CSE232 Final Exam

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Question 1.(10pt)

Determine whether each of these statements are true. If the truth value of the statement is true, write down **T**. If the truth value of the statement is false, write down **F**.

(a) $\mathbb{Q} \cup \mathbb{Z} = \mathbb{Q}$

(b) $(p \wedge q) \rightarrow p$ is a tautology

(c) $|\{2\} \times A| = 2|A|$ for all set A

(d) $(A \cap \bar{B}) \cup (A \cap B) = A$ for all sets A and B

(e) $\forall x \exists y (y > x)$ where the domain for x and y is $\mathbb{Z}$.

(f) $\exists x \forall y (y > x)$ where the domain for x and y is $\mathbb{Z}$.

(g) The sets $\mathbb{R}$ and $\mathbb{R} - \{0\}$ have the same cardinality.

(h) The sets $\{1, 2, 4, 5\}$, $\{3, 6, 7\}$, and $\{2, 3\}$ are a partition of $\{1, 2, 3, 4, 5, 6, 7\}$.

Answer. The answer is **T, T, F, T, T, F, T, F**.

(g) Yes. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R} - \{0\}$ defined by

$$f(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{N}, \\ x & \text{otherwise.} \end{cases}$$

Then it is easy to see that it is a bijection between $\mathbb{R}$ and $\mathbb{R} - \{0\}$. Hence, $|\mathbb{R}| = |\mathbb{R} - \{0\}|$.

(h) These subsets are not pairwise disjoint so they do not form a partition.

Question 2.(10pt)

Let $S = \{p, q, r\}$ be a set. Determine whether each of these statements are true. If the truth value of the statement is true, write down **T**. If the truth value of the statement is false, write down **F**.

(a) $\{p, q\} \notin \mathcal{P}(S)$

(b) $\{p, r\} \notin S$

(c) $\{q, r\} \subset \mathcal{P}(S)$

(d) $\{\emptyset\} \notin \mathcal{P}(S)$

(e) $\emptyset \in \mathcal{P}(S)$

(f) $\emptyset \subset \{p, q, r\}$

(g) $\emptyset \notin \{\emptyset\}$

(h) $\emptyset \subset \{\emptyset, \{\emptyset\}\}$

Answer. The answer is **F,T,F,T,T,T,F,T**.

We see that the power set $\mathcal{P}(S) = \{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, \{p, r\}, \{p, q, r\}\}$.
a. Since $\{p, q\} \subset S$ and the power set $\mathcal{P}(S)$ is the set of all the subsets of S, we have $\{p, q\} \in \mathcal{P}(S)$
b. T: $\{p, r\}$ is not an element of S
c. $\{q, r\}$ is not a subset of $\mathcal{P}(S)$.
d. T: $\{\emptyset\}$ is not an element of $\mathcal{P}(S)$.
g. F because $\emptyset \in \{\emptyset\}$

Question 3.

I. (5pt) What is the negation of the statement $\forall x \exists y\, (P(x,y) \to Q(x,y))$ so that all negation symbols immediately precede predicates?

(a) $\exists x \forall y\, (\neg P(x,y) \wedge Q(x,y))$

(b) $\forall x \exists y\, (\neg P(x,y) \wedge \neg Q(x,y))$

(c) $\exists x \forall y\, (P(x,y) \wedge \neg Q(x,y))$

(d) $\exists x \forall y\, (P(x,y) \vee \neg Q(x,y))$

II. (5pt) What is the negation of the statement $\exists y (\exists x R(x,y) \vee \forall x S(x,y))$ so that all negation symbols immediately precede predicates?

(a) $\forall y (\forall x \neg R(x,y) \vee \exists x \neg S(x,y))$

(b) $\forall y (\forall x \neg R(x,y) \wedge \exists x \neg S(x,y))$

(c) $\forall y (\exists x \neg R(x,y) \wedge \forall x \neg S(x,y))$

(d) $\forall y (\exists x \neg R(x,y) \vee \forall x \neg S(x,y))$

For each problem, write down an alphabet among **a**, **b**, **c** or **d**.

Answer I. The answer is (c) because

$$\begin{aligned} \neg\left(\forall x \exists y\, (P(x,y) \to Q(x,y))\right) &\equiv \exists x \forall y\, \neg(P(x,y) \to Q(x,y)) \\ &\equiv \exists x \forall y\, \neg(\neg P(x,y) \vee Q(x,y)) \\ &\equiv \exists x \forall y\, (P(x,y) \wedge \neg Q(x,y)) \end{aligned}$$

Answer II. The answer is (b) because

$$\begin{aligned} \neg \exists y (\exists x R(x,y) \vee \forall x S(x,y)) &= \forall y \neg(\exists x R(x,y) \vee \forall x S(x,y)) \\ &= \forall y (\neg \exists x R(x,y) \wedge \neg \forall x S(x,y)) \\ &= \forall y (\forall x \neg R(x,y) \wedge \exists x \neg S(x,y)) \end{aligned}$$

Question 4.

(a) (5pt) Assume that there are ten identical books. In how many ways can ten books be put in five labeled boxes, if one or more of the boxes can be empty?

(b) (5pt) How many ways are there to choose 12 cookies if there are five varieties, including chocolate chip, and at least four chocolate chip cookies must be chosen?

Answer. (a) The question can be rewritten as "how many different strings can be formed from 10 x's and 4 w's". This is to choose 4 positions between 14 and fill them with w's, the remainder 10 are x's. Therefore, the answer is $C(14, 4) = \mathbf{1001}$.
(b) Since at least four chocolate chip cookies must be chosen, this is equivalent to determining the number of ways to choose eight cookies from five varieties. Consequently the answer is given by the number of 8-combinations with repetition of five objects. Therefore there are $C(8+5-1, 5-1) = \mathbf{495}$ ways.

Question 5. (10pt)

Two identical urns contain balls. One of the urns has 6 red balls and 4 blue balls. The other urn has 4 red balls and 8 blue balls. An urn is chosen at random and two balls are drawn at random from this urn, without replacement.
Let p_1 be the probability that both balls are red. Let p_2 the probability that the second ball is red, given that the first ball is red. What is the value of $3 \times (p_1 + p_2)$?

Answer. Half the time we select the first urn, in which case the probability that the two balls are both red is $(6/10) \cdot (5/9) = 1/3$. Half the time we select the second urn, in which case the probability that the two balls are both red is $(4/12) \cdot (3/11) = 1/11$. Therefore the answer is

$$\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{11} = \frac{7}{33}.$$

Let F be the event that the first ball is red, and let S be the event that the second ball is red. We are asked for $p(S \mid F)$. By definition, this is $p(S \cap F)/p(F)$. From (a), we know that $p(S \cap F) = \frac{7}{33}$. By a simpler calculation, we see that

$$p(F) = \frac{1}{2} \cdot \frac{6}{10} + \frac{1}{2} \cdot \frac{4}{12} = \frac{7}{15}.$$

Thus the answer is

$$\frac{7/33}{7/15} = \frac{15}{33}.$$

Then

$$p_1 + p_2 = \frac{7}{33} + \frac{15}{33} = \frac{22}{33} = \frac{2}{3}.$$

Therefore, the answer is $3 \times (p_1 + p_2) = \mathbf{2}$.

Question 6. (10pt)

A flag consists of a 1×3 rectangle, divided into three equal squares, each square being painted in one of 10 possible colors. Moreover, two adjacent squares cannot be painted with the same color. Finally, a flag may have a colored circle at the center of the middle square: in this case, the color of the circle must be different from the other colors used to paint the squares. How many different flags can be obtained, following the rules above?

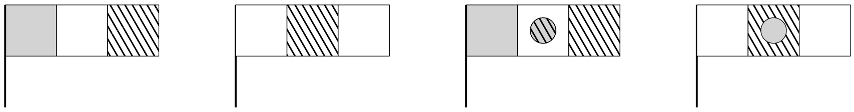


Figure 1: Examples of possible flags.

Answer. We split into 2 cases.

- Suppose that the flag has no circle. Then there are 10 choices for the color of the first square, 9 choices for the second square, and then 9 choices for the third square as it cannot be of the same color as the second one. So by the product rule, there are $10 \times 9 \times 9$ possible flags.
- Suppose that the flag has a circle. Then there are 10 choices for the color of the circle. This color is now forbidden, so the same argument as above shows that there are $9 \times 8 \times 8$ choices for the colors of the square. In total, we get $10 \times 9 \times 8 \times 8$ possibilities.

By the sum rule, the solution is

$$10 \times 9 \times (9 + 64) = 73 \times 10 \times 9 = \mathbf{6570}.$$

Question 7. (10pt)

A *run* is a maximal sequence of successes in a sequence of Bernoulli trials. For example, in the sequence S, S, S, F, S, S, F, F, S, where S represents success and F represents failure, there are three runs consisting of three successes, two successes, and one success, respectively. Let R denote the random variable on the set of sequences of 22 independent Bernoulli trials with probability of success 1/3, that counts the number of runs in this sequence. What is the expected value of R?

Answer. Let R_j, $j = 1, \dots, 22$ be a random variable such that $R_j = 1$ if a run starts at the jth trial, and $R_j = 0$ otherwise. So we have $R = \sum_{j=1}^{22} R_j$. By linearity of expectation, it follows that

$$E(R) = \sum_{j=1}^{22} E(R_j).$$

As $R_j = 0$ or 1, we have $E(R_j) = p(R_j = 1)$. A run starts at the first trial if and only if the result is S, so $E(R_1) = 1/3$. A run starts at the jth trial, where $j \geqslant 2$, if the $(j-1)$'th trial fails and the j'th trial is a success, so $E(R_j) = 1/3 \cdot (1 - 1/3) = 2/9$. Thus, we have

$$E(R) = E(R_1) + \sum_{j=2}^{22} E(R_j) = (1/3) + 21 \cdot (2/9) = \mathbf{5}$$

Question 8. (10pt)

Suppose that Ann selects a ball by first picking one of two boxes at random and then selecting a ball from this box. The first box contains three orange balls and four black balls, and the second box contains five orange balls and six black balls. Let p be the probability that Ann picked a ball from the second box if she has selected an orange ball. Assume that p is written as a fraction a/b of two integers a and b such that $\gcd(a, b) = 1$. What is the value of $(a + b)$?

Answer. Let E be the event that Ann selected an orange ball, and let F be the event that she picked a ball in the second box. Our goal is to determine $p(F \mid E)$.

As the second box contains 5 orange balls and 6 black balls, we know that $p(E \mid F) = 5/11$. As the first box contains 3 orange balls and four black, we have $p(E \mid \overline{F}) = 3/7$. As the box is chosen at random, we have $p(F) = p(\overline{F}) = 1/2$.

Applying Bayes's theorem, we obtain

$$\begin{aligned} p(F \mid E) &= \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})} \\ &= \frac{\frac{5}{11} \cdot \frac{1}{2}}{\frac{5}{11} \cdot \frac{1}{2} + \frac{3}{7} \cdot \frac{1}{2}} \\ &= \frac{\frac{5}{11}}{\frac{5}{11} + \frac{3}{7}} = \frac{\frac{35}{77}}{\frac{68}{77}} = \frac{35}{68} \end{aligned}$$

The answer is **103**.

Question 9. (10pt)

Let $g = \gcd(201, 111)$. Then g can be expressed as a linear combination of 201 and 111 in a way that $g = 201 \times s + 111 \times t$ for some $s, t \in \mathbb{Z}$. What are the values of g, s, and t?

Answer. We first compute $\gcd(201, 111)$ with the Euclidean algorithm.

$$\begin{aligned}
201 &= 111 \times 1 + 90 \\
111 &= 90 \times 1 + 21 \\
90 &= 21 \times 4 + 6 \\
21 &= 6 \times 3 + 3 \\
6 &= 3 \times 2
\end{aligned}$$

Therefore, $g = 3$. From the calculation above,

$$\begin{aligned}
3 &= 21 - 6 \times 3 \\
&= 21 - (90 - 21 \times 4) \times 3 \\
&= 21 \times 13 - 90 \times 3 \\
&= (111 - 90 \times 1) \times 13 - 90 \times 3 \\
&= 111 \times 13 - 90 \times 16 \\
&= 111 \times 13 - (201 - 111 \times 1) \times 16 \\
&= 201 \times (-16) + 111 \times 29
\end{aligned}$$

and thus $s = -16$, $t = 29$.

Question 10. (10pt)

Let a be the last digit of 7^{100}.
Find an integer $x \in [0, 210)$ such that $x \equiv 1 \pmod 5$, $x \equiv 2 \pmod 6$, and $x \equiv 3 \pmod 7$.
What is the value of $(a \times x)$?

Answer.

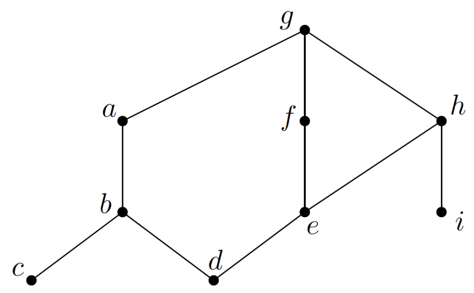
(a) $7^{100} \equiv (7^2)^{50} \equiv 49^{50} \equiv (-1)^{50} \equiv 1 \pmod{10}$, so we have $a = 1$.

(b) $x = 206$.

So, the answer is **206**.

Question 11.

Consider the poset whose Hasse diagram is drawn below.



(a) (2.5pt) Let n_1 be the number of its minimal elements. What is the value of n_1?

(b) (2.5pt) Let n_2 be the number of its maximal elements. What is the value of n_2?

(c) (2.5pt) Let n_3 be the number of its least element. What is the value of n_3?

(d) (2.5pt) Let n_4 be the number of its greatest element. What is the value of n_4?

Answer.

(a) c, d, and i; so $\mathbf{n_1 = 3}$

(b) g; so $\mathbf{n_2 = 1}$

(c) None; so $\mathbf{n_3 = 0}$

(d) g; so $\mathbf{n_4 = 1}$

Question 12.

Determine whether each of these statements are true. If the truth value of the statement is true, write down **T**. If the truth value of the statement is false, write down **F**.

(a) (2.5pt) $(\mathbb{Z}, \not|)$ is not a poset.

(b) (2.5pt) $(\mathbb{Z}, \geqslant)$ is not a poset.

(c) (2.5pt) If a relation R on a set A that is symmetric and transitive, then R is reflexive.

(d) (2.5pt) The relation $R = \{(x, y) \mid x = y^2\}$ on $\mathbb{Z}$ is antisymmetric.

Answer. The answer is **T,F,F,T**.

a. It is not transitive: $2 \not| 3$, $3 \not| 2$, but $2|2$.

b. $(\mathbb{Z}, \geqslant)$ is a poset.

c. The empty relation $R = \emptyset$ is symmetric and transitive, but it is not reflexive. Another example is the relation over $\{0, 1\}$ that consists of the single pair $\{(0, 0)\}$.

d Suppose that $y = x^2$ and $x = y^2$. Then we have $y = y^4$ and $x = x^4$. As x and y are integers, it implies that $x \in \{0, 1\}$ and $y \in \{0, 1\}$. If $x = 0$, as $x = y^2$, we also have $y = 0$. If $x = 1$, for the same reason, we have $y = 1$. So in any case $x = y$.

Question 13.

(a) (2.5pt) Let n_1 be the number of different spanning trees of the graph K_3. What is the value of n_1?

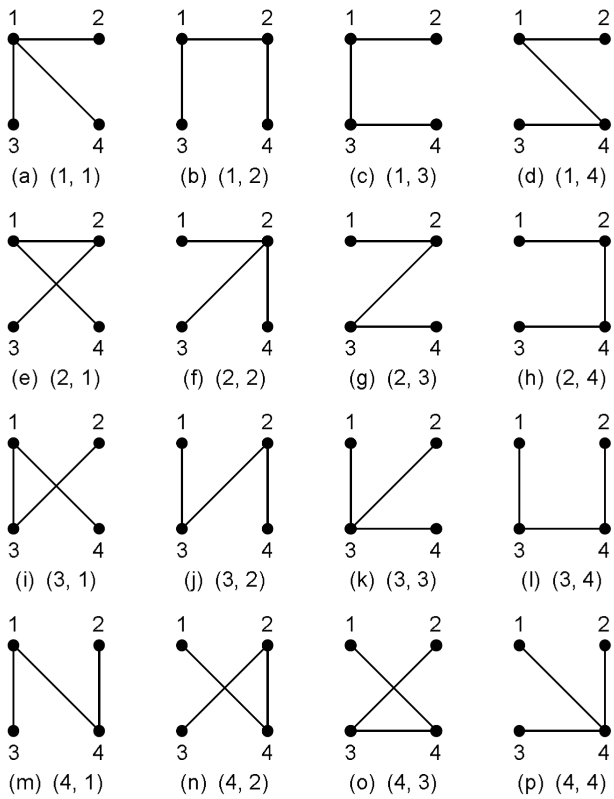
(b) (2.5pt) Let n_2 be the number of different spanning trees of the graph K_4. What is the value of n_2?

(c) (2.5pt) Let n_3 be the number of different spanning trees of the graph $K_{2,2}$. What is the value of n_3?

(d) (2.5pt) Let n_4 be the number of different spanning trees of the graph C_5. What is the value of n_4?

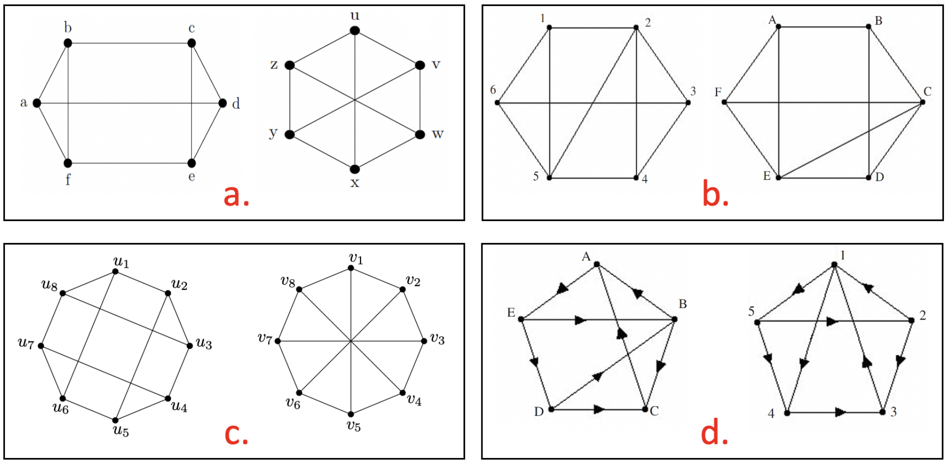
Answer 1. $n_1 = 3, n_2 = 16, n_3 = 4, n_4 = 5.$

The following is the different spanning trees of K_4.



Question 14. (10pt)

Which sets of graphs are not isomorphic?



Answer. The answer is $\mathbf{a}, \mathbf{c}, \mathbf{d}$. (I gave a partial credit for each set of graphs)

(a) These graphs **are not isomorphic**, since the left graph contains a subgraph isomorphic to K_3 but the right graph does not. (In fact, the right graph is bipartite.)

(b) The graphs are **isomorphic**. In the graph on the left, only vertices 2 and 5 have degree four. In the graph on the right, only vertices C and E have degree four. Therefore, if the two graphs are to be isomorphic, we must have 2 and 5 correspond to C and E as either 2-C, 5-E, or as 2-E, 5-C. Either correspondence gives rise to an isomorphism: 1-F; 2-C; 3-B; 4-D; 5-E; 6-A: 1-D; 2-E; 3-A; 4-F; 5-C; 6-B:

(c) The left graph is bipartite because all edges are between a vertex with odd index (u_1, u_3, u_5 and u_7) and a vertex with even index (u_2, u_4, u_6 and u_8). On the other hand, the right graph is not bipartite because it has a circuit of length 5, for instance $(v_1, v_2, v_3, v_4, v_5)$. However, any bipartite graph can only have circuits of even length, because its vertices must alternate between the sets V_1 and V_2 of the bipartition. Since the left is bipartite and the right is not, they **cannot be isomorphic**.

(d) Even though the graphs have many features in common (such as the same number of vertices, the same number of edges, matching in-degrees and out-degrees), the digraphs are not isomorphic. Here is one reason: Vertex B must correspond to vertex 1 because they are the only vertices with in-degree 2 and outdegree 2. Vertices D and E each have in-degree 1 and out-degree 2. If the two graphs are to be isomorphic, then D and E must correspond to 2 and 5 (in some order). Because there is an edge from E to D, there must be a corresponding edge in the digraph on the right—this forces D to correspond to 2 and E to correspond to 5. However in the left graph there is an edge from E to B, but no edge from 5 to 1 (the vertices corresponding to B and E) in the right graph. Therefore, the two digraphs are **not isomorphic**.

Question 15.

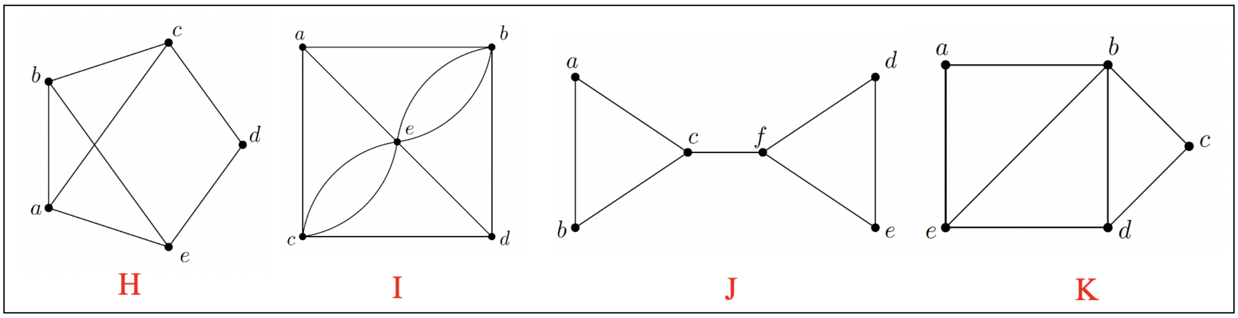
For a graph G, we define e_G as follows:

$$e_G = \begin{cases} 0 & \text{when } G \text{ has } \textbf{neither} \text{ an Euler path nor an Euler circuit.} \\ 1 & \text{when } G \text{ has an Euler path, but does not have any Euler circuit.} \\ 2 & \text{when } G \text{ has } \textbf{both} \text{ an Euler path and an Euler circuit.} \end{cases}$$

Similarly, we define h_G as follows:

$$h_G = \begin{cases} 0 & \text{when } G \text{ has } \textbf{neither} \text{ a Hamilton path nor a Hamilton circuit.} \\ 1 & \text{when } G \text{ has a Hamilton path, but does not have any Hamilton circuit.} \\ 2 & \text{when } G \text{ has } \textbf{both} \text{ a Hamilton path and a Hamilton circuit.} \end{cases}$$

For the graphs H, I, J and K,



(a) (2.5pt) what is the value of e_H?

(b) (2.5pt) what is the value of e_I?

(c) (2.5pt) what is the value of h_J?

(d) (2.5pt) what is the value of h_K?

Answer. The answer is **0, 1, 1, 2**.

(a) The graph H has 4 vertices of degree 3, so it does not have any Euler path or circuit. Thus, $e_H = 0$.

(b) In the graph I, the vertices a and d have degree 3, and the other vertices have even degree 4 or 6. So I has an Euler path from a to d, but no Euler circuit. Thus, $e_I = 1$.

(c) The graph J has a Hamilton path: a, b, c, f, e, d. But it has no Hamilton circuit. If it were the case, since a has degree 2 and its neighbors are b and c, the Hamilton circuit would contain the edges $\{a, b\}$ and $\{a, c\}$. Similarly, it would contain $\{b, c\}$, $\{d, f\}$, $\{d, e\}$ and $\{e, f\}$. In addition, the Hamilton circuit would need to contain $\{c, f\}$ because it is the only edge connecting the left side $\{a, b, c\}$ with the right side $\{d, e, f\}$. So the Hamilton circuit would need to contain all edges of G_3, but it cannot be a Hamilton circuit because the degrees of c and f are 3, and a vertex can be incident to only 2 edges of a Hamilton circuit. Thus, $h_J = 1$.

(d) The graph K has a Hamilton circuit a, b, c, d, e, a. So it also has a Hamilton path a, b, c, d, e. Thus $h_K = 2$.