1. **Consider the following system of linear algebraic equations :**

$$\begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -38 \\ -34 \\ -20 \end{bmatrix}$$

a) Under what condition does the solution converge with iterative methods?

Coefficient matrix should be diagonally dominant. +5

b) Considering the condition of a), solve the equations using the Gauss-Seidel method with the starting value $\boldsymbol{x}^{(0)} = [\mathbf{0\ 0\ 0}]^T$. Compute $\boldsymbol{x}^{(1)}$and $\boldsymbol{x}^{(2)}$

Step 1: Pivoting $\mathbf{Ax} = \mathbf{b}$

$$\begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -38 \\ -34 \\ -20 \end{bmatrix} \Rightarrow \begin{bmatrix} -8 & 1 & -2 \\ 2 & -6 & -1 \\ -3 & -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20 \\ -38 \\ -34 \end{bmatrix}$$ +5

Step 2: Compute the first iteration $\boldsymbol{x}^{(1)}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{-8}(-20 - 0 + 2 * 0) \\ \frac{1}{-6}(-38 - 2 * (2.5) + 0) \\ \frac{1}{7}(-34 + 3 * (2.5) + 7.1667) \end{bmatrix} = \begin{bmatrix} 2.5 \\ 7.1667 \\ -2.7619 \end{bmatrix}$$ +3

Step 3: Compute the second iteration $\boldsymbol{x}^{(2)}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{-8}(-20 - 7.1667 + 2 * (-2.7619)) \\ \frac{1}{-6}(-38 - 2 * (4.0863) + (-2.7619)) \\ \frac{1}{7}(-34 + 3 * (4.0863) + 8.1558) \end{bmatrix} = \begin{bmatrix} 4.0863 \\ 8.1558 \\ -1.9408 \end{bmatrix}$$ +2

c) In this time, solve the equation using the Gauss-Seidel method with relaxation, ω=1.5 for the relaxation factore and the starting value $\boldsymbol{x}^{(0)} = [\mathbf{0\ 0\ 0}]^T$. Compute $\boldsymbol{x}^{(1)}$and $\boldsymbol{x}^{(2)}$

Step 1: Compute the first iteration $\boldsymbol{x}^{(1)}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1.5}{-8}(-20 - 0 + 2 * 0) + \big(1 - (1.5)\big) * 0 \\ \frac{1.5}{-6}(-38 - 2 * (3.75) + 0) + \big(1 - (1.5)\big) * 0 \\ \frac{1.5}{7}(-34 + 3 * (3.75) + 11.3750) + \big(1 - (1.5)\big) * 0 \end{bmatrix} = \begin{bmatrix} 3.75 \\ 11.3750 \\ -2.4375 \end{bmatrix}$$ +3

Step 2: Compute the second iteration $\boldsymbol{x}^{(2)}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1.5}{-8}(-20 - 11.3750 + 2 * (-2.4375)) + (1 - (1.5)) * 3.75 \\ \frac{1.5}{-6}(-38 - 2 * (4.9219) + (-2.4375)) + (1 - (1.5)) * 11.3750 \\ \frac{1.5}{7}(-34 + 3 * (4.9219) + 6.8828) + (1 - (1.5)) * (-2.4375) \end{bmatrix}$$

$$= \begin{bmatrix} 4.9219 \\ 6.8828 \\ -1.4280 \end{bmatrix}$$

+2

2. Given the following data points, determine the natural cubic spline.

x	-2	-1	0	1	2
y	4	-1	2	1	8

a) Write the equation for the curvatures

$$k_{i-1} + 4 * k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2 * y_i + y_{i+1}) \quad (\mathrm{i} = 1,2,3) \quad \text{or}$$

$$f''_{0,1}(x) = k_1(x+2),\ f''_{i,i+1}(x) = \frac{k_i(x - x_{i+1}) - k_{i+1}(x - x_i)}{x_i - x_{i+1}} (\mathrm{i} = 1,2),$$

$$f''_{3,4}(x) = k_3(-x+2)$$

+5

b) Since $k_0 = 0$ and $k_4 = 0$, the resulting coefficient matrix for k_1, k_2, and k_3 is a tridiagonal matrix. Use the Doolittle's decomposition and find L and U for the coefficient matrix

Step 1: Express the given problem to the matrix form.

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 6 * \begin{bmatrix} y_0 - 2 * y_1 + y_2 \\ y_1 - 2 * y_2 + y_3 \\ y_2 - 2 * y_3 + y_4 \end{bmatrix} = \begin{bmatrix} 48 \\ -24 \\ 48 \end{bmatrix}$$

+5

Step 2: Apply Doolittle's LU decomposition($\mathbf{Ly} = \mathbf{b}$, where $\mathbf{y} = \mathbf{Ux}$).

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\lambda_{21}=0.25} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 3.75 & 1 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\lambda_{32}=0.2667} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 3.75 & 1 \\ 0 & 0 & 3.7333 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0.2667 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 3.75 & 1 \\ 0 & 0 & 3.7333 \end{bmatrix}$$

+10

c) Use the forward and backward substitution and find the values of k_1, k_2, and k_3

Step 1: Evaluate $\mathbf{Ly} = \mathbf{b}$ by forward substitution,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0.2667 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 48 \\ -24 \\ 48 \end{bmatrix}$$

$$y_1 = 48$$

$$0.25y_1 + y_2 = -24 \quad \rightarrow \quad y_2 = -36$$

$$0.2667y_2 + y_3 = 48 \quad \rightarrow \quad y_3 = -57.6$$

$$\mathbf{y} = \begin{bmatrix} 48 \\ -36 \\ 57.6 \end{bmatrix}$$

Step 2: Evaluate $\mathbf{Ux} = \mathbf{y}$ by backward substitution,

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 3.75 & 1 \\ 0 & 0 & 3.7333 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 48 \\ -36 \\ 57.6 \end{bmatrix}$$

$$3.7333x_3 = 57.6 \quad \rightarrow \quad x_3 = 15.4286$$

$$3.75x_2 + x_3 = -36 \quad \rightarrow \quad x_2 = -13.7143$$

$$4x_1 + x_2 = 48 \quad \rightarrow \quad x_1 = 15.4286$$

$$\mathbf{x} = \begin{bmatrix} \mathrm{k}_1 \\ \mathrm{k}_2 \\ \mathrm{k}_3 \end{bmatrix} = \begin{bmatrix} 15.4286 \\ -13.7143 \\ 15.4286 \end{bmatrix}$$

+5

d) Write the cubic spline interpolant equation between knots 3 and 4 and find the cubic spline interpolant at x= 1.5

Cubic spline interpolant equation is

$$f_{3,4}(x) = \frac{k_3}{6}\left[\frac{(x-x_4)^3}{x_3-x_4} - (x-x_4)(x_3-x_4)\right] - \frac{k_4}{6}\left[\frac{(x-x_3)^3}{x_3-x_4} - (x-x_3)(x_3-x_4)\right] + \frac{y_3(x-x_4) - y_4(x-x_3)}{x_3-x_4}$$

Therefore the solution is

$$f_{3,4}(1.5) = \frac{15.4286}{6}\left[\frac{(1.5-2)^3}{1-2} - (1.5-2)(1-2)\right] + \frac{1(1.5-2) - 8(1.5-1)}{1-2} = 3.5357$$

+5

3. Solve the following nonlinear system using the Newton-Raphson method

$$x^2 + y^2 - y + z^2 - 4 = 0$$
$$x^2 - x + y^2 + z^2 - 5 = 0$$
$$x^2 + y^2 + z^2 + z - 6 = 0$$

a) Find the function vector F(x,y,z) and the Jacobian matrix J(x,y,z) for the system

$$\mathbf{F}(x,y,z) = \begin{bmatrix} x^2 + y^2 - y + z^2 - 4 \\ x^2 - x + y^2 + z^2 - 5 \\ x^2 + y^2 + z^2 + z - 6 \end{bmatrix}$$

+2

$$\mathbf{J}(x,y,z) = \begin{bmatrix} 2x & 2y-1 & 2z \\ 2x-1 & 2y & 2z \\ 2x & 2y & 2z+1 \end{bmatrix}$$

+5

b) Write the algorithm of the Newton-Raphson method for finding the solution of a system of equation

Step 1. Choose initial guess of **x** = (x,y,z)

Step 2. Evaluate **F**(**x**)

Step 3. Compute **J**(**x**)

Step 4. Compute Δ**x** using **J**(**x**) Δ**x** = -**F**(**x**)

Step 5. Set **x** ← **x** + Δ**x**

Repeat 2-5 until |Δ**x**| ≤ Error tolerance

+10

c) Using Newton-Raphson method with the starting value $x^{(0)}=(1,2,0)$, compute $x^{(1)}$ and $x^{(2)}$, where x = (x,y,z). For this purpose, use the Gauss elimination method.

Step 1. First Iteration

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad \mathbf{J}(\mathbf{x}) = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

Gauss Elimination

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ 1 & 4 & 0 & 1 \\ 2 & 4 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ 0 & 5/2 & 0 & 1/2 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ 0 & 2.5 & 0 & 0.5 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ 0 & 5/2 & 0 & 1/2 \\ 0 & 0 & 1 & -1/5 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ 0 & 2.5 & 0 & 0.5 \\ 0 & 0 & 1 & -0.2 \end{array}\right]$$

Back substitution

$2\Delta x + 3\Delta y = 1$

$\frac{5}{2}\Delta y = 1/2$

$\Delta z = -1/5$

$$\Delta\mathbf{x} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/5 \\ -1/5 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \\ -0.2 \end{bmatrix}$$

$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \Delta\mathbf{x} = (6/5,\ 11/5,\ -1/5) = (1.2,\ 2.2,\ -0.2)$

+10

Step 2. Second Iteration

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 3/25 \\ 3/25 \\ 3/25 \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.12 \\ 0.12 \end{bmatrix} \qquad \mathbf{J}(\mathbf{x}) = \begin{bmatrix} 2.4 & 3.4 & -0.4 \\ 1.4 & 4.4 & -0.4 \\ 2.4 & 4.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 12/5 & 17/5 & -2/5 \\ 17/5 & 22/5 & -2/5 \\ 12/5 & 22/5 & 3/5 \end{bmatrix}$$

+2

Gauss Elimination

$$\left[\begin{array}{ccc|c} 12/5 & 17/5 & -2/5 & -3/25 \\ 7/5 & 22/5 & -2/5 & -3/25 \\ 12/5 & 22/5 & 3/5 & -3/25 \end{array}\right] = \left[\begin{array}{ccc|c} 2.4 & 3.4 & -0.4 & -0.12 \\ 1.4 & 4.4 & -0.4 & -0.12 \\ 2.4 & 4.4 & 0.6 & -0.12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 12/5 & 17/5 & -2/5 & -3/25 \\ 0 & 29/12 & -1/6 & -1/20 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2.4 & 3.4 & -0.4 & -0.12 \\ 0 & 2.416667 & -0.166667 & -0.05 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 12/5 & 17/5 & -2/5 & -3/25 \\ 0 & 29/12 & -1/6 & -1/20 \\ 0 & 0 & 31/29 & 3/145 \end{array}\right] = \left[\begin{array}{ccc|c} 2.4 & 3.4 & -0.4 & -0.12 \\ 0 & 2.416667 & -0.166667 & -0.05 \\ 0 & 0 & 1.068966 & 0.206897 \end{array}\right]$$

Back substitution

$$\frac{12}{5}\Delta x + \frac{17}{5}\Delta y - \frac{2}{5}\Delta z = -\frac{3}{25}$$

$$\frac{29}{12}\Delta y - \frac{1}{6}\Delta z = -\frac{1}{20}$$

$$\frac{31}{29}\Delta z = \frac{3}{145}$$

$$\Delta\mathbf{x} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -3/155 \\ -3/155 \\ 3/155 \end{bmatrix} = \begin{bmatrix} -0.019355 \\ -0.019355 \\ 0.019355 \end{bmatrix}$$

$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \Delta\mathbf{x}$ = (183/155, 338/155, -28/155) = (1.1806, 2.1806, -0.1806) +1

4. Linear regression can be extended to data that depend on two or more variables (called multiple linear regression). If the dependent variable is z and independent variables are x and y, the data to be fitted have the form

x_1	y_1	z_1
x_2	y_2	z_2
x_3	y_3	z_3
…	…	…
x_n	y_n	z_n

Instead of a straight line, the fitting function now represents a plane: f(x,y) = a + bx + cy + dxy

a) Write the function S(a,b,c,d) to be minimized

S(a,b,c,d) = $\sum_{i=1}^{n}[z_i - a - bx_i - cy_i - dx_i y_i]^2$ +5

b) Drive the normal equations for the coefficients and explain how to solve the equations.

$$\frac{\partial S}{\partial a} = \sum -2[z_i - a - bx_i - cy_i - dx_i y_i] = 0$$

$\rightarrow\ na + b\sum x_i + c\sum y_i + d\sum x_i y_i = \sum z_i$ +2

$$\frac{\partial S}{\partial b} = \sum -2[z_i - a - bx_i - cy_i - dx_i y_i]x_i = 0$$

$\rightarrow\ a\sum x_i + b\sum x_i^2 + c\sum x_i y_i + d\sum x_i^2 y_i = \sum x_i z_i$ +2

$$\frac{\partial S}{\partial c} = \sum -2[z_i - a - bx_i - cy_i - dx_i y_i]y_i = 0$$

$\rightarrow\ a\sum y_i + b\sum x_i y_i + c\sum y_i^2 + d\sum x_i y_i^2 = \sum y_i z_i$ +2

$$\frac{\partial S}{\partial d} = \sum -2[z_i - a - bx_i - cy_i - dx_i y_i]x_i y_i = 0$$

$\rightarrow\ a\sum x_i y_i + b\sum x_i^2 y_i + c\sum x_i y_i^2 + d\sum x_i^2 y_i^2 = \sum x_i y_i z_i$ +2

$$na + b\sum x_i + c\sum y_i + d\sum x_i y_i = \sum z_i$$

$$a\sum x_i + b\sum x_i^2 + c\sum x_i y_i + d\sum x_i^2 y_i = \sum x_i z_i$$

$$a\sum y_i + b\sum x_i y_i + c\sum y_i^2 + d\sum x_i y_i^2 = \sum y_i z_i$$

$$a\sum x_i y_i + b\sum x_i^2 y_i + c\sum x_i y_i^2 + d\sum x_i^2 y_i^2 = \sum x_i y_i z_i$$

This is a system of linear equations consists of four equations and 4 unknowns.

Using matrix form,

$$\begin{bmatrix} n & \sum x_i & \sum y_i & \sum x_i y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i & \sum x_i^2 y_i \\ \sum y_i & \sum x_i y_i & \sum y_i^2 & \sum x_i y_i^2 \\ \sum x_i y_i & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i^2 y_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \sum z_i \\ \sum x_i z_i \\ \sum y_i z_i \\ \sum x_i y_i z_i \end{bmatrix}$$

+5

The derived normal equation is $\mathbf{Ax} = \mathbf{b}$ form, therefore it can be solved by using inverse matrix, Gauss elimination or LU decomposition, etc. +2