Calculus explained in 25 pages

Excerpt from the **No Bullshit Guide to Math and Physics** by Ivan Savov

Abstract—**This tutorial introduces the key ideas of calculus, which are used in businesses, engineering, and science. We'll learn about limits, derivatives, integrals, sequences, and series. This material is normally taught in two separated university courses: differential calculus and integral calculus, but we choose to present both topics together so that we can highlight the connections between them. The explanations in this tutorial combine words, math formulas, visualizations and Python code examples. Visit the URL `bit.ly/calctut3` to follow along the calculations in this tutorial in an interactive computational environment.**

Contents

I. Introduction

Calculus is the study of functions and their properties. The two calculus techniques we'll learn in this tutorial are *derivatives*, which tell us how functions *change* over time, and *integrals*, which compute the total *accumulation* of functions over time. Derivatives and integrals might sound like fancy math jargon, but actually they are common-sense concepts that you're already familiar with, as you'll see in the following example.

A. Example: file download

Suppose you're downloading a 720[MB] file from the internet to your computer. At $t = 0$ you click "Save as" to start the download. Consider the function $f(t)$ that describes the amount of disk space occupied by the partially-downloaded file.

Download rate: The *derivative function* $f'(t)$, pronounced "f prime," describes how the function $f(t)$ changes over time. In our example $f'(t)$ is the download speed. If your download speed is $f'(t) = 2$[MB/s], then the file size $f(t)$ will increase by 2[MB] each second. If you maintain this download speed, the file size will grow at a constant rate: $f(0) = 0$[MB], $f(1) = 2$[MB], $f(2) = 4$[MB], ..., $f(100) = 200$[MB], and so on until $t = 360$[s] when the download will be done.

To estimate the time that remains before the download completes, assuming the current speed stays roughly constant, we can divide the amount of data that remains to be downloaded by the current download speed:

$$\text{time remaining at } t = \tfrac{720-f(t)}{f'(t)} \quad [\text{s}].$$

The bigger the derivative $f'(t)$, the faster the download will finish. If your internet connection were 10 times faster, the download would finish 10 times more quickly.

Inverse problem: Let's now consider the download scenario from the point of view of the modem that connects your computer to the internet. Any data you download comes through the modem, so the modem knows the download rate $f'(t)$[MB/s] at all times during the download. The modem is separate from your computer, so it doesn't know the file size $f(t)$. Nevertheless, the modem can infer the file size at time t from the transmission rate $f'(t)$. Think about it—if the modem sees data flowing through at the rate of $f'(t) = 2$[MB/s], then it knows that the data accumulated on your computer is growing at the rate of 2[MB] each second. In calculus, we describe the total file size accumulated until time $t = \tau$ (the Greek letter *tau*) as the *integral* of the download rate $f'(t)$ between $t = 0$ and $t = \tau$:

$$f(\tau) \;=\; \int_{t=0}^{t=\tau} f'(t)\,dt.$$

The symbol $\int$ is an elongated S that stands for *sum*. Indeed, the "integral of $f'(t)$ between 0 and τ" is in some sense the sum of $f'(t)$ during each time instant dt between $t = 0$ and $t = \tau$. To calculate the total accumulated file size, we split the time interval between $t = 0$ and $t = \tau$ into many short time intervals dt of length 1[s]. During each second, the file size grows by $f'(t)\,dt$, where $f'(t)$ is the download rate measured in [MB/s], and dt is a time interval of duration 1[s].

The situation described in the above example shows that calculus concepts are not some theoretical constructs reserved for math specialists, but something you encounter every day. The derivative $q'(t)$ describes the rate of change of the quantity $q(t)$. The integral $\int_a^b q(t)dt$ measures the total accumulation of the quantity $q(t)$ during the time period from $t = a$ to $t = b$.

B. Doing calculus: then and now

The key ideas of calculus were developed by Isaac Newton and Gottfried W. Leibniz in the 17$^{\text{th}}$ century using symbolic calculations performed with pen and paper. Today we have computers at our disposal that are extremely good at doing numerical calculations. This tutorial combines both symbolic and numerical methods of doing calculus to give a complete picture of all the tools available to you and their use cases.

Symbolic calculations: The pen-and-paper approach is still a good way to learn calculus, because manipulating math symbols "by hand" develops your intuitive understanding of calculus procedures. Writing math on paper allows you to use high-level abstractions and arrive at exact symbolic answers.

Symbolic calculations using SymPy: The Python library SymPy allows you to do symbolic math calculations on a computer. Using a computer algebra system like SymPy extends the reach of symbolic calculus operations you can do by automating some of the tedious steps and jumping straight to answers. You can also use SymPy to check the answers you obtain from pen-and-paper calculations.

Numerical computing using NumPy and SciPy: Most practical applications of calculus don't require exact symbolic answers. Engineers don't care about the exact value of the square root of two $\sqrt{2}$, and instead represent $\sqrt{2}$ approximately the floating-point number 1.4142135623730951 on a computer. This numerical approximation is good enough for most engineering and scientific use cases. What we give up in mathematical exactitude, we gain manyfold in computational power: modern computers can perform trillions (10^{12}) of floating point operations per second! The Python libraries NumPy and SciPy make it easy to do numerical calculus operations on a computer, as we'll demonstrate in code examples throughout this tutorial.

C. Applications of calculus

We use calculus concepts to describe various quantities in physics, chemistry, biology, engineering, business, economics and other domains where quantitative analysis is used. Many laws of nature are expressed in terms of derivatives and integrals, so it's essential that you learn the language of calculus if you want to study science. In all these areas, the quantities of interest are described by functions and we use derivatives and integrals to do various useful calculations based on these functions. For example, derivatives are used for optimization, and integrals are used to compute probabilities in statistics and machine learning. This is the power of mathematical abstraction: the calculus techniques you learn for analyzing the rates of change of functions apply to solving real-world problems in many different domains.

II. Math prerequisites

Let's start with a review of key ideas from high school math, which we'll need to use as building blocks for calculus.

A. Set notation

Sets are collections of math objects. Many math ideas are expressed in the language of sets, so it's worth knowing the notation for sets.

- { definition }: we use curly brackets to define sets. The definition in the curly brackets is either a description of the set's contents, or a list of the elements in the set.
- $\mathbb{N}$: the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$.
- $\mathbb{N}_+$: the positive natural numbers $\mathbb{N}_+ = \{1, 2, 3, 4, 5, \ldots\}$.
- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}_+$: the set of nonnegative real numbers.
- $x \in S$: this statement is read "x is an element of S." We use this notation to indicate the "type" of the variable x. For example, writing "$x \in \mathbb{R}$" tells us x is a real number.

We can use the *set-builder* notation $\{\cdot \mid \cdot\}$ to define new sets. Inside the curly brackets, we first describe the general kind of mathematical objects we are talking about, followed by the symbol "$|$" (which stands for "such that"), followed by the conditions that identify the elements of the set. For example, the set of nonnegative real numbers $\mathbb{R}_+$ is defined as "all real numbers x such that $x \geqslant 0$," which is expressed compactly as $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$ using the set-builder notation.

The number line: The *number line* is a visual representation of the set of real numbers $\mathbb{R}$, as shown in Figure 1.

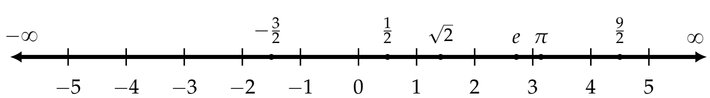


Figure 1. The real numbers $\mathbb{R}$ cover the entire number line.

The set of real numbers includes the natural numbers $\{0, 1, 2, 3, \ldots\}$, the integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, rational numbers like $-\frac{3}{2}$, 0.5, and $\frac{9}{2}$, as well as irrational numbers like $\sqrt{2}$, e, and π. All the numbers you will run into when doing math can be visualized as a point on the number line.

Infinity: The math symbol ∞ describes the concept of *infinity*. We use the symbol ∞ to represent an infinitely large quantity, that is greater than any number you can think of. Geometrically speaking, we can imagine the number line extends to the right forever towards infinity, as illustrated in Figure 1. The number line also extends forever to the left, which we denote as negative infinity $-\infty$.

Infinity is not a number but a *process*. When we use the symbol $+\infty$, we're describing the process of moving to the right on the number line *forever*. We go past larger and larger positive numbers and never stop.

Infinity is a key concept in calculus, so it's important that we develop a precise language to talk about infinitely large numbers and procedures with an infinite number of steps, which we'll do in Section III.

B. Functions

A *function* is a mathematical object that takes numbers as inputs and produces numbers as outputs. The output of the function f for the input x is denoted $f(x)$. We can define a function by writing an expression for the output of the function when the input is x. For example, the quadratic function is defined as $f(x) \stackrel{\text{def}}{=} x^2$. We use the "is defined as" symbol $\stackrel{\text{def}}{=}$ in the definition instead of the regular equals sign $=$. The function $f(x)$ takes any number x as input, and produces the square of this number as output. For example, when the input is $x = 3$, the output of the function is $f(3) = 3^2 = 9$.

In this tutorial, we'll often show code examples that mirror the math calculations. For example, here is the Python code that defines the function `f` and evaluates it for the input $x = 3$.

```
>>> def f(x):
        return x**2
>>> f(3)
9.0
```

Note the Python syntax for evaluating the function `f` for the input `3` is the same as the math syntax $f(3)$.

Function graphs: The *graph* of a function is a curve that passes through all input-output pairs of a function. Each input-output pair corresponds to a point $(x, f(x))$ in a Cartesian coordinate system. We obtain the graph of the function by varying the input coordinate x and plotting all the points $(x, f(x))$, as illustrated in Figure 2. The graph of the function f allows us to see at a glance the behaviour of the function for many inputs. Function graphs are an essential tool for calculus.

We can use the Python to plot the graph of the function $f(x) \stackrel{\text{def}}{=} x^2$ that we defined earlier. We start by importing the module `numpy` under the alias `np`. Next, we use the function `np.linspace` to create an array (a list of numbers) `xs` that contains 1000 x-values that range between $x = -3$ and $x = 3$. We then evaluate the function for all inputs `xs` and store the outputs of the function in an array called `fxs`.

```
>>> import numpy as np
>>> xs = np.linspace(-3, 3, 1000)
>>> fxs = f(xs)
```

At this point, the array `xs` contains 1000 x-inputs, while the array `fxs` contains the corresponding 1000 outputs of $f(x)$. To generate the graph of $f(x)$, we just need to trace a line passing through the 1000 coordinate pairs $(x, f(x))$. We can do this by importing the `seaborn` module (alias `sns`) and calling the function `sns.lineplot`.

```
>>> import seaborn as sns
>>> sns.lineplot(x=xs, y=fxs)
See Figure 2 for the output.
```

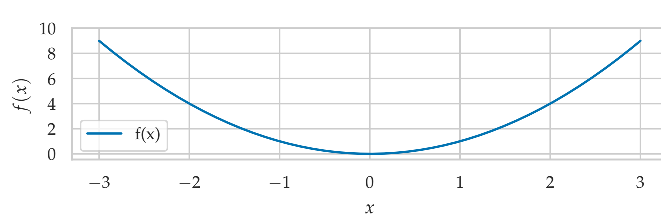


Figure 2. Graph of the function $f(x) = x^2$ from $x = -3$ to $x = +3$. The graph of the function f passes through the coordinate pairs $(x, f(x))$ for all x-values between $x = -3$ and $x = 3$.

We'll use this combination of `np.linspace`, function evaluation, and `sns.lineplot` to plot function graphs.

Inverse functions: The inverse function f^{-1} performs the *inverse operation* of the function f. If you start from some x, apply f, then apply f^{-1}, you'll arrive—full circle—back to the original input x:

$$f^{-1}\big(f(x) \big) = x.$$

In Figure 3, the function f is represented as a forward arrow, and the inverse function f^{-1} is represented as a backward arrow that puts the value $f(x)$ back to the x it came from.

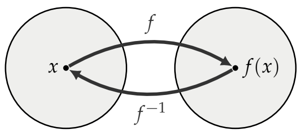


Figure 3. The inverse f^{-1} undoes the operation of the function f.

For example, when $x \geqslant 0$, the inverse of the function $f(x) = x^2$ is the function $f^{-1}(x) = \sqrt{x}$. Earlier we computed $f(3) = 9$. If we apply the inverse function $f^{-1}(x) = \sqrt{x}$ to 9, we get back to the number 3 that we started from $f^{-1}(9) = \sqrt{9} = 3$.

```
>>> from math import sqrt
>>> sqrt(9)
3
```

Function properties: We often think about the possible inputs and outputs of functions. We use the notation $f\colon A \to B$ to denote a function from the input set A to the output set B. The set of allowed inputs is called the *domain* of the function, while the set of possible outputs is called the *image* of the function. For example, the domain of the function $f(x) = x^2$ is $\mathbb{R}$ (any real number) and its image is $\mathbb{R}_+$ (nonnegative real numbers), so we say f is a function of the form $f\colon \mathbb{R} \to \mathbb{R}_+$.

C. Function inventory

Your function "vocabulary" determines which math expressions you'll be able to read and understand in the same way your English vocabulary determines which English sentences you'll be able to read and understand. Figure 4 shows the graphs of six important functions that are used in many areas of mathematical modelling.

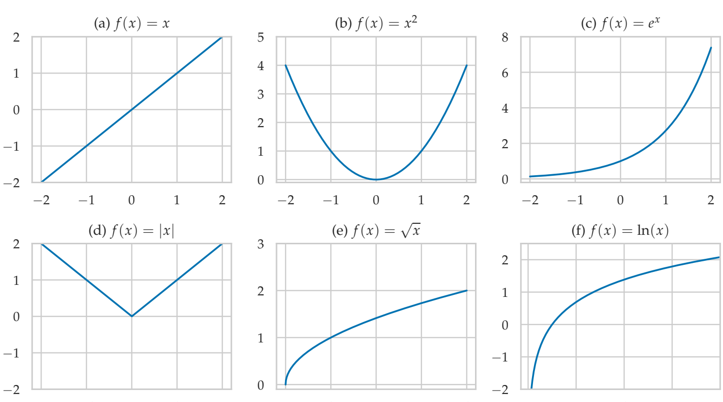


Figure 4. Graph of six math functions that you should know about.

Linear function: The linear function $f(x) \stackrel{\text{def}}{=} mx$ describes an input-output relationship where the output value $f(x)$ is *proportional* to the input value x, and the constant of proportionality is m. Geometrically, m is the slope in the graph of $f(x)$. Figure 4 shows the graph of $f(x) \stackrel{\text{def}}{=} x$ which is the linear function with $m = 1$ for which the output $f(x)$ is equal to the input x. More generally, we can define the line $f(x) \stackrel{\text{def}}{=} mx + b$, where m describes the slope of the line, and b is the value of the function when $x = 0$.

Quadratic function: The quadratic function $f(x) \stackrel{\text{def}}{=} x^2$ calculates the square of the input x. The name "quadratic" comes from the Latin *quadratus* for square. Geometrically, x^2 is the area of a square with side length x. See Figure 4 (b) for the graph. The outputs of the quadratic function are always nonnegative numbers since $x^2 \geqslant 0$, for all real numbers x.

Polynomial functions: We can combine different powers of x to obtain the polynomial $f(x) \stackrel{\text{def}}{=} a_2x^2 + a_1x + a_0$, where a_2, a_1, a_0 are arbitrary constants. This is called a second degree polynomial, since the highest power of x it contains is x^2. The general equation for a polynomial function of degree n is

$$P_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n.$$

The constants $a_0, a_1, a_2, \ldots, a_n$ are called the *coefficients* of the polynomial. Polynomials are a very useful family of functions, because they can take on different shapes depending on the choice of the coefficients. In Section VI-G, we'll learn how to obtain polynomial approximations to any function $f(x)$.

Exponential function: The exponential function base e is defined as $f(x) \stackrel{\text{def}}{=} e^x = \exp(x)$, where $e = 2.7182818\ldots$ is Euler's number. Figure 4 (c) shows the graph of the exponential function $f(x) = e^x$, which passes through the following points: $(-2, \frac{1}{e^2})$, $(-1, \frac{1}{e})$, $(0, 1)$, $(1, e)$, and $(2, e^2)$.

Absolute value function: The *absolute value* function tells us the size of numbers without paying attention to whether the number is positive or negative. We compute the absolute value of the number x by *forgetting* the sign of x. Geometrically, $|x|$ corresponds to the distance between x and the origin of the number line. We see the absolute values whenever we apply the combination of squaring followed by square root on some number, $\sqrt{x^2} = |x|$, since squaring destroys the sign.

Square root function: The square root function is denoted $f(x) \stackrel{\text{def}}{=} \sqrt{x}$. The square root $\sqrt{x}$ is the inverse function of the square function x^2, when $x \geqslant 0$. The symbol $\sqrt{c}$ refers to the *positive* solution to the equation $x^2 = c$. Note that $-\sqrt{c}$ is also a solution of $x^2 = c$. Another notation for the square root function is $f(x) \stackrel{\text{def}}{=} x^{\frac{1}{2}}$, where the fractional exponent $\frac{1}{2}$ makes sense since if we square $x^{\frac{1}{2}}$, we get back to x: $\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{2}{2}} = x^1 = x$. In addition to *square* root, there is also the *cube root* function $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, which is the inverse function for the cubic function $f(x) = x^3$. For example, $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$.

Logarithmic function: The natural logarithm function is denoted $f(x) \stackrel{\text{def}}{=} \ln(x) = \log_e(x)$. The function $\ln(x)$ is the inverse function of the exponential e^x. The graph of the function $\ln(x)$ passes through the following coordinate pairs: $(\frac{1}{e^2}, -2)$, $(\frac{1}{e}, -1)$, $(1, 0)$, $(e, 1)$, $(e^2, 2)$, $(e^3, 3)$, $(e^4, 4)$, etc.

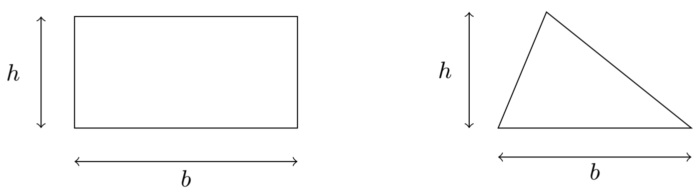
There are many other functions worth knowing about, but if you're familiar with the six function shown in Figure 4, you're doing well.

D. Functions with discrete inputs

Later in this tutorial, we'll study functions with discrete inputs, $a_k : \mathbb{N} \to \mathbb{R}$, which are called *sequences*. We often express sequences by writing explicitly the first few values the sequence $[a_0, a_1, a_2, a_3, \ldots]$, which correspond to evaluating a_k for $k = 0$, $k = 1$, $k = 2$, $k = 3$, etc. We'll learn more about sequences and their properties in Section VI.

E. Geometry of rectangles and triangles

The area of a rectangle of base b and height h is $A = bh$, as illustrated in Figure 5 (a). The area of a triangle is equal to $\frac{1}{2}$ times the length of its base b times its height h: $A = \frac{1}{2}bh$, as shown in Figure 5 (b).



(a) $A_{\square} = bh$ (b) $A_{\triangle} = \frac{1}{2}bh$

Figure 5. Formulas for calculating the area of a rectangle and a triangle.

F. Trigonometric functions

The *unit circle* is a circle of radius one centred at the origin, as illustrated in Figure 6. The unit circle consists of all points (x, y) that satisfy the equation $x^2 + y^2 = 1$. A point on the unit circle has coordinates $(\cos\theta, \sin\theta)$, where θ is the angle the point makes with the x-axis.

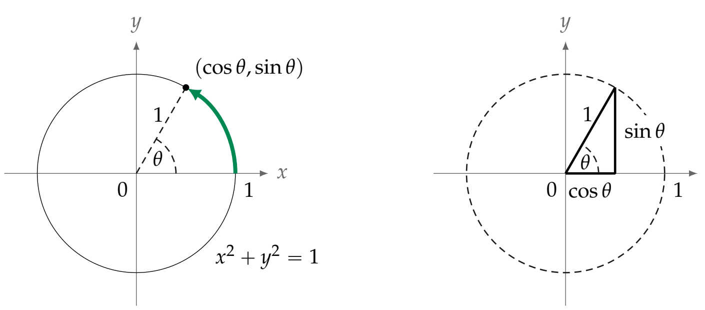


Figure 6. The coordinates of the point on the unit circle are $\cos\theta$ and $\sin\theta$.

In math, we use *radians* to measure angles instead of degrees $^\circ$. One full circle is 360° or 2π radians. Some common angle measures are $30^\circ = \frac{\pi}{6}$, $45^\circ = \frac{\pi}{4}$, $60^\circ = \frac{\pi}{3}$, and $90^\circ = \frac{\pi}{2}$. The trigonometric functions sin and cos take inputs in radians, so we often convert angles from degrees to radians.

Sine function: The graph of the sine function $f(\theta) \stackrel{\text{def}}{=} \sin(\theta)$ *oscillates* up and down and crosses the x-axis multiple times, as shown in Figure 7 (a). This graph corresponds to the vertical position of the point turning around on the unit circle, as illustrated in Figure 6 (a). We also use the sine function to find the y-component of a unit length, as shown in Figure 6 (b).

Cosine function: The cosine function is the same as the sine function shifted by $\frac{\pi}{2}$ to the left: $f(\theta) = \cos(\theta) = \sin(\theta + \frac{\pi}{2})$, as shown in Figure 7 (b). The cosine function represents the horizontal position of a point on the unit circle, and the x-component of a unit length, as illustrated in Figure 6.

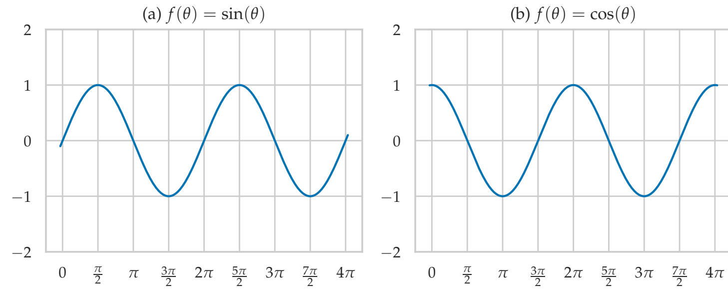


Figure 7. Graphs of the trigonometric functions $\sin(\theta)$ and $\cos(\theta)$ showing two full cycles. The graphs represent the vertical and horizontal position of a point that completes two full circles around the unit circle.

We use the trigonometric functions $\sin(\theta)$ and $\cos(\theta)$ to compute components of vectors in different directions. The sine and cosine functions are also used to describe waves and periodic motion in physics. In this tutorial, we won't discuss these applications too much, and instead focus on the trigonometric functions' graphs and rates of change.

Overview and a look ahead

The goal of this tutorial is to introduce you to the language of calculus. The concept map in Figure 8 shows an overview of the calculus ideas you'll learn in the next few pages.

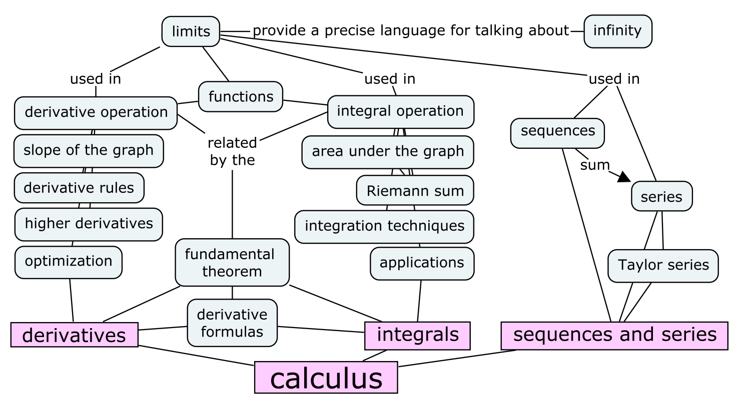


Figure 8. The calculus concepts and topics you'll learn in this tutorial.

We'll start by introducing limits in Section III. Limits give us a precise language to talk about infinity. Limits are a cornerstone idea in calculus, because they allow us to define the calculus operations: derivatives, integrals, and series. We'll discuss derivatives in Section IV and integrals in Section V. We'll then talk about sequences and series in Section VI, and conclude with a brief intro to multivariable calculus in Section VII.

Throughout the tutorial, we'll explain concepts using text, formulas, graphs, and code examples. My intention is for you to understand the key ideas of calculus in theory, but also learn practical skills you can use to solve real-world problems.

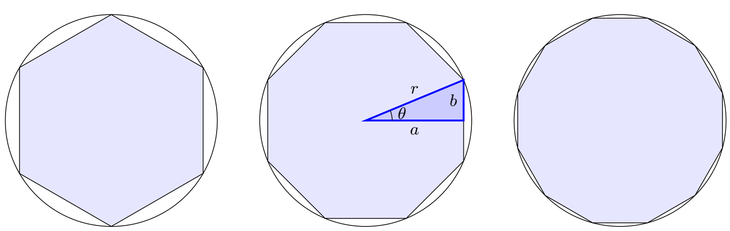
III. Limits

Limits are a precise mathematical language for talking about infinitely large numbers, infinitely small lengths, and procedures with an infinite number of steps. We use the shorthand "lim" to denote limit expressions. For example, the expression $\lim_{x\to\infty} f(x)$, read "the limit of $f(x)$ as x goes to infinity," describes what happens to $f(x)$ when x gets larger and larger.

A. Example 1: Archimedes' approximation to π

We'll start by looking at a visual example of a math procedure that was invented by Archimedes of Syracuse around 250 BCE. Archimedes wanted to calculate the area of a circle of radius $r = 1$. Today we know the formula for the area of the circle is $A_\circ = \pi r^2$, so the area of a circle with radius $r = 1$ is π. Try to place yourself in Archimedes's shoes (sandals?) and suppose that you don't know the formula.

Archimedes had the clever idea to approximate the circle as a regular polygon with n sides inscribed inside the circle. Figure 9 shows the hexagonal (6-sides), octagonal (8-sides), and dodecagonal (12-sides) approximations to the circle.



(a) $n = 6$. (b) $n = 8$. (c) $n = 12$.

Figure 9. Approximations to the area of a circle using a hexagon, an octagon, and a dodecagon inscribed inside a circle of radius r.

Archimedes split the area of the n-sided regular polygons into $2n$ triangular slices, like the one shown in Figure 9 (b). He then computed the area of each slice using the formula for the area of a triangle, and added up the areas of the $2n$ triangles to obtain the total area of the n-sided polygon. Let's denote $A(n)$ the area approximation computed from a n-sided polygon. Looking at Figure 9, we see the approximations to the area of the circle using six-sided and eight-sided polygons are underestimates for the total area. However, the polygon with $n = 12$ is starting to look almost like a circle. We can use our imagination to see that the approximation $A(n) \approx A_\circ$ will become increasingly accurate as n becomes larger and larger. Archimedes only managed to compute an approximation with a 96-sided polygon, but thanks to computers we can push the approximation to much higher values of n. For example, using a 1000-sided polygon gives us an approximation that is accurate to four decimals $A(1000) = 3.1415\ldots$. Using $n = 10\,000$ we get an approximation to π that is accurate to six decimals $A(10000) = 3.141592\ldots$. See the computational notebook `bit.ly/calctut3` for the details of these calculations.

In the limit as $n \to \infty$, the approximation $A(n)$ becomes *exactly* equal to $\pi = 3.141592653589793\ldots$, which we can express as the limit expression $\lim_{n\to\infty} A(n) = \pi$. Note that $A(n) \neq \pi$ for any finite number n, no matter how large n is. It is only in the limit as n goes to infinity that the n-sided polygon approximation becomes exactly equal to a circle.

Let's look at another example of a simple math procedure with n steps that produces a useful approximation.

B. Example 2: Euler's number

Suppose you take out a loan with 100% nominal interest rate. This is a very bad loan that nobody would agree to in the real world, but we'll use it for this example to make the math come out simpler. An interest of 100% calculated yearly means at the end of one year, you'll owe the bank $(1 + 100\%) = (1 + 1) = 2$ times the amount you borrowed initially.

However, most banks don't calculate the interest only once per year. If the bank calculates the interest twice per year, during the first six months you'll have accrued $\frac{100\%}{2} = 50\%$ of interest, so you'll owe them $(1 + 50\%) = (1 + \frac{1}{2}) = 1.5$ times the initial amount. Then during the second six months, the amount owed will grow by an additional $(1 + 50\%) = (1 + \frac{1}{2}) = 1.5$, so at the end of the year, you'll owe them $(1 + \frac{1}{2})(1 + \frac{1}{2}) = 2.25$.

If the bank computes the interest three times per year, the amount owed after one year is $(1 + \frac{1}{3})(1 + \frac{1}{3})(1 + \frac{1}{3}) = 2.370$. If they compute the interest four times per year (quarterly), then you'll owe $(1 + \frac{1}{4})(1 + \frac{1}{4})(1 + \frac{1}{4})(1 + \frac{1}{4}) = 2.441$. Note the amount owed after one year keeps changing, as the compounding is performed more frequently. In general, when the compounding is performed n times per year, the amount owed at the end of one year will be

$$\underbrace{\left(1 + \tfrac{1}{n}\right)\left(1 + \tfrac{1}{n}\right)\cdots\left(1 + \tfrac{1}{n}\right)}_{n \text{ times}} = \left(1 + \tfrac{1}{n}\right)^n.$$

With monthly compounding ($n = 12$), the amount owed will be $(1 + \frac{1}{12})^{12} = 2.613$ at the end of one year. With daily compounding, the amount would be $(1 + \frac{1}{365})^{365} = 2.715$. If computing the interest $n = 1000$ times per year, the amount will be $(1 + \frac{1}{1000})^{1000} = 2.717$. The amount owed keeps increasing, but it seems to "stabilize" around the value 2.71.

What happens if we perform the compounding even more frequently? Specifically, we want to know what happens if the interest is compounded infinitely often. The infinitely-often calculation corresponds to computing the *limit* of expression $(1 + \frac{1}{n})^n$, as n goes to infinity, which is written as follows using math notation:

$$\lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n = e = 2.718281828\ldots.$$

This limit expression *converges* to the value $e = 2.71828\ldots$, which is known as *Euler's number*. If we borrow \$1000, we'll owe \$1000e = \$2718.28 at the end of one year.

We defined the number π as the limit $\lim_{n\to\infty} A(n)$ and the number e as the limit $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$. The definitions of the numbers π and e as limits go beyond the regular math operations we learn in high school math. The limit expression $\lim_{n\to\infty}$ doesn't describe any particular number n, but the *process* of plugging in larger and larger values of n.

C. Limits at infinity

We can use limit expressions to describe what happens to a function when its input variable tends to infinity. Does $f(x)$

approach a finite number, or does it keep growing to ∞? The function $f(x)$ *converges* to L if the function approaches the value L for large values of x:

$$\lim_{x\to\infty} f(x) = L.$$

We say "the limit of $f(x)$ as x goes to infinity is L." See Figure 10 for an illustration.

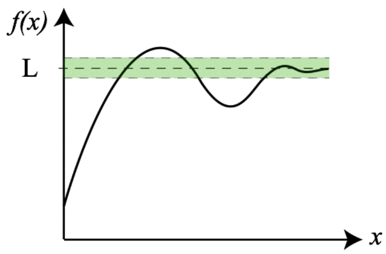


Figure 10. The function $f(x)$ oscillates up and down initially, but then it "settles down" close to the value L for large values of x.

Example 3: Consider the limit of the function $f(x) = \frac{1}{x}$ as x goes to infinity, which is illustrated in Figure 11 (a):

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0.$$

The function $\frac{1}{x}$ never *actually* reaches zero, so it would be wrong to write $f(x) = 0$ for any $x \in \mathbb{R}$. However, the expression $\frac{1}{x}$ gets closer and closer to 0 as x goes to infinity. Limits are useful because they allow us describe this tendency.

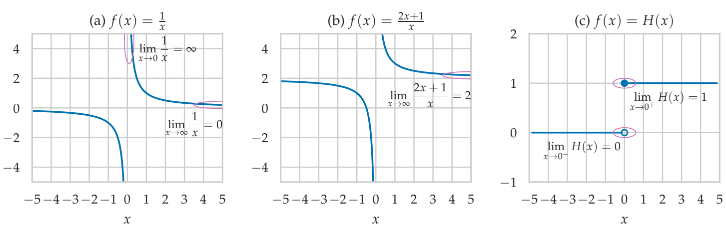


Figure 11. Visual representation of the limit calculations for three functions.

D. Limit formulas

The limit of the sum, difference, product, and quotient of two functions are computed as follows:

$$\begin{aligned}
\lim_{x\to\infty} (f(x) + g(x)) &= \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x), \\
\lim_{x\to\infty} (f(x) - g(x)) &= \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x), \\
\lim_{x\to\infty} f(x)g(x) &= \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x), \\
\lim_{x\to\infty} (f(x)/g(x)) &= \lim_{x\to\infty} f(x) \,/\, \lim_{x\to\infty} g(x).
\end{aligned}$$

In words, these formulas tell us we can bring the limit calculations "inside" basic arithmetic operations.

Example 4: Calculate $\lim_{x\to\infty} \frac{2x+1}{x}$. We're given the function $f(x) = \frac{2x+1}{x}$ and must determine what the function looks like for very large values of x. We can rewrite the function as $\frac{2x+1}{x} = 2 + \frac{1}{x}$ then apply the sum formula for limits:

$$\lim_{x\to\infty} \frac{2x+1}{x} = \lim_{x\to\infty} \left(2 + \frac{1}{x}\right) = \lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{1}{x} = 2 + 0 = 2.$$

As x goes to infinity, $\frac{1}{x}$ goes to 0, so the second term vanishes, leaving only the 2. See Figure 11 (b) for an illustration.

E. Limits to zero

The limit expression $\lim_{x\to 0} f(x)$ describes the behaviour of the function f for values of x very close to 0. The limit $\lim_{x\to 0} f(x)$, read "the limit of $f(x)$ as x goes to zero," asks us to evaluate the function f for inputs like $x = 0.1$, $x = 0.01$, $x = 0.001$, $x = 0.0001$, etc. to see the behaviour of the function for very small values of x.

For example, when $x > 0$, the limit $\lim_{x\to 0} \frac{1}{x} = \infty$. In words, the function $f(x) = \frac{1}{x}$ "blows up" to infinity as x goes to 0, as shown in Figure 11 (a).

F. Limits to a number

More generally, the limit of $f(x)$ approaching $x = a$ *from the right* is denoted $\lim_{x\to a^+} f(x) = \lim_{\delta\to 0} f(a+\delta)$. We use the symbol δ (the Greek letter *delta*) to describe a distance that gets smaller and smaller. This limit expression describes the value of the function f as the input x gets closer and closer to a with values like $a + 0.1$, $a + 0.01$, $a + 0.001$, $a + 0.0001$, etc. The limit of $f(x)$ when x approaches *from the left* is defined analogously, $\lim_{x\to a^-} f(x) = \lim_{\delta\to 0} f(a-\delta)$.

If both limits from the left and from the right at $x = a$ exist and are equal to each other, we say the limit as $x \to a$ exists:

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

For the two-sided limit of a function to exist at a point, both the limit from the left and the limit from the right must converge to the same number.

G. Continuity

If the function $f(x)$ obeys, $f(a) = \lim_{x\to a} f(x)$, we say the function $f(x)$ is *continuous* at $x = a$. Geometrically, the graph of the continuous function at $x = a$ is a "smooth" curve that doesn't have any hole or a jump at $x = a$. When a function is continuous, we can draw its graph using a single pen stroke without lifting the pen. In contrast, functions that blow up to infinity or make sudden jumps are not continuous.

Example 5: The *Heaviside step function* is an example of a function with a jump discontinuity. It is defined as follows:

$$H(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } x \geqslant 0 \\ 0, & \text{if } x < 0. \end{cases}$$

The function is zero for negative values of x, then suddenly jumps to one at $x = 0$, as shown in Figure 11 (c). The limit as x approaches $x = 0$ from the left is $\lim_{x\to 0^-} H(x) = 0$. The limit at $x = 0$ from the right is $\lim_{x\to 0^+} H(x) = 1$. The two limits are different, $\lim_{x\to 0^-} H(x) = 0 \neq 1 = \lim_{x\to 0^+} H(x)$, so the function is *discontinuous* at $x = 0$.

H. Computing limits using SymPy

We can use SymPy to compute limit expressions, which allows us to check the answers we obtain using pen-and-paper calculations. We'll start by importing the `sympy` module under the alias `sp`, defining the symbolic variable `n` $= n$, which we can then use to write various expressions.

```
>>> import sympy as sp
>>> n = sp.symbols("n")
```

To compute limit expressions, we use the SymPy function `sp.limit(expr,var,value)`, which returns $\lim_{\texttt{var}\to\texttt{value}} \texttt{expr}$. For limits to infinity, we use the symbol `sp.oo` (two lowercase os), which kind of looks like the infinity symbol ∞.

Euler's number is defined as the limit $e \overset{\text{def}}{=} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. To compute this limit using SymPy, we call `sp.limit` on the expression `(1+1/n)**n` as `n` goes to infinity $\infty =$ `sp.oo`:

```
>>> sp.limit((1+1/n)**n, n, sp.oo)
E
```

The result of `sp.limit` is the exact value e which is represented symbolically as `E`.

Let's now compute the limits $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to 0^+}\frac{1}{x}$. We first define the symbol `x` then call the function `sp.limit` to evaluate the two limits involving the expression `1/x` $= \frac{1}{x}$:

```
>>> x = sp.symbols("x")
>>> sp.limit(1/x, x, sp.oo)
0
>>> sp.limit(1/x, x, 0)
oo
```

SymPy confirms that $\lim_{x\to\infty}\frac{1}{x} = 0$ and $\lim_{x\to 0^+}\frac{1}{x} = \infty$, as calculated earlier. See Figure 11 (a) for an illustration.

Here is another example, that computes the limit of the fraction $\frac{2x+1}{x}$ as x goes to infinity, which is illustrated in Figure 11 (b).

```
>>> sp.limit((2*x+1)/x, x, sp.oo)
2
```

To calculate the limit from the left or the right of a number, we provide a fourth argument `"-"` or `"+"` to `sp.limit`.

```
>>> from sympy import Heaviside
>>> sp.limit(Heaviside(x,1), x, 0, "-")
0
>>> sp.limit(Heaviside(x,1), x, 0, "+")
1
```

I. *Applications of limits*

Limits are important because they are used in the formal definitions of derivatives, integrals, and series:

- The derivative function $f'(x)$ describes the rate of change of the function $f(x)$ at x. In Section IV, we'll calculate derivatives by evaluating limits of the form $\lim_{\delta\to 0}$.
- The integral $\int_a^b f(x)\,dx$ describes the area under the graph of the function $f(x)$ between $x = a$ and $x = b$. In Section V we'll learn how to compute integrals by splitting up areas into n rectangles, then taking the limit $\lim_{n\to\infty}$.
- The series $\sum_{k=1}^{n} a_k$ describes the sum of all the first n terms in the sequence a_k. In Section VI, we'll learn how to compute infinite series by evaluating limits like $\lim_{n\to\infty}$.

IV. Derivatives

The *derivative* function, denoted $f'(x)$, $\frac{d}{dx}f(x)$, or $\frac{df}{dx}$, describes the *rate of change* of the function $f(x)$. For example, the constant function $f(x) = c$ has derivative $f'(x) = 0$ since it doesn't change. Geometrically, the derivative function describes the *slope* of the graph of the function $f(x)$. The derivative of the line $f(x) = mx + b$ is $f'(x) = m$, since the slope of this line is equal to m. For general curves, the slope of a function will change at different values of x, so mathematicians invented the notation $f'(x)$ for describing "the slope of the function f at x."

Let's start by calculating the *average* slope of the function between two points. Consider the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ on the graph of the function. We'll denote the horizontal distance that separates the two points Δx (read *delta* x), and similarly denote the vertical distance between the points as $\Delta y = f(x + \Delta x) - f(x)$. We can obtain the average slope of the function using the rise-over-run formula: $m = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)\,-\,f(x)}{x+\Delta x\,-\,x}$. Figure 12 illustrates the result of the average slope calculations at $x = 1$ for different horizontal distance Δx. When $\Delta x = 2$ the average slope is $m = \frac{\Delta y}{\Delta x} = \frac{8}{2} = 4$. When $\Delta x = 1$ we get $m = \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$. When $\Delta x = 0.3$ the slope is $m = \frac{\Delta y}{\Delta x} = \frac{0.69}{0.3} = 2.3$. If we continue this process with even smaller Δx, we'll obtain the *instantaneous* slope at the point x.

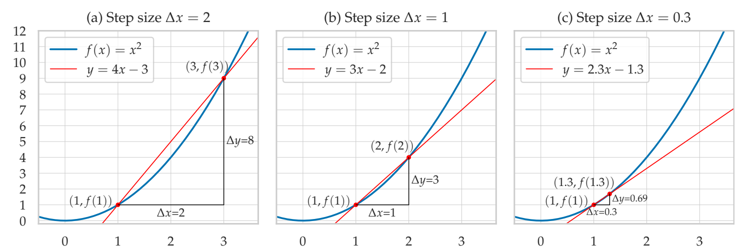


Figure 12. Calculating the slope of the function $f(x) = x^2$ by finding the line that passes through the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$.

The derivative function $f'(x)$ is defined as the following limit:

$$f'(x) \overset{\text{def}}{=} \lim_{\delta\to 0}\frac{f(x+\delta)\ -\ f(x)}{\delta}.$$

Here δ plays the same role as Δx, but goes to zero. In words, this derivative formula describes the rise-over-run calculation for an infinitely short horizontal distance δ.

The derivative is a function of the form $f' : \mathbb{R} \to \mathbb{R}$. It takes the value x as input and tells you the slope of the function f at that value. Figure 13 shows the slope of the function $f(x) = x^2$ at two different locations: at $x = -0.5$ and at $x = 1$.

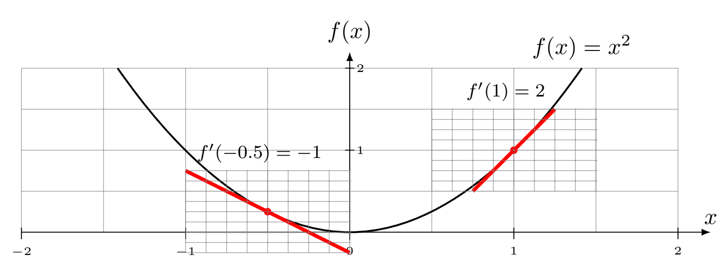


Figure 13. The derivative of the function at $x = a$ is denoted $f'(a)$ and describes the slope function at that point. You can visually confirm the slope calculation using the mini grids drawn near each point.

The derivative function $f'(x)$ is a property of the function $f(x)$. Indeed, this is where the name *derivative* comes from: $f'(x)$ is not an independent function—it is *derived* from the slope property of the original function $f(x)$. More generally, we can define the *derivative operation*, denoted $\frac{d}{dx}[\texttt{<f>}]$, which takes as input a function $f(x)$ and produces as output the derivative function $f'(x)$. Applying the derivative operation to the function is also called “taking the derivative” of a function. For example, the derivative of the function $f(x) = x^2$ is the function $f'(x) = 2x$. We can also describe this relationship as $(x^2)' = 2x$ or as $\frac{d}{dx}(x^2) = 2x$. Look at the graph in Figure 13 to convince yourself that the slope of $f(x) = x^2$ is indeed described by $f'(x) = 2x$ for any x. For example, when $x = 0$, we see the graph has zero slope and the derivative gives us the same thing: $f'(0) = 0$.

A. Numerical derivative calculations

Here is the Python code for computing a numerical approximation to the derivative of the function `f` at the point `x`:

```
>>> def differentiate(f, x, delta=1e-9):
        df = f(x+delta) - f(x)
        dx = delta
        return df / dx
```

The function `differentiate` calculates the derivative using a finite step `delta` $= 10^{-9}$ instead of the infinitely small step δ in the math definition of the derivative. This means the value returned by `differentiate` will be an approximation to the true derivative.

Let’s now define a Python function `f` that corresponds to the math function $f(x) = x^2$ and use `differentiate` to find the slope of f when $x = 1$:

```
>>> def f(x):
        return x**2
>>> differentiate(f, 1)
2.000000165480742
```

Using the numerical method, we obtain the approximation $f'(1) = 2.000000165480742$, which is not perfect, but pretty close to the true value $f'(1) = 2$. For most practical applications, this numerical approximation is good enough.

B. Derivative formulas

You don’t need to apply the complicated derivative formula $f'(x) \stackrel{\text{def}}{=} \lim_{\delta \to 0} \frac{f(x+\delta)-f(x)}{\delta}$ every time you need to find the derivative of a function. For each function $f(x)$, it’s enough to use the complicated formula once and record the formula you obtain for $f'(x)$, then you can reuse that formula whenever you need to compute $f'(x)$ in later calculations.

Table I shows the derivatives of several functions. I invite you to mentally bookmark this page so you can come back to it when you need to know the derivatives of some function.

Table I presents the results in terms of the derivative operator $\frac{d}{dx}[\texttt{<f>}]$, which takes as input some function $f(x)$ and produces as output its derivative function $f'(x)$.

C. Derivative rules

In addition to the table of derivative formulas, there are some important derivative rules that allow you to find derivatives of *composite* functions.

TABLE I
DERIVATIVE FORMULAS FOR COMMONLY USED FUNCTIONS

$f(x)$	$-$ derivative $\rightarrow$	$f'(x)$
a	$-\frac{d}{dx}\rightarrow$	0
x	$-\frac{d}{dx}\rightarrow$	1
$mx+b$	$-\frac{d}{dx}\rightarrow$	m
x^n, for $n \neq 0$	$-\frac{d}{dx}\rightarrow$	nx^{n-1}
$\frac{1}{x} = x^{-1}$	$-\frac{d}{dx}\rightarrow$	$\frac{-1}{x^2} = -x^{-2}$
$\sqrt{x} = x^{\frac{1}{2}}$	$-\frac{d}{dx}\rightarrow$	$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$
e^x	$-\frac{d}{dx}\rightarrow$	e^x
$\ln(x)$	$-\frac{d}{dx}\rightarrow$	$\frac{1}{x}$
$\sin(x)$	$-\frac{d}{dx}\rightarrow$	$\cos(x)$
$\cos(x)$	$-\frac{d}{dx}\rightarrow$	$-\sin(x)$

Constant multiple rule: The derivative of k times the function $f(x)$ is equal to k times the derivative of $f(x)$:

$$\left[kf(x)\right]' = kf'(x).$$

Sum rule: The derivative of the sum of two functions is the sum of their derivatives:

$$\left[f(x)+g(x)\right]' = f'(x) + g'(x).$$

Product rule: The derivative of a product of two functions is the sum of two contributions:

$$\left[f(x)g(x)\right]' = f'(x)g(x) + f(x)g'(x).$$

In each term, the derivative of one of the functions is multiplied by the value of the other function.

Quotient rule: This formula tells us how to obtain the derivative of a fraction of two functions:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Chain rule: If you encounter a situation that includes an inner function and an outer function, like $f(g(x))$, you can obtain the derivative by a two-step process:

$$\left[f(g(x))\right]' = f'(g(x))g'(x).$$

In the first step, we leave the inner function $g(x)$ alone and focus on taking the derivative of the outer function $f(x)$. This step gives us $f'(g(x))$, which is the value of f' evaluated at $g(x)$. In the second step, we multiply this expression by the derivative of the *inner* function $g'(x)$.

D. Higher derivatives

The second derivative of $f(x)$ is denoted $f''(x)$ or $\frac{d^2f}{dx^2}$. It is obtained by applying the derivative operation to $f(x)$ *twice*: $\frac{d}{dx}\left[\frac{d}{dx}[\texttt{<f>}]\right]$. Geometrically, the second derivative $f''(x)$ tells us the *curvature* of $f(x)$. Positive curvature means the function opens upward and looks like the bottom of a valley.

The function $f(x) = x^2$ shown in Figure 13 has derivative $f'(x) = 2x$ and second derivative $f''(x) = 2$, which means it has positive curvature. Negative curvature means the function opens downward and looks like a mountain peak. For example, the function $g(x) = -x^2$ has negative curvature.

E. Examples

Armed with the derivative formulas from Table I and the derivative rules from the previous section, you can find the derivative of any function, no matter how complicated. Let's look at some examples.

Example 6: To calculate the derivative of $f(x) = e^{x^2}$, we use the chain rule: $f'(x) = e^{x^2}[x^2]' = e^{x^2}2x$.

Example 7: To find the derivative of $f(x) = \sin(x)e^{x^2}$, we use the product rule and the chain rule: $f'(x) = \cos(x)e^{x^2} + \sin(x)2xe^{x^2}$.

Example 8: The derivative of $\sin(x^2)$ requires using the chain rule: $\left[\sin(x^2)\right]' = \cos(x^2)\left[x^2\right]' = \cos(x^2)2x$.

F. Computing derivatives using SymPy

The SymPy function `sp.diff` computes the derivative of any expression. For example, here is how to compute the derivative of the function $f(x) = mx + b$:

```
>>> m, x, b = sp.symbols("m x b")
>>> sp.diff(m*x + b, x)
m
```

Let's also verify the derivative formula $\frac{d}{dx}[x^n] = nx^{n-1}$:

```
>>> x, n = sp.symbols("x n")
>>> sp.diff(x**n, x)
n * x**(n - 1)
```

The exponential function $f(x) = e^x$ is special because it is the only function that is equal to its derivative:

```
>>> from sympy import exp
>>> sp.diff(exp(x), x)
exp(x)
```

Here is an example of the derivative of function that includes exponential, trigonometric, and logarithmic terms:

```
>>> from sympy import exp, sin, log
>>> sp.diff(exp(x) + sin(x) + log(x), x)
exp(x) + cos(x) + 1/x
```

Let's check the derivative calculations from the examples:

```
>>> sp.diff(sp.exp(x**2), x)
2*x*exp(x**2)
>>> sp.diff(sp.sin(x)*sp.exp(x**2), x)
2*x*exp(x**2)*sin(x) + exp(x**2)*cos(x)
>>> sp.diff(sp.sin(x**2), x)
2*x*cos(x**2)
```

As you can see, the function `sp.diff` gives the same answers.

G. Applications of derivatives

Derivatives are used in physics, chemistry, computing, biology, business, and many other areas of science. We need derivatives whenever we compute rates of change of quantities.

Tangent lines: The *tangent line* to the function $f(x)$ at $x = x_0$ is the line that passes through the point $(x_0, f(x_0))$ and has the same slope as the function at that point. The tangent line to the function $f(x)$ at the point $x = x_0$ is described by the equation

$$T_1(x) = f(x_0) \; + \; f'(x_0)(x - x_0).$$

For example, the tangent line to $f(x) = x^2$ at $x_0 = 1$ is $T_1(x) = f(1) + f'(1)(x-1) = 1 + 2(x-1) = 2x - 1$. Look at the right side of Figure 13 for an illustration of this tangent line.

The tangent line T_1 is also called a *first-order approximation* to the function f, since it has the same value and the same derivative as the function f, $T_1(1) = f(1)$ and $T_1'(1) = f'(1)$. In Section VI-G, we'll learn how to build a fancier approximation $T_n(x)$ that matches the second, third, and higher derivatives of $f(x)$.

H. Solving optimization problems using derivatives

We're often interested in finding the values x where some function $f(x)$ reaches its *minimum* value. Knowing the derivatives of the function $f(x)$ is very useful for solving optimization problems. For example, let's look the graph of the function $f(x) = x^2$ shown in Figure 13. The minimum of this function occurs when $x = 0$. We can make the following observations about the graph of the function at the minimum value:

(A) The slope of the function is negative on the left of the minimum, and positive on the right of the minimum.
(B) The slope of $f(x)$ is zero at the minimum: $f'(0) = 0$.
(C) The graph of the function looks locally like the bottom of a valley at $x = 0$. This means the second derivative of $f(x)$ is positive at that point $f''(0) > 0$.

We can use these observations to come up with general strategies for finding the minimum of any function. We'll describe two different strategies below: the first one based on math formulas, the second one based on numerical computations.

Analytical optimization: The values of x where the derivative is zero are called the *critical points* of the function and denoted x_1^*, x_2^*, etc. Observation (B) tells us that optimum values (maximum or minimum) occur at the critical points of the function. Observation (C) tells us that we can identify a critical point x_j^*, that corresponds to a minimum if the second derivative is positive at that point $f''(x_j^*) > 0$ (positive curvature). In contrast, a critical point x_k^* where $f''(x_k^*) < 0$ (negative curvature) is a maximum. These observations lead us to the following analytical procedure for finding minima and maxima of the function $f(x)$:

(1) Solve $f'(x) = 0$ to find the critical points $[x_1^*, x_2^*, x_3^*, \ldots]$.
(2) For each critical point x_i^*, check to see if it is a maximum or a minimum by evaluating $f''(x_i^*)$:
    - If $f''(x_i^*) < 0$ then x_i^* is a max (mountain top)
    - If $f''(x_i^*) > 0$ then x_i^* is a min (bottom of a valley).

We can also perform the check in step (2) visually by looking at the graph of the function, or by evaluating the slope of the function on the left and the right of the critical point. If $f'(x^* - 0.01)$ is negative and $f'(x^* + 0.01)$ is positive, the point

x^* is a minimum (like near $x* = 0$ in Figure 13). If $f'(x^* - 0.01)$ is positive and $f'(x^* + 0.01)$ is negative, then the point x^* is a maximum. If $f'(x^* - 0.01)$ and $f'(x^* + 0.01)$ have the same sign, the value x^* is a *stationary inflection point* that is neither a minimum nor a maximum.

Example 9: Let's apply the analytical optimization procedure to find the minimum value of the function $q(x) = (x - 5)^2$. The derivative of the function is $q'(x) = 2(x - 5)$. Next, we find the critical point(s) by solving the equation $q'(x) = 0$, which has a single solution $x_1^* = 5$. Is the critical value $x_1^* = 5$ a minimum or a maximum? To find out, we compute the second derivative $q''(x) = 2$, and check its sign at the critical value: $q''(5) = 2 > 0$. The second derivative is positive (bottom of a valley), so this means $x_1^* = 5$ is a minimum.

Example 10: What are the minimum and maximum values of the function $r(x) = x^3 - 2x^2 + x$. The derivative function is $r'(x) = 3x^2 - 4x + 1 = 3(x - 1)(x - \frac{1}{3})$. We find the critical points by solving the equation $r'(x) = 0$, which leads us to two critical points $x_1^* = \frac{1}{3}$ and $x_2^* = 1$. The second derivative of the function is $r''(x) = 6x - 4$. For the critical value $x_1^* = \frac{1}{3}$, we find $r''(\frac{1}{3}) = -2 < 0$, which tells us $x_1^* = \frac{1}{3}$ is a maximum. For $x_2^* = 1$, we find $r''(1) = 2$, so $x_2^* = 1$ is a minimum.

Numerical optimization: Observation (A) suggests another way to find the minimum of a function: if we repeatedly take steps in the "downhill" direction, we'll end up at the bottom of a valley. This is the idea behind the *gradient descent algorithm*, which allows us to find the minimum of any function. We start at some point $x = x_0$ and repeatedly take steps in the direction where the function is decreasing.

```
>>> def gradient_descent(f,x0=0,alpha=0.05,tol=1e-10):
        current_x = x0
        change = 1
        while change > tol:
            df_at_x = differentiate(f, current_x)
            next_x = current_x - alpha * df_at_x
            change = abs(next_x - current_x)
            current_x = next_x
        return current_x
```

The `gradient_descent` procedure takes two arguments as inputs: the function we want to minimize `f`, and an initial value x_0 where to start the minimization process. The procedure then visits the points x_1, x_2, x_3, etc., by repeatedly taking steps in the direction opposing the derivative at the current x. The formula $x_{i+1} = x_i - \alpha f'(x_i)$ is used to find the next point, where the step size is determined by the parameter α and the slope of the function.

Here is how to use `gradient_descent` to find the minimum of the functions $q(x) = (x - 5)^2$ and $r(x) = x^3 - 2x^2 + x$, using the value `x0` $= 10$ as the starting point of the gradient descent.

```
>>> def q(x):
        return (x - 5)**2
>>> gradient_descent(q, x0=10)
5.000000000396651
>>> def r(x):
        return x**3 - 2*x**2 + x
>>> gradient_descent(r, x0=10)
1.0000000932587236
```

The `while` loop in the `gradient_descent` procedure ran many times, and in each iteration took a small downhill step until we got to the minimum (the bottom of the valley). The optimization procedure returned the values $x = 5.000000000396651$ and $x = 1.0000000932587236$, which are close to true minimum values of the functions $q(x)$ and $r(x)$.

Numerical optimization using SciPy: The Python module SciPy provides a high-performance numerical optimization procedure called `minimize` that runs much faster than the `gradient_descent` procedure that we defined above. Here is a demonstration that shows how we use the function `minimize` to find the minima of the functions $q(x)$ and $r(x)$.

```
>>> from scipy.optimize import minimize
>>> minimize(q, x0=10)["x"][0]
4.9999999737
>>> minimize(r, x0=10)["x"][0]
1.0000004142283734
```

Once more, we obtain approximate values that are very close to the true minimum values of the functions $q(x)$ and $r(x)$.

V. Integrals

Integration is the process of computing the "total" of some function $f(x)$ accumulated over a range of inputs. The symbol $\int$, which we use to denote integrals, is an elongated letter S, for the Latin *summa*. This should give you a hint that integration is some kind of summation.

A. Act 1: Integrals as area calculations

Figure 14 shows a shaded region enclosed between the graph of $f(x)$ from above, the x-axis from below, and vertical lines at $x = a$ and $x = b$. The calculation of the *area* of this region is described by the following integral calculation:

$$A_f(a, b) = \int_{x=a}^{x=b} f(x)\, dx.$$

The numbers a and b are called the *limits of integration*. We refer to this type of integral as a *definite integral* since both limits of integration are defined.

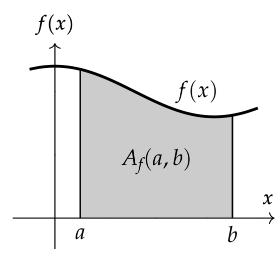


Figure 14. The integral of the function $f(x)$ between $x = a$ and $x = b$ corresponds to the area of the shaded region $A_f(a, b) = \int_a^b f(x)\, dx$.

We often use the simplified notation $\int_a^b f(x)\, dx$ as shorthand for $\int_{x=a}^{x=b} f(x)\, dx$ and read this expression as "the integral of $f(x)$ between a and b." If this is the first time you're seeing the notation for integrals, it might seem very intimidating and complicated, but don't freak out and bear with me for two more pages. You'll see this fancy-looking math notation is nothing to worry about! It's just the calculus way to denote a particular calculation that involves the function $f(x)$. You can think of $\int_a^b$ <f> dx as a "template" that you fill in by replacing <f> with the function $f(x)$ you're interested in, whenever you need to compute the area $A_f(a, b)$.

B. Properties of integrals

We'll now state some properties of integrals that follow from their interpretation as area calculations.

- The sum of the integral from a to b and the integral from b to c is equal to the integral starting from a going all the way to c: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
- The integral of k times the function $f(x)$ is equal to k times the integral of $f(x)$: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.
- The integral of the sum of two functions is the sum of their integrals: $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
- An integral over a region with zero width has zero value: $\int_a^a f(x)\,dx = 0$. Geometrically, this integral defines a region with height $f(x)$ and width 0, so it has zero area.

Let's look at some examples.

Example 11. Integral of a constant function: Consider the constant function $f(x) = 3$. We can easily find the area under the graph of this function because the region has a rectangular shape. The area under $f(x)$ between $x = 0$ and $x = 5$ is described by the following integral calculation:

$$A_f(0,5) = \int_0^5 f(x)\,dx = 3 \cdot 5 = 15.$$

The area under the graph of $f(x)$ is a rectangle with height 3 and width 5, so its area is $3 \cdot 5 = 15$, as shown in Figure 15.

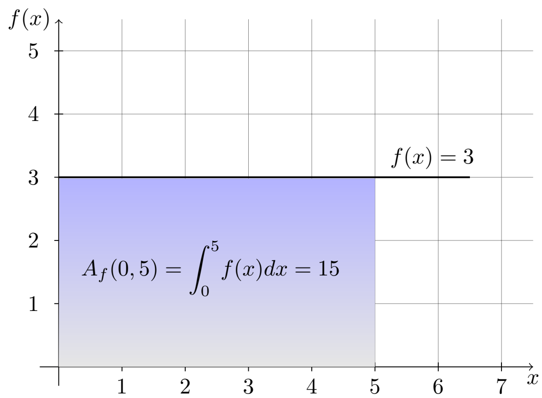


Figure 15. The area of a rectangle of height 3 and width 5 equals 15.

Example 12. Integral of a linear function: Consider now the area under the graph of the line $g(x) = x$ between $x = 0$ and $x = 5$, as shown in Figure 16. This area is described by the following integral calculation:

$$A_g(0,5) = \int_0^5 g(x)\,dx = \tfrac{1}{2}5 \cdot 5 = \tfrac{1}{2}5^2 = \tfrac{25}{2} = 12.5.$$

The region under the graph of $g(x)$ has a triangular shape, so we can compute its area using the formula for the area of a triangle: base times height divided by 2.

I hope these two examples are starting to convince you that the scary-looking integral notation is not that complicated after all. It's just a fancy way to describe the "area under the graph of the function" calculation.

Example 13. Integral of a polynomial: Consider now the function $h(x) = 4 - x^2$. We want to know the area under the graph of

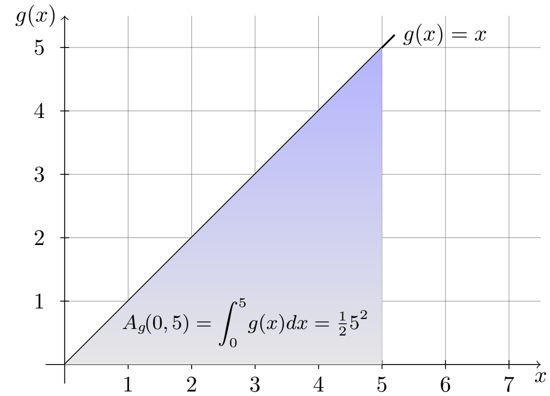


Figure 16. The area of a triangle with base 5 and height 5 is $\frac{1}{2}5^2 = \frac{25}{2} = 12.5$.

$h(x)$ between $x = 0$ and $x = 2$, as illustrated in Figure 17. We need to calculate the following integral:

$$A_h(0,2) = \int_0^2 h(x)\,dx \; = ???.$$

The region under the graph of $h(x)$ is curved and not a simple recognizable geometric shape with a known area formula. How could we compute the area in this case?

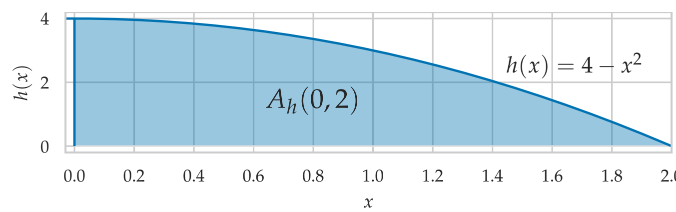


Figure 17. The area under the graph of $h(x)$ between $x = 0$ and $x = 2$.

One way to approximate the area under the graph of $h(x)$ is to compute the *Riemann sum* approximation, which splits the region into a bunch of vertical rectangular strips of fixed width. The height of each rectangular strip varies depending on $h(x)$. Look ahead to figures 18 and 19 to see where we're going with this. Splitting up the area $A_h(0,2)$ into $n = 10$ strips, calculating the area of the individual strips, and summing them together produces the approximation $A_h(0,2) \approx 4.92$. If we split the area $A_h(0,2)$ into $n = 20$ strips, we obtain the more accurate approximation $A_h(0,2) \approx 5.13$. The approximation with $n = 1000$ rectangular strips gives us $A_h(0,2) \approx 5.329$, and using $n = 1\,000\,000$ rectangles, we get $A_h(0,2) \approx 5.333329$. The more finely we chop up the region into rectangular strips, the closer we get to the *exact* value of the integral, which is $\int_0^2 h(x)\,dx = 5\frac{1}{3} = 5.\overline{3} = 5.3333333333333333\ldots$.

In the next section, we'll learn more about the split-area-into-rectangles calculation (a.k.a. *integration*). Don't worry, I won't make you calculate sums with $n = 10$ or $n = 20$ terms by hand, let alone the sum with $n = 1\,000\,000$ terms! Instead, we'll write a computer program that performs the integration procedure for us. Modern computers are really good at this stuff. Indeed early computers were often called "numerical integrators" since they were built primarily to evaluate integrals.

C. Computing integrals numerically

Computing the integral $\int_a^b f(x)dx$ *numerically* means using a computer to compute the Riemann sum approximation to $A_f(a,b)$ by splitting the region into many (think millions) of strips, computing the areas of each strip, then adding up the areas to get the total area under the graph of $f(x)$. The key step is to come up with a general mathematical expression that describes the approximate area calculation with n rectangular strips, then evaluate this expression for very large values of n.

Let's start by looking at the math required to calculate the approximation to $\int_0^2 h(x)dx$ using $n = 10$ rectangles, which is illustrated in Figure 18 (a). The width of each rectangle is $\Delta x = \frac{b-a}{n} = \frac{2-0}{10} = 0.2$. The x-coordinates of the right endpoints of the 10 rectangles are $[0.2, 0.4, 0.6, \ldots, 1.8, 2.0]$. To find the area of the rectangles, we need to know the height of the function h at these x-coordinates: $[h(0.2), h(0.4), h(0.6), \ldots, h(1.8), h(2.0)]$. The area of each rectangle is given by the height-times-width formula, and we sum together all of them to compute the approximation to the total area:

$$A_h(0,2) \approx h(0.2)\cdot 0.2 + h(0.4)\cdot 0.2 + \cdots + h(2.0)\cdot 0.2 = 4.92.$$

Looking at Figure 18 (a), we can clearly see that the area of the rectangles underestimates the true area under the graph, but let's keep going. We have to trust the process: the quality of the approximation will improve when we split the region into thinner and thinner strips.

The procedure we used for $n = 10$ works more generally for any n. In the general case, the rectangles have width $\Delta x = \frac{b-a}{n} = \frac{2}{n}$, which gets smaller and smaller as n grows. The x-coordinates of the right endpoints of the n rectangles are located at $[\Delta x, 2\Delta x, 3\Delta x, \ldots, (n-1)\Delta x, n\Delta x]$. The heights of the rectangles are $[h(\Delta x), h(2\Delta x), h(3\Delta x), \ldots, h((n-1)\Delta x), h(n\Delta x)]$. To find the area under the graph of $h(x)$, we sum together the individual height-times-width contributions of the n rectangular strips:

$$A_h(0,2) \approx h(\Delta x)\Delta x + h(2\Delta x)\Delta x + h(3\Delta x)\Delta x + \cdots + h(n\Delta x)\Delta x.$$

Observe that all the terms in this summation follow the same pattern: the k^{th} term in this summation is $h(k\Delta x)\Delta x$, and k goes from 1 to n. Mathematicians use the symbol $\sum$ (the capital Greek letter *sigma*) to describe long summations. The approximation to the area under $h(x)$ between $x = a$ and $x = b$ using n rectangular strips corresponds to the following sum: $A_h(0,2) \approx \sum_{k=1}^{n} h(k\Delta x)\,\Delta x$. The labels above and below the summation symbol $\sum$ play the same role as the superscript and subscript in integral notation: the label $k = 1$ tells us where to start the summation, and label $k = n$ tells us where to stop the summation.

We can take what we learned from the particular example above to write a general formula for approximating the area under the graph of any function $f(x)$ between $x = a$ and $x = b$ using n rectangular strips:

$$A_f(a,b) \approx \sum_{k=1}^{n} f(a + k\Delta x)\,\Delta x, \quad \text{where} \quad \Delta x = \tfrac{b-a}{n}.$$

This is known as the *Riemann sum* formula for computing areas. We'll now turn this math formula into a Python procedure that performs the n-rectangle area approximation calculation.

```
>>> def integrate(f, a, b, n):
        dx = (b - a) / n
        xs = [a + k*dx for k in range(1,n+1)]
        fxs = [f(x) for x in xs]
        area = sum([fx*dx for fx in fxs])
        return area
```

The code implements the operations described by the summation $A_f(a,b) \approx \sum_{k=1}^{n} f(a + k\Delta x)\,\Delta x$. We first compute the width of the rectangles `dx` $= \Delta x = \frac{b-a}{n}$, and create the list `xs` that contains the x-coordinates of the right endpoints of the rectangles, `xs` $= [a + \Delta x, a + 2\Delta x, a + 3\Delta x, \ldots, a + n\Delta x]$. We then evaluate the function `f` at these x-values to obtain `fxs` $= [f(a+\Delta x), f(a+2\Delta x), f(a+3\Delta x), \ldots, f(a+n\Delta x)]$. We calculate the areas of the rectangles by multiplying the heights `fxs` by the width `dx`, and sum everything together to obtain the total `area`, which we `return` as the output of the procedure.

Example 13 continued: Let's use the `integrate` procedure to compute the integral of the function $h(x) = 4 - x^2$. Recall we previously defined the Python function `h` that implements the same operation as the math function h:

```
>>> def h(x):
        return 4 - x**2
```

To calculate the $n = 10$ approximation to the area under the graph of $h(x)$ between $x = 0$ and $x = 2$, we call the `integrate` procedure with the desired arguments.

```
integrate(h, a=0, b=2, n=10)
4.92
```

We can compute the approximation with $n = 20$ rectangles just as easily:

```
>>> integrate(h, a=0, b=2, n=20)
5.13
```

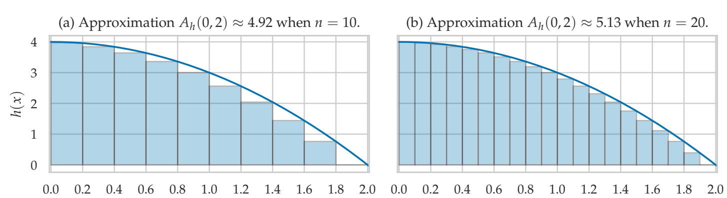


Figure 18. Approximations to the area under the graph of the function $h(x) = 4 - x^2$ computed using $n = 10$ and $n = 20$ rectangles.

Let's keep going to see what happens with $n = 50$ and $n = 100$:

```
>>> integrate(h, a=0, b=2, n=50)
5.2528
>>> integrate(h, a=0, b=2, n=100)
5.2932
```

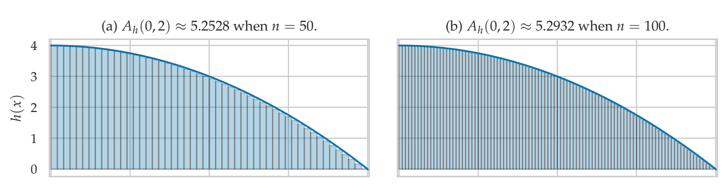


Figure 19. Approximations to the area under the graph of $h(x) = 4 - x^2$ computed using $n = 50$ and $n = 100$ rectangles.

The approximations get better and better as we increase the number of rectangles n.

```
>>> integrate(h, a=0, b=2, n=1000)
5.329332
>>> integrate(h, a=0, b=2, n=10000)
5.33293332
>>> integrate(h, a=0, b=2, n=1_000_000)
5.333329333332
```

The approximation computed using $n = 1M$ rectangles is accurate to 4 decimals. The exact value of the area $A_h(0,2)$ is $\frac{16}{3} = 5\frac{1}{3} = 5.\overline{3} = 5.33333333333333\ldots$. To obtain the exact value, we have to **split up the region into infinitely many rectangular strips**, as we'll learn next.

D. Formal definition of the integral

In the limit as the number of rectangles n approaches ∞, the approximation to the area under the graph of $f(x)$ becomes *arbitrarily close* to the true area.

The integral between $x = a$ and $x = b$ is *defined* as the limit as n goes to infinity of the Riemann sum:

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

In words, the integral is a Riemann sum that consists of infinitely thin rectangular strips. We previously defined the integral $\int_a^b f(x)\,dx$ geometrically as the area under the graph of $f(x)$, but now you know the formal math definition for the integral that mathematicians use.

Note the structural similarity between the summation formula on the right and the integral notation on the left: in both cases we evaluate f at different x values, multiply by a width, and add all these contributions together to get the total. Perhaps now the weird notation we use for integrals will start to make more sense to you. In the limit as $n \to \infty$, the summation sign $\sum$ becomes an integral sign $\int$, and the step size Δx becomes an infinitely small step dx.

The integral $\int_a^b f(x)\,dx$ is defined as a *procedure* with infinitely many steps ($\lim_{n\to\infty}$) that we perform on the function f. Recall that the formal definition of the derivative is also a procedure, specifically $f'(x) \overset{\text{def}}{=} \lim_{\delta\to 0} \frac{f(x+\delta)\ -\ f(x)}{\delta}$, which corresponds to rise-over-run calculation with an infinitely short run δ. These two procedures are the foundations of calculus. The limits $\lim_{n\to\infty}$ and $\lim_{\delta\to 0}$ are essential tools that allow us to perform these calculus operations on functions.

E. Act 2: Integrals as functions

The *integral function* $F_0(b)$ corresponds to the area calculation with a variable upper limit of integration $A_f(0,b)$:

$$F_0(b) \ \overset{\text{def}}{=}\ A_f(0,b) = \int_{x=0}^{x=b} f(x)\,dx\,.$$

As a matter of convention, we denote the integral function using the capitalized letter used to denote the original function. In the above definition, the starting point of the integral function $x = 0$ is an arbitrary choice. We can obtain another integral function if we use $x = a$ as the starting point, $F_a(b) \overset{\text{def}}{=} \int_a^b f(x)\,dx$. The integral functions F_a and F_0 differ only by a constant term: $F_0(b) = F_a(b) + C$, where $C = \int_0^a f(x)\,dx$.

The integral function $F_0(b)$ contains the "precomputed" information about the area under the graph of $f(x)$. Knowing F_0 allows us to compute the area under $f(x)$ between $x = a$ and $x = b$ as the *change* in the integral function:

$$A_f(a,b) = \int_a^b f(x)\,dx = F_0(b) - F_0(a).$$

Intuitively, this formula computes the area $A_f(a,b)$ as the difference between the areas of two regions: the area until $x = b$ minus the area until $x = a$, as illustrated in Figure 20.

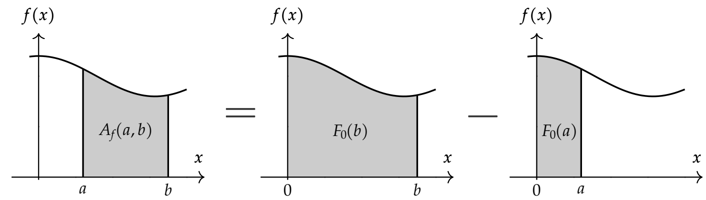


Figure 20. The area under $f(x)$ between $x = a$ and $x = b$ is computed using the formula $A_f(a,b) = F_0(b) - F_0(a)$, which describes the change in the output of $F_0(x)$ between $x = a$ and $x = b$.

Example 11 revisited: We can easily find the integral function for the constant function $f(x) = 3$ because the region under the curve is rectangular. Choosing $x = 0$ as the starting point, we obtain the integral function $F_0(b)$ that corresponds to the area under $f(x)$ between $x = 0$ and $x = b$ as follows:

$$F_0(b) = A_f(0,b) = \int_0^b f(x)\,dx = 3b.$$

The integral function corresponds to the area of a rectangle of height 3 and with width b, as shown in Figure 21.

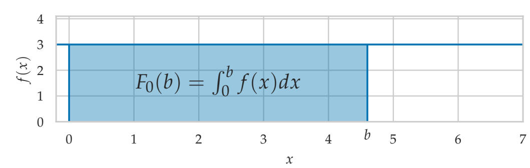


Figure 21. The integral function of the function $f(x) = 3$ is $F_0(b) = 3b$.

Knowing the function $F_0(b)$ allows us to compute the area under the graph of $f(x)$ between $x = 0$ and $x = 5$ as the difference $A_f(0,5) = F_0(5) - F_0(0) = 3 \cdot 5 - 3 \cdot 0 = 15$.

Example 12 revisited: Consider now the area under the graph of the line $g(x) = x$, starting from $x = 0$. Since the region is triangular, we can compute its area using the formula for the area of a triangle: base times height divided by two. The integral function of $g(x)$ is:

$$G_0(b) = A_g(0,b) = \int_0^b g(x)\,dx = \tfrac{1}{2}b^2.$$

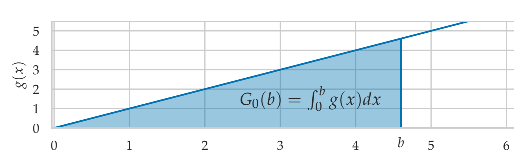


Figure 22. The integral function of the function $g(x) = x$ is $G_0(b) = \frac{1}{2}b^2$.

Knowing the integral function $G_0(b)$ allows us to compute the area under the graph of $g(x)$ between $x = 0$ and $x = 5$ as the difference $A_g(0,5) = G_0(5) - G_0(0) = \frac{1}{2}5^2 - \frac{1}{2}0^2 = 12.5$.

Example 13 revisited: The area under $h(x) = 4 - x^2$ from $x = 0$ until $x = b$ is described by the following integral calculation:

$$H_0(b) = A_h(0,b) = \int_0^b h(x)\,dx = ???.$$

We were able to compute the integral functions $F_0(b)$ and $G_0(b)$ thanks to the simple geometries of the areas under the graphs, but $h(x)$ is a curve so it requires some new integration methods. In the next few pages, we'll learn about symbolic integration techniques that will allow us to find the integral function $H_0(b)$.

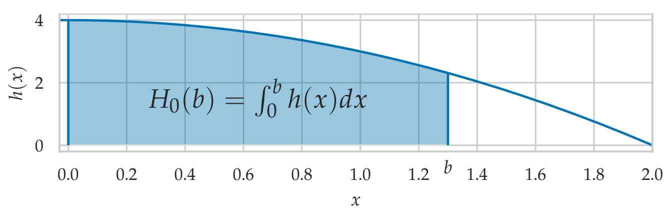


Figure 23. The integral of the function $h(x) = 4 - x^2$ from $x = 0$ to $x = b$.

F. Intermission

The integral in Example 13 was chosen to motivate the need for more advanced methods for integration. Is there a math formula that describes the area $A_h(0,b)$ of the region shown in Figure 23? We previously used numerical methods to compute the particular area $A_h(0,2)$ for $b = 2$, but now we're looking for a general math formula that computes the integral function $H_0(b) = A_h(0,b)$ for any b. In the next section, we'll learn about the *fundamental theorem of calculus*, which will allow us to find the exact formula for $H_0(b)$.

The section is like the intermission in the calculus show. As in a real-world intermission, this is also your chance to skip the rest of the show. Perhaps you have better things to do right now than learning about advanced calculus concepts. I won't get offended—no worries! Feel free to skip ahead to Section V-I for more technical details about high-performance numerical integration, or jump straight to Section VI to learn about sequences and series.

As a teacher, I'm happy that you know that integrals compute areas under the graph of functions, and can be approximated numerically using Riemann sums. These are the two key ideas related to integrals, so I feel I've done my job! I think you should stay though—you might enjoy the *knowledge buzz* moments that are coming your way in the next few pages. My calculus teacher in college described the realization you get after understanding the fundamental theorem of calculus as similar to the feeling you get when smoking some of that "funny stuff."

Still with me? Okay, with your consent, let's continue with the calculus show.

G. Act 3: Fundamental theorem of calculus

Note the pattern in the formulas for the integral functions $F_0(b)$ and $G_0(b)$. The integral function of the constant function $f(x) = 3$ is a linear function $F_0(b) = 3b$. The integral of the linear function $g(x) = x$ is a quadratic function $G_0(b) = \frac{1}{2}b^2$. In both cases, the integral function seems to increase the degree of the function. What is up with that? Is this a coincidence, or is there some fundamental math pattern we could follow to "guess" integral functions?

The fundamental theorem of calculus (FTC) describes the inverse relation that exists between the integration operation $\int \texttt{<f>}\,dx$ and the differentiation operation $\frac{d}{dx}[\texttt{<f>}]$. A priori, there is no reason to suspect that integrals would be related to derivatives. The integral corresponds to the computation of an area, whereas the derivative operation computes the slope of a function. Yet behold:

Theorem (fundamental theorem of calculus): Let $f(x)$ be a continuous function, and let $a \in \mathbb{R}$ be a constant. Define the function $F_a(x)$ as follows:

$$F_a(x) \stackrel{\text{def}}{=} A_f(a,x) = \int_a^x f(u)\,du.$$

Then, the derivative of $F_a(x)$ with respect to x is equal to $f(x)$:

$$\frac{d}{dx}\big[F_a(x)\big] = f(x).$$

Note we use the new variable u inside the integral since x is already used to denote the upper limit of integration.

To understand the inverse relationship between integration and differentiation, we can draw an analogy with the relationship between a function f and its inverse function f^{-1}, which *undoes* the effects of f. See Figure 3 on page 4. Given some initial value x, if we apply the function f to obtain the number $f(x)$, then apply the inverse function f^{-1} on the number $f(x)$, we get back to the initial value x we started from:

$$f^{-1}(f(x)) = x.$$

Similarly, **the derivative operation is the inverse of the integral operation**. If you perform the integral operation $\int \texttt{<f>}\,dx$ followed by the derivative operation $\frac{d}{dx}[\texttt{<f>}]$ on any function `<f>`, you'll get back to the original function:

$$\frac{d}{dx}\int_c^x f(u)\,du = f(x).$$

Let's use SymPy to verify that the fundamental theorem of calculus is true. We'll start with the function `fx` $= f(x) = x^2$, compute its integral function `Fx` using `sp.integrate`, then take the derivative of `Fx` using `sp.diff`.

```
>>> fx = x**2
>>> Fx = sp.integrate(fx, x)
>>> Fx
x**3/3              # + C
>>> sp.diff(Fx, x)
x**2
```

We confirm that that the sequence of operations $\frac{d}{dx}\int_0^x f(u)\,du =$ `sp.diff(sp.integrate(`$f(x)$`))` brings us back to the original $f(x)$ we started from.

For ordinary math functions, we know that if the function f^{-1} is the undo action for the function f, then f is also the undo action for f^{-1}: $f\big(f^{-1}(y)\big) = y$. Similarly, the inverse relationship between integrals and derivatives holds in the other direction too. **The integral operation is the inverse operator of the derivative operation**. If we start with some function $G(x)$, calculate its derivative function $G'(x)$, then compute the integral of the derivative function $G'(x)$, we arrive back at the original function $G(x)$ (up to an additive constant):

$$\int_c^x G'(u)\,du = G(x) + C.$$

Let's use SymPy to verify this formula. We'll start with the function `Gx` $= G(x) = x^3$, compute its derivative `dGdx` $= G'(x)$ using `sp.diff`, then use `sp.integrate` to compute the integral function of $G'(x)$.

```
>>> Gx = x**3
>>> dGdx = sp.diff(Gx, x)
>>> dGdx
3*x**2
>>> sp.integrate(dGdx, x)
x**3     # + C
```

We see the operations `integrate(diff(`$G(x)$`))` $= \int_0^x G'(u)\,du$ bring us back to the original $G(x)$ we started from.

Using antiderivatives to compute integrals: The fundamental theorem of calculus gives us a way to compute integrals and integral functions by "reverse engineering" derivatives. In order to explain this idea, we'll introduce a new concept.

Given some function $f(x)$, any function $F(x)$ that satisfies the equation $F'(x) = f(x)$ is called an *antiderivative* of $f(x)$.

In words, an antiderivative of $f(x)$ is a function whose derivative is $f(x)$. There is no single antiderivative function, since adding any constant C to an antiderivative function still satisfies the definition $\frac{d}{dx}\big[F(x)+C\big] = f(x)$, because the derivative of the constant C is zero.

The fundamental theorem of calculus tells us that antiderivative functions $F(x) + C$ are closely related to integral functions $F_c(x) = \int_c^x f(u)\,du$. The integral function $F_0(x)$ is equal to an antiderivative function $F(x) + C$, for some additive constant $+C$.

This equivalence gives us an analytical shortcut for obtaining the integral function $F_0(b) = \int_0^b f(x)\,dx$ by finding the antiderivative function of $f(x)$. To find an antiderivative of $f(x)$, we look for a function $F(x)$ whose derivative is $f(x)$. We can use the table of derivative formulas (Table I on page 9) in the reverse direction to find antiderivatives. For example, to find the antiderivative of the function $f(x) = m$, we look for a row where this function appears on the right side of the table, and then look at the corresponding function on the left side of the table, which is the function $F(x) = mx + b$ in this case. We can verify that $F'(x) = f(x)$, so indeed $F(x) = mx + b$ is an antiderivative of $f(x) = m$. Furthermore, the equivalence between antiderivatives and integral functions tells us that the integral function of $f(x)$ is $F_a(x) = \int_a^x m\,du = mx + b$, for some constant b.

Let's use the antiderivative reverse engineering procedure to find the integral function $H_0(b)$ in Example 13.

Example 13 continued: We're given the function $h(x) = 4 - x^2$ and we want to find its integral function $H_0(b) = \int_0^b h(x)\,dx$. This fundamental theorem of calculus tells us this problem is equivalent to finding a function $H(x)$ whose derivative is $h(x)$. The function $h(x) = 4 - x^2$ has two terms. The first term is a constant 4. We can guess that the corresponding term in the antiderivative function $H(x)$ will be $4x$, since $\frac{d}{dx}[4x] = 4$. Now for the quadratic term $-x^2$. Remembering the derivative formulas for polynomials, we can guess that antiderivative $H(x)$ must contain a x^3 term, because taking the derivative of a cubic term results in a quadratic term. Therefore, the antiderivative function has the form $H(x) = 4x - kx^3$, for some multiplicative constant k. Pick the constant k that makes this equation true: $H'(x) = 4 - 3kx^2 = 4 - x^2$. Solving $3k = 1$, we find $k = \frac{1}{3}$ and so the antiderivative function we're looking for is $H(x) = 4x - \frac{1}{3}x^3 + C$. The equivalence between antiderivatives and integral functions tells us that the integral function is $H_0(b) = 4b - \frac{1}{3}b^3 + C$ for some constant C. We know from the geometric definition of the integral that when $b = 0$ the integral function must have value zero, so $C = 0$ in this case. The integral function we're looking for is therefore $H_0(b) = 4b - \frac{1}{3}b^3$.

Using derivative formulas in reverse: Computing integral functions by finding antiderivatives is very powerful. We can use it to find the integral functions for all the functions listed in the table of derivative formulas (see page 9). For example, the table tells us that the derivative of the linear function $f(x) = mx + b$ is the constant function $f'(x) = m$. This means the integral of a constant function is a linear function $\int m\,dx = mx + C$. The integral function of an exponential is also an exponential $\int e^x\,dx = e^x + C$, since $\frac{d}{dx}[e^x] = e^x$. The derivative of $\log_e(x)$ is $\frac{1}{x}$, therefore the integral of $\frac{1}{x}$ is $\log(x)$. Similarly for the trigonometric functions $\int \cos(x)\,dx = \sin(x)$ and $\int -\sin(x)\,dx = \cos(x)$. For economy of space, we'll verify all these integral formulas by computing the integral of the function $f(x) = m + e^x + \frac{1}{x} + \cos(x) - \sin(x)$ that contains the mix of several functions on the right side of Table I.

```
>>> fx = m + sp.exp(x) + 1/x + sp.cos(x) - sp.sin(x)
>>> sp.integrate(fx, x)
m*x + exp(x) + log(x) + sin(x) + cos(x)
```

SymPy tells us the integral function F_0 is $F_0(x) = mx + e^x + \log(x) + \sin(x) + \cos(x)$, which are all the corresponding terms on the left side of the table of derivative formulas.

Okay, but what do we do if the function we want to integrate doesn't appear in Table I?

H. Act 4: Techniques of integration

Okay we're getting into the fourth act of the calculus show, and I want to remind you that you can "tap out" at any time. The material in this act is some of the most boring stuff. If you're taking an integral calculus class, then you need to know this stuff because it is going to be your final exam. Everyone else, feel free to skip ahead to the next section.

There are a bunch of tricks that extend the reach of analytical integration methods (anti-differentiation) to more complicated functions. We don't have space to discuss all these tricks in this tutorial, but we'll show the two most important tricks.

Substitution trick: Suppose the function we want to integrate has the structure $f(u(x))u'(x)$, which consists of an inner function wrapped in an outer function multiplied by the derivative of the inner function. We can use the *substitution trick* to rewrite this integral in terms of the function $f(u)$ using u as the variable of integration:

$$\int_{x=a}^{x=b} f(u(x))\,u'(x)\,dx \quad = \quad \int_{u=u_a}^{u=u_b} f(u)\,du.$$

The substitution trick is sometimes called *change of variable*, since we're replacing the variable x with the variable u, just like the search-and-replace operation in a text editor. Because we're doing the substitution "inside" an integral operation, we must change the limits integration (from a and b to u_a and u_b), and also change the "step" parameter (from dx to du).

Follow these three steps to apply the substitution trick:

1) Replace all occurrences of $u(x)$ with u.
2) Compute $u'(x)$ and replace dx with $\frac{1}{u'(x)}du$.
3) Replace the x-limits of integration $x = a$ and $x = b$ with u-limits of integration: $u_a = u(a)$ and $u_b = u(b)$.

Example 14: Let's compute the integral $\int_a^b \frac{1}{x-\sqrt{x}}\,dx$. This looks like a scary formula, but we can use the substitution trick to compute this integral. We'll apply the substitution $u = \sqrt{x}$, which implies $u'(x) = \frac{1}{2\sqrt{x}}$, and $dx = 2\sqrt{x}\,du = 2u\,du$. The new limits of integration are $u_a = \sqrt{a}$ and $u_b = \sqrt{b}$.

Performing the three steps of the substitution trick gives us:

$$\int_{x=a}^{x=b} \frac{1}{x-\sqrt{x}}\,dx = \int_{u=u(a)}^{u=u(b)} \frac{1}{u^2-u} 2u\,du = \int_{u=\sqrt{a}}^{u=\sqrt{b}} \frac{1}{u^2-u} 2u\,du.$$

We're simply doing the search-and-replace on $u = \sqrt{x}$, but to do this right, we need to also replace dx with the equivalent expression involving du, and use the new limits of integration.

We can now simplify the expression inside the integral:

$$\int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{u^2-u}\,2u\,du = \int_{\sqrt{a}}^{\sqrt{b}} \frac{2}{u-1}\,du = 2\int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{u-1}\,du.$$

The function inside the integral, $f(u) = \frac{1}{u-1}$, is similar to the inverse function $f(u) = \frac{1}{u}$ whose antiderivative is $\ln(u)$. Replacing u by $u-1$ gives us the following antiderivative formula $\int \frac{1}{u-1}\,du = \ln(u-1)$, which leads us to the answer:

$$2\int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{u-1}\,du \;=\; 2\ln(\sqrt{b}-1) - 2\ln(\sqrt{a}-1).$$

I know this sequence of steps went quickly, and there are a lot of integral symbols, but if you read each step carefully, you'll see we're just doing search-and-replace.

The substitution trick for integrals comes from the chain rule for derivatives $[F(u(x))]' = F'(u(x))u'(x)$. We can use substitution only when computing the integral of a function that has the special structure $f(u(x))u'(x)$.

Integration by parts: The integration by parts trick can be used when the function we're integrating is the product of two factors, $\int f(x)g'(x)\,dx$, where $f(x)$ is some arbitrary function, and $g'(x)$ is the derivative of some other function.

$$\int_a^b f(x)g'(x)\,dx \;=\; \big[f(b)g(b) - f(a)g(a)\big] \;-\; \int_a^b f'(x)g(x)\,dx.$$

Example 15: Let's calculate $\int_0^5 xe^x\,dx$ using the integration by parts procedure. The expression xe^x consists of two factors: x and e^x. We'll identify x with $f(x)$ and e^x as $g'(x)$. This means $f'(x) = 1$ and $g(x) = \int g'(x)\,dx = e^x$. We now know all the parts we need to apply the integration by parts formula:

$$\begin{aligned}
\int_0^5 \underbrace{x}_{f(x)}\underbrace{e^x}_{g'(x)}\,dx &= \big[f(5)g(5) - f(0)g(0)\big] - \int_0^5 f'(x)g(x)\,dx \\
&= \big[5e^5 \qquad - 0e^0\big] \qquad - \int_0^5 1\cdot e^x\,dx \\
&= \big[5e^5 - 0e^0\big] - \big[e^5 - e^0\big] \\
&= 5e^5 - e^5 + 1 = 4e^5 + 1.
\end{aligned}$$

Other tricks: Substitution and integration by parts are only two of the multitude of integration techniques. There are tricks for trigonometric functions, square roots, fractions that involve $x^2 + a^2$, etc. There is an entire course, called integral calculus, that is dedicated to learning integration tricks. If you want to pursue advanced studies in physics or engineering, you should definitely take this course to learn more integration tricks. Section 5.15 in **No Bullshit Guide to Math and Physics** is a good introduction to the material.

Despite all the formulas and integration techniques, not all functions are *integrable*. There are many functions that don't have an antiderivative function, and hence no "closed form" integral function. For example, the function $f(x) = e^{-x^2}$ doesn't have an antiderivative: there is no function $F(x)$ such that $F'(x) = e^{-x^2}$. For such functions, we can't use the "reverse engineering" analytical shortcut to find the integral, and we must use numerical integration (the split-the-area-into-thin-rectangular-strips procedure). Speaking of which...

I. *Computing integrals numerically using SciPy*

The Python function `integrate` that we showed in Section V-C is a useful teaching tool, but it would be much too slow to use for practical scientific computing tasks. The function `quad` from the `scipy.integrate` module is a much more powerful tool for computing numerical integrals. The name `quad` is short for "quadrature," which is a historical name for integrals.

Let's revisit the examples 11, 12, and 13 using the function `quad`. To compute the integral $\int_0^5 f(x)dx$, we call the function `quad` with inputs `f` as the first argument, and the limits of integration $a = 0$ and $b = 5$ as the second and third arguments.

```
>>> from scipy.integrate import quad
>>> quad(f, 0, 5)
(15.0, 1.1102230246251565e-13)
```

The function `quad` returns two numbers as output: $\big(A_f(0,5), \epsilon\big)$. The first number is the value of the area we're interested in. The second number ϵ tells us the accuracy of the procedure used to calculate the area. In the above calculation, the output tells us the definite integral $\int_0^5 f(x)dx$ is equal to `15.0` up to a precision on the order of 10^{-13}. Since we're usually only interested in the area $A_f(0,5)$ and not the precision ϵ, we often select the first element of `quad`'s output.

```
>>> quad(f, 0, 5)[0]
15.0
```

We can similarly use quad to calculate the integrals $\int_0^5 g(x)\,dx$ and $\int_0^2 h(x)\,dx$ from the other two examples.

```
>>> quad(g, 0, 5)[0]
12.5
>>> quad(h, 0, 2)[0]
5.3333333333333333
```

The answers we obtain match the results we obtained earlier. The main takeaway message is that the quad function is your friend whenever you need to compute integrals. All the scary-looking math equations that contain the $\int$ symbol can be approximated numerically using one or two lines of Python code. Specifically, whenever you see $\int_a^b$ `<f>` dx in a math formula, you can replace that with `quad(f,a,b)[0]`.

J. Computing integral functions using SymPy

We can use Python to do *symbolic* integration using variables (symbols) instead of numbers. The SymPy function `sp.integrate` allows us to obtain the formulas for integrals and integral functions. We'll now revisit the integral calculations from the three examples using symbolic math. We start by defining three symbols `x`, `a`, and `b`.

```
>>> import sympy as sp
>>> x, a, b = sp.symbols("x a b")
```

We'll use these symbols to express the functions and the limits of integration.

Example 11S: Constant function: Consider the constant function $f(x) = 3$, which we can define as follows:

```
>>> fx = 3
>>> fx
3
```

To compute the integral $\int_a^b f(x)dx$, we call the SymPy function `sp.integrate`, passing in the function as the first argument, and the triple `(x,a,b)` as the second argument, which specifies the variable of integration and the limits of integration a and b.

```
>>> sp.integrate(fx, (x,a,b))   # = A_f(a,b)
3*(b-a)
```

Since a and b are arbitrary constants, the answer we obtain for $A_f(a,b) = \int_a^b f(x)dx$ is a general formula that works for all possible limits of integration a and b. Geometrically, we recognize the result `3*(b-a)` as the height-times-width formula for the area of a rectangle, which we have seen several times already.

To compute the definite integral $\int_0^5 f(x)\,dx$, we specify the numerical limits of integration instead of the symbols `a` and `b`.

```
>>> sp.integrate(fx, (x,0,5))
15
```

This result matches the value we obtained using geometrical calculation in Figure 15, and the approximation we obtained using numerical integration `quad(f,0,5)[0]`.

We can also compute the integral function $F_0(b)$, which is defined as $F_0(b) \stackrel{\text{def}}{=} \int_0^b f(x)dx$, for the function `fx` $= f(x) = 3$.

```
>>> F0b = sp.integrate(fx, (x,0,b))
>>> F0b
3*b
```

Recall that the integral function F_0 is the area-under-the-graph calculation with a variable upper limit of integration b. See Figure 21 for an illustration of the integral function $F_0(b)$.

Given $F_0(b)$, we can compute the definite integral between $a = 0$ and $b = 5$ using the formula $\int_0^5 f(x)dx = F_0(5) - F_0(0)$. We'll use the method `subs` (short for substitute) on the expression `F0b` to "plug in" the values $b = 5$ and $b = 0$.

```
>>> F0b.subs({b:5}) - F0b.subs({b:0})
15
```

Example 12S: Linear function: Let's now compute the integral function of the linear function $g(x) = x$, which corresponds to the following SymPy expression:

```
>>> gx = 1*x
>>> gx
x
```

To compute the integral function $G_0(b) \stackrel{\text{def}}{=} \int_0^b g(x)dx$, we call `sp.integrate` using the symbol `b` for the upper limit of integration:

```
>>> G0b = sp.integrate(gx, (x,0,b))
>>> G0b
b**2 / 2
```

The expression $G_0(b) = \frac{1}{2}b^2$ we obtain is identical to the formula we obtained from the geometric calculation in Figure 22.

Given $G_0(b)$ = `G0b`, we can compute the definite integral $\int_0^5 g(x)\,dx$ using the formula $\int_0^5 g(x)\,dx = G_0(5) - G_0(0)$. We plug in $b = 5$ and $b = 0$ using the `subs` method:

```
>>> G0b.subs({b:5}) - G0b.subs({b:0})
25/2
```

SymPy computed the exact answer for us as a fraction $\frac{25}{2}$. This answer matches the value we obtained earlier using numerical integration, `quad(g,0,5)[0]` = `12.5`.

Example 13S: Polynomial function: We start by defining a SymPy expression that corresponds to the function $h(x) = 4 - x^2$.

```
>>> hx = 4 - x**2
>>> hx
4 - x**2
```

We can now call `sp.integrate` to make SymPy compute the integral function $H_0(b) = \int_0^b h(x)\,dx$:

```
>>> H0b = sp.integrate(hx, (x,0,b))
>>> H0b
4*b - b**3/3
```

The integral function $H_0(b) = 4b - \frac{1}{3}b^3$ corresponds to the area calculation under $h(x) = 4 - x^2$ starting at $x = 0$.

K. Applications of integration

Intuitively, we use integrals whenever we want to compute the "total" of some quantity that varies over time or space.

Kinematics: Calculus was originally invented to describe the equations of motion $x(t)$, $v(t)$, and $a(t)$, which correspond to the object's *position*, *velocity*, and *acceleration* at time t. We call these the *kinematics* equations, from the Greek word *kinema* for motion. The velocity function $v(t)$ is the derivative of the position function, and the acceleration $a(t)$ is the derivative of the velocity, which we can summarize as follows:

$$a(t) \xleftarrow{\frac{d}{dt}} v(t) \xleftarrow{\frac{d}{dt}} x(t).$$

The starting point of kinematics is Newton's second law, which tells us that the acceleration of an object of mass m that has a net force F_{net} acting on it is $a = \frac{1}{m}F_{\text{net}}$. Given the knowledge of acceleration over time $a(t)$, we can predict the position of the object $x(t)$ at any time t by "undoing" the derivative operations using integration:

$$\tfrac{1}{m}F_{\text{net}}(t) \;=\; \underbrace{a(t) \;\xrightarrow{v_i+\int dt}\; v(t) \;\xrightarrow{x_i+\int dt}\; x(t)}_{\text{kinematics}}.$$

We integrate $a(t)$ to obtain $v(t)$ and choose the initial velocity v_i as the integration constant so that $v(0) = v_i$. We then use integration a second time to obtain $x(t)$ from $v(t)$, using the initial position $x_i = x(0)$ as the integration constant.

The case of *uniform accelerated motion* (UAM) is of particular interest. Consider an object that experiences a constant acceleration $a(t) = a$. We can use integration to find the velocity of this object at a later time $t = \tau$:

$$v(\tau) = v_i + \int_0^\tau a(t)\,dt = v_i + \int_0^\tau a\,dt = v_i + a\tau.$$

Knowing the velocity as a function of time $v(t)$, we can use integration a second time to find its position at time τ:

$$x(\tau) = x_i + \int_0^\tau v(t)\,dt = x_i + \int_0^\tau (v_i + at)\,dt = x_i + v_i\tau + \tfrac{1}{2}a\tau^2.$$

These two simple calculus steps allow us to obtain the famous kinematics equation $x(t) = x_i + v_i t + \frac{1}{2}at^2$ for descibing the motion of objects undergoing constant acceleration a. Students taking a physics class are normally presented with this equation and it seems to come out of nowhere, but if you understand calculus you'll know where it comes from: the integration operation applied to the acceleration function $a(t) = a$ and the initial conditions $x_i \stackrel{\text{def}}{=} x(0)$ and $v_i \stackrel{\text{def}}{=} v(0)$.

Solving differential equations: Many important laws in science are described by *differential equations* that specify an unknown function $f(t)$ in terms of their derivatives $f'(t)$, $f''(t)$, etc.

Here are some examples of differential equations and their solutions:

- The kinematics equations when the acceleration is constant come from the differential equation $x''(t) = a$. We use integration twice to find the unknown function $x(t) = x_i + v_i t + \frac{1}{2}at^2$. We can verify that $x(t)$ is a solution to the differential equation $x''(t) = a$ by computing the second derivative of $x(t)$.
- In biology, unconstrained bacterial growth is described by the equation $b'(t) = kb(t)$, where $b(t)$ is the number of bacteria at time t. Intuitively, the bacterial growth rate $b'(t)$ is proportional to the number of existing bacteria. The solution to this equation is $b(t) = b_0 e^{kt}$, where b_0 describes the number of bacteria at time $t = 0$.
- Radioactive decay is described by the differential equation $r'(t) = -\lambda r(t)$, where $r(t)$ describes the number of atoms of some radioactive element. The solution is $r(t) = r_0 e^{-\lambda t}$.
- Simple harmonic motion is described by the differential equation $x''(t) + \omega^2 x(t) = 0$, which has solution $x(t) = c_1\sin(\omega t) + c_2\cos(\omega t)$, for some constants c_1 and c_2.

If you take a course on differential equations, you'll learn all kind of tricks and techniques for solving differential equations. Integration plays a key role in all these techniques, since it allows us to "undo" the derivative operation.

Probability calculations: Integration is a key tool for computing probabilities of continuous random variables. A continuous random variable X is described by its *probability density function* f_X, and the probability of the event $\{a \leqslant X \leqslant b\}$ is given by the integral $\Pr(\{a \leqslant X \leqslant b\}) \stackrel{\text{def}}{=} \int_a^b f_X(x)\,dx$. For example, the standard normal random variable Z is described by the probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$. To calculate the probability of the event $\{-1 \leqslant Z \leqslant 1\}$, we must evaluate the integral $\int_{-1}^1 f_Z(z)\,dz$, which is easy to do using SciPy helper function `quad(`f_Z`,a=-1,b=1)[0]` $= 0.68269$.

VI. Sequences and series

A *sequence* a_k is a function that takes natural numbers as inputs and produces real numbers as outputs: $a_k : \mathbb{N} \to \mathbb{R}$. The *series* $\sum a_k$ describes the sum of all the terms in the sequence a_k. Sequences and series are the third pillar of the calculus knowledge that I want you to have because they are powerful computational tools that allow us to describe procedures with an infinite number of steps.

A. Sequences are functions with discrete inputs

We use the notation $f : \mathbb{R} \to \mathbb{R}$ to describe functions that take real numbers $x \in \mathbb{R}$ as inputs and produce real numbers as outputs $f(x) \in \mathbb{R}$. When studying functions that take natural numbers $k \in \mathbb{N}$ as inputs, we use a different notation: $a_k : \mathbb{N} \to \mathbb{R}$, where a_k describes the k^{th} term in the sequence. The sequence's input variable is usually denoted k and corresponds to the *index* within the sequence. Usually k is a natural number $k \in \mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, 4, \ldots\}$ but some sequences are only defined for positive natural numbers $k \in \mathbb{N}_+ \stackrel{\text{def}}{=} \{1, 2, 3, 4, \ldots\}$. Note the change in notation: we use subscript to denote the input variable of a sequence a_k instead of the usual notation for functions $a(k)$.

We can define a sequence by specifying the formula for the k^{th} term in the sequence. For example, the *harmonic sequence* is defined by the formula $h_k \stackrel{\text{def}}{=} \frac{1}{k}$. Another way to define a sequence is by listing the first few values in the sequence: $[h_0, h_1, h_2, h_3, \ldots]$, which correspond to evaluating the formula h_k for $k = 0$, $k = 1$, $k = 2$, $k = 3$, etc. We'll now look at some examples of sequences, specifying both their formulas and showing the first few values of each sequence.

The natural numbers: The simplest possible example of a sequence is the identity function, which returns the index input k as output:

$$n_k \stackrel{\text{def}}{=} k, \text{ for } k \in \mathbb{N} \quad \Leftrightarrow \quad [0, 1, 2, 3, 4, 5, 6, 7, \ldots].$$

This is the fundamental counting sequence that describes the process of taking a "unit step" to the right on the number line, starting at the origin.

Squares of natural numbers: The sequence-equivalent of the quadratic function $f(x) = x^2$ is the sequence of squares of the natural numbers:

$$q_k \stackrel{\text{def}}{=} k^2, \text{ for } k \in \mathbb{N} \quad \Leftrightarrow \quad [0, 1, 4, 9, 16, 25, 36, 49, \ldots].$$

Harmonic sequence: We obtain another useful sequence by computing the fractions $\frac{1}{k}$ for each $k \in \{1, 2, 3, \ldots\} = \mathbb{N}_+$:

$$h_k \stackrel{\text{def}}{=} \tfrac{1}{k}, \text{ for } k \in \mathbb{N}_+ \quad \Leftrightarrow \quad \left[1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \ldots\right].$$

This is called the *harmonic sequence* because it describes the wavelengths of harmonic frequencies produced by musical instruments. When we play the note that corresponds to the frequency f, we also hear notes with frequencies that are integer multiple of the "dominant" frequency: $2f$, $3f$, $4f$, etc., which are called the harmonics. The harmonic sequence describes the wavelengths of the harmonics frequencies. On a string instrument, the harmonic sequence tells you where to place your fingers if you want to play higher harmonics.

The alternating harmonic sequence: Consider now a harmonic sequence with alternating positive and negative terms:

$$a_k \stackrel{\text{def}}{=} \tfrac{(-1)^{k+1}}{k}, \text{ for } k \in \mathbb{N}_+ \quad \Leftrightarrow \quad \left[1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{7}, \ldots\right].$$

The factor $(-1)^{k+1}$ is positive for all odd inputs $k \in \{1, 3, 5, 7, \ldots\}$ since $(-1)^m = +1$ for any even number m. The factor $(-1)^{k+1}$ is negative for all even indices $k \in \{2, 4, 6, 8, \ldots\}$, hence the values in the sequence oscillate between positive and negative.

Inverse factorial sequence: The factorial function is denoted $k!$ and describes the product of the first k positive natural numbers: $k! \stackrel{\text{def}}{=} k \cdot (k-1) \cdots 3 \cdot 2 \cdot 1$, and we define $0! = 1$. We'll see factorials in several formulas in this section. In particular, the following sequence will be of interest:

$$f_k \stackrel{\text{def}}{=} \tfrac{1}{k!}, \text{ for } k \in \mathbb{N} \quad \Leftrightarrow \quad \left[1, 1, \tfrac{1}{2!}, \tfrac{1}{3!}, \tfrac{1}{4!}, \tfrac{1}{5!}, \tfrac{1}{6!}, \tfrac{1}{7!}, \ldots\right].$$

The values in the inverse factorial sequence quickly become very small because the factorial function grows very quickly: $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, ..., $10! = 3628800$, ..., $13! \approx 6.2 \times 10^9$, ..., $70! \approx 1.2 \times 10^{100}$, etc.

Geometric sequence: The sequence-equivalent of the exponential function $f(x) = e^x$ is the *geometric sequence* where the k^{th} value in the sequence is the k^{th} powers of some number r:

$$g_k \stackrel{\text{def}}{=} r^k, \text{ for } k \in \mathbb{N} \quad \Leftrightarrow \quad \left[1, r, r^2, r^3, r^4, r^5, r^6, r^7, \ldots\right].$$

Each term in the sequence equals r times the previous term, which describes a *geometric* process that repeatedly grows/shrinks by the amount r. When $r < 1$, the values in the sequence g_k quickly go to zero, similar to how exponential function e^{-x} goes to zero for large values of x. When $r > 1$ the sequence g_k increases quickly, similar to how exponential function e^x increases for large value of x.

Powers of two: We'll also use the label b_k for the special case of the geometric sequence with $r = 2$:

$$b_k \stackrel{\text{def}}{=} 2^k, \text{ for } k \in \mathbb{N} \quad \Leftrightarrow \quad [1, 2, 4, 8, 16, 32, 64, 128, \ldots].$$

This sequence comes up all over the place in computer science because it describes the number of different numbers we can store in k bits of memory.

B. Convergence of sequences

What happens to a sequence as k goes to infinity? We can use the limit notation $\lim_{k\to\infty}$ to describe this process. There are two behaviours we're interested in: sequences that blow up to infinity, and sequences that approach some fixed number as k goes to ∞.

For example, the sequences $n_k \stackrel{\text{def}}{=} k$, $q_k \stackrel{\text{def}}{=} k^2$, and $b_k \stackrel{\text{def}}{=} 2^k$ keep getting larger and larger as k goes to infinity:

$$\lim_{k\to\infty} k = \infty, \qquad \lim_{k\to\infty} k^2 = \infty, \qquad \lim_{k\to\infty} 2^k = \infty.$$

We say these sequences are *divergent*. In contrast, the values in the sequences $h_k \stackrel{\text{def}}{=} \frac{1}{k}$, $a_k \stackrel{\text{def}}{=} \frac{(-1)^{k+1}}{k}$, and $f_k \stackrel{\text{def}}{=} \frac{1}{k!}$ *converge* to the value 0 in the limit as k goes to infinity:

$$\lim_{k\to\infty} \tfrac{1}{k} = 0, \qquad \lim_{k\to\infty} \tfrac{(-1)^{k+1}}{k} = 0, \qquad \lim_{k\to\infty} \tfrac{1}{k!} = 0.$$

The geometric series $g_k \stackrel{\text{def}}{=} r^k$ converges only if the absolute value of r is less than one: $\lim_{k\to\infty} r^k = 0$, when $|r| < 1$.

The limit of a sequence as k goes to infinity is directly analogous to the limit of function $f(x)$ as x goes to infinity.

C. Summation notation

We're often interested in computing sums of values in a sequence. To describe the sum of 3^{rd}, 4^{th}, and 5^{th} elements of the sequence c_k, we turn to summation notation: $\sum_{k=3}^{5} c_k = c_3 + c_4 + c_5$. The capital Greek letter *sigma* stands in for the word *sum*, and the range of index values included in this sum is denoted below and above the summation sign. The sum of the values in the sequence c_k from $k = 0$ until $k = n$ is denoted as $\sum_{k=0}^{n} c_k = c_0 + c_1 + c_2 + \cdots + c_{n-1} + c_n$.

Since this is a calculus tutorial, you should expect that an infinity of some kind will show up, and indeed we'll soon learn about *infinite series* that describe the sum of *all* the values in the sequence c_k: $\sum c_k \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{k=0}^{n} c_k$. But before we get to infinite sums, we'll start by looking at some finite sums to gain some experience with the summation notation.

D. Exact formulas for finite summations

We'll now show some useful formulas for calculating sum of the terms in certain sequences. For example, here is a formula for the sum of the first n terms in the geometric sequence:

$$G_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

We can use this formula to find the sum of the powers of 2:

$$\sum_{k=0}^{n} 2^k = 1 + 2 + 4 + 8 + \cdots + 2^n = \frac{1 - 2^{n+1}}{1 - 2} = 2^{n+1} - 1.$$

The sum of the first n positive integers and the sum of their squares are described by the following formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

E. Series

Series are defined as the sums computed from the terms in the sequence c_k. The *finite series* $\sum_{k=1}^{n} c_k$ computes the first n terms of the sequence:

$$C_n = \sum_{k=1}^{n} c_k = c_1 + c_2 + c_3 + c_4 + c_5 + \cdots + c_{n-1} + c_n.$$

The *infinite series* $\sum c_k$ computes *all* the terms in the sequence:

$$C_\infty = \sum c_k = \lim_{n\to\infty} \sum_{k=1}^{n} c_k = c_1 + c_2 + c_3 + c_4 + c_5 + \cdots.$$

The infinite series $\sum c_k$ of the sequence $c_k : \mathbb{N} \to \mathbb{R}$ is analogous to the integral $\int_0^\infty f(x)\,dx$ of a function $f : \mathbb{R} \to \mathbb{R}$.

Some infinite series converge to a finite value. For example, when $|r| < 1$, the limit as $n \to \infty$ of the geometric series converges to the following value:

$$G_\infty = \lim_{n\to\infty} G_n = \sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

This expression describes an infinite sum, which is not possible to compute in practice, but we can see the truth of this equation using our mind's eye. The formula for first n terms is the geometric series is $G_n = \frac{1-r^{n+1}}{1-r}$. The term r^{n+1} goes to zero as $n \to \infty$, so the only part of the formula that remains is $\frac{1}{1-r}$.

Example 16: sum of a geometric series: Let's use the formula to compute infinite series of the geometric sequence with $r = \frac{1}{2}$:

$$\sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

Figure 24 shows a visualization for this infinite sum.

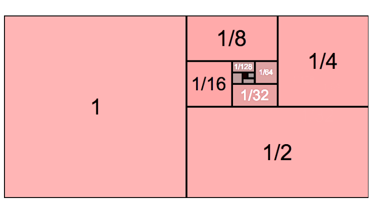


Figure 24. A graphical representation of the infinite sum of the geometric series with $r = \frac{1}{2}$. The area of each region corresponds to one of the terms in the series. The total area is equal to $\sum_{k=0}^{\infty}(\frac{1}{2})^k = \frac{1}{1-\frac{1}{2}} = 2$.

That's kind of cool, no? We're able to compute the value of a summation with infinitely many terms, because we have the general pattern G_n for the sum with n terms then evaluate the limit as n goes to infinity.

Convergent and divergent series: We say the geometric series $G_\infty = \sum g_k = \sum_{k=0}^{\infty} r^k$ *converges* to the value $\frac{1}{1-r}$. We can also say that the infinite geometric series $\sum g_k$ *is convergent*, meaning it has a finite value and doesn't blow up. Another example of a converging infinite series is $F_\infty = \sum f_k$, which converges to the number e, as we'll see in Example 17 below.

In contrast, the harmonic series $\sum h_k$ *diverges*. When we sum together more and more terms of the sequences h_k, the total computed keeps growing and the infinite series blows up to infinity $\sum h_k = \infty$. We say that the harmonic series *is divergent*.

Using convergent series for practical calculations: We can use infinite series to compute irrational numbers.

Example 17: Euler's number: The infinite sum of the sequence $f_k \stackrel{\text{def}}{=} \frac{1}{k!}$ converges to Euler's number $e = 2.71828182845905\ldots$:

$$F_\infty = \lim_{n\to\infty} F_n = \sum_{k=0}^{\infty} \tfrac{1}{k!} = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{720} + \cdots = e.$$

The calculation above is not just cool math fact, but a useful computational procedure that we can use to approximate the value of $e = 2.71828\ldots$ using only basic arithmetic operations like repeated multiplication (factorial), division, and addition.

Let's look at some practical calculations where we compute the first $n = 10$ and $n = 15$ terms in the series $\sum_{k=0}^{n} f_k$:

```
>>> import math
>>> def f_k(n):
        return 1 / math.factorial(n)
>>> sum([f_k(k) for k in range(0,10)])
2.718281...
>>> sum([f_k(k) for k in range(0,15)])
2.71828182845...
```

Summing together the first 10 terms in the series gives us an approximation to e that is accurate to six decimals. With 15 terms, we get an approximation that is accurate to 11 decimals. The more terms we include in the summation, the closer we get to the true value of e, which is $2.71828182845905\ldots$.

If we want to compute the *exact* value of e, we would need to compute the infinite series $\sum_{k=0}^{\infty} \frac{1}{k!}$. We can do this using SymPy by calling the function `sp.summation` whose syntax is similar to the function `sp.integrate` we used to compute integrals. The first argument is an the expression for the k^{th} term in the sequence, then we specify the index variable, the starting point, and the end point of the summation:

```
>>> import sympy as sp
>>> k = sp.symbols("k")
>>> sp.summation(1/sp.factorial(k), (k, 0, sp.oo))
E
```

We used `sp.oo` to make SymPy compute the infinite sum, which produced the exact symbolic answer `E` $= e$.

There are other series we can use to compute values of interest.

Example 18: We can calculate the value $\ln(2)$ by computing the infinite sum of the alternating harmonic sequence $a_k \stackrel{\text{def}}{=} \frac{(-1)^{k+1}}{k}$:

$$A_\infty = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = \ln(2).$$

To obtain the exact value $\ln(2)$, we need to sum together an infinite number of terms in the series $\sum a_k$, but we can obtain successively better approximations to $\ln(2)$ using finite sums.

```
>>> def a_k(k):
        return (-1)**(k+1) / k
>>> sum([a_k(k) for k in range(1,100+1)])
0.6...
>>> sum([a_k(k) for k in range(1,1000+1)])
0.69...
>>> sum([a_k(k) for k in range(1,1_000_000+1)])
0.69314...
```

The series approximation to $\ln(2)$ converges more slowly than the series approximation to e we saw in the previous example. We need to sum 1M terms in the series to obtain an approximation that is accurate to five decimals. Nevertheless, if we keep calculating sums with more and more terms, we can obtain an approximation that is arbitrarily close to the true value $\ln(2) = 0.6931471805599453\ldots$.

To get the exact value $\ln(2)$, we can make SymPy compute the infinite series:

```
>>> sp.summation((-1)**(k+1)/k, (k, 1, sp.oo))
log(2)
```

We can come up with all kinds of other infinite series expression for calculating other numbers. Instead of showing you other series for approximating numbers, I'll show you an even more powerful calculus technique: a way to approximate *functions* as infinite series.

F. Power series

The term *power series* describes a series whose terms contain different powers of the variable x. The k^{th} term in a power series consists of some coefficient c_k and the k^{th} power of x:

$$P_n(x) = \sum_{k=0}^{n} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n.$$

The math expression we obtain in this way is a *polynomial* of degree n in x, which we denote $P_n(x)$. Depending on the choice of the coefficients $(c_0, c_1, c_2, c_3, \ldots, c_n)$ we can make the polynomial function $P_n(x)$ *approximate* some other function $f : \mathbb{R} \to \mathbb{R}$. To find such approximations, we need some way to choose the coefficients c_k of the power series, so that the resulting polynomial approximates the function: $P_n(x) \approx f(x)$.

G. Taylor series

The *Taylor series approximation* to the function $f(x)$ is a power series whose coefficients c_k are computed by evaluating the k^{th} derivative of the function $f(x)$ at $x = 0$, which we denote $f^{(k)}(0)$. Specifically, the k^{th} coefficient in the Taylor series approximation for the function $f(x)$ is $c_k \stackrel{\text{def}}{=} \frac{f^{(k)}(0)}{k!}$, where $k!$ is the factorial function. The finite series with n terms produces the following approximation:

$$f(x) \approx f(0) + f'(0)x + \tfrac{f''(0)}{2!}x^2 + \cdots + \tfrac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \tfrac{f^{(k)}(0)}{k!}x^k.$$

In the limit as n goes to infinity, the Taylor series approximation becomes exactly equal to the function $f(x)$:

$$f(x) = f(0) + f'(0)x + \tfrac{f''(0)}{2!}x^2 + \tfrac{f^{(3)}(0)}{3!}x^3 + \cdots = \sum_{k=0}^{\infty} \tfrac{f^{(k)}(0)}{k!}x^k.$$

Using this formula and your knowledge of derivative formulas, you can compute the Taylor series of any function $f(x)$. For example, let's find the Taylor series of the function $f(x) = e^x$ at $x = 0$. The first derivative of $f(x) = e^x$ is $f'(x) = e^x$. The second derivative of $f(x) = e^x$ is $f''(x) = e^x$. In fact, all the derivatives of $f(x)$ will be e^x because the derivative of e^x is equal to e^x. The k^{th} coefficient in the power series of $f(x) = e^x$ at the point $x = 0$ is equal to the value of the k^{th} derivative of $f(x)$ evaluated at $x = 0$ divided by $k!$. In the case of $f(x) = e^x$, we have $f^{(k)}(0) = e^0 = 1$, so the coefficient of the k^{th} term is $c_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}$. The Taylor series of $f(x) = e^x$ is

$$e^x = \sum_{k=0}^{\infty} \tfrac{1}{k!}x^k = 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \tfrac{x^6}{6!} + \tfrac{x^7}{7!} + \cdots$$

Taylor series are a powerful computational tool for approximating functions. As we compute more terms from the above series, the polynomial approximation to the function $f(x) = e^x$ becomes more accurate.

Table II shows the Taylor series obtained using the formula $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ for several important functions.

TABLE II
TAYLOR SERIES EXPANSIONS FOR COMMONLY USED FUNCTIONS

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots \\
\frac{1}{1+x} &= \sum_{k=0}^{\infty} (-x)^k = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 + \cdots \\
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\ln(x+1) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots
\end{aligned}$$

Readers who are familiar with the concept of a basis from linear algebra can think of the Taylor series shown in Table II as *representations* of the corresponding functions with respect to the basis of polynomial functions: $(1, x, x^2, x^3, x^4, x^5, \ldots)$. The Taylor series coefficients $c_k = \frac{f^{(k)}(0)}{k!}$ are the *coordinates* of the function $f(x)$ in the *polynomial basis*.

H. Obtaining Taylor series using SymPy

The `SymPy` function `sp.series` is a convenient way to obtain the Taylor series of any function. Calling `series(fun,var,x0,n)` will show you the series expansion of any function `fun` near `var=x0` up to powers of `n`. We can quickly fact-check the Taylor series given in Table II using SymPy.

```
>>> import sympy as sp
```

```
>>> x = sp.symbols("x")
>>> sp.series(1/(1-x), x, x0=0, n=7)
1 + x + x**2 + x**3 + x**4 + x**5 + x**6 + O(x**7)
>>> sp.series(1/(1+x), x, x0=0, n=7)
1 - x + x**2 - x**3 + x**4 - x**5 + x**6 + O(x**7)
>>> sp.series(sp.E**x, x, x0=0, n=6)
1 + x + x**2/2 + x**3/6 + x**4/24 + x**5/120 + O(x**6)
>>> sp.series(sp.sin(x), x, x0=0, n=9)
x - x**3/6 + x**5/120 - x**7/5040 + O(x**9)
>>> sp.series(sp.cos(x), x, x0=0, n=8)
1 - x**2/2 + x**4/24 - x**6/720 + O(x**8)
>>> sp.series(sp.ln(x+1), x, x0=0, n=6)
x - x**2/2 + x**3/3 - x**4/4 + x**5/5 + O(x**6)
```

The "big-O" notation `O(x**n)` appears in all the above outputs as a reminder that the exact Taylor series contain additional terms that are on the *order of* x^n.

I. Applications of series

Series allow us to compute numbers like e, π, $\ln(2)$, etc. Taylor series allow us to approximate functions. The Taylor series representation for the function $f(x)$ also provides an easy way to compute its integral function $F_0(x) \stackrel{\text{def}}{=} \int_0^x f(u)\,du$. The Taylor series of $f(x)$ consists only of polynomial terms of the form $c_n x^n$. To compute the integral function $F_0(x) \stackrel{\text{def}}{=} \int_0^x f(u)\,du$, we can compute the integrals of the individual terms, which gives us $\frac{c_n}{n+1}x^{n+1}$.

VII. Multivariable calculus

In multivariable calculus, we extend the ideas of differential and integral calculus to functions with multiple input variables. If you understood the concepts of single-variable calculus, then you'll also be able to understand multivariable calculus: it's essentially the same stuff but in more dimensions!

A. Multivariable functions

A *single-variable function* $f : \mathbb{R} \to \mathbb{R}$ takes a real number $x \in \mathbb{R}$ as input and produces a real number $f(x) \in \mathbb{R}$ as output. A *multivariable function* takes multiple real numbers as inputs. For example, a bivariate function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ takes two real numbers as inputs $(x, y) \in \mathbb{R} \times \mathbb{R}$ and produces a real number $f(x, y) \in \mathbb{R}$ as output.

Consider the bivariate function $f(x, y) = 4 - x^2 - \frac{1}{4}y^2$. We can plot this function as a *surface* in a three-dimensional space, as shown in Figure 25. The height of the surface above the point (x, y) is function output $f(x, y)$.

Surface plots are very good for visualizing multivariable functions, but they can be difficult to draw by hand. Another approach for representing the function $f(x, y)$ is to use a two-dimensional plot that shows the "view from above" of the surface $f(x, y)$. We can trace *level curves* in the surface, to produce a "topographic map" of the surface where each level curve shows the points that are at a certain height. The curve labeled 0.0 you see in Figure 26 represents the solution to the equation $f(x, y) = 0$, which is where the function $f(x, y)$ intersects the xy-plane.

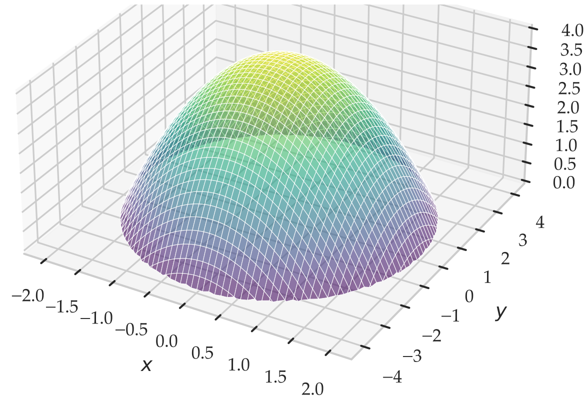


Figure 25. The 3D surface plot of the function $f(x, y) = 4 - x^2 - \frac{1}{4}y^2$.

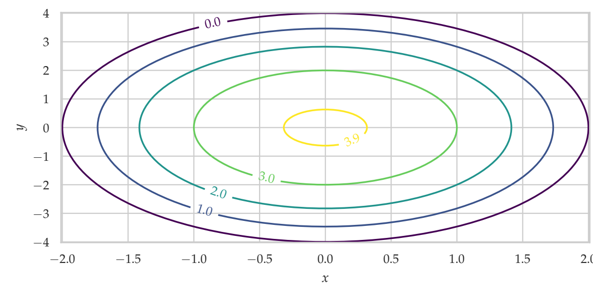


Figure 26. Topographic map that shows the function $f(x, y)$ as level curves.

B. Partial derivatives

For a function of two variables $f(x, y)$, there is an "x-derivative" operator $\frac{\partial}{\partial x}$ and a "y-derivative" operator $\frac{\partial}{\partial y}$. The operation $\frac{\partial}{\partial x} f(x, y)$ describes taking the derivative of $f(x, y)$ with respect to the input variable x, while keeping the input variable y constant. Taking the derivative of a multivariable function with respect to one of its input variables is called a *partial derivative* and denoted with the symbol ∂.

The partial derivative of $f(x, y)$ with respect to x is

$$\frac{\partial}{\partial x} f(x, y) \quad = \quad \frac{\partial f}{\partial x} \quad \stackrel{\text{def}}{=} \quad \lim_{\delta \to 0} \frac{f(x + \delta, y) - f(x, y)}{\delta}.$$

Similarly the partial derivative of with respect to y is

$$\frac{\partial}{\partial y} f(x, y) \quad = \frac{\partial f}{\partial y} \quad \stackrel{\text{def}}{=} \quad \lim_{\delta \to 0} \frac{f(x, y + \delta) - f(x, y)}{\delta}.$$

Note that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are functions of x and y.

Intuitively, the partial derivative $\frac{\partial f}{\partial x}$ tells us the slope of the function $f(x, y)$ in the x-direction, and $\frac{\partial f}{\partial y}(x, y)$ tells us the slope of $f(x, y)$ in the y-direction.

Example: The partial derivatives of $f(x, y) = 4 - x^2 - \frac{1}{4}y^2$ are $\frac{\partial f}{\partial x} = -2x$ and $\frac{\partial f}{\partial y} = -\frac{1}{2}y$.

C. The gradient operator

The *gradient* of the function is a vector that combines the x and y partial derivatives:

$$\nabla f(x, y) \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x} f(x, y), \ \frac{\partial}{\partial y} f(x, y) \right) = \left(\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y} \right).$$

We use the symbol ∇ (*nabla*) for the gradient operation because it looks like an upside Δ, which is the symbol for change. The direction of the gradient vector tells us the direction of the function's maximum increase—the "uphill" direction at the surface of graph of $f(x,y)$ at the point (x,y). The gradient vector is always perpendicular to the *level curve* at that point.

Example: The gradient of the function $f(x,y) = 4 - x^2 - \frac{1}{4}y^2$ is $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(-2x, -\frac{1}{2}y\right)$. The gradient vector at coordinates $(x,y) = (0,-4)$ is $\nabla f(0,-4) = (0,2)$, which is a vector pointing in the positive y-direction. Try to identify the point with coordinates $(0,-4)$ in figures 25 and 26 and confirm that the "uphill" direction at that point is indeed in the positive y-direction.

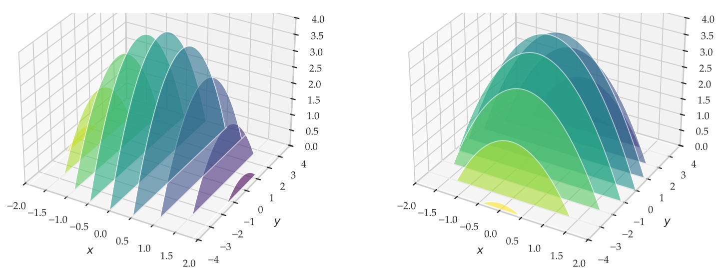
The other half of multivariable calculus involves computing integrals of multivariable functions.

D. Partial integration

We can integrate over one of the input variables to produce a function that depends only on the other input variable:

$$f(y) = \int f(x,y)\,dx \quad \text{and} \quad f(x) = \int f(x,y)\,dy.$$

The functions $f(y)$ and $f(x)$ are called the *partial integrals* or *marginals* of the function $f(x,y)$. For example, the partial integrals of the function $f(x,y) = 4 - x^2 - \frac{1}{4}y^2$ are obtained by computing the areas of "slices" throughout the function $f(x,y)$, as illustrated in Figure 27.



(a) Slices at different values of x. (b) Slices at different values of y.

Figure 27. Visualization of the partial integration procedures.

E. Double integrals

The multivariable generalization of the integral $\int_a^b f(x)\,dx$ that computes the "total" amount of $f(x)$ between a and b is the multivariable integral of the form:

$$\iint_{(x,y)\in R} f(x,y)\,dxdy,$$

where R is called the *region of integration* and corresponds to some subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$. The idea behind multivariable integrals is the same as for single variable integrals—to compute the total amount of some function accumulated over a range of input values. For single-variable functions, we integrate by splitting the region into thin rectangular strips of width dx. For double integrals, we split the two-dimensional region of integration into small squares of area $dxdy$, and compute the total volume of many rectangular columns with base $dxdy$ and height $f(x,y)$.

F. Applications of multivariable calculus

Multivariable functions appear all the time in machine learning, engineering, physics, and other sciences. The optimization techniques we discussed in Section IV-H (see page 10) readily generalize to functions with multiple variables. Indeed, the *gradient descent algorithm* is often used to optimize functions with hundreds or thousands of variables. Many of the cutting-edge machine learning models "learn" the model parameters by minimizing the value of some multivariable function and repeatedly taking steps in the "downhill" direction, as indicated by the gradient vector. Multivariable integrals are used a lot in probability theory and statistics, where they are used to compute probabilities and expectations. Your basic knowledge of derivatives and integrals concepts we discussed earlier in this tutorial will be very useful for understanding multivariable calculus topics.

VIII. Vector calculus

A discussion on vector calculus is out of scope for an introductory tutorial. Like, *waaaaay* out of scope. However, I want to show you one picture and explain where you might encounter vector calculus concepts in your future studies. Vector calculus is the study of *vector fields* and their properties. In a three dimensional space, vector fields are functions of the form $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$. The vector field $\mathbf{F}$ assigns a three-dimensional output vector $\mathbf{F} = (F_x, F_y, F_z)$ for each point $\mathbf{r} = (x,y,z)$ in $\mathbb{R}^3$. Note we use boldface to denote vectors.

A. Example: electric field around a positive charge

Figure 28 shows the *electric field* $\mathbf{E}$ around a positive charge q Coulombs located at the origin of the three-dimensional coordinate system. The electric field is strongest close to the charge, and gets weaker as you move away from the charge.

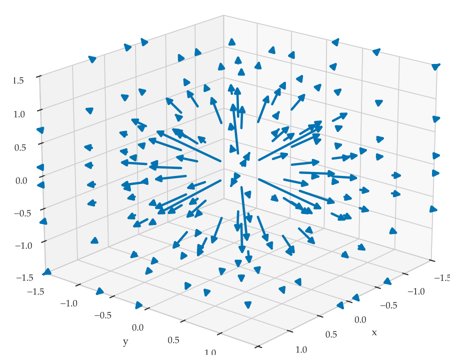


Figure 28. The electric field $\mathbf{E}(x,y,z) = \frac{kq}{r^2}\hat{\mathbf{r}}$ around a positive charge q.

The strength of the electric field $\mathbf{E}$ at the point $\mathbf{r} = (x,y,z)$ is

$$\mathbf{E}(x,y,z) = \frac{kq}{(x^2+y^2+z^2)^{3/2}}(x,y,z) = \frac{kq}{r^3}\mathbf{r} = \frac{kq}{r^2}\hat{\mathbf{r}},$$

where k is Coulomb's constant, $r \stackrel{\text{def}}{=} |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ is the distance from the origin, and $\hat{\mathbf{r}} \stackrel{\text{def}}{=} \frac{\mathbf{r}}{r}$ is a *unit vector* in the same direction as $\mathbf{r}$. Electric fields are used in the study of electromagnetism. Specifically, the electric field $\mathbf{E}(x,y,z)$ describes the strength and the direction of the *electric force* that a charged particle would experience if placed at (x,y,z).

B. Vector calculus derivatives

There are two derivative operations for vector fields, and these are written in terms of the vector derivative operator ∇ (*nabla*), which is defined as $\nabla \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. The *divergence* of the vector field $\mathbf{F}$ is computed by taking the "dot product" of ∇ and the vector field $\mathbf{F} = (F_x, F_y, F_z)$:

$$\nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

The divergence tells us if the field $\mathbf{F}$ is acting as a "source" or a "sink" at the point (x,y,z).

The *curl* of the vector field $\mathbf{F}$ is defined as the "cross product" of ∇ and the vector field $\mathbf{F} = (F_x, F_y, F_z)$:

$$\nabla \times \mathbf{F}(x,y,z) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right).$$

The curl tells us the rotational tendency of the vector field $\mathbf{F}$.

C. Vector calculus integrals

There are several different kinds of integral operations you can use with vector fields, depending on the type of "total" you want to compute, and the region of integration. The concept of a vector path integral is denoted $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$, where C is some curve in three dimensional space, and $d\mathbf{r}$ describes a short step along this curve. This integral computes the total action of the vector field $\mathbf{F}$ in the direction along the curve C. The vector surface integral is denoted $\iint_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S}$, where S is some surface in three dimensional space, and $d\mathbf{S}$ is a small "piece of the surface." This integral computes the total *flux* of the vector field $\mathbf{F}$ flowing through the surface S.

The two main results in vector calculus are *Gauss' divergence theorem* and *Stokes' theorem*. Both theorems can be understood as extensions of the fundamental theorem of calculus (FTC), since they show equivalences between certain vector derivative and integral operations.

D. Applications of vector calculus

Vector calculus is the math machinery used in physics (electricity and magnetism, mechanics, thermodynamics) and electrical engineering. If you're not planning to work in these fields, you can probably skip vector calculus: it's just derivatives and integrals applied to vector quantities.

IX. Practice problems

Learning calculus requires calculating lots of limits, derivatives, and integrals. Here are some practice problems for you.

P1 Calculate the following limit expressions:

$$\text{(a)}\ \lim_{x\to\infty} \tfrac{7}{x+4} \qquad \text{(b)}\ \lim_{x\to\infty} \tfrac{4x^2-7x+1}{x^2} \qquad \text{(c)}\ \lim_{x\to 0^-} \tfrac{1}{x}.$$

P2 Assuming $\lim\limits_{x\to\infty} f(x) = 2$ and $\lim\limits_{x\to\infty} g(x) = 3$, compute:

$$\text{(a)}\ \lim_{x\to\infty} \big(2f(x) - g(x)\big) \quad \text{(b)}\ \lim_{x\to\infty} f(x)g(x) \quad \text{(c)}\ \lim_{x\to\infty} \tfrac{4f(x)}{g(x)+1}.$$

P3 Find the derivative with respect to x of the functions:

$$\text{(a)}\ f(x) = x^{13} \qquad \text{(b)}\ g(x) = \sqrt[3]{x} \qquad \text{(c)}\ h(x) = ax^2 + bx + c.$$

P4 Calculate the derivatives of the following functions:

$$\text{(a)}\ p(x) = \tfrac{2x+3}{3x+2} \qquad \text{(b)}\ q(x) = \sqrt{x^2+1} \qquad \text{(c)}\ r(\theta) = \sin^3\theta.$$

P5 Find the maximum and the minimum of $f(x) = x^5 - 5x$.

P6 Calculate the integral function $F_0(b) = \int_0^b f(x)\,dx$ for the polynomial $f(x) = 4x^3 + 3x^2 + 2x + 1$.

P7 Find the area under $f(x) = 8 - x^3$ from $x = 0$ to $x = 2$.

P8 Find the area under the graph of the function $g(x) = \sin(x)$ between $x = 0$ to $x = \pi$.

P9 Compute $\int_0^1 \frac{4x}{(1+x^2)^3}\,dx$ using the substitution $u = 1 + x^2$. Check your answer numerically using the SciPy function `quad`.

P10 Calculate the value of the infinite series $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$.

P11 Find the Taylor series for the function $f(x) = e^{-x}$.

Answers: **P1** (a) 0. (b) 4. (c) $-\infty$. **P2** (a) 1. (b) 6. (c) 2. **P3** $f'(x) = 13x^{12}$. $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. $h'(x) = 2ax + b$. **P4** $p'(x) = \frac{-5}{(3x+2)^2}$. $q'(x) = \frac{x}{\sqrt{x^2+1}}$. $r'(\theta) = 3\sin^2\theta\cos\theta$. **P5** Max at $x = -1$; min at $x = 1$. **P6** $F_0(b) = b^4 + b^3 + b^2 + b$. **P7** $A_f(0,2) = 12$. **P8** $A_g(0,\pi) = 2$. **P9** $\frac{3}{4}$. **P10** 3. **P11** $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$.

X. Links

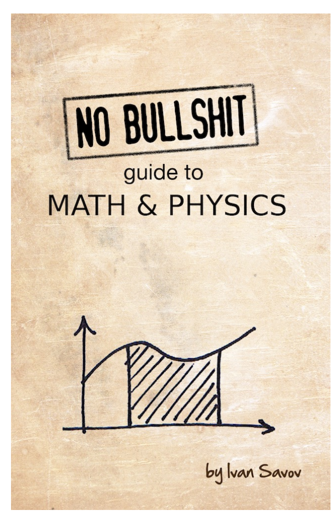
Here are some additional resources for learning about calculus.

[*Essence of calculus* series by 3Blue1Brown]
`https://tinyurl.com/CALCess`

[*Calculus made simple* by Silvanus P. Thompson]
`https://gutenberg.org/ebooks/33283`

Check out the **No bullshit guide to math and physics** for further calculus explanations and lots of practice problems.

This book contains short lessons on mechanics and calculus written in a style that is jargon-free and to the point. The main focus of the book is to show the intricate connections between the concepts of mechanics and calculus. This textbook covers both subjects in an integrated manner and aims to highlight the connections between them.



Contents:

- HIGH SCHOOL MATH
- VECTORS
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- INTEGRAL CALCULUS
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For more info, see the book's website: `minireference.com`.