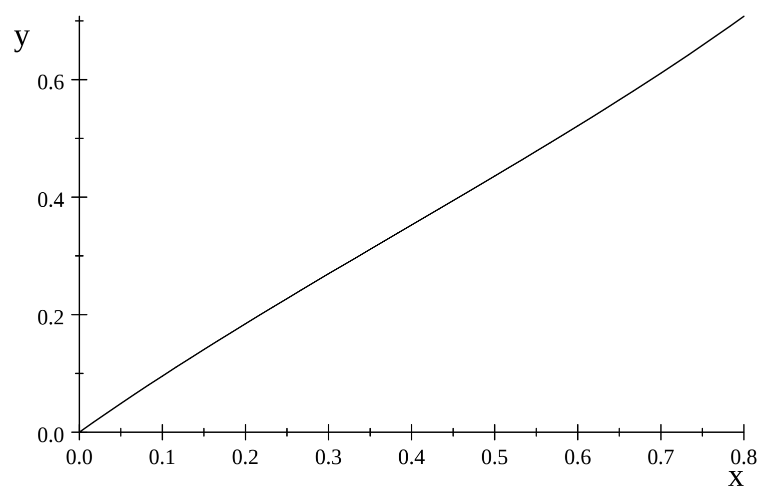
PROBLEM SET 6.1

Problem 1

$$I = \int_0^{\pi/4} \ln(1 + \tan x)dx$$

$$\begin{aligned}
I_1 &= \{\ln[1 + \tan(0)] + \ln[1 + \tan(\pi/4)]\}\frac{\pi}{8} = 0.272\,19\,8 \\
I_2 &= \frac{1}{2}(0.272\,19\,8) + \frac{\pi}{8}\ln[1 + \tan(\pi/8)] = 0.272\,19\,8 \\
I_3 &= \frac{1}{2}(0.272\,19\,8) + \frac{\pi}{16}\{\ln[1 + \tan(\pi/16)] + \ln[1 + \tan(3\pi/16)]\} \\
&= 0.272\,19\,8
\end{aligned}$$

It seems that a single panel (I_1) yields 6-figure accuracy. This fortuitous circumstance can be explained by the plot the function, which is practically a straight line in the region of integration.



Problem 2

i	1	2	3	4	5	6	7
v_i (m/s)	1.0	1.8	2.4	3.5	4.4	5.1	6.0
P_i (kW)	4.7	12.2	19.0	31.8	40.1	43.8	43.2
$(v/P)_i$ (kN^{-1})	0.2128	0.1475	0.1263	0.1101	0.1097	0.1164	0.1389

$$\Delta t = m \int_{1\text{s}}^{6\text{s}} (v/P)\, dv$$

Uneven spacing of data points on the v-axis precludes the use of Simpson's rule or Romberg integration. The best we can do is to apply the trapezoidal rule to each panel and sum the results:

$$I = \int_1^6 (v/P)\, dv \approx \frac{1}{2} \sum_{i=1}^{6} \left[(v/P)_i + (v/P)_{i+1}\right] (v_{i+1} - v_i)$$

i	$(v/P)_i + (v/P)_{i+1}$	$(v_{i+1} - v_i)$	$\left[(v/P)_i + (v/P)_{i+1}\right] (v_{i+1} - v_i)$
1	0.3603	0.8	0.2882
2	0.2738	0.6	0.1643
3	0.2364	1.1	0.2600
4	0.2198	0.9	0.1978
5	0.2261	0.7	0.1583
6	0.2553	0.9	0.2298
$\sum$			1.2984

$$\begin{aligned} I &= \frac{1.2984}{2} = 0.6492 \text{ m/(kN}\cdot\text{s)} = 0.6492 \times 10^{-3} \text{ m/(N}\cdot\text{s)} \\ \Delta t &= mI = 2000(0.6492 \times 10^{-3}) = 1.2984 \text{ s} \blacktriangleleft \end{aligned}$$

Problem 3

$$I = \int_{-1}^{1} f(x)\, dx \qquad f(x) = \cos(2\cos^{-1} x)$$

Two panels ($h = 1$):

x	-1	0	1
$f(x)$	1.0	-1.0	1.0

$$I = [2\,(1.0) + 4(-1.0)]\,\frac{1}{3} = -0.6667 \blacktriangleleft$$

Four panels ($h = 1/2$):

x	-1	$-1/2$	0	$1/2$	1
$f(x)$	1.0	-0.5	-1.0	-0.5	1.0

$$I = [2\,(1.0) + 8(-0.5) + 2(-1.0)]\,\frac{1}{6} = -0.6667 \blacktriangleleft$$

Six panels ($h = 1/3$):

x	-1	$-2/3$	$-1/3$	0	$1/3$	$2/3$	1
$f(x)$	1.0	-0.1111	-0.7778	-1.0	-0.7778	-0.1111	1.0

$$I = [2(1.0) + 8(-0.1111) + 4\,(-0.7778) + 4(-1.0)]\,\frac{1}{9} = -0.6667 \blacktriangleleft$$

The function $f(x)$ appears to be a quadratic, which can be integrated exactly with Simpson's rule. Indeed, it can be shown that $\cos(2\cos^{-1} x) = -1 + 2x^2$.

Problem 4

$$I = \int_1^\infty (1 + x^4)^{-1} dx$$

$$\begin{aligned} x^3 &= \frac{1}{t} \qquad 3x^2 dx = -\frac{dt}{t^2} \\ dx &= -\frac{dt}{3x^2t^2} = -\frac{dt}{3(1/t)^{2/3}t^2} = -\frac{dt}{3t^{4/3}} \end{aligned}$$

$$I = \int_1^0 \left(1 + \frac{1}{t^{4/3}}\right)^{-1} \left(-\frac{1}{3t^{4/3}}\right) dt = \int_0^1 \frac{dt}{3(t^{4/3}+1)}$$

t	0	0.2	0.4	0.6	0.8	1.0
$\left[3(t^{4/3}+1)\right]^{-1}$	0.3333	0.2984	0.2575	0.2214	0.1913	0.1667

$$\begin{aligned} I &\approx [0.3333 + 2(0.2984 + 0.2575 + 0.2214 + 0.1913) + 0.1667]\,0.1 \\ &= 0.2437 \blacktriangleleft \end{aligned}$$

Problem 5

x (m)	0.00	0.05	0.10	0.15	0.20	0.25
F (N)	0	37	71	104	134	161
x (m)	0.30	0.35	0.40	0.45	0.50	
F (N)	185	207	225	239	250	

$$U = \frac{1}{2}mv^2 = \int_0^{0.5\text{ m}} F\,dx$$

Using Simpson's rule:

$$\begin{aligned} U &\approx \left[\begin{array}{c} 0 + 4(37 + 104 + 161 + 207 + 239) \\ +\, 2(71 + 134 + 185 + 225) + 250 \end{array} \right] \frac{0.05}{3} \\ &= 74.53 \text{ N} \cdot \text{m} \end{aligned}$$

$$v = \sqrt{\frac{2U}{m}} = \sqrt{\frac{2(74.53)}{0.075}} = 44.58 \text{ m/s} \blacktriangleleft$$

Problem 6

$$f(x) = x^5 + 3x^3 - 2 \qquad I = \int_0^2 f(x)\,dx$$

Recursive trapezoidal rule:

$$\begin{aligned} R_{1,1} &= [f(0) + f(2)]\,\frac{H}{2} = (-2 + 54)\frac{2}{2} = 52 \\ R_{2,1} &= \frac{1}{2}R_{1,1} + \frac{H}{2}f(1) = \frac{52}{2} + \frac{2}{2}(2) = 28 \\ R_{3,1} &= \frac{1}{2}R_{2,1} + \frac{H}{4}\,[f(0.5) + f(1.5)] \\ &= \frac{1}{2}(28) + \frac{2}{4}(-1.593\,75 + 15.718\,75) = 21.0625 \end{aligned}$$

Romberg extrapolation:

$$\mathbf{R} = \begin{bmatrix} 52 & & \\ 28 & 20 & \\ 21.0625 & 18.75 & 18.6667 \end{bmatrix}$$

Because the error in $R_{3.3}$ is $\mathcal{O}(h^6)$, the result is exact for a polynomial of degree 5. Therefore,

$$I = 18.6667 \blacktriangleleft$$

is the "exact" integral.

Problem 7

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$f(x)$	1.0000	0.3431	0.2500	0.3431	1.0000

Romberg integration:

$$\begin{aligned}
R_{1,1} &= [f(0)+f(\pi)]\,\frac{H}{2} = (1+1)\frac{\pi}{2} = 3.1416 \\
R_{2,1} &= \frac{1}{2}R_{1,1} + \frac{H}{2}f(\pi/2) = \frac{\pi}{2} + \frac{\pi}{2}(0.25) = 1.9635 \\
R_{3,1} &= \frac{1}{2}R_{2,1} + \frac{H}{4}\left[f(\pi/4)+f(3\pi/4)\right] \\
&= \frac{1}{2}(1.9635) + \frac{\pi}{4}(0.3431+0.3431) = 1.5207
\end{aligned}$$

$$\mathbf{R} = \begin{bmatrix} 3.1416 & & \\ 1.9635 & 1.5708 & \\ 1.5207 & 1.3732 & 1.3600 \end{bmatrix}$$

$$I = \int_0^\pi f(x)\,dx \approx 1.3600 \quad \blacktriangleleft$$

This result has an error $\mathcal{O}(h^6)$. Note that trapezoidal rule would result in $I = 1.5207$ with an error $\mathcal{O}(h^2)$, and Simpson's rule would yield $I = 1.3732$ with an error $\mathcal{O}(h^4)$.

Problem 8

$$I = \int_0^1 \frac{\sin x}{\sqrt{x}}dx$$

$$x = t^2 \qquad dx = 2t\,dt$$

$$I = \int_0^1 \frac{\sin(t^2)}{t}2t\,dt = \int_0^1 2\sin(t^2)\,dt = \int_0^1 f(t)\,dt$$

Romberg integration:

$$\begin{aligned}
R_{1,1} &= [f(0)+f(1)]\,\frac{H}{2} = (0+1.6829)\frac{1}{2} = 0.8415 \\
R_{2,1} &= \frac{1}{2}R_{1,1} + \frac{H}{2}f(0.5) = \frac{0.8415}{2} + \frac{1}{2}(0.4948) = 0.6682 \\
R_{3,1} &= \frac{1}{2}R_{2,1} + \frac{H}{4}\left[f(0.25)+f(0.75)\right] \\
&= \frac{0.6682}{2} + \frac{1}{4}(0.1249+1.0667) = 0.6320 \\
R_{4,1} &= \frac{1}{2}R_{3,1} + \frac{H}{8}\left[f(0.125)+f(0.375)+f(0.625)+f(0.875)\right] \\
&= \frac{0.6320}{2} + \frac{1}{8}(0.0312+0.2803+0.7615+1.3860) = 0.6234
\end{aligned}$$

$$\mathbf{R} = \begin{bmatrix} 0.8415 & & & \\ 0.6682 & 0.6104 & & \\ 0.6320 & 0.6199 & 0.6205 & \\ 0.6234 & 0.6205 & 0.6205 & 0.6205 \blacktriangleleft \end{bmatrix}$$

Problem 9

According to Eq. (3.10) the cubic spline interpolant is

$$\begin{aligned} f_{i,i+1}(x) &= \frac{k_i}{6}\left[\frac{(x-x_{i+1})^3}{x_i - x_{i+1}} - (x-x_{i+1})(x_i - x_{i+1})\right] \\ &\quad -\frac{k_{i+1}}{6}\left[\frac{(x-x_i)^3}{x_i - x_{i+1}} - (x-x_i)(x_i - x_{i+1})\right] \\ &\quad +\frac{y_i(x-x_{i+1}) - y_{i+1}(x-x_i)}{x_i - x_{i+1}} \end{aligned}$$

Integrating over the panel between yields

$$\begin{aligned} \int_{x_i}^{x_{i+1}} f_{i,i+1}(x)\,dx &= -\frac{k_i}{24}(x_i - x_{i+1})^3 + \frac{k_i}{12}(x_i - x_{i+1})^3 \\ &\quad -\frac{k_{i+1}}{24}(x_i - x_{i+1})^3 + \frac{k_{i+1}}{12}(x_i - x_{i+1})^3 \\ &\quad -\frac{1}{2}(x_i - x_{i+1})(y_i + y_{i+1}) \end{aligned}$$

Substituting $x_i - x_{i+1} = -h$, this becomes

$$\int_{x_i}^{x_{i+1}} f_{i,i+1}(x)\,dx = -\frac{h^3}{24}(k_i + k_{i+1}) + \frac{h}{2}(y_i + y_{i+1})$$

Therefore,

$$\begin{aligned} I &= \int_{x_0}^{x_n} y(x)\,dx = \sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} f_{i,i+1}(x)\,dx\right] \\ &= -\frac{h^3}{24}(k_0 + 2k_1 + 2k_2 + \cdots + 2k_{n-1} + k_n) \\ &\quad +\frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \text{ Q.E.D.} \end{aligned}$$

Problem 10

$$\begin{aligned} \sin x &= t^2 \qquad \cos x\,dx = 2t\,dt \\ \sqrt{1-\sin^2 x}dx &= 2tdt \qquad dx = \frac{2t}{\sqrt{1-t^4}}dt \end{aligned}$$

$$\int_0^{\pi/4} \frac{dx}{\sqrt{\sin x}} = \int_0^{2^{-1/4}} \frac{2t}{\sqrt{1-t^4}} dt$$

```
% problem6_1_10
f = inline('2*x/sqrt(1 - x^4)','x');
a = 0; b = 1/sqrt(sqrt(2));
I = romberg(f,a,b)

>> I =
    0.7854
```

Problem 11

$$h(\theta_0) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2(\theta_0/2)\sin^2\theta}}$$

```
% problem6_1_11
global AMPL
theta0 = [15 30 45]*pi/180;
fprintf('Amplitude (deg)     h\n')
for AMPL = theta0
    h = romberg(@p6_1_11,0,pi/2);
    fprintf('%12.1f %14f\n',AMPL*180/pi,h)
end

function y = p6_1_11(x)
% Function used in Problem 11, Problem Set 6.1.
global AMPL
y = 1/sqrt(1-(sin(AMPL/2)*sin(x))^2);

Amplitude (deg)     h
        15.0       1.577552
        30.0       1.598142
        45.0       1.633586
```

In comparison, the small amplitude approximation is $h = \pi/2 = 1.570\,79\,6$.

Problem 12

$$w(r) = w_0 \int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{(r/a)^2 - \sin^2\theta}} d\theta$$

When $r = 2a$, we have

$$\frac{w}{w_0} = \int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{4 - \sin^2\theta}} d\theta$$

```
% problem6_1_12
f = inline('cos(x)^2/sqrt(4 - sin(x)^2)','x');
romberg(f,0,pi/2)

>> ans =
    0.4063
```

Problem 13

$$f(x) = \mu g + \frac{k}{m}(\mu b + x)\left(1 - \frac{b}{\sqrt{b^2 + x^2}}\right)$$

$$\begin{aligned} \mu g &= 0.3(9.81) = 2.943 \text{ m/s}^2 \\ \frac{k}{m} &= \frac{80}{0.8} = 100 \text{ s}^{-2} \\ \mu b &= 0.3(0.4) = 0.12 \text{ m} \end{aligned}$$

$$\begin{aligned} f(x) &= 2.943 + 100(0.12 + x)\left(1 - \frac{0.4}{\sqrt{0.16 + x^2}}\right) \\ I &= \int_0^{0.4} f(x)\, dx \qquad v_0 = \sqrt{2I} \end{aligned}$$

```
% problem6_1_13
f = inline('2.943 + 100*(0.12 + x)*(1 - 0.4/sqrt(0.16 + x^2))'...
          ,'x');
I = romberg(f,0,0.4);
v0 = sqrt(2*I)

>> v0 =
    2.4977
```

Problem 14

$$g(u) = u^3 \int_0^{1/u} \frac{x^4 e^x}{(e^x - 1)^2} dx$$

Note that

$$\int_0^\infty \frac{x^4 e^x}{(e^x-1)^2}dx$$

is finite, so that $g(0) = 0$. Also

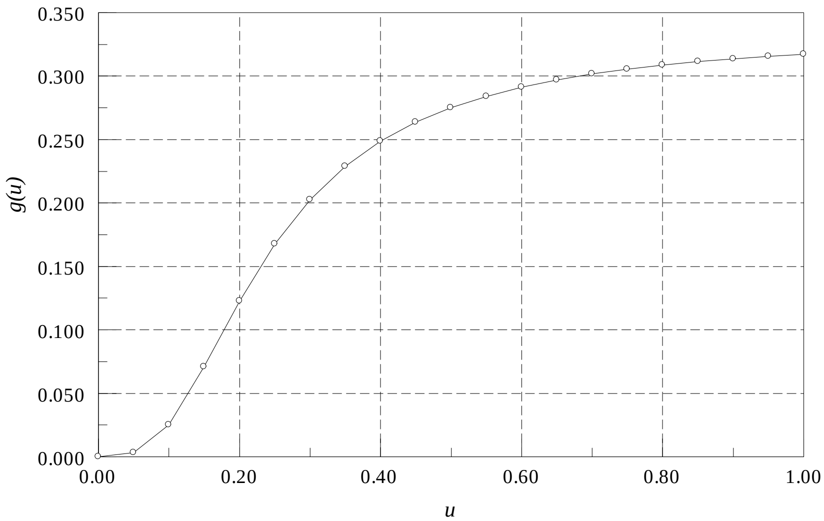
$$\frac{x^4 e^x}{(e^x-1)^2} \to 0 \text{ as } x \to 0$$

```
% problem6_1_14
fprintf(' u          g(u)\n')
for u = 0:0.05:1
    if u == 0; g = 0;
    else
        I = romberg(@p6_1_14,0,1/u);
        g = I*u^3;
    end
    fprintf('%6.2f %13.6f\n',u,g)
end

function y = p6_1_14(x)
% Function used in Problem 14, Problem Set 6.1.
if x == 0; y = 0;
else y = x^4*exp(x)/(exp(x) - 1)^2; end
```

Here is a partial printout of the results

```
    u          g(u)
  0.00       0.000000
  0.05       0.003247
  0.10       0.025274
  0.15       0.070997
  0.20       0.122878
  0.25       0.167686
  0.30       0.202568
  0.35       0.228858
  0.40       0.248618
```



Problem 15

$$i(t) = i_0 e^{-t/t_0} \sin(2t/t_0) \qquad E = \int_0^\infty R\,[i(t)]^2\,dt$$

$$\begin{aligned} i_0 &= 100 \text{ A} \qquad R = 0.5\ \Omega \qquad t_0 = 0.01 \text{ s} \\ Ri_0^2 &= 0.5(100)^2 = 5000 \end{aligned}$$

Since we cannot deal with infinite integration limits, we must change the upper limit from ∞ to τ, where τ is a time during which the current *just* reaches negligible magnitude. If τ is too large, Romberg integration will not work—it converges prematurely to $E = 0$. In the program below we tried $\tau = 0.05$ s:

```
% problem6_1_15
f = inline('0.5*(100*exp(-t/0.01)*sin(2*t/0.01))^2','t');
E = romberg(f,0,0.05)

>> E =
    9.9993
```

Is is prudent to try another value of τ and compare the results. Running the program with $\tau = 0.1$ s we get

```
>> E =
   10.0000
```

It is safe to conclude that the solution is $E = 10.0$ W·s ◀

Problem 16

The following program uses Romberg integration:

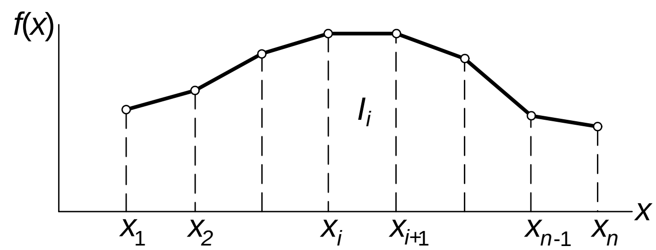
```
% problem6_1_16
func = @(t) ((sin(pi*t/0.05) - 0.2*sin(2*pi*t/0.05))^2);
[Integral,numEval] = romberg(func,0,0.05);
iRMS = sqrt(Integral/0.05)

iRMS =
    0.7211
```

The root-mean-square current is 0.7211 A ◄

Problem 17

(a)



The trapezoidal rule for a single panel is

$$I_i = \frac{1}{2}(f_i + f_{i+1})(x_{i+1} - x_i)$$

so that the composite trapezoidal rule becomes

$$\begin{aligned}
I &= \sum_{i=1}^{n-1} I_i \\
&= \frac{1}{2}[(f_1 + f_2)(x_2 - x_1) + (f_2 + f_3)(x_3 - x_2) + (f_3 + f_4)(x_4 - x_3) \\
&\quad + \cdots + (f_{n-1} + f_n)(x_n - x_{n-1})] \\
&= \frac{1}{2}[f_1(x_2 - x_1) + f_2(x_3 - x_1) + f_3(x_4 - x_2) + \cdots + f_i(x_{i+1} - x_{i-1}) \\
&\quad + \cdots + f_{n-1}(x_n - x_{n-2}) + f_n(x_n - x_{n-1})] \quad \blacktriangleleft
\end{aligned}$$

(b)

```
% problem6_2_17
stress = [586 662 765 841 814 689 600]'*10^6;    % Stress in Pa
strain = [1   25  45  68  89  122 150]'*0.001;
n = length(stress);
% Trapezoidal rule
modulus = stress(1)*(strain(2) - strain(1));
for i = 2:n-1
    modulus = modulus + stress(i)*(strain(i+1) - strain(i-1));
end
modulus = modulus + stress(n)*(strain(n) - strain(n-1));
modulus_in Pa = modulus/2

modulus_in_Pa =
   107938000
```

Thus the modulus of toughness is 108 MPa ◂

PROBLEM SET 6.2

Problem 1

$$f(x) = \frac{\ln x}{x^2 - 2x + 2} \qquad I = \int_1^\pi f(x)dx$$

$$x_i = \frac{b+a}{2} + \frac{b-a}{2}\xi_i \qquad I \approx \frac{b-a}{2}\sum_{i=1}^{n} A_i f(x_i)$$

(a) 2-node quadrature:

$$\begin{aligned}
x_1 &= \frac{\pi+1}{2} + \frac{\pi-1}{2}(-0.577\,350) = 1.452\,572 \\
x_2 &= \frac{\pi+1}{2} + \frac{\pi-1}{2}(0.577\,350) = 2.689\,021 \\
A_1 &= A_2 = 1
\end{aligned}$$

$$I \approx \frac{\pi-1}{2}(0.256\,743 + 0.309\,868) = 0.6067 \quad \blacktriangleleft$$

(b) 4-node quadrature:

$$\begin{aligned}
x_1 &= \frac{\pi+1}{2} + \frac{\pi-1}{2}(-0.861\,136) = 1.148\,69\,5 \\
x_2 &= \frac{\pi+1}{2} + \frac{\pi-1}{2}(-0.339\,981) = 1.706\,74\,6 \\
x_3 &= \frac{\pi+1}{2} + \frac{\pi-1}{2}(0.339\,981) = 2.434\,84\,7 \\
x_4 &= \frac{\pi+1}{2} + \frac{\pi-1}{2}(0.861\,136) = 2.992\,89\,8
\end{aligned}$$

i	x_i	$f(x_i)$	A_i	$A_i f(x_{i(}$
1	1.148 69 5	0.135 628	0.347 855	0.047 179
2	1.706 74 6	0.356 514	0.652 145	0.232 499
3	2.434 84 7	0.290 927	0.652 145	0.189 727
4	2.992 89 8	0.220 499	0.347 855	0.076 702
$\sum$				0.546 107

$$I \approx \frac{\pi-1}{2}(0.546\,107) = 0.5848 \quad \blacktriangleleft$$

Problem 2

$$f(x) = (1 - x^2)^3 \qquad I = \int_0^\infty e^{-x} f(x)\, dx \approx \sum_{i=1}^{n} A_i f(x_i)$$

Since $f(x)$ is a polynomial of degree 6, we use 4-node quadrature for an exact result:

i	x_i	$f(x_i)$	A_i	$A_i f(x_i)$
1	0.322 548	0.719 234	0.603 154	0.434
2	1.745 761	−8.586 927	0.357 418	−3.069
3	4.536 620	$-7.507\,569 \times 10^3$	$0.388\,791 \times 10^{-1}$	−291.954
4	9.395 071	$-6.645\,926 \times 10^5$	$0.539\,295 \times 10^{-3}$	−358.411
$\sum$				−653.000

$$I = -653.0 \blacktriangleleft$$

Problem 3

$$I = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$$

$$\begin{aligned} \sin x &= t^2 \qquad \cos x\, dx = 2t\, dt \qquad \sqrt{1 - t^4} dx = 2t\, dt \\ dx &= \frac{2t}{\sqrt{(1 - t^2)(1 + t^2)}} dt \end{aligned}$$

$$\begin{aligned} I &= 2\int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 + t^2)}} = \int_{-1}^1 \frac{dt}{\sqrt{(1 - t^2)(1 + t^2)}} \\ I &= \int_{-1}^1 \frac{g(t)}{\sqrt{1 - t^2}} dt \approx \frac{\pi}{n} \sum_{i=1}^{n} g(t_i) \qquad g(t) = \frac{1}{\sqrt{1 + t^2}} \qquad n = 5 \end{aligned}$$

i	$t_i = \cos \dfrac{(2i + 1)\pi}{2n + 2}$	$g(t_i)$
1	0.965 926	0.719 255
2	0.707 107	0.816 497
3	0.258 819	0.968 100
4	−0.258 819	0.968 100
5	−0.707 107	0.816 497
6	−0.965 926	0.719 255
$\sum$		5.007 703

$$I \approx \frac{\pi}{6}(5.007\,703) = 2.622\,03 \blacktriangleleft$$

Problem 4

$$I = \int_0^{\pi} f(x)\,dx \qquad f(x) = \sin x$$

The truncation error is

$$E = \frac{(b-a)^{2n+1}\,[(n)!]^4}{(2n+1)\,[(2n)!]^3} f^{(2n)}(c), \quad a < c < b$$

where

$$a = 0 \qquad b = \pi \qquad n = 3$$

$$f^{(2n)}(x) = f^{(8)}(x) = \sin x$$

Thus

$$\begin{aligned} E_{\min} &= \frac{(\pi-0)^9(4!)^4}{9(8!)^3}\sin 0 = 0 \;◀ \\ E_{\max} &= \frac{(\pi-0)^9(4!)^4}{9(8!)^3}\sin\frac{\pi}{2} = 1.6764\times10^{-5} \;◀ \end{aligned}$$

Problem 5

$$I = \int_0^{\infty} e^{-x} f(x)\,dx \qquad f(x) = \sin x$$

The truncation error is

$$E = \frac{(n!)^2}{(2n)!} f^{(2n)}(c), \quad 0 < c < \infty$$

Noting that

$$f^{(2n)}_{\min} = -1 \qquad f^{(2n)}_{\max} = 1$$

we have

$$E_{\min,\max} = \pm\frac{(n!)^2}{(2n)!}$$

n	$\lvert E_{\min,\max}\rvert$
7	2.914×10^{-4}
8	7.7704×10^{-5}
9	2.057×10^{-5}
10	5.413×10^{-6}
11	1.418×10^{-6}
12	3.698×10^{-7}

To be be sure of 6 decimal place accuracy, one should use 12 nodes ◀

Problem 6

$$I = \int_0^1 \frac{2x+1}{\sqrt{x(1-x)}} dx$$

$$x = \frac{1}{2}(1+t) \qquad dx = \frac{1}{2} dt$$

$$\begin{aligned} I &= \int_{-1}^1 \frac{2+t}{\sqrt{(1-t^2)}} dt \\ I &= \int_{-1}^1 \frac{f(t)}{\sqrt{(1-t^2)}} dt \qquad f(t) = 2+t \end{aligned}$$

Since $f(t)$ is linear in t, Gauss-Chebyshev quadrature will give the exact integral with a single node. Substituting $n = 1$ into the quadrature formulas

$$I = \frac{\pi}{n} \sum_{i=1}^n f(t_i) \qquad t_i = \cos \frac{(2i-1)\pi}{2n}$$

we get

$$I = \pi f\left(\cos \frac{\pi}{2}\right) = \pi(2+0) = 2\pi \quad \blacktriangleleft$$

Problem 7

$$I = \int_0^\pi \sin x \ln x \, dx$$

Let

$$x = \pi z \qquad dx = \pi \, dz$$

$$\begin{aligned} I &= \pi \int_0^1 \sin(\pi z) \ln(\pi z) \, dz \\ &= \pi \ln \pi \int_0^1 \sin(\pi z) \, dz + \pi \int_0^1 \sin(\pi z) \ln z \, dz \end{aligned}$$

The first term is

$$I_1 = \pi \ln \pi \int_0^1 \sin(\pi z) \, dz = \pi \ln \pi \left(\frac{2}{\pi}\right) = 2 \ln \pi = 2.289\,46$$

The second term

$$I_2 = \pi \int_0^1 f(z) \ln z \, dz \qquad f(z) = \sin(\pi z)$$

can be evaluated with Gauss quadrature with logarithmic singularity. Using $n = 4$ we get

i	z_i	$f(z_i)$	A_i	$A_i f(z_i)$
1	0.041 449	0.129 848	0.383 464	0.049 792
2	0.245 275	0.696 533	0.386 875	0.269 471
3	0.556 165	0.984 473	0.190 435	0.187 478
4	0.848 982	0.456 838	0.039 226	0.017 920
$\sum$				0.524 661

$$I_2 = -\pi(0.524661) = -1.648\,27$$

$$I = I_1 + I_1 = 2.289\,46 - 1.648\,27 = 0.6412 \quad \blacktriangleleft$$

The true value of the integral is 0.641 182.

Problem 8

$$I = \int_0^\pi f(x)\,dx \qquad f(x) = x\sin x$$

$$E = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)\left[(2n)!\right]^3} f^{(2n)}(c), \quad a < c < b$$

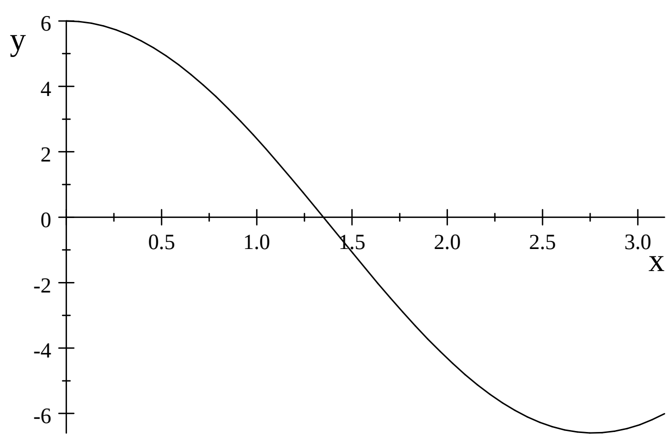
With $n = 3$ the error becomes

$$E = \frac{(\pi - 0)^7(3!)^4}{7(6!)^3} f^{(6)}(c) = 1.498 \times 10^{-3} f^{(6)}(c)$$

where

$$f^{(6)}(x) = \frac{d^6}{dx^6}(x\sin x) = 6\cos x - x\sin x$$

is plotted below.



The plot shows that $f^{(6)}_{\max} = 6$ and $f^{(6)}_{\min}$ occurs at $x \approx 2.75$, so that

$$f^{(6)}_{\min} \approx 6\cos 2.75 - 2.75 \sin 2.75 = -6.60$$

Therefore,

$$\begin{aligned} E_{\min} &= 1.498 \times 10^{-3}(-6.60) = -9.89 \times 10^{-3} \quad \blacktriangleleft \\ E_{\max} &= 1.498 \times 10^{-3}(6) = 8.99 \times 10^{-3} \quad \blacktriangleleft \end{aligned}$$

To find the actual error, we must evaluate the integral with Gauss-Legendre quadrature:

$$\begin{aligned} x_1 &= \frac{\pi}{2}(1 - 0.774\,597) = 0.354\,06\,2 \\ x_2 &= \frac{\pi}{2}(1 - 0) = 1.570\,79\,6 \\ x_3 &= \frac{\pi}{2}(1 + 0.774\,597) = 2.787\,530 \end{aligned}$$

i	x_i	$f(x_i)$	A_i	$A_i f(x_i)$
1	0.354 06 2	0.068 199	0.555 556	0.068 199
2	1.570 79 6	1.570 79 6	0.888 889	1.396 264
3	2.787 530	0.536 927	0.555 556	0.536 927
$\sum$				2.001 390

$$I \approx \frac{\pi}{2}(2.001\,390) = 3.143\,77\,6$$

The exact integral is $I = \pi = 3.141\,593$, so that the actual error is

$$E = 3.143\,77\,6 - 3.141\,593 = 2.18 \times 10^{-3} \quad \blacktriangleleft$$

Problem 9

$$I = \int_0^2 f(x)\,dx \qquad f(x) = \frac{\sinh x}{x}$$

Try Gauss-Legendre quadrature with 3 nodes:

$$\begin{aligned} x_1 &= 1 - 0.774\,597 = 0.225\,40\,3 \\ x_2 &= 1 \\ x_3 &= 1 + 0.774\,597 = 1.774\,597 \end{aligned}$$

i	x_i	$f(x_i)$	A_i	$A_i f(x_i)$
1	0.225 40 3	1.008 489	0.555 556	0.560 272
2	1	1.175 201	0.888 889	1.044 623
3	1.774 597	1.613 987	0.555 556	0.896 660
$\sum$				2.501 555

$$I \approx 2.502 \blacktriangleleft$$

The true value of the integral is $I = 2.501\,567$.

Problem 10

$$I = \int_0^\infty \frac{x\,dx}{e^x + 1}$$

$$e^x = \frac{1}{t} \qquad e^x dx = -\frac{1}{t^2}dt \qquad dx = -\frac{1}{t}dt \qquad x = -\ln t$$

$$I = \int_1^0 \frac{-\ln t}{(1/t + 1)(-t)}dt = -\int_0^1 \frac{\ln t}{1+t}dt$$

Use Gauss 4-node quadrature with logarithmic singularity.

$$I = \int_0^1 f(t) \ln t\,dt \qquad f(x) = \frac{1}{1+t}$$

i	t_i	$f(t_i)$	A_i	$A_i f(x_i)$
1	0.041 449	−0.960 201	0.383 464	−0.368 203
2	0.254 275	−0.797 273	0.386 875	−0.310674
3	0.556 165	−0.642 60 5	0.190 435	−0.122 375
4	0.848 982	−0.540 838	0.039 226	−0.021214
$\sum$				−0.822 466

$$I \approx 0.822\,466 \blacktriangleleft$$

The true value of the integral is $I = 0.822\,467$. The discrepancy is due to unavoidable roundoff errors.

Problem 11

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy = 0 \qquad \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y} = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}$$

$$S = 2\int_{-a}^{a}\sqrt{1 + (dy/dx)^2}dx = 2\int_{-a}^{a}\sqrt{1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}}dx$$

Because the integrand is singular at $x = a$, the integral in its present form is not well-suited for quadrature. But with a change of variable

$$x = a\xi \qquad dx = a\,d\xi$$

$$\begin{aligned} S &= 2a\int_{-1}^{1}\sqrt{1 + \frac{b^2}{a^2}\frac{\xi^2}{1 - \xi^2}}d\xi = 2\int_{-1}^{1}\frac{\sqrt{(1 - \xi^2)a^2 + b^2\xi^2}}{\sqrt{1 - \xi^2}}d\xi \\ S &= 2\int_{-1}^{1}\frac{f(\xi)}{\sqrt{1 - \xi^2}}d\xi \qquad f(\xi) = \sqrt{(1 - \xi^2)a^2 + b^2\xi^2} \end{aligned}$$

the integral can be evaluated with Gauss-Chebyshev quadrature.
We found by experimentation that the number of nodes required to achieve the specified accuracy increases with the eccentricity of the ellipse. Consequently, we chose $n = 6 \times \max(a/b, b/a)$ which appears to give 5 decimal point accuracy over a wide range of eccentricities.

```
% problem6_2_11
A = input('a ==> ');
B = input('b ==>');
n = round(6*max(A/B,B/A));
S = 0;
for i =1:n
    x = cos((i - 0.5)*pi/n);
    S = S + sqrt((1 - x^2)*A^2 + (B*x)^2);
end
S = 2*pi*S/n

a ==> 2
b ==>1
S =
    9.6884
```

The true value of circumference is $S = 9.688\,448$

Problem 12

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Here the required number of nodes is dependent on the value of x (larger x requires more nodes). In the following program we overcome this problem by brute force: we apply Gauss-Legendre quadrature with $n = 4, 5, \ldots, 11$ until successive results are in agreement within 10^{-6}.

```
% problem6_2_12
format long
func = inline('exp(-t^2)','t');
x = input('x ==> ');
if x > 5; erf = 1
else
    Iold = gaussQuad(func,0,x,2);
    for n = 3:12
        Inew = gaussQuad(func,0,x,n);
        if abs(Inew - Iold) < 1.0e-6; break; end
        Iold = Inew;
    end
    erf = 2/sqrt(pi)*Inew
end

x ==> 1
erf =
   0.84270078612733
```

Problem 13

$$C = \int_0^1 \left(\left(\sqrt{2} - 1\right)^2 - \left(\sqrt{1 + z^2} - 1\right)^2 \right)^{-1/2} dz$$

This can be written in the form

$$\begin{aligned} C &= \frac{1}{2} \int_{-1}^{1} \frac{f(z)}{\sqrt{1 - z^2}} dz \\ f(z) &= \sqrt{\frac{1 - z^2}{\left(\sqrt{2} - 1\right)^2 - \left(\sqrt{1 + z^2} - 1\right)^2}} \end{aligned}$$

where $f(z)$ is free of singularities. The integral can be now evaluated with Gauss-Chebyshev quadrature ($n = 11$ is required for 6 decimal place accuracy).

```
% problem6_2_13
format long
func = inline('sqrt((1-z^2)/((sqrt(2)-1)^2-(sqrt(1+z^2)-1)^2))'...
             ,'z');
n = 11; C = 0;
for i = 1:n
    z = cos((i - 0.5)*pi/n);
    C = C + feval(func,z);
end
C = 0.5*pi/n*C

>> C =
   3.26670231592733
```

Problem 14

$$C\left(\frac{h}{b}\right) = \int_0^1 z^2 \sqrt{1 + \left(\frac{2h}{b}z\right)^2}\, dz$$

We use Gauss-Legendre quadrature with 6 nodes, which was found to be sufficient for 4 decimal point accuracy.

```
% problem6_2_14
global RATIO
while 1
    RATIO = input('h/b ==> ');
    if isempty (RATIO); fprintf('Done'); break; end
    C = gaussQuad(@p6_2_14,0,1,6)
end

function y = p6_2_14(z)
% Function used in Problem 14, Problem Set 6.2.
global RATIO
y = z^2*sqrt(1+(2*RATIO*z)^2);

h/b ==> 0.5
C =
    0.4202
h/b ==> 1.0
C =
    0.6063
h/b ==> 2.0
C =
```

```
    1.0589
h/b ==>
Done
```

Problem 15

$$I = \int_0^{\pi/2} \ln(\sin x) dx = I_1 + I_2 + I_3$$

$$\begin{aligned} I_1 &= \int_0^{0.01} \ln(\sin x) dx \approx \int_0^{0.01} \ln x \, dx = [x \ln x - x]_0^{0.01} \\ &= 0.01(\ln 0.01 - 1) \end{aligned}$$

$$I_2 = \int_{0.01}^{0.2} \ln(\sin x) dx \qquad I_3 = \int_{0.2}^{\pi/2} \ln(\sin x) dx$$

To guarantee 6-decimal point accuracy, we compute both I_2 and I_3 with $n = 2, 3, \ldots, 30$ until successive results are in agreement within 10^{-6}.

```
% problem6_2_15
format long
f = inline('log(sin(x))','x');
lowLim = [0.01 0.2]; upLim = [0.2 pi/2];
I = zeros(length(lowLim),1);
for i =1:length(lowLim)
    a = lowLim(i); b = upLim(i);
    Iold = gaussQuad(f,a,b,2);
    for n = 3:30
        Inew = gaussQuad(f,a,b,n);
        if abs(Inew - Iold) < 0.1e-6;
            I(i) = Inew; break;
        end
        Iold = Inew;
    end
end

>> Integral =
  -1.08879293368231
```

Problem 16

h (m)	0	15	35	52	80	112
p (Pa)	310	425	530	575	612	620

```
% problem6_2_16
global C
hData = [0 15 35 52 80 112];
pData = [310 425 530 575 612 620];
C = polynFit(hData,pData,4);
resultant = gaussQuad(@p6_2_16a,0,112,2);
moment    = gaussQuad(@p6_2_16b,0,112,3);
h_pressure_center = moment/resultant

function y = p6_2_16a(h)
% Function used in Problem 16, Problem Set 6.2.
global C
y = C(1)*h^3 + C(2)*h^2 + C(3)*h + C(4);

function y = p6_2_16b(h)
% Function used in Problem 16, Problem Set 6.2.
global C
y = (C(1)*h^3 + C(2)*h^2 + C(3)*h + C(4))*h;

>> h_pressure_center =
   60.5730
```

Problem 17

Since the spline in each segment is cubic, integration order of 2 is sufficient in the Gauss-Legendre quadrature (recall that quadrature with 2 integration points is exact for a cubic).

```
function integral = integrate(xData,yData)
% Integration of unevenly spaced data using cubic spline
% and Gauss-Legendre quadrature of order 2.
m = length(yData);
k = splineCurv(xData,yData);
integral = 0;
for i = 1:m-1                          % Loop over segments
    c1 = (xData(i+1) + xData(i))/2;
    c2 = (xData(i+1) - xData(i))/2;
```

```
    x1 = c1 - c2/sqrt(3);               % x-coord. of node 1
    x2 = c1 + c2/sqrt(3);               % x-coord. of node 2
    y1 = splineEval(xData,yData,k,x1);  % Interpolant at node 1
    y2 = splineEval(xData,yData,k,x2);  % Interpolant at node 2
    integral = integral + c2*(y1 + y2); % Eq. (6.29)
end

% problem6_2_17
stress = [586;662;765;841;814;689;600]*10^6;
strain = [1;  25; 45; 68; 89; 122;150]*0.001;
modulus = integrate(strain,stress)

modulus =
  1.0817e+008
```

This is approximately the same value as calculated in Problem 17, Problem Set 6.1.

PROBLEM SET 6.3

Problem 1

$$I = \int_{-1}^{1}\int_{-1}^{1}(1-x^2)(1-y^2)\,dx\,dy$$

As the integral is biquadratic, second-order quadrature is exact. The region of integration is a "standard" rectangle. All 4 integration points contribute the same amount to the integral:

$$I = 4(1 - 0.577350^2)^2 = 1.777\,8 \quad \blacktriangleleft$$

Problem 2

$$I = \int_{y=0}^{2}\int_{x=0}^{3} f(x,y)\,dx\,dy \qquad f(x,y) = x^2y^2$$

Since the integrand is biquadratic, second-order quadrature is exact. The coordinates of the integration points are

$$\begin{aligned} x_{1,2} &= \frac{3+0}{2} \pm \frac{3-0}{2}(0.577350) = \begin{cases} 2.366\,025 \\ 0.633\,975 \end{cases} \\ y_{1,2} &= \frac{2+0}{2} \pm \frac{2-0}{2}(0.577350) = \begin{cases} 1.577350 \\ 0.422\,650 \end{cases} \end{aligned}$$

The area scale factor (constant in this case) is

$$|\mathbf{J}| = \frac{\text{area of rectangle}}{\text{area of "std". rectangle}} = \frac{3 \times 2}{2 \times 2} = 1.5$$

$$\begin{aligned} I &= \sum_{i=1}^{2}\sum_{j=1}^{2} A_iA_jf(x_i,y_j)\,|\mathbf{J}| \\ &= 1.5\begin{bmatrix} (2.366\,025)^2\,(1.577350)^2 + (2.366\,025)^2(0.422\,650)^2 \\ +(0.633\,975)^2(1.577350)^2 + (0.633\,975)^2(0.422\,650)^2 \end{bmatrix} \\ &= 24.0000 \quad \blacktriangleleft \end{aligned}$$

Problem 3

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy \qquad f(x,y) = e^{-(x^2+y^2)}$$

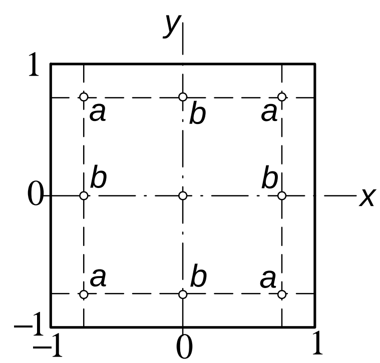
The integration is over a "standard" rectangle.

(a)

All four integration points contribute the same amount. Therefore,

$$I = \sum_{i=1}^{2}\sum_{j=1}^{2} A_i A_j f(x_i, y_j) = 4\exp\left[-2(0.577350)^2\right] = 2.0537$$

(b)



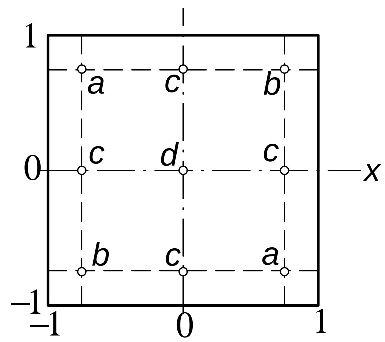
Values of $f(x,y)$ at the integration points are

$$\begin{aligned} f_a &= \exp\left[-2(0.774\,597)^2\right] = 0.301\,194 \\ f_b &= \exp\left[-(0.774\,597)^2\right] = 0.548\,811 \\ f_{\text{center}} &= 1 \end{aligned}$$

$$\begin{aligned} I &\approx 4(0.555\,556)^2(0.301\,194) \\ &\quad +4(0.555\,556)(0.888\,889)(0.548\,811) + (0.888\,889)^2 \\ &= 2.2460 \ \blacktriangleleft \end{aligned}$$

Problem 4

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dx\,dy \qquad f(x,y) = \cos\frac{\pi(x-y)}{2}$$



Values of $f(x,y)$ at the integration points are

$$\begin{aligned}
f_a &= \cos\frac{2\pi(-0.774\,597)}{2} = -0.759\,583 \\
f_b &= \cos(0) = 1 \\
f_c &= \cos\frac{\pi(0.774\,597)}{2} = 0.346\,711 \\
f_d &= \cos(0) = 1
\end{aligned}$$

$$\begin{aligned}
I &\approx 2(0.555\,556)^2(-0.759\,583) + 2(0.555\,556)^2(1) \\
&\quad +4(0.555\,556)(0.888\,889)(0.346\,711) + (0.888\,889)^2(1) \\
&= 1.6234 \blacktriangleleft
\end{aligned}$$

Problem 5

$$I = \int\int_A xy\,dx\,dy$$

$$\mathbf{x} = \begin{bmatrix} 0 & 2 & 4 & 0 \end{bmatrix}^T \qquad \mathbf{y} = \begin{bmatrix} 0 & 0 & 4 & 4 \end{bmatrix}^T$$

$$\begin{aligned}
x(\xi,\eta) &= \sum_{k=1}^{4} N_k(\xi,\eta)x_k \\
&= \frac{1}{4}(1+\xi)(1-\eta)(2) + \frac{1}{4}(1+\xi)(1+\eta)(4) \\
&= \frac{1}{2}(1+\xi)(3+\eta)
\end{aligned}$$

$$
\begin{aligned}
y(\xi,\eta) &= \sum_{k=1}^{4} N_k(\xi,\eta) y_k \\
&= \frac{1}{4}(1+\xi)(1+\eta)(4) + \frac{1}{4}(1-\xi)(1+\eta)(4) \\
&= 2(1+\eta)
\end{aligned}
$$

$$
\mathbf{J}(\xi,\eta) = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} (3+\eta)/2 & 0 \\ (1+\xi)/2 & 2 \end{bmatrix} \qquad |\mathbf{J}(\xi,\eta)| = 3+\eta
$$

$$
\begin{aligned}
I &= \int_{-1}^{1}\int_{-1}^{1} x(\xi,\eta) y(\xi,\eta)\,|\mathbf{J}(\xi,\eta)|\; d\eta\, d\xi \\
&= \int_{-1}^{1}\int_{-1}^{1} \left[\frac{1}{2}(1+\xi)(3+\eta)\right] [2(1+\eta)]\,(3+\eta)\, d\eta\, d\xi \\
&= \int_{-1}^{1}\int_{-1}^{1} (9 + 15\eta + 7\eta^2 + \eta^3 + 9\xi + 15\xi\eta + 7\xi\eta^2 + \xi\eta^3)\, d\eta\, d\xi \\
&= 4\int_{0}^{1}\int_{0}^{1} (9 + 7\eta^2)\, d\eta\, d\xi = 4\left(9 + \frac{7}{3}\right) = \frac{136}{3} \quad \blacktriangleleft
\end{aligned}
$$

Problem 6

$$
I = \int\int_A x\, dx\, dy
$$

$$
\mathbf{x} = \begin{bmatrix} -1 & 1 & 4 & 0 \end{bmatrix}^T \qquad \mathbf{y} = \begin{bmatrix} 0 & 0 & 3 & 3 \end{bmatrix}^T
$$

$$
\begin{aligned}
x(\xi,\eta) &= \sum_{k=1}^{4} N_k(\xi,\eta) x_k \\
&= \frac{1}{4}(1-\xi)(1-\eta)(-1) + \frac{1}{4}(1+\xi)(1-\eta)(1) + \frac{1}{4}(1+\xi)(1+\eta)(4) \\
&= \frac{1}{2}(2 + 3\xi + 2\eta + \xi\eta) \\
y(\xi,\eta) &= \sum_{k=1}^{4} N_k(\xi,\eta) y_k \\
&= \frac{1}{4}(1+\xi)(1+\eta)(3) + \frac{1}{4}(1-\xi)(1+\eta)(3) = \frac{3}{2}(1+\eta)
\end{aligned}
$$

$$
\mathbf{J}(\xi,\eta) = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} (3+\eta)/2 & 0 \\ (2+\xi)/2 & 3/2 \end{bmatrix} \qquad |\mathbf{J}(\xi,\eta)| = \frac{3}{4}(3+\eta)
$$

$$
\begin{aligned}
I &= \int_{-1}^{1}\int_{-1}^{1} x(\xi,\eta)\,|\mathbf{J}(\xi,\eta)|\;d\eta\,d\xi \\
&= \int_{-1}^{1}\int_{-1}^{1}\left[\frac{1}{2}(2+3\xi+2\eta+\xi\eta)\right]\left[\frac{3}{4}(3+\eta)\right]\,dx\,dy \\
&= \int_{-1}^{1}\int_{-1}^{1}\left(\frac{9}{4}+3\eta+\frac{3}{4}\eta^2+\frac{27}{8}\xi+\frac{9}{4}\xi\eta+\frac{3}{8}\xi\eta^2\right)dx\,dy \\
&= 4\int_{0}^{1}\int_{0}^{1}\left(\frac{9}{4}+\frac{3}{4}\eta^2\right)dy\,dx = 4\left(\frac{9}{4}+\frac{1}{4}\right) = 10 \blacktriangleleft
\end{aligned}
$$

Problem 7

$$\int\int_A x^2 dx\,dy$$

The quadratic triangle formula (3 integration points) is exact for this integral. Referring to Fig. 6.10 in the text, the coordinates of the integration points are

$$x_a = 0 \qquad x_b = x_c = 1.5$$

$$I = A\sum_{k=a}^{c} W_k f(x_k, y_k) = 9\left(\frac{1}{3}\right)(0^2 + 1.5^2 + 1.5^2) = 13.5 \blacktriangleleft$$

Problem 8

$$\int\int_A x^3 dx\,dy$$

We must use the cubic triangle formula for exact result. The corner x-coordinates are

$$\mathbf{x} = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}^T$$

and the x-coordinates of the integration points become

$$
\begin{aligned}
x_a &= \frac{1}{3}(0+3+0) = 1 \\
x_b &= \frac{1}{5}(0+3) + \frac{3}{5}(0) = 0.6 \\
x_c &= \frac{3}{5}(0) + \frac{1}{5}(3+0) = 0.6 \\
x_d &= \frac{1}{5}(0+0) + \frac{3}{5}(3) = 1.8
\end{aligned}
$$

$$
\begin{aligned}
I &= A\sum_{k=a}^{d} W_k f(x_k, y_k) \\
&= 9\left[-\frac{27}{48}(1)^3 + \frac{25}{48}\left(0.6^3 + 0.6^3 + 1.8^3\right)\right] = 24.3 \quad \blacktriangleleft
\end{aligned}
$$

Problem 9

$$
\int\int_A (3-x)y\,dx\,dy
$$

Quadratic triangle formula is exact in this case. The integration points are located at

$$
\begin{aligned}
x_a &= 0 \qquad x_b = x_c = 1.5 \\
y_a &= y_c = 2 \qquad y_b = 0
\end{aligned}
$$

$$
\begin{aligned}
I &= A\sum_{k=a}^{c} W_k f(x_k, y_k) \\
&= 6\left(\frac{1}{3}\right)[(3-0)(2) + (3-1.5)(0) + (3-1.5)(2)] = 18 \quad \blacktriangleleft
\end{aligned}
$$

Problem 10

$$
I = \int\int_A x^2 y\,dx\,dy
$$

$$
\mathbf{x} = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}^T \qquad \mathbf{y} = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}^T
$$

The integrand is cubic, requiring 4 integration points, which are located at

$$
\begin{aligned}
x_a &= \frac{1}{3}(0+3+0) = 1 \\
x_b &= \frac{1}{5}(0+3) + \frac{3}{5}(0) = \frac{3}{5} \\
x_c &= \frac{3}{5}(0) + \frac{1}{5}(3+0) = \frac{3}{5} \\
x_d &= \frac{1}{5}(0+0) + \frac{3}{5}(3) = \frac{9}{5}
\end{aligned}
$$

$$
\begin{aligned}
y_a &= \frac{1}{3}(0+0+4) = \frac{4}{3} \\
y_b &= \frac{1}{5}(0+0) + \frac{3}{5}(4) = \frac{12}{5} \\
y_c &= \frac{3}{5}(0) + \frac{1}{5}(0+4) = \frac{4}{5} \\
y_d &= \frac{1}{5}(0+4) + \frac{3}{5}(0) = \frac{4}{5}
\end{aligned}
$$

$$
\begin{aligned}
I &= A\sum_{k=a}^{d} W_k f(x_k, y_k) \\
&= 6\left\{-\frac{27}{48}(1)^2\frac{4}{3} + \frac{25}{48}\left[\left(\frac{3}{5}\right)^2\frac{12}{5} + \left(\frac{3}{5}\right)^2\frac{4}{5} + \left(\frac{9}{5}\right)^2\frac{4}{5}\right]\right\} \\
&= 7.2 \blacktriangleleft
\end{aligned}
$$

Problem 11

$$
I = \int\int_A f(x,y)\,dx\,dy \qquad f(x,y) = xy(2-x^2)(2-xy)
$$

The integrand $f(x,y)$ is a 4th degree polynomial in x. In addition, $|\mathbf{J}(\xi,\eta)|$ is generally a quadratic, so that the integrand of $\int\int_A f(\xi,\eta)\,|\mathbf{J}(\xi,\eta)|\,d\xi\,d\eta$ is a polynomial of degree 6, requiring 4th-order quadrature ($n = 4$).

```
% problem6_3_11
func = inline('x*y*(2 - x^2)*(2 - x*y)','x','y');
x = [-3 1 3 -1]; y = [-2 -2 2 2];
Integral = gaussQuad2(func,x,y,4)

>> Integral =
   41.8540
```

Problem 12

$$
I = \int\int_A f(x,y)\,dx\,dy \qquad f(x,y) = xy\exp(-x^2)
$$

As $f(x,y)$ is not a polynomial, quadrature is not exact. Of course, the accuracy increases with the order n of integration, but it is difficult to determine beforehand the relationship between n and the error. The following program prompts for n, which helps to determine its proper value by experimentation:

```
% problem6_3_12
x = [-3 1 3 -1]; y = [-2 -2 2 2];
func = inline('x*y*exp(-x^2)','x','y');
while 1
    n = input('Integration order ==> ');
    if isempty(n); fprintf('Done'); break; end
    Integral = gaussQuad2(func,x,y,n)
end

Integration order ==> 6
Integral =
    0.3788
Integration order ==> 8
Integral =
    0.3796
Integration order ==> 10
Integral =
    0.3796
Integration order ==>
Done
```

It seems that $I = 0.3796$ ◀ is achievable with 8th order quadrature.

Problem 13

$$I = \int\int_A f(x,y)\,dx\,dy \qquad f(x,y) = (1-x)(y-x)y$$

The program below uses `triangleQuad` (the cubic integration formulas for a triangle). Because the integrand is a cubic, the result is exact.

```
% problem6_3_13
format short e
x = [0 1 1]'; y = [0 0 1]';
func = inline('(1 - x)*(y - x)*y','x','y');
Integral = triangleQuad(func,x,y)

>> Integral =
 -8.3333e-003
```

Problem 14

$$I = \int\int_A f(x, y)\,dx\,dy \qquad f(x, y) = \sin \pi x$$

The quadrature will not be exact because $f(x, y)$ is not a polynomial.

```
% problem6_3_14
format short e
x = [0 1 1]'; y = [0 0 1]';
func = inline('sin(pi*x)','x','y');
Integral = triangleQuad(func,x,y)

>> Integral =
  3.1024e-001
```

In comparison, the true value of the integral is $I = 0.318\,310$.

Problem 15

$$I = \int\int_A f(x, y)\,dx\,dy \qquad f(x, y) = \sin \pi x \sin \pi (y - x)$$

We used the program below, which prompts for the integration order m, to evaluate the integral with increasing m until the desired 6-digit accuracy was reached).

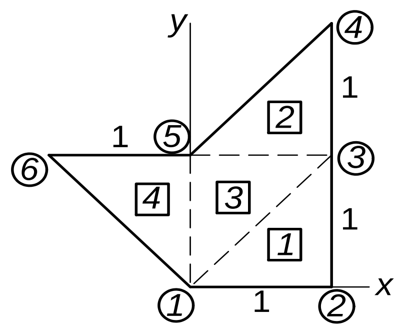
```
% problem6_3_15
format long
x = [0 1 1 1]'; y = [0 0 1 1]';
func = inline('sin(pi*x)*sin(pi*(y - x))','x','y');
Iold = gaussQuad2(func,x,y,2);
for n = 3:20
    Inew = gaussQuad2(func,x,y,n);
    if abs(Iold - Inew) < 1.0e-6; break
    else Iold = Inew; end
end
Integration_order = n
Integral = Inew

>> Integration_order =
     7
Integral =
  -0.20264236754791
```

The last result agrees with the true value of the integral $I = -2/\pi^2 = -0.202\,642$ ◄

Problem 16

The figure shows the numbering of the corner points and the elements. The data used by the program (the arrays `x`, `y` and `cornerID`) are derived from this figure.




```
% problem6_3_16
func = inline('x*y*(y - x)','x','y');
% Coordinates of corner points
x = [0 1 1 1 0 -1]; y = [0 0 1 2 1 1];
% Corner numbers of each element
cornerID = [1 2 3; 3 4 5; 1 3 5; 5 6 1];
% Coordinates of corners of an element
xCorner = zeros(3,1); yCorner = zeros(3,1);
I = 0;
for i = 1:size(cornerID,1)
    for j = 1:3
        xCorner(j) = x(cornerID(i,j));
        yCorner(j) = y(cornerID(i,j));
    end
    I = I + triangleQuad(func,xCorner,yCorner);
end
integral = I

>> integral =
    0.1333
```