Problem 1

$$f(x-h_1) = f(x) - f'(x)h_1 + \frac{1}{2}f''(x)h_1^2 - \frac{1}{6}f'''(x)h_1^3 + \cdots$$

$$f(x+h_2) = f(x) + f'(x)h_2 + \frac{1}{2}f''(x)h_2^2 + \frac{1}{6}f'''(x)h_2^3 + \cdots$$

Multiplying the first expression by h_2/h_1 and adding it to the second expression yields

$$\frac{h_2}{h_1}f(x-h_1) = \frac{h_2}{h_1}f(x) - f'(x)h_2 + \frac{1}{2}f''(x)\frac{h_2}{h_1}h_1^2 - \frac{1}{6}f'''(x)\frac{h_2}{h_1}h_1^3 + \cdots$$

$$f(x+h_2) = f(x) + f'(x)h_2 + \frac{1}{2}f''(x)h_2^2 + \frac{1}{6}f'''(x)h_2^3 + \cdots$$

$$\frac{h_2}{h_1}f(x-h_1) + f(x+h_2) = \left(\frac{h_2}{h_1}+1\right)f(x) + \frac{1}{2}f''(x)\left(\frac{h_2}{h_1}h_1^2 + h_2^2\right) + \frac{1}{6}f'''(x)\left(h_2^3 - \frac{h_2}{h_1}h_1^3\right) + \cdots$$

$$f''(x) = \frac{\frac{h_2}{h_1}f(x-h_1) - \left(\frac{h_2}{h_1}+1\right)f(x) + f(x+h_2)}{\frac{h_2}{h_1}\left(1+\frac{h_2}{h_1}\right)\frac{h_1^2}{2}} + \mathcal{O}(h)$$

$$h = h_2 - h_1$$

Problem 3

Central difference approximations for $f''(x)$ of $\mathcal{O}(h^2)$ are

$$\begin{aligned} g(h) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \\ g(2h) &= \frac{f(x+2h) - 2f(x) + f(x-2h)}{(2h)^2} \end{aligned}$$

Richardson's extrapolation gives us an approximation of $\mathcal{O}(h^4)$:

$$\begin{aligned} f''(x) &\approx \frac{4g(h) - g(2h)}{4-1} \\ &= \frac{16}{12}\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{1}{12}\frac{f(x+2h) - 2f(x) + f(x-2h)}{h^2} \\ &= \frac{-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)}{12h^2} \quad \blacktriangleleft \end{aligned}$$

Problem 5

Taylor series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \cdots \qquad \text{(a)}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \cdots \qquad \text{(b)}$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + \cdots \qquad \text{(c)}$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2!}f''(x) - \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f^{(4)}(x) - \cdots \qquad \text{(d)}$$

Eq. (a) + Eq. (b):

$$f(x+2h) + f(x-2h) = 2f(x) + 4h^2f''(x) + \frac{4h^4}{3}f^{(4)}(x) + \cdots \qquad \text{(e)}$$

Eq. (c) + Eq. (d):

$$f(x+h) + f(x-h) = 2f(x) + h^2f''(x) + \frac{h^4}{12}f^{(4)}(x) + \cdots \qquad \text{(f)}$$

Eq. (e) $-4\times$ Eq. (f):

$$f(x+2h) + f(x-2h) - 4[f(x+h) + f(x-h)] = -6f(x) + h^4f^{(4)}(x) + \cdots$$

$$f^{(4)}(x) \approx \frac{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)}{h^4}$$

Problem 8

x	0.84	0.92	1.00	1.08	1.16
$f(x)$	0.431711	0.398519	0.367879	0.339596	0.313486

Central difference approximations for $f''(x)$ of $\mathcal{O}(h^2)$ are

$$g(h) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$g(2h) = \frac{f(x+2h) - 2f(x) + f(x-2h)}{(2h)^2}$$

At $x = 1.0$ we get

$$g(0.08) = \frac{0.339596 - 2(0.367879) + 0.398519}{0.08^2} = 0.36828$$

$$g(0.16) = \frac{0.313486 - 2(0.367879) + 0.431711}{0.16^2} = 0.36871$$

Richardson's extrapolation gives us an approximation of $\mathcal{O}(h^4)$:

$$f''(x) \approx \frac{4g(0.08) - g(0.16)}{4-1} = \frac{4(0.36828) - 0.36871}{3} = 0.36814$$