PROBLEM SET 2.1

Problem 1

(a)

$$\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\
&= 1(-1) - 2(-2) + 3(-1) = 0 \quad \text{Singular} \ \blacktriangleleft
\end{aligned}$$

(b)

$$\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 2.11 & -0.80 & 1.72 \\ -1.84 & 3.03 & 1.29 \\ -1.57 & 5.25 & 4.30 \end{vmatrix} \\
&= 2.11\begin{vmatrix} 3.03 & 1.29 \\ 5.25 & 4.30 \end{vmatrix} + 0.80\begin{vmatrix} -1.84 & 1.29 \\ -1.57 & 4.30 \end{vmatrix} + 1.72\begin{vmatrix} -1.84 & 3.03 \\ -1.57 & 5.25 \end{vmatrix} \\
&= 2.11(6.2565) + 0.80(-5.8867) + 1.72(-4.9029) \\
&= 0.058\,867 \quad \text{Ill conditioned} \ \blacktriangleleft
\end{aligned}$$

(c)

$$\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} \\
&= 2(3) + 1(-2) = 4 \quad \text{Well-conditioned} \ \blacktriangleleft
\end{aligned}$$

(d)

$$\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 4 & 3 & -1 \\ 7 & -2 & 3 \\ 5 & -18 & 13 \end{vmatrix} = 4\begin{vmatrix} -2 & 3 \\ -18 & 13 \end{vmatrix} - 3\begin{vmatrix} 7 & 3 \\ 5 & 13 \end{vmatrix} - 1\begin{vmatrix} 7 & -2 \\ 5 & -18 \end{vmatrix} \\
&= 4(28) - 3(76) - 1(-116) = 0 \quad \text{Singular} \ \blacktriangleleft
\end{aligned}$$

Problem 2

(a)

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 21 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 7 & 39 \end{bmatrix} \quad \blacktriangleleft$$

$$|\mathbf{A}| = |\mathbf{L}|\,|\mathbf{U}| = (1 \times 1 \times 1)(1 \times 3 \times 0) = 0 \quad \blacktriangleleft$$

(b)

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -4 \\ 2 & -4 & 11 \end{bmatrix} \quad \blacktriangleleft$$

$$|\mathbf{A}| = |\mathbf{L}|\,|\mathbf{U}| = (2 \times 1 \times 1)(2 \times 1 \times 1) = 4 \quad \blacktriangleleft$$

Problem 3

First solve $\mathbf{Ly} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 11/13 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} y_1 &= 1 \\ \frac{3}{2}(1) + y_2 &= -1 \qquad y_2 = -\frac{5}{2} \\ \frac{1}{2}(1) + \frac{11}{13}\left(\frac{-5}{2}\right) + y_3 &= 2 \qquad y_3 = \frac{47}{13} \end{aligned}$$

Then solve $\mathbf{Ux} = \mathbf{y}$:

$$\begin{bmatrix} 2 & -3 & -1 \\ 0 & 13/2 & -7/2 \\ 0 & 0 & 32/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5/2 \\ 47/13 \end{bmatrix}$$

$$\begin{aligned} \frac{32}{13}x_3 &= \frac{47}{13} \qquad x_3 = \frac{47}{32} \quad \blacktriangleleft \\ \frac{13}{2}x_2 - \frac{7}{2}\left(\frac{47}{32}\right) &= -\frac{5}{2} \qquad x_2 = \frac{13}{32} \quad \blacktriangleleft \\ 2x_1 - 3\left(\frac{13}{32}\right) - \frac{47}{32} &= 1 \qquad x_1 = \frac{59}{32} \quad \blacktriangleleft \end{aligned}$$

Problem 4

The augmented coefficient matrix is

$$[\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 2 & -3 & -1 & 3 \\ 3 & 2 & -5 & -9 \\ 2 & 4 & -1 & -5 \end{bmatrix}$$

Elimination phase:

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - \frac{3}{2} \times \text{row } 1 \\ \text{row. } 3 &\leftarrow \text{row } 3 - \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 2 & -3 & -1 & 3 \\ 3-(3/2)(2) & 2-(3/2)(-3) & -5-(3/2)(-1) & -9-(3/2)(3) \\ 2-2 & 4-(-3) & -1-(-1) & -5-3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 7 & 0 & -8 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - \frac{14}{13} \times \text{row } 2$$

$$\begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 7-(14/13)(13/2) & 0-(14/13)(-7/2) & -8-(14/13)(-27/2) \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 0 & 49/13 & 85/13 \end{bmatrix}$$

Solution by back substitution:

$$\begin{aligned} \frac{49}{13}x_3 &= \frac{85}{13} \qquad x_3 = \frac{85}{49} = 1.7347 \quad \blacktriangleleft \\ \frac{13}{2}x_2 - \frac{7}{2}\left(\frac{85}{49}\right) &= -\frac{27}{2} \qquad x_2 = -\frac{8}{7} = -1.1429 \quad \blacktriangleleft \\ 2x_1 - 3\left(-\frac{8}{7}\right) - \frac{85}{49} &= 3 \qquad x_1 = \frac{32}{49} = 0.6531 \quad \blacktriangleleft \end{aligned}$$

Problem 5

The augmented coefficient matrix is

$$[\mathbf{A}|\mathbf{B}] = \left[\begin{array}{rrrrrr} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

Before elimination, we exchange rows 2 and 3 in order to reduce the amount of algebra:

$$\left[\begin{array}{rrrrrr} 2 & 0 & -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

Elimination phase:

$$\text{row 2} \leftarrow \text{row 2} + \frac{1}{2} \times \text{row 1}$$

$$\left[\begin{array}{rrrrrr} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

$$\text{row 3} \leftarrow \text{row 3} - \frac{1}{2} \times \text{row 2}$$

$$\left[\begin{array}{rrrrrr} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 9/4 & -1/2 & -1/4 & -1/2 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

$$\text{row 4} \leftarrow \text{row 4} - \frac{4}{9} \times \text{row 3}$$

$$\left[\begin{array}{rrrrrr} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 9/4 & -1/2 & -1/4 & -1/2 \\ 0 & 0 & 0 & -16/9 & 1/9 & 2/9 \end{array}\right]$$

First solution vector by back substitution:

$$\begin{aligned} -\frac{16}{9}x_4 &= \frac{1}{9} & x_4 &= -\frac{1}{16} \\ \frac{9}{4}x_3 - \frac{1}{2}\left(-\frac{1}{16}\right) &= -\frac{1}{4} & x_3 &= -\frac{1}{8} \\ 2x_2 - \frac{1}{2}\left(-\frac{1}{8}\right) + \left(-\frac{1}{16}\right) &= \frac{1}{2} & x_2 &= \frac{1}{4} \\ 2x_1 - \left(-\frac{1}{8}\right) &= 1 & x_1 &= \frac{7}{16} \end{aligned}$$

Second solution vector:

$$\begin{aligned}
-\frac{16}{9}x_4 &= \frac{2}{9} & x_4 = -\frac{1}{8} \\
\frac{9}{4}x_3 - \frac{1}{2}\left(-\frac{1}{8}\right) &= -\frac{1}{2} & x_3 = -\frac{1}{4} \\
2x_2 - \frac{1}{2}\left(-\frac{1}{4}\right) + \left(-\frac{1}{8}\right) &= 1 & x_2 = \frac{1}{2} \\
2x_1 - \left(-\frac{1}{4}\right) &= 0 & x_1 = -\frac{1}{8}
\end{aligned}$$

Therefore,

$$\mathbf{X} = \begin{bmatrix} 7/16 & -1/8 \\ 1/4 & 1/2 \\ -1/8 & -1/4 \\ -1/16 & -1/8 \end{bmatrix} \quad \blacktriangleleft$$

Problem 6

After reordering rows, the augmented coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

Elimination phase:

$$\text{row } 3 \leftarrow \text{row } 3 - \text{row } 2$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\text{row } 4 \leftarrow \text{row } 4 - 2 \times \text{row } 3$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\text{row } 5 \leftarrow \text{row } 5 - \text{row } 4$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Back substitution:

$$\begin{aligned} -x_5 &= 1 & x_5 = -1 \blacktriangleleft \\ -x_4 + 2(-1) &= -3 & x_4 = 1 \blacktriangleleft \\ x_3 + 1 &= 2 & x_3 = 1 \blacktriangleleft \\ x_2 - 1 + 1 - (-1) &= -1 & x_2 = -2 \blacktriangleleft \\ x_1 + 2(-2) - 2(1) &= -4 & x_1 = 2 \blacktriangleleft \end{aligned}$$

Problem 7

(a)

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Use Gauss elimination storing each multiplier in the location occupied by the element that was eliminated (the multipliers are enclosed in boxes $\boxed{\text{thus}}$:

$$\text{row 2} \leftarrow \text{row 2} - \left(-\frac{1}{4}\right) \times \text{row 2}$$

$$\begin{bmatrix} 4 & -1 & 0 \\ \boxed{-1/4} & 15/4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - \left(-\frac{4}{15}\right) \times \text{row 2}$$

$$\begin{bmatrix} 4 & -1 & 0 \\ \boxed{-1/4} & 15/4 & -1 \\ 0 & \boxed{-4/15} & 56/15 \end{bmatrix}$$

Thus

$$\mathbf{U} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 15/4 & -1 \\ 0 & 0 & 56/15 \end{bmatrix} \blacktriangleleft \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & -4/15 & 1 \end{bmatrix} \blacktriangleleft$$

(b)

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T$$

Substituting for $\mathbf{L}\mathbf{L}^T$ from Eq. (2.16), we get

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$$

Equating matrices term-by term:

$$\begin{aligned} L_{11}^2 &= 4 & L_{11} = 2 \\ 2L_{21} &= -1 & L_{21} = -\frac{1}{2} \\ 2L_{31} &= 0 & L_{31} = 0 \\ \left(-\frac{1}{2}\right)^2 + L_{22}^2 &= 4 & L_{22} = \frac{\sqrt{15}}{2} \\ -\frac{1}{2}(0) + \frac{\sqrt{15}}{2}L_{32} &= -1 & L_{32} = -\frac{2}{\sqrt{15}} \\ 0^2 + \left(-\frac{2}{\sqrt{15}}\right)^2 + L_{33}^2 &= 4 & L_{32} = 2\sqrt{\frac{14}{15}} \end{aligned}$$

Therefore,

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{15}/2 & 0 \\ 0 & -2/\sqrt{15} & 2\sqrt{14/15} \end{bmatrix} \blacktriangleleft$$

Problem 8

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -4 \\ 9 & -8 & 24 \\ -12 & 24 & -26 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 65 \\ -42 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row 2} &\leftarrow \text{row 2} - (-3) \times \text{row 1} \\ \text{row 3} &\leftarrow \text{row 3} - 4 \times \text{row 1} \end{aligned}$$

$$\begin{bmatrix} -3 & 6 & -4 \\ \boxed{-3} & 10 & 12 \\ \boxed{4} & 0 & -10 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} -3 & 6 & -4 \\ 0 & 10 & 12 \\ 0 & 0 & -10 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= -3 \\ -3(-3) + y_2 &= 65 \qquad y_2 = 56 \\ 4(-3) + y_3 &= -42 \qquad y_3 = -30 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} -10x_3 &= -30 \qquad x_3 = 3 \ \blacktriangleleft \\ 10x_2 + 12(3) &= 56 \qquad x_2 = 2 \ \blacktriangleleft \\ -3x_1 + 6(2) - 4(3) &= -3 \qquad x_1 = 1 \ \blacktriangleleft \end{aligned}$$

Problem 9

$$\mathbf{A} = \begin{bmatrix} 2.34 & -4.10 & 1.78 \\ -1.98 & 3.47 & -2.22 \\ 2.36 & -15.17 & 6.18 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0.02 \\ -0.73 \\ -6.63 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row 2} &\leftarrow \text{row 2} - (-0.846\,154) \times \text{row 1} \\ \text{row 3} &\leftarrow \text{row 3} - 1.008\,547 \times \text{row 1} \end{aligned}$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 \\ \boxed{-0.846\,154} & 0.000\ 769 & -0.713\,846 \\ \boxed{1.008\,547} & -11.034\,96 & 4.384\,786 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - (-14349.75) \times \text{row 2}$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 \\ \boxed{-0.846\,154} & 0.000\ 769 & -0.713\,846 \\ \boxed{1.008\,547} & \boxed{-14349.75} & -10239.13 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 2.34 & -4.10 & 1.78 \\ 0 & 0.000\ 769 & -0.713\,846 \\ 0 & 0 & -10\,239.1 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.846\,154 & 1 & 0 \\ 1.008\,547 & -14349.7 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 0.02 \\ -0.846\,154(0.02) + y_2 &= -0.73 \qquad y_2 = -0.713\,07\,7 \\ 1.008\,547(0.02) - 14349.7(-0.713\,07\,7) + y_3 &= -6.63 \qquad y_3 = -10\,239.1 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} -10\,239.1x_3 &= -10\,239.1 \qquad x_3 = 1.0 \ \blacktriangleleft \\ 0.000\ 769x_2 - 0.713\,846 &= -0.713\,07\,7 \qquad x_2 = 1.0 \ \blacktriangleleft \\ 2.34x_1 - 4.10 + 1.78 &= 0.02 \qquad x_1 = 1.0 \ \blacktriangleleft \end{aligned}$$

Problem 10

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 6 \\ 8 & -3 & 10 \\ -4 & 12 & -10 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - 2 \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - (-1) \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 4 & -3 & 6 \\ \boxed{2} & 3 & -2 \\ \boxed{-1} & 9 & -4 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - 3 \times \text{row } 2$$

$$\begin{bmatrix} 4 & -3 & 6 \\ \boxed{2} & 3 & -2 \\ \boxed{-1} & \boxed{3} & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -3 & 6 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

<u>First solution vector</u>
Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 1 \\ 2(1) + y_2 &= 0 \qquad y_2 = -2 \\ -1 + 3(-2) + y_3 &= 0 \qquad y_3 = 7 \end{aligned}$$

Solution of $\mathbf{Uy} = \mathbf{x}$:

$$\begin{aligned} 2x_3 &= 7 \qquad x_3 = \frac{7}{2} \\ 3x_2 - 2\left(\frac{7}{2}\right) &= -2 \qquad x_2 = \frac{5}{3} \\ 4x_1 - 3\left(\frac{5}{3}\right) + 6\left(\frac{7}{2}\right) &= 1 \qquad x_1 = -\frac{15}{4} \end{aligned}$$

<u>Second solution vector</u>
Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 0 \\ 2(0) + y_2 &= 1 \qquad y_2 = 1 \\ -1(0) + 3(1) + y_3 &= 0 \qquad y_3 = -3 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned}
2x_3 &= -3 \qquad x_3 = -\frac{3}{2} \\
3x_2 - 2\left(-\frac{3}{2}\right) &= 1 \qquad x_3 = -\frac{2}{3} \\
4x_1 - 3\left(-\frac{2}{3}\right) + 6\left(-\frac{3}{2}\right) &= 0 \qquad x_1 = \frac{7}{4}
\end{aligned}$$

Therefore,

$$\mathbf{X} = \begin{bmatrix} 7/2 & -3/2 \\ 5/3 & -2/3 \\ -15/4 & 7/4 \end{bmatrix} \blacktriangleleft$$

Problem 11

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3/2 \\ 3 \end{bmatrix}$$

Substituting for $\mathbf{LL}^T$ from Eq. (2.16), we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$$

Equating matrices term-by term:

$$\begin{aligned}
L_{11} &= 1 \qquad L_{21} = 1 \qquad L_{31} = 1 \\
1^2 + L_{22}^2 &= 2 \qquad L_{22} = 1 \\
(1)(1) + (1)L_{32} &= 2 \qquad L_{32} = 1 \\
1^2 + 1^2 + L_{33}^2 &= 3 \qquad L_{33} = 1
\end{aligned}$$

Thus

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{L}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned}
y_1 &= 1 \\
1 + y_2 &= \frac{3}{2} \qquad y_2 = \frac{1}{2} \\
1 + \frac{1}{2} + y_3 &= 3 \qquad y_3 = \frac{3}{2}
\end{aligned}$$

Solution of $\mathbf{L}^T\mathbf{x} = \mathbf{y}$:

$$\begin{aligned}
x_3 &= \frac{3}{2} & & \blacktriangleleft \\
x_2 + \frac{3}{2} &= \frac{1}{2} & x_2 &= -1 \quad \blacktriangleleft \\
x_1 - 1 + \frac{3}{2} &= 1 & x_1 &= \frac{1}{2} \quad \blacktriangleleft
\end{aligned}$$

Problem 12

$$A = \begin{bmatrix} 4 & -2 & -3 \\ 12 & 4 & -10 \\ -16 & 28 & 18 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1.1 \\ 0 \\ -2.3 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned}
\text{row 2} &\leftarrow \text{row 2} - 3 \times \text{row 1} \\
\text{row 3} &\leftarrow \text{row 3} - (-4) \times \text{row 1}
\end{aligned}$$

$$\begin{bmatrix} 4 & -2 & -3 \\ \boxed{3} & 10 & -1 \\ \boxed{-4} & 20 & 6 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - 2 \times \text{row 2}$$

$$\begin{bmatrix} 4 & -2 & -3 \\ \boxed{3} & 10 & -1 \\ \boxed{-4} & \boxed{2} & 8 \end{bmatrix}$$

Therefore

$$\mathbf{U} = \begin{bmatrix} 4 & -2 & -3 \\ 0 & 10 & -1 \\ 0 & 0 & 8 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned}
y_1 &= 1.1 \\
3(1.1) + y_2 &= 0 & y_2 &= -3.3 \\
-4(1.1) + 2(-3.3) + y_3 &= -2.3 & y_3 &= 8.7
\end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned}
8x_3 &= 8.7 & x_3 &= 1.0875 \quad \blacktriangleleft \\
10x_2 - 1.0875 &= -3.3 & x_2 &= -0.221\,25 \quad \blacktriangleleft \\
4x_1 - 2(-0.221\,25) - 3(1.0875) &= 1.1 & x_1 &= 0.98 \quad \blacktriangleleft
\end{aligned}$$

Problem 13

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \cdots \\ 0 & 0 & \alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Since the banded structure of a matrix is preserved during decomposition, $\mathbf{L}$ must be a diagonal matrix. Therefore,

$$\mathbf{L}\mathbf{L}^T = \begin{bmatrix} L_{11}^2 & 0 & 0 & \cdots \\ 0 & L_{22}^2 & 0 & \cdots \\ 0 & 0 & L_{33}^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It follows from $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ that

$$\mathbf{L} = \begin{bmatrix} \sqrt{\alpha_1} & 0 & 0 & \cdots \\ 0 & \sqrt{\alpha_2} & 0 & \cdots \\ 0 & 0 & \sqrt{\alpha_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \blacktriangleleft$$

Problem 14

```
function [X,det] = gauss2(A,B)
% Solves A*X = B by Gauss elimination and computes det(A).
% USAGE: [X,det] = gauss(A,B)

[n,m] = size(B);
for k = 1:n-1                    % Elimination phase
    for i= k+1:n
        if A(i,k) ~= 0
            lambda = A(i,k)/A(k,k);
            A(i,k+1:n) = A(i,k+1:n) - lambda*A(k,k+1:n);
            B(i,:)= B(i,:) - lambda*B(k,:);
         end
     end
end
if nargout == 2; det = prod(diag(A)); end
for k = n:-1:1                   % Back substitution phase
    for i = 1:m
        B(k,i) = (B(k,i) - A(k,k+1:n)*B(k+1:n,i))/A(k,k);
```

```
    end
end
X = B;
```

Testing gauss2:

```
>> A = [2 -1 0; -1 2 -1; 0 -1 1];
>> B = [1 0 0; 0 1 0; 0 0 1];
>> [X,detA] = gauss2(A,B)
X =
    1.0000    1.0000    1.0000
    1.0000    2.0000    2.0000
    1.0000    2.0000    3.0000
detA =
     1
```

Problem 15

```
function hilbert(n)
% Solves A*x = b by LU decomposition, where
% [A] is an n x n Hilbert matrix and b(i) is the sum of
% the elemnts in the ith row of [A].
% USAGE: hibert(n)

A = zeros(n); b = zeros(n,1);
for i = 1:n
    for j = 1:n
        A(i,j)= 1/(i + j - 1);
        b(i) = b(i) + A(i,j);
    end
end
A = LUdec(A); x = LUsol(A,b)
```

The largest n for which 6-figure accuracy is achieved seems to be 8:

```
>> format long
>> hilbert(8)
x =
   0.99999999996352
   1.00000000197488
   0.99999997404934
   1.00000014104434
   0.99999961908263
```

```
1.00000054026798
0.99999961479776
1.00000010884699
```

Problem 16

Forward sustitution The kth equation of $\mathbf{Ly} = \mathbf{b}$ is

$$L_{k1}y_1 + L_{k2}y_2 + \cdots + L_{kk}y_k = b_k$$

Solving for y_k yields

$$\begin{aligned} y_k &= \frac{b_k - (L_{k,1}y_1 + L_{k,2}y_2 + \cdots + L_{k,k-1}y_{k-1})}{L_{kk}} \\ &= b_k - \frac{\left[\begin{array}{cccc} L_{k,1} & L_{k,2} & \cdots & L_{k,k-1} \end{array}\right] \cdot \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_{k-1} \end{array}\right]}{L_{k,k}} \end{aligned}$$

This expression, evaluated with $k = 1, 2, \ldots, n$ (in that order), constitutes the forward substitution phase In `choleskiSol` the b's are overwritten with y's during the computations.

Back substitution A typical (kth) equation of $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ is

$$L_{k,k}x_k + L_{k+1,k}x_{k+1} + L_{k+2,k}x_{k+2} + \cdots + L_{n,k}x_n = y_k$$

The solution for x_k is

$$\begin{aligned} x_k &= \frac{y_k - (L_{k+1,k}x_{k+1} + L_{k+2,k}x_{k+2} + \ldots L_{n,k}x_n)}{L_{k,k}} \\ &= \frac{y_k - \left[\begin{array}{cccc} L_{k+1,k} & L_{k+2,k} & \cdots & L_{n,k} \end{array}\right] \cdot \left[\begin{array}{cccc} x_{k+1} & x_{k+2} & \cdots & x_n \end{array}\right]}{L_{kk}} \end{aligned}$$

In back substitution we evaluate this expression in the order $k = n, n-1, \ldots, 1$. Note that in `choleskiSol` the vector $\mathbf{x}$ overwrites the vector $\mathbf{y}$.

Problem 17

```
% problem2_1_17
x = [0 1 3 4]'; y = [10 35 31 2]';
n = length(x);
A = zeros(n);
```

```
for i = 1:n; A(:,i) = x.^(i-1); end
L = LUdec(A);
coefficients = LUsol(L,y)

>> coefficients =
    10
    34
    -9
     0
```

Problem 18

```
% problem2_1_18
x = [0 1 3 5 6]'; y = [-1 1 3 2 -2]';
n = length(x);
A = zeros(n);
for i = 1:n; A(:,i) = x.^(i-1); end
L = LUdec(A);
coefficients = LUsol(L,y)

>> coefficients =
  -1.00000000000000
   2.68333333333333
  -0.87500000000000
   0.21666666666667
  -0.02500000000000
```

Problem 19

$$
\begin{aligned}
f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \\
f''(x) &= 2c_2 + 6c_3 x + 12 c_4 x^2
\end{aligned}
$$

The specified conditions result in the equations

$$\mathbf{Ac} = \mathbf{y}$$

```
% problem2_1_19
A = zeros(5);
A(1,:) = [1 0 0 0 0];                    % f(0)
```

```
A(2,:) = [1 0.75 0.75^2 0.75^3 0.75^4];  % f(0.75)
A(3,:) = [1 1 1 1 1];                     % f(1)
A(4,:) = [0 0 2 0 0];                     % f''(0)
A(5,:) = [0 0 2 6 12];                    % f''(1)
y = [1 -0.25 1 0 0]';
L = LUdec(A);
c = LUsol(L,y)

>> c =
   1.00000000000000
  -5.61403508771930
   0.00000000000000
  11.22807017543859
  -5.61403508771930
```

Therefore, the polynomial is

$$f(x) = 1 - 5.6140x + 11.2281x^3 - 5.6140x^4$$

Problem 20

$$\mathbf{A} = \begin{bmatrix} 3.50 & 2.77 & -0.76 & 1.80 \\ -1.80 & 2.68 & 3.44 & -0.09 \\ 0.27 & 5.07 & 6.90 & 1.61 \\ 1.71 & 5.45 & 2.68 & 1.71 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 7.31 \\ 4.23 \\ 13.85 \\ 11.55 \end{bmatrix}$$

```
% problem2_1_20
A = [3.50 2.77 -0.76  1.80
    -1.80 2.68  3.44 -0.09
     0.27 5.07  6.90  1.61
     1.71 5.45  2.68  1.71];
b = [7.31 4.23 13.85 11.55]';
n = length(b);
L = LUdec(A);
x = LUsol(L,b)
detA = prod(diag(L))
Ax = A*x

>> x =
   1.00000000000011
   1.00000000000002
   1.00000000000003
   0.99999999999976
```

```
detA =
  -0.22579734000001
Ax =
   7.31000000000000
   4.23000000000000
  13.85000000000000
  11.55000000000000
```

The determinant is a little smaller than the elements of **A**, indicating a mild case of ill-conditioning. From the results it appears that the solution is 12-figure accurate.

Problem 21

```
>> A = [1 -1 -1; 0 1 -2; 0 0 1];
>> inv(A)

ans =

     1     1     3
     0     1     2
     0     0     1
```

(a)

$$\begin{aligned}\|A\|_e &= \sqrt{1^2+1^2+1^2+1^2+2^2+1^2} = 3\\ \left\|A^{-1}\right\|_e &= \sqrt{1^2+1^2+3^2+1^2+2^2+1^2} = \sqrt{17}\\ \text{cond}_e &= \|A\|_e\left\|A^{-1}\right\|_e = 3\sqrt{17} = 12.37 \quad \blacktriangleleft\end{aligned}$$

(b)

$$\begin{aligned}\|A\|_\infty &= 3 \text{ (determined by row 1 or row 2)}\\ \left\|A^{-1}\right\|_\infty &= 5 \text{ (determined by row 1)}\\ \text{cond}_\infty &= \|A\|_\infty\left\|A^{-1}\right\|_\infty = 3(5) = 15 \quad \blacktriangleleft\end{aligned}$$

Problem 22

```
function eCond = cond(A)
% Returns condition number of [A]
```

```
eCond = norm(A)*norm(inv(A));

function eNorm = norm(A)
% Returns euclidean norm of [A]
n = size(A,1);
eNorm = 0;
for i = 1:n
    eNorm = eNorm + sum(A(i,:).^2);
end
eNorm = sqrt(eNorm);

>> A = [ 1  4  9 16
         4  9 16 25
         9 16 25 36
        16 25 36 49];

>> cond(A)
Warning: Matrix is close to singular or badly scaled.

ans =

  3.0371e+016
```

PROBLEM SET 2.2

Problem 1

$$\mathbf{A} = \begin{bmatrix} 3 & -3 & 3 \\ -3 & 5 & 1 \\ 3 & 1 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 9 \\ -7 \\ 12 \end{bmatrix}$$

Noting that $\mathbf{A}$ is symmetric, we have two choices: (1) the Gauss elimination scheme that stores the multipliers in the *upper* portion of the matrix and results in $\mathbf{A} \to \left[\mathbf{0}\backslash\mathbf{D}\backslash\mathbf{L}^T\right]$ (see Example 2.10); or (2) the regular Gauss elimination that produces an upper triangular matrix $\mathbf{A} \to \mathbf{U}$. We choose the latter, which is somewhat simpler to implement in hand computation.

$$\begin{aligned} \text{row 2} &\leftarrow \text{row 2} + \text{row 1} \\ \text{row 3} &\leftarrow \text{row 3} - \text{row 1} \end{aligned}$$

$$\begin{bmatrix} 3 & -3 & 3 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - 2 \times \text{row 2}$$

$$\mathbf{U} = \begin{bmatrix} 3 & -3 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{bmatrix}$$

We obtain L^T by dividing each row of U by its diagonal element. Thus

$$\mathbf{L}^T = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 9 \\ -9 + y_2 &= -7 \qquad y_2 = 2 \\ 9 + 2(2) + y_3 &= 12 \qquad y_3 = -1 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} -6x_3 &= -1 \qquad x_3 = \frac{1}{6} \blacktriangleleft \\ 2x_2 + 4\left(\frac{1}{6}\right) &= 2 \qquad x_2 = \frac{2}{3} \blacktriangleleft \\ 3x_1 - 3\left(\frac{2}{3}\right) + 3\left[\frac{1}{6}\right] &= 9 \qquad x_1 = \frac{7}{2} \blacktriangleleft \end{aligned}$$

Problem 2

$$\mathbf{A} = \begin{bmatrix} 4 & 8 & 20 \\ 8 & 13 & 16 \\ 20 & 16 & -91 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 24 \\ 18 \\ -119 \end{bmatrix}$$

Since $\mathbf{A}$ is symmetric, we could employ the same Gauss elimination $\mathbf{A} \to \mathbf{U}$ that was used in Problem 1. However, we chose the form $\mathbf{A} \to \left[\mathbf{0}\backslash\mathbf{D}\backslash\mathbf{L}^T\right]$ obtained by storing the multipliers (shown enclosed in boxes) in the *upper* half of the matrix.

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - 2 \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - 5 \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 4 & \boxed{2} & \boxed{5} \\ 0 & -3 & -24 \\ 0 & -24 & -191 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - 8 \times \text{row } 2$$

$$\left[\mathbf{0}\backslash\mathbf{D}\backslash\mathbf{L}^T\right] = \begin{bmatrix} 4 & \boxed{2} & \boxed{5} \\ 0 & -3 & \boxed{8} \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\mathbf{L}^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 8 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{U} = \mathbf{D}\mathbf{L}^T = \begin{bmatrix} 4 & 6 & 20 \\ 0 & -3 & -24 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 24 \\ 2(24) + y_2 &= 18 \qquad y_2 = -30 \\ 5(24) + 8(-30) + y_3 &= -119 \qquad y_3 = 1 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} x_3 &= 1 \;\blacktriangleleft \\ -3x_2 - 24(1) &= -30 \qquad x_2 = 2 \;\blacktriangleleft \\ 4x_1 + 8(2) + 20(1) &= 24 \qquad x_1 = -3 \;\blacktriangleleft \end{aligned}$$

Problem 3

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 5 & -6 & 0 & 0 \\ 0 & -6 & 16 & 12 & 0 \\ 0 & 0 & 12 & 39 & -6 \\ 0 & 0 & 0 & -6 & 14 \end{bmatrix}$$

Noting that $\mathbf{A}$ is symmetric, we use the reduction $\mathbf{A} \to \left[\mathbf{0} \backslash \mathbf{D} \backslash \mathbf{L}^T\right]$ obtained by storing the multipliers (shown enclosed in boxes) in the *upper* half of the matrix during Gauss elimination.

$$\text{row 2} \leftarrow \text{row 2} - (-1) \times \text{row 1}$$

$$\begin{bmatrix} 2 & \boxed{-1} & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & -6 & 16 & 12 & 0 \\ 0 & 0 & 12 & 39 & -6 \\ 0 & 0 & 0 & -6 & 14 \end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - (-2) \times \text{row 2}$$

$$\begin{bmatrix} 2 & \boxed{-1} & 0 & 0 & 0 \\ 0 & 3 & \boxed{-2} & 0 & 0 \\ 0 & 0 & 4 & 12 & 0 \\ 0 & 0 & 12 & 39 & -6 \\ 0 & 0 & 0 & -6 & 14 \end{bmatrix}$$

$$\text{row 4} \leftarrow \text{row 4} - 3 \times \text{row 3}$$

$$\begin{bmatrix} 2 & \boxed{-1} & 0 & 0 & 0 \\ 0 & 3 & \boxed{-2} & 0 & 0 \\ 0 & 0 & 4 & \boxed{3} & 0 \\ 0 & 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & -6 & 14 \end{bmatrix}$$

$$\text{row 5} \leftarrow \text{row 5} - (-2) \times \text{row 4}$$

$$\left[\mathbf{0} \backslash \mathbf{D} \backslash \mathbf{L}^T\right] = \begin{bmatrix} 2 & \boxed{-1} & 0 & 0 & 0 \\ 0 & 3 & \boxed{-2} & 0 & 0 \\ 0 & 0 & 4 & \boxed{3} & 0 \\ 0 & 0 & 0 & 3 & \boxed{-2} \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Thus

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \blacktriangleleft \quad \mathbf{L}^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix} \blacktriangleleft$$

Problem 4

$$\mathbf{A} = \begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ -1 & 7 & 2 & 0 & 0 \\ 0 & -2 & 8 & 2 & 0 \\ 0 & 0 & 3 & 7 & -2 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ -3 \\ 1 \end{bmatrix}$$

LU decomposition of $\mathbf{A}$:

$$\text{row } 2 \leftarrow \text{row } 2 - (-0.1667) \times \text{row } 1$$

$$\begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ \boxed{-0.1667} & 7.3333 & 2 & 0 & 0 \\ 0 & -2 & 8 & 2 & 0 \\ 0 & 0 & 3 & 7 & -2 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - (-0.2727) \times \text{row } 2$$

$$\begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ \boxed{-0.1667} & 7.3333 & 2 & 0 & 0 \\ 0 & \boxed{-0.2727} & 8.5454 & 2 & 0 \\ 0 & 0 & 3 & 7 & -2 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix}$$

$$\text{row } 4 \leftarrow \text{row } 4 - 0.3511 \times \text{row } 3$$

$$\begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ \boxed{-0.1667} & 7.3333 & 2 & 0 & 0 \\ 0 & \boxed{-0.2727} & 8.5454 & 2 & 0 \\ 0 & 0 & \boxed{0.3511} & 6.2978 & -2 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix}$$

$$\text{row } 5 \leftarrow \text{row } 5 - 0.4764 \times \text{row } 4$$

$$[\mathbf{L} \backslash \mathbf{U}] = \begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ \boxed{-0.1667} & 7.3333 & 2 & 0 & 0 \\ 0 & \boxed{-0.2727} & 8.5454 & 2 & 0 \\ 0 & 0 & \boxed{0.3511} & 6.2978 & -2 \\ 0 & 0 & 0 & \boxed{0.4764} & 5.9528 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 2 \\ -0.1667(2) + y_2 &= -3 \qquad y_2 = -2.6667 \\ -0.2727(-2.6667) + y_3 &= 4 \qquad y_3 = 3.2728 \\ 0.3511(3.2728) + y_4 &= -3 \qquad y_4 = -4.1491 \\ 0.4764(-4.1491) + y_5 &= 1 \qquad y_5 = 2.9766 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} 5.9528x_5 &= 2.9766 \qquad x_5 = 0.5000 \blacktriangleleft \\ 6.2978x_4 - 2(0.5000) &= -4.1491 \qquad x_4 = -0.5000 \blacktriangleleft \\ 8.5454x_3 + 2(-0.5000) &= 3.2728 \qquad x_3 = 0.5000 \blacktriangleleft \\ 7.3333x_2 + 2(0.5000) &= -2.6667 \qquad x_2 = -0.5000 \blacktriangleleft \\ 6x_1 + 2(-0.5000) &= 2 \qquad x_1 = 0.5000 \blacktriangleleft \end{aligned}$$

Problem 5

$$[\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 4 & -2 & 1 & 2 \\ -2 & 1 & -1 & -1 \\ -2 & 3 & 6 & 0 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 1/3 \end{bmatrix}$$

No need to pivot here.

$$\begin{aligned} \text{row 2} &\leftarrow \text{row 2} + \frac{1}{2} \times \text{row 1} \\ \text{row 3} &\leftarrow \text{row 3} + \frac{1}{2} \times \text{row 1} \end{aligned}$$

$$\begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 0 & -1/2 & 0 \\ 0 & 2 & 13/2 & 1 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} * \\ 1/2 \\ 13/2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} * \\ 0 \\ 4/13 \end{bmatrix}$$

Exchanging rows 2 and 3 triangulizes the coefficient matrix:

$$\begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 2 & 13/2 & 1 \\ 0 & 0 & -1/2 & 0 \end{bmatrix}$$

Back substitution:

$$\begin{aligned} x_3 &= 0 \blacktriangleleft \\ 2x_2 + \frac{13}{2}(0) &= 1 \qquad x_2 = \frac{1}{2} \blacktriangleleft \\ 4x_1 - 2\left(\frac{1}{2}\right) + 0 &= 2 \qquad x_1 = \frac{3}{4} \blacktriangleleft \end{aligned}$$

Problem 6

$$[\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 2.34 & -4.10 & 1.78 & 0.02 \\ -1.98 & 3.47 & -2.22 & -0.73 \\ 2.36 & -15.17 & 6.81 & -6.63 \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} 4.10 \\ 3.47 \\ 15.17 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 2.34/4.10 \\ 1.98/3.47 \\ 2.36/15.17 \end{bmatrix} = \begin{bmatrix} 0.5707 \\ 0.5706 \\ 0.1556 \end{bmatrix}$$

No need to pivot here.

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 + (1.98/2.34) \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - (2.36/2.34) \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 & 0.02 \\ 0 & 0.0008 & -0.7138 & -0.7131 \\ 0 & -11.0350 & 5.0148 & -6.6502 \end{bmatrix}$$

Without computing $\mathbf{r}$, it is clear that row 3 must be the next pivot row. We do not physically interchange rows 2 and 3, but carry out the elimination "in place":

$$\text{row } 2 \leftarrow \text{row } 2 + (0.0008/11.0350) \times \text{row } 3$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 & 0.02 \\ 0 & 0 & -0.7134 & -0.7136 \\ 0 & -11.0350 & 5.0148 & -6.6502 \end{bmatrix}$$

Back substitution:

$$\begin{aligned} -0.7134x_3 &= -0.7136 \qquad x_3 = 1.0003 \blacktriangleleft \\ -11.0350x_2 + 5.0148(1.0003) &= -6.6502 \qquad x_2 = 1.0572 \blacktriangleleft \\ 2.34x_1 - 4.10(1.0572) + 1.78(1.0003) &= 0.02 \qquad x_1 = 1.1000 \blacktriangleleft \end{aligned}$$

Problem 7

We do not physically interchange rows, but eliminate "in place".

$$[\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$\text{row } 4 \leftarrow \text{row } 4 + \frac{1}{2} \times \text{row } 1$$

$$
\begin{bmatrix} 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 3/2 & -1 & 0 & 1/2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} * \\ 0 \\ 1/2 \\ 1 \end{bmatrix}
$$

$$
\text{row } 3 \leftarrow \text{row } 3 + \frac{2}{3} \times \text{row } 4
$$

$$
\begin{bmatrix} 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 4/3 & -1 & 1/3 \\ 0 & 3/2 & -1 & 0 & 1/2 \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} * \\ 1 \\ 1 \\ * \end{bmatrix}
$$

$$
\text{row } 2 \leftarrow \text{row } 2 + \frac{3}{4} \times \text{row } 3
$$

$$
\begin{bmatrix} 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 4/3 & -1 & 1/3 \\ 0 & 3/2 & -1 & 0 & 1/2 \end{bmatrix}
$$

Note that by rearranging rows, the coefficient matrix could be given an upper triangular form. There is no need for this rearrangement, since back substitution can be carried out just as easily on the matrix as it is:

$$
\begin{aligned}
\frac{1}{4}x_4 &= \frac{1}{4} & x_4 &= 1 \blacktriangleleft \\
\frac{4}{3}x_3 - 1 &= \frac{1}{3} & x_3 &= 1 \blacktriangleleft \\
\frac{3}{2}x_2 - 1 &= \frac{1}{2} & x_2 &= 1 \blacktriangleleft \\
2x_1 - 1 &= 1 & x_1 &= 1 \blacktriangleleft
\end{aligned}
$$

Problem 8

We chose Gauss elimination with pivoting (pivoting is essential here due to the zero element in the top left corner of the coefficient matrix).

```
% problem2_2_8
A = [0  2  5 -1
     2  1  3  0
    -2 -1  3  1
     3  3 -1  2];
b = [-3 3 -2 5]';
x = gaussPiv(A,b)

>> x =
```

```
  2.0000
 -1.0000
  0.0000
  1.0000
```

Problem 9

As the coefficient matrix is tridiagonal, it is very unlikely to benefit from pivoting. Hence we use the non-pivoting LU decomposition functions written for tridiagonal matrices.

```
% problem2_2_9
n = 10;
c = ones(n-1,1)*(-1.0); e = c;
d = ones(n,1)*4.0;
b = ones(n,1)*5.0; b(1) = 9.0;
[c,d,e] = LUdec3(c,d,e);
x = LUsol3(c,d,e,b)

>> x =
    2.9019
    2.6077
    2.5288
    2.5075
    2.5011
    2.4971
    2.4873
    2.4519
    2.3205
    1.8301
```

Problem 10

Unless there are obvious reasons to do otherwise, play it safe by using pivoting. Here we chose LU decomposition with pivoting.

```
% problem2_2_10
A = [1.3174 2.7250 2.7250 1.7181
     0.4002 0.8278 1.2272 2.5322
     0.8218 1.5608 0.3629 2.9210
```

```
     1.9664 2.0011 0.6532 1.9945];
b = [8.4855 4.9874 5.6665 6.6152]';
[L,perm] = LUdecPiv(A);
x = LUsolPiv(L,b,perm)

>> x =
    1.0000
    1.0000
    1.0000
    1.0000
```

Problem 11

We use LU decomposition with pivoting:

```
% problem2_2_11
A = [10 -2 -1  2  3   1 -4  7
     5  11  3 10 -3   3  3 -4
     7  12  1  5  3 -12  2  3
     8   7 -2  1  3   2  2  4
     2 -15 -1  1  4  -1  8  3
     4   2  9  1 12  -1  4  1
    -1   4 -7 -1  1   1 -1 -3
    -1   3  4  1  3  -4  7  6];
b = [0 12 -5 3 -25 -26 9 -7]';
[L,perm] = LUdecPiv(A);
x = LUsolPiv(L,b,perm)

>> > x =
   -1.0000
    1.0000
   -1.0000
    1.0000
   -1.0000
    1.0000
   -1.0000
    1.0000
```

Problem 12

As the coefficient matrix is tridiagonal, we use the LU decomposition routines for tridiagonal matrices:

```
% problem2_2_12
k = [10 10 10 5 5]';
W = [100 100 100 50 50]';
n = length(k);
c = zeros(n-1,1); d = zeros(n,1);
c = -k(2:n); e = c;
d(1:n-1) = k(1:n-1) + k(2:n); d(n) = k(n);
[c,d,e] = LUdec3(c,d,e); x = LUsol3(c,d,e,W)
```

The units of the computed displacements are mm:

```
>> x =
   40.0000
   70.0000
   90.0000
  110.0000
  120.0000
```

Problem 13

The given equilibrium equations have the form $\mathbf{A}(k_1, k_2, \cdots) \cdot \mathbf{x} = \mathbf{b}(W_1, W_2, W_3)$. Since the ratios k_i/k and W_j/W are specified, the equations must be rewritten as

$$\mathbf{A}\left(\frac{k_1}{k}, \frac{k_2}{k}, \cdots\right) \cdot \mathbf{x} = \frac{W}{k}\mathbf{b}\left(\frac{W_1}{W}, \frac{W_2}{W}, \frac{W_3}{W}\right)$$

before they can be solved numerically. In the program below `k[i]` stands for k_i/k and `W[j]` represents W_j/W. As the coefficient matrix is diagonally dominant, Gauss elimination without pivoting is used.

```
% problem2_2_13
k = [1 2 1 1 2]';
W = [2 1 2]';
A = [k(1) + k(2) + k(3) + k(5)  -k(3)          -k(5)
    -k(3)                        k(3) + k(4)   -k(4)
    -k(5)                       -k(4)           k(4) + k(5)];
x = gauss(A,W)
>> x =
```

```
    1.6667
    2.6667
    2.6667
```

Note that the units of $\mathbf{x}$ are W/k; that is, displacement$_i = (W/k)x_i$.

Problem 14

Here the coefficient matrix is close to being diagonally dominant, so that pivoting is not needed. We chose LU decomposition as the method of solution.

```
% problem2_2_14
K = [27.58   7.004 -7.004 0.0    0.0
      7.004 29.57  -5.253 0.0  -24.32
     -7.004 -5.253 29.57  0.0    0.0
      0.0    0.0    0.0  27.58  -7.004
      0.0  -24.32   0.0  -7.004 29.57];
p = [0 0 0 0 -45]';
L = LUdec(K); u = LUsol(L,p)
```

The computed displacements $\mathbf{u}$ are in mm:

```
>> u =
    1.4404
   -6.4825
   -0.8104
   -1.8518
   -7.2920
```

Problem 15

(a)

We use LU decomposition with pivoting (pivoting is a must here):

```
% problem2_2_15
c = -1/sqrt(2);
A = [-1  1 -c  0  0  0
      0  0  c  1  0  0
      0 -1  0  0 -c  0
      0  0  0  0  c  0
      0  0  0  0  c  1
```

```
      0  0  0 -1 -c  0];
b = [0 18 0 12 0 0]';
[L,perm] = LUdecPiv(A); P = LUsolPiv(L,b,perm)
```

The forces **in** the members are (in kN):

```
>> P =
  -42.0000
  -12.0000
  -42.4264
  -12.0000
  -16.9706
  -12.0000
```

(b)

After rearranging rows, we get

$$\begin{bmatrix} -1 & 1 & -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1/\sqrt{2} & 1 \\ 0 & 0 & 0 & 0 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 18 \\ 0 \\ 0 \\ 12 \end{bmatrix}$$

Interchanging columns 5 and 6 yields reduces the coefficient matrix to triangular form:

$$\begin{bmatrix} -1 & 1 & -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_6 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 18 \\ 0 \\ 0 \\ 12 \end{bmatrix}$$

Back substitution:

$$\begin{aligned}
\frac{1}{\sqrt{2}}P_5 &= 12 & P_5 &= 16.971 \text{ kN} \blacktriangleleft \\
P_6 + \frac{1}{\sqrt{2}}(16.971) &= 0 & P_6 &= -12.000 \text{ kN} \blacktriangleleft \\
-P_4 - \frac{1}{\sqrt{2}}(16.971) &= 0 & P_4 &= -12.000 \text{ kN} \blacktriangleleft \\
\frac{1}{\sqrt{2}}P_3 + (-12.000) &= 18 & P_3 &= 42.426 \text{ kN} \blacktriangleleft \\
-P_2 - \frac{1}{\sqrt{2}}(16.971) &= 0 & P_2 &= -12.000 \text{ kN} \blacktriangleleft \\
-P_1 + (-12.000) - \frac{1}{\sqrt{2}}(42.426) &= 0 & P_1 &= -42.000 \text{ kN} \blacktriangleleft
\end{aligned}$$

Problem 16

We could rearrange the rows and columns of the coefficient matrix so as to arrive at an upper triangular matrix, as was done in Problem 15. This would definitely facilitate hand computations, but is hardly worth the effort when a computer is used. Therefore, we solve the equations as they are, using Gauss elimination with pivoting. The following program prompts for θ:

```
% problem2_2_16
theta = input('theta in degrees = ');
s = sin(theta*pi/180); c = cos(theta*pi/180);
A = [c  1  0  0 0
     0  s  0  0 1
     0  0 2*s 0 0
     0 -c  c  1 0
     0  s  s  0 0];
 b = [0 0 1 0 0]'; P = gaussPiv(A,b)

theta in degrees = 53
P =
    1.0403
   -0.6261
    0.6261
   -0.7536
    0.5000
```

Problem 17

The equations are

$$\begin{bmatrix} 20 & 0 & -15 \\ 0 & R+15 & -1 \\ -15 & -1 & R+35 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 220 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is diagonally dominant, so that pivoting is unnecessary. Gauss elimination was chosen for the method of solution.

```
% problem2_2_17
r = [5 10 20];
b = [220 0 0]';
for R = r
    R
    A = [20  0  -15; 0  15+R  -1; -15  -1  35+R];
    i = gauss(A,b)'
end
```

In the output R is in ohms and $\mathbf{i}$ is in amperes:

```
>> R =
     5
i =
   15.3118    0.2875    5.7491
R =
    10
i =
   14.6710    0.1958    4.8947
R =
    20
i =
   13.8304    0.1078    3.7739
```

Problem18

Kirchoff's equations for the 4 loops are

$$\begin{aligned} 50(i_1 - i_2) + 30(i_1 - i_3) &= -120 \\ 50(i_2 - i_1) + 15i_2 + 25(i_2 - i_4) + 10(i_2 - i_3) &= 0 \\ 30(i_3 - i_1) + 10(i_3 - i_2) + 20(i_3 - i_4) + 5i_3 &= 0 \\ 20(i_4 - i_3) + 25(i_4 - i_2) + (10 + 30 + 15)\, i_4 &= 0 \end{aligned}$$

or

$$\begin{bmatrix} 80 & -50 & -30 & 0 \\ -50 & 100 & -10 & -25 \\ -30 & -10 & 65 & -20 \\ 0 & -25 & -20 & 100 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} -120 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix is diagonally dominant, Gauss elimination without pivoting can be used safely:

```
% problem2_2_18
A = [80 -50 -30   0
    -50 100 -10 -25
    -30 -10  65 -20
      0 -25 -20 100];
b = [-120 0 0 0]';
i = gauss(A,b)

>> i =
   -4.1824
   -2.6646
   -2.7121
   -1.2086
```

Problem 19

This program, which prompts for n, uses Gauss elimination with pivoting.

```
% problem2_2_19
N = [2 3 4];
for n = N
    A = zeros(n); b = zeros(n,1);
    for i = 1:n
        for j = 1:n
            A(i,j) = (i + j - 2)^2;
            A(j,i) = A(i,j);
            b(i) = b(i) + A(i,j);
        end
    end
    n
    x = gaussPiv(A,b)
end

>> n =
     2
```

```
x =
     1
     1
n =
     3
x =
     1
     1
     1
n =
     4
Warning: Divide by zero.
```

The determinant of the coefficient matrix becomes zero for $n \geq 4$. Although a solution of the equations is $x = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^T$ for all n, this solution is not unique if $n \geq 4$.

Problem 20

We apply the conservation equation

$$\Sigma\,(Qc)_{\text{in}} + \Sigma\,(Qc)_{\text{out}} = 0$$

to each vessel, where Q is the flow rate of water, and c is the concentration. The results are

$$\begin{array}{ll} 1 & -8c_1 + 4c_2 + 4(20) = 0 \\ 2 & 8c_1 - 10c_2 + 2c_3 = 0 \\ 3 & 6c_2 - 11c_3 + 5c_4 = 0 \\ 4 & 3c_3 - 7c_4 + 4c_5 = 0 \\ 5 & 2c_4 - 4c_5 + 2(15) = 0 \end{array}$$

Since these equations are tridiagonal, we solve them with `LUdec3` and `LUsol3`:

```
% problem2_2_20
d1 = [8 6 3 2]';
d2 = [-8 -10 -11 -7 -4]';
d3 = [4 2 5 4]';
rhs = [-80 0 0 0 -30]';
[d1,d2,d3] = LUdec3(d1,d2,d3);
c = LUsol3(d1,d2,d3,rhs)
```

The solution for the concentrations is (units are mg/m^3):

```
c =
```

```
   19.7222
   19.4444
   18.3333
   17.0000
   16.0000
```

Problem 21

The conservation equations for the four tanks are

$$\begin{array}{lr} 1 & -6c_1 + 4c_2 + 2(25) = 0 \\ 2 & -7c_2 + 3c_3 + 4c_4 = 0 \\ 3 & 4c_1 - 4c_3 = 0 \\ 4 & 2c_1 + c_3 - 4c_4 + 1(50) = 0 \end{array}$$

The solution is obtained with the MATLAB commands

```
>> A = [-6  4  0  0;
         0 -7  3  4;
         4  0 -4  0;
         2  0  1 -4];
>> b = [-50 0 0 -50]';
>> c = gaussPiv(A,b)
```

The concentrations are (in $\mathrm{mg/m^3}$)

```
c =

   30.5556
   33.3333
   30.5556
   35.4167
```

PROBLEM SET 2.3

Problem 1

The inverse of $\mathbf{B}$ is obtained by interchanging the first two *columns* of $\mathbf{A}^{-1}$:

$$\mathbf{B}^{-1} = \begin{bmatrix} 0 & 0.5 & 0.25 \\ 0.4 & 0.3 & 0.45 \\ 0.2 & -0.1 & -9.16 \end{bmatrix} \quad \blacktriangleleft$$

Problem 2

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 6 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$

Solve $\mathbf{AX} = \mathbf{I}$ by back substitution, one column of $\mathbf{X}$ at a time.
Solution of $\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$(column 1 of $\mathbf{X}$):

$$\begin{aligned} 2x_3 &= 0 & x_3 &= 0 \\ 6x_2 + 5(0) &= 0 & x_2 &= 0 \\ 2x_1 + 4(0) + 3(0) &= 1 & x_1 &= \frac{1}{2} \end{aligned}$$

Solution of $\mathbf{Ax} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$(column 2 of $\mathbf{X}$):

$$\begin{aligned} 2x_3 &= 0 & x_3 &= 0 \\ 6x_2 + 5(0) &= 1 & x_2 &= \frac{1}{6} \\ 2x_1 + 4\left(\frac{1}{6}\right) + 3(0) &= 0 & x_1 &= -\frac{1}{3} \end{aligned}$$

Solution of $\mathbf{Ax} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$(column 3 of $\mathbf{X}$):

$$\begin{aligned} 2x_3 &= 1 & x_3 &= \frac{1}{2} \\ 6x_2 + 5\left(\frac{1}{2}\right) &= 0 & x_2 &= -\frac{5}{12} \\ 2x_1 + 4\left(-\frac{5}{12}\right) + 3\left(\frac{1}{2}\right) &= 0 & x_1 &= \frac{1}{12} \end{aligned}$$

$$\mathbf{A}^{-1} = \mathbf{X} = \begin{bmatrix} 1/2 & -1/3 & 1/12 \\ 0 & 1/6 & -5/12 \\ 0 & 0 & 1/2 \end{bmatrix} \blacktriangleleft$$

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Solve $\mathbf{BX} = \mathbf{I}$ by forward substitution, one column of $\mathbf{X}$ at a time.
Solution of $\mathbf{Bx} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$(column 1 of $\mathbf{X}$):

$$\begin{aligned} 2x_1 &= 1 \qquad x_1 = \frac{1}{2} \\ 3\left(\frac{1}{2}\right) + 4x_2 &= 0 \qquad x_2 = -\frac{3}{8} \\ 4\left(\frac{1}{2}\right) + 5\left(-\frac{3}{8}\right) + 6x_3 &= 0 \qquad x_3 = -\frac{1}{48} \end{aligned}$$

Solution of $\mathbf{Bx} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$(column 2 of $\mathbf{X}$):

$$\begin{aligned} 2x_1 &= 0 \qquad x_1 = 0 \\ 3(0) + 4x_2 &= 1 \qquad x_2 = \frac{1}{4} \\ 4(0) + 5\left(\frac{1}{4}\right) + 6x_3 &= 0 \qquad x_3 = -\frac{5}{24} \end{aligned}$$

Solution of $\mathbf{Bx} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$(column 3 of $\mathbf{X}$):

$$\begin{aligned} 2x_1 &= 0 \qquad x_1 = 0 \\ 3(0) + 3x_2 &= 0 \qquad x_2 = 0 \\ 4(0) + 5(0) + 6x_3 &= 1 \qquad x_3 = \frac{1}{6} \end{aligned}$$

$$\mathbf{B}^{-1} = \mathbf{X} = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/8 & 1/4 & 0 \\ -1/48 & -5/24 & 1/6 \end{bmatrix} \blacktriangleleft$$

Problem 3

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 0 & 1 & 1/3 & 1/9 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve $\mathbf{AX} = \mathbf{I}$ by back substitution, one column of $\mathbf{X}$ at a time.

Solution of $\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$(column 1 of $\mathbf{X}$):

$$\begin{aligned} x_4 &= 0 \\ x_3 + \frac{1}{4}(0) &= 0 \qquad x_3 = 0 \\ x_2 + \frac{1}{3}(0) + \frac{1}{9}(0) &= 0 \qquad x_2 = 0 \\ x_1 + \frac{1}{2}(0) + \frac{1}{4}(0) + \frac{1}{8}(0) &= 1 \qquad x_1 = 1 \end{aligned}$$

Solution of $\mathbf{Ax} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$(column 2 of $\mathbf{X}$):

$$\begin{aligned} x_4 &= 0 \\ x_3 + \frac{1}{4}(0) &= 0 \qquad x_3 = 0 \\ x_2 + \frac{1}{3}(0) + \frac{1}{9}(0) &= 1 \qquad x_2 = 1 \\ x_1 + \frac{1}{2}(1) + \frac{1}{8}(0) &= 0 \qquad x_1 = -\frac{1}{2} \end{aligned}$$

Solution of $\mathbf{Ax} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$(column 3 of $\mathbf{X}$):

$$\begin{aligned} x_4 &= 0 \\ x_3 + \frac{1}{4}(0) &= 1 \qquad x_3 = 1 \\ x_2 + \frac{1}{3}(1) + \frac{1}{9}(0) &= 0 \qquad x_2 = -\frac{1}{3} \\ x_1 + \frac{1}{2}\left(-\frac{1}{3}\right) + \frac{1}{4}(1) + \frac{1}{8}(0) &= 0 \qquad x_1 = -\frac{1}{12} \end{aligned}$$

Solution of $\mathbf{Ax} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$(column 4 of $\mathbf{X}$):

$$\begin{aligned} x_4 &= 1 \\ x_3 + \frac{1}{4}(1) &= 1 \qquad x_3 = -\frac{1}{4} \\ x_2 + \frac{1}{3}\left(-\frac{1}{4}\right) + \frac{1}{9}(1) &= 0 \qquad x_2 = -\frac{1}{36} \\ x_1 + \frac{1}{2}\left(-\frac{1}{36}\right) + \frac{1}{4}\left(-\frac{1}{4}\right) + \frac{1}{8}(1) &= 0 \qquad x_1 = -\frac{7}{144} \end{aligned}$$

$$\mathbf{A}^{-1} = \mathbf{X} = \begin{bmatrix} 1 & -1/2 & -1/12 & -7/144 \\ 0 & 1 & -1/3 & -1/36 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \blacktriangleleft$$

Problem 4

We solve $\mathbf{AX} = \mathbf{I}$ by Gauss elimination (LU decomposition could also be used, but it takes more space in hand computation).
The augmented coefficient matrix is

$$(\mathbf{A}|\mathbf{I}) = \begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 2 & 12 & -1 & 0 & 1 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - 2 \times \text{row } 2$$

$$[\mathbf{U}|\mathbf{Y}] = \begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ux} = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$(column 1 of $\mathbf{X}$):

$$\begin{aligned} 2x_3 &= 1 \qquad x_3 = 0.5 \\ x_2 + 5(0.5) &= -1 \qquad x_2 = -3.5 \\ x_1 + 2(-3.5) + 4(0.5) &= 1 \qquad x_1 = 6 \end{aligned}$$

Solution of $\mathbf{Ux} = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}^T$(column 2 of $\mathbf{X}$):

$$\begin{aligned} 2x_3 &= -2 \qquad x_3 = -1 \\ x_2 + 5(-1) &= 1 \qquad x_2 = 6 \\ x_1 + 2(6) + 4(-1) &= 0 \qquad x_1 = -8 \end{aligned}$$

Solution of $\mathbf{Ux} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$(column 1 of $\mathbf{X}$):

$$\begin{aligned} 2x_3 &= 1 \qquad x_3 = 0.5 \\ x_2 + 5(0.5) &= 0 \qquad x_2 = -2.5 \\ x_1 + 2(-2.5) + 4(0.5) &= 0 \qquad x_1 = 3 \end{aligned}$$

$$\mathbf{A}^{-1} = \mathbf{X} = \begin{bmatrix} 6.0 & -8.0 & 3.0 \\ -3.5 & 6.0 & -2.5 \\ 0.5 & -1.0 & 0.5 \end{bmatrix} \quad \blacktriangleleft$$

Use Gauss elimination. The augmented coefficient matrix is

$$[\mathbf{B}|\mathbf{I}] = \begin{bmatrix} 4 & -1 & 0 & 1 & 0 & 0 \\ -1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{row } 2 \leftarrow \text{row } 2 + 0.25 \times \text{row } 1$$

$$\begin{bmatrix} 4 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3.75 & -1 & 0.25 & 1 & 0 \\ 0 & -1 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 + \frac{1}{3.75} \times \text{row } 2$$

$$[\mathbf{U}|\mathbf{Y}] = \begin{bmatrix} 4 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3.75 & -1 & 0.25 & 1 & 0 \\ 0 & 0 & 3.7333 & 0.06667 & 0.2667 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ux} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$(column 1 of $\mathbf{X}$):

$$\begin{aligned} 3.7333x_3 &= 1 \qquad x_3 = 0.267\,9 \\ 3.75x_2 - 0.2679 &= 0 \qquad x_2 = 0.071\,43 \\ 4x_1 - 0.0714 &= 0 \qquad x_1 = 0.017\,86 \end{aligned}$$

Solution of $\mathbf{Ux} = \begin{bmatrix} 0 & 1 & 0.2667 \end{bmatrix}^T$ (column 2 of $\mathbf{X}$):

$$\begin{aligned} 3.7333x_3 &= 0.2667 \qquad x_3 = 0.071\,43 \\ 3.75x_2 - 0.07134 &= 1 \qquad x_2 = 0.2857 \\ 4x_1 - 0.2857 &= 0 \qquad x_1 = 0.071\,43 \end{aligned}$$

Solution of $\mathbf{Ux} = \begin{bmatrix} 1 & 0.25 & 0.06667 \end{bmatrix}^T$ (gives column 1 of $\mathbf{X}$):

$$\begin{aligned} 3.7333x_3 &= 0.06667 \qquad x_3 = 0.017\,86 \\ 3.75x_2 - 0.01786 &= 0.25 \qquad x_2 = 0.07143 \\ 4x_1 - 0.07143 &= 1 \qquad x_1 = 0.2679 \end{aligned}$$

$$\mathbf{B}^{-1} = \mathbf{X} = \begin{bmatrix} 0.2679 & 0.071\,4 & 0.017\,9 \\ 0.0714 & 0.2857 & 0.071\,4 \\ 0.017\,9 & 0.071\,4 & 0.267\,9 \end{bmatrix} \quad \blacktriangleleft$$

Problem 5

Solve $\mathbf{AX} = \mathbf{I}$ by Gauss elimination. The augmented coefficient matrix is

$$[\mathbf{A}|\mathbf{I}] = \begin{bmatrix} 4 & -2 & 1 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 + 0.5 \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - 0.25 \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 4 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -0.5 & 0.5 & 1 & 0 \\ 0 & -1.5 & 3.75 & -0.25 & 0 & 1 \end{bmatrix}$$

This completes the elimination stage (by switching rows 2 and 3, the coefficient matrix would have upper triangular form).
Solving $\mathbf{Ux} = \begin{bmatrix} 1 & 0.5 & -0.25 \end{bmatrix}^T$ (column 1 of $\mathbf{X}$):

$$\begin{aligned} -0.5x_3 &= 0.5 \qquad x_3 = -1 \\ -1.5x_2 + 3.75(-1) &= -0.25 \qquad x_2 = -2.3333 \\ 4x_1 - 2(-2.3333) + (-1) &= 1 \qquad x_1 = -0.6667 \end{aligned}$$

Solving $\mathbf{Ux} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ (column 2 of $\mathbf{X}$):

$$\begin{aligned} -0.5x_3 &= 1 \qquad x_3 = -2 \\ -1.5x_2 + 3.75(-2) &= 0 \qquad x_2 = -5 \\ 4x_1 - 2(-5) + (-2) &= 0 \qquad x_1 = -2 \end{aligned}$$

Solving $\mathbf{Ux} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ (column 3 of $\mathbf{X}$):

$$\begin{aligned} -0.5x_3 &= 0 \qquad x_3 = 0 \\ -1.5x_2 + 3.75(0) &= 1 \qquad x_2 = -0.6667 \\ 4x_1 - 2(-0.6667) + 0 &= 0 \qquad x_1 = -0.3333 \end{aligned}$$

$$\mathbf{A}^{-1} = \mathbf{X} = \begin{bmatrix} -0.6667 & -2 & -0.3333 \\ -2.3333 & -5 & -0.6667 \\ -1 & -2 & 0 \end{bmatrix} \blacktriangleleft$$

Problem 6

The function `matInv` in Example 2.13 is used to compute $\mathbf{A}$. It uses LU cecomposition with pivoting.

```
% problem2_3_6a
A = [5 -3 -1  0
```

```
     -2  1  1  1
      3 -5  1  2
      0  8 -4 -3];
Ainv = matInv(A)

>> Ainv =
    1.1250    1.0000   -0.8750   -0.2500
    0.8125    1.0000   -0.6875   -0.1250
    2.1875    2.0000   -2.3125   -0.8750
   -0.7500         0    1.2500    0.5000
```

We wrote the function `matInv3` listed below to invert a tridiagonal matrix, such as **B**. It is similar to `matInv` in Example 2.13.

```
function Ainv = matInv3(c,d,e)
% Inverts a tridiagonal matrix [A] stored in
% the form [A] = [c\d\e] by LU decomposition.
% USAGE: Ainv = matInv3(c,d,e)

n = length(d);
Ainv = eye(n);
[c,d,e] = LUdec3(c,d,e);
for i = 1:n
    Ainv(:,i) = LUsol3(c,d,e,Ainv(:,i));
end
```

Here is the program that inverts **B** using `matInv3`:

```
% problem2_3_6b
format short e
d = ones(4,1)*4; c = -ones(3,1); e = c;
Binv = matInv3(c,d,e)

>> Binv =
  2.6794e-001  7.1770e-002  1.9139e-002  4.7847e-003
  7.1770e-002  2.8708e-001  7.6555e-002  1.9139e-002
  1.9139e-002  7.6555e-002  2.8708e-001  7.1770e-002
  4.7847e-003  1.9139e-002  7.1770e-002  2.6794e-001
```

Problem 7

We used the function `matInv` in Example 2.13 to invert **A**. The program also computes $\mathbf{A}\mathbf{A}^{-1}$ (the esult should be **I**).

```
% problem2_3_7
A = [1  3 -9  6 4
     2 -1  6  7 1
     3  2 -3 15 5
     8 -1  1  4 2
    11  1 -2 18 7];
Ainv = matInv(A)
Check = A*Ainv

>> Ainv =
  1.0e+016 *
   -0.2226   -0.2226   -0.2226   -0.4453    0.4453
   -1.0576   -1.0576   -1.0576   -2.1152    2.1152
   -0.1771   -0.1771   -0.1771   -0.3542    0.3542
         0         0    0.0000    0.0000   -0.0000
    0.4504    0.4504    0.4504    0.9007   -0.9007
Check =
    -8    -4     2     8    -8
     0     1     1    -1     1
     0     4     0     8    -8
     0     0     0     0     0
    -8     0     0     8    -8
```

The very large elements of $\mathbf{A}^{-1}$ are harbingers of ill-conditioning. The bad news is confirmed by checking $\mathbf{AA}^{-1}$, which has no resemblance to the identity matrix. Hence the computed value of $\mathbf{A}^{-1}$ is totally unreliable.

Problem 8

As $\mathbf{K}$ exhibits diagonal dominance, pivoting is not necessary in the inversion process. But since we have the inversion function `matInv` (see Example 2.13) that employes pivoting, we might as well use it:

```
% problem2_3_8
format short e
K = [27.58   7.004 -7.004 0.0    0.0
      7.004 29.57  -5.253 0.0  -24.32
     -7.004 -5.253 29.57  0.0    0.0
      0.0    0.0    0.0  27.58  -7.004
      0.0  -24.32   0.0  -7.004 29.57];
P = [0 0 0 0 -45]';
Flexibility_matrix = matInv(K)
Displacements = Flexibility_matrix*P
```

```
>> Flexibility_matrix =
  4.6707e-002 -3.6579e-002  4.5650e-003 -8.1289e-003 -3.2010e-002
 -3.6579e-002  1.6462e-001  2.0580e-002  3.6583e-002  1.4406e-001
  4.5650e-003  2.0580e-002  3.8555e-002  4.5734e-003  1.8009e-002
 -8.1289e-003  3.6583e-002  4.5734e-003  4.6709e-002  4.1151e-002
 -3.2010e-002  1.4406e-001  1.8009e-002  4.1151e-002  1.6204e-001
Displacements =
  1.4404e+000
 -6.4825e+000
 -8.1041e-001
 -1.8518e+000
 -7.2920e+000
```

Problem 9

We use the function `matInv` for both matrices.

```
% problem2_3_9a
A = [3 -7  45  21
    12 11  10  17
     6 25 -80 -24
    17 55  -9   7];
Ainv = matInv(A)
Check = A*Ainv

>> Ainv =
   -2.7305    1.4853   -1.3466   -0.0325
    0.4824   -0.2787    0.2332    0.0293
   -0.6540    0.3243   -0.3406    0.0067
    2.0000   -1.0000    1.0000         0
Check =
    1.0000         0         0    0.0000
         0    1.0000         0         0
   -0.0000         0    1.0000   -0.0000
    0.0000   -0.0000    0.0000    1.0000
```

```
% problem2_3_9b
B = [1 1 1 1; 1 2 2 2; 2 3 4 4; 4 5 6 7];
Binv = matInv(B)
Check = B*Binv
```

```
>> Binv =
     2    -1     0     0
     0     2    -1     0
     1    -1     2    -1
    -2     0    -1     1
Check =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Problem 10.

```
function Linv = invertL(L)
% Inverts a lower triangualr matrix [L].
% USAGE: Linv = invertL(L)
n = size(L,1);
Linv = eye(n);
for i =1:n
    Linv(:,i) = solve(L,Linv(:,i));
end

function b = solve(L,b)
% Solves [L]{x} = {b} where [L] is lower triangular
n = size(L,1);
b(1)=b(1)/L(1,1);
for i = 2:n
    b(i) = (b(i) - dot(L(i,1:i-1),b(1:i-1)))/L(i,i);
end
```

Here is the test program:

```
% problem2_3_10
format short e
L = [30  0  0  0
     18 36  0  0
      9 12 36  0
      5  4  9 36];
Linv = invertL(L)
```

The program output is

```
>> Linv =
  3.3333e-002             0             0             0
 -1.6667e-002  2.7778e-002             0             0
 -2.7778e-003 -9.2593e-003  2.7778e-002             0
 -2.0833e-003 -7.7160e-004 -6.9444e-003  2.7778e-002
```

We now check the result using the command window:

```
>> Check = L*Linv
Check =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Problem 11

We first rearrange the equations so that the diagonal terms dominate:

$$\begin{bmatrix} 7 & 1 & 1 \\ -3 & 7 & -1 \\ -2 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -26 \\ 1 \end{bmatrix}$$

The iterative equations are

$$\begin{aligned} x_1 &= \frac{6 - x_2 - x_3}{7} \\ x_2 &= \frac{-26 + 3x_1 + x_3}{7} \\ x_3 &= \frac{1 + 2x_1 - 5x_2}{9} \end{aligned}$$

Starting with $x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, successive iterations yield

$$\begin{aligned} x_1 &= \frac{6 - 1 - 1}{7} = 0.571 \\ x_2 &= \frac{-26 + 3(0.571) + 1}{7} = -3.327 \\ x_3 &= \frac{1 + 2(0.571) - 5(-3.327)}{9} = 2.086 \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{6 - (-3.327) - 2.086}{7} = 1.034 \\ x_2 &= \frac{-26 + 3(1.034) + 2.086}{7} = -2.973 \\ x_3 &= \frac{1 + 2(1.034) - 5(-2.973)}{9} = 1.993 \end{aligned}$$

$$
\begin{aligned}
x_1 &= \frac{6-(-2.973)-1.993}{7} = 0.997 \\
x_2 &= \frac{-26+3(0.997)+1.993}{7} = -3.002 \\
x_3 &= \frac{1+2(0.997)-5(-3.002)}{9} = 2.000
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \frac{6-(-3.002)-2.000}{7} = 1.000 \ \blacktriangleleft \\
x_2 &= \frac{-26+3(1.000)+2.000}{7} = -3.000 \ \blacktriangleleft \\
x_3 &= \frac{1+2(1.000)-5(-3.000)}{9} = 2.000 \ \blacktriangleleft
\end{aligned}
$$

Problem 12

The equations are already in optimal order. The formulas for the iterations are

$$
\begin{aligned}
x_1 &= \frac{2x_2-3x_3-x_4}{12} \\
x_2 &= \frac{2x_1-6x_3+3x_4}{15} \\
x_3 &= \frac{20-x_1-6x_2+4x_4}{20} \\
x_4 &= \frac{3x_2-2x_3}{9}
\end{aligned}
$$

Starting with $x_1 = x_2 = x_3 = x_4$, we get

$$
\begin{aligned}
x_1 &= \frac{2(1)-3(1)-1}{12} = -0.167 \\
x_2 &= \frac{2(-0.167)-6(1)+3(1)}{15} = -0.222 \\
x_3 &= \frac{20-(-0.167)-6(-0.222)+4(1)}{20} = 1.275 \\
x_4 &= \frac{3(-0.222)-2(1.275)}{9} = -0.357
\end{aligned}
$$

$$\begin{aligned}
x_1 &= \frac{2(-0.222) - 3(1.275) - (-0.357)}{12} = -0.326 \\
x_2 &= \frac{2(-0.326) - 6(1.275) + 3(-0.357)}{15} = -0.625 \\
x_3 &= \frac{20 - (-0.326) - 6(-0.625) + 4(-0.357)}{20} = 1.132 \\
x_4 &= \frac{3(-0.625) - 2(1.132)}{9} = -0.460
\end{aligned}$$

Subsequent iterations yield

Iteration	x_1	x_2	x_3	x_3
3	-0.349	-0.591	1.103	-0.442
4	-0.337	-0.575	1.101	-0.436
5	-0.335	-0.572	1.101	-0.435
6	-0.334	-0.572	1.101	-0.435

Thus $\mathbf{x} = \begin{bmatrix} -0.334 & -0.572 & 1.101 & -0.435 \end{bmatrix}^T$ ◀

Problem 13

With $\omega = 1.1$, the iterative equations become

$$\begin{aligned}
x_1 &= 1.1\frac{15 + x_2}{4} - 0.1x_1 \\
x_2 &= 1.1\frac{10 + x_1 + x_3}{4} - 0.1x_2 \\
x_3 &= 1.1\frac{10 + x_2 + x_4}{4} - 0.1x_3 \\
x_4 &= 1.1\frac{10 + x_3}{3} - 0.1x_3
\end{aligned}$$

The starting values are

$$\begin{aligned}
x_1 &= \frac{15}{4} = 3.75 \qquad x_2 = \frac{10}{4} = 2.5 \\
x_3 &= \frac{10}{4} = 2.5 \qquad x_4 = \frac{10}{3} = 3.33
\end{aligned}$$

Iterations yield

$$\begin{aligned}
x_1 &= 1.1\frac{15 + 2.5}{4} - 0.1(3.75) = 4.44 \\
x_2 &= 1.1\frac{10 + 4.44 + 2.5}{4} - 0.1(2.5) = 4.41 \\
x_3 &= 1.1\frac{10 + 4.41 + 3.33}{4} - 0.1(2.5) = 4.63 \\
x_4 &= 1.1\frac{10 + 4.63}{3} - 0.1(3.33) = 5.03
\end{aligned}$$

$$
\begin{aligned}
x_1 &= 1.1\frac{15 + 4.41}{4} - 0.1(4.44) = 4.89 \\
x_2 &= 1.1\frac{10 + 4.89 + 4.63}{4} - 0.1(4.41) = 4.93 \\
x_3 &= 1.1\frac{10 + 4.93 + 5.03}{4} - 0.1(4.63) = 5.03 \\
x_4 &= 1.1\frac{10 + 5.03}{3} - 0.1(5.03) = 5.01
\end{aligned}
$$

The next two iterations yield

Iteration	x_1	x_2	x_3	x_3
3	4.99	5.01	5.00	5.00
4	5.00	5.00	5.00	5.00

Thus the solution is $x_1 = x_2 = x_3 = x_4 = 5$ ◀

Problem 14

Starting with $\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, the first iteration is

$$
\begin{aligned}
\mathbf{r}_0 &= \mathbf{b} - \mathbf{A}\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
\mathbf{s}_0 &= \mathbf{r}_0 \\
\mathbf{A}\mathbf{s}_0 &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\alpha_0 &= \frac{\mathbf{s}_0^T \mathbf{r}_0}{\mathbf{s}_0^T \mathbf{A}\mathbf{s}_0} = \frac{3}{1} = 3 \\
\mathbf{x}_1 &= \mathbf{x}_0 + \alpha_0 \mathbf{s}_0 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}
\end{aligned}
$$

Second iteration:

$$
\begin{aligned}
\mathbf{r}_1 &= \mathbf{b} - \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
\beta_0 &= -\frac{\mathbf{r}_1^T \mathbf{A}\mathbf{s}_0}{\mathbf{s}_0^T \mathbf{A}\mathbf{s}_0} = -\frac{-2}{1} = 2 \\
\mathbf{s}_1 &= \mathbf{r}_1 + \beta_0 \mathbf{s}_0 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \\
\mathbf{A}\mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} \\
\alpha_1 &= \frac{\mathbf{s}_1^T \mathbf{r}_1}{\mathbf{s}_1^T \mathbf{A}\mathbf{s}_1} = \frac{0+3+3}{0+9+0} = \frac{2}{3} \\
\mathbf{x}_2 &= \mathbf{x}_1 + \alpha_1 \mathbf{s}_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}
\end{aligned}
$$

Third and final iteration:

$$
\begin{aligned}
\mathbf{r}_2 &= \mathbf{b} - \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\
\beta_1 &= -\frac{\mathbf{r}_2^T \mathbf{A}\mathbf{s}_1}{\mathbf{s}_1^T \mathbf{A}\mathbf{s}_1} = -\frac{-3}{9} = \frac{1}{3} \\
\mathbf{s}_2 &= \mathbf{r}_2 + \beta_1 \mathbf{s}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\
\mathbf{A}\mathbf{s}_2 &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \\
\alpha_2 &= \frac{\mathbf{r}_2^T \mathbf{s}_2}{\mathbf{s}_2^T \mathbf{A}\mathbf{s}_2} = \frac{2}{4} = \frac{1}{2} \\
\mathbf{x} &= \mathbf{x}_2 + \alpha_2 \mathbf{s}_2 = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \quad \blacktriangleleft
\end{aligned}
$$

Problem 15

Starting with $\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, the first iteration is

$$
\begin{aligned}
\mathbf{r}_0 &= \mathbf{b} - \mathbf{A}\mathbf{x}_0 = \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix} \\
\mathbf{s}_0 &= \mathbf{r}_0 \\
\mathbf{A}\mathbf{s}_0 &= \begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 22 \\ 60 \\ -74 \end{bmatrix} \\
\alpha_0 &= \frac{\mathbf{s}_0^T \mathbf{r}_0}{\mathbf{s}_0^T \mathbf{A}\mathbf{s}_0} = \frac{4^2 + 10^2 + (-10)^2}{4(22) + 10(60) + (-10)(-74)} = 0.15126 \\
\mathbf{x}_1 &= \mathbf{x}_0 + \alpha_0 \mathbf{s}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0.15126 \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 0.60504 \\ 1.51261 \\ -1.51261 \end{bmatrix}
\end{aligned}
$$

Second iteration:

$$
\begin{aligned}
\mathbf{r}_1 &= \mathbf{b} - \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 0.60504 \\ 1.51261 \\ -1.51261 \end{bmatrix} \\
&= \begin{bmatrix} 0.67227 \\ 0.92434 \\ 1.19331 \end{bmatrix} \\
\beta_0 &= -\frac{\mathbf{r}_1^T \mathbf{A}\mathbf{s}_0}{\mathbf{s}_0^T \mathbf{A}\mathbf{s}_0} = -\frac{0.67227(22) + 0.92434(60) + 1.19331)(-74)}{4(22) + 10(60) + (-10)(-74)} \\
&= 0.012643 \\
\mathbf{s}_1 &= \mathbf{r}_1 + \beta_0 \mathbf{s}_0 = \begin{bmatrix} 0.67227 \\ 0.92434 \\ 1.19331 \end{bmatrix} + 0.012643 \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 0.72284 \\ 1.05077 \\ 1.06688 \end{bmatrix} \\
\mathbf{A}\mathbf{s}_1 &= \begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 0.72284 \\ 1.05077 \\ 1.06688 \end{bmatrix} = \begin{bmatrix} 1.10164 \\ 2.06932 \\ 2.51002 \end{bmatrix} \\
\alpha_1 &= \frac{\mathbf{s}_1^T \mathbf{r}_1}{\mathbf{s}_1^T \mathbf{A}\mathbf{s}_1} = \frac{0.72284(0.67227) + 1.05077(0.92434) + 1.06688(1.19331)}{0.72284(1.10164) + 1.05077(2.06932) + 1.06688(2.51002)} \\
&= 0.48337 \\
\mathbf{x}_2 &= \mathbf{x}_1 + \alpha_1 \mathbf{s}_1 = \begin{bmatrix} 0.60504 \\ 1.51261 \\ -1.51261 \end{bmatrix} + 0.48337 \begin{bmatrix} 0.72284 \\ 1.05077 \\ 1.06688 \end{bmatrix} = \begin{bmatrix} 0.95444 \\ 2.02052 \\ -0.99691 \end{bmatrix}
\end{aligned}
$$

Third and final iteration:

$$\begin{aligned}
\mathbf{r}_2 &= \mathbf{b} - \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 0.95444 \\ 2.02052 \\ -0.99691 \end{bmatrix} \\
&= \begin{bmatrix} 0.13977 \\ -0.07590 \\ -0.01997 \end{bmatrix} \\
\beta_1 &= -\frac{\mathbf{r}_2^T \mathbf{A}\mathbf{s}_1}{\mathbf{s}_1^T \mathbf{A}\mathbf{s}_1} \\
&= -\frac{0.13977(1.10164) + (-0.07590)(2.06932) + (-0.01997)(2.51002)}{0.72284(1.10164) + 1.05077(2.06932) + 1.06688(2.51002)} \\
&= 9.4201 \times 10^{-3} \\
\mathbf{s}_2 &= \mathbf{r}_2 + \beta_1 \mathbf{s}_1 = \begin{bmatrix} 0.13977 \\ -0.07590 \\ -0.01997 \end{bmatrix} + (9.4201 \times 10^{-3}) \begin{bmatrix} 0.72284 \\ 1.05077 \\ 1.06688 \end{bmatrix} \\
&= \begin{bmatrix} 0.14658 \\ -0.06600 \\ -0.00992 \end{bmatrix} \\
\mathbf{A}\mathbf{s}_2 &= \begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & -2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 0.14658 \\ -0.06600 \\ -0.00992 \end{bmatrix} = \begin{bmatrix} 0.44966 \\ -0.24416 \\ -0.06418 \end{bmatrix} \\
\alpha_2 &= \frac{\mathbf{r}_2^T \mathbf{s}_2}{\mathbf{s}_2^T \mathbf{A}\mathbf{s}_2} \\
&= \frac{0.13977(0.14658) + (-0.07590)(-0.06600) + (-0.01997)(-0.00992)}{0.14658(0.44966) + (-0.06600)(-0.24416) + (-0.00992)(-0.06418)} \\
&= 0.31084 \\
\mathbf{x} &= \mathbf{x}_2 + \boldsymbol{\alpha}_2 \mathbf{s}_2 = \begin{bmatrix} 0.95444 \\ 2.02052 \\ -0.99691 \end{bmatrix} + 0.31084 \begin{bmatrix} 0.14658 \\ -0.06600 \\ -0.00992 \end{bmatrix} \\
&= \begin{bmatrix} 1.0000 \\ 2.0000 \\ -1.0000 \end{bmatrix} \blacktriangleleft
\end{aligned}$$

Problem 16

The function gaussSeidel is essentially a book-keeping program. The actual iterations are carried out in a user-supplied function routine. If the coefficient matrix is sparse, is is feasible to specify the iteration formulas explicitly, as we did in the function `fex2_17` of Example 2.17; there is no need to input the coefficient matrix. In cases where the coefficient matrix is not sparse, it is more convenient (but computationally less efficient) to specify the coefficient matrix and a general formula for performing the iterations.

```
function x = p2_3_16a(x,omega)
% Iteration routine for Problem 16a, Problem Set 2.3
A = [3 -2  1  0  0  0
    -2  4 -2  1  0  0
     1 -2  4 -2  1  0
     0  1 -2  4 -2  1
     0  0  1 -2  4 -2
     0  0  0  1 -2  3];
b = [10 -8 10 10 -8 10]';
for i =1:length(b)   % General iteration formula
    x(i) = (-dot(A(i,:),x) + A(i,i)*x(i)...
        + b(i))*omega/A(i,i)...
        + (1 - omega)*x(i);
end
```

The command

```
>> [x,numIter,omega] = gaussSeidel(@p2_3_16a,zeros(6,1))
```

results in

```
x =
  2.0000e+000
  3.2940e-010
  4.0000e+000
  4.0000e+000
 -1.9795e-010
  2.0000e+000
numIter =
    34
omega =
  1.1988e+000
```

Note that $x_2 = x_5 = 0$.

```
function x = p2_3_16b(x,omega)
% Iteration formulas for Problem 16b, Problem Set 2.3
B = [3 -2  1  0  0  1
    -2  4 -2  1  0  0
     1 -2  4 -2  1  0
     0  1 -2  4 -2  1
     0  0  1 -2  4 -2
```

```
     1  0  0  1 -2  3];
b = [10 -8 10 10 -8 10]';
for i =1:length(b)
    x(i) = (-dot(B(i,:),x) + B(i,i)*x(i)...
        + b(i))*omega/B(i,i)...
        + (1 - omega)*x(i);
end

>> [x,numIter,omega] = gaussSeidel(@p2_3_16b,zeros(6,1))
x =
  1.3000e+000
 -3.0000e-001
  4.2000e+000
  4.2000e+000
 -3.0000e-001
  1.3000e+000
numIter =
    34
omega =
  1.3035e+000
```

Problem 17

```
function x = p2_3_17(x,omega)
% Iteration formulas Eq. for Problem 17, Problem Set 2.3.
n = length(x);
x(1) = omega*(x(2) - x(n))/4 + (1-omega)*x(1);
for i = 2:n-1
    x(i) = omega*(x(i-1) + x(i+1))/4 + (1-omega)*x(i);
end
x(n) = omega *(1 - x(1) + x(n-1))/4 + (1-omega)*x(n);

>> format short e
>> [x,numIter,omega] = gaussSeidel(@p2_3_17,zeros(20,1))
x =
 -7.7350e-002
 -2.0726e-002
 -5.5535e-003
 -1.4881e-003
 -3.9872e-004
 -1.0683e-004
 -2.8617e-005
```

```
 -7.6311e-006
 -1.9078e-006
 -1.2644e-011
  1.9078e-006
  7.6311e-006
  2.8616e-005
  1.0683e-004
  3.9872e-004
  1.4881e-003
  5.5535e-003
  2.0726e-002
  7.7350e-002
  2.8868e-001
numIter =
    17
omega =
  1.0977e+000
```

It took 259 iterations in Example 2.17. This illustrates the profound effect that diagonal dominance has on the rate of convergence in the Gauss-Seidel method.

Problem 18

```
function Av = p2_3_18(v)
% Computes the product A*v in Problem 18, Problem Set 2.3.
n = length(v);
Av = zeros(n,1);
Av(1) = 4*v(1) - v(2) + v(n);
Av(2:n-1) = -v(1:n-2) + 4*v(2:n-1) - v(3:n);
Av(n) = -v(n-1) + 4*v(n) + v(1);

% problem2_3_18
n = 20;
x = zeros(n,1);
b = zeros(n,1); b(n) = 100;
[x,numIter] = conjGrad(@p2_3_18,x,b)

>> x =
 -7.7350e+000
 -2.0726e+000
 -5.5535e-001
```

```
 -1.4881e-001
 -3.9872e-002
 -1.0683e-002
 -2.8616e-003
 -7.6311e-004
 -1.9078e-004
            0
  1.9078e-004
  7.6311e-004
  2.8616e-003
  1.0683e-002
  3.9872e-002
  1.4881e-001
  5.5535e-001
  2.0726e+000
  7.7350e+000
  2.8868e+001
numIter =
    10
```

Problem 19

```
function Av = p2_3_19(v)
% Computes the product A*v in Problem 19, Problem Set 2.3.
Av = zeros(9,1);
Av(1) =             - 4.0*v(1) + v(2) + v(4);
Av(2) =        v(1) - 4.0*v(2) + v(3) + v(5);
Av(3) =        v(2) - 4.0*v(3)        + v(6);
Av(4) = v(1)        - 4.0*v(4) + v(5) + v(7);
Av(5) = v(2) + v(4) - 4.0*v(5) + v(6) + v(8);
Av(6) = v(3) + v(5) - 4.0*v(6)        + v(9);
Av(7) = v(4)        - 4.0*v(7) + v(8);
Av(8) = v(5) + v(7) - 4.0*v(8) + v(9);
Av(9) = v(6) + v(8) - 4.0*v(9);
```

The equations are solved with the commands

```
>> b = -[0 0 100 0 0 100 200 200 300]';
>> x = zeros(9,1);
>> [x,numIter] = conjGrad(@p2_3_19,x,b)
```

The result is

```
x =
  21.4286
  38.3929
  57.1429
  47.3214
  75.0000
  90.1786
  92.8571
 124.1071
 128.5714

numIter =
     5
```

Problem 20

(a) The equations can be written as

$$\begin{aligned}
x_1 &= 0.6x_2 + 16 \\
x_2 &= 0.5x_1 + 0.5x_3 \\
x_3 &= 0.5x_2 + 0.5x_4 \\
x_4 &= 0.5x_3 + 0.5x_5 - 10 \\
x_5 &= 0.6x_4
\end{aligned}$$

```
% problem2_3_20
epsilon = 0.0001;  % Required precision
x = zeros(5,1);
for i = 1:100      %100 iteration should be enough
    xOld = x;
    x(1) = 0.6*x(2) + 16;
    x(2) = 0.5*(x(1) + x(3));
    x(3) = 0.5*(x(2) + x(4));
    x(4) = 0.5*(x(3) + x(5)) - 10;
    x(5) = 0.6*x(4);
    dx = sqrt(dot((x - xOld),(x - xOld)));
    if dx < epsilon
        numIter = i
        displacements = x
        return
    end
end
error('Too many iterations')
```

```
numIter =
    49

displacements =
   20.7144
    7.8573
   -4.9998
  -17.8570
  -10.7142
```

(b) The function that returns improved **x** is:

```
function x = p2_3_21(x,omega)
%Iteration formulas for Prob. 2.3.21
x(1) = omega*(0.6*x(2) + 16) + (1 - omega)*x(1);
x(2) = omega*(0.5*(x(1) + x(3))) + (1 - omega)*x(2);
x(3) = omega*(0.5*(x(2) + x(4))) + (1 - omega)*x(3);
x(4) = omega*(0.5*(x(3) + x(5)) - 10) + (1 - omega)*x(4);
x(5) = omega*(0.6*x(4)) + (1 - omega)*x(5);
```

The MATLAB command

```
>> [x,numIter,omega] = gaussSeidel(@p2_3_21,zeros(5,1),100,0.0001)
```

produces the following results:

```
x =
   20.7143
    7.8572
   -5.0000
  -17.8571
  -10.7143

numIter =
    25

omega =
    1.3824
```

We see that relaxation about halves the number of required iterarions.

Problem 21

The following function supplies the product **Av**:

```
function Av = p2_3_21(v)
%Iteration formulas for Prob. 2.3.21
Av = zeros(5,1);
Av(1) = -5*v(1) + 3*v(2);
Av(2) = 3*v(1) - 6*v(2) + 3*v(3);
Av(3) = 3*v(2) - 6*v(3) + 3*v(4);
Av(4) = 3*v(3) - 6*v(4) + 3*v(5);
Av(5) = 3*v(4) - 5*v(5);
```

The MATLAB commands that produce the solution are

```
>> b = zeros(5,1); b(1) = -80; b(4) = 60;
>> [x,numIter] = conjGrad(@p2_3_21,zeros(5,1),b,0.0001)

x =
   20.7143
    7.8571
   -5.0000
  -17.8571
  -10.7143

numIter =
     5
```