account, and not only the given discrete values. Eventually, a simple numerical measure we are concerned with in the present method is the maximal magnitude of the vector field in a cell containing a critical point of index +1. This permits to remove singularities that are due to numerical noise and lead to misinterpretation.

Practically, we have adopted two complementary criteria: On one hand we apply a threshold on the euclidean distance of both points of a pair and preserve the pairs with sufficient lengths. On the other hand, we choose to maintain every source or sink lying in a cell with minimum magnitude above a second threshold. With this definition, critical points belonging to several valid pairs will be simplified concurrently: We process the pairs in increasing length's order and skip those that contain singularities that have been removed already.

8.3 Local Deformation

Once a pair of critical points has been identified that fulfills our criteria, it must be removed. To do this, we start a local deformation of the vector field in a small area around the considered singular points. To preserve both the interpolation scheme and the grid structure, we only modify vector values at grid vertices. In the following, we detail how we determine which vertices have to get a new vector value and how we set the new values in order to ensure the absence of a singularity in the incident cells after processing. At last, we illustrate the continuity of this deformation.

8.3.1 Cell-wise Connection

Consider the situation shown in Fig. 8.3. We first compute the intersections of the straight line connecting the first critical point to the second with the edges of the triangulation. For each intersection point, we insert the grid vertex closest to the second critical point (see vertices surrounded by a circle) in a list. After this, we compute the bounding box of all vertices in the list and

include all grid vertices contained in this box. This obviously includes every vertex marked in the former step.

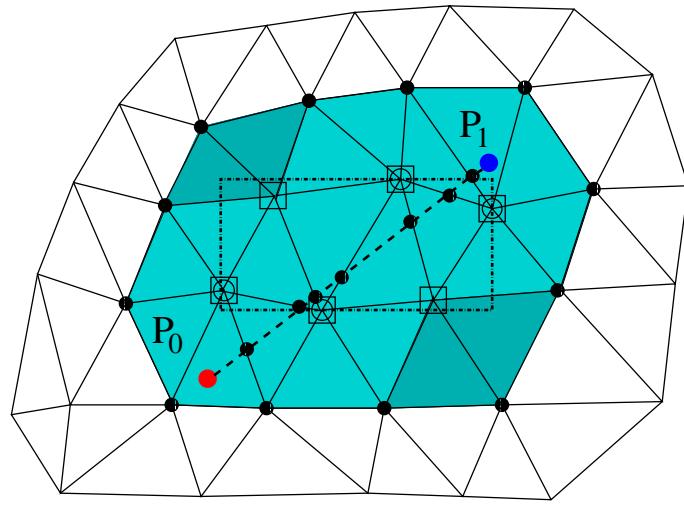


Figure 8.3: Cell-wise connection

The use of a bounding box is intended to ensure a well shaped deformation domain, especially useful if many cells separate both singular points. This configuration occurs if the threshold has been assigned a large value to obtain a high simplification rate. The vertices concerned with modification are surrounded by squares. We call them *internal vertices* in the following. Since the modification of a vertex vector value has repercussions on the indices of all incident triangles, we include every cell incident to one of the selected vertices in the cell pad. These cells are colored in gray. Further processing will have to associate the internal vertices with vector values that ensure the absence of any singular point in the cell group with respect to the vector values defined at the *boundary vertices* (marked by big dots in Fig. 8.3) that will not be changed. Note that the connection may fail if one of the included cells contains a critical point that does not belong to the current pair: In this case, the global index of the cell group is no longer zero. If it occurs, we interrupt the processing of this pair. Nevertheless, such cases can be mostly avoided by simplifying pairs of increasing distance. Moreover, if a pair cannot be further simplified because of its cell-wise connection, it can be reinserted at the end of the pairs' list to be retried after processing of all remaining pairs (with lower priorities).

8.3.2 Angular Constraints

To give insight into our deformation strategy, we first consider a single internal vertex and its incident cells as shown in Fig. 8.4. Suppose that every position marked in black is associated with a constant vector value and that the corresponding global index of all triangles (in the sense of theorem 11, p. 28) is zero. The problem consists then in determining a new vector value at the internal vertex (in white) such that no incident cell contains a critical point. According to property 9 p. 72, this is equivalent to the fact that every incident triangle has index 0.

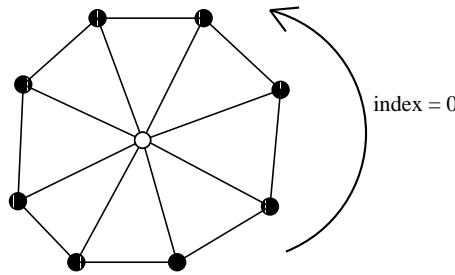


Figure 8.4: Configuration with single intern vertex and incident cells

Now, recall the expression for the computation of the Poincaré index in the special case of a linear vector field along the edges of a triangular cell (see also chapter 4, p. 70): Let ϕ_0 , ϕ_1 and ϕ_2 be the angle coordinates ($\in [0, 2\pi[$) of the vectors \mathbf{v}_0 , \mathbf{v}_1 and \mathbf{v}_2 defined at the vertices of the triangle. The index of this triangle T is given by

$$\text{index}(T) = \frac{1}{2\pi} (\Delta(\phi_0, \phi_1) + \Delta(\phi_1, \phi_2) + \Delta(\phi_2, \phi_0)) \quad (8.1)$$

$$\text{where } \Delta(\phi_i, \phi_j) = \begin{cases} \phi_j - \phi_i + 2\pi & \text{if } \phi_j - \phi_i < -\pi, \\ \phi_j - \phi_i & \text{if } |\phi_j - \phi_i| < \pi, \\ \phi_j - \phi_i - 2\pi & \text{if } \phi_j - \phi_i > +\pi, \end{cases}$$

Consequently, in each triangle the angle coordinates of the vectors defined at the black vertices entail an angular constraint for the new vector. (The index definition implies that the index does not depend on the magnitude of the vectors defined at the vertices.) If $\Delta(\phi_0, \phi_1)$ is already assigned a value strictly smaller than π , the two missing terms must induce a global angle change smaller than 2π (for an index is an integer). It will be the case if and only if the new vector value has angle coordinate in $\] \phi_1 + \pi, \phi_0 + \pi [$, with

$[\phi_0, \phi_1]$ being an interval with width smaller than π , i.e. the angle change along a linear edge occurs from ϕ_0 to ϕ_1 (see Fig. 8.5). This provides a constraint on the new value for a single triangle. Intersecting the intervals induced by all incident triangles, one is eventually able to determine an interval that fulfills all the constraints. Note that this interval may be empty. In this case, the simplification is (at least temporarily) impossible. As far as the magnitude of the new vector is concerned, one simply takes the mean value of the field magnitude on the exterior edges. Once again, the linear interpolant defined on these edges facilitates the computation of this quantity.

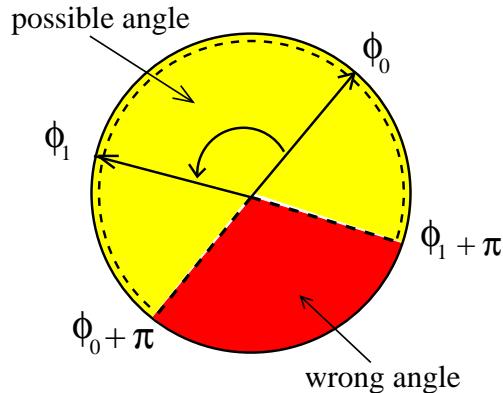


Figure 8.5: Angular constraint in a triangle cell

8.3.3 Iterative Solution

When considering all internal vertices as shown in Fig. 8.3, one must find for each of them a new vector value that fulfills all the constraints induced by the edges connecting their incident vertices. These incident vertices are of two types: internal or boundary vertices. Edges linking boundary vertices are considered constant and induce therefore fixed constraints. Internal vertices on the contrary, must be provided a final vector value and consequently provide the flexibility required by the simplification scheme (see Fig. 8.6).

Our method is then as follows.

```
// initialisation
foreach (intern vertex)
    if (no fixed constraints)
        interval = [0,2PI[
    else
        interval = fixed constraints
```

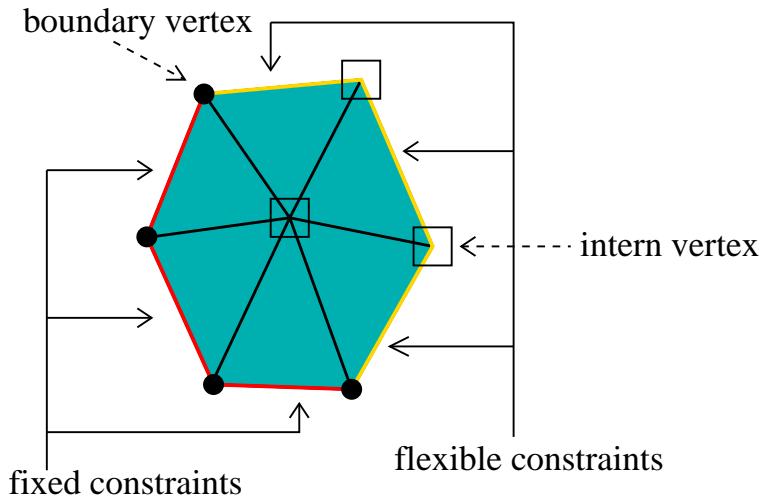


Figure 8.6: Different types of constraints for an intern vertex

```

        endif
        if (interval is empty)
            interrupt
        endif
    end foreach

    // iterations
    nb_iterations = 0
    repeat
        succeeded = true
        nb_iterations++
        foreach intern vertex
            compute mean vector of defined incident vertices
            if (interval is not empty)
                if (mean vector in interval)
                    current_value = mean vector
                else
                    current_value =
                        best approximation of mean vector in interval
                end if
            else
                succeeded = false
                if (mean vector in fixed constraints)
                    current_value = mean vector

```

```

    else
        current_value = best approximation of mean vector
                      in interval
    end if
end foreach
until (succeeded or nb_iterations > MAX_NB_ITERATIONS)

```

That is, we iteratively modify the vector values at all internal vertices by selecting angles that fulfill the current requirement induced by the neighbor vertices and by taking the mean value of these neighbors as predictor. This predictor permits to overcome provisory impossibilities due to flexible constraints.

If one of the internal vertices has incompatible fixed constraints, our scheme cannot succeed. Therefore we interrupt the process during initialization and move to the next pair. If the iterative process failed at determining compatible angular constraints for every internal vertex, we maintain the current pair and move to the next as well.

8.4 Results

We show next the results of our method applied to a swirling jet simulation. The grid is rectilinear and has 124 x 101 vertices ranging from 0 to 9.84 in x and from -3.864 to 3.864 in y . The triangulation has 24600 linearly interpolated cells. The original topology is shown in Fig. 8.7 together with the underlying grid structure. (Fig. 8.4 offers a depiction of the topology over a LIC representation.) There are 94 critical points and 134 corresponding pairs. We first simplify without magnitude control. The only threshold is therefore the graphical distance between critical points. We apply increasing thresholds ranging from 1% to 50% of the grid width to select the pairs to simplify. The table proposed next puts the corresponding results together.

threshold	satisfying pairs	connected pairs	removed pairs	removed sing.
1%	13 (10%)	10 (7%)	10 (7%)	20 (21%)
5%	24 (18%)	19 (14%)	19 (14%)	38 (40%)
10%	40 (30%)	27 (20%)	27 (20%)	54 (57%)
20%	65 (49%)	36 (27%)	34 (25%)	68 (72%)
50%	90 (67%)	40 (43%)	38 (40%)	76 (81%)

Remark that the number of connected pairs is typically far smaller than the number of pairs satisfying the numerical criteria. The reason for it is double: First, the critical points involved in the satisfying pairs are redundant

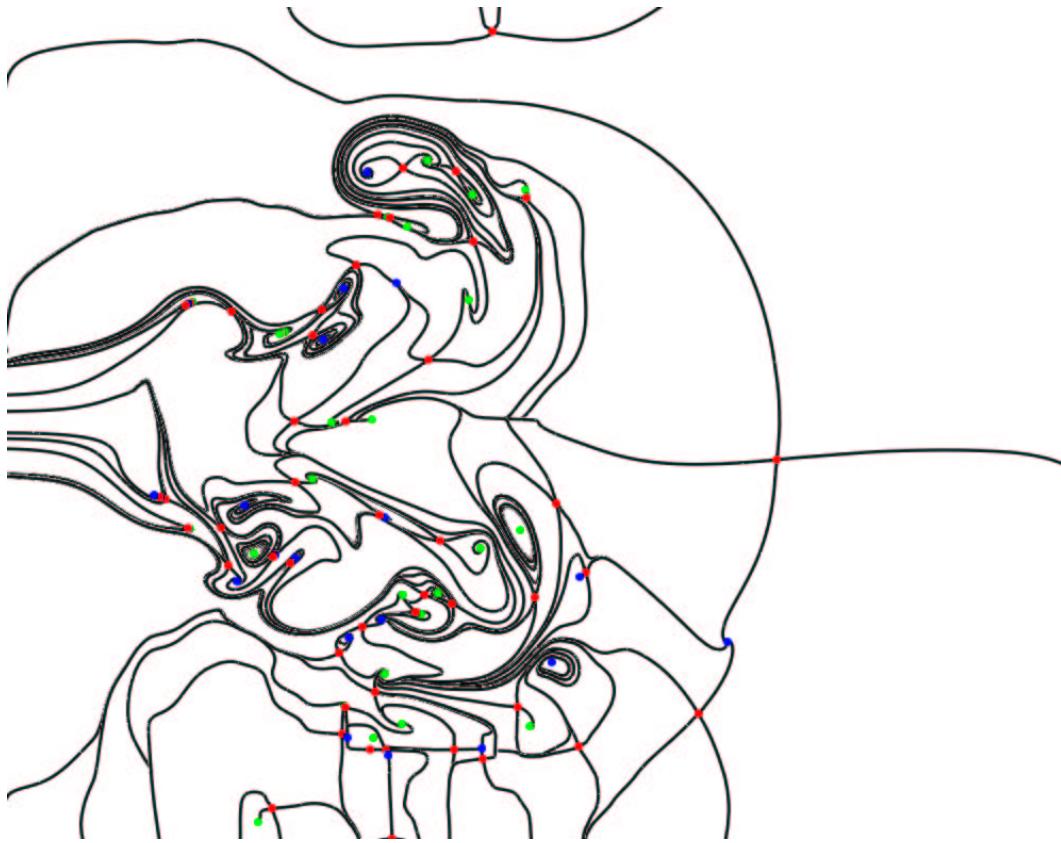


Figure 8.7: Original topology

(a critical point will often belong to several satisfying pairs) which prohibits the connection of a pair once one of its singularity has already been processed and removed. Second, the cell-wise connection of two linked critical point may fail, even if this is rarely the case (since the pairs are processed in proximity order). The pictures associated with the thresholds 5% and 50% are shown in Fig. 8.8 and Fig. 8.9 respectively. The first topology contains 56 critical points whereas there are only 18 singularities remaining in the second one. If we focus on a small part of the topology, we observe how features of small scale are removed: Compare Fig. 8.10(a) and Fig. 8.10(b).



Figure 8.8: Simplified topology: Small graphic threshold (5%)

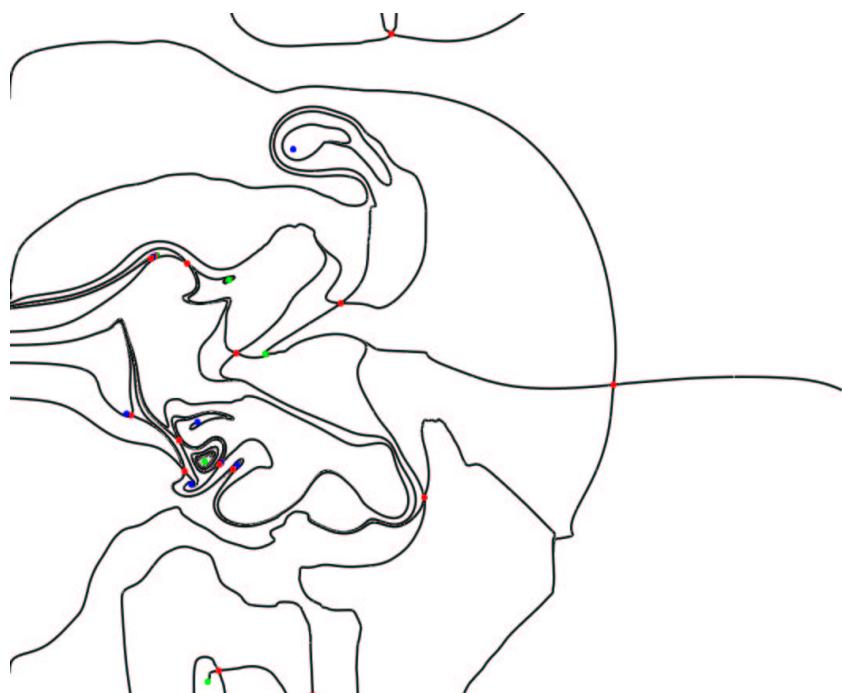


Figure 8.9: Simplified topology: Large graphic threshold (50%)

If we choose, on the contrary, to restrict the simplification to a filtering of computational noise by the use of a threshold on the field magnitude, we get the results presented in the following table (the threshold is expressed with respect to the largest norm of the vector field).

threshold	satisfying pairs	connected pairs	removed pairs	removed sing.
0.5%	25 (19%)	8 (6%)	8 (6%)	16 (17%)
1%	30 (22%)	11 (8%)	11 (8%)	22 (23%)
5%	47 (35%)	15 (11%)	15 (11%)	30 (32%)
10%	77 (57%)	21 (16%)	21 (16%)	42 (45%)
20%	95 (71%)	28 (21%)	26 (19%)	52 (55%)
50%	115 (86%)	36 (27%)	33 (25%)	66 (70%)

The picture shown in Fig. 8.11 illustrates the topology obtained after simplification with a very low threshold (0.5%) on the magnitude. The graph presents then 78 critical points.

At last, we propose in Fig. 8.12 the topology obtained when the simplification is applied without distance nor norm threshold. This obviously leads to the highest simplification rate that can be reached by our method for this particular dataset. The 14 critical points remaining correspond to configurations that cannot be resolved by the method. Nevertheless, in this depiction, no visual clutter is present and the original structural complexity has been greatly simplified. A LIC texture is proposed in the background to show the effect on the flow.

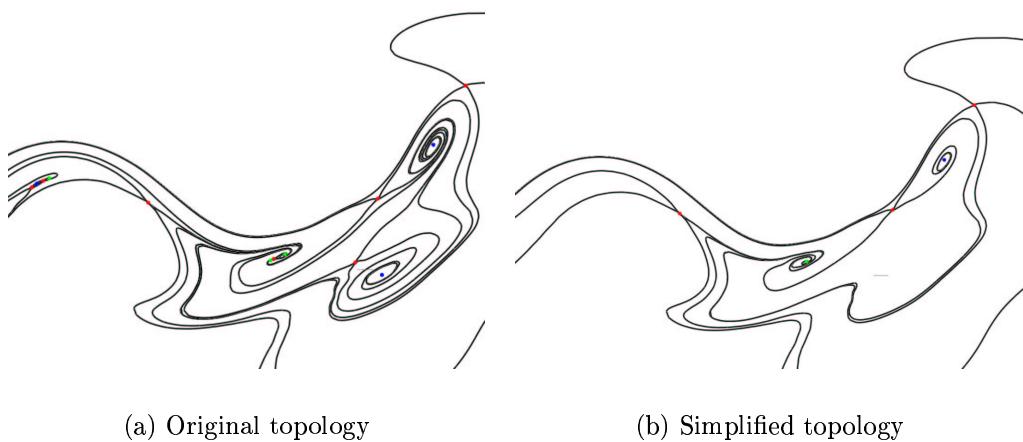


Figure 8.10: Enlargements

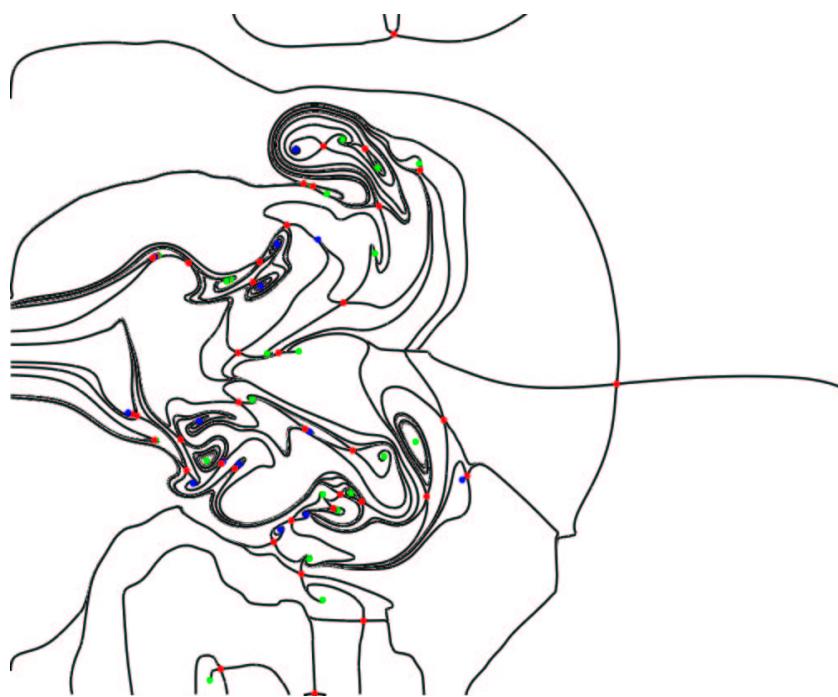


Figure 8.11: Simplified topology: Low noise filter (0.5% of maximal magnitude)

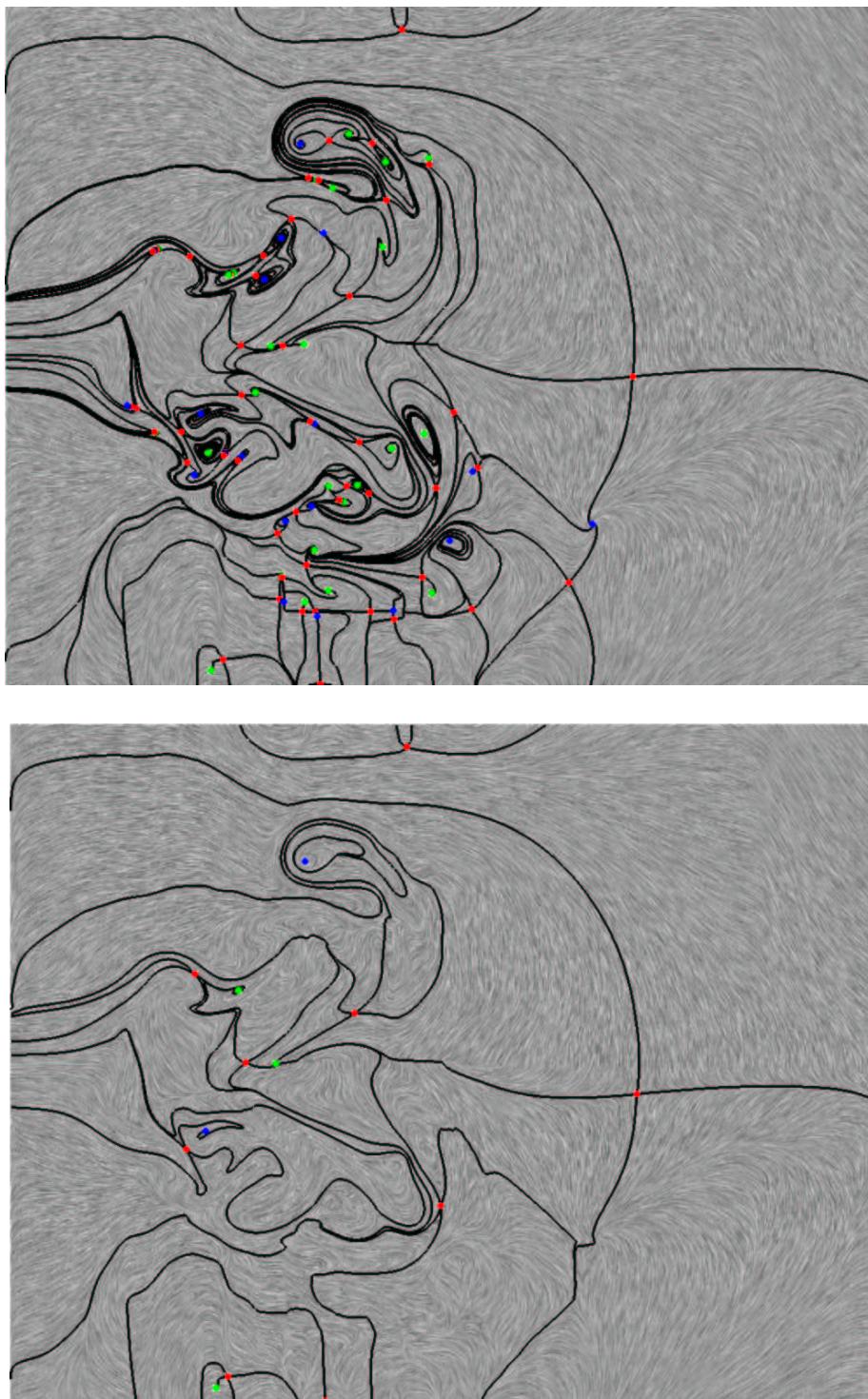


Figure 8.12: Original and simplified topology with LIC

Appendix A

Remarks on Numerical Integration in Vector and Tensor Fields

All methods presented previously rely on the integration of tangential curves in either vector or line fields. The computation of such curves is thus an essential task for visualization purposes. More precisely, two aspects must be taken into account when considering this numerical problem. On one hand the integration must be carried out efficiently to ensure satisfying speed for the results. This is particularly important if interactivity is required. On the other hand, to be of any use the computation must guaranty a prescribed accuracy. Clearly, a trade off is necessary between these requirements. We present next the choices made in this context and make few comments on these numerical issues.

A.1 Vector Fields

The integration of ordinary differential equations in the study of dynamical systems has been a large field of research for numerics. For this reason, a large number of numerical schemes exist. They offer different precision orders and the choice of a particular scheme must be made according to the considered application. For our purpose we chose the famous Runge-Kutta method and more precisely the scheme of fourth-order with adaptive step size. Corresponding definitions are given next. An overview of Runge-Kutta as well as various other integration methods can be found in [PTV92].

Let $a \in \mathbb{R}^2$ be the initial condition of the Cauchy problem and \mathbf{v} the vector field to integrate. The formulation of the Runge-Kutta method of fourth-order

is as follows. The η_i s are the successive steps of the method and h is the stepsize.

$$\begin{aligned}\eta_0 &:= a \\ k_1 &:= h\mathbf{v}(t_i, \eta_i) \\ k_2 &:= h\mathbf{v}(t_i + \frac{1}{2}h, \eta_i + \frac{1}{2}k_1) \\ k_3 &:= h\mathbf{v}(t_i + \frac{1}{2}h, \eta_i + \frac{1}{2}k_2) \\ k_4 &:= h\mathbf{v}(t_i + h, \eta_i + k_3) \\ \eta_{i+1} &:= \eta_i + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^4)\end{aligned}$$

It is thus of precision order 4 in h . Note that this is a single-step method i.e. of the form $\eta_{i+1} := \eta_i + h_i\Phi(h_i, \eta_i, \mathbf{v})$ for some function Φ .

The above formula is parametrized by the stepsize h . A simple but inefficient choice is to use a constant value. This might be too inaccurate in some regions and too expansive in some others. This argument motivated the design of methods that adaptively evaluate the stepsize. In our implementation, this evaluation is based on an embedded Runge-Kutta scheme invented by Fehlberg [Feh69]. The method starts with a general fifth-order Runge-Kutta formula

$$\begin{aligned}\eta_0 &:= a \\ k_1 &:= h\mathbf{v}(t_i, \eta_i) \\ k_2 &:= h\mathbf{v}(t_i + a_2h, \eta_i + b_{21}k_1) \\ k_3 &:= h\mathbf{v}(t_i + a_3h, \eta_i + b_{31}k_1 + b_{32}k_2) \\ k_4 &:= h\mathbf{v}(t_i + a_4h, \eta_i + b_{41}k_1 + b_{42}k_2 + b_{43}k_3) \\ k_5 &:= h\mathbf{v}(t_i + a_5h, \eta_i + b_{51}k_1 + b_{52}k_2 + b_{53}k_3 + b_{54}k_4) \\ k_6 &:= h\mathbf{v}(t_i + a_6h, \eta_i + b_{61}k_1 + b_{62}k_2 + b_{63}k_3 + b_{64}k_4 + b_{65}k_5) \\ \eta_{i+1} &:= \eta_i + c_1k_1 + c_2k_2 + c_3k_3 + c_4k_4 + c_5k_5 + c_6k_6 + O(h^5).\end{aligned}$$

Fehlberg found an alternative combination of these intermediate steps that results in an embedded fourth-order formula:

$$\eta_{i+1}^* := \eta_i + c_1^*k_1 + c_2^*k_2 + c_3^*k_3 + c_4^*k_4 + c_5^*k_5 + c_6^*k_6 + O(h^4).$$

The error estimate is therefore obtained from the comparison of the two results

$$\Delta = \eta_{i+1} - \eta_{i+1}^* = \sum_{i=1}^6 (c_i - c_i^*)k_i.$$

The parameters used in our implementation are those from Cash and Karp (refer to [PTV92, p. 717]). Remark that they are preferred to the original values proposed by Fehlberg. They are shown below.

i	a_i	b_{i1}	b_{i2}	b_{i3}	b_{i4}	b_{i5}	c_i	c_i^*
1							$\frac{37}{378}$	$\frac{2825}{27648}$
2	$\frac{1}{5}$	$\frac{1}{5}$					0	0
3	$\frac{3}{10}$	$\frac{3}{40}$	$\frac{9}{40}$				$\frac{250}{621}$	$\frac{18575}{48384}$
4	$\frac{3}{5}$	$\frac{3}{10}$	$-\frac{9}{10}$	$\frac{6}{5}$			$\frac{125}{594}$	$\frac{13525}{55296}$
5	1	$-\frac{11}{54}$	$\frac{5}{2}$	$-\frac{70}{27}$	$\frac{35}{27}$		0	$\frac{277}{14336}$
6	$\frac{7}{8}$	$\frac{1631}{55296}$	$\frac{175}{512}$	$\frac{575}{13824}$	$\frac{44275}{110592}$	$\frac{253}{4096}$	$\frac{512}{1771}$	$\frac{1}{4}$

Stepsize adaptivity is then obtained by the following algorithm (h_i is the current stepsize at the i -th step).

- (1) $h_1 := h_i$
- (2) Compute $\eta_{i+1}^*, \eta_{i+1}, \Delta$.
- (3) $h_0 := h_1 |\frac{\epsilon}{\Delta}|^{\frac{1}{5}}$ (estimated optimal stepsize)
- (4) if $\Delta > \epsilon$ then $h_i := h_0$; goto (2)
- (5) take η_{i+1} ; $h_{i+1} := h_0$

Implementation details are given in [PTV92, pp. 719-722].

A.2 Tensor Fields

As underlined previously, symmetric second-order tensor fields are by definition numerically more complicated to handle than vector fields. As a matter of fact, the actual analysis deals with eigenvector fields that provide a tangency information reduced to a line direction. This was detailed in section 3.2. Unfortunately the existing schemes for numerical integration are designed to handle vector fields, i.e. with additional norm and orientation information.

Practically, the challenging aspect of the integration in this case is the correlation of the integration orientation as one moves along the curve. This is better understood if one recalls that, except at possible degenerate points, the tensor lines can always be locally computed as the integral curves of a vector field that is tangent to the line field there, see section 3.3. Hence, when proceeding the computation of a tensor line, the task consists in determining a unit vector at each step that is equal to the unit vector obtained by local, smooth transformation of the last unit vector obtained so far along the path. This difficulty is illustrated in Fig. A.1.

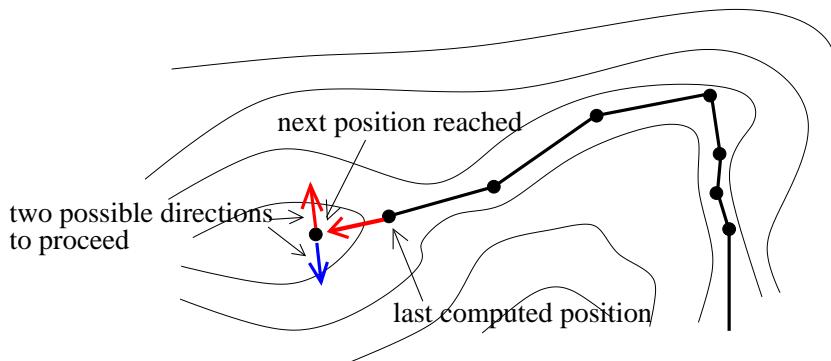


Figure A.1: Correlation of the orientation along a tensor line

A configuration as in the situation depicted above actually occurs in the vicinity of a singular point. There, the line field rotates around the singularity which inconveniences the correlation of successive directions. For instance, no reliable choice can be made if the new direction reached is orthogonal to the previous one (or close to orthogonality). In fact this problem is not specific to the tensor fields: The integration of streamlines in the neighborhood of a critical point also has to deal with rapidly rotating tangency directions. Nevertheless, this problem is properly handled by standard numerical schemes like fourth-order Runge-Kutta. The explanation is provided by the norm of the vector field in the vicinity of a singularity: By continuity of the field in this region and by definition of a critical point the norm of the vectors becomes arbitrarily small if one gets close to the singularity. So, if one remembers the iterative scheme of our Runge-Kutta method in the light of this remark, it is clear that the integration will move on very slow (i.e. the integration steps are very small) when approaching a critical point. This obviously ensures better accuracy for the integration. Consequently, one needs to define some norm for an eigenvector field that is consistent with the topology (which means that it is continuous and zero at the singular points). This norm must be seen

as a computational artifact that is not inherent to the data but conveys its structural properties. This reflexion is in fact an additional argument for the use of deviator fields in practice. As a matter of fact, the properties described in section 3.1 imply that any norm of the deviator field itself can serve as norm for both eigenvector fields (it is zero if and only if the considered point is singular). For our purpose we identify this norm with the scalar function $(x, y) \mapsto \alpha^2(x, y) + \beta^2(x, y)$ (keeping the same notations as in section 3.1) and apply it to each unit vector returned by our eigensystem solver. This gives very good results and used in combination with the fourth-order Runge-Kutta, it improves the efficiency of stepsize adaptivity.

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