CSE232: Discrete Mathematics
Assignment 3

November 8, 2020

Q1. How many bit strings of length 10 have

(a) exactly three 0s?

(b) more 0s than 1s?

(c) at least seven 1s?

(d) at least three 1s?

Answer (a). There are $C(10,3)$ ways to choose the positions for the 0s, and that is the only choice to be made, so the answer is $C(10,3) = \mathbf{120}$.

Answer (b). There are more 0s than 1s if there are fewer than five 1s. Using the same reasoning as in the previous part, together with the sum rule, we obtain the answer

$$C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4) = 1 + 10 + 45 + 120 + 210 = \mathbf{386}.$$

Alternatively, by symmetry, half of all cases in which there are not five 0s have more 0s than 1s; therefore the answer is $(210 - C(10,5))/2 = (1024 - 252)/2 = 386$.

Answer (c). We want the number of bit strings with $7, 8, 9$, or 10 1s. By the same reasoning as above, there are

$$C(10,7) + C(10,8) + C(10,9) + C(10,10) = 120 + 45 + 10 + 1 = \mathbf{176}$$

such strings.

Answer (d). If a string does not have at least three 1s, then it has $0, 1$, or 2 1s. There are $C(10,0) + C(10,1) + C(10,2) = 1 + 10 + 45 = 56$ such strings. There are $210 = 1024$ strings in all. Therefore there are $1024 - 56 = \mathbf{968}$ strings with at least three 1.

Q2. Prove the following statements.

(a) Prove that in any group of three positive integers, there are at least two whose sum is even.

(b) Let S be the set of integers $\{3, 4, 5, 6, 7, 8, 9, 10\}$. Suppose that you choose 5 integers in S. Prove that the sum of two of them is 13. For instance, if you choose $\{4, 6, 7, 9, 10\}$, then two of them, 4 and 9, satisfy 4+9=13.

Answer (a). Consider two pigeonholes, labeled *EVEN* and *ODD*. If three positive integers are placed in these pigeonholes, one of the pigeonholes must have at least two integers (say a and b) in it. Thus, a and b are either both even or both odd. In either case, $a + b$ is even.

Answer (b). We will apply the pigeonhole principle. We use 4 boxes, each box consisting of two numbers summing up to 13: $\{3, 10\}$, $\{4, 9\}$, $\{5, 8\}$ and $\{6, 7\}$. As there are 4 boxes and we choose 5 numbers, two of them must fall in one box. Then these two numbers sum up to 13.

Q3. Let $S = \{1, 2, \ldots, 19\}$. Find the number of subsets of S with equal numbers of odd integers and even integers.

Answer. Note that there are 10 odd integers and 9 even integers in S. The subsets to be counted must consist of k odd integers and k even integers, where $k = 0, 1, 2, \ldots, 9$. Therefore, by the product rule, the number of each type is $C(10, k) \cdot C(9, k)$. Therefore, by the sum rule the answer is

$$\sum_{k=0}^{9} C(10, k) \cdot C(9, k) = \mathbf{92, 378}.$$

Q4. In how many ways can ten books be put in four labeled boxes, if one or more of the boxes can be empty? Assume that the books are:

(a) distinct.

(b) identical.

Answer (a). Each book can be placed in any of the four boxes. By the product rule, this can be done in $\mathbf{4^{10}}$ ways.

Answer (b). The question can be rewritten as "how many different strings can be formed from 10 x's and 3 w's" (3 w's are the walls between boxes). This is to choose 3 positions between 13 and fill them with w's, the remainder 10 are x's. Therefore, the answer is $C(13, 3) = \mathbf{286}$.

Q5. In this problem, we consider strings of letters chosen from the set $\{a, b, c, d\}$. A *good* string is a string such that no two adjacent letters are the same. For instance, the string *acbadc* is a good string, but the string *acbbad* is not a good string. Suppose that you draw a string of six letters in $\{a, b, c, d\}$ at random. What is the probability that it is a good string?

Answer. Our sample space S is the set of strings of six letters in $\{a, b, c, d\}$. So its cardinality is $|S| = 4^6$ by the product rule.

Let E be the event that we draw a good string. There are 4 ways of choosing the first letter of this string. For the second letter, there are only 3 ways because it must be different from the first letter. Then for each following letter, there are again only three ways of choosing it. By the product rule, it means that $|E| = 4 \times 3^5$. So the answer is:

$$p(E) = \frac{|E|}{|S|} = \frac{4 \times 3^5}{4^6} = \left(\frac{\mathbf{3}}{\mathbf{4}}\right)^{\mathbf{5}}.$$

Q6. We use a standard deck of 52 cards (see Lecture 11). What is the probability that a hand of 5 cards contains at least one card of each suit? Justify your answer.

Answer. Our sample space S is the set of hands of 5 cards. So we have $|S| = C(52, 5)$. Let E be the event that the hand contains at least one card of each suit. We now determine $|E|$.

We first choose the suit that is represented by 2 cards. There are 4 possible choices for this suit, and then $\binom{13}{2}$ choices for the two cards in this suit. For each of the three other suits, we choose one card, so there are 13 possibilities. Therefore, by the product rule, we have:

$$|E| = 4 \times \frac{13 \times 12}{2} \times 13^3 = 24 \times 13^4$$

Thus the probability is:

$$p(E) = \frac{|E|}{|S|} = \frac{24 \times 13^4}{\binom{52}{5}} = \frac{24 \times 13^4 \times 5!}{52 \times 51 \times 50 \times 49 \times 48} = \frac{\mathbf{2197}}{\mathbf{8330}}.$$

Q7. In UNIST, there are 4500 students, and 300 of them are in the CSE track. 20% of the CSE students use Linux, and only 5% of the other students use Linux. What is the probability that a student is in the CSE track, given that he uses Linux?

Answer. Let L be the event that a student uses Linux, and C be the event that he is a CSE student. The problem statement tells us that:

$$p(L \mid C) = \frac{1}{5} \qquad p(L \mid \bar{C}) = \frac{1}{20} \qquad p(C) = \frac{300}{4500} = \frac{1}{15}$$

We want to determine $p(C \mid L)$. We apply Baye's theorem:

$$\begin{aligned} p(C \mid L) &= \frac{p(L \mid C)p(C)}{p(L \mid C)p(C) + p(L \mid \bar{C})p(\bar{C})} \\ &= \frac{\frac{1}{5} \cdot \frac{1}{15}}{\frac{1}{5} \cdot \frac{1}{15} + \frac{1}{20} \cdot \frac{14}{15}} = \frac{1}{1 + \frac{1}{4} \cdot 14} = \frac{\mathbf{2}}{\mathbf{9}} \end{aligned}$$

Q8. Give a combinatorial proof of the identity below:

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

(proofs by calculation will not get full mark).

Answer. Let A and B be two disjoint sets of m and n elements, respectively (see Figure 1). We will count in two different ways the number of subsets of $A \cup B$ of size k.

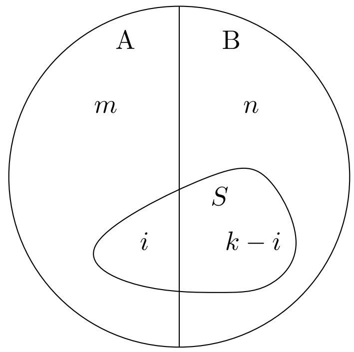

Figure 1: Combinatorial proof.

- As $|A \cup B|$ has $m+n$ elements, then it has $\binom{m+n}{k}$ subsets of size k.
- We can also count these subsets in a different way. Let $S \subset A \cup B$, such that $|S| = k$. In order to construct such subsets, we can first choose the number $i = |S \cap A|$ of its elements that are in A. Then there are $\binom{m}{i}$ ways of choosing $S \cap A$. After that, there are $\binom{n}{k-i}$ ways of choosing $S \cap B$. So for a fixed $i = |S \cap A|$, there are $\binom{m}{i}\binom{n}{k-i}$ ways of choosing S. Summing up over all possible sizes $i = \{0, \ldots, k\}$ of $S \cap A$, we obtain that the number of subsets of $A \cup B$ of size k is $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$.

Q9. Let E and F be two events in a sample space S. Prove that $p(\bar{E} \mid F) = 1 - p(E \mid F)$.

Answer. First observe that

$$\begin{aligned}
p(F) &= p(S \cap F) \\
&= p((E \cup \bar{E}) \cap F) \\
&= p((E \cap F) \cup (\bar{E} \cap F)) \\
&= p(E \cap F) + p(\bar{E} \cap F) - p((E \cap F) \cap (\bar{E} \cap F)) \\
&= p(E \cap F) + p(\bar{E} \cap F) - p(E \cap \bar{E} \cap F) \\
&= p(E \cap F) + p(\bar{E} \cap F) - p(\emptyset \cap F) \\
&= p(E \cap F) + p(\bar{E} \cap F) - p(\emptyset) \\
&= p(E \cap F) + p(\bar{E} \cap F).
\end{aligned}$$

After dividing both sides by $p(F)$, we obtain

$$1 = \frac{p(E \cap F)}{p(F)} + \frac{p(\bar{E} \cap F)}{p(F)} = p(E \mid F) + p(\bar{E} \mid F)$$

and thus $p(\bar{E} \mid F) = 1 - p(E \mid F)$.

Q10. Two identical urns contain balls. One of the urns has 6 red balls and 3 blue balls. The other urn has 5 red balls and 8 blue balls. An urn is chosen at random and two balls are drawn at random from this urn, without replacement.

(a) What is the probability that both balls are red?

(b) What is the probability that the second ball is red, given that the first ball is red?

Answer (a). Half the time we select the first urn, in which case the probability that the two balls are both red is $(6/9) \cdot (5/8) = 5/12$. Half the time we select the second urn, in which case the probability that the two balls are both red is $(5/13) \cdot (4/12) = 5/39$. Therefore the answer is

$$\frac{1}{2} \cdot \frac{5}{12} + \frac{1}{2} \cdot \frac{5}{39} = \frac{\mathbf{85}}{\mathbf{312}}.$$

Answer (b). Let F be the event that the first ball is red, and let S be the event that the second ball is red. We are asked for $p(S \mid F)$. By definition, this is $p(S \cap F)/p(F)$. In part (a) we found that $p(S \cap F) = 85/312$. By a simpler calculation, we see that

$$p(F) = \frac{1}{2} \cdot \frac{6}{9} + \frac{1}{2} \cdot \frac{5}{13} = \frac{123}{234}.$$

Thus the answer is

$$\frac{85/312}{123/234} = \frac{\mathbf{85}}{\mathbf{164}}.$$