Q1. a) The coefficient matrix should be diagonal - dominant

+5

b) Rearrange the equations to make the diagonal term dominate.

$$\begin{bmatrix} 7 & 1 & 1 \\ -3 & 7 & -1 \\ -2 & 5 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -26 \\ 1 \end{bmatrix}$$

+5

Gauss-Seidel method.

$$x_1 = \frac{1}{7}(6 - x_2 - x_3)$$

$$x_2 = \frac{1}{7}(-26 + 3x_1 + x_3)$$

$$x_3 = \frac{1}{9}(1 + 2x_1 - 5x_2)$$

+3

Starting with $x = [1\,1\,1]^T$

① $\vec{x} = \begin{pmatrix} 0.571 \\ -3.327 \\ 2.086 \end{pmatrix}$ ② $\vec{x} = \begin{pmatrix} 1.034 \\ -2.973 \\ 1.993 \end{pmatrix}$ ③ $\begin{pmatrix} 0.997 \\ -3.002 \\ 2.000 \end{pmatrix}$

④ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.000 \\ -3.000 \\ 2.000 \end{pmatrix}$

+2

c) SOR $\omega = 1.1$

$$x_1 = \frac{\omega}{7}(6 - x_2 - x_3) + (1-\omega)x_1$$

$$x_2 = \frac{\omega}{7}(-26 + 3x_1 + x_3) + (1-\omega)x_2$$

$$x_3 = \frac{\omega}{9}(1 + 2x_1 - 5x_2) + (1-\omega)x_3$$

+3

① $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.5286 \\ -3.7794 \\ 2.4611 \end{pmatrix}$ ② $\begin{pmatrix} 1.0972 \\ -2.8038 \\ 1.8577 \end{pmatrix}$ ③ $\begin{pmatrix} 0.9818 \\ -3.0506 \\ 2.0407 \end{pmatrix}$

④ $\begin{pmatrix} 1.0034 \\ -2.9870 \\ 1.9888 \end{pmatrix}$ . . .

+2

2. a) $k_{i-1} + 4k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$ $i = 1, 2, 3$

+5

b) $k_0 + 4k_1 + k_2 = -3 \Rightarrow 5k_1 + k_2 = -3$

$k_1 + 4k_2 + k_3 = 0$

$k_2 + 4k_3 + k_4 = 0 \Rightarrow k_2 + 5k_3 = 0.$

$$\Rightarrow \begin{bmatrix} 5 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

+5

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ 0 & \frac{5}{19} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 5 & 1 & 0 \\ 0 & \frac{19}{5} & 1 \\ 0 & 0 & \frac{90}{19} \end{bmatrix}$$

w/ procedure

+6

c). $Ly = b.$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ 0 & \frac{5}{19} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -3 \\ y_2 &= -\frac{y_1}{5} = \frac{3}{5} \\ y_3 &= -\frac{5}{19} y_2 = -\frac{3}{19} \end{aligned}$$

+2

$Ux = y.$

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & \frac{19}{5} & 1 \\ 0 & 0 & \frac{90}{19} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{3}{5} \\ -\frac{3}{19} \end{bmatrix} \Rightarrow \begin{aligned} x_3 &= -\frac{1}{30} \\ x_2 &= \frac{1}{6} \\ x_1 &= -\frac{19}{30} \end{aligned}$$

+2

d)

$$f_{i,i+1}(x) = \frac{k_i}{6}\left[\frac{(x_i - x_{i+1})^3}{x_i - x_{i+1}} - (x - x_{i+1})(x_i - x_{i+1})\right]$$

$$- \frac{k_{i+1}}{6}\left[\frac{(x - x_i)^3}{x_i - x_{i+1}} - (x - x_i)(x_i - x_{i+1})\right]$$

$$+ \frac{y_i(x - x_{i+1}) - y_{i+1}(x - x_i)}{x_i - x_{i+1}}$$

+5

$$y(3.4) = f_{3,4}(3.4) = -0.196$$

+5

~~Problem 17~~ Problem 3

The function to be minimized is

$$S(a,b,c) = \sum_{i=1}^{n}(z_i - a - bx_i - cy_i)^2$$

+5

which yields

$$\begin{aligned}
\frac{\partial S}{\partial a} &= -2\sum_{i=1}^{n}(z_i - a - bx_i - cy_i) = 0 \\
\frac{\partial S}{\partial b} &= -2\sum_{i=1}^{n}x_i(z_i - a - bx_i - cy_i) = 0 \\
\frac{\partial S}{\partial c} &= -2\sum_{i=1}^{n}y_i(z_i - a - bx_i - cy_i) = 0
\end{aligned}$$

+9.

This can be written as

$$\begin{aligned}
na + b\sum_{i=1}^{n}x_i + c\sum_{i=1}^{n}y_i &= \sum_{i=1}^{n}z_i \\
a\sum_{i=1}^{n}x_i + b\sum_{i=1}^{n}x_i^2 + c\sum_{i=1}^{n}x_iy_i &= \sum_{i=1}^{n}x_iz_i \\
a\sum_{i=1}^{n}y_i + b\sum_{i=1}^{n}x_iy_i + c\sum_{i=1}^{n}y_i^2 &= \sum_{i=1}^{n}y_iz_i
\end{aligned}$$

Q.E.D.

Problem 18

The normal equations to be solved are

$$\begin{bmatrix} n & \Sigma x_i & \Sigma y_i \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_iy_i \\ \Sigma y_i & \Sigma x_iy_i & \Sigma y_i^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \Sigma z_i \\ \Sigma x_iz_i \\ \Sigma y_iz_i \end{bmatrix}$$

+5

From the given data we have

$$\begin{aligned}
n &= 6 \qquad \Sigma x_i = 7 \qquad \Sigma y_i = 4 \qquad \Sigma z_i = 5.88 \\
\Sigma x_i^2 &= 13 \qquad \Sigma y_i^2 = 6 \qquad \Sigma x_iy_i = 6 \\
\Sigma x_iz_i &= 4.44 \qquad \Sigma y_iz_i = 4.55
\end{aligned}$$

+9 ~~5~~

Thus the normal equations are

$$\begin{bmatrix} 6 & 7 & 4 \\ 7 & 13 & 6 \\ 4 & 6 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5.88 \\ 4.44 \\ 4.55 \end{bmatrix}$$ ~~+2~~

The solution is

$$a = 1.413 \qquad b = -0.621 \qquad c = 0.438$$ +2

so that the fitting function becomes

$$f(x, y) = 1.413 - 0.621x + 0.438y \blacktriangleleft$$

4. a) $\vec{F} = \begin{bmatrix} x^2+y^2-1 \\ x+y-1 \end{bmatrix}$ +2

$\mathbb{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix}$ + 3

b) ① Estimate the solution vector $\vec{x}$

② Evaluate $\vec{F}(\vec{x})$

③ Compute the Jacobian matrix $J(\vec{x})$

④ Solve for $\Delta\vec{x}$ from $J(\vec{x})\Delta\vec{x} = -\vec{F}(\vec{x})$

⑤ Let $\vec{x} \leftarrow \vec{x} + \Delta\vec{x}$ and repeat steps ② ~ ⑤ until $|\Delta\vec{x}| < \varepsilon$.

+10

c) $(x_0, y_0) = (0.5, 1.5)^T$

$\vec{F} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$, $\vec{J} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, $\Delta\vec{x} = -J^{-1}(\vec{x}) \cdot \vec{F}(\vec{x})$

$= \begin{bmatrix} -0.75 \\ -0.25 \end{bmatrix}$

+ 2

$(x_1, y_1) = (x_0, y_0) + (-0.75, -0.25)$

$= (-0.25, 1.25)$

In the same way, $(x_2, y_2) = (-0.0417, 1.0417)$

$(x_3, y_3) = (-0.0016, 1.0016)$

+ 3