PROBLEM SET 9.1

Problem 1

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 1 \\ 3 & 9 & 6 \\ 1 & 6 & 8 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

From Eqs. (9.26):

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{L}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 7/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1 \\ 1/4 & 1 & 2 \end{bmatrix} \quad \blacktriangleleft$$

$$\mathbf{x} = (\mathbf{L}^{-1})^T \mathbf{z} = \begin{bmatrix} z_1/2 \\ z_2/3 \\ z_3/2 \end{bmatrix} \quad \blacktriangleleft$$

Problem 2

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Choleski's decomposition of **B**:

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{array}{lll}
L_{11}^2 = 2 & & L_{11} = \sqrt{2} \\
L_{11}L_{21} = -1 & \sqrt{2}L_{21} = -1 & L_{21} = -1/\sqrt{2} \\
L_{31}L_{11} = 0 & & L_{31} = 0 \\
L_{21}^2 + L_{22}^2 = 2 & 1/2 + L_{22}^2 = 2 & L_{22} = \sqrt{3/2} \\
L_{31}L_{21} + L_{32}L_{22} = -1 & L_{32}\sqrt{3/2} = -1 & L_{32} = -\sqrt{2/3} \\
L_{31}^2 + L_{32}^2 + L_{33}^2 = 1 & 2/3 + L_{33}^2 = 1 & L_{33} = \sqrt{1/3}
\end{array}$$

$$\mathbf{L} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{1/2} & \sqrt{3/2} & 0 \\ 0 & -\sqrt{2/3} & \sqrt{1/3} \end{bmatrix}$$

Inversion of $\mathbf{L}$:

$$\begin{bmatrix} L_{11}^{-1} & 0 & 0 \\ L_{21}^{-1} & L_{22}^{-1} & 0 \\ L_{31}^{-1} & L_{32}^{-1} & L_{33}^{-1} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{1/2} & \sqrt{3/2} & 0 \\ 0 & -\sqrt{2/3} & \sqrt{1/3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{lll}
L_{11}^{-1}\sqrt{2} = 1 & & L_{11}^{-1} = \sqrt{1/2} \\
L_{22}^{-1}\sqrt{3/2} = 1 & & L_{22}^{-1} = \sqrt{2/3} \\
L_{21}^{-1}\sqrt{2} - L_{22}^{-1}\sqrt{1/2} = 0 & L_{21}^{-1}\sqrt{2} - \sqrt{2/3}\sqrt{1/2} = 0 & L_{21}^{-1} = \sqrt{1/6} \\
L_{33}^{-1}\sqrt{1/3} = 1 & & L_{33}^{-1} = \sqrt{3} \\
L_{32}^{-1}\sqrt{3/2} - L_{33}^{-1}\sqrt{2/3} = 0 & L_{32}^{-1}\sqrt{3/2} - \sqrt{3}\sqrt{2/3} = 0 & L_{32}^{-1} = \sqrt{4/3} \\
L_{31}^{-1}\sqrt{2} - L_{32}^{-1}\sqrt{1/2} = 0 & L_{31}^{-1}\sqrt{2} - \sqrt{4/3}\sqrt{1/2} = 0 & L_{31}^{-1} = \sqrt{1/3}
\end{array}$$

$$\mathbf{L}^{-1} = \begin{bmatrix} \sqrt{1/2} & 0 & 0 \\ \sqrt{1/6} & \sqrt{2/3} & 0 \\ \sqrt{1/3} & \sqrt{4/3} & \sqrt{3} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{H} &= \begin{bmatrix} \sqrt{1/2} & 0 & 0 \\ \sqrt{1/6} & \sqrt{2/3} & 0 \\ \sqrt{1/3} & \sqrt{4/3} & \sqrt{3} \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} \sqrt{1/2} & \sqrt{1/6} & \sqrt{1/3} \\ 0 & \sqrt{2/3} & \sqrt{4/3} \\ 0 & 0 & \sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} 2 & \sqrt{1/3} & \sqrt{2/3} \\ \sqrt{1/3} & 8/3 & 5\sqrt{2}/3 \\ \sqrt{2/3} & 5\sqrt{2}/3 & 40/3 \end{bmatrix} = \begin{bmatrix} 2.0000 & 0.5774 & 0.8165 \\ 0.5774 & 2.6667 & 2.3570 \\ 0.8165 & 2.3570 & 13.3333 \end{bmatrix} \blacktriangleleft
\end{aligned}$$

Problem 3

$$\begin{aligned}
\mathbf{A}^* &= \mathbf{A} - s\mathbf{B} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} - 2.5 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1.0 & 1.5 & 0 \\ 1.5 & -1.0 & 1.5 \\ 0 & 1.5 & 1.5 \end{bmatrix}
\end{aligned}$$

<u>First iteration</u>

$$\mathbf{Bv} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Solve $\mathbf{A}^*\mathbf{z} = \mathbf{Bv}$:

$$\begin{bmatrix} -1.0 & 1.5 & 0 \\ 1.5 & -1.0 & 1.5 \\ 0 & 1.5 & 1.5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} -14 \\ -8 \\ 8 \end{bmatrix}$$

$$v = \frac{\mathbf{z}}{|\mathbf{z}|} = \begin{bmatrix} -14 \\ -8 \\ 8 \end{bmatrix} \frac{1}{18} = \begin{bmatrix} -0.7778 \\ -0.4444 \\ 0.4444 \end{bmatrix}$$

$$\lambda = s - \frac{1}{|\mathbf{z}|} = 2.5 - \frac{1}{18} = 2.4444$$

<u>Second iteration</u>

$$\mathbf{Bv} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -0.7778 \\ -0.4444 \\ 0.4444 \end{bmatrix} = \begin{bmatrix} -1.1112 \\ -0.5554 \\ 0.8888 \end{bmatrix}$$

Solve $\mathbf{A}^*\mathbf{z} = \mathbf{Bv}$:

$$\begin{bmatrix} -1.0 & 1.5 & 0 \\ 1.5 & -1.0 & 1.5 \\ 0 & 1.5 & 1.5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1.1112 \\ -0.5554 \\ 0.8888 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 19.778 \\ 12.444 \\ -11.852 \end{bmatrix}$$

$$v = \frac{\mathbf{z}}{|\mathbf{z}|} = \begin{bmatrix} 19.778 \\ 12.444 \\ -11.852 \end{bmatrix} \frac{1}{26.20} = \begin{bmatrix} 0.7549 \\ 0.4750 \\ -0.4524 \end{bmatrix}$$

$$\lambda = s - \frac{1}{|\mathbf{z}|} = 2.5 - \frac{1}{26.20} = 2.4618 \blacktriangleleft$$

Problem 4

$$\mathbf{S} = \begin{bmatrix} 150 & -60 & 0 \\ -60 & 120 & 0 \\ 0 & 0 & 80 \end{bmatrix} \text{MPa}$$

The characteristic equation is

$$|\mathbf{S}-\lambda\mathbf{I}| = 0 \qquad \begin{vmatrix} 150-\lambda & -60 & 0 \\ -60 & 120-\lambda & 0 \\ 0 & 0 & 80-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (80-\lambda)\left[(150-\lambda)(120-\lambda) - 60^2\right] &= 0 \\ (80-\lambda)(14400 - 270\lambda + \lambda^2) &= 0 \end{aligned}$$

The solution (principal stresses) is

$$\lambda_1 = 73.15 \text{ MPa} \qquad \lambda_2 = 80 \text{ MPa} \qquad \lambda_3 = 196.85 \text{ MPa} \blacktriangleleft$$

Problem 5

$$\begin{aligned} kL(\theta_2-\theta_1)-mg\theta_1 &= mL\ddot{\theta}_1 \\ -kL(\theta_2-\theta_1)-2mg\theta_2 &= 2mL\ddot{\theta}_2 \end{aligned}$$

Substituting $\theta_i = x_i \sin\omega t$ we get

$$\begin{aligned} [kL(x_2-x_1)-mgx_1]\sin\omega t &= -\omega^2 mLx_1\sin\omega t \\ [-kL(x_2-x_1)-2mgx_2]\sin\omega t &= -2\omega^2 mLx_2\sin\omega t \end{aligned}$$

$$\begin{bmatrix} kL+mg & -kL \\ -kL & kL+2mg \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \omega^2 mL\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1+mg/(kL) & -1 \\ -1 & 1+2mg/(kL) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \omega^2\frac{m}{k}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using

$$\frac{mg}{kL} = \frac{0.25(9.80665)}{20(0.75)} = 0.16344 \qquad \lambda = \omega^2\frac{m}{k}$$

the equations of motion become

$$\begin{bmatrix} 1.16344-\lambda & -1 \\ -1 & 1.32688-2\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{a}$$

The characteristic equation is

$$\begin{aligned} \begin{vmatrix} 1.16344-\lambda & -1 \\ -1 & 1.32688-2\lambda \end{vmatrix} &= 0 \\ (1.16344-\lambda)(1.32688-2\lambda)-1 &= 0 \\ 0.54375-3.65376\lambda+2\lambda^2 &= 0 \end{aligned}$$

$$\lambda_1 = 0.16344 \qquad \lambda_2 = 1.66344$$

The circular frequencies are

$$\begin{aligned} \omega_1 &= \sqrt{\lambda_1\frac{k}{m}} = \sqrt{0.16344\frac{20}{0.25}} = 3.616 \text{ rad/s} \blacktriangleleft \\ \omega_2 &= \sqrt{\lambda_2\frac{k}{m}} = \sqrt{1.66344\frac{20}{0.25}} = 11.536 \text{ rad/s} \blacktriangleleft \end{aligned}$$

Substituting $\lambda = \lambda_1$, Eq. (a) becomes

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

yielding $x_1 = x_2$. Hence the (normalized) relative amplitudes of the first mode are

$$x_1 = x_2 = \frac{1}{\sqrt{2}} \quad \blacktriangleleft$$

With $\lambda = \lambda_2$, Eq. (a) is

$$\begin{bmatrix} -0.5 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $x_1 = -2x_2$, so that the relative amplitudes of the second mode are

$$x_1 = \frac{1}{\sqrt{5}} \qquad x_2 = -\frac{2}{\sqrt{5}} \quad \blacktriangleleft$$

Problem 6

$$\begin{aligned} 3i_1 - i_2 - i_3 &= -LC\frac{d^2 i_1}{dt^2} \\ -i_1 + i_2 &= -LC\frac{d^2 i_2}{dt^2} \\ -i_1 + i_3 &= -LC\frac{d^2 i_3}{dt^2} \end{aligned}$$

Let $i_k = x_k \sin \omega t$. Then the equations become (after cancelling $\sin \omega t$)

$$\begin{aligned} 3x_1 - x_2 - x_3 &= \omega^2 LC x_1 \\ -x_1 + x_2 &= \omega^2 LC x_2 \\ -x_1 + x_3 &= \omega^2 LC x_3 \end{aligned}$$

or

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{a}$$

where $\lambda = \omega^2 LC$. The characteristic equation is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} 1 - 5\lambda + 5\lambda^2 - \lambda^3 &= 0 \\ (1-\lambda)\left(\lambda^2 - 4\lambda + 1\right) &= 0 \end{aligned}$$

$$\lambda_1 = 0.2679 \qquad \lambda_2 = 1 \qquad \lambda_3 = 3.7321$$

The circular frequencies are

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{\lambda_1}{LC}} = \sqrt{\frac{0.2679}{LC}} = \frac{0.5176}{\sqrt{LC}} \\
\omega_2 &= \sqrt{\frac{\lambda_2}{LC}} = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}} \\
\omega_3 &= \sqrt{\frac{\lambda_3}{LC}} = \sqrt{\frac{3.7321}{LC}} = \frac{1.9319}{\sqrt{LC}}
\end{aligned}$$

<u>First mode</u>: substitute λ_1 into Eqs. (a):

$$\begin{bmatrix} 2.7321 & -1 & -1 \\ -1 & 0.7321 & 0 \\ -1 & 0 & 0.7321 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Choosing $x_1 = 1$, the second and third equations yield

$$\begin{aligned}
-1 + 0.7321 x_2 &= 0 \qquad x_2 = 1.3659 \\
-1 + 0.7321 x_3 &= 0 \qquad x_3 = 1.3659
\end{aligned}$$

After normalizing we have

$$\mathbf{x} = \begin{bmatrix} 0.4597 & 0.6280 & 0.6280 \end{bmatrix}^T \quad \blacktriangleleft$$

<u>Second mode</u>: substituting λ_2 into Eqs. (a) we get

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0 \qquad x_2 = x_3$$

$$\mathbf{x} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T \quad \blacktriangleleft$$

<u>Third mode</u>: with $\lambda = \lambda_3$ Eqs. (a) are

$$\begin{bmatrix} -0.7321 & -1 & -1 \\ -1 & -2.7321 & 0 \\ -1 & 0 & -2.7321 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Choosing $x_1 = 1$, the second and third equations yield

$$\begin{aligned}
-1 - 2.7321 x_2 &= 0 \qquad x_2 = -0.3660 \\
-1 - 2.7321 x3 &= 0 \qquad x_3 = -0.3660
\end{aligned}$$

After normalizing we have

$$\mathbf{x} = \begin{bmatrix} 0.8881 & -0.3250 & -0.3250 \end{bmatrix}^T \quad \blacktriangleleft$$

Problem 7

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 & 1 \\ -1 & 6 & -2 & 0 \\ 0 & -2 & 3 & 2 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

From Eqs. (9.15)-(9.17) and (9.19):

$$\begin{aligned}
\phi &= -\frac{A_{11} - A_{44}}{2A_{14}} = -\frac{4-4}{2(1)} = 0 \\
t &= \frac{\operatorname{sgn}(\phi)}{|\phi| + \sqrt{\phi^2+1}} = \frac{-1}{|0| + \sqrt{0^2+1}} = -1 \\
c &= \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+(-1)^2}} = 0.7071 \qquad s = tc = -0.7071 \\
\tau &= \frac{s}{1+c} = \frac{-0.7071}{1+0.7071} = -0.4142
\end{aligned}$$

From Eq. (9.18):

$$\begin{aligned}
A_{11}^* &= A_{11} - tA_{14} = 4 - (-1)(1) = 5 \\
A_{44}^* &= A_{44} + tA_{14} = 4 + (-1)(1) = 3 \\
A_{12}^* &= A_{12} - s(A_{42} + \tau A_{12}) = -1 - (-0.7071)\,[0 + (-0.4142)(-1)] \\
&= -0.7071 \\
A_{13}^* &= A_{13} - s(A_{43} + \tau A_{13}) = 0 - (-0.7071)\,[2 + (-0.4142)(0)] = 1.4142 \\
A_{42}^* &= A_{42} + s(A_{12} - \tau A_{42}) = 0 + (-0.7071)\,[-1 - (-0.4142)(0)] = 0.7071 \\
A_{43}^* &= A_{43} + s(A_{13} - \tau A_{43}) = 2 + (-0.7071)\,[0 - (-0.4142)(2)] = 1.4142
\end{aligned}$$

$$\mathbf{A}^* = \begin{bmatrix} 5.0000 & -0.7071 & 1.4142 & 0 \\ -0.7071 & 6.0000 & -2.0000 & 0.7071 \\ 1.4142 & -2.0000 & 3.0000 & 1.4142 \\ 0 & 0.7071 & 1.4142 & 3.0000 \end{bmatrix} \blacktriangleleft$$

Problem 8

```
% problem9_1_8
A = [4 -1 -2; -1 3 3; -2 3 1];
[eVals,eVecs] = jacobi(A);
eigenvalues = eVals'
eigenvectors = eVecs(:,1:3)
```

```
>> eigenvalues =
    2.6916    6.6956   -1.3872
eigenvectors =
    0.7636   -0.6102    0.2114
    0.6168    0.5923   -0.5184
    0.1911    0.5262    0.8286
```

Problem 9

```
% problem9_1_9
A = [4 -2  1 -1
    -2  4 -2  1
     1 -2  4 -2
    -1  1 -2  4];
[eVals,eVecs] = jacobi(A);
eigenvalues = eVals'
eigenvectors = eVecs(:,1:4)

>> eigenvalues =
    3.6180    1.3820    8.5414    2.4586
eigenvectors =
    0.6015    0.3717    0.4571   -0.5395
   -0.3717    0.6015   -0.5395   -0.4571
   -0.3717    0.6015    0.5395    0.4571
    0.6015    0.3717   -0.4571    0.5395
```

Problem 10

```
function [eVal,eVec] = powerMethod(A,maxIter,tol)
% Power method for finding the largest eigenvalue of A and
% the corresponding eigenvector.
% USAGE: [eVal,eVec] = invPower(A,maxIter,tol)
% maxIter = limit on number of iterations (default is 50).
% tol = error tolerance (default is 1.0e-6).

if nargin < 3; tol = 1.0e-6; end
if nargin < 2; maxIter = 50; end
n = size(A,1);
v = rand(n,1);
```

```
vMag = v/sqrt(dot(v,v));
for i = 1:maxIter
    z = A*v;
    zMag = sqrt(dot(z,z)); z = z/zMag;
    if sqrt(dot(v - z,v - z)) < tol
        if dot(v,z) > 0; eVal = zMag;
        else; eVal = -zMag; end
        eVec = z; return
    end
    v = z;
end
error('Too many iterations')

% problem9_1_10
A = [4 -2  1 -1
    -2  4 -2  1
     1 -2  4 -2
    -1  1 -2  4];
[eVal,eVec] = powerMethod(A);
eigenvalue = eVal
eigenvector = eVec

>> eigenvalue =
    8.5414
eigenvector =
   -0.4571
    0.5395
   -0.5395
    0.4571
```

Problem 11

```
% problem9_1_11
A = [4 -2  1 -1
    -2  4 -2  1
     1 -2  4 -2
    -1  1 -2  4];
[eVal,eVec] = invPower(A,0);
eigenvalue = eVal
eigenvector = eVec

>> eigenvalue =
```

```
    1.3820
eigenvector =
    0.3717
    0.6015
    0.6015
    0.3717
```

Problem 12

```
% problem9_1_12 (Jacobi method)
A = [1.4 0.8 0.4
     0.8 6.6 0.8
     0.4 0.8 5.0];
B = [0.4 -0.1  0.0
    -0.1  0.4 -0.1
     0.0 -0.1  0.4];
[H,T] = stdForm(A,B);           % Convert to std. form
[eVal,Z] = jacobi(H);           % Solve by Jacobi method.
X = T*Z;                        % Eigenvector of orig. prob.
for i = 1:size(A,1)             % Normalize eigenvector
    xMag = sqrt(dot(X(:,i),X(:,i)));
    X(:,i) = X(:,i)/xMag;
end
[eVals,X] = sortEigen(eVals,X); % Sort in ascending order
eigenvalues = eVal'
eigenvectors = X

>> eigenvalues =
    2.9277    9.9029   25.5980
eigenvectors =
    0.9810   -0.1871    0.3225
   -0.1876   -0.4614    0.7854
   -0.0489    0.8672    0.5283
```

Problem 13

```
% problem9_1_13 (Inverse power method)
A = [1.4 0.8 0.4
     0.8 6.6 0.8
```

```
     0.4 0.8 5.0];
B = [0.4 -0.1  0.0
    -0.1  0.4 -0.1
     0.0 -0.1  0.4];
[H,T] = stdForm(A,B);       % Convert to std. form
[eVal,z] = invPower(H,0);   % Solve by inv. power mthd.
x = T*z;                    % Eigenvector of orig. prob.
x = x/sqrt(dot(x,x));       % Normalize eigenvector
eigenvalue = eVal'
eigenvector = x

>> eigenvalue =
    2.9277
eigenvector =
   -0.9810
    0.1876
    0.0489
```

Problem 14

```
% problem9_1_14 (Jacobi)
A = [ 11.0, 2.0,  3.0,  1.0,  4.0, 2.0
       2.0, 9.0,  3.0,  5.0,  2.0, 1.0
       3.0, 3.0, 15.0,  4.0,  3.0, 2.0
       1.0, 5.0,  4.0, 12.0,  4.0, 3.0
       4.0, 2.0,  3.0,  4.0, 17.0, 5.0
       2.0, 1.0,  2.0,  3.0,  5.0, 8.0];
[eVals,eVecs] = jacobi(A);
[eVals,eVecs] = sortEigen(eVals,eVecs);
eigenvalues = eVals'
eigenvectors = eVecs

>> eigenvalues =
    4.4636    5.9889    8.7119   10.9767   13.8675   27.9913
eigenvectors =
   -0.2380    0.1537    0.7247   -0.5314   -0.1213    0.3121
    0.6234   -0.3858    0.4368    0.3089    0.3001    0.2938
    0.0251   -0.0554   -0.4416   -0.4987    0.5901    0.4521
   -0.5653    0.2082    0.0785    0.6054    0.2871    0.4266
   -0.0416   -0.4034   -0.2640    0.0328   -0.6522    0.5826
    0.4825    0.7864   -0.1147    0.0769   -0.1981    0.3009
```

Problem 15

Because $\mathbf{B}$ is not positive definite, the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{Bx}$ cannot be transformed into the standard form, since Choleski's decomposition $\mathbf{B} = \mathbf{LL}^T$ would fail. We can, however, interchange the roles of $\mathbf{A}$ and $\mathbf{B}$ by dividing both sides of the problem by λ. The result is the eigenvalue problem $\mathbf{Bx} = (1/\lambda)\mathbf{Ax}$. As $\mathbf{A}$ is positive definite, we have no trouble decomposing it.

```
% problem9_1_15
A = [6 -4  1  0
    -4  6 -4  2
     1 -4  6 -4
     0  1 -4  7];
 B = [1 -2  3 -1
     -2  6 -2  3
      3 -2  6 -2
     -1  3 -2  9];
H = stdForm(B,A);
eVals = jacobi(H);
eigenvalues = 1./eVals

>> eigenvalues =
   -7.2961
    1.3171
    0.9289
    0.1040
```

Problem 16

(a)

Here the eigenvalue problem is $\mathbf{Ax} = \lambda\mathbf{Bx}$, where $\mathbf{B}$ is a diagonal matrix. Using the notation in Eq. (9.25), the diagonal terms of $\mathbf{B}$ are $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = 1$, $\beta_n = 1/2$. Equation (9.26b) is

$$H_{ij} = \frac{A_{ij}}{\sqrt{\beta_i \beta_j}}$$

The differences between $\mathbf{H}$ and $\mathbf{A}$ are confined to the last row and column:

$$\begin{aligned} H_{in} &= H_{ni} = \sqrt{2}A_{in}, \quad i = 1, 2, \ldots, n-1 \\ H_{nn} &= 2A_{nn} \end{aligned}$$

Thus

$$\mathbf{H} = \begin{bmatrix} 7 & -4 & 1 & 0 & 0 & \cdots & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -4 & 6 & -4 & \sqrt{2} \\ 0 & \cdots & 0 & 1 & -4 & 5 & -2\sqrt{2} \\ 0 & \cdots & 0 & 0 & \sqrt{2} & -2\sqrt{2} & 2 \end{bmatrix} \blacktriangleleft$$

The transformation is

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{2} \end{bmatrix} \mathbf{z} \blacktriangleleft$$

(b)

```
% problem9_1_16
format short e
n = 10;
% Jacobi needs only the upper half of [H].
H = zeros(n,n);
for i = 1:n-2
    H(i,i) = 6; H(i,i+1) = -4; H(i,i+2) = 1;
end
H(1,1) = 7; H(n-1,n-1) = 5; H(n,n) = 2;
H(n-1,n) = -2*sqrt(2); H(n-2,n) = sqrt(2);
% Solve with Jacobis's method & sort.
[eVals,X] = jacobi(H);
[eVals,X] = sortEigen(eVals,X);
% Extract the lowest two eigenvectors.
eVecs = X(:,1:2);
% Recover eigenvectors of original matrix
% (only the last element of ea. vector changes).
eVecs(n,1:2) = eVecs(n,1:2)*sqrt(2);
eigenvalues = eVals(1:2)'
eigenvectors = eVecs

>> eigenvalues =
  1.2156e-003  4.4663e-002
eigenvectors =
  1.0962e-002  6.4238e-002
  4.0841e-002  1.9507e-001
  8.6641e-002  3.3347e-001
```

```
  1.4542e-001  4.2913e-001
  2.1433e-001  4.4665e-001
  2.9072e-001  3.6977e-001
  3.7217e-001  2.0221e-001
  4.5665e-001 -3.5827e-002
  5.4255e-001 -3.1509e-001
  6.2883e-001 -6.0793e-001
```

The circular frequencies are

$$\omega_i = \sqrt{\frac{\lambda_i EI}{\gamma}\frac{n^2}{L^2}}$$

so that

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{(1.2156 \times 10^{-3})EI}{\gamma}\frac{10^2}{L^2}} = 3.487\sqrt{\frac{EI}{\gamma}\frac{1}{L^2}} \blacktriangleleft \\
\omega_2 &= \sqrt{\frac{(4.4663 \times 10^{-2})EI}{\gamma}\frac{10^2}{L^2}} = 21.134\sqrt{\frac{EI}{\gamma}\frac{1}{L^2}} \blacktriangleleft
\end{aligned}$$

Problem 17

The following function solves the eigenvalue problem is $A\mathbf{u} = \lambda\mathbf{B}\mathbf{u}$, where $\mathbf{B}$ is a diagonal matrix and $\mathbf{A}$ is tridiagonal.

```
function eVal = inversePower3(d,c,b)
% Finds smallest eigenvalue of [A]{x} = eVal[B]{x}
% by the invers power method.
% USAGE: eVal = inversePower3(d,c,b)
% [A] must be tridiagonal: [A] = [c\d\c];
% [B] must be diagonal: [B] =[\b\].

n = length(d); e = c;
z = zeros(n,1);
v = rand(n,1); v = v/sqrt(dot(v,v));
[c,d,e] = LUdec3(c,d,e);
for i = 1:50
    z = b.*v;             % {z} = [B]{v}
    z = LUsol3(c,d,e,z);
    zMag = sqrt(dot(z,z)); z = z/zMag;
    if sqrt(dot(v - z,v - z)) < 1.0e-6
        if dot(v,z) > 0; eVal = 1/zMag;
```

```
        else; eVal = -1/zMag; end
        return
    else; v = z; end
end
error('Too many iterations')
```

For the problem at hand the calling program is

```
% problem9_1_17
d = ones(10,1)*2; d(10) = 1;
c = -ones(9,1);
b = ones(10,1);
b(5) = 2/3; b(6:9) = 0.5; b(10) = 0.25;
eVal = inversePower3(d,c,b);
Lowest_eigenvalue = eVal

>> Lowest_eigenvalue =
  4.1785e-002
```

The buckling load is

$$P = 400\lambda \frac{EI_0}{L^2} = 400(4.1785 \times 10^{-2})\frac{EI_0}{L^2} = 16.714\frac{EI_0}{L^2} \quad \blacktriangleleft$$

Problem 18

$$\begin{bmatrix} 6 & 5 & 3 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \frac{P}{kL} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

The problem can be transformed into standard form by the operations

$$\begin{aligned} \text{row 1} &\leftarrow \text{row 1} - \text{row 2} \\ \text{row 3} &\leftarrow \text{row 2} - \text{row 3} \end{aligned}$$

This yields

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \frac{P}{kL} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

```
% problem 9_1_18
A = [3 2 1; 2 2 1; 1 1 1];
[eVal,eVec] = invPower(A,0);
eigenvalue = eVal
eigenvector = eVec
```

```
>> eigenvalue =
  3.0798e-001
eigenvector =
 -3.2799e-001
  7.3698e-001
 -5.9101e-001
```

The buckling load is

$$P = 0.3080\,kL \quad \blacktriangleleft$$

Problem 19

$$\begin{aligned} k\left(-2u_1 + u_2\right) &= m\ddot{u}_1 \\ k(u_1 - 2u_2 + u_3) &= 3m\ddot{u}_2 \\ k(u_2 - 2u_3) &= 2m\ddot{u}_3 \end{aligned}$$

Substituting $u_i = x_i \sin \omega t$, the equations of motion become (after cancelling $\sin \omega t$)

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \omega^2 \frac{m}{k} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

```
% problem9_1_19
A = [2 -1 0; -1 2 -1; 0 -1 2];
b = [1 3 2];                % Diagonal elements of [B].
b = sqrt(b);                % Diagonal elements of [L].
for i = 1:3                 % Convert to standard form.
    for j = 1:3
        H(i,j) = A(i,j)/b(i)/b(j);
    end
end
[eVals,Z] = jacobi(H);      % Solve by Jacobi method.
for i = 1:3                           % Recover eigenvectors
    X(i,:) = Z(i,:)/b(i);             % of original problem.
end
for i = 1:3
    xMag = sqrt(dot(X(:,i),X(:,i))); % Normalize eigenvecs.
    X(:,i) = X(:,i)/xMag;
end
eigenvalues = eVals'
eigenvectors = X
```

```
>> eigenvalues =
  2.2329e+000  2.5282e-001  1.1809e+000
eigenvectors =
  9.6983e-001  4.2955e-001 -3.8362e-001
 -2.2590e-001  7.5050e-001 -3.1422e-001
  9.1611e-002  5.0222e-001  8.6839e-001
```

The circular frequencies are

$$\omega_i = \sqrt{\lambda_i \frac{k}{m}}$$

$$\begin{aligned}
\omega_1 &= \sqrt{0.252\,8\frac{k}{m}} = 0.5028\sqrt{\frac{k}{m}} \quad \blacktriangleleft \\
\omega_2 &= \sqrt{1.1809\frac{k}{m}} = 1.0867\sqrt{\frac{k}{m}} \quad \blacktriangleleft \\
\omega_3 &= \sqrt{2.2329\frac{k}{m}} = 1.4943\sqrt{\frac{k}{m}} \quad \blacktriangleleft
\end{aligned}$$

Problem 20

$$\begin{aligned}
L\frac{d^2 i_1}{dt^2} + \frac{1}{C}i_1 + \frac{2}{C}(i_1 - i_2) &= 0 \\
L\frac{d^2 i_2}{dt^2} + \frac{2}{C}(i_2 - i_1) + \frac{3}{C}(i_2 - i_3) &= 0 \\
L\frac{d^2 i_3}{dt^2} + \frac{3}{C}(i_3 - i_2) + \frac{4}{C}(i_3 - i_4) &= 0 \\
L\frac{d^2 i_4}{dt^2} + \frac{4}{C}(i_4 - i_3) + \frac{5}{C}i_4 &= 0
\end{aligned}$$

Substituting $i_k = x_k \sin\omega t$, we get (after cancelling $\sin\omega t$)

$$\begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 5 & -3 & 0 \\ 0 & -3 & 7 & -4 \\ 0 & 0 & -4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \omega^2 LC \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

```
% problem9_1_20
A = [3 -2  0  0
    -2  5 -3  0
     0 -3  7 -4
     0  0 -4  9];
```

```
[eVals,X] = jacobi(A);
circular_frequencies = sqrt(eVals)'

>> circular_frequencies =
    1.8412    2.6579    0.9514    3.5554
```

$$\begin{aligned}
\omega_1 &= 0.9514\sqrt{\frac{1}{LC}} \blacktriangleleft \qquad \omega_2 = 1.8412\sqrt{\frac{1}{LC}} \blacktriangleleft \\
\omega_3 &= 2.6579\sqrt{\frac{1}{LC}} \blacktriangleleft \qquad \omega_4 = 3.5554\sqrt{\frac{1}{LC}} \blacktriangleleft
\end{aligned}$$

Problem 21

$$\begin{aligned}
L\frac{d^2i_1}{dt^2} + L\left(\frac{d^2i_1}{dt^2} - \frac{d^2i_2}{dt^2}\right) + \frac{1}{C}i_1 &= 0 \\
L\left(\frac{d^2i_2}{dt^2} - \frac{d^2i_1}{dt^2}\right) + L\left(\frac{d^2i_2}{dt^2} - \frac{d^2i_3}{dt^2}\right) + \frac{2}{C} &= 0 \\
L\left(\frac{d^2i_3}{dt^2} - \frac{d^2i_2}{dt^2}\right) + L\left(\frac{d^2i_3}{dt^2} - \frac{d^2i_4}{dt^2}\right) + \frac{3}{C}i_3 &= 0 \\
L\left(\frac{d^2i_4}{dt^2} - \frac{d^2i_3}{dt^2}\right) + L\frac{d^2i_4}{dt^2} + \frac{4}{C}i_4 &= 0
\end{aligned}$$

After substituting $i_k = x_k \sin \omega t$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \omega^2 LC \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

This can be written as

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where $\lambda = (\omega^2 LC)^{-1}$.

```
% problem9_1_21
A = [2 -1 0 0; -1 2 -1 0; 0 -1 2 -1; 0 0 -1 2];
b = [1 2 3 4];             % Diagonal elements of [B].
```

```
b = sqrt(b);                % Diagonal elements of [L].
for i = 1:4                 % Convert to standard form.
    for j = 1:4
        H(i,j) = A(i,j)/b(i)/b(j);
    end
end
[eVals,Z] = jacobi(H);    % Solve by Jacobi method.
circular_frequencies = 1./sqrt(eVals)'

>> circular_frequencies =
    0.6471    2.6170    0.9628    1.3437
```

$$\omega_1 = 0.6471\sqrt{\frac{1}{LC}} \blacktriangleleft \qquad \omega_2 = 0.9628\sqrt{\frac{1}{LC}} \blacktriangleleft$$
$$\omega_2 = 1.3437\sqrt{\frac{1}{LC}} \blacktriangleleft \qquad \omega_4 = 2.6170\sqrt{\frac{1}{LC}} \blacktriangleleft$$

Problem 22

```
function eVals = LRmethod(A)
% Computes the eigenvalues of [A] with the LR method.
% USAGE: eVals = LRmethod(A)

eValsOld = diag(A);
for i = 1:50
    L = choleski(A);
    A = L'*L;
    eVals = diag(A);
    err = dot(eVals - eValsOld,eVals - eValsOld);
    if abs(err) < 1.0e-6; return
    else; eValsOld = eVals; end
end
error('LR method did not converge')

% problem9_1_22
A = [4 3 1; 3 4 2; 1 2 3];
eigenvalues = LRmethod(A)'

>> eigenvalues =
    7.9268    2.3856    0.6875
```

PROBLEM SET 9.2

Problem 1

(a)

$$\mathbf{A} = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 6 \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} 10 \\ 2 \\ 6 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 4+1 \\ 4+3 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \lambda_{\min} &\geq \min_i (a_i - r_i) = 2 - 7 = -5 \ \blacktriangleleft \\ \lambda_{\max} &\leq \max_i (a_i + r_i) = 10 + 5 = 15 \ \blacktriangleleft \end{aligned}$$

(The actual eigenvalues are $\lambda_{\min} = -1.066$ and $\lambda_{\max} = 11.667$).

(b)

$$\mathbf{B} = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 2+2 \\ 2+3 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix}$$

$$\begin{aligned} \lambda_{\min} &\geq \min_i (a_i - r_i) = 4 - 5 = -1 \ \blacktriangleleft \\ \lambda_{\max} &\leq \max_i (a_i + r_i) = 5 + 5 = 10 \ \blacktriangleleft \end{aligned}$$

(The actual eigenvalues are $\lambda_{\min} = -0.365$ and $\lambda_{\max} = 7.565$).

Problem 2

$$P_4(\lambda) = |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 5-\lambda & -2 & 0 & 0 \\ -2 & 4-\lambda & -1 & 0 \\ 0 & -1 & 4-\lambda & -2 \\ 0 & 0 & -2 & 5-\lambda \end{vmatrix}$$

Using $\lambda = 2$:

$$\begin{aligned}
P_0(2) &= 1 \\
P_1(2) &= d_1 - \lambda = 5 - 2 = 3 \\
P_2(2) &= (d_2 - \lambda)P_1(2) - c_1^2 P_0(2) = (4-2)3 - (-2)^2 1 = 2 \\
P_3(2) &= (d_3 - \lambda)P_2(2) - c_2^2 P_1(2) = (4-2)2 - (-1)^2 3 = 1 \\
P_4(2) &= (d_4 - \lambda)P_3(2) - c_3^2 P_2(2) = (5-2)1 - (-2)^2 2 = -5
\end{aligned}$$

There is one sign change in this sequence. Therefore, one eigenvalue is less than 2.

Using $\lambda = 4$:

$$\begin{aligned}
P_0(4) &= 1 \\
P_1(4) &= d_1 - \lambda = 5 - 4 = 1 \\
P_2(4) &= (d_2 - \lambda)P_1(4) - c_1^2 P_0(4) = (4-4)3 - (-2)^2 1 = -4 \\
P_3(4) &= (d_3 - \lambda)P_2(4) - c_2^2 P_1(4) = (4-4)(-4) - (-1)^2 1 = -1 \\
P_4(4) &= (d_4 - \lambda)P_3(4) - c_3^2 P_2(4) = (5-4)(-1) - (-2)^2(-4) = 15
\end{aligned}$$

Since there are 2 sign changes in this sequence, there are 2 eigenvalues are smaller than 4.

It follows that there is one eigenvalue between 2 and 4.

Problem 3

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Find global bounds with Gerschgorin's theorem:

$$\mathbf{a} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\lambda_{\min} &\geq \min_i (a_i - r_i) = 4 - 2 = 2 \\
\lambda_{\max} &\leq \max_i (a_i + r_i) = 4 + 2 = 6
\end{aligned}$$

Use Sturm sequence to determine intermediate bounds:

With $\lambda = 4$:

$$\begin{aligned}
P_0(4) &= 1 \\
P_1(4) &= d_1 - \lambda = 4 - 4 = 0 \\
P_2(4) &= (d_2 - \lambda)P_1(4) - c_1^2 P_0(4) = (4-4)(0) - (-1)^2 1 = -1 \\
P_3(4) &= (d_3 - \lambda)P_2(4) - c_2^2 P_1(4) = (4-3)(-1) - (-1)^2 0 = 0
\end{aligned}$$

The zero result indicates that $\lambda = 4$ is an eigenvalue.
With $\lambda = 3$:

$$\begin{aligned}
P_0(3) &= 1 \\
P_1(3) &= d_1 - \lambda = 4 - 3 = 1 \\
P_2(3) &= (d_2 - \lambda)P_1(3) - c_1^2 P_0(3) = (4-3)1 - (-1)^2 1 = 0 \\
P_3(3) &= (d_3 - \lambda)P_2(3) - c_2^2 P_1(3) = (4-3)0 - (-1)^2 1 = -1
\end{aligned}$$

There is one sign change in this sequence; hence one of the eigenvalues is smaller than 3.
In conclusion:

$$2 \le \lambda_1 \le 4 \qquad \lambda_2 = 4 \qquad 4 \le \lambda_3 \le 6 \ \blacktriangleleft$$

Problem 4

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 0 \\ 1 & 8 & 2 \\ 0 & 2 & 9 \end{bmatrix}$$

Find global bounds with Gerschgorin's theorem:

$$\mathbf{a} = \begin{bmatrix} 6 \\ 8 \\ 9 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{aligned}
\lambda_{\min} &\ge \min_i (a_i - r_i) = 6 - 1 = 5 \\
\lambda_{\max} &\le \max_i (a_i + r_i) = 8 + 3 = 11
\end{aligned}$$

Use Sturm sequence to determine intermediate bounds:
With $\lambda = 8$:

$$\begin{aligned}
P_0(8) &= 1 \\
P_1(8) &= d_1 - \lambda = 6 - 8 = -2 \\
P_2(8) &= (d_2 - \lambda)P_1(8) - c_1^2 P_0(8) = (8-8)(-2) - 1^2\,(1) = -1 \\
P_3(8) &= (d_3 - \lambda)P_2(8) - c_2^2 P_1(8) = (9-8)(-1) - 2^2(-2) = 7
\end{aligned}$$

The sign changes indicate that there are 2 eigenvalues smaller than 8.
With $\lambda = 6$:

$$\begin{aligned}
P_0(6) &= 1 \\
P_1(6) &= d_1 - \lambda = 6 - 6 = 0 \\
P_2(6) &= (d_2 - \lambda)P_1(6) - c_1^2 P_0(6) = (8-6)0 - 1^2\,(1) = -1 \\
P_3(6) &= (d_3 - \lambda)P_2(6) - c_2^2 P_1(6) = (9-6)(-1) - 2^2(0) = -3
\end{aligned}$$

There is one sign change in this sequence; hence one of the eigenvalues is smaller than 6.
In conclusion:

$$5 \le \lambda_1 \le 6 \qquad 6 \le \lambda_2 \le 8 \qquad 8 \le \lambda_3 \le 11 \quad \blacktriangleleft$$

Problem 5

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Find global bounds with Gerschgorin's theorem:

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \lambda_{\min} &\ge \min_i (a_i - r_i) = 2 - 2 = 0 \\ \lambda_{\max} &\le \max_i (a_i + r_i) = 2 + 2 = 4 \end{aligned}$$

Use Sturm sequence to determine intermediate bounds:
<u>With $\lambda = 2$</u>:

$$\begin{aligned} P_0(2) &= 1 \\ P_1(2) &= d_1 - \lambda = 2 - 2 = 0 \\ P_2(2) &= (d_2 - \lambda)P_1(2) - c_1^2 P_0(2) = (2-2)(0) - (-1)^2\, 1 = -1 \\ P_3(2) &= (d_3 - \lambda)P_2(2) - c_2^2 P_1(2) = (2-2)(1) - (-1)^2(0) = 0 \\ P_4(2) &= (d_4 - \lambda)P_3(2) - c_3^2 P_2(2) = (1-2)(0) - (-1)^2 1 = 1 \end{aligned}$$

The two sign changes indicate that there are 2 eigenvalues smaller than 2.
<u>With $\lambda = 3$</u>:

$$\begin{aligned} P_0(3) &= 1 \\ P_1(3) &= d_1 - \lambda = 2 - 3 = -1 \\ P_2(3) &= (d_2 - \lambda)P_1(3) - c_1^2 P_0(3) = (2-3)(-1) - (-1)^2\,(1) = 0 \\ P_3(3) &= (d_3 - \lambda)P_2(3) - c_2^2 P_1(3) = (2-3)(0) - (-1)^2(-1) = 1 \\ P_4(3) &= (d_4 - \lambda)P_3(3) - c_3^2 P_2(3) = (1-3)(1) - (-1)^2(0) = -2 \end{aligned}$$

There are 3 sign change in this sequence; hence 3 of the eigenvalues are smaller than 3.

With $\lambda = 1$:

$$\begin{aligned}
P_0(1) &= 1 \\
P_1(1) &= d_1 - \lambda = 2 - 1 = 1 \\
P_2(1) &= (d_2 - \lambda)P_1(1) - c_1^2 P_0(1) = (2-1)(1) - (-1)^2(1) = 0 \\
P_3(1) &= (d_3 - \lambda)P_2(1) - c_2^2 P_1(1) = (2-1)(0) - (-1)^2(1) = -1 \\
P_4(1) &= (d_4 - \lambda)P_3(1) - c_3^2 P_2(1) = (1-1)(-1) - (-1)^2(0) = 0
\end{aligned}$$

$\lambda = 1$ is an eigenvalue
In conclusion:

$$0 \le \lambda_1 \le 1 \qquad \lambda_2 = 1 \qquad 2 \le \lambda_3 \le 3 \qquad 3 \le \lambda_4 \le 4 \quad \blacktriangleleft$$

Problem 6

$$\mathbf{A} = \begin{bmatrix} 12 & 4 & 3 \\ 4 & 9 & 3 \\ 3 & 3 & 15 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} 9 & 3 \\ 3 & 15 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad k = |\mathbf{x}| = 5 \text{ (note that } x_1 > 0)$$

$$\mathbf{u} = \begin{bmatrix} k + x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} \qquad \mathbf{u}\mathbf{u}^T = \begin{bmatrix} 81 & 27 \\ 27 & 9 \end{bmatrix} \qquad H = \frac{1}{2}|\mathbf{u}|^2 = 45$$

$$\mathbf{Q} = \mathbf{I} - \frac{\mathbf{u}\mathbf{u}^T}{H} = \begin{bmatrix} 1 - 81/45 & -27/45 \\ -27/45 & 1 - 9/45 \end{bmatrix} = \begin{bmatrix} -0.8 & -0.6 \\ -0.6 & 0.8 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{Q}^T\mathbf{A}' &= \begin{bmatrix} -0.8 & -0.6 \\ -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 9 & 3 \\ 3 & 15 \end{bmatrix} = \begin{bmatrix} -9.0 & -11.4 \\ -3.0 & 10.2 \end{bmatrix} \\
\mathbf{Q}^T\mathbf{A}'\mathbf{Q} &= \begin{bmatrix} -9.0 & -11.4 \\ -3.0 & 10.2 \end{bmatrix} \begin{bmatrix} -0.8 & -0.6 \\ -0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} 14.04 & -3.72 \\ -3.72 & 9.96 \end{bmatrix}
\end{aligned}$$

$$\mathbf{A} \leftarrow \begin{bmatrix} A_{11} & (\mathbf{Qx})^T \\ \mathbf{Qx} & \mathbf{Q}^T\mathbf{A}'\mathbf{Q} \end{bmatrix} = \begin{bmatrix} 12 & -5 & 0 \\ -5 & 14.04 & -3.72 \\ 0 & -3.72 & 9.96 \end{bmatrix} \quad \blacktriangleleft$$

Problem 7

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 1 & -1 \\ -2 & 4 & -2 & 1 \\ 1 & -2 & 4 & -2 \\ -1 & 1 & -2 & 4 \end{bmatrix}$$

First round:

$$\mathbf{A}' = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \qquad k = -|\mathbf{x}| = -\sqrt{6} = -2.4495$$

$$\mathbf{u} = \begin{bmatrix} k + x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4.4495 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{u}\mathbf{u}^T = \begin{bmatrix} 19.7981 & -4.4495 & 4.4495 \\ -4.4495 & 1 & -1 \\ 4.4495 & -1 & 1 \end{bmatrix}$$

$$H = \frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}(4.4495^2 + 2) = 10.8990$$

$$\mathbf{Q} = \mathbf{I} - \frac{\mathbf{u}\mathbf{u}^T}{H} = \begin{bmatrix} -0.8165 & 0.4082 & -0.4082 \\ 0.4082 & 0.9082 & 0.0918 \\ -0.4082 & 0.0918 & 0.9082 \end{bmatrix}$$

$$\begin{aligned} \mathbf{Q}^T\mathbf{A}' &= \begin{bmatrix} -4.4906 & 4.0822 & -3.2657 \\ -0.0918 & 2.6328 & -1.0410 \\ -0.9082 & -0.6328 & 3.0410 \end{bmatrix} \\ \mathbf{Q}^T\mathbf{A}'\mathbf{Q} &= \begin{bmatrix} 6.6660 & 1.5746 & -0.7581 \\ 1.5746 & 2.2581 & -0.6663 \\ -0.7581 & -0.6663 & 3.0745 \end{bmatrix} \end{aligned}$$

$$\mathbf{A} \leftarrow \begin{bmatrix} A_{11} & (\mathbf{Q}\mathbf{x})^T \\ \mathbf{Q}\mathbf{x} & \mathbf{Q}^T\mathbf{A}'\mathbf{Q} \end{bmatrix} = \begin{bmatrix} 4 & 2.4495 & 0 & 0 \\ 2.4495 & 6.6660 & 1.5746 & -0.7581 \\ 0 & 1.5746 & 2.2581 & -0.6663 \\ 0 & -0.7581 & -0.6663 & 3.0745 \end{bmatrix}$$

Second round:

$$\begin{aligned} \mathbf{A}' &= \begin{bmatrix} 2.2581 & -0.6663 \\ -0.6663 & 3.0745 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1.5746 \\ -0.7581 \end{bmatrix} \\ k &= |\mathbf{x}| = \sqrt{1.5746^2 + 0.7581^2} = 1.7476 \end{aligned}$$

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} k + x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.3222 \\ -0.7581 \end{bmatrix} \qquad \mathbf{u}\mathbf{u}^T = \begin{bmatrix} 11.0370 & -2.5186 \\ -2.5186 & 0.5747 \end{bmatrix} \\ H &= \frac{1}{2}|\mathbf{u}|^2 = \frac{1}{2}(3.3222^2 + 0.7581^2) = 5.8059 \end{aligned}$$

$$\mathbf{Q} = \mathbf{I} - \frac{\mathbf{u}\mathbf{u}^T}{H} = \begin{bmatrix} -0.9010 & 0.4338 \\ 0.4338 & 0.9010 \end{bmatrix}$$

$$\mathbf{Q}^T\mathbf{A}' = \begin{bmatrix} -2.3236 & 1.9341 \\ 0.3792 & 2.4811 \end{bmatrix} \qquad \mathbf{Q}^T\mathbf{A}'\mathbf{Q} = \begin{bmatrix} 2.9326 & 0.7346 \\ 0.7346 & 2.4000 \end{bmatrix}$$

$$\mathbf{A} \leftarrow \begin{bmatrix} A_{11} & A_{12} & \mathbf{0}^T \\ A_{21} & A_{22} & (\mathbf{Q}\mathbf{x})^T \\ \mathbf{0} & \mathbf{Q}\mathbf{x} & \mathbf{Q}^T\mathbf{A}'\mathbf{Q} \end{bmatrix} = \begin{bmatrix} 4 & 2.4495 & 0 & 0 \\ 2.4495 & 6.6660 & -1.7476 & 0 \\ 0 & -1.7476 & 2.9326 & 0.7346 \\ 0 & 0 & 0.7346 & 2.4000 \end{bmatrix} \quad \blacktriangleleft$$

Problem 8

$$\mathbf{A} = \begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ 2 & 5 & 2 & 0 & 0 \\ 0 & 2 & 7 & 4 & 0 \\ 0 & 0 & 4 & 6 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

```
% problem9_2_8
c = [2 2 4 1];
d = [6 5 7 6 3];
eigenvalues = eigenvals3(c,d,5)'

>> eigenvalues =
    1.4356    2.9378    4.1515    7.4473   11.0277
```

Problem 9

```
% problem9_2_9
A = [4 -1  0  1
    -1  6 -2  0
     0 -2  3  2
     1  0  2  4];
[c,d]= householder(A);
eigenvalues = eigenvals3(c,d,2)'

>> eigenvalues =
    0.6939    3.8056
```

Problem 10

```
% problem9_2_10
m = 3;
A = [7 -4  3 -2  1  0
    -4  8 -4  3 -2  1
     3 -4  9 -4  3 -2
    -2  3 -4 10 -4  3
     1 -2  3 -4 11 -4
     0  1 -2  3 -4 12];
```

```
eVecMat = zeros(size(A,1),m);       % Init. eigenvector matrix.
[c,d,P] = householder(A);           % Tridiagonalize A.
eVals = eigenvals3(c,d,m);          % Find lowest m eigenvals.
for i = 1:m                         % Compute corresponding
    s = eVals(i)*1.0000001;         %    eigenvectors by inverse
    [eVal,eVec] = invPower3(c,d,s); %    power method with
    eVecMat(:,i) = eVec;            %    eigenvalue shifting.
end
eVecMat = P*eVecMat;                % Eigenvecs. of orig. A.
eigenvalues = eVals'
eigenvectors = eVecMat

>> eigenvalues =
    3.3768    4.7593    6.0368
eigenvectors =
    0.6904   -0.3780    0.2697
    0.7151    0.2675   -0.2486
    0.0951    0.8386   -0.0374
   -0.0241    0.2781    0.8030
    0.0257   -0.0428    0.4640
   -0.0406    0.0558   -0.0636
```

Problem 11

The elements of an $n \times n$ Hilbert matrix are

$$A_{ij} = \frac{1}{i+j-1}, \quad i, j = 1, 2, \ldots, n$$

```
% problem9_2_11
format short e
A = zeros(6,6);
for i = 1:6
    for j = 1:6
        A(i,j) = 1/(i + j - 1);
    end
end
A = householder(A);
d = diag(A); c = diag(A,1);
eigenvalues = eigenvals3(c,d,2)'

>> eigenvalues =
  1.0828e-007  1.2571e-005
```

Due to ill-conditioning, we expect the computed eigenvalues to be inaccurate.

Problem 12

The program lines of the function `eValBrackets2` that differ from `eValBrackets` are marked by `% <==`.

```
function r = eValBrackets2(c,d,m)
% Brackets each of the m largest eigenvalues of A = [c\d\c]
% so that here is one eigenvalue in [r(i), r(i+1)].
% USAGE: r = eValBrackets2(c,d,m).

n = length(d);
[eValMin,eValMax]= gerschgorin(c,d);  % Find global limits
r = ones(m+1,1);
r(m+1) = eValMax; % <==
% Search for eigenvalues in ascending order
for k = 1:m  % <==
    % First bisection of interval (eValMin,eValMax)
    eVal = (eValMax + eValMin)/2;
    h = (eValMax - eValMin)/2;
    for i = 1:100
        % Find number of eigenvalues less than eVal
        num_eVals = count_eVals(c,d,eVal);
        % Bisect again & find the half containing eVal
        h = h/2;
        if num_eVals < n - m + k - 1; % <==
            eVal = eVal + h;
        elseif num_eVals > n - m + k - 1; % <==
            eVal = eVal - h;
        else; break
        end
    end
    % If eigenvalue located, change lower limit of
    % search and record result in {r}
    ValMin = eVal;
    r(k) = eVal;  % <==
end    (c,d,m)
% Brackets each of the m largest eigenvalues of A = [c\d\c]
% so that here is one eigenvalue in [r(i), r(i+1)].
% USAGE: r = eValBrackets2(c,d,m).
```

```
n = length(d);
[eValMin,eValMax]= gerschgorin(c,d);  % Find global limits
r = ones(m+1,1);
r(m+1) = eValMax; % <==
% Search for eigenvalues in ascending order
for k = 1:m  % <==
    % First bisection of interval (eValMin,eValMax)
    eVal = (eValMax + eValMin)/2;
    h = (eValMax - eValMin)/2;
    for i = 1:100
        % Find number of eigenvalues less than eVal
        num_eVals = count_eVals(c,d,eVal);
        % Bisect again & find the half containing eVal
        h = h/2;
        if num_eVals < n - m + k - 1; % <==
            eVal = eVal + h;
        elseif num_eVals > n - m + k - 1; % <==
            eVal = eVal - h;
        else; break
        end
    end
    % If eigenvalue located, change lower limit of
    % search and record result in {r}
    ValMin = eVal;
    r(k) = eVal;  % <==
end
```

This is the calling program:

```
% problem9_2_12
format short e
A = zeros(6,6);
for i = 1:6
    for j = 1:6
        A(i,j) = 1/(i + j - 1);
    end
end
[c,d] = householder(A);
brackets = eValBrackets2(c,d,2)

>> brackets =
  1.5921e-001
  8.1998e-001
  1.7010e+000
```

The largest two eigenvalues of are 0.242 and 1.619, so that the results are O.K.

Problem 13

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = m\omega^2/k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

```
% problem9_2_13
d = [2 2 2]; c = [-1 -1];
b = [1 3 2]';                % Diagonal elements of [B].
b = sqrt(b);                 % Diagonal elements of [L].
X = zeros(3,3);
% Transform into standard form.
for i = 1:3; d(i) = d(i)/b(i)^2; end
for i = 1:2; c(i) = c(i)/b(i)/b(i+1); end
eVals = eigenvals3(c,d,3); % Compute eigenvalues
for i=1:3                    % Compute eigenvectors
    s = eVals(i)*1.0000001;
    [eVal,eVec] = invPower3(c,d,s);
    eVec = eVec./b;        % Eigenvectors of orig. prob.
    X(:,i) = eVec/sqrt(dot(eVec,eVec));
end
eigenvalues = eVals'
eigenvectors = X

>> eigenvalues =
  2.5282e-001  1.1809e+000  2.2329e+000
eigenvectors =
  4.2955e-001  3.8362e-001  9.6983e-001
  7.5050e-001  3.1422e-001 -2.2590e-001
  5.0222e-001 -8.6839e-001  9.1611e-002
```

The circular frequencies are

$$\omega_i = \sqrt{\lambda_i \frac{k}{m}}$$

$$\begin{aligned} \omega_1 &= \sqrt{0.252\,8\frac{k}{m}} = 0.5028\sqrt{\frac{k}{m}} \blacktriangleleft \\ \omega_2 &= \sqrt{1.1809\frac{k}{m}} = 1.0867\sqrt{\frac{k}{m}} \blacktriangleleft \\ \omega_3 &= \sqrt{2.2329\frac{k}{m}} = 1.4943\sqrt{\frac{k}{m}} \blacktriangleleft \end{aligned}$$

Problem 14

```
% problem9_2_14
m = 4;
k = [400 400 400 0.2 400 400 400];
n = length(k);
d = zeros(n,1); c = zeros(n-1,1);
X = zeros(n,m);
% Transform into standard form.
for i = 1:n-1
    d(i) = k(i) + k(i+1); c(i) = -k(i+1);
end
d(n) = k(n);
eVals = eigenvals3(c,d,m); % Compute eigenvalues
for i = 1:m                % Compute eigenvectors
    s = eVals(i)*1.0000001;
    [eVal,eVec] = invPower3(c,d,s);
    X(:,i) = eVec/sqrt(dot(eVec,eVec));
end
eigenvalues = eVals'
eigenvectors = X

>> eigenvalues =
  4.9903e-002  7.9334e+001  2.3440e+002  6.2205e+002
eigenvectors =
  2.4970e-004  3.2808e-001 -3.2676e-004  7.3696e-001
  4.9938e-004  5.9109e-001 -4.6204e-004  3.2785e-001
  7.4898e-004  7.3687e-001 -3.2656e-004 -5.9111e-001
  4.9978e-001  9.2929e-007  6.5335e-001 -9.5073e-008
  4.9997e-001 -3.6769e-004  2.7082e-001  2.9561e-004
  5.0009e-001 -6.6338e-004 -2.7042e-001  1.3160e-004
  5.0016e-001 -8.2750e-004 -6.5319e-001 -2.3706e-004
```

The first mode reflects the weak coupling: the four masses on the right move in unison as the remaining three masses are almost stationary.

Problem 15

(a)

This is a non-standard eigenvalue problem $\mathbf{Ax} = \lambda \mathbf{Bx}$. The matrix $\mathbf{B}$ is empty except for its diagonal $\boldsymbol{\beta} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1/2 \end{bmatrix}^T$. According to Eq. (9.26b) the matrix $\mathbf{H}$ of the standard problem is

$$H_{ij} = \frac{A_{ij}}{\sqrt{\beta_i \beta_j}}$$

In this case, the differences between $\mathbf{H}$ and $\mathbf{A}$ are confined to the last row and column:

$$\begin{aligned} H_{in} &= H_{ni} = \sqrt{2} A_{in}, \quad i = 1, 2, \ldots, n-1 \\ H_{nn} &= 2A_{nn} \end{aligned}$$

Thus

$$\mathbf{H} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -\sqrt{2} \\ & & & -\sqrt{2} & 2 \end{bmatrix}$$

(b)

```
% problem9_2_15
for n = [10 100 1000];
    d = ones(n,1)*2;
    c = -ones(n-1,1);
    c(n-1) = -sqrt(2);
    [eVal,eVec] = invPower3(c,d,0);
    n
    circular_frequency = sqrt(eVal)*n
end
```

```
>> n =
    10
circular_frequency =
  1.5692e+000
n =
   100
circular_frequency =
  1.5708e+000
n =
        1000
circular_frequency =
  1.5708e+000
```

The analytical solution is

$$\omega_1 = \frac{\pi}{2}\sqrt{\frac{E}{\rho}\frac{1}{L}} = 1.570\,79\,6\sqrt{\frac{E}{\rho}\frac{1}{L}}$$

Problem 16

```
% problem9_2_16
m = 3; n = 25;
beta = 1000;                        % beta = kL^4/(EI)
alpha = beta/(n + 1)^4;
A = diag(ones(n,1)*(6 + alpha));
A = A + diag(ones(n-1,1)*(-4),1);
A = A + diag(ones(n-1,1)*(-4),-1);
A = A + diag(ones(n-2,1),2);
A = A + diag(ones(n-2,1),-2);
A(1,1) = 5 + alpha; A(n,n) = 5 + alpha;
B = diag(ones(n,1)*2);
B = B + diag(ones(n-1,1)*(-1),1);
B = B + diag(ones(n-1,1)*(-1),-1);
X = zeros(n,m);
[H,T] = stdForm(A,B);               % Transform to std. form.
[c,d,P] = householder(H);           % Tridiagonalize A..
eVals = eigenvals3(c,d,m);          % Find lowest m eigenvals.
for i = 1:m                         % Compute corresponding
    s = eVals(i)*1.0000001;         %    eigenvectors by inverse
    [eVal,eVec] = invPower3(c,d,s); %    power method with
    X(:,i) = eVec;                  %    eigenvalue shifting.
end
X = P*X; X = T*X;                   % Eigenvecs. of orig. prob.
buckling_loads = eVals'*(n + 1)^2
eigenvectors = X

>> buckling_loads =
  6.4741e+001  9.9240e+001  1.1130e+002

eigenvectors =
 -2.7533e-001 -2.7281e-001  2.7684e-001
 -5.3465e-001 -5.1016e-001  5.4965e-001
 -7.6291e-001 -6.8121e-001  8.1444e-001
 -9.4683e-001 -7.6372e-001  1.0674e+000
 -1.0757e+000 -7.4697e-001  1.3047e+000
 -1.1421e+000 -6.3314e-001  1.5230e+000
 -1.1421e+000 -4.3703e-001  1.7192e+000
 -1.0757e+000 -1.8411e-001  1.8902e+000
 -9.4683e-001  9.2732e-002  2.0337e+000
 -7.6291e-001  3.5752e-001  2.1475e+000
 -5.3465e-001  5.7585e-001  2.2300e+000
```

```
-2.7533e-001  7.1933e-001  2.2800e+000
 5.9092e-014  7.6933e-001  2.2968e+000
 2.7533e-001  7.1933e-001  2.2800e+000
 5.3465e-001  5.7585e-001  2.2300e+000
 7.6291e-001  3.5752e-001  2.1475e+000
 9.4683e-001  9.2732e-002  2.0337e+000
 1.0757e+000 -1.8411e-001  1.8902e+000
 1.1421e+000 -4.3703e-001  1.7192e+000
 1.1421e+000 -6.3314e-001  1.5230e+000
 1.0757e+000 -7.4697e-001  1.3047e+000
 9.4683e-001 -7.6372e-001  1.0674e+000
 7.6291e-001 -6.8121e-001  8.1444e-001
 5.3465e-001 -5.1016e-001  5.4965e-001
 2.7533e-001 -2.7281e-001  2.7684e-001]
```

The buckling loads are

$$P_1 = 64.74\frac{EI}{L^2} \blacktriangleleft \qquad P_2 = 99.24\frac{EI}{L^2} \blacktriangleleft \qquad P_3 = 111.30\frac{EI}{L^2} \blacktriangleleft$$

Problem 17

```
% problem9_2_17
m = 5; n = 20;
A = diag(ones(n,1)*2);
A = A + diag(ones(n-1,1),1);
A = A + diag(ones(n-1,1),-1);
A(n,1) = 1; A(1,n) = 1;
[c,d] = householder(A);            % Tridiagonalize A.
eVals = eigenvals3(c,d,m);         % Find lowest m eigenvals.
eigenvalues = eVals

>> eigenvalues =
  1.6974e-015
  9.7887e-002
  9.7887e-002
  3.8197e-001
  3.8197e-001
```

Problem 18

Substituting $\xi = x/L$, the differential equation

$$\frac{d^2\theta}{dx^2} + \gamma^2 \left(1 - \frac{x}{L}\right)^2 \theta = 0$$

becomes

$$\frac{d^2\theta}{d\xi^2} + \gamma^2 L^2 (1-\xi)^2 \theta = 0$$

According to Eqs. (8.11) the finite difference approximation is (m is the number of intervals)

$$\begin{aligned} \theta_0 &= 0 \\ \theta_{i-1} - 2\theta_i + \theta_{i+1} + h^2\gamma^2 L^2 (1-\xi_i)^2 \theta_i &= 0 \quad i = 1, 2, \ldots, m-1 \\ 2\theta_{m-1} - 2\theta_m - h^2\gamma^2 L^2 (1-\xi_m)^2 \theta_m + 2h(0) &= 0 \end{aligned}$$

But $1-\xi_m = 0$, so that the last equation is simply $\theta_m = \theta_{m-1}$. Substituting the first and last equations into the remaining equations, we obtain the following $(m-1) \times (m-1)$ matrix eigenvalue problem

$$\begin{aligned} 2\theta_1 - \theta_2 &= \lambda(1-\xi_1)^2 \theta_1 \\ -\theta_{i-1} + 2\theta_i - \theta_{i+1} &= \lambda(1-\xi_i)^2 \theta_i \quad i = 2, 3, \ldots, m-2 \\ -\theta_{m-2} + \theta_{m-1} &= \lambda(1-\xi_{m-1})^2 \theta_{m-1} \end{aligned}$$

where

$$\lambda = h^2\gamma^2 L^2 = \frac{P^2 L^4 h^2}{(GJ)(EI_z)}$$

Note that the problem is tridiagonal but not of standard form.

```
% problem9_2_18
format long
m = 50;           % Use 50 intervals
h = 1/m;
n = m - 1;        % Size of matrix after boundary
                  % condition are applied
% --- Set up matrices ---
d = ones(n,1)*2; c = -ones(n-1,1); b = zeros(n,1);
d(n) = 1;
for i = 1:n
    b(i) = (1 - h*i)^2;
end
% --- Transform to standard form using Eq. (9.26b) ---
d(n) = d(n)/b(n);
for i = 1:n-1
```

```
    d(i) = d(i)/b(i); c(i) = c(i)/sqrt(b(i)*b(i+1));
end
% --- Compute lowest eigenvalue ---
[eVal,eVec] = invPower3(c,d,0);
eigenvalue = eVal

eigenvalue =
   0.006439102291348
```

$$
\begin{aligned}
P_{cr} &= \frac{\sqrt{\lambda(GJ)(EI_z)}}{hL^2} = \frac{\sqrt{0.006\ 439\ 1(GJ)(EI_z)}}{0.02L^2} \\
&= 4.012\frac{\sqrt{(GJ)(EI_z)}}{L^2}
\end{aligned}
$$

which agrees well with the analytical solution.