CSE232: Discrete Mathematics
Assignment 5: Suggested answers

December 10, 2020

Q1. For any integer $n \geqslant 2$, let

$$a_n = \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right).$$

Prove by mathematical induction that $a_n = \frac{1}{n}$ for all $n \geqslant 2$.

Answer. Let $P(n)$ be the property that $a_n = 1/n$. We want to prove that $P(n)$ is true for all $n \geqslant 2$.

- **Basis step.** We observe that

$$a_2 = \prod_{i=2}^{2} \left(1 - \frac{1}{i}\right) = 1 - \frac{1}{2} = \frac{1}{2},$$

so $P(2)$ is true.

- **Inductive step.** Suppose that $P(k)$ is true for $k \geqslant 2$. It follows that

$$\begin{aligned}
a_{k+1} = \prod_{i=2}^{k+1} \left(1 - \frac{1}{i}\right) &= \left(1 - \frac{1}{k+1}\right) \cdot \prod_{i=2}^{k} \left(1 - \frac{1}{i}\right) \\
&= \frac{k}{k+1} a_k && \text{by definition of } a_k \\
&= \frac{k}{k+1} \cdot \frac{1}{k} && \text{by the inductive hypothesis} \\
&= \frac{1}{k+1}
\end{aligned}$$

Therefore, $P(k+1)$ is true. $\square$

Q2. Prove the following statements.

(a) Prove that $10^k \equiv 1 \pmod 3$ for all $k \in \mathbb{N}$.

(b) Let $n = (a_k \dots a_1 a_0)_{10}$ be an integer with $k+1$ digits $a_k, \dots a_1$ and a_0. Prove that $3 \mid n$ if and only if $3 \mid \sum_{i=0}^{k} a_i$.

Answer a. We prove it by mathematical induction. So $P(k)$ is the property $10^k \equiv 1 \pmod 3$.

- **Basis step.** $10^0 = 1$ so $10^0 \equiv 1 \pmod 3$. So $P(0)$ is true.
- **Inductive step.** Suppose that $P(k)$ is true. Then

$$\begin{aligned} 10^{k+1} &\equiv 10 \times 10^k \pmod 3 & \\ &\equiv 10 \times 1 \pmod 3 & \text{by the inductive hypothesis} \\ &\equiv 1 \times 1 \pmod 3 & \because 10 \equiv 1 \pmod 3 \\ &\equiv 1 \pmod 3 & \end{aligned}$$

Answer b. We have $n = \sum_{i=0}^{k} a_i 10^i$. As we proved in (a) that $10^i \equiv 1 \pmod 3$, we have $n \equiv \sum_{i=0}^{k} a_i \pmod 3$. So $3 \mid n$ if and only if $n \equiv 0 \pmod 3$, which means that $\sum_{i=0}^{k} a_i \equiv 0 \pmod 3$, or equivalently, $3 \mid \sum_{i=0}^{k} a_i$.

Q3. Let S be the subset of the set of ordered pairs of integers defined recursively by:

- Basis step: $(0, 0) \in S$.
- Recursive step: If $(a, b) \in S$, then $(a, b+1) \in S$, $(a+1, b+1) \in S$, and $(a+2, b+1) \in S$.

(a) List the elements of S produced by the first four applications of the recursive definition (Hint: there are 24 pairs in total).

(b) Use strong induction on the number of applications of the recursive step of the definition to show that $a \leqslant 2b$ whenever $(a, b) \in S$.

Answer a. The list of the elements produced by four applications of the recursive definition is:

$$\begin{aligned} \{&(0,1),(1,1),(2,1),\\ &(0,2),(1,2),(2,2),(3,2),(4,2),\\ &(0,3),(1,3),(2,3),(3,3),(4,3),(5,3),(6,3),\\ &(0,4),(1,4),(2,4),(3,4),(4,4),(5,4),(6,4),(7,4),(8,4)\} \end{aligned}$$

Answer b. Let $P(n)$ be the statement that $a \leqslant 2b$ whenever $(a, b) \in S$ is obtained by n applications of the recursive step.

- **Basis step**: $P(0)$ is true, because the only element of S obtained with no applications of the recursive step is $(0, 0)$, and indeed $0 \leqslant 2 \cdot 0$.
- **Inductive step**: Assume that $a \leqslant 2b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step. We consider an element obtained with $k+1$ applications of the recursive step. Because the final application of the recursive step to an element (a, b) must be applied to an element obtained with fewer applications of the recursive step, we know that $a \leqslant 2b$. Add $0 \leqslant 2, 1 \leqslant 2$, and $2 \leqslant 2$, respectively, to obtain $a \leqslant 2(b+1)$, $a+1 \leqslant 2(b+1)$, and $a+2 \leqslant 2(b+1)$, as desired.

Q4. Use structural induction to show that $n(T) \geqslant 2h(T) + 1$, where T is a full binary tree, $n(T)$ equals the number of vertices of T, and $h(T)$ is the height of T.

Answer.

- **Basis step:** It holds for the tree T consisting of a single node, because by definition it has height $h(T) = 0$ and it has $n(T) = 1$ vertex.
- **Inductive step:** If it holds for two full binary trees T_1, T_2, then the proof below shows that it holds for the tree $T = T_1 \cdot T_2$ obtained by applying the recursive step.

$$\begin{aligned}
n(T_1 \cdot T_2) &= 1 + n(T_1) + n(T_2) && \text{by the recursive definition of } n(T) \\
&\geqslant 1 + 2h(T_1) + 1 + 2h(T_2) + 1 && \text{by IH} \\
&= 1 + 2(h(T_1) + h(T_2) + 1) \\
&\geqslant 1 + 2\big(\max\{h(T_1), h(T_2)\} + 1\big) && \because h(T_1) \geqslant 0 \text{ and } h(T_2) \geqslant 0 \\
&= 1 + 2h(T_1 \cdot T_2) && \text{by the recursive definition of } h(T)
\end{aligned}$$

□

Q5. Does there exist a graph with the following vertex set V? If there exists, you describe or draw such a graph; otherwise, explain why there is not.

(a) $V = \{a, b, c, d\}$ such that $\deg(a) = \deg(b) = 3$ and $\deg(c) = \deg(d) = 2$?

(b) $V = \{a, b, c, d\}$ such that $\deg(a) = \deg(b) = \deg(c) = 3$ and $\deg(d) = 2$?

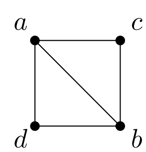


Figure 1: Answer to Question 5(a).

Answer a.

Answer b. No, because the sum of the degrees is 11, but the handshaking theorem implies that it should be even.

Q6. Determine if the following graphs are isomorphic.

(a) Are graph G_1 and G_2 in Figure 2 isomorphic? If so, provide an isomorphism, and if not, give a proof that they are not isomorphic.

(b) Are graph G_3 and G_4 in Figure 2 isomorphic? If so, provide an isomorphism, and if not, give a proof that they are not isomorphic.

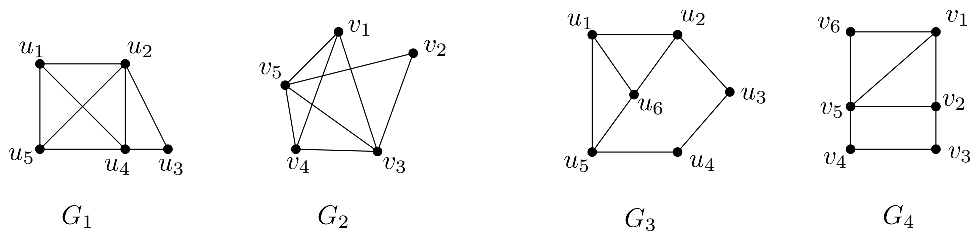


Figure 2: Graphs in Question 6.

Answer a. G_1 and G_2 are isomorphic. An isomorphism is given by

$$f(u_1) = v_1 \quad f(u_2) = v_5 \quad f(u_3) = v_2 \quad f(u_4) = v_3 \quad f(u_5) = v_4.$$

Answer b. G_3 and G_4 are not isomorphic, because $\deg(v_5) = 4$, but no node of G_3 has degree 4.

Q7. The complementary graph $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in G if and only if they are not adjacent in G.
Describe or draw each of these graphs of (a),(b), and (c). Prove the statement of (d).

(a) $\overline{K_5}$

(b) $\overline{K_{3,2}}$

(c) $\overline{C_5}$

(d) Show that if G is a simple graph with n vertices, then the union of G and $\overline{G}$ is K_n.

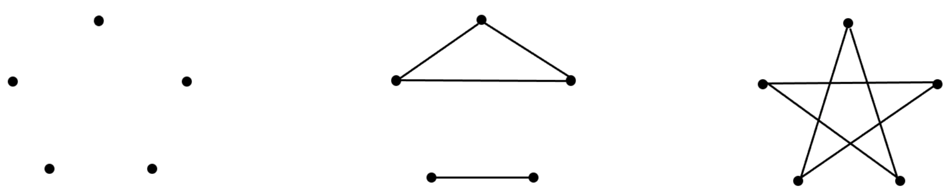


Figure 3: Answer to Question 7(a),(b),(c)

Answer a. The graph with 5 vertices and no edges.

Answer b. The disjoint union of K_3 and K_2.

Answer c. The graph with vertices $\{v_1, \ldots, v_5\}$ with an edge between v_i and v_j unless $i \equiv j \pm 1 \pmod{n}$.

Answer d. The union of G and $\overline{G}$ contains an edge between each pair of the n vertices. Hence, this union is K_n.

Q8. Answer the following questions.

(a) Draw an undirected graph represented by the given adjacency matrix:

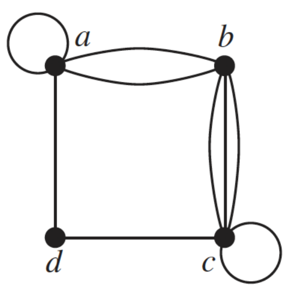
$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(b) Draw a directed graph represented by the given adjacency matrix:

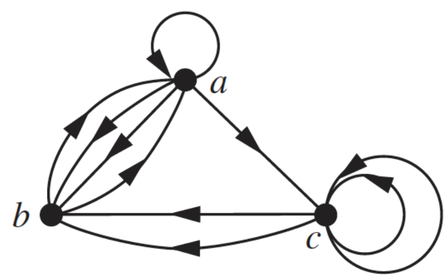
$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

(c) What is the sum of the entries in a row of the adjacency matrix for a directed graph?

(d) What is the sum of the entries in a column of the adjacency matrix for a directed graph?



(a) Graph of 8(a)



(b) Graph of 8(b)

Figure 4: Answer to Question 8(a) and 8(b).

Answer c. $\deg^+(v)$ for any vertex v

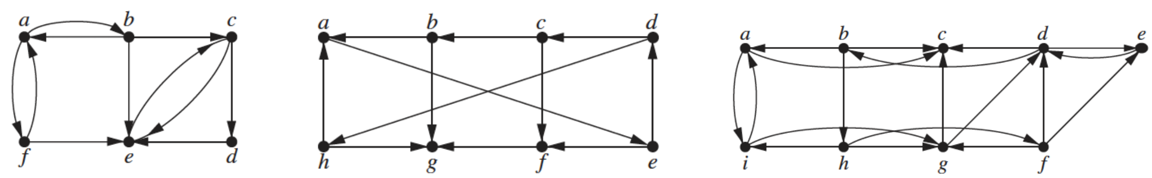
Answer d. $\deg^-(v)$ for any vertex v

Q9. Find the strongly connected components of each of these graphs.

Answer a. $\{a, b, f\}$, $\{c, d, e\}$

Answer b. $\{a, b, c, d, e, h\}$, $\{f\}$, $\{g\}$

Answer c. $\{a, b, d, e, f, , g, h, i\}$, $\{c\}$



(a) Graph of 9(a) (b) Graph of 9(b) (c) Graph of 9(c)

Figure 5: Graphs in Question 9(a),9(b), and 9(c).