1. (20 points) We can derive the central finite difference approximation for $f'(x)$ accurate to $O(h^4)$.

① Derive a central difference approximation of $f'(x)$ accurate to $O(h^4)$ using Taylor series expansion only

(hint : $f(x+h)=f(x)+hf'(x)+\cdots,\quad f(x-h)=f(x)-hf'(x)+\cdots,$

$f(x+2h)=f(x)+2hf'(x)+\cdots,$ and $f(x-2h)=f(x)-2hf'(x)+\cdots$).

Sol)

$$f(x+h)=f(x)+hf'(x)+\frac{1}{2!}h^2f''(x)+\frac{1}{3!}h^3f'''(x)+\frac{1}{4!}h^4f^{(iv)}(x)+\cdots \quad \text{- Eq.①}$$

$$f(x-h)=f(x)-hf'(x)+\frac{1}{2!}h^2f''(x)-\frac{1}{3!}h^3f'''(x)+\frac{1}{4!}h^4f^{(iv)}(x)+\cdots \quad \text{- Eq.②}$$

$$f(x+2h)=f(x)+2hf'(x)+\frac{1}{2!}(2h)^2f''(x)+\frac{1}{3!}(2h)^3f'''(x)+\frac{1}{4!}(2h)^4f^{(iv)}(x)+\cdots \quad \text{- Eq.③}$$

$$f(x-2h)=f(x)-2hf'(x)+\frac{1}{2!}(2h)^2f''(x)-\frac{1}{3!}(2h)^3f'''(x)+\frac{1}{4!}(2h)^4f^{(iv)}(x)+\cdots \quad \text{- Eq.④}$$

+4

Eq.① - Eq.② becomes

$$f(x+h)-f(x-h)=2hf'(x)+\frac{2}{3!}h^3f'''(x)+O(h^5)+\cdots \quad \text{- Eq.⑤}$$

Eq.③ - Eq.④ becomes

$$f(x+2h)-f(x-2h)=4hf'(x)+\frac{16}{3!}h^3f'''(x)+O(h^5)+\cdots \quad \text{- Eq.⑥}$$

+4

Therefore, 8* Eq.⑤ - Eq.⑥ is

$$-f(x+2h)+8f(x+h)-8(x-h)+f(x-2h)=12hf'(x)+O(h^5)+\cdots$$

$$\therefore f'(x)=\frac{-f(x+2h)+8f(x+h)-8(x-h)+f(x-2h)}{12h}+O(h^4)+\cdots$$

+2

② **In this time, derive the central difference approximation for $f'(x)$ accurate to $O(h^4)$ by applying Richardson extrapolation to the central difference approximation of $O(h^2)$**

Sol)

$$g(h) = \frac{f(x+h) + f(x-h)}{2h}$$

$$g(2h) = \frac{f(x+2h) + f(x-2h)}{2(2h)}$$

+4

Richardson extrapolation is

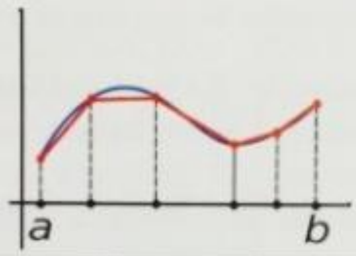
$$G = \frac{(2)^2 g(h) - g(2h)}{(2)^2 - 1} = \frac{4g(h)}{3} - \frac{g(2h)}{3}$$
$$= \frac{4f(x+h) + 4f(x-h)}{6h} - \frac{f(x+2h) + f(x-2h)}{12h}$$
$$= \frac{8f(x+h) + 8f(x-h)}{12h} - \frac{f(x+2h) + f(x-2h)}{12h}$$

+4

$$\therefore f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8(x-h) + f(x-2h)}{12h} + O(h^4) + \cdots$$

+2

2. (20 points) Let's consider the following data set with a non-uniform grid.



x	$x_1 = a$	x_2	$\cdots$	x_i	$\cdots$	$x_N = b$
f	f_1	f_2	$\cdots$	f_i	$\cdots$	f_N

① Derive the approximate area of a typical (i-th) panel using the trapezoidal rule.

$$I_i = \int_{x_i}^{x_{i+1}} f(x_i)l_i(x) + f(x_{i+1})l_{i+1}(x)dx = f(x_i)\int_{x_i}^{x_{i+1}} l_i(x) + f(x_{i+1})\int_{x_i}^{x_{i+1}} l_{i+1}(x) = f(x_i)\frac{1}{2}h_i + f(x_{i+1})\frac{1}{2}h_i$$

$$= [f(x_i) + f(x_{i+1})]\frac{h_i}{2}$$

$$\therefore I_i = [f(x_i) + f(x_{i+1})]\frac{(x_{i+1} - x_i)}{2}$$

+5

② Based on the above area, derive the total area representing $\int_a^b f(x)dx$.

$$\therefore \int_a^b f(x)dx \approx I = \sum_{i=1}^{N-1} I_i = \sum_{i=1}^{N-1}[f(x_i) + f(x_{i+1})]\frac{(x_{i+1} - x_i)}{2}$$

+5

③ (Programming)

+10

```
data =

         0    0.1000    0.1500    0.3000    0.4000    0.5500    0.6000    0.7000    0.7500    0.8000    0.9000    0.9500    1.0000
         0    0.0100    0.0225    0.0900    0.1600    0.3025    0.3600    0.4900    0.5625    0.6400    0.8100    0.9025    1.0000


ans =

    0.3352
```

3. (30 points) A simple predator-prey model is often used to simulate biological populations. One of the simplest such model is

$$\frac{dy_1}{dt} = \alpha y_1 - \beta y_1 y_2 \qquad \frac{dy_2}{dt} = -\gamma y_2 + \eta y_1 y_2$$

Where y_1 represents the prey population and y_2 is the predator population. The behavior of the solution to this system of equations depends on the set of constants chosen and the initial conditions. Assume α =1, β =0.01, γ =1 and η =0.001. Initial conditions are y_1(0)=1100 and y_2(0)=120.

① Use the modified Euler's method (RK2) to integrate the system of equations from t=0 to 0.2 in steps of h=0.1

Sol)

Modified Euler's method is

$$\left.\begin{aligned} \vec{y}^{n+1} &= \vec{y}^{n} + \vec{K}_2 \\ \vec{K}_1 &= h\vec{F}(t,\vec{y}^{n}) \\ \vec{K}_2 &= h\vec{F}(t+\frac{h}{2},\vec{y}^{n}+\frac{\vec{K}_1}{2}) \end{aligned}\right\} +4$$

i) First step

$$\vec{y}(0) = \begin{bmatrix} 1100 \\ 120 \end{bmatrix}$$

$$\vec{K}_1 = h\vec{F}(t,\vec{y}) = 0.1 * \begin{bmatrix} 1*1100 - 0.01*1100*120 \\ -1*120 + 0.001*1100*120 \end{bmatrix} = \begin{bmatrix} -22 \\ 1.2 \end{bmatrix}$$

$$\vec{K}_2 = h\vec{F}(t+\frac{h}{2},\begin{bmatrix} 1089 \\ 120.6 \end{bmatrix}) = 0.1 * \begin{bmatrix} 1*1089 - 0.01*1089*120.6 \\ -1*120.6 + 0.001*1089*120.6 \end{bmatrix} = \begin{bmatrix} -22.43 \\ 1.07 \end{bmatrix}$$

$$\therefore \vec{y}(0.1) = \vec{y}(0) + \vec{K}_2 = \begin{bmatrix} 1100 \\ 120 \end{bmatrix} + \begin{bmatrix} -22.43 \\ 1.07 \end{bmatrix} = \begin{bmatrix} 1077.57 \\ 121.07 \end{bmatrix} \qquad +2$$

ii) Second step

$$\vec{y}(0.1) = \begin{bmatrix} 1077.57 \\ 121.07 \end{bmatrix}$$

$$\vec{K}_1 = h\vec{F}(t, \vec{y}) = 0.1 * \begin{bmatrix} 1*1077.57 - 0.01*1077.57*121.07 \\ -1*121.07 + 0.001*1077.57*121.07 \end{bmatrix} = \begin{bmatrix} -22.70 \\ 0.94 \end{bmatrix}$$

$$\vec{K}_2 = h\vec{F}(t + \frac{h}{2}, \begin{bmatrix} 1089 \\ 120.6 \end{bmatrix}) = 0.1 * \begin{bmatrix} 1*1089 - 0.01*1089*120.6 \\ -1*120.6 + 0.001*1089*120.6 \end{bmatrix} = \begin{bmatrix} -22.97 \\ 0.80 \end{bmatrix}$$

$$\therefore \vec{y}(0.2) = \vec{y}(0.1) + \vec{K}_2 = \begin{bmatrix} 1077.57 \\ 121.07 \end{bmatrix} + \begin{bmatrix} -22.97 \\ 0.80 \end{bmatrix} = \begin{bmatrix} 1054.6 \\ 121.87 \end{bmatrix}$$

+2

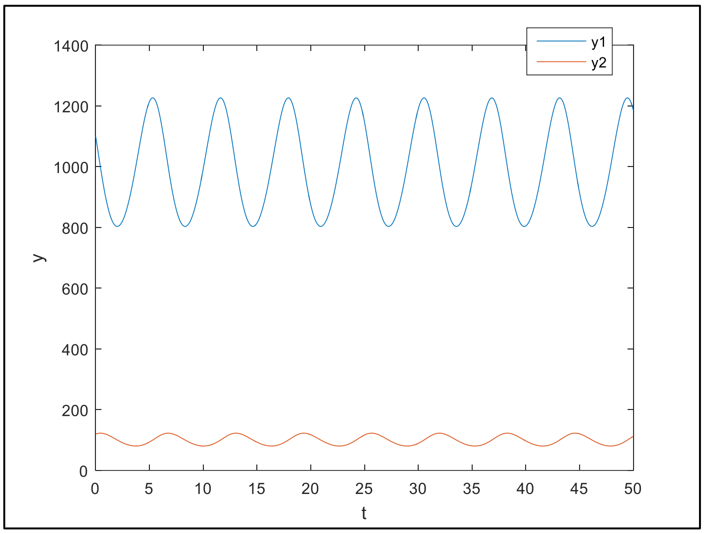
② In this time, solve the equation using the backward Euler method (hint : $\vec{y}^{n+1} = \vec{y}^{n} + \Delta t\, \vec{f}(t, \vec{y}^{n+1})$). Derive two nonlinear algebraic equations at time of n+1 as functions of y_1^{n+1}, y_2^{n+1}, y_1^{n} and y_2^{n}. Explain how to solve the equations.

Sol)

$$\left.\begin{aligned} y_1^{n+1} &= y_1^{n} + \Delta t(\alpha y_1^{n+1} - \beta y_1^{n+1} y_2^{n+1}) \\ y_2^{n+1} &= y_2^{n} + \Delta t(-\gamma y_2^{n+1} + \eta y_1^{n+1} y_2^{n+1}) \end{aligned}\right\} +4$$

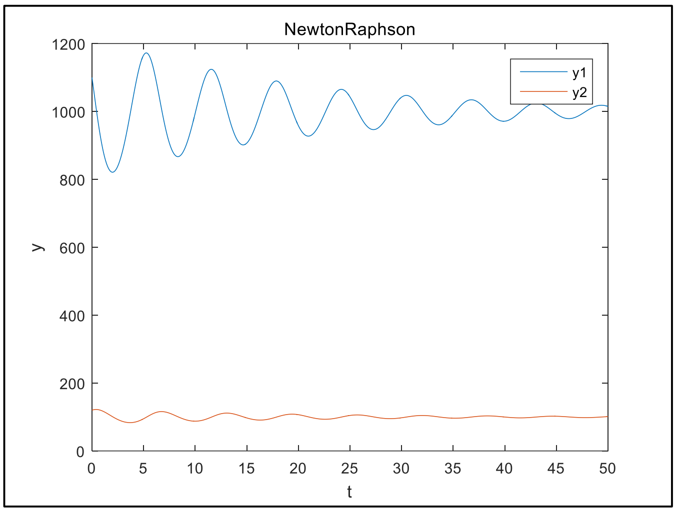
These are nonlinear algebraic equations such that we can find solutions of y_1^{n+1} and y_2^{n+1} using a root-finding technique (e.g. the Newton-Raphson method) for given y_1^{n} and y_2^{n} +3

③ **(Programming) Using runKut4, integrate the system from t=0 to 50 in steps of h=0.1. Plot y_1 and y_2 as a function of time using 'plot' function.**



+5

④ **(Programming) Using the backward Euler method, integrate the system from t=0 to 50 in steps of h=0.1. Plot y_1 and y_2 as a function of time using 'plot' function.**



+10

4. ①

a)

$$\begin{aligned} y_1 &= f \\ y_2 &= f' \\ y_3 &= f'' \\ y_4 &= g \\ y_5 &= \theta \\ y_6 &= \theta' \end{aligned} \quad \rightarrow \quad \left.\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -\frac{3}{5}y_1y_3 + \frac{1}{5}y_2{}^2 + \frac{2}{5}(y_4 - \eta y_5) \\ y_4' &= y_5 \\ y_5' &= y_6 \\ y_6' &= -\frac{3}{5}\mathrm{Pr}\, y_1y_6 \end{aligned}\right\} +6$$

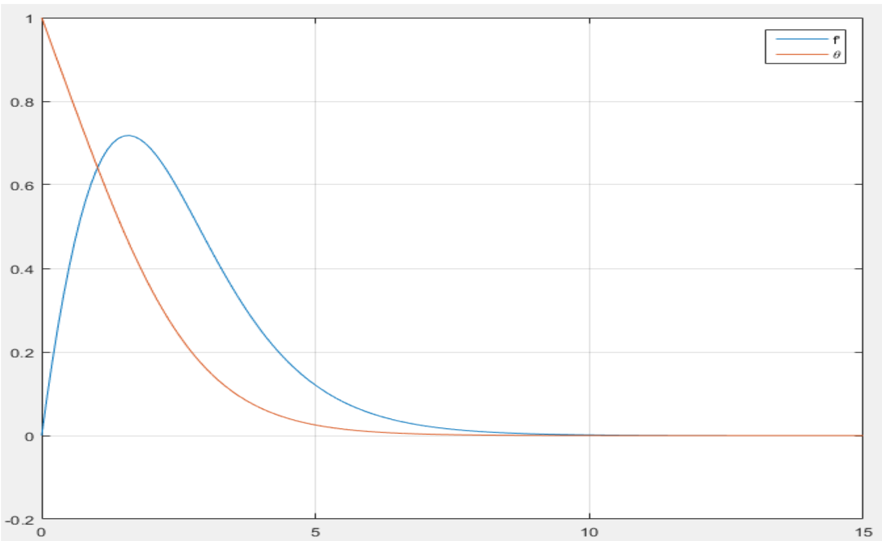
b)

$$\left.\begin{aligned} y_1(0) &= 0 \\ y_2(0) &= 0 \\ y_3(0) &= u(1) \\ y_4(0) &= u(2) \\ y_5(0) &= 1 \\ y_6(0) &= u(3) \end{aligned}\right\} +6$$

c)

$$\left.\begin{aligned} r(1) &= y_2(15) - 0 \\ r(2) &= y_4(15) - 0 \\ r(3) &= y_5(15) - 0 \end{aligned}\right\} +3$$

② **(Programming)**



+15