CSE232: Discrete Mathematics
Assignment 2

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Each question is worth 10 Marks, so the total is 100 marks.

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1. Prove that $(p \to q) \lor (q \to p)$ is a tautology.

(a) using a truth table,

(b) using logical equivalences.

Answer a. The last column of the table is entirely true, so it is a tautology.

p	q	$p \to q$	$q \to p$	$(p \to q) \lor (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Answer b. We show that the statement is always true using logical equivalences.

$$\begin{aligned}
(p \to q) \lor (q \to p) &\equiv (\neg p \lor q) \lor (\neg q \lor p) && \text{by conditional-disjunction equivalence} \\
&\equiv \neg p \lor q \lor \neg q \lor p && \text{by associative law} \\
&\equiv \neg p \lor p \lor \neg q \lor q && \text{by commutative law} \\
&\equiv \mathrm{T} \lor \neg q \lor q && \text{by negation law} \\
&\equiv \mathrm{T} && \text{by domination law}
\end{aligned}$$

2. For each of the two statements below, determine whether it is true or false. Justify your answers.

(a) $(p \oplus q) \land r \equiv (p \land r) \oplus (q \land r)$

(b) $(p \oplus q) \lor r \equiv (p \lor r) \oplus (q \lor r)$

Answer a. This statement is **true**. Proof: In the truth table below, the third column and the last column are identical.

p	q	r	$p \oplus q$	$(p \oplus q) \wedge r$	$p \wedge r$	$q \wedge r$	$(p \wedge r) \oplus (q \wedge r)$
T	T	T	F	F	T	T	F
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

Answer b. This statement is **false**. Proof: If p, q and r are true, then $(p \oplus q) \vee r$ is true, but $(p \vee r) \oplus (q \vee r)$ is false.

3. Let $F(x)$ be the statement "x plays football," let $V(x)$ be the statement "x plays video games," and let $P(x)$ be the statement "x eats popcorn." Express each of these statements in terms of $F(x)$, $V(x)$, $P(x)$, quantifiers, and logical connectives. Let the domain consist of all the students in your class.

(a) Each student in your class plays video games or football.

(b) Some student in your class plays video games and football.

(c) Each student in your class who plays video games eats popcorn.

(d) No student in your class plays football and eats popcorn.

(e) There is exactly one student in your class who eats popcorn, and plays video games and football.

Answer.

(a) $\forall x(F(x) \vee V(x))$

(b) $\exists x(F(x) \wedge V(x))$

(c) $\forall x(V(x) \to P(x))$

(d) $\forall x \neg(F(x) \wedge P(x))$ or $\neg \exists x(F(x) \wedge P(x))$

(e) $\exists! x(P(x) \wedge V(x) \wedge F(x))$

4. Determine whether the following compound propositions are satisfiable.

(a) $(\neg p \vee \neg q) \wedge (p \to q)$.

(b) $(p \to q) \wedge (q \to \neg p) \wedge (p \vee q)$.

Answer a. Setting $p = \mathrm{F}$ and $q = \mathrm{T}$ makes the compound proposition true; therefore it is satisfiable.

Answer b. Setting $q = \mathrm{T}$ and $p = \mathrm{F}$ makes the compound proposition true; therefore it is satisfiable.

5. What is the truth value of $\forall n \exists p \left(p^2 \leqslant n < (p+1)^2\right)$ where the domain of the quantifiers is $\mathbb{N}$? Justify your answer.

Answer. Yes, it is **true**. Let $n \in \mathbb{N}$. Then consider the largest integer p such that $p^2 \leqslant n$. Then we must have $n < (p+1)^2$. So for each $n \in \mathbb{N}$, we can find $p \in \mathbb{N}$ such that $p^2 \leqslant n < (p+1)^2$.

6. Find a proposition with three variables p, q, and r

(a) that is true when p and r are true and q is false, and false otherwise.

(b) that is never true.

Answer a. The answer is $p \wedge \neg q \wedge r$.

Answer b. The answer is $(p \wedge \neg p) \vee (q \wedge \neg q) \vee (r \wedge \neg r)$.

7. Express the negations of each statements so that all negation symbols immediately precede predicates (for example, the negation of the statement $\forall x(P(x))$ can be expressed as $\exists x(\neg P(x))$).

(a) $\forall x \exists y \forall z\, T(x, y, z)$

(b) $\forall x \exists y\, P(x, y) \vee \forall x \exists y\, Q(x, y)$

(c) $\forall x \exists y\, (P(x, y) \wedge \exists\, z R(x, y, z))$

(d) $\forall x \exists y\, (P(x, y) \rightarrow Q(x, y))$

Answer a.

$$\begin{aligned}\neg \forall x \exists y \forall z\, T(x, y, z) &\equiv \exists x \neg \exists y \forall z\, T(x, y, z) \\ &\equiv \exists x \forall y \neg \forall z\, T(x, y, z) \\ &\equiv \exists x \forall y \exists z\, \neg T(x, y, z)\end{aligned}$$

Answer b.

$$\begin{aligned}\neg(\forall x \exists y\, P(x, y) \vee \forall x \exists y\, Q(x, y)) &\equiv (\neg \forall x \exists y\, P(x, y)) \wedge (\neg \forall x \exists y\, Q(x, y)) \\ &\equiv (\exists x \forall y\, \neg P(x, y)) \wedge (\exists x \forall y\, \neg Q(x, y))\end{aligned}$$

Answer c.

$$\begin{aligned}\neg\forall x\exists y\,(P(x,y)\wedge\exists z R(x,y,z)) &\equiv \exists x\forall y\,\neg(P(x,y)\wedge\exists z\,R(x,y,z))\\ &\equiv \exists x\forall y\,(\neg P(x,y)\vee\neg\exists z\,R(x,y,z))\\ &\equiv \exists x\forall y\,(\neg P(x,y)\vee\forall z\,\neg R(x,y,z))\end{aligned}$$

Answer d.

$$\begin{aligned}\neg\forall x\exists y\,(P(x,y)\to Q(x,y)) &\equiv \exists x\forall y\,\neg(P(x,y)\to Q(x,y))\\ &\equiv \exists x\forall y\,(P(x,y)\wedge\neg Q(x,y))\end{aligned}$$

8. Prove that for any integer n, the relation below is true.

$$\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor=n$$

Answer. We make a proof by case. We have two cases: n is even or n is odd.

First suppose that n is even. Then there exists $k\in\mathbb{Z}$ such that $n=2k$. It follows that

$$\begin{aligned}\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor &= \left\lfloor\frac{2k}{2}\right\rfloor+\left\lfloor\frac{2k+1}{2}\right\rfloor\\ &= \lfloor k\rfloor+\left\lfloor k+\frac{1}{2}\right\rfloor\\ &= k+k+\left\lfloor\frac{1}{2}\right\rfloor && \text{because } k \text{ is an integer}\\ &= k+k\\ &= n.\end{aligned}$$

Now suppose that n is odd. Then there exists $k\in\mathbb{Z}$ such that $n=2k+1$. It follows that

$$\begin{aligned}\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor &= \left\lfloor\frac{2k+1}{2}\right\rfloor+\left\lfloor\frac{2k+2}{2}\right\rfloor\\ &= \left\lfloor k+\frac{1}{2}\right\rfloor+\lfloor k+1\rfloor\\ &= k+(k+1) && \text{because } k \text{ is an integer}\\ &= n.\end{aligned}$$

9. Given two integers u and v, prove the following.

(a) uv is odd if and only if u and v are odd.

(b) $u+v$ is odd if and only if exactly one of the two numbers u and v is odd.

Answer a. Suppose u and v are odd. Then there exists $k, m \in \mathbb{Z}$ such that $u = 2k + 1$ and $v = 2m + 1$. Hence $uv = (2k + 1)(2m + 1) = 4km + 2k + 2m + 1 = 2(2km + k + m) + 1$. Therefore uv is odd.

We now prove that, if uv is odd, then u and v must be odd. We make a proof by contraposition. So we assume that u or v is even. Without loss of generality, suppose u is even. Then there exists $k \in \mathbb{Z}$ such that $u = 2k$. So $uv = 2(kv)$ is even.

Answer b. We first prove that, if $u + v$ is odd, then exactly one of the two numbers u and v is odd. We make a proof by contraposition. So we assume that either u and v are even, or u and v are odd.

- In the first case, there exist $k, m \in \mathbb{Z}$ such that $u = 2k$ and $v = 2m$. So $u + v = 2k + 2m = 2(k + m)$ is even.
- In the second case, we have $u = 2k + 1$ and $v = 2m + 1$ for some $m, k \in \mathbb{Z}$. So $u + v = 2k + 1 + 2m + 1 = 2(k + m + 1)$ is even.

We now prove that if exactly one of u and v is odd, then $u + v$ is odd. Without loss of generality, we assume that u is even and v is odd. So there exist $k, m \in \mathbb{Z}$ such that $u = 2k$ and $v = 2m + 1$. Then $u + v = 2k + 2m + 1 = 2(k + m) + 1$ is odd.

10. Suppose that a, b and c are odd integers. Assume that a real number x satisfies the equation $ax^2 + bx + c = 0$. Prove that x is irrational. [Hint: Use Question 9.]

Answer. We make a proof by contradiction. So suppose that x is rational. Then it can be written $x = p/q$, where p and q are integers with no common factor. Therefore, p and q cannot both be even, and we have

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0.$$

Multiplying this equation by q^2, we obtain

$$ap^2 + bpq + cq^2 = 0.$$

Now there are two cases.

- If p and q are odd, then by Question 6, ap^2, bpq and cq^2 are odd. So their sum is odd. But this is impossible because this sum is equal to 0.
- Now suppose that exactly one of p and q is odd. WLOG, assume p is even and q is odd. Then by Question 6, ap^2 and bpq are even, and cq^2 is odd. So their sum is odd. We reach the same contradiction.