CSE232: Discrete Mathematics
Assignment 6: Suggested answers

December 20, 2020

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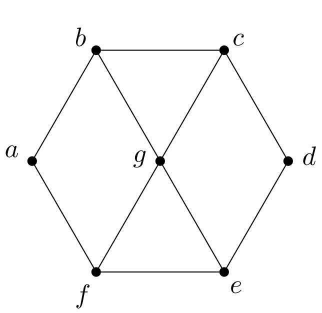
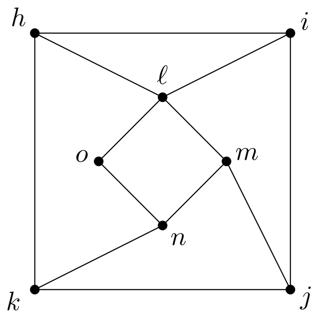
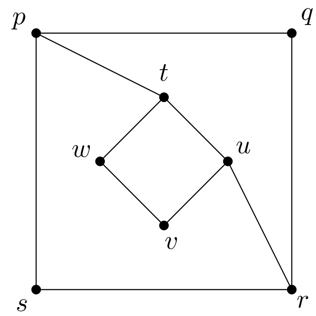
1. [10pts] For each graph F, G and H, determine whether it has a Hamilton circuit. Justify your answers. (If it has a Hamilton circuit, demonstrate the exact circuit in the graph. Otherwise, explain the reason in detail).



F

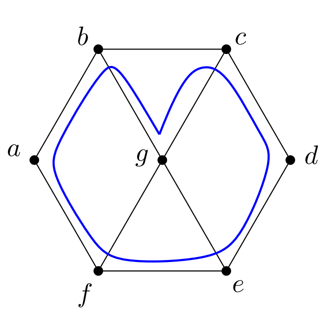


G

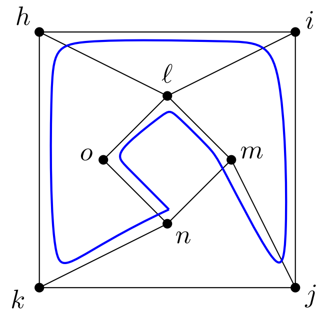


H

Answer. The graphs F and G have Hamilton circuits. (See Figure 1.)



F



G

Figure 1: F and G have Hamilton circuits.

The graph H does not have any Hamilton circuit. We prove it by contradiction. So suppose that there is a Hamilton circuit C. As $\deg(s) = 2$, the edges $\{s, p\}$ and $\{s, r\}$ must be in C. Similarly, as $\deg(q) = 2$, the edges $\{q, p\}$ and $\{q, r\}$ must be in C. But in addition, C must contain the edge $\{p, t\}$ or the edge $\{u, r\}$, because H must visit the vertices of the inner square

$\{t, u, v, w\}$. It means that p or r is visited more than once, which is impossible because C is a Hamilton circuit.

2. [10pts] Let T be an ordered binary tree with vertex set $\{A, B, C, D, E, F, G\}$. Suppose that the inorder traversal of T is $ABCDEFG$ and its preorder traversal is $EBADCFG$. Draw T and explain how you constructed it.

Answer. As the preorder traversal starts with E, vertex E must be the root. Then the inorder traversal tells us that the left subtree has vertices A, B, C and D, and the right subtree has vertices F and G. (See Figure 2a.)

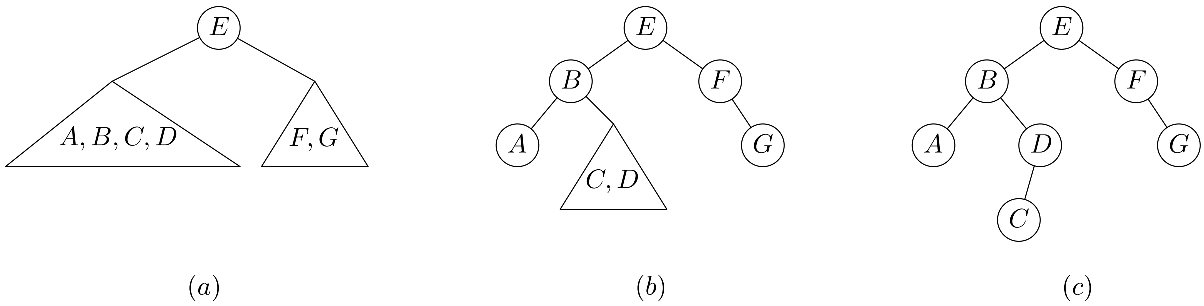


Figure 2: Solution to Problem 3.

The preorder traversal of the left subtree of E is $BADC$ so its root is B. Its inorder traversal is $ABCD$ so the left subtree of B consists of A only, and its right subtree has vertices F and G. (See Figure 2b.) The preorder traversal of the right subtree at F is FG, so the root is F. The inorder traversal is FG so G is the right child.

Finally, the preorder traversal of the right subtree of B is DC so D is its root. The inorder traversal is CD so C is the left child. (See Figure 2c.) So we have obtained the tree T.

3. Consider the following relation R over $\mathbb{Z}$: $\quad R = \{(x, y) \mid x + y \text{ is even}\}$.

(a) [5pts] Is R a partial order? Justify your answer.

(b) [5pts] Is R an equivalence relation? Justify your answer.

Answer a. No, it is not a partial order, because it is not antisymmetric. Counterexample: $x = 0$, $y = 2$. Then $(x, y) \in R$ and $(y, x) \in R$, but $x \neq y$.

Answer b. Yes, it is an equivalence relation, because it is:

- Reflexive: For any $x \in \mathbb{Z}$, $x + x = 2x$ is even, so $(x, x) \in \mathbb{Z}$.
- Symmetric: Suppose $(x, y) \in R$. Then $x + y$ is even. It means that $y + x = x + y$ is even. So $(y, x) \in R$.
- Transitive: Suppose $(x, y) \in R$ and $(y, z) \in R$. Then $x + y \equiv 0 \pmod 2$ and $y + z \equiv 0 \pmod 2$. So

$$\begin{aligned} x + z &\equiv x + 2y + z \pmod 2 && \text{because } 2y \equiv 0 \pmod 2 \\ &\equiv (x + y) + (y + z) \pmod 2 \\ &\equiv 0 + 0 \pmod 2 \\ &\equiv 0 \pmod 2 \end{aligned}$$

so $(x, z) \in R$.

4. Consider the following relation over $\mathbb{Z}^+$:

$$(p, q) \in R \text{ if and only if } \exists k \in \mathbb{Z}^+(q = p^k).$$

(a) [5pts] Prove that R is a partial order.

(b) [5pts] Determine the set of all the upper bounds of $\{2, 3\}$ according to R.

Answer a. We need to check that R is reflexive, antisymmetric and transitive. the proofs are below.

- Let $p \in R$. Then $p = p^1$ so $(p, p) \in R$. Thus R is **reflexive**.
- Suppose that $(p, q) \in R$ and $(q, p) \in R$. Then there exist positive integers k and ℓ such that $q = p^k$ and $p = q^\ell$. Therefore $p = q^\ell = p^{k^\ell} = p^{k\ell}$. Since $k \geqslant 1$ and $\ell \geqslant 1$, and since $p \geqslant 1$, there are two cases:
  - If $p = 1$, then $q = 1$ because $q = p^k$, and thus $p = q$.
  - If $p \geqslant 2$, since $p = p^{k\ell}$, we must have $k = \ell = 1$. Since $q = p^k$, it implies that $p = q$.

  In any case, we have $p = q$, which proves that R is **antisymmetric**.
- Suppose that $(p, q) \in R$ and $(q, r) \in R$. Then there exist positive integers k and ℓ such that $q = p^k$ and $r = q^\ell$. Therefore $r = q^\ell = (p^k)^\ell = p^{k\ell}$, so $(r, q) \in R$. Hence R is **transitive**.

Answer b. Suppose that q is an upper bound of $\{2, 3\}$. Then we must have $(2, q) \in R$ and $(3, q) \in R$. So we have $q = 2^k = 3^\ell$ for some positive integers k and ℓ. So 2^k and 3^ℓ are two different prime factorizations of q. But this is impossible by the fundamental theorem of arithmetic.

Conclustion: $\{2, 3\}$ has no upper bound, so the set of upper bounds is **the empty set**.