Problem 2

$$y' = F(x, y) = -4y + x^2$$

(a)

$$K_1 = hF(x, y) = 0.1(-4) = -0.4$$

$$K_2 = hF\left(x + \frac{h}{2}, y + \frac{1}{2}K_1\right) = 0.1(-4) = -0.4$$

$$= 0.1\left[-4\left(1 + \frac{-0.4}{2}\right) + \left(0 + \frac{0.1}{2}\right)^2\right] = -0.31975$$

$$y(0.1) = y(0) + K_2 = 1 + (-0.31975) = 0.68025$$

(b)

$$K_1 = hF(x, y) = -0.4$$

$$K_2 = hF\left(x + \frac{h}{2}, y + \frac{K_1}{2}\right) = -0.31975$$

$$K_3 = hF\left(x + \frac{h}{2}, y + \frac{K_2}{2}\right)$$

$$= 0.1\left[-4\left(1 + \frac{-0.31975}{2}\right) + \left(0 + \frac{0.1}{2}\right)^2\right] = -0.3358$$

$$K_4 = hF(x + h, y + K_3)$$

$$= 0.1[-4(1 - 0.3358) + (0 + 0.1)^2] = -0.26468$$

$$y(0.1) = y(0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}[-0.4 + 2(-0.31975) + 2(-0.3358) + (-0.26468)]$$

$$= 0.6707$$

The result agrees with the analytical solution.

Problem 3

The integration formula is

$$y(x+h) = y(x) + y'(x)h + \frac{1}{2}y''(x)h^2$$

where

$$y'(x) = \sin y \qquad y''(x) = \cos y \cdot y' = \cos y \sin y \qquad h = 0.1$$

Step 1:

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= \sin(1) = 0.841\,471 \\ y''(0) &= \cos(1)\sin(1) = 0.454\,649 \end{aligned}$$

$$y(0.1) = 1 + (0.841\,471)\,(0.1) + \frac{1}{2}\,(0.454\,649)\,(0.1)^2 = 1.086\,42$$

Step 2:

$$\begin{aligned} y'(0.1) &= \sin(1.086\,42) = 0.884\,966 \\ y''(0.1) &= \cos(1.086\,42)\sin(1.086\,42) = 0.412\,09 \end{aligned}$$

$$y(0.2) = 1.086\,42 + (0.884\,966)(0.1) + \frac{1}{2}(0.412\,09)(0.1)^2 = 1.176\,977$$

Step 3:

$$\begin{aligned} y'(0.2) &= \sin(1.176\,977) = 0.923\,45 \\ y''(0.2) &= \cos(1.176\,977)\sin(1.176\,977) = 0.354\,345 \end{aligned}$$

$$y(0.3) = 1.176\,977 + (0.923\,45)(0.1) + \frac{1}{2}(0.354\,345)(0.1)^2 = 1.271\,094$$

Step 4:

$$\begin{aligned} y'(0.3) &= \sin(1.271\,094) = 0.955\,424 \\ y''(0.3) &= \cos(1.271\,094)\sin(1.271\,094) = 0.282\,076 \end{aligned}$$

$$y(0.4) = 1.271\,094 + (0.955\,424)(0.1) + \frac{1}{2}(0.282\,076)(0.1)^2 = 1.368\,047$$

Step 5:

$$\begin{aligned} y'(0.4) &= \sin(1.368\,047) = 0.979\,517 \\ y''(0.4) &= \cos(1.368\,047)\sin(1.368\,047) = 0.197\,239 \end{aligned}$$

$$y(0.5) = 1.368\,047 + (0.979\,517)(0.1) + \frac{1}{2}(0.197\,239)(0.1)^2 = 1.4670 \quad \blacktriangleleft$$

Using the 2nd-order Runge-Kutta method in Example 7.3 we had $y(0.5) = 1.4664$, which is correct to 4 decimal places. In this problem, the Taylor series method is somewhat less accurate.

Problem 6

We use the notation $x = y_1$, $y = y_2$, $\dot{x} = y_3$ and $\dot{y} = y_4$

(a)

$$\ddot{y} = x - 2y \qquad \ddot{x} = y - x$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ y_2 - y_1 \\ y_1 - 2y_2 \end{bmatrix} \quad \blacktriangleleft$$

(b)

$$\ddot{y} = -y(\dot{y}^2 + \dot{x}^2)^{1/4} \qquad \ddot{x} = -x(\dot{y}^2 + \dot{x}\ )^{1/4} - 32$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ -y_1(y_4^2 + y_3^2)^{1/4} - 32 \\ -y_2(y_4^2 + y_3)^{1/4} \end{bmatrix} \quad \blacktriangleleft$$

(c)

$$\ddot{y} = (4\dot{x} - t\sin y)^{1/2} \qquad \ddot{x} = (4\dot{y} - t\cos y)/x$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ (4y_4 - t\cos y_2)/y_1 \\ (4y_3 - t\sin y_2)^{1/2} \end{bmatrix} \quad \blacktriangleleft$$