PROBLEM SET 7.1

Problem 1

The integration formula is

$$y(x+h) = y(x) + y'(x)h + \frac{1}{2}y''(x)h^2$$

where

$$y'(x) = -4y + x^2 \qquad y''(x) = -4y' + 2x \qquad h = 0.05$$

First integration step:

$$y(0) = 1 \qquad y'(0) = -4 \qquad y''(0) = 16$$

$$y(0.05) = 1 + (-4)(0.05) + \frac{1}{2}(16)(0.05)^2 = 0.82$$

Second integration step:

$$\begin{aligned} y'(0.05) &= -4(0.82) + 0.05^2 = -3.2775 \\ y''(0.05) &= -4(-3.2775) + 2(0.05) = 13.21 \end{aligned}$$

$$y(0.1) = 0.82 + (-3.2775)(0.05) + \frac{1}{2}(13.21)(0.05)^2 = 0.6726 \blacktriangleleft$$

Using a single step of the 4th-order method in Example 1 we had $y(0.1) = 0.6707$. Comparing with the true solution 0.6706, we see that one step of the 4th-order method is more accurate than two steps of the 2nd-order method.

Problem 2

$$y' = F(x, y) = -4y + x^2$$

(a)

$$\begin{aligned} K_0 &= hF(x, y) = 0.1(-4) = -0.4 \\ K_1 &= hF\left(x + \frac{h}{2}, y + \frac{1}{2}K_0\right) \\ &= 0.1\left[-4\left(1 + \frac{-0.4}{2}\right) + \left(0 + \frac{0.1}{2}\right)^2\right] = -0.31975 \\ y(0.1) &= y(0) + K_1 = 1 + (-0.31975) = 0.68025 \blacktriangleleft \end{aligned}$$

(b)

$$\begin{aligned}
K_0 &= hF(x,y) = -0.4 \\
K_1 &= hF\left(x+\frac{h}{2}, y+\frac{1}{2}K_0\right) = -0.31975 \\
K_2 &= hF\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right) \\
&= 0.1\left[-4\left(1+\frac{-0.31975}{2}\right)+\left(0+\frac{0.1}{2}\right)^2\right] = -0.3358 \\
K_3 &= hF\left(x+h, y+K_2\right) \\
&= 0.1\left[-4(1-0.3358)+(0+0.1)^2\right] = -0.26468 \\
y(0.1) &= y(0)+\frac{1}{6}(K_0+2K_1+2K_2+K_3) \\
&= 1+\frac{1}{6}\left[-0.4+2(-0.31975)+2(-0.3358)+(-0.26468)\right] \\
&= 0.6707 \blacktriangleleft
\end{aligned}$$

The result agrees with the analytical solution.

Problem 3

The integration formula is

$$y(x+h) = y(x) + y'(x)h + \frac{1}{2}y''(x)h^2$$

where

$$y'(x) = \sin y \qquad y''(x) = \cos y \cdot y' = \cos y \sin y \qquad h = 0.1$$

Step 1:

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= \sin(1) = 0.841\,471 \\
y''(0) &= \cos(1)\sin(1) = 0.454\,649
\end{aligned}$$

$$y(0.1) = 1 + (0.841\,471)\,(0.1) + \frac{1}{2}\,(0.454\,649)\,(0.1)^2 = 1.086\,42$$

Step 2:

$$\begin{aligned}
y'(0.1) &= \sin(1.086\,42) = 0.884\,966 \\
y''(0.1) &= \cos(1.086\,42)\sin(1.086\,42) = 0.412\,09
\end{aligned}$$

$$y(0.2) = 1.086\,42 + (0.884\,966)(0.1) + \frac{1}{2}(0.412\,09)(0.1)^2 = 1.176\,977$$

Step 3:

$$\begin{aligned} y'(0.2) &= \sin(1.176\,977) = 0.923\,45 \\ y''(0.2) &= \cos(1.176\,977)\sin(1.176\,977) = 0.354\,345 \end{aligned}$$

$$y(0.3) = 1.176\,977 + (0.923\,45)(0.1) + \frac{1}{2}(0.354\,345)(0.1)^2 = 1.271\,094$$

Step 4:

$$\begin{aligned} y'(0.3) &= \sin(1.271\,094) = 0.955\,424 \\ y''(0.3) &= \cos(1.271\,094)\sin(1.271\,094) = 0.282\,076 \end{aligned}$$

$$y(0.4) = 1.271\,094 + (0.955\,424)(0.1) + \frac{1}{2}(0.282\,076)(0.1)^2 = 1.368\,047$$

Step 5:

$$\begin{aligned} y'(0.4) &= \sin(1.368\,047) = 0.979\,517 \\ y''(0.4) &= \cos(1.368\,047)\sin(1.368\,047) = 0.197\,239 \end{aligned}$$

$$y(0.5) = 1.368\,047 + (0.979\,517)(0.1) + \frac{1}{2}(0.197\,239)(0.1)^2 = 1.4670 \quad \blacktriangleleft$$

Using the 2nd-order Runge-Kutta method in Example 7.3 we had $y(0.5) = 1.4664$, which is correct to 4 decimal places. In this problem, the Taylor series method is somewhat less accurate.

Problem 4

$$y' = y^{1/3} \qquad y(0) = 0$$

One solution is clearly $y = 0$. To prove that $y = (2x/3)^{3/2}$ is also a solution, we compute

$$y' = \frac{d}{dx}\left(\frac{2x}{3}\right)^{3/2} = \frac{3}{2}\left(\frac{2x}{3}\right)^{1/2}\frac{2}{3} = \left(\frac{2x}{3}\right)^{1/2} = y^{1/3} \text{ Q.E.D.}$$

(a)

If $y(0) = 0$, the solution $y = 0$ would be produced. Let us try integrating with the 4th-order Runge-Kutta method from $x = 0$ to 1 (only the initial and final values are printed):

```
% problem7_1_4
func = inline('y(1)^(1/3)','x','y');
x = 0; y = 0; xStop = 1; h = 0.01;
[xSol,ySol] = runKut4(func,x,y,xStop,h);
printSol(xSol,ySol,0)
```

```
>>       x              y1
   0.0000e+000   0.0000e+000
   1.0000e+000   0.0000e+000
```

(b)

If $y(0)$ is any non-zero number, the solution $y' = y^{1/3}$ would be produced. With the initial condition $y(0) = 10^{-16}$ the above program results in

```
>>       x              y1
   0.0000e+000   1.0000e-016
   1.0000e+000   5.4025e-001
```

The analytical solution is $y(1) = (2/3)^{3/2} = 0.5443$, so the the numerical solution is not very accurate. The discrepancy is caused by singularity of y'' and higher derivatives at $x = 0$, which in results in a large truncation error in the first integration step.

Problem 5

We use the notation $y = y_1$, $y' = y_2$, $y'' = y_3$ etc.

(a)

$$\ln y' + y = \sin x \qquad y' = \exp(\sin x - y)$$

$$y_1' = \exp(\sin x - y_1) \quad \blacktriangleleft$$

(b)

$$y''y - xy' - 2y^2 = 0 \qquad y'' = \frac{xy'}{y} + 2y$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ xy_2/y_1 + 2y_1 \end{bmatrix} \quad \blacktriangleleft$$

(c)

$$y^{(4)} - 4y'' \left(1 - y^2\right)^{1/2} = 0 \qquad y^{(4)} = 4y'' \left(1 - y^2\right)^{1/2}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ 4y_3 \left(1 - y_1^2\right)^{1/2} \end{bmatrix} \quad \blacktriangleleft$$

(d)

$$(y'')^2 = \left|32y'x - y^2\right| \qquad y'' = \left|32y'x - y^2\right|^{1/2}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ |32y_2x - y_1^2|^{1/2} \end{bmatrix} \quad \blacktriangleleft$$

Problem 6

We use the notation $x = y_1$, $y = y_2$, $\dot{x} = y_3$ and $\dot{y} = y_4$

(a)

$$\ddot{y} = x - 2y \qquad \ddot{x} = y - x$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ y_2 - y_1 \\ y_1 - 2y_2 \end{bmatrix} \blacktriangleleft$$

(b)

$$\ddot{y} = -y(\dot{y}^2 + \dot{x}^2)^{1/4} \qquad \ddot{x} = -x(\dot{y}^2 + \dot{x}^2)^{1/4} - 32$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ -y_1(y_4^2 + y_3^2)^{1/4} - 32 \\ -y_2(y_4^2 + y_3^2)^{1/4} \end{bmatrix} \blacktriangleleft$$

(c)

$$\ddot{y} = (4\dot{x} - t\sin y)^{1/2} \qquad \ddot{x} = (4\dot{y} - t\cos y)/x$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ (4y_4 - t\cos y_2)/y_1 \\ (4y_3 - t\sin y_2)^{1/2} \end{bmatrix} \blacktriangleleft$$

Problem 7

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta$$

With the notation $\theta = y_1$, $\dot{\theta} = y_2$ the equivalent first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -\sin y_1 \end{bmatrix}$$

We release the pendulum from rest at $\theta = 1$, $\tau = 0$ and determine the time it takes for it to return to the starting point for the first time. Hence the initial conditions are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To assure that the integration covers one period, we stop at $\tau = 2.2\pi$ (this is 10% larger than the period for small amplitudes). We use the 4th-order Runge-Kutta method with $h = 0.25$.

```
% problem7_1_7
F = inline('[y(2) -sin(y(1))]','x','y');
x = 0; y = [1 0]; xStop = 2.2*pi; h = 0.25;
[xSol,ySol] = runKut4(F,x,y,xStop,h);
printSol(xSol,ySol,1)
```

The part of the printout that spans the return of the pendulum to the release position is (note the change in the sign of the velocity y_2):

```
>>      x              y1             y2
   6.5000e+000    9.8315e-001    1.6776e-001
   6.7500e+000    9.9892e-001   -4.1972e-002
```

The value of τ at the instant when $d\theta/d\tau = 0$ can be estimated from two-term Taylor series expansion

$$\begin{aligned}
\left.\frac{d\theta}{d\tau}\right|_{6.75+\Delta\tau} &= \left.\frac{d\theta}{d\tau}\right|_{6.75} + \left.\frac{d^2\theta}{d\tau^2}\right|_{6.75} \Delta\tau \\
0 &= -0.041\,972 + (-\sin 0.99892)\,\Delta\tau \\
\Delta\tau &= -0.04991 \\
\tau &= 6.75 - 0.04991 = 6.700 \quad \blacktriangleleft
\end{aligned}$$

Thus the period is $6.700\sqrt{L/g}$ ◀

Problem 8

$$\ddot{y} = g - \frac{c_D}{m}\dot{y}^2$$

With the notation $\theta = y_1$, $\dot{\theta} = y_2$ the equivalent first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ g - (c_D/m)y_2^2 \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Without air resistance it takes approximately 10 s for a 500 m fall (obtained from $t = \sqrt{2g/h}$). With air resistance the time should be considerably longer; we estimate 15 s. The program below uses the 4th-order Runge-Kutta method with $h = 0.5$ s.

```
% problem7_1_8
F = inline('[y(2)  9.80665-0.2028/80*y(2)^2]','x','y');
x = 0; y = [0 0]; xStop = 15; h = 0.5;
[xSol,ySol] = runKut4(F,x,y,xStop,h);
printSol(xSol,ySol,1)
```

Here is a portion of the output:

```
>>       x              y1              y2
   1.2000e+001   4.8180e+002   5.9433e+001
   1.2500e+001   5.1162e+002   5.9828e+001
```

The time t of the 500 m fall is estimated Taylor series:

$$\begin{aligned} y(12.5+\Delta t) &= y(12.5)+y'(12.5)\,\Delta t \\ 500 &= 511.62+59.828\,\Delta t \\ \Delta t &= -0.194 \\ t &= 12.5-0.194=12.306 \text{ s} \quad \blacktriangleleft \end{aligned}$$

Problem 9

$$\ddot{y}=\frac{P(t)}{m}-\frac{k}{m}y \qquad y(0)=\dot{y}(0)=0$$

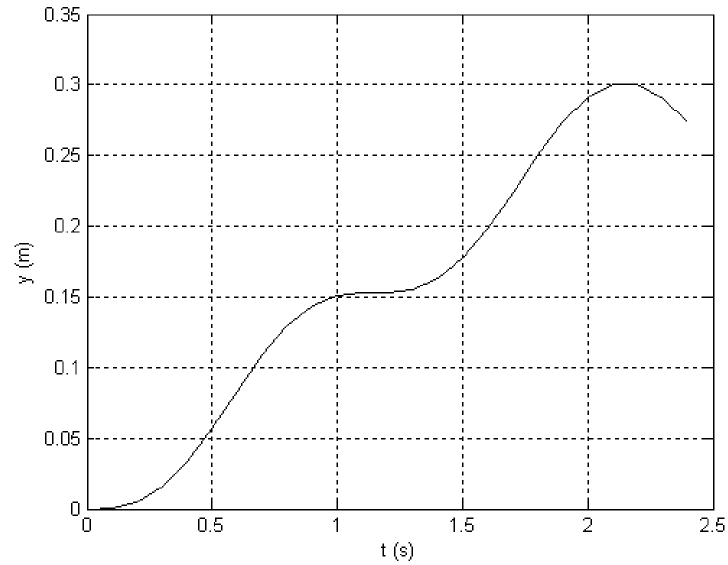
$$P(t)=\begin{cases} 10t \text{ N} & \text{when } t<2 \text{ s} \\ 20 \text{ N} & \text{when } t\geq 2 \text{ s} \end{cases} = 20-(x<2)\left[10(2-x)\right] \text{ N}$$

The equivalent first-order differential equations are

$$\mathbf{F}=\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}=\begin{bmatrix} y_1 \\ P/m-(k/m)\,y_0 \end{bmatrix}=\begin{bmatrix} y_1 \\ 0.4P-30y_0 \end{bmatrix}$$

The maximum displacement should occur soon after P reaches its full value of 20 N at $t=2$ s. We guessed this time to be about 2.4 s.

```
% problem7_1_9
F = inline('[y(2) 0.4*(20-(x<2)*10*(2-x))-30*y(1)]','x','y');
x = 0; y = [0 0]; xStop = 2.4; h = 0.1;
[xSol,ySol] = runKut4(F,x,y,xStop,h);
printSol(xSol,ySol,1)
plot(xSol,ySol(:,1),'k-'); grid on
xlabel('t (s)'); ylabel('y (m)')
```



Here is the printout of the two points spanning the maximum displacement:

```
>>       x              y1              y2
   2.1000e+000   3.0063e-001   5.0283e-002
   2.2000e+000   3.0044e-001  -5.3863e-002
```

At the time when the maximum displacement occurs the velocity is zero. We find this time from the Taylor series

$$\begin{aligned}\dot{y}(2.1+\Delta t) &= \dot{y}(2.1)+\ddot{y}(2.1)\,\Delta t \\ 0 &= 0.050\,283+\frac{20-75(0.3006\,3)}{2.5}\Delta t \\ \Delta t &= 0.049\,35 \\ t &= 2.1+0.049\,35=2.149\text{ s}\end{aligned}$$

Here is the computation for the maximum displacement:

$$\begin{aligned}y_{\max} &= y(2.149)=y(2.1)+\frac{1}{2}\dot{y}(2.1)\Delta t \\ &= 0.300\,63+\frac{1}{2}(0.050\,283)(0.049\,35)=0.3019\text{ m} \quad \blacktriangleleft\end{aligned}$$

Note that $\dot{y}(2.1)$ was multiplied by 1/2. This factor takes into account the fact that $\dot{y}$ cannot be considered as constant during the time interval Δt, since it varies from $\dot{y}(2.1)$ to zero. Therefore, we must use the *average velocity* during this period, which is $\dot{y}(2.1)/2$.

The computed displacement somewhat bigger than the static displacement (which assumes that the load is applied very slowly) $y_{\text{static}} = P_{\max}/k = 20/75 = 0.2667$ m.

Problem 10

$$\ddot{y} = g(1 - ay^3) \qquad y(0) = 0.1 \text{ m} \qquad \dot{y}(0) = 0$$

The equivalent first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ g(1 - ay_1^3) \end{bmatrix}$$

To get a very rough idea of the period, let us consider the linear version of the problem

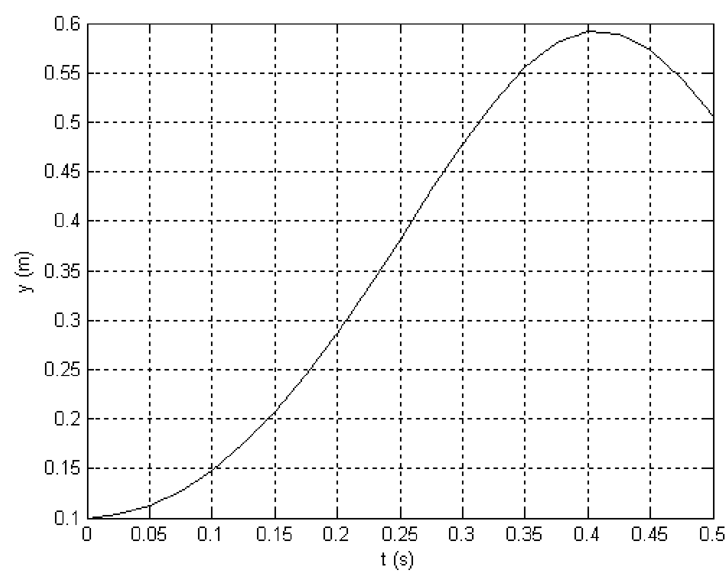
$$\ddot{y} = g(1 - ay)$$

the solution of which is

$$y = 1/a + C \sin \sqrt{ag} t$$

The period of this motion is $2\pi/\sqrt{ag} = 2\pi/\sqrt{16 \times 9.8} \approx 0.5$ s, which we specified as the integration period. It turns out that 0.5 s covers just over half the period of the original problem, but this is all we need.

```
% problem7_1_10
F = inline('[y(2) 9.80665*(1-16*y(1)^3)]','x','y');
x = 0; y = [0.1 0]; xStop = 0.5; h = 0.025;
[xSol,ySol] = runKut4(F,x,y,xStop,h);
printSol(xSol,ySol,1)
plot(xSol,ySol(:,1),'k-'); grid on
xlabel('t (s)'); ylabel('y (m)')
```



From the plot of the output we see that the peak displacement occurs at about $t = 4$ s, which represents half the period. Here are two lines of the output than span the peak displacement:

```
>>      x              y1              y2
   4.0000e-001   5.9175e-001   1.8123e-001
   4.2500e-001   5.8917e-001  -3.8596e-001
```

The value of the half-period t can be refined with Taylor series:

$$\begin{aligned}
\dot{y}(0.4+\Delta t) &= \dot{y}(0.4)+\ddot{y}(0.4)\,\Delta t \\
0 &= 0.18123+9.80665\left[1-16(0.59175)^3\right]\,\Delta t \\
\Delta t &= 0.007\,98\text{ s} \\
t &= 0.4+0.007\,982 = 0.4080\text{ s}
\end{aligned}$$

Therefore, the period is $2(0.4080)=0.8160$ s ◀
The maximum displacement can be estimated in the same manner:

$$\begin{aligned}
y_{\max} &= y(0.4080) = y(0.4)+\frac{1}{2}\dot{y}(0.4)\,\Delta t \\
&= 0.59175+\frac{1}{2}0.18123(0.00798) = 0.5925\text{ m}
\end{aligned}$$

The reason for the factor 1/2 was explained in Problem 9: $\dot{y}(0.4)/2$ is the average velocity during the time interval Δt. The *trough-to-peak amplitude* is

$$y_{\max}-y_{\min} = 0.5925-0.1 = 0.4925\text{ m} ◀$$

Problem 11

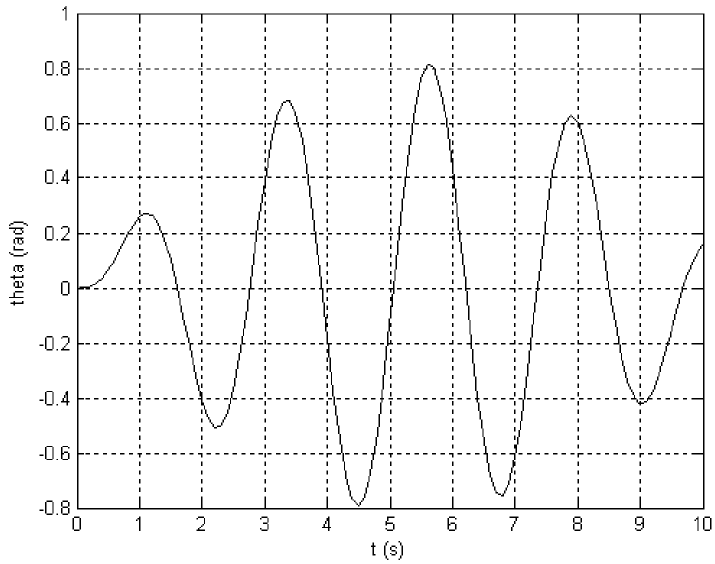
$$\ddot{\theta} = -\frac{g}{L}\sin\theta+\frac{\omega^2}{L}Y\cos\theta\sin\omega t \qquad \theta(0)=\dot{\theta}(0)=0$$

With the notation $y_0=\theta$, $y_1=\dot{\theta}$ the equivalent first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -\dfrac{g}{L}\sin y_1+\dfrac{\omega^2}{L}Y\cos y_1\sin\omega t \end{bmatrix}$$

Since we have no idea what the solution looks like, some experimentation is required to determine the step size h. We found $h=0.1$ to be satisfactory.

```
% problem7_1_11
F = inline('[y(2) -9.80665*sin(y(1))+1.5625*cos(y(1))*sin(2.5*x)]'...
          ,'x','y');
x = 0; y = [0 0]; xStop = 10; h = 0.1;
[xSol,ySol] = runKut4(F,x,y,xStop,h);
printSol(xSol,ySol,1)
plot(xSol,ySol(:,1),'k-'); grid on
xlabel('t (s)'); ylabel('theta (rad)')
```



The peak value of θ is between the following two points:

```
>>        x              y1             y2
   5.6000e+000    8.1249e-001    1.9457e-001
   5.7000e+000    8.0165e-001   -4.0988e-001
```

The time $t = 5.6+\Delta t$ when the peak occurs can be estimated from the two-term Taylor series

$$\dot\theta(5.6+\Delta t) = \dot\theta(5.6) + \ddot\theta(5.6)\,\Delta t \tag{a}$$

where

$$\begin{aligned}\ddot\theta(5.6) &= -\frac{9.806\,65}{1.0}\sin(0.812\ 49)\\ &\quad +\frac{2.5^2}{1.0}(0.25)\cos(0.812\ 49)\sin(2.5\times 5.6)\\ &= -6.055\ 22\ \text{rad/s}^2\end{aligned}$$

Hence Eq. (a) becomes

$$\begin{aligned}0 &= 0.019\ 457 + (-6.055\ 22)\,\Delta t\\ \Delta t &= 0.003213\ \text{s}\\ \therefore t &= 5.6 + 0.003213 = 5.632\ \text{s}\end{aligned}$$

Recognizing that the average velocity in the time period Δt is $\dot\theta(5.6)/2$, the estimate of the maximum displacement is

$$\begin{aligned}\theta_{\max} &= \theta(5.6+\Delta t) = \theta(5.6) + \frac{1}{2}\dot\theta(5.6)\Delta t\\ &= 0.812\ 49 + \frac{1}{2}(0.19457)(0.032\ 13) = 0.8155\ \text{rad} \ \blacktriangleleft\end{aligned}$$

Problem 12

$$\ddot{r} = \left(\frac{\pi^2}{12}\right)^2 r\sin^2 \pi t - g\sin\left(\frac{\pi}{12}\cos\pi t\right) \qquad r(0) = \dot{r}(0) = 0$$

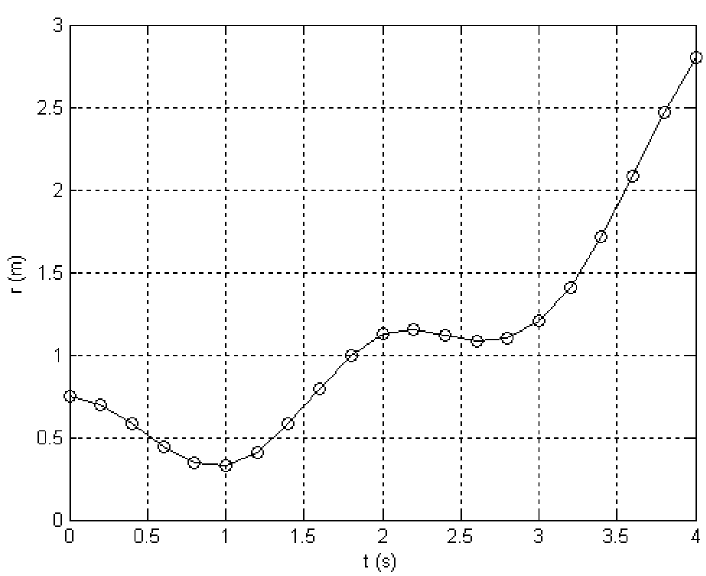
With the notation $y_0 = r$, $y_1 = \dot{r}$ the equivalent first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ (\pi^2/12)^2 y_1 \sin\pi t - g\sin[(\pi/12)\cos\pi t] \end{bmatrix}$$

The period of integration (`xStop = 4.0`) and step size (`h = 0.2`) were obtained by trial-and error. The plotting capability of MATLAB proved to be very helpful here.

```
% problem7_1_12
x = 0; y = [0.75 0]; xStop = 4; h = 0.2;
[xSol,ySol] = runKut4(@p7_1_12,x,y,xStop,h);
printSol(xSol,ySol,1)
plot(xSol,ySol(:,1),'ko-'); grid on
xlabel('t (s)'); ylabel('r (m)')

function F = p7_1_12(x,y)
% Differential equations used in Problem 12, Problem Set 7.1.
g = 9.80665; F = zeros(1,2);
F(1) = y(2);
F(2) = (pi^2/12)^2*y(1)*sin(pi*x)^2 - g*sin(pi/12*cos(pi*x));
```



From the plot we see that the slider reaches the end of the rod (at $r = 2$ m) in about 3.5 s. The two points spanning this event are printed below.

```
>>       x                y1               y2
   3.4000e+000    1.7123e+000    1.7134e+000
   3.6000e+000    2.0843e+000    1.9615e+000
```

Two-term Taylor series expansion about $t = 3.6$ s yields

$$\begin{aligned} r(3.6 + \Delta t) &= r(3.6) + \dot{r}(3.6)\,\Delta t \\ 2 &= 2.0843 + 1.9615\,\Delta t \\ \Delta t &= -0.04298 \text{ s} \end{aligned}$$

Therefore, the time when the slider leaves the rod is

$$t = 3.6 - 0.04298 = 3.557 \text{ s} \quad \blacktriangleleft$$

Problem 13

$$\begin{aligned} \ddot{x} &= -\frac{C_D}{m}\dot{x}v^{1/2} \qquad \ddot{y} = -\frac{C_D}{m}\dot{y}v^{1/2} - g \\ x(0) &= y(0) = 0 \qquad \dot{x}(0) = 50\cos 30^\circ \qquad \dot{y}(0) = 50\sin 30^\circ \end{aligned}$$

Letting

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix}$$

the first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ -(C_D/m)y_2v^{1/2} \\ y_4 \\ -(C_D/m)y_4v^{1/2} - g \end{bmatrix}$$

Without air resistance, the time of flight is $2(v_0 \sin 30^\circ)/g = 2(25)/9.8 \approx 5$ s. Since air resistance reduces the flight time, we guessed `xStop = 4.0`. The time increment h can be quite large here because the trajectory is a smooth curve; `h = 0.2` was considered satisfactory.

```
% problem7_1_13
x = 0; xStop = 4; h = 0.2;
y =[0 50*cos(pi/6) 0 50*sin(pi/6)];
[xSol,ySol] = runKut4(@p7_1_13,x,y,xStop,h);
printSol(xSol,ySol,1)
```

```
function F = p7_1_13(x,y)
% Differential eqs. used in Problem 13, Problem Set 7.1.
g = 9.80665; C = 0.03; m = 0.25;
sqrtv = sqrt(sqrt(y(2)^2 + y(4)^2));
F = zeros(1,4);
F(1) = y(2); F(2) = -C/m*y(2)*sqrtv;
F(3) = y(4); F(4) = -C/m*y(4)*sqrtv - g;
```

Here is the printout of the two points spanning the instant when $y = 0$:

```
>>    x             y1            y2            y3            y4
3.4000e+000  6.1289e+001  7.0962e+000  9.3838e-001 -1.2671e+001
3.6000e+000  6.2645e+001  6.4720e+000 -1.6732e+000 -1.3430e+001
```

The time of flight can be estimated from the two-term Taylor series expansion of y about $t = 3.4$ s:

$$\begin{aligned} y(3.4+\Delta t) &= y(3.4) + \dot{y}(3.4)\,\Delta t \\ 0 &= 0.93838 + (-12.671)\,\Delta t \\ \Delta t &= 0.07406 \text{ s} \\ t &= 3.4 + 0.07406 = 3.474 \text{ s} \quad \blacktriangleleft \end{aligned}$$

The range is obtained from the Taylor series expansion of x:

$$\begin{aligned} R &= x(3.4+\Delta t) = x(3.4) + \dot{x}(3.4)\,\Delta t \\ &= 61.289 + 7.0962(0.07406) = 61.81 \text{ m} \quad \blacktriangleleft \end{aligned}$$

Problem 14

$$\ddot{\theta} = \frac{a(b-\theta) - \theta\dot{\theta}^2}{1+\theta^2} \qquad \theta(0) = 2\pi \qquad \dot{\theta}(0) = 0$$

With the notation $\theta = y_1$, $\dot{\theta} = y_2$, the equivalent first-order differential equations are

$$F = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ \left[a(b-y_1) - y_1 y_2^2\right]/(1+y_1^2) \end{bmatrix}$$

Here the time increment h is hard to predict. Using trial-and-error, we found that $h = 0.05$ was small enough to yield 4 decimal point accuracy.

```
% problem7_1_14
F = inline('[y(2) (100*(15-y(1))-y(1)*y(2)^2)/(1+y(1)^2)]'...
          ,'x','y');
x = 0; xStop = 0.5; h = 0.05;
y =[2*pi 0];
[xSol,ySol] = runKut4(F,x,y,xStop,h);
printSol(xSol,ySol,0)
```

By specifying `freq = 0` in `printSol` only the first and last points are printed:

```
>>      x              y1              y2
   0.0000e+000   6.2832e+000   0.0000e+000
   5.0000e-001   8.3768e+000   6.7175e+000
```

$$\theta(0.5) = 8.377 \text{ rad} \blacktriangleleft \qquad \dot{\theta}(0.5) = 6.718 \text{ rad/s} \blacktriangleleft$$

Problem 15

$$\ddot{r} = r\dot{\theta}^2 + g\cos\theta - \frac{k}{m}(r - L) \qquad \ddot{\theta} = \frac{-2\dot{r}\dot{\theta} - g\sin\theta}{r}$$

$$r(0) = 0.5 \text{ m} \qquad \dot{r}(0) = 0 \qquad \theta(0) = \frac{\pi}{3} \qquad \dot{\theta}(0) = 0$$

Using the notation

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

the differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_1 y_4^2 + g\cos y_3 - (k/m)(y_1 - L) \\ y_4 \\ -(2y_2 y_4 + g\sin y_3)/y_1 \end{bmatrix}$$

A pendulum with a stiff arm has a period $\tau = 2\pi\sqrt{L/g}$ for small amplitudes. Although our problem is far removed from a simple pendulum, this formula can still give us a very rough estimate of the time of integration (a quarter of the period):

$$t = \frac{\pi}{2}\sqrt{\frac{0.5}{9.8}} = 0.35 \text{ s}$$

To be on the safe side, the period of integration should be somewhat longer; say, 0.5 s.

```
% problem7_1_15
x = 0; xStop = 0.5; h = 0.025;
y = [0.5 0 pi/3 0];
[xSol,ySol] = runKut4(@p7_1_15,x,y,xStop,h);
printSol(xSol,ySol,1)

function F = p7_1_15(x,y)
```

```
% Differential eqs. used in Problem15, Problem Set 7.1.
g = 9.80665; k = 40; L = 0.5; m = 0.25;
F = zeros(1,4);
F(1) = y(2);
F(2) = y(1)*y(4)^2 + g*cos(y(3)) - k/m*(y(1) - L);
F(3) = y(4);
F(4) = -(2*y(2)*y(4) + g*sin(y(3)))/y(1);
```

The two points spanning the position $\theta = 0$ are

```
>>       x             y1             y2            y3            y4
   4.2500e-001   6.1870e-001  -1.3669e-001   5.4800e-002  -3.5924e+000
   4.5000e-001   6.1499e-001  -1.5723e-001  -3.5664e-002  -3.6398e+000
```

Letting $t = 4.5 + \Delta t$ s be the time when $\theta = 0$, we obtain from the Taylor series

$$\begin{aligned}\theta(4.5 + \Delta t) &= \theta(4.5) + \dot{\theta}(4.5)\,\Delta t \\ 0 &= -0.035\,664 + (-3.6398)\,\Delta t \\ \Delta t &= -0.009\,798\end{aligned}$$

The length of the cord at the instant when $\theta = 0$ is given by

$$\begin{aligned}r(4.5 + \Delta t) &= r(4.5) + \dot{r}(4.5)\,\Delta t \\ &= 0.614\,99 + (-0.157\,23)(-0.009\,798) \\ &= 0.6165 \text{ m} \quad \blacktriangleleft\end{aligned}$$

Problem 16

Changing the initial condition to $r(0) = 0.575$ m in Problem 15, the points spanning $\theta = 0$ are

```
>>     x             y1            y2            y3            y4
 4.0000e-001   6.6414e-001   5.8568e-001   6.0208e-002  -2.8607e+000
 4.2500e-001   6.7509e-001   2.8260e-001  -1.0184e-002  -2.7776e+000
```

The computations now yield

$$\begin{aligned}\theta(4.25 + \Delta t) &= \theta(4.25) + \dot{\theta}(4.25)\,\Delta t \\ 0 &= -0.010\,184 + (-2.7776)\,\Delta t \\ \Delta t &= -0.003\,666\end{aligned}$$

$$\begin{aligned}r(4.25 + \Delta t) &= r(4.25) + \dot{r}(4.25)\,\Delta t \\ &= 0.675\,09 + 0.028\,260(-0.003\,666) \\ &= 0.6750 \text{ m} \quad \blacktriangleleft\end{aligned}$$

Problem 17

$$\ddot{y} = -\frac{k}{m}y - \mu g\frac{\dot{y}}{|\dot{y}|} \qquad y(0) = 0.1 \text{ m} \qquad \dot{y}(0) = 0$$

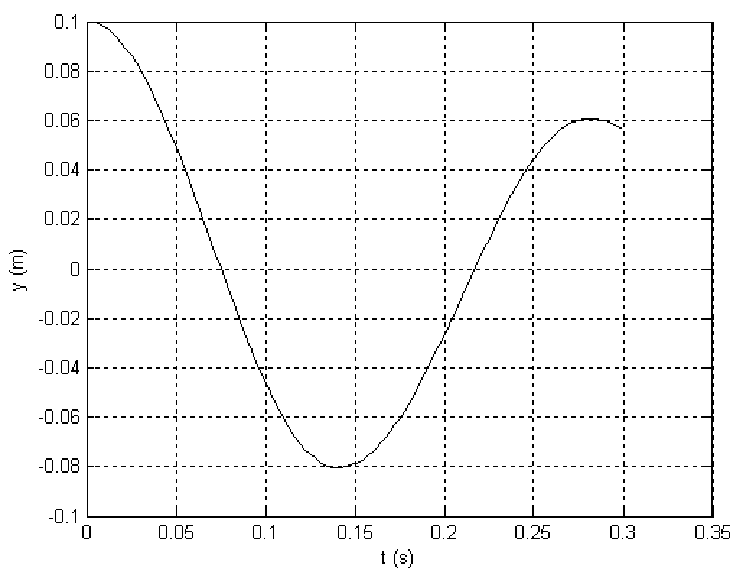
Using the notation $y = y_1$, $\dot{y} = y_2$, the first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -(k/m)y_1 - \mu g y_2/|y_2| \end{bmatrix}$$

A rough idea of the period of the motion can obtained by removing the friction term from the differential equation. Without friction, the period is $\tau = 2\pi/\sqrt{k/m} = 2\pi/\sqrt{3000/6} \approx 0.28$ s, so that 0.3 s seems to be a reasonable period of integration. We chose for the time increment $h = 0.025$ s, printing out every 4th point.

```
% problem7_1_17
x = 0; xStop = 0.3; h = 0.0025;
y = [0.1 0];
[xSol,ySol] = runKut4(@p7_1_17,x,y,xStop,h);
printSol(xSol,ySol,4)
plot(xSol,ySol(:,1),'k-'); grid on
xlabel('t (s)'); ylabel('y (m)')

function F = p7_1_17(x,y)
k = 3000; m = 6; mu = 0.5; g = 9.80665;
F = zeros(1,2);
F(1) = y(2);
if y(2) > 0; F(2) = -k/m*y(1) - mu*g;
else; F(2) = -k/m*y(1) + mu*g; end
```



Here is a printout of the two points spanning the peak displacement:

```
>>       x              y1             y2
   2.8000e-001   6.0754e-002   3.5807e-002
   2.9000e-001   5.9730e-002  -2.2971e-001
```

The peak displacement occurs at time $t = 0.28 + \Delta t$ when the velocity vanishes. We can compute Δt from the Taylor series

$$\dot{y}(0.28 + \Delta t) = \dot{y}(0.28) + \ddot{y}(0.28)\,\Delta t \tag{a}$$

where

$$\begin{aligned}\ddot{y}(0.28) &= -\frac{k}{m}y(0.28) - \mu g \\ &= -\frac{3000}{6}(0.060\,754) - 0.5(9.806\,65) = -35.280 \ \text{m/s}^2\end{aligned}$$

Substitution into Eq. (a) yields

$$\begin{aligned}0 &= 0.035\,807 + (-35.280)\,\Delta t \\ \Delta t &= 0.001\,014\,9 \text{ s}\end{aligned}$$

The peak displacement is given by

$$\begin{aligned}y(0.28 + \Delta t) &= y(0.28) + \frac{1}{2}\dot{y}(0.28)\,\Delta t \\ &= 0.060\,754 + \frac{1}{2}0.035807(0.001\,014\,9) \\ &= 0.06077 \text{ m}\end{aligned}$$

The analytical formula gives for the peak displacement

$$y(0) - 4\frac{\mu m g}{k} = 0.1 - 4\frac{0.5(6)(9.80665)}{3000} = 0.06077 \text{ m} \quad \text{Checks}$$

Problem 18

We use the notation $y = y_1$, $y' = y_2$ in both problems. Being unable to determine a suitable time increment h beforehand, we let the program do it for us. Starting with an initial guess for h, the program integrates the differential equations with h, $h/2$, $h/4$, etc. until the results of two successive integrations agree within a prescribed tolerance.

(a)

$$y'' + 0.5(y^2 - 1)\dot{y} + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

This is a version of the well-known Van der Pol equation. The equivalent first-order differential equations are

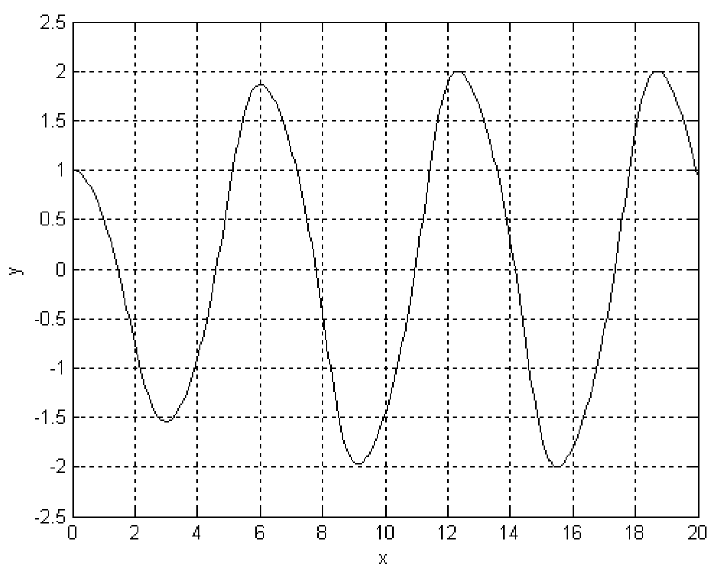
$$\mathbf{F} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} - \begin{bmatrix} y_2 \\ -0.5(y_1^2 - 1)y_2 - y_1 \end{bmatrix}$$

```
% problem7_1_18a
F = inline('[y(2) -0.5*(y(1)^2-1)*y(2)-y(1)]','x','y');
x = 0; xStop = 20; h = 0.2; y = [1 0];
yOld = 0;
while 1
    [xSol,ySol] = runKut4(F,x,y,xStop,h);
    yNew = ySol(size(ySol,1));
    if abs(yNew - yOld) < 1.0e-4; break
    else; h = h/2; yOld = yNew; end
end
h
plot(xSol,ySol(:,1),'k-'); grid on
xlabel('x'); ylabel('y')
```

The output is

```
>> h = 0.050000
```

Note that the initial increment $h = 0.2$ was reduced to 0.05 in the last run of the program.



(b)

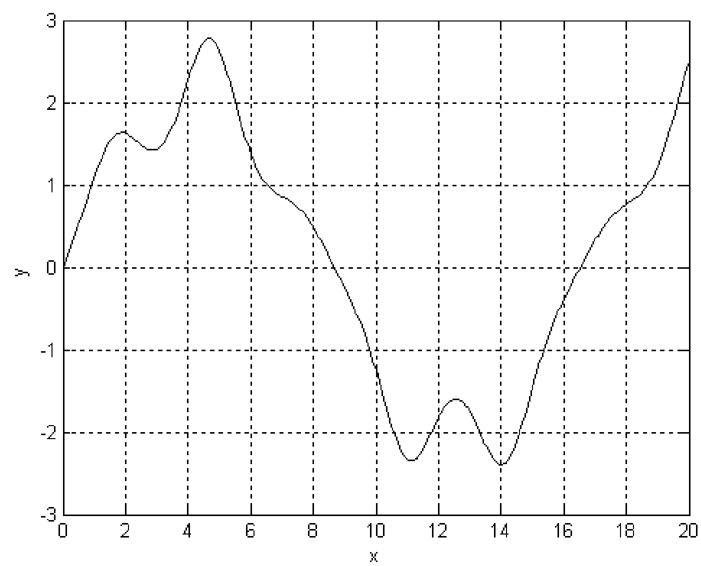
$$y'' = y \cos 2x \qquad y(0) = 0 \qquad y'(0) = 1$$

This differential equation is called Mathieu's equation. The equivalent first-order equations are

$$\mathbf{F} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} - \begin{bmatrix} y_2 \\ y_1 \cos 2x \end{bmatrix}$$

We used the program listed in Part (a); only $\mathbf{F}$ and the initial conditions were changed.

```
>> h = 0.050000
```



Problem 19

$$y'' + \frac{1}{x}y' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

With the notation $y = y_1$, $y' = y_2$, the first-order equations become

$$\mathbf{F} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} - \begin{bmatrix} y_2 \\ -y_2/x - y_1 \end{bmatrix}$$

The program below keeps halving h until the results of two successive integrations agree within the prescribed tolerance.

```
% problem7_1_19
F = inline('[y(2) -y(2)/x-y(1)]','x','y');
x = 1.0e-12; xStop = 5; h = 0.2; y = [1 0];
yOld = 0;
while 1
    [xSol,ySol] = runKut4(F,x,y,xStop,h);
    yNew = ySol(size(ySol,1));
    if abs(yNew - yOld) < 1.0e-4; break
    else; h = h/2; yOld = yNew; end
end
fprintf('h = %8.6f\n',h)
printSol(xSol,ySol,0)

>> h = 0.025000
       x              y1              y2
   1.0000e-012    1.0000e+000    0.0000e+000
   5.0000e+000   -1.7759e-001    3.2756e-001
```

The final increment $h = 0.025$ yields satisfactory agreement with the tabulated value $y(5) = -0.17760$.

Problem 20

(a)

$$y'' = 16.81y \qquad y(0) = 1.0 \qquad y'(0) = -4.1$$

Solution of the differential equation is (note that $\sqrt{16.81} = 4.1$)

$$y = Ae^{4.1x} + Be^{-4.1x}$$

The initial conditions yield

$$\begin{aligned} y(0) &= A + B = 1 \qquad y'(0) = 4.1(A - B) = -4.1 \\ A &= 0 \qquad B = 1 \end{aligned}$$

Therefore,

$$y = e^{-4.1x} \quad \blacktriangleleft$$

(b)

As numerical integration proceeds, the dormant term $Ae^{4.1x}$ will become alive and eventually dominates the solution. This is a case numerical instability caused by sensitivity of the solution to initial conditions. Numerical integration will not work here.

(c)

Using the 4th-order Runge-Kutta method with $h = 0.1$, the initial and final points of the solution are

```
      x             y1             y2
  0.0000e+000   1.0000e+000   0.0000e+000
  8.0000e+000   8.7386e+013   3.5828e+01
```

Clearly the numerical solution is unstable.

Problem 21

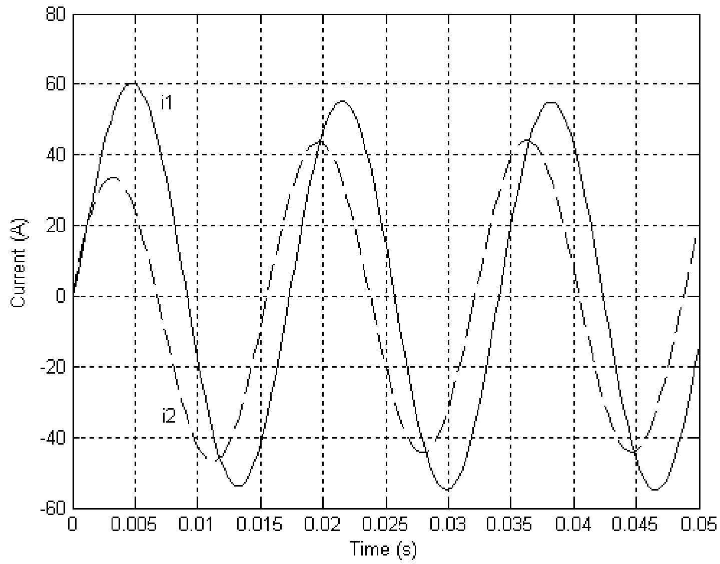
$$\begin{aligned}
\frac{di_1}{dt} &= \frac{-3Ri_1 - 2Ri_2 + E}{L} \\
\frac{di_2}{dt} &= -\frac{2}{3}\frac{di_1}{dt} - \frac{i_2}{3RC} + \frac{\dot{E}}{3R} \\
i_1(0) &= i_2(0) = 0
\end{aligned}$$

Using the notation $i_1 = y_1$, $i_2 = y_2$, the differential equations are

$$F = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} (-3Ry_1 - 2Ry_2 + E)/L \\ \left[-2\dot{y}_1 - y_2/(RC) + \dot{E}/R\right]/3 \end{bmatrix}$$

```
% problem7_1_21
x = 0; xStop = 0.05; h = 0.00025; y = [0 0];
[xSol,ySol] = runKut4(@p7_1_21,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol(:,1),'k-'); hold on
plot(xSol,ySol(:,2),'k--'); grid on
xlabel('Time (s)'); ylabel('Current (A)')
gtext('i1'); gtext('i2')

function F = p7_1_21(x,y)
% Differential eqs. used in Problem 21, Problem Set 7.1.
R = 1; L = 0.2e-3; C = 3.5e-3;
E = 240*sin(120*pi*x);
dE = 240*120*pi*cos(120*pi*x);
F = zeros(1,2);
F(1) = (-3*R*y(1) - 2*R*y(2) + E)/L;
F(2) = (-2*F(1) -y(2)/R/C + dE/R)/3;
```



Problem 22

$$
\begin{aligned}
L\frac{di_1}{dt} + Ri_1 + \frac{q_1 - q_2}{C} &= E \\
L\frac{di_2}{dt} + Ri_2 + \frac{q_2 - q_1}{C} + \frac{q_2}{C} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{dq_1}{dt} &= i_1 \qquad \frac{dq_2}{dt} = i_2 \\
q_1(0) &= q_2(0) = i_1(0) = i_2(0) = 0
\end{aligned}
$$

With the notation

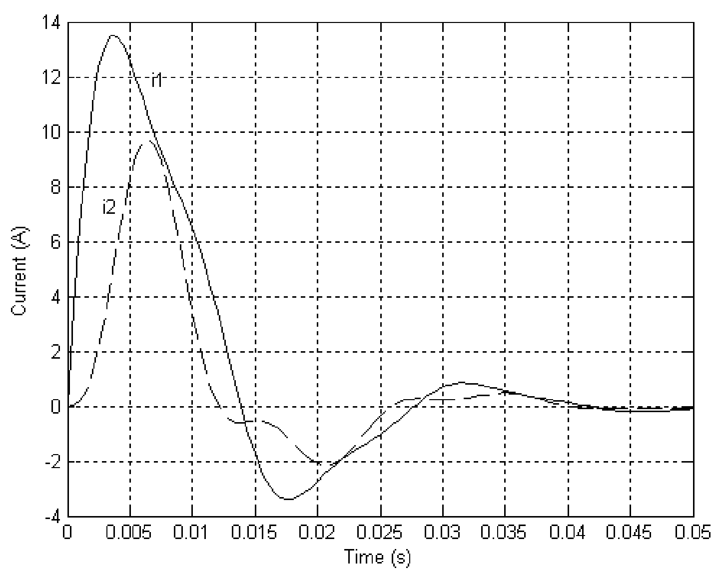
$$
\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ i_1 \\ i_2 \end{bmatrix}
$$

the first-order differential equations are

$$
\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ \left[E - Ry_3 - (y_1 - y_2)/C\right]/L \\ \left[-Ry_4 - (y_2 - y_1)/C - y_2/C\right]/L \end{bmatrix}
$$

```
% problem7_1_22
x = 0; xStop = 0.05; h = 0.0005; y = [0 0 0 0];
[xSol,ySol] = runKut4(@p7_1_22,x,y,xStop,h);
plot(xSol,ySol(:,3),'k-'); hold on
plot(xSol,ySol(:,4),'k--'); grid on
xlabel('Time (s)'); ylabel('Current (A)')
gtext('i1'); gtext('i2')

function F = p7_1_22(x,y)
% Differential eqs. used in Problem 22, Problem Set 7.1.
R = 0.25; L = 1.2e-3; C = 5e-3; E = 9;
F = zeros(1,4);
F(1) = y(3); F(2) = y(4);
F(3) = (E - R*y(3) - (y(1) - y(2))/C)/L;
F(4) = (-R*y(4) - (y(2) - y(1))/C - y(2)/C)/L ;
```



Problem 23

```
function [xSol,ySol] = runKut2(dEqs,x,y,xStop,h)
% 2nd-order Runge-Kutta integration.
if size(y,1) > 1 ; y = y'; end  % y must be row vector
xSol = zeros(2,1); ySol = zeros(2,length(y));
xSol(1) = x; ySol(1,:) = y;
i = 1;
while x < xStop
```

```
    i = i + 1;
    h = min(h,xStop - x);
    K1 = h*feval(dEqs,x,y);
    K2 = h*feval(dEqs,x + h/2,y + K1/2);
    y = y + K2;
    x = x + h;
    xSol(i) = x; ySol(i,:) = y;  % Store current soln.
end
```

```
>> [x,y] = runKut2(@fex7_4,0,[0 1],2,0.25);
>> printSol(x,y,1)
     x            y1            y2
   0.0000e+000   0.0000e+000   1.0000e+000
   2.5000e-001   2.4688e-001   9.4406e-001
   5.0000e-001   4.7213e-001   8.2779e-001
   7.5000e-001   6.6086e-001   6.5266e-001
   1.0000e+000   7.9855e-001   4.2014e-001
   1.2500e+000   8.7103e-001   1.3165e-001
   1.5000e+000   8.6446e-001  -2.1145e-001
   1.7500e+000   7.6539e-001  -6.0779e-001
   2.0000e+000   5.6065e-001  -1.0561e+000
```

In Example 7.2 it was pointed out that the analytical solution yields $y(2) = 0.543\ 45$, $y'(2) = -1.0543$. We also found in Example 7.4 that the 4th-order Runge-Kutta method agreed with the analytical solution. The results of the 2nd-order method, on the other hand, have an error of about 3% in $y(2)$ and 0.2% in $y'(2)$.

PROBLEM SET 7.2

Problem 1

$$y'' = 380y - y' \qquad y(0) = 1 \qquad y'(0) = -20$$

With $y = y_1$, $y' = y_2$ the equivalent first-order differential equations are

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 380 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

These equations are of the form $\dot{\mathbf{y}} = -\mathbf{\Lambda y}$, where

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & -1 \\ -380 & 1 \end{bmatrix}$$

The eigenvalues of $\mathbf{\Lambda}$ are the roots of

$$\begin{vmatrix} 0-\lambda & -1 \\ -380 & 1-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - \lambda - 380 = 0$$

which yields $\lambda_1 = -19$, $\lambda_2 = 20$. Therefore,

$$y = C_1 e^{-\lambda_1 x} + C_2 e^{-\lambda_2 x} = C_1 e^{19x} + C_2 e^{-20x}$$

From the initial conditions we get

$$\begin{aligned} y(0) &= 1: \qquad C_1 + C_2 = 0 \\ y'(0) &= -20: \qquad 19C_1 - 20C_2 = -20 \\ C_1 &= 0 \qquad C_2 = 1 \end{aligned}$$

so that $y = e^{-20x}$ ◀

It would be difficult to obtain the solution numerically due to the dormant term $C_1 e^{19x}$.

Problem 2

$$y' = x - 10y \qquad y(0) = 10$$

(a)

$$\begin{aligned}
y &= 0.1x - 0.01 + 10.01e^{-10x} \\
y' &= 0.1 - 100.1e^{-10x} \\
x - 10y &= x - 10(0.1x - 0.01 + 10.01e^{-10x}) \\
&= 0.1 - 100.1e^{-10x} = y' \text{ Q.E.D.}
\end{aligned}$$

$$y(0) = 0 - 0.01 + 10.01 = 10 \text{ Checks}$$

(b)

$$h < \frac{2}{\lambda} \qquad h < \frac{2}{10} = 0.2 \blacktriangleleft$$

Problem 3

The analytical solution is

$$y(5) = 0.1(5) - 0.01 + 10.01e^{-10(5)} = 0.4900$$

```
% problem7_2_3
F = inline('[x-10*y(1)]','x','y');
for h = [0.1 0.25 0.5];
    x = 0; xStop = 5; y = [10];
 [xSol,ySol] = runKut4(F,x,y,xStop,h);
    fprintf('\nh = %6.3f\n',h)
 printSol(xSol,ySol,0)
end
```

```
h =  0.100
      x            y1
   0.0000e+000   1.0000e+001
   5.0000e+000   4.9000e-001
h =  0.250
      x            y1
   0.0000e+000   1.0000e+001
   5.0000e+000   4.9173e-001
h =  0.500
      x            y1
   0.0000e+000   1.0000e+001
   5.0000e+000   2.3457e+012
```

In Problem 2 the stable range of h was estimated as $h < 0.2$. Thus $h = 0.1$ is stable and $h = 0.5$ is unstable, as verified by the numerical results. On the other hand. $h = 0.25$ is close to the borderline—it is stable in the specified range of integration, but not accurate.

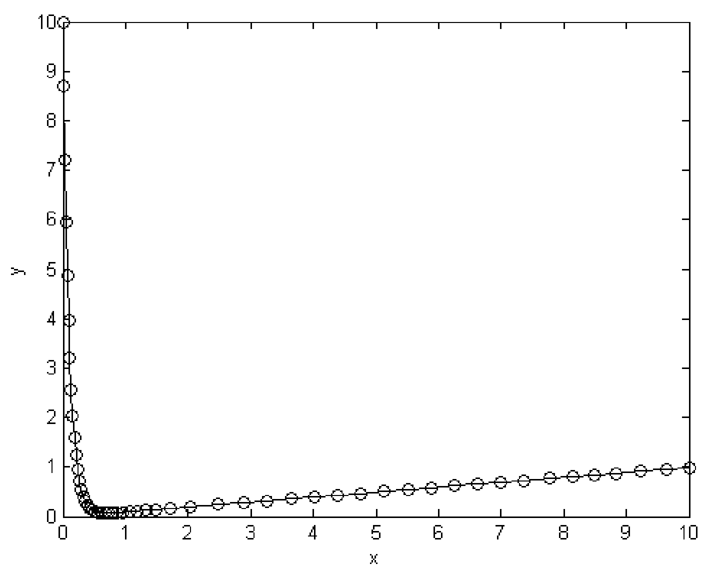
Problem 4

The analytical solution is

$$y(10) = 0.1(10) - 0.01 + 10.01e^{-10(10)} = 0.9900$$

```
% problem7_2_4
F = inline('[x-10*y(1)]','x','y');
x = 0; xStop = 10; y = [10];
[xSol,ySol] = runKut5(F,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol,'k-o')
xlabel('x'); ylabel('y')
>>       x              y1
   0.0000e+000   1.0000e+001
   1.0000e+001   9.9000e-001
```

The plot of the solution shows the integration points as circles. Note the greater density points where y varies rapidly.



Problem 5

$$\ddot{y} = -\frac{c}{m}\dot{y} - \frac{k}{m}y \qquad y(0) = 0.01 \text{ m} \qquad \dot{y}(0) = 0$$

(a)

With $y = y_1$, $\dot{y} = y_2$ the equivalent first-order differential equations are

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

These equations are of the form $\dot{\mathbf{y}} = -\mathbf{\Lambda}\mathbf{y}$, where

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & -1 \\ k/m & c/m \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 450/2 & 460/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 225 & 230 \end{bmatrix}$$

The eigenvalues of $\mathbf{\Lambda}$ are the roots of

$$\begin{vmatrix} 0 - \lambda & -1 \\ 225 & 230 - \lambda \end{vmatrix} = 0 \qquad \lambda^2 - 230\lambda + 225 = 0$$

The solution is $\lambda_1 = 0.982\,458$, $\lambda_2 = 229.0175$. Since there is a large disparity in the eigenvalues, the problem is stiff. Numerical integration requires

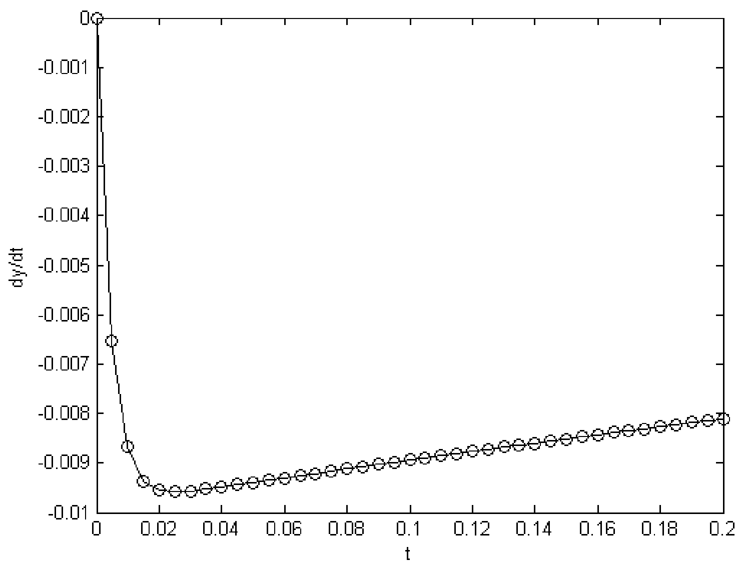
$$h < \frac{2}{\lambda_2} = \frac{2}{229.0175} = 0.008\,733$$

A reasonable choice would be $h = 0.005$ ◀

(b)

```
% problem7_2_5
F = inline('[y(2) -225*y(1)-230*y(2)]','x','y');
x = 0; xStop = 0.2; y = [0.01 0];
h = 0.005;
[xSol,ySol] = runKut4(F,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol(:,2),'k-o')
xlabel('t'); ylabel('dy/dt')
```

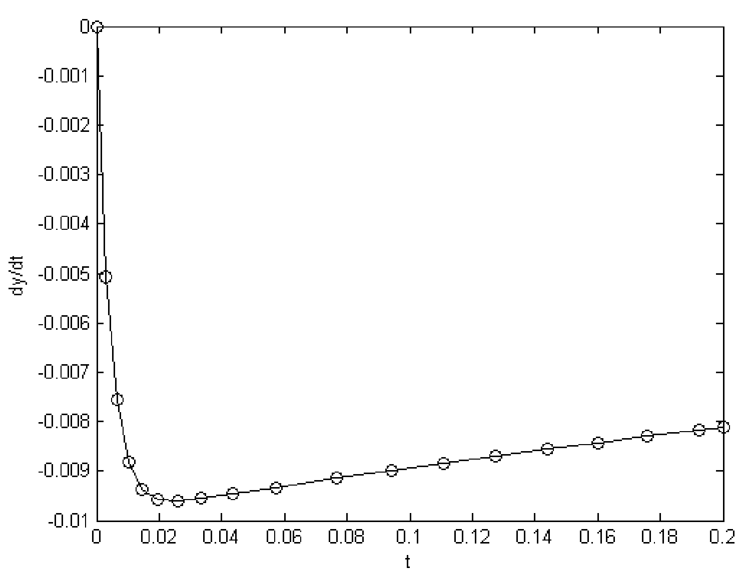
```
>>       x              y1              y2
   0.0000e+000    1.0000e-002    0.0000e+000
   2.0000e-001    8.2515e-003   -8.1067e-003
```



Problem 6

The program in Problem 5 was used with the function call `runKut4` replaced by `runKut5`.

```
>>      x              y1            y2
   0.0000e+000   1.0000e-002   0.0000e+000
   2.0000e-001   8.2515e-003  -8.1065e-003
```



Note the larger h used in the region $t > 0.05$ s than in Problem 5.

Problem 7

$$y'' = 16.81y$$

This problem was integrated (unsuccessfully) with the non-adaptive Runge-Kutta method in Problem 20, Problem Set 7.1. The analytical solution is

$$y = Ae^{4.1x} + Be^{-4.1x} \tag{a}$$

(a)

The initial conditions $y(0) = 1$, $y'(0) = -4.1$ yield $A = 0$, $B = 1$, so the the analytical solution becomes

$$y = e^{-4.1x}$$

(b)

The initial conditions are $y(0) = 1$, $y'(0) = -4.11$. Substituting these conditions into Eq. (a) gives us $A = -1.2195 \times 10^{-3}$ and $B = 1.0012$. Thus the analytical solution is

$$y = -1.2195 \times 10^{-3} e^{4.1x} + 1.0012e^{-4.1x}$$

Here is a program that computes and plots both solutions:

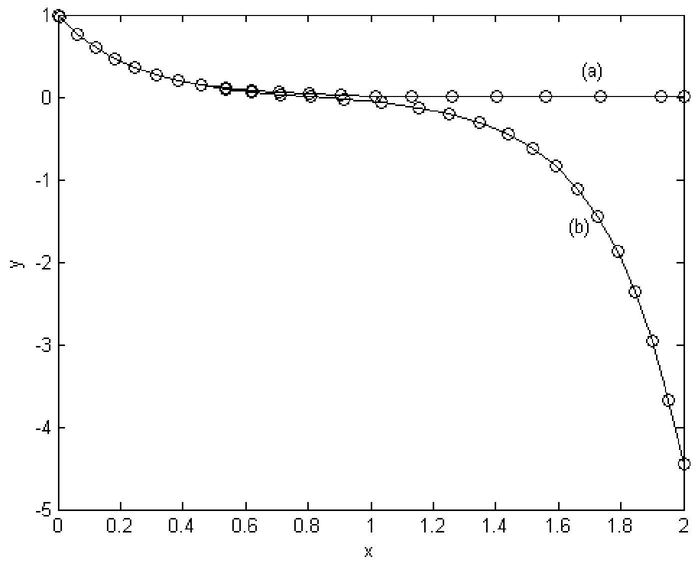
```
% problem7_2_7
F = inline('[y(2) 16.81*y(1)]','x','y');
y = zeros(1,2);
for dy = [-4.1 -4.11]
    x = 0; xStop = 2; y = [1 dy];
    h = 0.005;
    [xSol,ySol] = runKut5(F,x,y,xStop,h);
    printSol(xSol,ySol,2)
    fprintf('\n\n')
    plot(xSol,ySol(:,1),'k-o'); hold on
end
xlabel('x'); ylabel('y')
gtext('(a)'); gtext('(b)')
```

The numerical results for Part (a), shown below, reproduce the analytical solution quite closely. The effect of the dormant term $Ae^{4.1x}$ has not yet appeared in the numerical solution.

```
>>      x            y1            y2
   0.0000e+000   1.0000e+000  -4.1000e+000
   6.4507e-002   7.6761e-001  -3.1472e+000
   1.8336e-001   4.7152e-001  -1.9333e+000
   3.1406e-001   2.7592e-001  -1.1313e+000
   4.5873e-001   1.5247e-001  -6.2513e-001
   6.2051e-001   7.8544e-002  -3.2203e-001
   8.0372e-001   3.7059e-002  -1.5194e-001
   1.0144e+000   1.5620e-002  -6.4041e-002
   1.2617e+000   5.6687e-003  -2.3242e-002
   1.5593e+000   1.6732e-003  -6.8601e-003
   1.9304e+000   3.6539e-004  -1.4981e-003
   2.0000e+000   2.7465e-004  -1.1261e-003
```

The solution for Part (b) is initially dominated by the term $Be^{-4.1x}$, but the term $Ae^{4.1x}$ rapidly gains prominence beyond $x = 1$:

```
        x            y1            y2
   0.0000e+000   1.0000e+000  -4.1100e+000
   6.4505e-002   7.6696e-001  -3.1576e+000
   1.8336e-001   4.6953e-001  -1.9463e+000
   3.1409e-001   2.7180e-001  -1.1506e+000
   4.5893e-001   1.4452e-001  -6.5818e-001
   6.2141e-001   6.2765e-002  -3.8513e-001
   8.0756e-001   3.0949e-003  -2.8681e-001
   1.0336e+000  -7.0012e-002  -4.0558e-001
   1.2519e+000  -2.0075e-001  -8.7153e-001
   1.4382e+000  -4.4095e-001  -1.8304e+000
   1.5935e+000  -8.3727e-001  -3.4448e+000
   1.7277e+000  -1.4531e+000  -5.9647e+000
   1.8465e+000  -2.3662e+000  -9.7056e+000
   1.9535e+000  -3.6684e+000  -1.5043e+001
   2.0000e+000  -4.4399e+000  -1.8206e+001
```



Problem 8

$$y'' = -y' + y^2 \qquad y(0) = 1 \qquad y'(0) = 0$$

```
% problem7_2_8
F = inline('[y(2) -y(2)+y(1)^2]','x','y');
x = 0; xStop = 3.5; y = [1 0];
h = 0.05; tol = 1.0e-6;
[xSol,ySol] = runKut5(F,x,y,xStop,tol);
printSol(xSol,ySol,20)
```

With the per-step error tolerance (`tol`) set to 1.0e-6, the numerical solution yielded (only every 20th step is printed):

```
>>       x             y1            y2
   0.0000e+000   1.0000e+000   0.0000e+000
   2.8065e+000   1.3707e+001   3.6452e+001
   3.2987e+000   1.3638e+002   1.2475e+003
   3.4040e+000   5.1743e+002   9.4064e+003
   3.4424e+000   1.2261e+003   3.4568e+004
   3.4624e+000   2.3793e+003   9.3813e+004
   3.4744e+000   4.0949e+003   2.1232e+005
   3.4824e+000   6.4956e+003   4.2486e+005
   3.4879e+000   9.7088e+003   7.7722e+005
   3.4920e+000   1.3866e+004   1.3276e+006
```

```
   3.4951e+000   1.9104e+004   2.1483e+006
   3.4975e+000   2.5561e+004   3.3265e+006
   3.4994e+000   3.3381e+004   4.9664e+006
   3.5000e+000   3.6472e+004   5.6725e+006
```

Note that the step size rapidly diminishes as x approaches 3.5. At the same time, y appears to approach infinity. If this is caused by numerical instability, the results should be sensitive to the per-step error tolerance used in the integration (tighter tolerance reduces the truncation error thus delaying the onset of instability). We tested for instability by re-running the program with `tol` set to 10^{-4} and 10^{-8}. Here are the results:

`tol`	$y(3.5)$
10^{-4}	3.6394×10^4
10^{-6}	3.6472×10^4
10^{-8}	3.6475×10^4

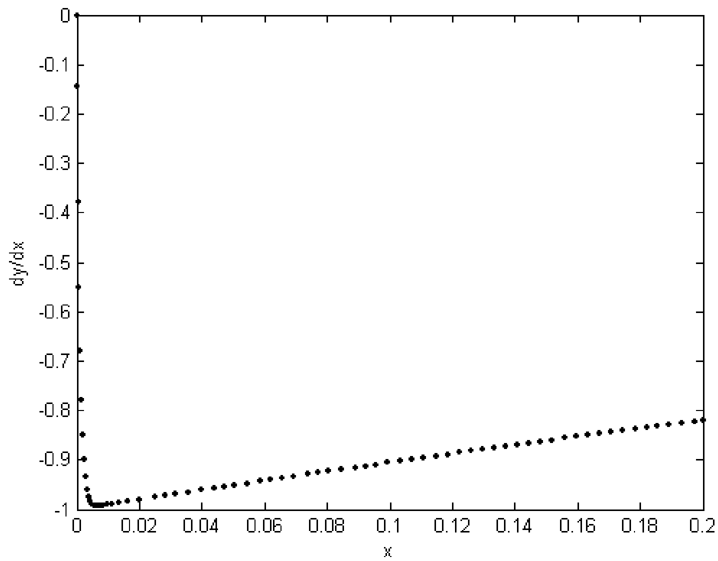
Since changing `tol` made no significant difference, we conclude that the sudden increase in y is real.

Problem 9

$$y'' = -1001y' - 1000y \qquad y(0) = 1 \qquad y'(0) = 0$$

```
% problem7_2_9
F = inline('[y(2) -1001*y(2)-1000*y(1)]','x','y');
x = 0; xStop = 0.2; y = [1 0];
h = 0.05;
[xSol,ySol] = runKut5(F,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol(:,2),'k.')
xlabel('x'); ylabel('dy/dy')
```

```
>>       x             y1             y2
   0.0000e+000   1.0000e+000   0.0000e+000
   2.0000e-001   8.1955e-001  -8.1955e-001
```



The adaptive quadrature had no trouble overcoming the stiffness of this problem.

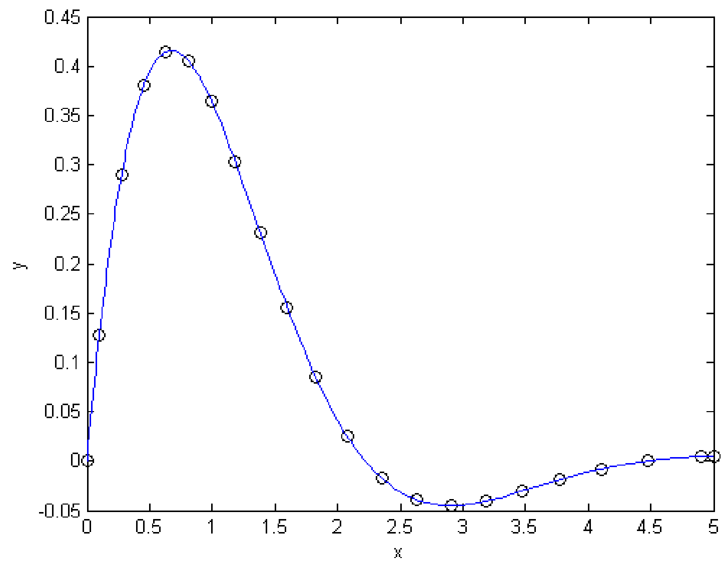
Problem 10

$$y'' = -2y' - 3y \qquad y(0) = 0 \qquad y'(0) = \sqrt{2}$$

```
% problem7_2_10
F = inline('[y(2) -2*y(2)-3*y(1)]','x','y');
yExact = inline('exp(-x)*sin(sqrt(2)*x)','x');
x = 0; xStop = 5; y = [0 sqrt(2)];
h = 0.1;
[xSol,ySol] = runKut5(F,x,y,xStop,h);
printSol(xSol,ySol,2)
plot(xSol,ySol(:,1),'ko'); hold on
fplot(yExact,[0 5])
xlabel('x'); ylabel('y')
>>       x              y1              y2
   0.0000e+000   0.0000e+000   1.4142e+000
   2.7762e-001   2.8986e-001   7.0001e-001
   6.3152e-001   4.1427e-001   5.7260e-002
   9.9524e-001   3.6472e-001  -2.7972e-001
   1.3858e+000   2.3143e-001  -3.6560e-001
   1.8271e+000   8.5150e-002  -2.7820e-001
   2.3537e+000  -1.7666e-002  -1.1437e-001
```

2.9075e+000	-4.5055e-002	1.4137e-003
3.4695e+000	-3.0547e-002	3.9044e-002
4.1027e+000	-7.6492e-003	2.8370e-002
4.9004e+000	4.4869e-003	3.9116e-003
5.0000e+000	4.7759e-003	1.9454e-003

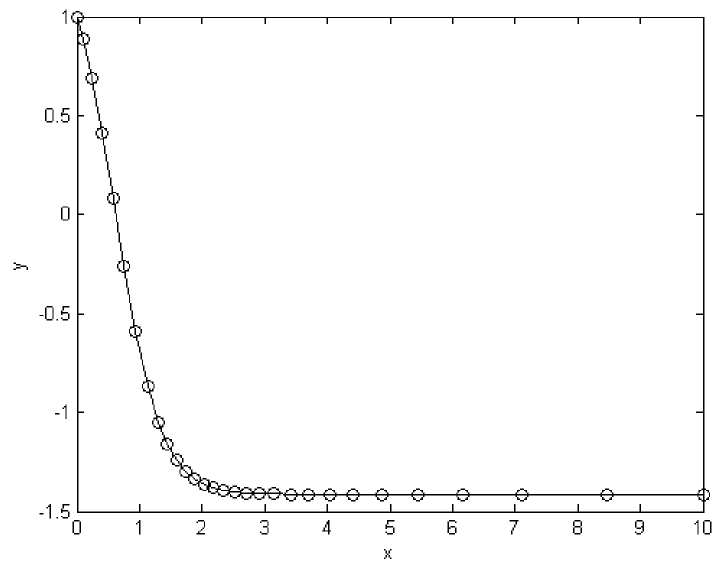
In the plot below, the open circles represent the numerical solution and solid line is the analytical solution.



Problem 11

$$y'' = 2yy' \qquad y(0) = 1 \qquad y'(0) = -1$$

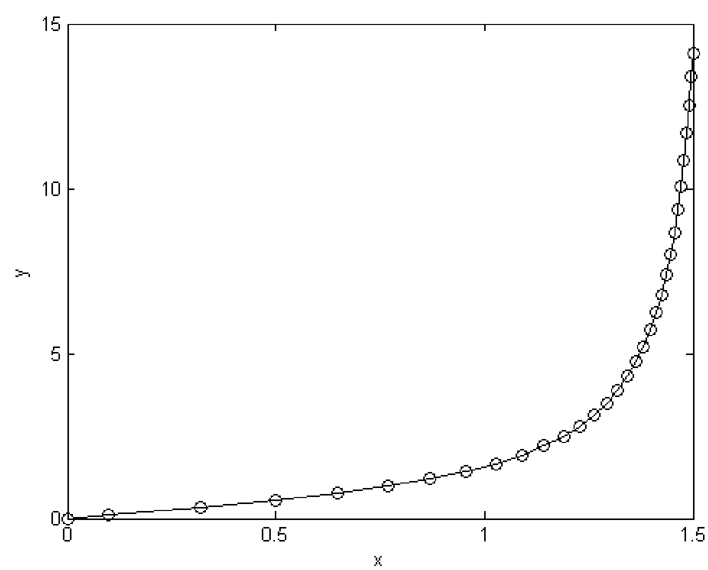
```
% problem7_2_11
F = inline('[y(2) 2*y(1)*y(2)]','x','y');
x = 0; xStop = 10; y = [1 -1];
h = 0.1;
[xSol,ySol] = runKut5(F,x,y,xStop,h);
plot(xSol,ySol(:,1),'k-o')
xlabel('x'); ylabel('y')
```



Problem 12

$$y'' = 2yy' \qquad y(0) = 0 \qquad y'(0) = 1$$

We used the program in Problem 11; only the initial values of and the range of integration were changed.



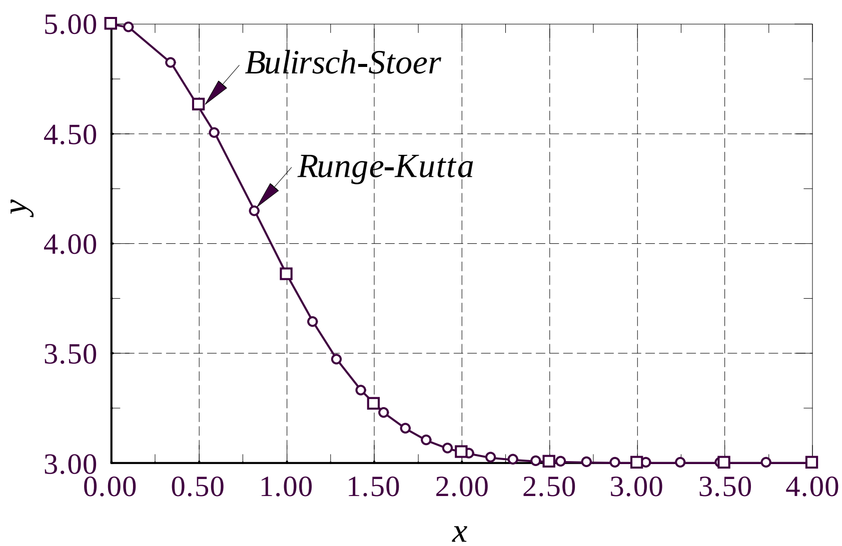
The analytical solution is $y = \tan x$.

Problem 13

$$y' = \left(\frac{9}{y} - y\right)x \qquad y(0) = 5$$

```
 problem7_2_13
F = inline('[(9/y(1)-y(1))*x]','x','y');
x = 0; xStop = 4; y = [5];
h = 0.1;
[xSol,ySol] = runKut5(F,x,y,xStop,h);
printSol(xSol,ySol,2);
>>      x             y1
   0.0000e+000   5.0000e+000
   3.4132e-001   4.8208e+000
   8.1895e-001   4.1451e+000
   1.1506e+000   3.6412e+000
   1.4276e+000   3.3294e+000
   1.6806e+000   3.1543e+000
   1.9219e+000   3.0656e+000
   2.1668e+000   3.0243e+000
   2.4268e+000   3.0074e+000
   2.7155e+000   3.0017e+000
   3.0525e+000   3.0002e+000
   3.4747e+000   3.0000e+000
   4.0000e+000   3.0000e+000
```

The plot of the solution also shows points obtained by the Bulirsch-Stoer method in Problem 14.



Problem 14

$$y' = \left(\frac{9}{y} - y\right)x \qquad y(0) = 5$$

```
% problem7_2_14
F = inline('[(9/y(1)-y(1))*x]','x','y');
x = 0; xStop = 4; y = [5];
H = 0.5;
[xSol,ySol] = bulStoer(F,x,y,xStop,H);
printSol(xSol,ySol,1);

>>      x            y1
   0.0000e+000   5.0000e+000
   5.0000e-001   4.6326e+000
   1.0000e+000   3.8582e+000
   1.5000e+000   3.2690e+000
   2.0000e+000   3.0485e+000
   2.5000e+000   3.0051e+000
   3.0000e+000   3.0003e+000
   3.5000e+000   3.0000e+000
   4.0000e+000   3.0000e+000
```

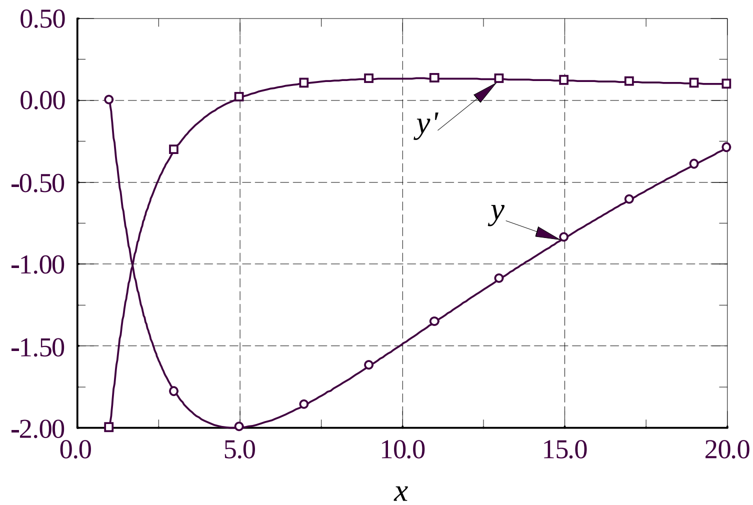
The plot is shown in Problem 13.

Problem 15

$$y'' = -\frac{1}{x}y' - \frac{1}{x^2}y \qquad y(1) = 0 \qquad y'(1) = -2$$

```
% problem7_2_15
F = inline('[y(2) -y(2)/x-y(1)/x^2]','x','y');
x = 1; xStop = 20; y = [0 -2];
H = 2;
[xSol,ySol] = bulStoer(F,x,y,xStop,H);
printSol(xSol,ySol,1)
>>      x            y1            y2
   1.0000e+000   0.0000e+000  -2.0000e+000
   3.0000e+000  -1.7812e+000  -3.0322e-001
   5.0000e+000  -1.9985e+000   1.5451e-002
   7.0000e+000  -1.8609e+000   1.0468e-001
   9.0000e+000  -1.6203e+000   1.3028e-001
   1.1000e+001  -1.3540e+000   1.3381e-001
```

```
1.3000e+001  -1.0904e+000   1.2897e-001
1.5000e+001  -8.4019e-001   1.2100e-001
1.7000e+001  -6.0704e-001   1.1210e-001
1.9000e+001  -3.9177e-001   1.0322e-001
2.0000e+001  -2.9070e-001   9.8938e-002
```



Problem 16

$$\ddot{x} = \frac{c}{m}\frac{1}{x^2} - \frac{k}{m}(x - L) \qquad x(0) = L \qquad \dot{x}(0) = 0$$

where

$$\frac{c}{m} = \frac{5}{1.0} = 5 \qquad \frac{k}{m} = \frac{120}{1.0} = 120 \qquad L = 0.2$$

In the program we use the notation $x = y_1$ and $\dot{x} = y_2$. An estimate of the period is $\tau = 2\pi/\sqrt{k/m} = 2\pi/\sqrt{120/1} = 0.57$ s, which is the period of mass-spring system. We played is safe and chose $\tau = 0.6$ s as the upper integration limit.

```
% problem7_2_16
F = inline('[y(2) 5/y(1)^2-120*(y(1)-0.2)]','x','y');
x = 0; xStop = 0.6; y = [0.2 0];
h = 0.1;
[xSol,ySol] = runKut5(F,x,y,xStop,h);
printSol(xSol,ySol,1)
```

The printout below shows the two points that span the instant when the mass returns to the starting position $x = 0.2$ m for the first time.

```
>>        x              y1              y2
   3.7567e-001   2.0045e-001  -3.3681e-001
   3.8225e-001   2.0094e-001   4.8405e-001
```

Letting $t = 0.375\,67 + \Delta t$ be the time when $\dot{x} = 0$, we obtain from Taylor series

$$\dot{x}(0.375\,67 + \Delta t) = \dot{x}(0.375\,67) + \ddot{x}(0.375\,67)\,\Delta t = 0 \tag{a}$$

But

$$\ddot{x}(0.375\,67) = \frac{5}{1}\frac{1}{0.200\,45^2} - \frac{120}{1}(0.2 - 0.200\,45) = 124.49 \text{ m/s}^2$$

so the Eq. (a) is

$$0 = -0.33681 + 124.49t \qquad \Delta t = 0.002\,706 \text{ s}$$

Therefore, the period is $\tau = 0.375\,67 + 0.002\,706 = 0.3784$ s ◄

Problem 17

$$\ddot{\theta} = \dot{\phi}^2 \sin\theta\cos\theta \qquad \ddot{\phi} = -2\dot{\theta}\dot{\phi}\cot\theta$$
$$\theta(0) = \frac{\pi}{12} \qquad \dot{\theta}(0) = 0 \qquad \phi(0) = 0 \qquad \dot{\phi} = 20 \text{ rad/s}$$

With the notation

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

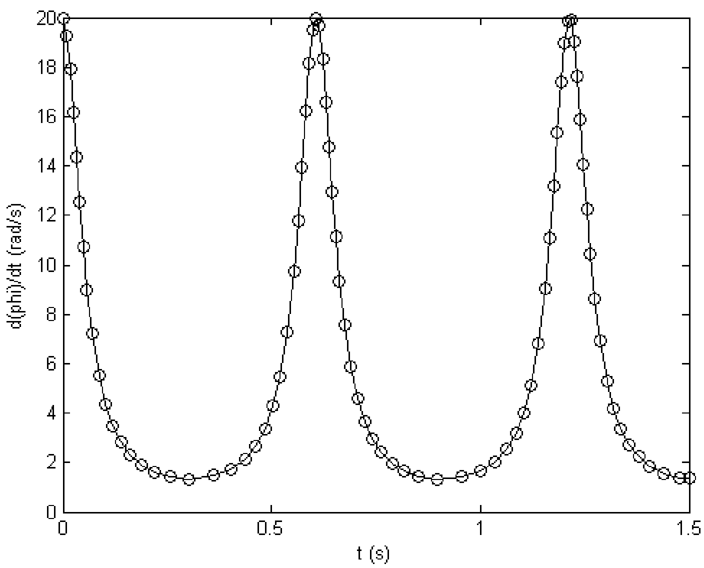
the equivalent first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_4^2 \sin y_1 \cos y_1 \\ y_4 \\ -2y_2 y_4 \cot y_1 \end{bmatrix}$$

```
% problem7_2_17
x = 0; xStop = 1.5; y = [pi/12 0 0 20];
h = 0.1;
[xSol,ySol] = runKut5(@p7_2_17,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol(:,4),'k-o')
xlabel('t (s)'); ylabel('d(phi)/dt (rad/s)')

function F = p7_2_17(x,y)
```

```
% Differential eqs. used in Problem 17, Problem Set 7.2.
F = zeros(1,4);
F(1) = y(2); F(2) = y(4)^2*sin(y(1))*cos(y(1));
F(3) = y(4); F(4) = -2*y(2)*y(4)*cot(y(1));
```



Problem 18

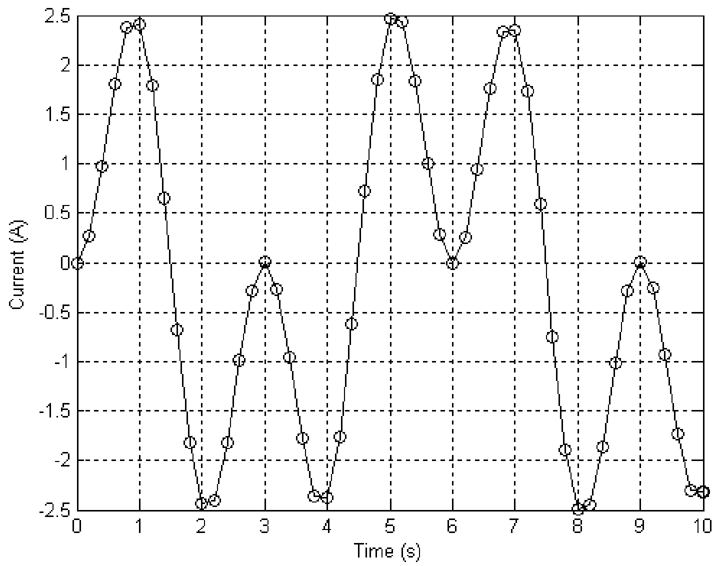
The equations in Example 7.11 were

$$\frac{di}{dt} = \left[-Ri - \frac{q}{C} + E(t)\right]\frac{1}{L} \qquad \frac{dq}{dt} = i \qquad i(0) = q(0) = 0$$

Substituting $y_1 = q$, $y_1 = i$, $R = 0$, $C = 0.45$, $L = 2$ and $E(t) = 9\sin\pi t$ V, the equivalent first-order differential equations become

$$\mathbf{F} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix} = \begin{bmatrix} y_2 \\ \left[-y_1/0.45 + 9\sin\pi x\right]/2 \end{bmatrix}$$

```
% problem7_2_18
F = inline('[y(2) (-y(1)/0.45+9*sin(pi*x))/2]','x','y');
x = 0; xStop = 10; y = [0 0];
H = 0.2;
[xSol,ySol] = bulStoer(F,x,y,xStop,H);
plot(xSol,ySol(:,2),'k-o'); grid on
xlabel('Time (s)'); ylabel('Current (A)')
```



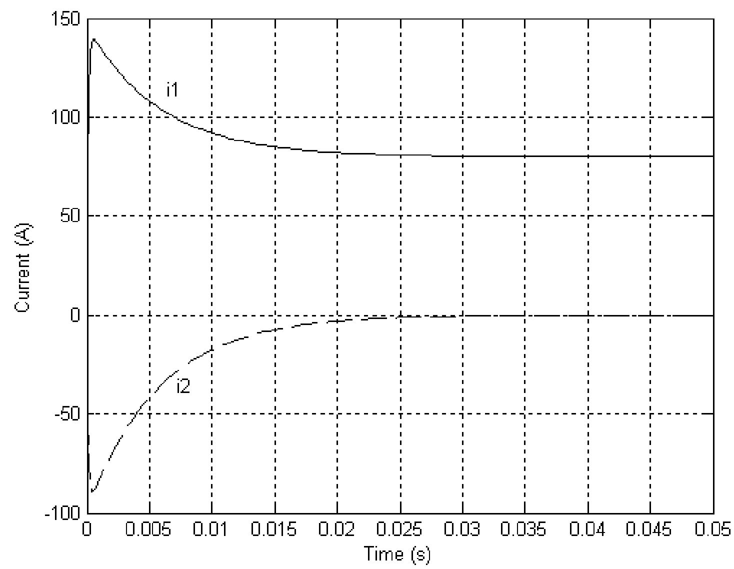
Problem 19

With constant E the equations in Problem 21, Problem Set 1 become

$$\begin{aligned}
\frac{di_1}{dt} &= (-3Ri_1 - 2Ri_2 + E)\frac{1}{L} \\
\frac{di_2}{dt} &= \left(-2\frac{di_1}{dt} - \frac{i_2}{RC}\right)\frac{1}{3} \\
i_1(0) &= i_2(0) = 0
\end{aligned}$$

```
% problem7_2_19
x = 0; xStop = 0.05; h = 0.001; y = [0 0];
[xSol,ySol] = runKut5(@p7_2_19,x,y,xStop,h);
plot(xSol,ySol(:,1),'k-'); hold on
plot(xSol,ySol(:,2),'k--'); grid on
xlabel('Time (s)'); ylabel('Current (A)')
gtext('i1'); gtext('i2')

function F = p7_2_19(x,y)
% Differential eqs. used in Problem 19, Problem Set 7.2.
R = 1; L = 0.2e-3; C = 3.5e-3; E = 240;
F = zeros(1,2);
F(1) = (-3*R*y(1) - 2*R*y(2) + E)/L;
F(2) = (-2*F(1) - y(2)/R/C)/3;
```



Problem 20

$$\begin{aligned} L\frac{di_1}{dt} + R_1 i_1 + R_2(i_1 - i_2) &= E(t) \\ L\frac{di_2}{dt} + R_2(i_2 - i_1) + \frac{q_2}{C} &= 0 \\ \frac{dq_2}{dt} &= i_2 \end{aligned}$$

$$q_2(0) = i_1(0) = i_2(0) = 0$$

Using the notation

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} q_2 \\ i_1 \\ i_2 \end{bmatrix}$$

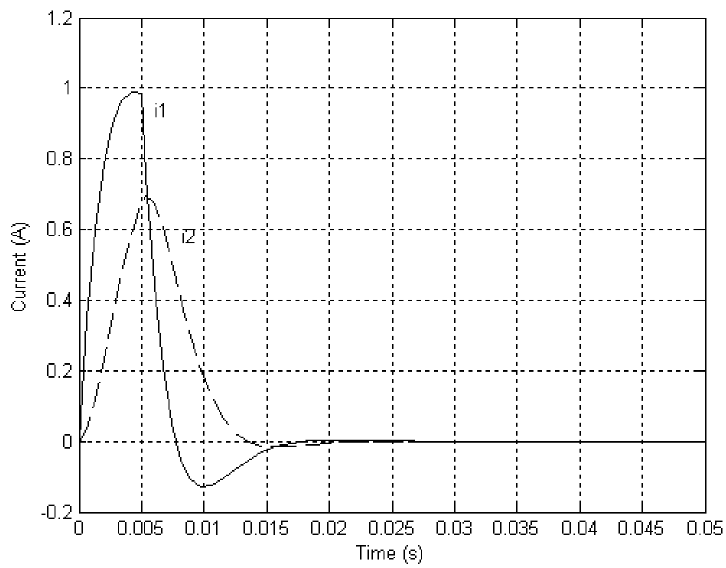
the equivalent first-order differential equations become

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} y_3 \\ \left[-(R_1 + R_2)y_2 - R_2 y_3 + E\right]/L \\ \left[-y_1/C - R_2(y_3 - y_2)\right]/L \end{bmatrix}$$

```
% problem7_2_19
x = 0; xStop = 0.05; h = 0.001; y = [0 0 0];
[xSol,ySol] = runKut5(@p7_2_19,x,y,xStop,h);
plot(xSol,ySol(:,2),'k-'); hold on
```

```
plot(xSol,ySol(:,3),'k--'); grid on
xlabel('Time (s)'); ylabel('Current (A)')
gtext('i1'); gtext('i2')

function F = p7_2_19(x,y)
% Differential eqs. used in Problem 19, Problem Set 7.2.
R1 = 4; R2 = 10; L = 0.032; C = 0.53;
if x < 0.005; E = 20;
else E = 0; end
F = zeros(1,3);
F(1) = y(3);
F(2) = (-(R1 + R2)*y(2) - R2*y(3) + E)/L;
F(3) = (-y(1)/C - R2*(y(3) - y(2)))/L;
```

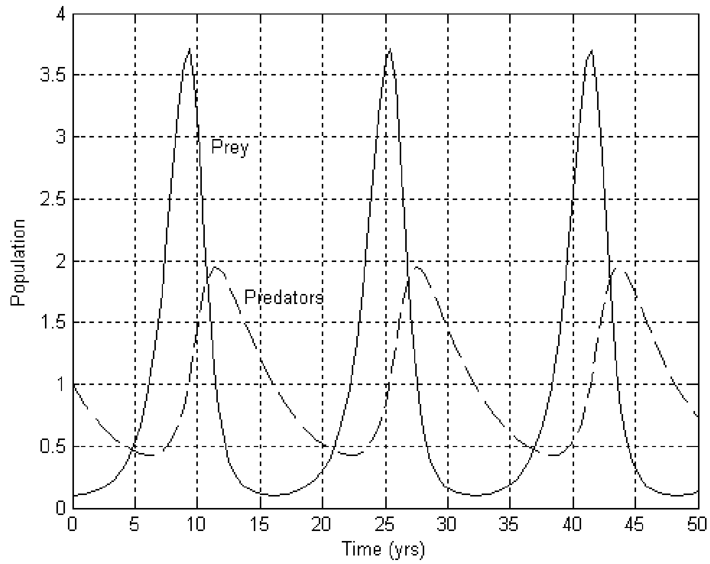


Problem 21

$$\dot{y}_1 = 1.0(y_1 - y_1 y_2) \qquad \dot{y}_2 = 0.2(-y_2 + y_1 y_2)$$
$$y_1(0) = 0.1 \qquad y_2(0) = 1$$

```
% problem7_2_21
F = inline('[y(1)-y(1)*y(2) 0.2*(-y(2)+y(1)*y(2))]','x','y');
x = 0; xStop = 50; y = [0.1 1];
h = 1.0;
[xSol,ySol] = runKut5(F,x,y,xStop,h);
plot(xSol,ySol(:,1),'k-'); hold on
```

```
plot(xSol,ySol(:,2),'k--'); grid on
xlabel('Time (yrs)'); ylabel('Population')
gtext('Prey'); gtext('Predators')
```



Problem 22

$$\dot{u} = -au + av \qquad \dot{v} = cu - v - uw \qquad \dot{w} = -bw + uv$$

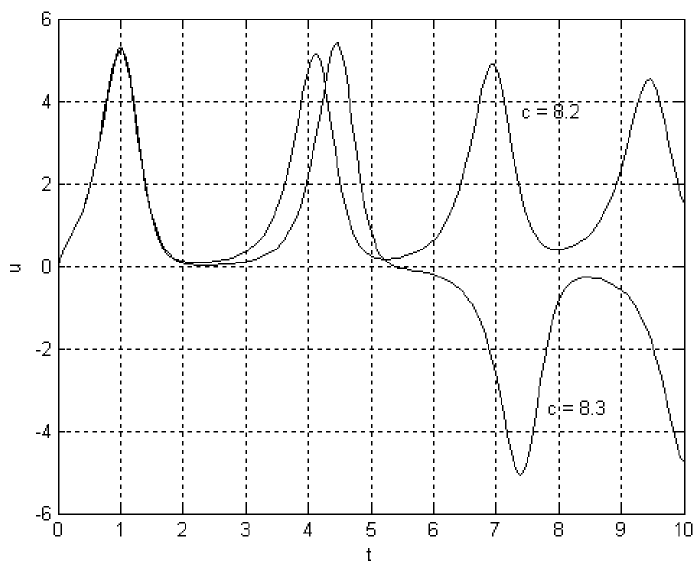
$$u(0) = 0 \qquad v(0) = 1 \qquad w(0) = 2$$

We use the notation $u = y_1$, $v = y_2$ and $w = y_3$.

```
% problem7_2_22
global C
for C = [8.2 8.3];
    x = 0; xStop = 10; y = [0 1 2];
    h = 0.1;
    [xSol,ySol] = runKut5(@p7_2_22,x,y,xStop,h);
    plot(xSol,ySol(:,1),'k-'); hold on
end
grid on
xlabel('t'); ylabel('u')
gtext('c = 8.2'); gtext('c = 8.3')

function F = p7_2_22(x,y)
% Differential eqs. used in Prob. 22, Problem Set 7.2.
```

```
global C
a = 5; b = 0.9;
F = zeros(1,3);
F(1) = -a*y(1) + a*y(2);
F(2) = C*y(1) - y(2) - y(1)*y(3);
F(3) = -b*y(3) + y(1)*y(2);
```



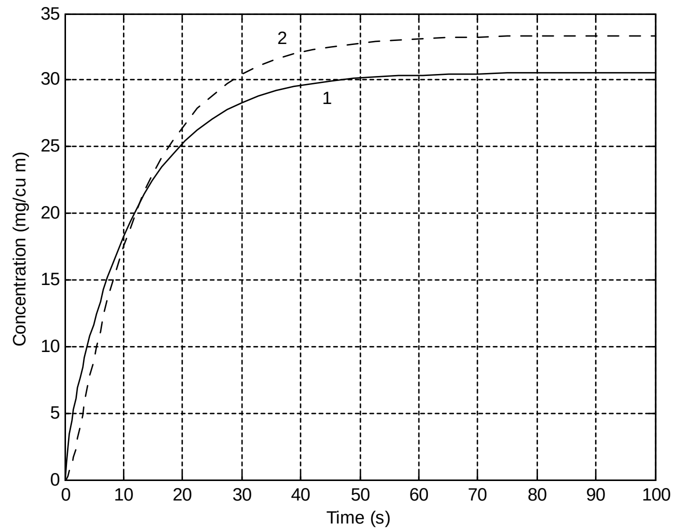
The solution is very sensitive to the values of the parameters.

Problem 23

```
% problem7_2_23
F = @(t,c) [-0.6*c(1) + 0.4*c(2) + 5;
            -0.7*c(2) + 0.3*c(3) + 0.4*c(4);
             0.4*c(1) - 0.4*c(3);
             0.2*c(1) + 0.1*c(3) - 0.4*c(4) + 5;];
[xSol,ySol] = runKut5(F,0,[0;0;0;0],100,1);
printSol(xSol,ySol,0)
plot(xSol,ySol(:,1),'k-'); hold on
plot(xSol,ySol(:,2),'k--'); grid on
xlabel ('Time (s)'); ylabel('Concentration (mg/cu m)')
gtext('1');gtext('2')
```

Only the initial and final values of the concentration are printed:

```
   x            y1           y2           y3           y4
0.0000e+000  0.0000e+000  0.0000e+000  0.0000e+000  0.0000e+000
1.0000e+002  3.0549e+001  3.3325e+001  3.0548e+001  3.5410e+001
```



In Prob. 21, Problem Set 2.2 the steady-state values were found to be

$$
\begin{aligned}
c_1 &= 30.556 \text{ mg/m}^3 \qquad c_2 = 33.333 \text{ mg/m}^3 \\
c_3 &= 30.556 \text{ mg/m}^3 \qquad c_4 = 35.417 \text{ mg/m}^3
\end{aligned}
$$

Comparing these values with the printout, we conclude that the system is very close to the steady state after 100 s.