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1.

(a) Diagonally dominant. +3

(b) $$\begin{bmatrix} 7 & 1 & 1 \\ -3 & 7 & -1 \\ -2 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -26 \\ 1 \end{bmatrix} \qquad x^{(0)} = [1 \ \ 1 \ \ 1]^T$$ +3

$$x_i = \frac{1}{A_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} A_{ij}x_j\right)$$ +3

1st Iteration.

$$x_1 = \frac{1}{A_{11}}(b_1 - A_{12}x_2 - A_{13}x_3) = \frac{1}{7}(6 - 1\cdot(1) - 1\cdot(1)) = 0.571$$

$$x_2 = \frac{1}{A_{22}}(b_2 - A_{21}x_1 - A_{23}x_3) = \frac{1}{7}(-26 - (-3)\cdot(0.571) - (-1)\cdot(1)) = -3.32$$

$$x_3 = \frac{1}{A_{33}}(b_3 - A_{31}x_1 - A_{32}x_2) = \frac{1}{9}(1 - (-2)\cdot(0.571) - 5\cdot(-3.32)) = 2.08$$

$$x^{(1)} = [0.571 \ \ -3.32 \ \ 2.08]^T$$ +2

2nd Iteration

$$x_1 = \frac{1}{7}(6 - 1\cdot(-3.32) - 1\cdot(2.08)) = 1.03$$

$$x_2 = \frac{1}{7}(-26 - (-3)\cdot(1.03) - (-1)\cdot(2.08)) = -2.98$$

$$x_3 = \frac{1}{9}(1 - (-2)\cdot(1.03) - 5\cdot(-2.98)) = 1.99$$

$$x^{(2)} = [1.03 \ \ -2.98 \ \ 1.99]^T$$ +2

(c)

$$x_i = \frac{w}{A_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} A_{ij}x_j\right) + (1-w)\cdot x_i \ , \quad w = 1.1$$ +3

1st Iteration.

$$x_1 = \frac{1.1}{7}\cdot(6 - 1\cdot(1) - 1\cdot(1)) + (1-1.1)\cdot 1 = 0.528$$

$x_2 = \frac{1.1}{7}(-26-(-3)\cdot(0.528)-(-1)\cdot(1)) + (1-1.1)\cdot 1 = -3.77$

$x_3 = \frac{1.1}{9}(1-(-2)\cdot(0.528)-5\cdot(-3.77)) + (1-1.1)\cdot 1 = 2.46$

$x^{(1)} = [0.528 \quad -3.77 \quad 2.46]^T$

+2

2nd Iteration

$x_1 = \frac{1.1}{7}(6-1\cdot(-3.77)-1\cdot(2.46)) + (1-1.1)\cdot 0.528 = 1.09$

$x_2 = \frac{1.1}{7}(-26-(-3)\cdot(1.09)-(-1)\cdot(2.46)) + (1-1.1)\cdot(-3.77) = -2.81$

$x_3 = \frac{1.1}{9}(1-(-2)\cdot(1.09)-5\cdot(-2.81)) + (1-1.1)\cdot 2.46 = 1.86$

$x^{(2)} = [1.09 \quad -2.81 \quad 1.86]^T$

+2

2. (a) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

i) Symmetric. ($A = A^T$)

$A^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = A$. $\therefore$ A is Symmetric matrix

+2

ii) Positive definite

$x^TAx > 0$ then A is positive definite with $\forall x$ (x is non-zero column vector)

+2

let $x = [x_1 \; x_2 \; x_3]^T$

$$x^TA = [x_1 \; x_2 \; x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = [2x_1 - x_2 \quad -x_1 + 2x_2 - x_3 \quad -x_2 + x_3]$$

$$x^TAx = [2x_1 - x_2 \quad -x_1 + 2x_2 - x_3 \quad -x_2 + x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 - x_2x_3 - x_2x_3 + x_3^2$$

$$= (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_1^2 > 0.$$

+3.

$\therefore$ A is positive definite.

(b)

Choose x_0

$r_0 \leftarrow b - Ax_0$

$S_0 \leftarrow r_0$

do with $k = 0, 1, 2, \cdots$

$\alpha_k \leftarrow \dfrac{S_k^T r_k}{S_k^T A S_k}$

$x_{k+1} \leftarrow x_k + \alpha_k S_k$

$r_{k+1} \leftarrow b - Ax_{k+1}$

If $|r_{k+1}| \le \varepsilon$, then exit loop.

$\beta_k \leftarrow -\dfrac{r_{k+1}^T A S_k}{S_k^T A S_k}$

$S_{k+1} \leftarrow r_{k+1} + \beta_k S_k$

end do.

+5

(c) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $x^{(0)} = [0 \ \ 0 \ \ 0]^T$

$r_0 = b - Ax_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$S_0 = r_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\alpha_0 = \frac{S_0^T r_0}{S_0^T A S_0} = \frac{3}{1} = 3$

$x_1 = x_0 + \alpha_0 S_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ $\therefore x^{(1)} = [3 \ \ 3 \ \ 3]^T$

+4

$r_1 = b - Ax_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\beta_0 = -\frac{r_1^T A S_0}{S_0^T A_0 S_0} = -\frac{-2}{1} = 2$

$S_1 = r_1 + \beta_0 S_0 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$

$\alpha_1 = \frac{S_1^T r_1}{S_1^T A S_1} = \frac{6}{9} = \frac{2}{3}$

$x_2 = x_1 + \alpha_1 S_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$ $\therefore x^{(2)} = [3 \ \ 5 \ \ 5]^T$

+4.

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3.

a) $k_{i-1}+4k_i+k_{i+1}=\frac{6}{h^2}(y_{i-1}-2y_i+y_{i+1})$ $i=1,2,3$ +5

b)
$$k_0+4k_1+k_2 = 6(1-2\times1+0.5) = -3$$
$$k_1+4k_2+k_3 = 6(1-2\times0.5+0) = 0$$
$$k_2+4k_3+k_4 = 6(0.5-2\times0+(0.5)) = 0$$
+3

$k_0=k_1,\ k_4=k_3$

$$\Rightarrow \begin{cases} 5k_1+k_2 = -3 \\ k_1+4k_2+k_3 = 0 \\ k_2+5k_3 = 0 \end{cases} \Rightarrow \begin{bmatrix} 5 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$
+5

Use Doolittle's decomposition

$$\begin{vmatrix} 5 & 1 & 0 \\ 0 & 4-\frac{1}{5} & 1 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 5 & 1 & 0 \\ 0 & 19/5 & 1 \\ 0 & 0 & 5-5/19 \end{vmatrix} \quad \therefore [L \backslash U] = \begin{bmatrix} 5 & 1 & 0 \\ 1/5 & 19/5 & 1 \\ 0 & 5/19 & 18\cdot5/19 \end{bmatrix}$$

$\lambda_{21}=L_{21}=\frac{1}{5}$ $\lambda_{32}=L_{32}=\frac{5}{19}$

$\lambda_{31}=L_{31}=0$

+5

c) $LUx=b$

i) $Ly=b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 0 & 5/19 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \quad \begin{cases} y_1 = -3 \\ y_2 = 3/5 \\ y_3 = -5/19\,y_2 = -3/19 \end{cases}$$
+3

ii) $Ux=y$

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & 19/5 & 1 \\ 0 & 0 & 18\times5/19 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3/5 \\ -3/19 \end{bmatrix} \quad \begin{cases} k_1 = \frac{1}{5}(-3-k_2) \\ k_2 = \frac{5}{19}\left(\frac{3}{5}+\frac{3}{18\cdot5}\right) \\ k_3 = -3/18\times5 \end{cases}$$

$\therefore k_1 = -0.6333$

$k_2 = 0.1667$

$k_3 = -0.0333$ +3

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d) $f_{3,4}(x) = -\frac{k_3}{6}\left[(x-x_4)^3-(x-x_4)\right]+\frac{k_4}{6}\left[(x-x_3)^3-(x-x_3)\right]$

$-\left[y_3(x-x_4)-y_4(x-x_3)\right]$ $\quad (\because x_3-x_4=-1)$

+ 4

$\therefore f_{3,4}(3.4) = -0.1960$

+ 2

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4. a) $F(x, y) = \begin{bmatrix} \ln(x^2+y) - 1 + y \\ \sqrt{x} + xy \end{bmatrix}$ +5

$J(x, y) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{2x}{x^2+y} & \frac{1}{x^2+y} + 1 \\ \frac{1}{2\sqrt{x}} + y & x \end{bmatrix}$

+5

b) 1. Estimate the solution vector x

2. Evaluate $F(x, y)$

3. Compute the Jacobian matrix $J(x, y)$

4. Set up the simultaneous equation in $J(x,y)\Delta x = -F(x,y)$ and solve for Δx

+10

5. Let $x \leftarrow x + \Delta x$ and repeat steps 2-5 until $|\Delta x| < \varepsilon$

c) $J(x, y)\Delta x = -F(x, y)$

$$\begin{bmatrix} 0.9302 & 1.1938 \\ -0.2773 & 2.4 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -0.0409 \\ -0.1092 \end{bmatrix}$$ +3

$x^{(0)} = (2.4, -0.6)$

$\Delta x = 0.01258,\ \Delta y = -0.04406$ +3

$x^{(1)} = (2.4 + \Delta x, -0.6 + \Delta y) = (2.41258, -0.64406)$ +4

5. a) $$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}f'''(x) + \cdots$$

$$\therefore f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

+5

b) $$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \cdots$$
$$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2!}f''(x) + \cdots$$

$$f(x+2h) - f(x-2h) = 4hf'(x) + \frac{8h^3}{3}f'''(x) + \cdots$$

$$\therefore f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} + O(h^2)$$

+5

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c) $g_1 = g(h) = \dfrac{f(x+h) - f(x-h)}{2h}$, $h_1 = h$

$g_2 = g(2h) = \dfrac{f(x+2h) - f(x-2h)}{4h}$, $h_2 = 2h$

$h_2/h_1 = 2$. $P = 2$ ($\because O(h^2)$)

$$G = \frac{(h_2/h_1)^P g(h_1) - g(h_2)}{(h_2/h_1)^P - 1}$$

$$= \frac{2^P g_1 - g_2}{2^P - 1}$$

$$= \frac{4g_1 - g_2}{3}$$

+5

$$f'(x) = \frac{4}{3}\left(\frac{f(x+h) - f(x-h)}{2h}\right) - \frac{1}{3}\left(\frac{f(x+2h) - f(x-2h)}{4h}\right) + O(h^4)$$

$$= \frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)}{12h} + O(h^4)$$

+5