ARIMA (AutoRegressive Integrated Moving Average)

The ARIMA (AutoRegressive Integrated Moving Average) model is a popular time series forecasting method that combines three key components: AutoRegression (AR), Differencing (I), and Moving Average (MA). Let's break down the mathematics behind ARIMA and how it works.

1. Components of ARIMA

ARIMA is denoted as **ARIMA(p, d, q)**, where:

- **p** = order of the AutoRegressive (AR) part
- **d** = degree of differencing (I)
- **q** = order of the Moving Average (MA) part

(A) AutoRegressive (AR) Model (p)

The AR part predicts future values based on past values (lags).

- **AR(p)** model equation:

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

  where:

  - X_t = current value
  - c = constant (bias)
  - $\phi_1, \phi_2, \ldots, \phi_p$ = coefficients of lagged terms
  - ϵ_t = white noise (error term)

(B) Differencing (I) (d)

- Makes the time series **stationary** (constant mean & variance).
- **First-order differencing**:

$$\nabla X_t = X_t - X_{t-1}$$

- **Second-order differencing**:

$$\nabla^2 X_t = (X_t - X_{t-1}) - (X_{t-1} - X_{t-2})$$

- **d = 0**: No differencing needed (already stationary).

(C) Moving Average (MA) Model (q)

The MA part models the error terms of past predictions.

- **MA(q)** model equation:

$$X_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

  where:

  - μ = mean of the series
  - $\theta_1, \theta_2, \ldots, \theta_q$ = coefficients of past errors
  - $\epsilon_t, \epsilon_{t-1}, \ldots$ = white noise terms

2. Combining AR and MA: ARMA(p, q)

Before differencing, ARMA combines AR and MA:

$$X_t = c + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}$$

3. Full ARIMA(p, d, q) Model

ARIMA = AR + I (differencing) + MA.

- If **d > 0**, we first difference the series **d times** to make it stationary.
- Then apply ARMA(p, q) on the differenced series.

Equation for ARIMA(1,1,1):

$$(1 - \phi_1 B)(1 - B)X_t = c + (1 + \theta_1 B)\epsilon_t$$

where:

- B = backshift operator ($BX_t = X_{t-1}$)
- Expanded form:

$$X_t = (1 + \phi_1)X_{t-1} - \phi_1 X_{t-2} + c + \epsilon_t + \theta_1 \epsilon_{t-1}$$

4. How ARIMA Works Step-by-Step

1. **Check Stationarity** (ADF test, KPSS test).
   - If not stationary, apply differencing (**d**).
2. **Determine p (AR) and q (MA)**:
   - Use **ACF (AutoCorrelation Function)** and **PACF (Partial ACF)** plots.
   - ACF helps find **q (MA terms)**.
   - PACF helps find **p (AR terms)**.

3. **Estimate coefficients** (ϕ_i, θ_j) using **Maximum Likelihood Estimation (MLE)**.

4. **Validate the model**:

   - Check residuals (should be white noise).

   - Use metrics like **AIC, BIC, RMSE**.

5. **Forecast future values** recursively.

5. Seasonal ARIMA (SARIMA)

For seasonal patterns, SARIMA extends ARIMA with seasonal terms:
SARIMA(p, d, q)(P, D, Q, s)

- **s** = seasonal period (e.g., 12 for monthly data).
- **P, D, Q** = seasonal AR, differencing, MA terms.

Equation Example (SARIMA(1,1,1)(1,1,1,12)):

$$(1 - \phi_1 B)(1 - \Phi_1 B^{12})(1 - B)(1 - B^{12})X_t = (1 + \theta_1 B)(1 + \Theta_1 B^{12})\epsilon_t$$