PROBLEM SET 10.1

Problem 1

$$V = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

Letting $x = \sigma/r$, the function to be minimized is $f(x) = x^{12} - x^6$.

```
% problem10_1_1
func = inline('x^12 - x^6','x');
x = 0.0; h = 0.02;
[a,b] = goldBracket(func,x,h);
[xMin,fMin] = goldSearch(func,a,b)

>> xMin =
    0.8909
fMin =
   -0.2500
```

Analytical solution:

$$\begin{aligned}\frac{df}{dx} &= 12x^{11} - 6x^5 = 0 \qquad \left(2x^6 - 1\right)x = 0 \\ x &= 2^{-1/6} = 0.890\,899 \ \blacktriangleleft \ \text{Checks}\end{aligned}$$

Problem 2

We are to minimize the function $f(\sigma) = (27 - 18\sigma + 2\sigma^2)e^{-\sigma/3}$.

```
% problem10_1_2
func = inline('(27 - 18*x + 2*x^2)*exp(-x/3)','x');
x = 0.0; h = 0.01;
[a,b] = goldBracket(func,x,h);
[xMin,fMin] = goldSearch(func,a,b

>> xMin =
    3.5314
fMin =
   -3.5819
```

Analytical solution:

$$
\begin{aligned}
\frac{df}{d\sigma} &= 0 \\
-\frac{1}{3}(27-18\sigma+2\sigma^2)e^{-\sigma/3}+(-18+4\sigma)e^{-\sigma/3} &= 0 \\
-\frac{1}{3}\left(81-30\sigma+2\sigma^2\right)e^{-\sigma/3} &= 0
\end{aligned}
$$

$$
\sigma = \frac{30 \pm \sqrt{30^2-4(81)(2)}}{4} = \frac{30 \pm 15.87451}{4} = \begin{cases} 11.468\,63 \\ 3.531\,373 \quad \blacktriangleleft \text{ Checks} \end{cases}
$$

Problem 3

$$
f(p) = \int_0^{\pi} \sin x \cos px \, dx
$$

The program below uses the MATLAB function `quad` for numerical integration. The functions presented in the text, such as `romberg`, would not work here without changes. The reason is that the function `func` to be integrated contains the parameter `p`; that is, it is a function of `x` and `p`. As it is written, there is no way of passing the value of `p` to `romberg`. However, `quad` accepts any number of parameters. It achieves this by utilizing the `varargin` function of MATLAB that allows us to input any number of parameters to a function. So far, we managed to avoid `varargin` for the sake of keeping the programs simple, but in this problem we pay the price.

```
function p10_1_3
% Solution of Problem 3, Problem Set 10.1.

p = 0.0; h = 0.02;
[a,b] = goldBracket(@quadrature,p,h);
[p,integral] = goldSearch(@quadrature,a,b)

function I = quadrature(p)
func = inline('sin(x).*cos(p*x)','x','p');
I = quad(func,0,pi,1.0e-6,0,p);

>> p =
    1.6522
integral =
   -0.8441
```

Problem 4

$$\begin{aligned} R_1 i_1 + R_3 i_1 + R(i_1 - i_2) &= E \\ R_2 i_2 + R_4 i_2 + R_5 i_2 + R(i_2 - i_1) &= 0 \end{aligned}$$

In matrix notation the equations are

$$\begin{bmatrix} R_1 + R_3 + R & -R \\ -R & R_2 + R_4 + R_5 + R \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3.5 + R & -R \\ -R & 6.6 + R \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 120 \\ 0 \end{bmatrix}$$

The power dissipated by the resistor is $P(R) = R(i_1 - i_2)^2$. Since we want to maximize the power, we must minimize $-P(R)$.

```
function p10_1_4
% Solution of Problem 4, Problem Set 10.1.

[Rmin,Rmax] = goldBracket(@pwr,1,0.01);
[R,Pmin] = goldSearch(@pwr,Rmin,Rmax);
R
power = -Pmin

function P = pwr(R)
A = [3.5+R, -R; -R, 6.6+R];
b = [120, 0]';
i = A\b;
P = -R*(i(1) - i(2))^2;

>> R =
    2.2871
power =
  672.1358
```

Problem 5

$$T = \frac{q}{2\pi}\left(\frac{\ln(r/a)}{k} + \frac{1}{hr}\right) + T_\infty$$

```
function p10_1_5
% Solution of Problem 5, Problem Set 10.1.
```

```
format short e
[rMin,rMax] = goldBracket(@temp,0.006, 0.0001);
[r,Tmin] = goldSearch(@temp,rMin,rMax)

function T = temp(r)
a = 0.005; h = 20; k = 0.16; q = 50; Tinf = 280;
T = q/(2*pi)*(log(r/a)/k + 1/(h*r)) + Tinf;

>> r =
  8.0000e-003
Tmin =
  3.5311e+002
```

Problem 6

$$F(x, y) = (x-1)^2 + (y-1)^2 \qquad x + y \leq 1 \qquad x \geq 0.6$$

```
function p10_1_6
% Solution of Problem 6, Problem Set 10.1.
x = [0.5981; 0.4078];
[x,Fmin,nCyc] = powell(@func,x)

    function y = func(x);
    mu = 10000;
    c1 = max(0, x(1) + x(2) - 1);
    c2 = min(0,x(1) - 0.6);
    F = (x(1) - 1)^2 + (x(2) - 1)^2;
    y = F + mu*(c1^2 + c2^2);
    end
end

X =
  5.9998e-001
  4.0008e-001

Fmin =
  5.1996e-001

nCyc =
     3
```

Note that the constraints $x > 0.6$ and

$$x + y = 0.59998 + 0.40008 = 1.000\,06 < 1$$

are satisfied within close tolerances.

Problem 7

$$F(x, y) = 6x^2 + 3y^3 + xy \qquad y \geq 0$$

```
function p10_1_7
% Solution of Problem 7, Problem Set 10.1.

format short e
[x,Fmin,nCyc] = powell(@func,[-1;1])

    function y = func(x);
    mu = 100;
    c1 = min(0, x(2));
    F = 6*x(1)^2 + x(2)^3 + x(1)*x(2);
    y = F + mu*c1^2;
    end
end
```

```
x =
 -2.3148e-003
  2.7778e-002
Fmin =
 -1.0717e-005
nCyc =
     5
```

Note that the constraint $y \geq 0$ was not active at the optimal point.
Analytical solution:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 0 \qquad 12x + y = 0 \qquad y = -12x \\
\frac{\partial F}{\partial y} &= 0 \qquad 3y^2 + x = 0 \qquad 3(-12x)^2 + x = 0
\end{aligned}$$

$$432x^2 + x = 0 \qquad x = \begin{cases} 0 \\ -0.002\,314\,8 \end{cases} \qquad y = \begin{cases} 0 \\ 0.027\,778 \end{cases}$$

The solution $x = -0.002\,314\,8$, $y = 0.027\,778$ is the optimal one since it results in smaller F than $x = y = 0$.

Problem 8

We use the program in Problem 7 with changes to the penalty and the starting point (if we do not change the starting point, there is a good chance that we would finish up at the stationary point found in Problem 7).

```
function p10_1_8
% Solution of Problem 8, Problem Set 10.1.

format short e
X = [0.17207; -2.0648];
[x,Fmin,nCyc] = powell(@func,[0;2])

function y = func(x);
mu = 100;
c1 = min(0, x(2) + 2);
F = 6*x(1)^2 + x(2)^3 + x(1)*x(2);
y = F + mu*c1^2;
```

The result is

```
>> x =
  1.7207e-001
 -2.0648e+000
Fmin =
 -8.5608e+000
nCyc =
     3
```

Since the constraint $y \geq -2$ is violated significantly, we run the program again with the following changes:

```
mu = 1000000;
X = [0.17207; -2.0648];
```

obtaining

```
>> x =
  1.6667e-001
 -2.0000e+000
Fmin =
 -8.1667e+000
nCyc =
     3
```

Analytical solution:
As the constraint $y \geq -2$ is active at the optimal point, we set $y = -2$ in the merit function:

$$F = 6x^2 + (-2)^3 + x(-2) = 6x^2 - 8 - 2x$$

$$\frac{dF}{dx} = 0 \qquad 12x - 2 = 0 \qquad x = \frac{1}{6} = 0.166\,667$$

Problem 9

```
function p10_1_9
% Solution of Problem 9, Problem Set 10.1.

[x,Fmin,nCyc] = powell(@merit_func,[2;2]);
Intersection_point = x
Min_distance = sqrt(dist_squared(x))
Constraint = constraint(x)
Number_of_cycles = nCyc

    function d = dist_squared(X)
    d = (X(1) - 1)^2 + (X(2) - 2)^2;
    end

    function c = constraint(X)
    c = X(2) - X(1)^2;
    end

    function y = merit_func(X);
    mu = 100;
    y = dist_squared(X) + mu*constraint(X)^2;
    end
end

>> Intersection_point =
    1.3656
    1.8661
Min_distance =
    0.3893
Constraint =
    0.0013
Number_of_cycles =
     5
```

To get closer to the constant $y - x^2 = 0$, we run the program again with the data

```
mu = 1000000;
X = [1.3656, 1.8661];
```

This yields

```
>> Intersection_point =
    1.3660
    1.8660
Min_distance =
    0.3898
Constraint =
  1.3397e-007
Number_of_cycles =
     4
```

Problem 10

The location of the centroid is

$$\begin{aligned} d &= \frac{\sum A_i d_i}{\sum A_i} = \frac{(0.4)^2 (0.2) - (0.2x)(0.4 - x/2)}{(0.4)^2 - 0.2x} \\ &= \frac{0.0\,32 - 0.\,08x + 0.1x^2}{0.16 - 0.2x} \end{aligned}$$

```
function p10_1_10
% Solution of Problem 10, Problem Set 10.1.

[a,b] = goldBracket(@centroid,0.2,0.01);
[x,d_min] = goldSearch(@centroid,a,b)

function d = centroid(x)
d = (0.032 - 0.08*x + 0.1*x^2)/(0.16 - 0.2*x);

>> x =
    0.2343
d_min =
    0.1657
```

Problem 11

The mass of water in the vessel is

$$M_w = \pi r^2 x\gamma = \pi(0.25)^2 x(1000) = 62.5\pi x \text{ kg}$$

The height of the combined centroid above the floor of the vessel is

$$\begin{aligned} d &= \frac{\sum m_i d_i}{\sum m_i} = \frac{M(0.43H) + M_w(x/2)}{M + M_w} \\ &= \frac{115(0.43)(0.8) + 62.5\pi x^2/2}{115 + 62.5\pi x} = \frac{39.56 + 31.25\pi x^2}{115 + 62.5\pi x} \text{ m} \end{aligned}$$

```
function p10_1_11
% Solution of Problem 11, Problem Set 10.1.

[a,b] = goldBracket(@centroid,0.2,0.01);
[x,d_min] = goldSearch(@centroid,a,b)

function d = centroid(x)
d = (39.56 + 31.25*pi*x^2)/(115 + 62.5*pi*x);

>> x =
    0.2780
d_min =
    0.2780
```

Note that $x = d$ when d is minimized.

Problem 12

In the program below we use the notation $a = x_1$ and $b = x_2$.

```
function p10_1_12
% Solution of Problem 12, Problem Set 10.1.

x = powell(@func,[1;1]);
a = x(1)
b = x(2)
Area = area(x)
Volume = volume(x)
```

```
    function V = volume(x)
    V = x(1)*x(1)*x(2);
    end

    function A = area(x)
    A = x(1)^2 + 4*x(1)*x(2);
    end

    function y = func(x);
    mu = 100;
    y = area(x) + mu*(volume(x) - 1)^2;
    end
end
```

```
>> a =
    1.2532
b =
    0.6266
area =
    4.7114
volume =
    0.9840
```

Running the program again with the changes

```
mu = 1000000;
X = [1.2532; 0.6266];
```

results in

```
>> a =
    1.2599
b =
    0.6300
Area =
    4.7622
Volume =
    1.0000
```

The true solution is $a = 2b = \sqrt[3]{2} = 1.259\,921$

Problem 13

The following program uses the notation $u = x_1$, $v = x_2$.

```
function p10_1_13
% Solution of Problem 13, Problem Set 10.1.

x = powell(@pot_energy,[0;0]);
u = x(1)
v = x(2)

    function V = pot_energy(x)
    a = 150; b = 50; k = 0.6; P = 5;
    delAB = sqrt((a + x(1))^2 + x(2)^2) - a;
    delBC = sqrt((b - x(1))^2 + x(2)^2) - b;
    V = -P*x(2) + 0.5*k*(a + b)*(delAB^2/a + delBC^2/b);
    end
end
   28.3751
```

Problem 14

We use the notation $A = x_1$, $\theta = x_2$; the units are meters and newtons. Because of the large difference between the magnitudes of stress and displacement, we multiply the displacement penalty by 10^{12}.

```
function p10_1_14
% Solution of Problem 14, Problem Set 10.1.

x = powell(@merit_func,[5e-4;0.8],0.00001);
A = x(1)
Theta_in_deg = x(2)*180/pi
Stress = stress(x)
Displt = displt(x)

    function V = volume(x)
    V = 4*x(1)/cos(x(2));
    end

    function s = stress(x)
    s = 50e3/(2*x(1)*sin(x(2)));
    end

    function d = displt(x)
    d = 50e3*4/(2*200e9*x(1)*sin(2*x(2))*sin(x(2)));
    end
```

```
    function F = merit_func(x)
    mu = 1;
    c1 = max(0,stress(x) - 150e6);
    c2 = max(0,displt(x) - 5e-3)*1.0e6;
    F = volume(x) + mu*(c1^2 + c2^2);
    end
end
```

```
>> A =
  2.3236e-004
Theta_in_deg =
  4.5830e+001
Stress =
  1.5000e+008
Displt =
  3.0013e-003
```

Note that only the stress constraint is active.

Problem 15

Running the program in Problem 14 with the allowable displacement limit changed to 0.0025, we get

```
>> A =
  2.7910e-004
Theta_in_deg =
  4.5797e+001
Stress =
  1.2495e+008
Displt =
  2.5000e-003
```

Here the optimal design is governed by the displacement constraint.

Problem 16

```
function p10_1_16
% Solution of Problem 16, Problem Set 10.1.
```

```
x = powell(@merit_func,[0.1;0.1],0.001);
r1 = x(1)
r2 = x(2)
Max_stress = max(stress(x))
Displt = displt(x)

    function V = volume(x)
    V = pi*(x(1)^2 + x(2)^2);
    end

    function [s1,s2] = stress(x)
    s1 = 8*(10e3)/(pi*x(1)^3);
    s2 = 4*(10e3)/(pi*x(2)^3);
    end

    function d = displt(x)
    d = 4*10e3/(3*pi*200e9)*(7/x(1)^4 + 1/x(2)^4);
    end

    function F = merit_func(x)
    mu = 1;
    [s1,s2] = stress(x);
    c1 = max(0,s1 - 180e6);
    c2 = max(0,s2 - 180e6);
    c3 = max(0,displt(x) - 25e-3)*1.0e6;
    F = volume(x) + mu*(c1^2 + c2^2 + c3^2);
    end
end
```

```
>> r1 =
  5.2107e-002

r2 =
  4.5737e-002

Max_stress =
  1.8000e+008

Displt =
  2.5000e-002
```

Both the displacement and the stress constraints are active at the optimal point.

Problem 17

We use the downhill simplex method:

```
function p10_1_17
% Solution of Problem 17, Problem Set 10.1.

[x,Fmin] = downhill(@func,[0 0 0])

    function y = func(x)
    y = 2*x(1)^2 + 3*x(2)^2 + x(3)^2 + ...
        x(1)*x(2) + x(1)*x(3) - 2*x(2);
    end
end
```

```
x =
 -1.0000e-001  3.5000e-001  5.0000e-002

Fmin =
 -3.5000e-001
```

Analytical solution:

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0 \qquad 4x + y - z = 0 \\ \frac{\partial F}{\partial y} &= 0 \qquad x + 6y = 2 \\ \frac{\partial F}{\partial z} &= 0 \qquad x + 2z = 0 \end{aligned}$$

The solution of these equations is $x = -0.1$, $y = 0.35$, $z = 0.05$.

Problem 18

$$V = \pi r^2 \left(\frac{b}{3} + h\right) \qquad S = \pi r \left(2h + \sqrt{b^2 + r^2}\right)$$

The most efficient way to obtain the optimal design is to solve the volume constraint $V = 1$ for h and substitute the result into the expression for S:

$$\pi r^2 \left(\frac{b}{3} + h\right) = 1 \qquad h = \frac{1}{\pi r^2} - \frac{b}{3}$$

$$S = \pi r \left(\frac{2}{\pi r^2} - \frac{2b}{3} + \sqrt{b^2 + r^2}\right)$$

We now have an unconstrained optimization problem) in the two variables.

```
function p10_1_18
% Solution of Problem 18, Problem Set 10.1.

[x,S] = powell(@surface_area,[0.5;0.5],0.001);
r = x(1)
b = x(2)
h = 1/pi/x(1)^2 -x(2)/3
S

    function S = surface_area(x)
    S = pi*x(1)*(2/pi/x(1)^2 - 2/3*x(2) + sqrt(x(1)^2 + x(2)^2));
    end
end

>> r =
    0.7531
b =
    0.6736
h =
    0.3368
S =
    3.9838
```

Problem 19

```
function p10_1_19
% Solution of Problem 19, Problem Set 10.1.

A = powell(@merit_func, [0.001;0.001;0.001],0.000005)
Volume = volume(A)
Stresses = stresses(A)

    function s = stresses(x)
    % Sets up and solves simultaneous eqs. for forces
    B = [ 1   0.8  0
          0   0.6  1
         3.2/x(1) -5/x(2)  1.8/x(3)];
    b = [200; 200; 0]*1.0e3;
    P = B\b;   % Forces in members
    s = P./x;  % Stresses in members
```

```
    end

    function V = volume(x)
    V = 4*x(1) + 5*x(2) + 3*x(3);
    end

    function F = merit_func(x)
    mu = 11;
    s = stresses(x);
    c1 = max(0,s(1) - 150e6);
    c2 = max(0,s(2) - 150e6);
    c3 = max(0,s(3) - 150e6);
    F = volume(x) + mu*(c1^2 + c2^2 + c3^2);
    end
end

>> A =
  5.2048e-004
  1.0161e-003
  7.2369e-004
Volume =
  9.3333e-003
Stresses =
  1.5000e+008
  1.5000e+008
  1.5000e+008
```

In this problem the minimum weight is the fully-stressed design (all members are stressed to their allowable limits), but this is not always the case.
The numerical solution obtained above is *not unique*. It is not hard to show that the number of solutions is infinite, all of them being fully stressed and having the same weight. Therefore, the design produced by the program is completely dependent on the starting point.

Problem 20

```
function p10_1_19
% Solution of Problem 19, Problem Set 10.1.

A = powell(@merit_func, [0.001;0.001;0.001],0.000005)
Volume = volume(A)
Stresses = stresses(A)
```

```
    function s = stresses(x)
    % Sets up and solves simultaneous eqs. for forces
    B = [ 1   0.8  0
          0   0.6  1
         3.2/x(1) -5/x(2)  1.8/x(3)];
    b = [200; 200; 0]*1.0e3;
    P = B\b;   % Forces in members
    s = P./x;  % Stresses in members
    end

    function V = volume(x)
    V = 4*x(1) + 5*x(2) + 3*x(3);
    end

    function F = merit_func(x)
    mu = 11;
    s = stresses(x);
    c1 = max(0,s(1) - 150e6);
    c2 = max(0,s(2) - 150e6);
    c3 = max(0,s(3) - 150e6);
    F = volume(x) + mu*(c1^2 + c2^2 + c3^2);
    end
end
```

```
>> Angles_in_deg =
  2.1362e+001  1.2492e+000 -2.8021e+001
Constraints =
 -2.7076e-005  1.4410e-005
Pot_energy =
 -2.2837e+004
```

Usually a large penalty multiplier λ results in many iterative cycles. This is not the case here thanks to good starting values of the angles.

Alternative solution

As pointed out in Art. 10.1, solutions to optimization problems with equality constraints may also be obtained by the method of Lagrange multipliers.
We have the merit function

$$F = (-6.0 \sin\theta_1 - 4.5 \sin\theta_2) \times 10^4$$

and the constraints

$$\begin{aligned} g_1 &= 1.2\cos\theta_1 + 1.5\cos\theta_2 + \cos\theta_3 - 3.5 = 0 && \text{(a)} \\ g_2 &= 1.2\sin\theta_1 + 1.5\sin\theta_2 + \sin\theta_3 = 0 && \text{(b)} \end{aligned}$$

Since F can be scaled without affecting θ_1 and θ_2, we drop the factor 10^4. The equations $\nabla F^*(\mathbf{x}) = 0$ are

$$\begin{aligned} -6.0\cos\theta_1 - 1.2\lambda_1\sin\theta_1 + 1.2\lambda_2\cos\theta_1 &= 0 \\ -4.5\cos\theta_2 - 1.5\lambda_1\sin\theta_2 + 1.5\lambda_2\cos\theta_2 &= 0 \\ -\lambda_1\sin\theta_3 + \lambda_2\cos\theta_3 &= 0 \end{aligned}$$

These equations together with Eqs. (a) and (b) are coded in the function below using the notation $\mathbf{x} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \lambda_1 & \lambda_2 \end{bmatrix}^T$.

```
function y = fex10_20a(x)
% Function used is solving Prob. 20, Problem Set 10.1
% with Lagrange multipliers.
s = sin(x); c = cos(x);
y = [-6.0*c(1) - 1.2*x(4)*s(1) + 1.2*x(5)*c(1)
     -4.5*c(2) - 1.5*x(4)*s(2) + 1.5*x(5)*c(2)
     -x(4)*s(3) + x(5)*c(3)
     1.2*c(1) + 1.5*c(2) + c(3) - 3.5
     1.2*s(1) + 1.5*s(2) + s(3)];
```

The solution can obtained with the Newton-Raphson method as follows:

```
>> newtonRaphson2(@fex10_20a,[0.5; 0.1; -0.5; 0; 0])
ans =
   0.37280973858715
   0.02179694841184
  -0.48903379067170
  -5.41566992591836
   2.88193622377579
```

The first three lines are the angles; the last two lines represent the multipliers. Converting the angles from radians to degrees, we get

```
>> ans(1:3)*180/pi
ans =
  21.36042458241947
   1.24887315026265
 -28.01957224477242
```

These angles are close to the values obtained with Powell's method.

Problem 21

With $A_3 = A_1$ the displacement equations become

$$\frac{E}{4L}\begin{bmatrix} 6A_1 & 2\sqrt{3}A_1 \\ 2\sqrt{3}A_1 & 2A_1 + 8A_2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} P \\ 2P \end{bmatrix} \tag{a}$$

Introducing the dimensionless variables

$$x_i = \frac{E\delta}{PL}A_i \qquad u' = \frac{u}{\delta} \qquad v' = \frac{v}{\delta} \tag{b}$$

we can write Eqs. (a) as

$$\begin{bmatrix} 6x_1 & 2\sqrt{3}x_1 \\ 2\sqrt{3}x_1 & 2x_1 + 8x_2 \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \tag{c}$$

The structural volume is

$$V = 2LA_1 + 0.5LA_2 = \frac{PL^2}{2E\delta}(4x_1 + x_2)$$

Our task is to minimize $4x_1 + x_2$ subject to the constraints $u' \leq 1$ and $v' \leq 1$.

```
function problem10_1_21
mu = 10000;
xStart = [1,1];
x = downhill(@merit_func,xStart)
displacements = displ(x)
volume = vol(x)

    function V = vol(x);
        V = 4*x(1) + x(2);
    end

    function F = merit_func(x)
        y = displ(x);
        c1 = max([0,y(1) - 1]);
        c2 = max([0,y(2) - 1]);
        F = vol(x) + mu*(c1^2 + c2^2);
    end

    function y = displ(x);
        a = zeros(2,2);
        a(1,1) = 6*x(1); a(1,2) = 2*sqrt(3)*x(1);
        a(2,1) = a(1,2); a(2,2) = 2*x(1) + 8*x(2);
        b = [4;8];
```

```
        y = gauss(a,b);
    end
end
```

The output of the program is

```
x =
    0.4226    0.7113

displacements =
    1.0001
    1.0001

volume =
    2.4018
```

Note that u' and v' are close enough to the constraints $u' \leq 1$ and $v' \leq 1$. Therefore, the optimal design is

$$A_1 = 0.4226\frac{PL}{E\delta} \qquad A_2 = 0.7113\frac{PL}{E\delta} \blacktriangleleft$$

with the structural volume

$$V = \frac{PL^2}{2E\delta}(2.402) = 1.201\frac{PL^2}{E\delta}$$

Alternative Solution If we had known beforehand that both displacement constraints are active at optimal design (a pretty safe bet), we could have substituted $u' = v' = 1$ into Eqs. (c) and solved for x_1 and x_2, thereby bypassing the optimization procedure.

Problem 22

The displacement equations are

$$\frac{E}{4L}\begin{bmatrix} 3A_1 + 3A_3 & \sqrt{3}A_1 + \sqrt{3}A_3 \\ \sqrt{3}A_1 + \sqrt{3}A_3 & A_1 + 8A_2 + A_3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} P \\ 2P \end{bmatrix}$$

Using the nondimensional variables in Eqs. (b) of Problem 21, the these equations become

$$\begin{bmatrix} 3x_1 + 3x_3 & \sqrt{3}x_1 + \sqrt{3}x_3 \\ \sqrt{3}x_1 + \sqrt{3}x_3 & x_1 + 8x_2 + x_3 \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

The structural volume is

$$V = LA_1 + \frac{L}{2}A_2 + LA_3 = \frac{PL^2}{2E\delta}(2x_1 + x_2 + 2x_3)$$

The task of optimization is to minimize $4x_1 + x_2$ subject to the constraints $u' \leq 1$ and $v' \leq 1$. We also need the additional constraints $x_i \geq 0$ to avoid negative cross-sectional areas.

```
function problem10_1_22
mu = 10000;
xStart = [1,1,1];
x = downhill(@merit_func,xStart)
displacements = displ(x)
volume = vol(x)

    function V = vol(x);
        V = 2*x(1) + x(2) + 2*x(3);
    end

    function F = merit_func(x)
        c = zeros(5,1);
        y = displ(x);
        c(1) = max([0,y(1) - 1]);
        c(2) = max([0,y(2) - 1]);
        for i = 3:5
            c(i) = min([0,x(i-2)]);
        end
        F = vol(x) + mu*(sum(c.^2));
    end

    function y = displ(x);
        a = zeros(2,2);
        a(1,1) = 3*x(1) + 3*x(3); a(1,2) = sqrt(3)*(x(1) + x(3));
        a(2,1) = a(1,2);          a(2,2) = x(1) + 8*x(2) + x(3);
        b = [4;8];
        y = gauss(a,b);
    end
end
```

The output of the program is

```
x =
    0.8159    0.7113    0.0293

displacements =
    1.0001
```

```
    1.0001

volume =
    2.4018
```

Since the constraints $u' \leq 1$ and $v' \leq 1$ are very close to being satisfied, the design is satisfactory. The optimal cross-sectional areas are

$$A_1 = 0.8159\frac{PL}{E\delta} \qquad A_2 = 0.7113\frac{PL}{E\delta} \qquad A_3 = 0.0293\frac{PL}{E\delta} \blacktriangleleft$$

The structural volume is

$$V = \frac{PL^2}{2E\delta}(2.402) = 1.201\frac{PL^2}{tE\delta}$$

which is the same as in Problem 21.

Note Added Note that A_3 is quite small, which begs the question: can it be removed altogether? The answer is "yes". The result would be a statically determinate truss of about the same strucural volume as the one above.

Problem 23

We have to maximize $I = bh^3/12$ with the constraint $b^2 + h^2 = d^2$. Introducing the variables $x_1 = b/d$ and $x_2 = h/d$, the problem becomes:

$$\text{minimize } F = -x_1 x_2^3 \text{ subject to } x_1^2 + x_2^2 - 1 = 0$$

```
function problem10_1_23
mu = 10000;
xStart = [1,1];
x = downhill(@merit_func,xStart)
constraint = constr(x)

    function c = constr(x)
        c = x(1)^2 + x(2)^2 - 1;
    end

    function F = merit_func(x)
        F = -x(1)*(x(2)^3) + mu*(constr(x)^2);
    end
end

x =
```

```
    0.5000     0.8660

constraint =
  3.2493e-005
```

The optimal dimansions are

$$b = \frac{1}{2}d \qquad h = \frac{\sqrt{3}}{2}d \blacktriangleleft$$

the moment of inertia being

$$I = \frac{1}{12}\left(\frac{\sqrt{3}}{2}\right)^3 d^4 = \frac{\sqrt{3}}{32}d^4$$

<u>Alternative solution</u>:
The equations to be solved are—see Eqs. (10.2):

$$\begin{aligned}\frac{\partial}{\partial x_1}\left[F(x) + \lambda g(x)\right] &= \frac{\partial}{\partial x_1}\left[-x_1 x_2^3 + \lambda(x_1^2 + x_2^2 - 1)\right] \\ &= -x_2^3 + 2\lambda x_1 = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x_2}\left[F(x) + \lambda g(x)\right] &= \frac{\partial}{\partial x_2}\left[-x_1 x_2^3 + \lambda(x_1^2 + x_2^2 - 1)\right] \\ &= -3x_1 x_2^2 + 2\lambda x_2 = 0\end{aligned}$$

$$g(x) = x_1^2 + x_2^2 - 1 = 0$$

Letting $\lambda = x_3$ results in the following program:

```
function problem10_23_calc
xStart = [1;1;1];
x = newtonRaphson2(@eqs,xStart)

    function F = eqs(x)
        F = zeros(3,1);
        F(1) = -x(2)^3 + 2*x(3)*x(1);
        F(2) = -3*x(1)*x(2)^2 + 2*x(3)*x(2);
        F(3) = x(1)^2 + x(2)^2 - 1;
    end
end
```

The result is

```
x =
    0.5000
    0.8660
    0.6495
```

Recall that $x(3)$ is the Lagrangian multipier λ.