1. (a) Diagonally dominant. +5

(b) We should swap the row 1 and row 2.

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Using gauss-seidel with $x^{(0)} = [0 \ 0]^T$

1st Iteration.

$$x_1 = \frac{1}{4}(1 - 1 \cdot 0) = \frac{1}{4}$$

$$x_2 = \frac{1}{3}\left(3 - 1 \cdot \frac{1}{4}\right) = \frac{11}{12}$$

$$x^{(1)} = \begin{bmatrix} \frac{1}{4} \\ \frac{11}{12} \end{bmatrix} \leftarrow \text{Ans}$$

2nd Iteration

$$x_1 = \frac{1}{4}\left(1 - 1 \cdot \frac{11}{12}\right) = \frac{1}{48}$$

$$x_2 = \frac{1}{3}\left(1 - 1 \cdot \frac{1}{48}\right) = \frac{143}{144}$$

$$x^{(2)} = \begin{bmatrix} \frac{1}{48} \\ \frac{143}{144} \end{bmatrix} \leftarrow \text{Ans}$$

+5

(c) SOR with $\omega = 1.5$

1st Iteration.

$$x_1 = \frac{1.5}{4}(1 - 1 \cdot 0) + (1 - 1.5) \cdot 0 = \frac{3}{8}$$

$$x_2 = \frac{1.5}{3}\left(1 - 1 \cdot \frac{3}{8}\right) + (1 - 1.5) \cdot 0 = \frac{21}{16}$$

$$x^{(1)} = \begin{bmatrix} \frac{3}{8} \\ \frac{21}{16} \end{bmatrix} \leftarrow \text{Ans}$$

2nd Iteration.

$$x_1 = \frac{1.5}{4}\left(1 - 1 \cdot \frac{21}{16}\right) + (1 - 1.5)\frac{3}{8} = -\frac{39}{128}$$

$$x_2 = \frac{1.5}{3}\left(3 - 1 \cdot \left(-\frac{39}{128}\right)\right) + (1 - 1.5) \cdot \frac{21}{16} = \frac{255}{256}$$

$$x^{(2)} = \begin{bmatrix} -\frac{39}{128} \\ \frac{255}{256} \end{bmatrix} \leftarrow \text{Ans}$$

+5

(d) ① A is symmetric and positive definite

+5

② 1) Choose x_0

2) $r_0 \leftarrow b - Ax_0$

3) $s_0 \leftarrow r_0$

4) do with $k=0,1,2,\ldots$

$$\alpha_k \leftarrow \frac{s_k^T r_k}{s_k^T A s_k}$$

$$x_{k+1} \leftarrow x_k + \alpha_k s_k$$

$$r_{k+1} \leftarrow b - Ax_{k+1}$$

If $|r_{k+1}| \le \varepsilon$ exit loop.

$$\beta_k \leftarrow -\frac{r_{k+1}^T A s_k}{s_k^T A s_k}$$

$$s_{k+1} \leftarrow r_{k+1} + \beta_k s_k$$

5) end do.

+5

③ $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$r_0 = b - Ax_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$s_0 = r_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\alpha_0 = \frac{s_0^T r_0}{s_0^T A s_0} = \frac{1^2+3^2}{1\cdot 7+3\cdot 10} = \frac{10}{37}$$

$$x_1 = x_0 + \alpha_0 s_0 = \begin{bmatrix} \frac{10}{37} \\ \frac{30}{37} \end{bmatrix}$$ ← Ans

+5

2. (a)

$$k_{i-1} + 4k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \quad i = 1, 2, \ldots, n-1.$$ ← Ans.

+5

(b) $k_0 + 4k_1 + k_2 = \frac{6}{1}(4 - 2\cdot(-1) + 2) = 48$

$k_1 + 4k_2 + k_3 = \frac{6}{1}((-1) - 2\cdot 2 + 1) = -24$ $(k_0 = k_4 = 0)$

$k_2 + 4k_3 + k_4 = \frac{6}{1}(2 - 2\cdot 1 + 8) = 48$

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 48 \\ -24 \\ 48 \end{bmatrix}$$

+5

Using Doolittle's decomposition.

row2 ← row2 $- \frac{1}{4}\cdot$ row1

row3 ← row3 $- 0 \cdot$ row1.

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 3.75 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 48 \\ -36 \\ 48 \end{bmatrix}$$

$row3 \leftarrow row3 - \frac{4}{15} \cdot row2$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 3.75 & 1 \\ 0 & 0 & \frac{56}{15} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 48 \\ -36 \\ 57.6 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{15} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{15}{4} & 1 \\ 0 & 0 & \frac{56}{15} \end{bmatrix}$$ ← Ans.

+10

(C) $L\underbrace{Ux}_{y} = b. \Rightarrow Ly = b.$

forward Substitution.

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{15} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 48 \\ -24 \\ 48 \end{bmatrix} \rightarrow y = \begin{bmatrix} 48 \\ -36 \\ 57.6 \end{bmatrix}.$$

+4

backward substitution.

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 15/4 & 1 \\ 0 & 0 & 56/15 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 48 \\ -36 \\ 57.6 \end{bmatrix} \rightarrow k = \begin{bmatrix} 108/7 \\ -96/7 \\ 108/7 \end{bmatrix}$$ ← Ans

+4

(d) $f_{3,4}(x) = \frac{108}{7} \cdot \frac{1}{6}\left[-(x-2)^3 + (x-2)\right] - \left((x-2) - 8(x-1)\right)$ ← Ans.

+2

3. (a) $r_i = w_i[y_i - f(x_i)]$ +5

(b) $f(x) = a + bx$

$$S(a,b) = \sum_{i=1}^{n} w_i^2 (y_i - a - bx_i)^2$$ +5

(c) $$\frac{\partial S}{\partial a} = -2\sum_{i=1}^{n} w_i^2 (y_i - a - bx_i) = 0$$

$$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n} w_i^2 (y_i - a - bx_i) x_i = 0$$

$$\Downarrow$$

$$a\sum_{i=1}^{n} w_i^2 + b\sum_{i=1}^{n} w_i^2 x_i = \sum_{i=1}^{n} w_i^2 y_i$$

$$a\sum_{i=1}^{n} w_i^2 x_i + b\sum_{i=1}^{n} w_i^2 x_i^2 = \sum_{i=1}^{n} w_i^2 x_i y_i$$

$$\hat{x} = \frac{\sum w_i^2 x_i}{\sum w_i^2}, \quad \hat{y} = \frac{\sum w_i^2 y_i}{\sum w_i^2}$$

$$a = \hat{y} - b\hat{x}, \quad b = \frac{\sum w_i^2 y_i (x_i - \hat{x})}{\sum w_i^2 x_i (x_i - \hat{x})}$$

$$a = \frac{\sum w_i^2 y_i}{\sum w_i^2} - \frac{\sum w_i^2 y_i (x_i - \hat{x})}{\sum w_i^2 x_i (x_i - \hat{x})} \cdot \frac{\sum w_i^2 x_i}{\sum w_i^2}$$ ← Ans. +10.

(d) $f(x) = ae^{bx}$

$\ln(ae^{bx}) = \ln a + bx$, $z = \ln y$. $w_i = y_i$.

$$\hat{x} = \frac{\sum y_i^2 x_i}{\sum y_i^2} = \frac{737.5\times10^3}{98.67\times10^3} = 7.474$$

$$\hat{y} = \frac{\sum y_i^2 z_i}{\sum y_i^2} = \frac{528.2\times10^3}{98.67\times10^3} = 5.353$$

$$b = \frac{\sum y_i^2 z_i (x_i - \hat{x})}{\sum y_i^2 x_i (x_i - \hat{x})} = \frac{35.39\times10^3}{65.05\times10^3} = 0.5440.$$ ← Ans +4

$\ln a = \bar{z} - b\bar{x} = 5.353 - 0.5440 \cdot 7.477 = 1.287.$

$\therefore\ a = e^{\ln a} = e^{1.287} = 3.622.$ +4 ← Ans

e)

$\therefore\ f(x) = 3.622\, e^{0.544026}$

y	7.50	16.10	38.90	67.00	146.60	266.20
$f(x)$	6.96	16.61	37.56	68.33	146.33	266.20
$y-f(x)$	0.54	-0.51	1.34	-1.33	0.267	0.00

$S = \sum [y_i - f(x_i)]^2 = 4.186.$

$\sigma = \sqrt{\dfrac{S}{n-m}} = \sqrt{\dfrac{4.186}{6-2}} = 1.023$ ← Ans.

+2

4.

a) $$F(x,y,z) = \begin{pmatrix} -x+\sin(x+y)+9 \\ 8y-\cos^2(z-y)-1 \\ 12z+\sin z-1 \end{pmatrix}$$ +5

$$J(x,y,z) = \begin{pmatrix} -1+\cos(x+y) & \cos(x+y) & 0 \\ 0 & 8-2\cos(z-y)\sin(z-y) & 2\cos(z-y)\sin(z-y) \\ 0 & 0 & 12+\cos z \end{pmatrix}$$

b) +5

1. Estimate the solution vector x
2. Evaluate $f(x)$
3. Compute the Jacobian matrix $J(x)$ from $J_{ij} = \frac{\partial f_i}{\partial x_j}$
4. Set up the simultaneous equations in $J(x)\Delta x = -F(x)$ and solve for Δx
5. Let $x \leftarrow x+\Delta x$ and repeat steps 2-5

+10

c) $$x^{(0)} = (\underset{x_0}{0.1}, \underset{y_0}{0.25}, \underset{z_0}{0.08}) \qquad J(x)\Delta x = -F(x)$$

$$\begin{pmatrix} -1+\cos(x+y) & \cos(x+y) & 0 \\ 0 & 8-2\cos(z-y)\sin(z-y) & 2\cos(z-y)\sin(z-y) \\ 0 & 0 & 12+\cos z \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} x-\sin(x+y)-9 \\ -8y+\cos^2(z-y)+1 \\ -12z-\sin z+1 \end{pmatrix}$$

substitute x, y, z, and calculate $\Delta x, \Delta y, \Delta z$

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 152.4 \\ -3.558\times10^{-3} \\ -3.071\times10^{-3} \end{pmatrix}$$

step 5

$$x^{(1)} = \begin{pmatrix} x_0+\Delta x \\ y_0+\Delta y \\ z_0+\Delta z \end{pmatrix} = \begin{pmatrix} 152.5 \\ 0.2464 \\ 0.07693 \end{pmatrix} \longleftarrow \text{ans}$$

+10

5.

a)

$$9\times(f_{i+1}) = 9\times(f_i + hf_i' + \cancel{\frac{h^2}{2}f_i''} + \frac{h^3}{6}f_i''' + \cdots)$$

$$-)\quad f_{i+3} = f_i + 3hf_i' + \cancel{\frac{(3h)^2}{2}f_i''} + \frac{(3h)^3}{6}f_i''' + \cdots$$

$$9f_{i+1} - f_{i+3} = 8f_i + 6hf_i' + O(h^3)$$

$$\frac{26}{6} = \boxed{\frac{13}{3}h^3 \cdot f_i'''}$$

$$f_i' = \frac{-f_{i+3} + 9f_{i+1} - 8f_i}{6h} + O(h^2)$$

$$+\frac{13}{18}h^2 f_i''' + O(h^3)$$

+10

$\therefore$ The truncation error is $O(h^2)$ ← ans

b)

$$f_{i+1} = f_i + \cancel{hf_i'} + \frac{h^2}{2}f_i'' + \cancel{\frac{h^3}{6}f_i'''} + \frac{h^4}{24}f_i'''' + \cdots$$

$$+)\quad f_{i-1} = f_i - \cancel{hf_i'} + \frac{h^2}{2}f_i'' - \cancel{\frac{h^3}{6}f_i'''} + \frac{h^4}{24}f_i'''' + \cdots \quad ①$$

$$f_{i+2} = f_i + \cancel{2hf_i'} + \frac{(2h)^2}{2}f_i'' + \cancel{\frac{(2h)^3}{6}f_i'''} + \frac{(2h)^4}{24}f_i'''' + \cdots$$

$$+)\quad f_{i-2} = f_i - \cancel{2hf_i'} + \frac{(2h)^2}{2}f_i'' - \cancel{\frac{(2h)^3}{6}f_i'''} + \frac{(2h)^4}{24}f_i'''' + \cdots \quad ②$$

$$① \quad f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} - \frac{h^2}{12}f_i'' + O(h^4)$$

$$② \quad f_i'' = \frac{f_{i+2} - 2f_i + f_{i-2}}{(2h)^2} - \frac{(2h)^2}{12}f_i'' + O(h^4)$$

$\alpha ① + (1-\alpha) ②$

$$\Rightarrow f_i'' = \alpha\left(\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}\right) + (1-\alpha)\left(\frac{f_{i+2} - 2f_i + f_{i-2}}{(2h)^2}\right)$$

$$\underbrace{-\alpha\frac{h^2}{12}f_i'' - (1-\alpha)\frac{(2h)^2}{12}f_i''}_{\text{should be zero to maximize the accuracy.}} + O(h^4)$$

$$\alpha\cancel{\frac{h^2}{12}}\cancel{f_i''} + (1-\alpha)\frac{\cancel{(2h)^2}}{\cancel{12}}\cancel{f_i''} = 0$$

$$\alpha + (1-\alpha)4 = 0 \qquad \therefore \alpha = \frac{4}{3}$$ ← ans

+10

c)

$$2\times(f_{i-1}) = \cancel{2\times}f_i - \cancel{hf_i'} + \frac{h^2}{2}f_i'' - \frac{h^3}{6}f_i''' + \cdots)$$

$$+)\quad f_{i+1} = f_i + \cancel{2hf_i'} + \frac{(2h)^2}{2}f_i'' + \frac{(2h)^3}{6}f_i''' + \cdots$$

$$2f_{i-1} + f_{i+1} = 3f_i + 3h^2 f_i'' + O(h^3)$$

$$\therefore\ f_i'' = \frac{f_{i+1} - 3f_i + 2f_{i-1}}{3h^2} + O(h) \leftarrow \text{ans}$$

+10.