1 (10 points).

①

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \cdots \qquad +1$$

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(x) + \frac{4h^3}{3}f'''(x) + \cdots \qquad +1$$

$$\rightarrow f(x+2h) - 4f(x+h) = -3f(x) - 2hf'(x) + \frac{2h^3}{3}f'''(x) + \cdots$$

$$\rightarrow f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + \mathcal{O}(h^2) \qquad +2$$

②

$$\mathrm{g}(h) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} \qquad +1$$

$$\mathrm{g}(2h) = \frac{-f(x+4h) + 4f(x+2h) - 3f(x)}{4h} \qquad +1$$

Richardson extrapolation:

$$\begin{aligned}
G &= \frac{2^2\mathrm{g}(h) - \mathrm{g}(2h)}{2^2 - 1} \qquad +3 \\
&= \frac{1}{3}\left[\frac{-4f(x+2h) + 16f(x+h) - 12f(x)}{2h} - \frac{-f(x+4h) + 4f(x+2h) - 3f(x)}{4h}\right] \\
&= \frac{1}{3}\left[\frac{-8f(x+2h) + 32f(x+h) - 24f(x)}{4h} - \frac{-f(x+4h) + 4f(x+2h) - 3f(x)}{4h}\right] \\
&= \frac{f(x+4h) - 12f(x+2h) + 32f(x+h) - 21f(x)}{12h} \qquad +1
\end{aligned}$$

$$\therefore f'(x) = \frac{f(x+4h) - 12f(x+2h) + 32f(x+h) - 21f(x)}{12h} + \mathcal{O}(h^4)$$

2 (20 points).

①

Step 1. Choose initial guess of x = (x,y)

Step 2. Evaluate f(x)

Step 3. Compute J(x)

Step 4. Compute Δx using J(x) Δx = -f(x)

Step 5. Set x ← x + Δx

Repeat 2-5 until |Δx| ≤ Error tolerance

or

Estimate the solution vector $\mathbf{x}$.
Do until $|\mathbf{\Delta x}| < \varepsilon$:
  Compute the matrix $\mathbf{J}(\mathbf{x})$ from Eq. (4.8).
  Solve $\mathbf{J}(\mathbf{x})\,\mathbf{\Delta x} = -\mathbf{f}(\mathbf{x})$ for $\mathbf{\Delta x}$.
  Let $\mathbf{x} \leftarrow \mathbf{x} + \mathbf{\Delta x}$.

+6

②

Initial guess: $(x, y) = (2.0, 0.25)$

$$\mathbf{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 - 2x - y + 0.5 \\ x^2 + 4y^2 - 4 \end{bmatrix}$$ +3

$$\mathbf{J}(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x-2 & -1 \\ 2x & 8y \end{bmatrix}$$ +5

$$\begin{bmatrix} 2x-2 & -1 \\ 2x & 8y \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\begin{bmatrix} x^2 - 2x - y + 0.5 \\ x^2 + 4y^2 - 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -0.09375 \\ 0.0625 \end{bmatrix} \qquad \therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \end{bmatrix} = \begin{bmatrix} 1.90625 \\ 0.3125 \end{bmatrix} \qquad \leftarrow \text{Iteration 1}$$ +3

$$\begin{bmatrix} 2(1.90625) - 2 & -1 \\ 2(1.90625) & 8(0.3125) \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\begin{bmatrix} (1.90625)^2 - 2(1.90625) - (0.3125) + 0.5 \\ (1.90625)^2 + 4(0.3125)^2 - 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -0.0088 \\ -0.0244 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -0.00556 \\ -0.0013 \end{bmatrix} \qquad \therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \end{bmatrix} = \begin{bmatrix} 1.9007 \\ 0.3112 \end{bmatrix} \qquad \leftarrow \text{Iteration 2}$$ +3

3 (35 points).

①

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} \quad \color{red}{+1} \quad \text{and} \quad \boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y_2 \\ 2y_1y_2 \end{bmatrix} \quad \color{red}{+4}$$

②

$$\boldsymbol{y}(x+h) = \boldsymbol{y}(x) + \mathbf{F}[x, \boldsymbol{y}(x)]h = \boldsymbol{y}(x) + \boldsymbol{y}'(x)h$$

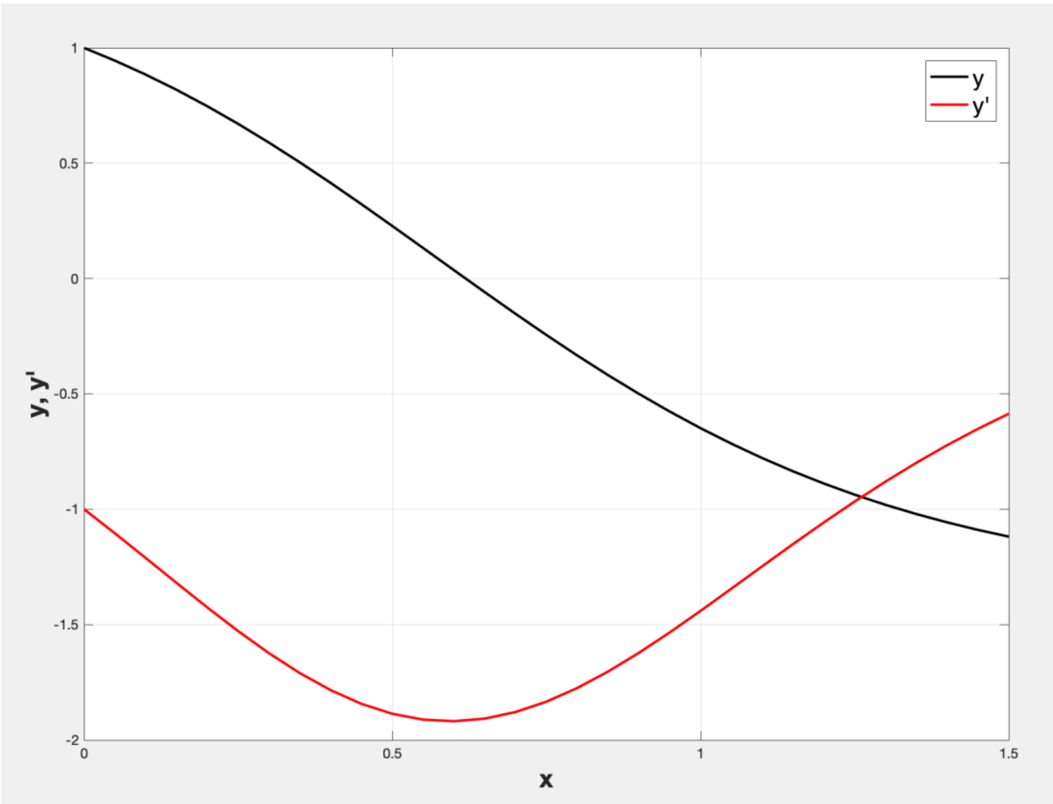
$$\rightarrow \begin{bmatrix} y_1(x+h) \\ y_2(x+h) \end{bmatrix} = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} y_2(x) \\ 2y_1(x)y_2(x) \end{bmatrix} h \quad \color{red}{+5}$$

③

$$\boldsymbol{y}(x+h) = \boldsymbol{y}(x) + \mathbf{F}[(x+h), \boldsymbol{y}(x+h)]h = \boldsymbol{y}(x) + \boldsymbol{y}'(x+h)h$$

$$\rightarrow \begin{bmatrix} y_1(x+h) \\ y_2(x+h) \end{bmatrix} = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} y_2(x+h) \\ 2y_1(x+h)y_2(x+h) \end{bmatrix} h \quad \color{red}{+5}$$

④ Programming



+20

4 (35 points).

①

$$\frac{y_{1,i+1}-2y_{1,i}+y_{1,i-1}}{h^2}=\frac{y_{1,i+1}-y_{1,i-1}}{h(1-x_i)}+\frac{y_{1,i}}{x_i^2(1-x_i)^2}\left[y_{1,i}^2-1+\frac{y_{3,i}^2-y_{2,i}^2}{(1-x_i)^2}\right]$$ +2

$$\frac{y_{2,i+1}-2y_{2,i}+y_{2,i-1}}{h^2}=\frac{2y_{2,i}y_{1,i}^2}{x_i^2(1-x_i)^2}$$ +2

$$\frac{y_{3,i+1}-2y_{3,i}+y_{3,i-1}}{h^2}=\frac{y_{3,i}}{x_i^2(1-x_i)^2}\left[2y_{1,i}^2+\frac{(y_{3,i}^2-x_i^2)}{(1-x_i)^2}\right]$$ +2

②

$$r(1)=y(1)-1 \;\;/\;\; r(n)=y(n)$$
$$r(n+1)=y(n+1) \;\;/\;\; r(2n)=y(2n)-0.5$$
$$r(2n+1)=y(2n+1) \;\;/\;\; r(3n)=y(3n)-1$$

Total +3
= +0.5 X 6

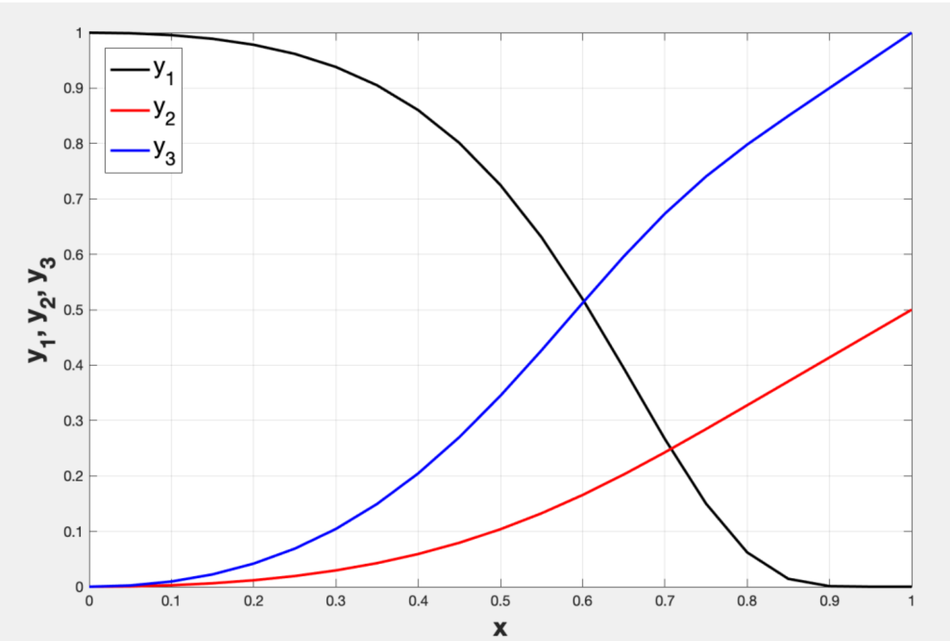
For $i=2\!:\!n-1$

$$r(i)=\frac{y(i+1)-2y(i)+y(i-1)}{h^2}-\frac{y(i+1)-y(i-1)}{h[1-x(i)]}-\frac{y(i)}{x(i)^2[1-x(i)]^2}\left[y(i)^2-1+\frac{y(2n+i)^2-y(n+i)^2}{[1-x(i)]^2}\right]$$ +2

$$r(n+i)=\frac{y(n+i+1)-2y(n+i)+y(n+i-1)}{h^2}-\frac{2y(n+i)y(i)^2}{x(i)^2[1-x(i)]^2}$$ +2

$$r(2n+i)=\frac{y(2n+i+1)-2y(2n+i)+y(2n+i-1)}{h^2}-\frac{y(2n+i)}{x(i)^2[1-x(i)]^2}\left[2y(i)^2+\frac{y(2n+i)^2-y(i)^2}{[1-x(i)]^2}\right]$$ +2

③ Programming



+20