PROBLEM SET 3.1

Problem 1

(a)

i	1	2	3
x_i	-1.2	0.3	1.1
$y_i = P_0[x_i]$	-5.76	-5.61	-3.69

At $x = 0$:

$$\begin{aligned} P_1[x_1, x_2] &= \frac{(0 - x_2)P_0[x_1] + (x_1 - 0)P[x_2]}{x_1 - x_2} \\ &= \frac{(-0.3)(-5.76) + (-1.2)(-5.61)}{-1.2 - 0.3} = -5.64 \end{aligned}$$

$$\begin{aligned} P_1[x_2, x_3] &= \frac{(0 - x_3)P_0[x_2] + (x_2 - 0)P_0[x_3]}{x_2 - x_3} \\ &= \frac{(-1.1)(-5.61) + 0.3(-3.69)}{0.3 - 1.1} = -6.33 \end{aligned}$$

$$\begin{aligned} P_2[x_1, x_2, x_3] &= \frac{(0 - x_3)P_1[x_1, x_2] + (x_1 - 0)P_1[x_2, x_3]}{x_0 - x_2} \\ &= \frac{(-1.1)(-5.64) + (-1.2)(-6.33)}{-1.2 - 1.1} = -6.0 \ \blacktriangleleft \end{aligned}$$

(b)

$$\begin{aligned} \ell_1(0) &= \frac{(0 - x_2)(0 - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(-0.3)(-1.1)}{(-1.2 - 0.3)(-1.2 - 1.1)} = 0.0957 \\ \ell_2(0) &= \frac{(0 - x_1)(0 - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{1.2(-1.1)}{[0.3 - (-1.2)]\,(0.3 - 1.1)} = 1.1000 \\ \ell_3(0) &= \frac{(0 - x_1)(0 - x_2)}{(x_3 - x_1)(x_3 - x_2)} = \frac{-1.2(-0.3)}{[1.1 - (-1.2)]\,(1.1 - 0.3)} = -0.1957 \end{aligned}$$

$$\begin{aligned} P_2(0) &= \sum_{i=1}^{3} y_i \ell_i(0) \\ &= -5.76(0.0957) + (-5.61)(1.1000) + (-3.69)(-0.1957) = 6.0 \ \blacktriangleleft \end{aligned}$$

Problem 2

i	1	2	3	4	5	6	7
x_i	0	0.5	1	1.5	2	2.5	3
y_i	1.8421	2.4694	2.4921	1.9047	0.8509	-0.4112	-1.5727

(a)

This is inverse interpolation: find x where $y = 0$. Using points 5-7:

$$\begin{aligned}
\ell_5(0) &= \frac{(0-y_6)(0-y_7)}{(y_5-y_6)(y_5-y_7)} \\
&= \frac{0.4112(1.5727)}{(0.8509-(-0.4112))(0.8509-(-1.5727))} = 0.2114 \\
\ell_6(0) &= \frac{(0-y_5)(0-y_7)}{(y_6-y_5)(y_6-y_7)} \\
&= \frac{-0.8509(1.5727)}{(-0.4112-0.8509)\,(-0.4112-(-1.5727))} = 0.9129 \\
\ell_7(0) &= \frac{(0-y_5)(0-y_6)}{(y_7-y_5)(y_7-y_6)} \\
&= \frac{-0.8509(0.4112)}{(-1.5727-0.8509)\,(-1.5727-(-0.4112))} = -0.1243 \\
x|_{y=0} &= \sum_{i=5}^{7} x_i \ell_i(0) \\
&= 2(0.2114) + 2.5(0.9129) + 3(-0.1243) = 2.332 \quad \blacktriangleleft
\end{aligned}$$

(b)

Use points 4-7:

$$\begin{aligned}
\ell_4(0) &= \frac{(0-y_5)(0-y_6)(0-y_7)}{(y_4-y_5)(y_4-y_6)(y_4-y_7)} \\
&= \frac{-0.8509(0.4112((1.5727)}{(1.9047-0.8509)\,(1.9047-(-0.4112))\,(1.9047-(-1.5727))} \\
&= -0.0648 \\
\ell_5(0) &= \frac{0-y_4}{y_5-y_4}\,[\ell_5(0)]_{\text{3-point}} = \frac{-1.9047}{0.8509-1.9047}(0.2114) = 0.3821 \\
\ell_6(0) &= \frac{0-y_4}{y_6-y_4}\,[\ell_6(0)]_{\text{3-point}} = \frac{-1.9047}{-0.4112-1.9047}(0.9129) = 0.7508 \\
\ell_7(0) &= \frac{0-y_4}{y_7-y_4}\,[\ell_7(0)]_{\text{3-point}} = \frac{-1.9129}{-1.5727-1.9129}(-0.1243) = -0.0682 \\
x\big|_{y=0} &= \sum_{i=4}^{7} x_i \ell_i(0) \\
&= 1.5(-0.0648) + 2(0.3821) + 2.5(0.7508) + 3(-0.0682) = 2.339 \quad \blacktriangleleft
\end{aligned}$$

Problem 3

Interpolationg at $x = 0.7679$:

$$\begin{aligned}
P_1[x_1,x_2] &= \frac{(x-x_2)P_0[x_1] + (x_1-x)P_0[x_2]}{x_1-x_2} \\
&= \frac{(0.7679-0.5)(1.8421) + (0-0.7679)(2.4694)}{0-0.5} = 2.8055 \\
P_1[x_2,x_3] &= \frac{(x-x_3)P_0[x_2] + (x_2-x)P_0[x_3]}{x_2-x_3} \\
&= \frac{(0.7679-1.0)(2.4694) + (0.5-0.7679)(2.4921)}{0.5-1.0} = 2.4816 \\
P_1[x_3,x_4] &= \frac{(x-x_4)P_0[x_3] + (x_3-x)P_0[x_4]}{x_3-x_4} \\
&= \frac{(0.7679-1.5)(2.4921) + (1.0-0.7679)(1.9047)}{1.0-1.5} = 2.7648
\end{aligned}$$

$$
\begin{aligned}
P_2[x_1,x_2,x_3] &= \frac{(x-x_3)P_1[x_1,x_2]+(x_1-x)P_1[x_2,x_3]}{x_1-x_3} \\
&= \frac{(0.7679-1.0)(2.8055)+(0-0.7679)(2.4816)}{0-1.0} = 2.5568 \\
P_2[x_2,x_3,x_4] &= \frac{(x-x_4)P_1[x_2,x_3]+(x_2-x)P_1[x_3,x_4]}{x_2-x_4} \\
&= \frac{(0.7679-1.5)(2.4816)+(0.5-0.7679)(2.7648)}{0.5-1.5} = 2.5575
\end{aligned}
$$

$$
\begin{aligned}
y_{\max} &= P_3[x_1,x_2,x_3,x_4] = \frac{(x-x_4)P_2[x_1,x_2,x_3]+(x_1-x)P_2[x_2,x_3,x_4]}{x_1-x_4} \\
&= \frac{(0.7679-1.5)(2.5568)+(0-0.7679)(2.5575)}{0-1.5} = 2.5572 \quad \blacktriangleleft
\end{aligned}
$$

Problem 4

Interpolating at $x = 0.25\pi$:

$$\begin{aligned}
P_1[x_1, x_2] &= \frac{(x - x_2)P_0[x_1] + (x_1 - x)P_0[x_2]}{x_1 - x_2} \\
&= \frac{(0.25\pi - 0.5))(-1.0) + (0 - 0.25\pi)(1.75)}{0 - 0.5} = 3.3197 \\
P_1[x_2, x_3] &= \frac{(x - x_3)P_0[x_2] + (x_2 - x)P_0[x_3]}{x_2 - x_3} \\
&= \frac{(0.25\pi - 1.0)(1.75) + (0.5 - 0.25\pi)(4.0)}{0.5 - 1.0} = 3.0343 \\
P_1[x_3, x_4] &= \frac{(x - x_4)P_0[x_3] + (x_3 - x)P_0[x_4]}{x_3 - x_4} \\
&= \frac{(0.25\pi - 1.5)4.0) + (1.0 - 0.25\pi)(5.75)}{1.0 - 1.5} = 3.2489 \\
P_1[x_4, x_5] &= \frac{(x - x_5)P_0[x_4] + (x_4 - x)P_0[x_5]}{x_4 - x_5} \\
&= \frac{(0.25\pi - 2.0)(5.75) + (1.5 - 0.25\pi)(7.0)}{1.5 - 2.0} 3.9635
\end{aligned}$$

$$\begin{aligned}
P_2[x_1, x_2, x_3] &= \frac{(x - x_3)P_1[x_1, x_2] + (x_1 - x)P_1[x_2, x_3]}{x_1 - x_3} \\
&= \frac{(0.25\pi - 1.0)(3.3197) + (0 - 0.25\pi)(3.0343)}{0 - 1.0} = 3.0955 \\
P_2[x_2, x_3, x_4] &= \frac{(x - x_4)P_1[x_2, x_3] + (x_2 - x)P_1[x_3, x_4]}{x_2 - x_4} \\
&= \frac{(0.25\pi - 1.5)(3.0343) + (0.5 - 0.25\pi)(3.2489)}{0.5 - 1.5} = 3.0955 \\
P_2[x_3, x_4, x_5] &= \frac{(x - x_5)P_1[x_3, x_4] + (x_4 - x)P_1[x_4, x_5]}{x_3 - x_5} \\
&= \frac{(0.25\pi - 2.0)(3.2489) + (1.0 - 0.25\pi)(3.9635)}{1.0 - 2.0} = 3.0955
\end{aligned}$$

There is no need to go futher. The tabulated function is clearly a quadratic (interpolating over any three points gives the same result). Hence $y(0.25\pi) = 3.0955$ ◄

Problem 5

Use Newton's method. The formulas

$$\begin{aligned}
\nabla y_i &= \frac{y_i - y_1}{x_i - x_1} \qquad \nabla^2 y_i = \frac{\nabla y_i - \nabla y_2}{x_i - x_2} \\
\nabla^3 y_i &= \frac{\nabla^2 y_i - \nabla^2 y_3}{x_i - x_3} \qquad \nabla^4 y_i = \frac{\nabla^3 y_i - \nabla^3 y_4}{x_i - x_4}
\end{aligned}$$

yield the following tableau:

i	x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
1	0	−0.7854				
2	0.5	0.6529	2.8766			
3	1.0	1.7390	2.5244	−0.7043		
4	1.5	2.2071	1.9950	−0.8816	−0.3546	
5	2.0	1.9425	1.3640	−1.0084	−0.3041	0.1009

The diagonal terms in the tableau are the coefficients of Newton's polynomial. We evaluate this polynomial at $x = 0.25\pi$ with the recurrence relations

$$\begin{aligned}
P_0(0.25\pi) &= 0.1009 \\
P_1(0.25\pi) &= -0.3546 + (0.25\pi - 1.5)(0.1009) = -0.4267 \\
P_2(0.25\pi) &= -0.7043 + (0.25\pi - 1.0)(-0.4267) = -0.6127 \\
P_3(0.25\pi) &= 2.8766 + (0.25\pi - 0.5)(-0.6127) = 2.7017 \\
P_4(0.25\pi) &= y|_{0.25\pi} = -0.7854 + (0.25\pi - 0)(2.7017) = 1.3365 \quad \blacktriangleleft
\end{aligned}$$

At $x = 0.5\pi$ the recurrence relations are

$$\begin{aligned}
P_0(0.5\pi) &= 0.1009 \\
P_1(0.5\pi) &= -0.3546 + (0.5\pi - 1.5)(0.1009) = -0.3475 \\
P_2(0.5\pi) &= -0.7043 + (0.5\pi - 1.0)(-0.3475) = -0.9027 \\
P_3(0.5\pi) &= 2.8766 + (0.5\pi - 0.5)(-0.9027) = 1.9100 \\
P_4(0.5\pi) &= y|_{0.5\pi} = -0.7854 + (0.5\pi - 0)(1.9100) = 2.2148 \quad \blacktriangleleft
\end{aligned}$$

Problem 6

The divided difference table is

i	x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$	$\nabla^5 y_i$
1	-2	-1					
2	1	2	1				
3	4	59	10	3			
4	-1	4	5	-2	1		
5	3	24	5	2	1	0	
6	4	-53	26	-5	1	0	0

The last nonzero diagonal term $\nabla^3 y_4$ is the coefficient of the cubic term in Newton's polynomial. Therefore, the data points lie on a *cubic* ◄.

Problem 7

Constructing the divided difference table:

i	x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
1	-3	0				
2	2	5	1			
3	-1	-4	-2	1		
4	3	12	2	1	0	
5	1	0	0	1	0	0

Hence the polynomial is

$$\begin{aligned}
P_2(x) &= a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) \\
&= 0 + [x - (-3)] + [x - (-3)]\,(x - 2) \\
&= (x + 3)(1 + x - 2) = (x + 3)(x - 1) \quad ◄
\end{aligned}$$

Problem 8

i	1	2	3
x_i	-1	1	3
$y_i = P_0[x_i]$	17	-7	-15

$$
\begin{aligned}
P_1[x_1, x_2] &= \frac{(x - x_2)P_0[x_1] + (x1 - x)P_0[x_2]}{x_1 - x_2} \\
&= \frac{(x-1)(17) + (-1-x)(-7)}{-1-1} = -12x + 5
\end{aligned}
$$

$$
\begin{aligned}
P_1[x_2, x_3] &= \frac{(x - x_3)P_0[x_2] + (x_2 - x)P_0[x_3]}{x_2 - x_3} \\
&= \frac{(x-3)(-7) + (1-x)(-15)}{1-3} = -4x - 3
\end{aligned}
$$

$$
\begin{aligned}
P_2[x_1, x_2, x_3] &= \frac{(x - x_3)P_1[x_1, x_2] + (x_1 - x)P_1[x_2, x_3]}{x_1 - x_3} \\
&= \frac{(x-3)(-12x+5) + (-1-x)(-4x-3)}{-1-3} \\
&= 2x^2 - 12x + 3 \blacktriangleleft
\end{aligned}
$$

Problem 9

i	1	2	3
h_i (km)	0	3	6
ρ_i (kg/m^3)	1.225	0.905	0.652

$$
\begin{aligned}
\ell_1 &= \frac{(h-h_2)(h-h_3)}{(h_1-h_2)(h_1-h_3)} = \frac{(h-3)(h-6)}{(0-3)(0-6)} = \frac{(h-3)(h-6)}{18} \\
\ell_2 &= \frac{(h-h_1)(h-h_3)}{(h_2-h_1)(h_2-h_3)} = \frac{(h-0)(h-6)}{(3-0)(3-6)} = -\frac{h(h-6)}{9} \\
\ell_3 &= \frac{(h-h_1)(h-h_2)}{(h_3-h_1)(h_3-h_2)} = \frac{(h-0)(h-3)}{(6-0)(6-3)} = \frac{h(h-3)}{18}
\end{aligned}
$$

$$
\begin{aligned}
\rho(h) &= \sum_{i=0}^{2} \rho_i \ell_i \\
&= 1.225\frac{(h-3)(h-6)}{18} - 0.905\frac{h(h-6)}{9} + 0.652\frac{h(h-3)}{18} \\
&= 0.003722h^2 - 0.1178h + 1.225 \blacktriangleleft
\end{aligned}
$$

Problem 10

i	1	2	3
x_i	0	1	2
y_i	0	2	1

For natural spline we have $k_1 = k_3 = 0$. The equation for k_1 is

$$\begin{aligned} k_1 + 4k_2 + k_3 &= \frac{6}{h^2}(y_1 - 2y_2 + y_3) \\ 0 + 4k_2 + 0 &= \frac{6}{1^2}\left[0 - 2(2) + 1\right] \qquad k_2 = -4.5 \end{aligned}$$

The interpolant in $0 \le x \le 1$ is

$$\begin{aligned} f_{1,2}(x) &= -\frac{k_2}{6}\left[\frac{(x-x_1)^3}{x_1 - x_2} - (x - x_1)(x_1 - x_2)\right] \\ &\quad + \frac{y_1(x - x_2) - y_1(x - x_1)}{x_1 - x_2} \\ &= \frac{4.5}{6}\left(\frac{(x-0)^3}{0-1} - (x-0)(0-1)\right) + \frac{0 - 2(x-0)}{0-1} \\ &= -0.75x^3 + 2.75x \quad \blacktriangleleft \end{aligned}$$

The interpolant in $1 \le x \le 2$ is

$$\begin{aligned} f_{2,3}(x) &= \frac{k_2}{6}\left[\frac{(x-x_3)^3}{x_2 - x_3} - (x - x_3)(x_2 - x_3)\right] \\ &\quad + \frac{y_2(x - x_3) - y_3(x - x_2)}{x_2 - x_3} \\ &= -\frac{4.5}{6}\left(\frac{(x-2)^3}{1-2} - (x-2)(1-2)\right) + \frac{2(x-2) - (x-1)}{1-2} \\ &= 0.75(x-2)^3 - 1.75x + 4.5 \quad \blacktriangleleft \end{aligned}$$

Check:

$$\begin{aligned} f'_{1,2}(x) &= -3(0.75)x^2 + 2.75 = -2.25x^2 + 2.75 \\ f'_{2,3}(x) &= 3(0.75)(x-2)^2 - 1.75 = 2.25(x-2)^2 - 1.75 \end{aligned}$$

$$\begin{aligned} f'_{1,2}(1) &= -2.25(1)^2 + 2.75 = 0.5 \\ f'_{2,3}(1) &= 2.25(1-2)^2 - 1.75 = 0.5 \qquad \text{O.K.} \end{aligned}$$

$$\begin{aligned} f''_{1,2}(1) &= -2.25(2) = -4.5 \\ f''_{2,3}(1) &= 2.25(2)(1-2) = -4.5 \qquad \text{O.K.} \end{aligned}$$

Problem 11

i	1	2	3	4	5
x_i	1	2	3	4	5
y_i	13	15	12	9	13

For equally spaced knots, the equations for the curvatures are

$$k_{i-1} + 4k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \quad i = 2, 3, 4$$

Noting that $k_1 = k_5$ and $h = 1$, we get

$$\begin{aligned}
4k_2 + k_3 &= 6\,[13 - 2(15) + 12] = -30 \\
k_2 + 4k_3 + k_4 &= 6\,[15 - 2(12) + 9] = 0 \\
k_3 + 4k_4 &= 6\,[12 - 2(9) + 13] = 42
\end{aligned}$$

Solution of these equations is

$$k_2 = -7.286 \qquad k_3 = -0.857 \qquad k_4 = 10.714$$

The interpolant between knots 2 and 3 is

$$\begin{aligned}
f_{3,4}(x) &= \frac{k_3}{6}\left[\frac{(x - x_4)^3}{x_3 - x_4} - (x - x_4)(x_3 - x_4)\right] \\
&\quad - \frac{k_4}{6}\left[\frac{(x - x_3)^3}{x_3 - x_4} - (x - x_3)(x_3 - x_4)\right] \\
&\quad + \frac{y_3(x - x_4) - y_4(x - x_3)}{x_3 - x_4}
\end{aligned}$$

Hence

$$\begin{aligned}
f_{3,4}(3.4) &= \frac{-0.857}{6}\left[\frac{(3.4 - 4)^3}{3 - 4} - (3.4 - 4)(3 - 4)\right] \\
&\quad - \frac{10.714}{6}\left[\frac{(3.4 - 3)^3}{3 - 4} - (3.4 - 3)(3 - 4)\right] \\
&\quad + \frac{12(3.4 - 4) - 9(3.4 - 3)}{3 - 4} \\
&= .0\,54848 - .\,59998\,4 + 10.8 = 10.255 \ \blacktriangleleft
\end{aligned}$$

Problem 12

After reordering, the data are

i	1	2	3	4	5
x_i	1.0	0.8	0.6	0.4	0.2
y_i	−1.049	−0.266	0.377	0.855	1.150

The equations for the curvatures at the interior knots are (note that the roles of x and y are interchanged and $k_1 = k_5 = 0$):

$$\begin{aligned} & k_{i-1}(y_{i-1}-y_i) + 2k_i(y_{i-1}-y_{i+1}) + k_{i+1}(y_i - y_{i+1}) \\ = \; & 6\left(\frac{x_{i-1}-x_i}{y_{i-1}-y_i} - \frac{x_i - x_{i+1}}{y_i - y_{i+1}}\right), \quad i = 2,3,4 \end{aligned}$$

These are simultaneous equations $\mathbf{Ak} = \mathbf{b}$, where $\mathbf{k} = \begin{bmatrix} k_2 & k_3 & k_4 \end{bmatrix}^T$ and

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 2\,(-1.049 - 0.377) & -0.266 - 0.377 & 0 \\ -0.266 - 0.377 & 2\,(-0.266 - 0.855) & 0.377 - 0.855 \\ 0 & 0.377 - 0.855 & 2(0.377 - 1.150) \end{bmatrix} \\ &= \begin{bmatrix} -2.852 & -0.643 & 0 \\ -0.643 & -2.242 & -0.478 \\ 0 & -0.478 & -1.546 \end{bmatrix} \end{aligned}$$

$$\mathbf{b} = 6\begin{bmatrix} \dfrac{1.0-0.8}{-1.049-(-0.266)} - \dfrac{0.8-0.6}{-0.266-0.377} \\ \dfrac{0.8-0.6}{-0.266-0.377} - \dfrac{0.6-0.4}{0.377-0.855} \\ \dfrac{0.6-0.4}{0.377-0.855} - \dfrac{0.4-0.2}{0.855-1.150} \end{bmatrix} = \begin{bmatrix} 0.3337 \\ 0.6442 \\ 1.5573 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} -0.1069 \\ -0.0449 \\ -0.9934 \end{bmatrix}$$

The interpolant between knots 2 and 3 is

$$\begin{aligned} f_{2,3}(y) = \; & \frac{k_2}{6}\left[\frac{(y-y_3)^3}{(y_2-y_3)} - (y-y_3)(y_2-y_3)\right] \\ & -\frac{k_3}{6}\left[\frac{(y-y_2)^3}{(y_2-y_3)} - (y-y_2)(y_2-y_3)\right] \\ & +\frac{x_2(y-y_3) - x_3(y-y_2)}{y_2-y_3} \end{aligned}$$

Evaluating at $y = 0$:

$$\begin{aligned} f_{2,3}(0) = \; & \frac{-0.1069}{6}\left(\frac{(-0.377)^3}{-0.266-0.377} + 0.377(-0.266-0.377)\right) \\ & +\frac{0.0449}{6}\left(\frac{(0.266)^3}{-0.266-0.377} - 0.266(-0.266-0.377)\right) \\ & +\frac{0.8(-0.377) - 0.6(0.266)}{-0.266-0.377} \\ = \; & 0.0028 + 0.0011 + 0.7173 = 0.7212 \blacktriangleleft \end{aligned}$$

Problem 13

i	1	2	3	4
x	0	1	2	3
y	1	1	0.5	0

With evenly spaced knots, the equations for the curvatures are

$$k_{i-1} + 4k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \quad i = 2, 3$$

With $k_1 = k_2$, $k_4 = k_3$ and $h = 1$ these equations are

$$\begin{aligned} 5k_2 + k_3 &= 6(1 - 2(1) + 0.5) = -3 \\ k_2 + 5k_3 &= 6\left[1 - 2(0.5) + 0\right] = 0 \end{aligned}$$

The solution is $k_2 = -5/8$, $k_3 = 1/8$. The interpolant can now be evaluated from

$$\begin{aligned} f_{i,i+1}(x) &= \frac{k_i}{6}\left[\frac{(x - x_{i+1})^3}{x_i - x_{i+1}} - (x - x_{i+1})(x_i - x_{i+1})\right] \\ &\quad -\frac{k_{i+1}}{6}\left[\frac{(x - x_i)^3}{x_i - x_{i+1}} - (x - x_i)(x_i - x_{i+1})\right] \\ &\quad +\frac{y_i(x - x_{i+1}) - y_{i+1}(x - x_i)}{x_i - x_{i+1}} \end{aligned}$$

Substituting $x_i - x_{i+1} = -1$ and $i = 3$, this reduces to

$$\begin{aligned} f_{3,4}(x) &= \frac{k_3}{6}\left[-(x - x_4)^3 + (x - x_{4)}\right] - \frac{k_4}{6}\left[-(x - x_3)^3 + (x - x_3)\right] \\ &\quad -y_3(x - x_4) + y_4(x - x_3) \end{aligned}$$

Therefore,

$$\begin{aligned} f_{3,4}(2.6) &= \frac{1/8}{6}\left[-(2.6 - 3)^3 + (2.6 - 3)\right] - \frac{1/8}{6}\left[-(2.6 - 2)^3 + (2.6 - 2)\right] \\ &\quad -0.5(2.6 - 3) + 0 \\ &= 0.185 \blacktriangleleft \end{aligned}$$

Problem 14

This program keeps prompting for x. To break the cycle, press the 'return' key.

```
% problem3_1_14
xData = [-2.0 -0.1 -1.5 0.5 -0.6 2.2 1.0 1.8];
yData = [2.2796 1.0025 1.6467 1.0635 1.0920...
         2.6291 1.2661 1.9896];
while 1
    x = input('x ==> ');
    if isempty(x); fprintf('Done'); break; end
    y = neville(xData,yData,x)
end
```

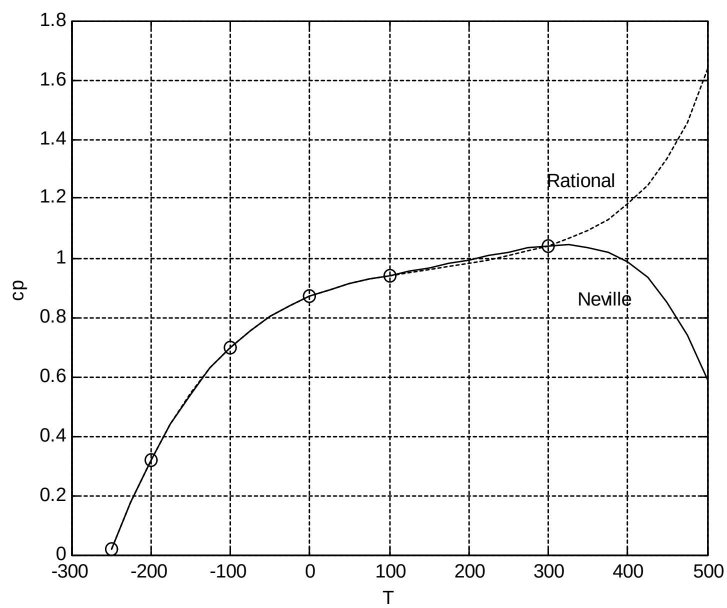
```
x ==> 1.1
y =
    1.3262
x ==> 1.2
y =
    1.3938
x ==> 1.3
y =
    1.4693
x ==>
Done
```

Problem 15

In the program listed below, Neville's method was chosen for the polynomial interpolant.

```
% problem3_1_15 (Rational interpolation)
xData = [-250 -200 -100 0 100 300];
yData = [0.0163 0.318 0.699 0.870 0.941 1.04];
i = 0;
for T = -250:25:500
    i = i + 1;
    temp(i)= T;
    cp_neville(i) = neville(xData,yData,T);
    cp_rational(i) = rational(xData,yData,T);
end
```

```
plot(xData,yData,'ko')
hold on
plot(temp,cp_neville,'k-')
hold on
plot(temp,cp_rational,'k:')
grid on
xlabel('T'); ylabel('cp')
gtext('Neville')
gtext('Rational')
```



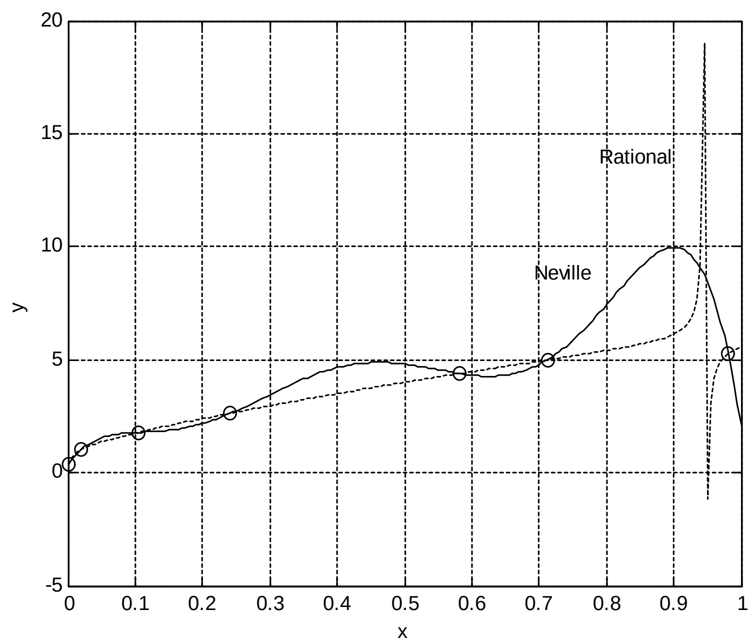
The two interpolants are equally satisfactory for interpolation, but fail in extrapolation.

Problem 16

Neville's algorithm was chosen for the polynomial interpolation in the program listed below.

```
% problem3_1_16 (Rational interpolation)
xData = [0 0.0204 0.1055 0.241 0.582 0.712 0.981];
yData = [0.385 1.04 1.79 2.63 4.39 4.99 5.27];
i = 0;
for t = 0:0.005:1
    i = i + 1;
```

```
    x(i)= t;
    y_neville(i) = neville(xData,yData,t);
    y_rational(i) = rational(xData,yData,t);
end
plot(xData,yData,'ko')
hold on
plot(x,y_neville,'k-')
hold on
plot(x,y_rational,'k:')
grid on
xlabel('x'); ylabel('y')
gtext('Neville')
gtext('Rational')
```



The rational function interpolation does a creditable job in $0 < x < 0.8$, but fails for $x > 0.8$. Polynomial interpolation over more than a few points is generally not recommended since a high-order interpolant has a tendency to develop spurious wiggles. This is clearly visible in the plot.

Problem 17

Since cubic spline resists spurious wiggles, it can be used with relative safety over numerous data points. This program does cubic spline interpolation on the logarithms of the data. It prompts for user input of x. In this case, $x = \mathrm{Re}$ and $y = c_D$.

```
% problem3_1_17
xData = log10([0.2 2 20 200 2000 20000]);
yData = log10([103 13.9 2.72 0.800 0.401 0.433]);
x = [5 50 500 5000];
k = splineCurv(xData,yData);
fprintf('   Re      cD\n')
for Re = x
    cD = 10^splineEval(xData,yData,k,log10(Re));
    fprintf('%8.0f %8.4f\n',Re,cD)
end

>>    Re       cD
       5   6.9016
      50   1.5908
     500   0.5574
    5000   0.3868
```

Problem 18

```
% problem3_1_18 (Rational interpolation)
xData = [0.2 2 20 200 2000 20000];
yData = [103 13.9 2.72 0.800 0.401 0.433];
x = [5 50 500 5000];
fprintf(' Re cD\n')
for i = 1:length(x)
 y = rational(xData,yData,x(i));
 fprintf('%8.0f %8.4f\n',x(i),y)
end

>>    Re       cD
       5   6.6622
      50   1.6524
     500   0.5452
    5000   0.3802
```

Problem 19

Since there are 7 data points, we could use global cubic spline interpolation or polynomial interpolation over nearest-neighour data points. The following program uses the cubic spline.

```
% problem3_1_19
xData = [0 21.1 37.8 54.4 71.1 87.8 100];
yData = [1.79 1.13 0.696 0.519 0.338 0.321 0.296];
x = [10 30 60 90];
k = splineCurv(xData,yData);
fprintf('Temperature  Viscosity\n')
for T = x
    mu = splineEval(xData,yData,k,T);
    fprintf('%9.0f %13.4f\n',T,mu)
end

>> Temperature  Viscosity
        10        1.4750
        30        0.8701
        60        0.4541
        90        0.3195
```

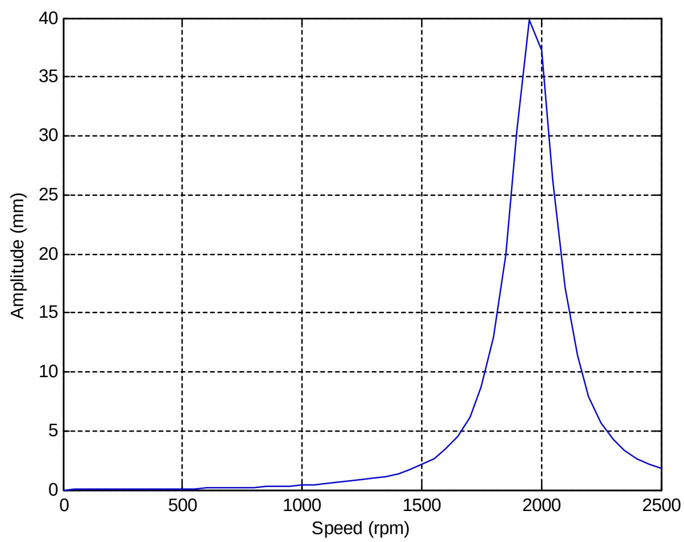
Problem 20

As we have extrapolation, polynomial interpolation over all the data points is dangerous. It is more prudent to use a few point at the high end of the data range. The following is essentially the program used in Problem 14; it interpolates over three points with Neville's method:

```
% problem3_1_20
xData = [6.1 7.625 9.150];
yData = [0.5328 0.4481 0.3741];
h = 10.5
rho = neville(xData,yData,h)

>> h =
   10.5000
rho =
    0.3175
```

Problem 21

```
% problem 3_1_21
xData = [0 400 800 1200 1600]';
yData = [0 0.072 0.233 0.712 3.40]';
x = 0:50:2500;
n = length(x);
y = zeros(n,1);
for i = 1:n
    y(i) = rational(xData,yData,x(i));
end
plot(x,y)
grid on
xlabel('Speed (rpm)');ylabel('Amplitude (mm)')
```



By inspection of the plot, resonance occurs at approximately 1950 rpm ◀

PROBLEM SET 3.2

Problem 1

The equation of the regression line is

$$f(x) = a + bx = (\bar{y} - \bar{x}b) + bx$$

Thus

$$f(\bar{x}) = \bar{y} \quad \text{Q.E.D.}$$

Problem 2

	x	y	$x - \bar{x}$	$x(x - \bar{x})$	$y(x - \bar{x})$	$f(x)$	$y - f(x)$
	-1.0	-1.00	-1.0	1.00	1.000	-1.02	0.02
	-0.5	-0.55	-0.5	0.25	0.275	-0.52	-0.03
	0.0	0.00	0.0	0.00	0.000	-0.02	0.02
	0.5	0.45	0.5	0.25	0.225	0.48	-0.03
	1.0	1.00	1.0	1.00	1.000	0.98	0.02
$\sum$	0.0	-0.100		2.50	2.500		

$$\bar{x} = \frac{1}{5}\sum x = 0 \qquad \bar{y} = \frac{1}{5}\sum y = \frac{-0.100}{5} = -0.02$$

$$\begin{aligned} b &= \frac{\sum y(x - \bar{x})}{\sum x(x - \bar{x})} = \frac{2.500}{2.50} = 1.0 \\ a &= \bar{y} - \bar{x}b = -0.02 - 0(1.0) = -0.02 \end{aligned}$$

The regression line is

$$f(x) = -0.02 + x \blacktriangleleft$$

$$S = \sum [y - f(x)]^2 = 3(0.02)^2 + 2(-0.03)^2 = 0.003$$

The standard deviation is ($n + 1 =$ the number of data points; $m =$ degree of interpolating polynomial)

$$\sigma = \sqrt{\frac{S}{n - m}} = \sqrt{\frac{0.003}{4 - 1}} = 0.0316 \quad \blacktriangleleft$$

Problem 3

	x (Stress)	y (Strain)	$x - \bar{x}$	$x(x - \bar{x})$	$y(x - \bar{x})$
	34.5	0.46	−51.75	−1785	−23.81
	69.0	0.95	−17.25	−1190	−16.39
	103.5	1.48	17.25	1785	25.53
	138.0	1.93	51.75	7142	99.88
	34.5	0.34	−51.75	−1785	−17.60
	69.0	1.02	−17.25	−1190	−17.60
	103.5	1.51	17.25	1785	26.05
	138.0	2.09	51.75	7142	108.16
	34.5	0.73	−51.75	−1785	−37.78
	69.0	1.10	−17.25	−1190	−18.98
	103.5	1.62	17.25	1785	27.95
	138.0	2.12	51.75	7142	109.71
$\sum$	1 035.0	15.35		17 854	265.12

$$\bar{x} = \frac{\sum x}{12} = \frac{1035.0}{12} = 86.25 \qquad \bar{y} = \frac{\sum y}{12} = \frac{15.35}{12} = 1.2792$$

Converting the strain y from mm/m to m/m, we have

$$b = \frac{\sum y(x - \bar{x})}{\sum x(x - \bar{x})} = \frac{265.12 \times 10^{-3}}{17\,854} (\mathrm{MPa})^{-1} = 1.4849 \times 10^{-5} (\mathrm{MPa})^{-1}$$

The modulus of elasticity is

$$E = \frac{1}{b} = \frac{1}{1.4849 \times 10^{-5}} \mathrm{MPa} = 67350 \ \mathrm{MPa} = 67.34 \ \mathrm{GPa} \blacktriangleleft$$

Problem 4

Let x = stress and y = strain.

	x	y	W	W^2x	W^2y	$x-\bar{x}$	$W^2x(x-\bar{x})$	$W^2y(x-\bar{x})$
	34.5	0.46	1.0	34.5	0.460	−51.75	−1785	−23.81
	69.0	0.95	1.0	69.0	0.950	−17.25	−1190	−16.39
	103.5	1.48	1.0	103.5	1.480	17.25	1785	25.53
	138.0	1.93	1.0	138.0	1.930	51.75	7142	99.88
	34.5	0.34	1.0	34.5	0.340	−51.75	−1785	−17.60
	69.0	1.02	1.0	69.0	1.020	−17.25	−1190	−17.60
	103.5	1.51	1.0	103.5	1.510	17.25	1785	26.05
	138.0	2.09	1.0	138.0	2.090	51.75	7142	108.16
	34.5	0.73	0.5	8.6	0.183	−51.75	−446	−9.44
	69.0	1.10	0.5	17.3	0.275	−17.25	−298	−4.74
	103.5	1.62	0.5	25.9	0.405	17.25	445	6.99
	138.0	2.12	0.5	34.5	0.530	51.75	1785	27.43
$\sum$				776.3	11.173		13 390	204.46

$$\sum W^2 = 8 + 4(0.25) = 9$$

$$\hat{x} = \frac{\sum W^2 x}{\sum W^2} = \frac{776.3}{9} = 86.25 \qquad \hat{y} = \frac{\sum W^2 y}{\sum W} = \frac{11.173}{9} = 1.2414$$

Converting the strain y from mm/m to m/m, we have

$$b = \frac{\sum W^2 y(x-\hat{x})}{\sum W^2 x(x-\hat{x})} = \frac{204.46 \times 10^{-3}}{13\,390}(\text{MPa})^{-1} = 1.5270 \times 10^{-5}(\text{MPa})^{-1}$$

The modulus of elasticity is

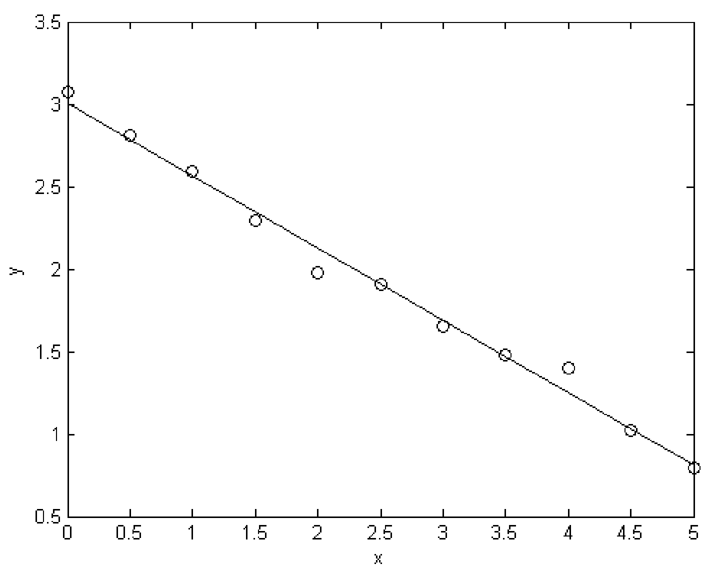
$$E = \frac{1}{b} = \frac{1}{1.5270 \times 10^{-5}}\text{MPa} = 65.49 \text{ MPa} = 65.49 \text{ GPa} \blacktriangleleft$$

Problem 5

In this and the following problems we use MATlAB's plotting capability to plot the data points and the fitting function.

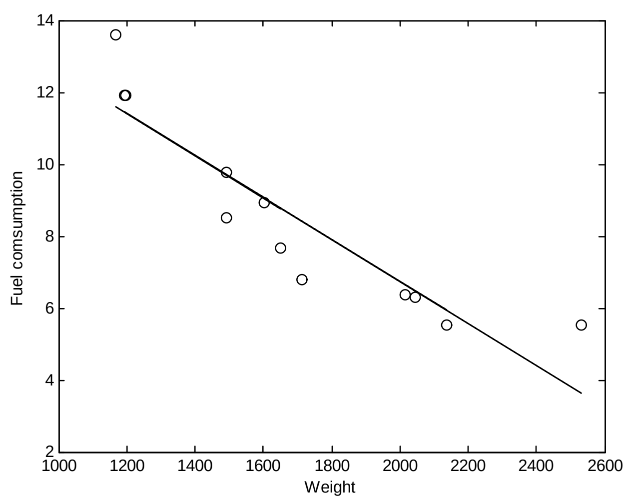
```
% problem3_2_5
xData = 0:0.5:5;
yData =[3.076 2.810 2.588 2.297 1.981 1.912 ...
        1.653 1.478 1.399 1.018 0.794];
format short e
c = polynFit(xData,yData,2)
std_deviation = stdDev(c,xData,yData)
y = c(2)*ones(length(xData),1) + c(1)*xData';  % Interpolant
plot(xData,yData,'ko'); hold on; plot(xData,y,'k-')
xlabel('x'); ylabel('y')

>> c =
 -4.3838e-001
  3.0056e+000
std_deviation =
  7.7499e-002
```

The equation of the interpolant is $y = 3.0056 - 0.43838x$ (solid line in the figure).



Problem 6

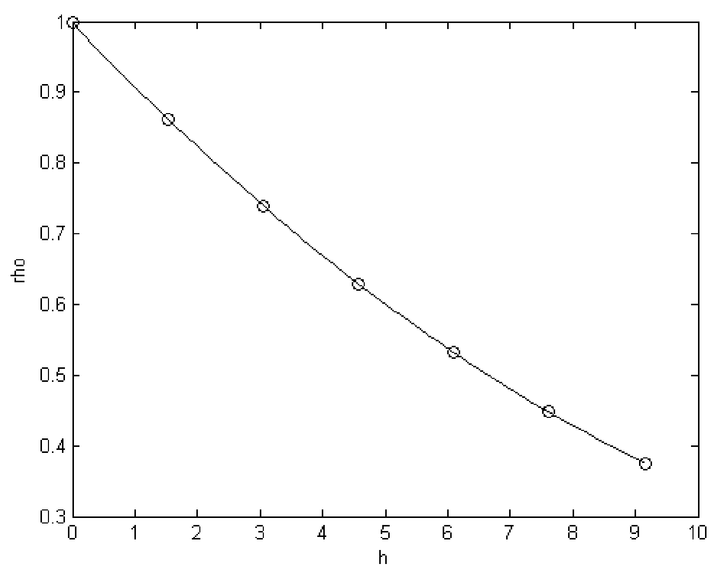
```
% problem3_2_6
xData = [1198 1715 2530 2014 2136 1492 1652 1168 ...
         1492 1602 1192 2045];
yData =[11.90 6.80 5.53 6.38 5.53 8.50 7.65 13.60 ...
        9.78 8.93 11.90 6.30];
format short e
c = polynFit(xData,yData,2)
std_deviation = stdDev(c,xData,yData)
y = c(2)*ones(length(xData),1) + c(1)*xData';
plot(xData,yData,'ko'); hold on; plot(xData,y,'k-')
xlabel('Weight'); ylabel('Fuel comsumption'))

c =
 -5.8476e-003
  1.8428e+001

std_deviation =
  1.1651e+000
```



Problem 7

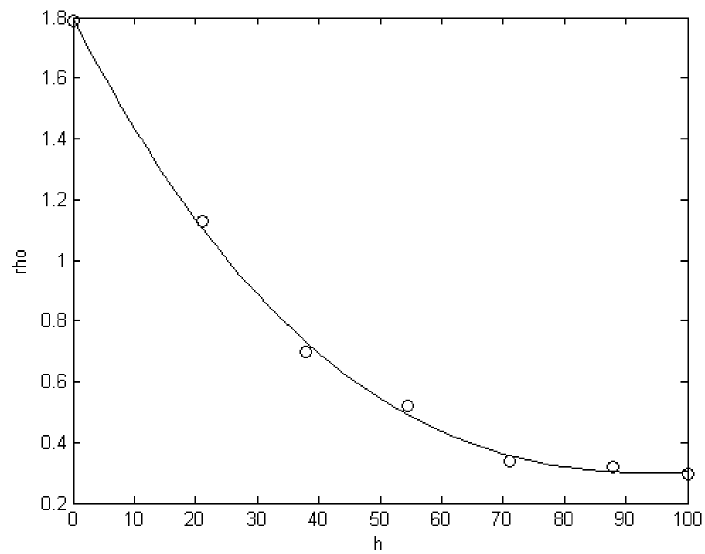
```
% problem3_2_7
xData = [0 1.525 3.050 4.575 6.100 7.625 9.150];
yData = [1 0.8617 0.7385 0.6292 0.5328 0.4481 0.3741];
format short e
c = polynFit(xData,yData,3)
std_deviation = stdDev(c,xData,yData)
x1 = min(xData); x2 = max(xData); dx = (x2 - x1)/50;
x = x1:dx:x2;
y = c(3)*ones(1,length(x)) + c(2)*x + c(1)*x.^2;
plot(xData,yData,'ko'); hold on; plot(x,y,'k-')
xlabel('h'); ylabel('rho')
```

```
>> c =
  2.7632e-003
 -9.3447e-002
  9.9890e-001
std_deviation =
  1.3473e-003
```



Problem 8

```
% problem3_2_8
xData = [0 21.1 37.8 54.4 71.1 87.8 100];
yData = [1.79 1.13 0.696 0.519 0.338 0.321 0.296];
format short e
c = polynFit(xData,yData,4)
std_deviation = stdDev(c,xData,yData)
fprintf('Temperature  Viscosity\n')
for T = [10 30 60 90]
    [mu,dmu,ddmu] = evalPoly(c,T);
    fprintf('%6.0f %13.4f\n',T,mu)
end
% Plotting the data and the fitting function
x1 = min(xData); x2 = max(xData); dx = (x2 - x1)/50;
x = x1:dx:x2;
y = c(4)*ones(1,length(x)) + c(3)*x + c(2)*x.^2 + c(1)*x.^3;
plot(xData,yData,'ko'); hold on; plot(x,y,'k-')
xlabel('h'); ylabel('rho')

>> c =
 -8.4589e-007
  3.2857e-004
 -3.9321e-002
  1.7957e+000
std_deviation =
  3.4005e-002
Temperature  Viscosity
    10        1.4345
    30        0.8889
    60        0.4366
    90        0.3016
```

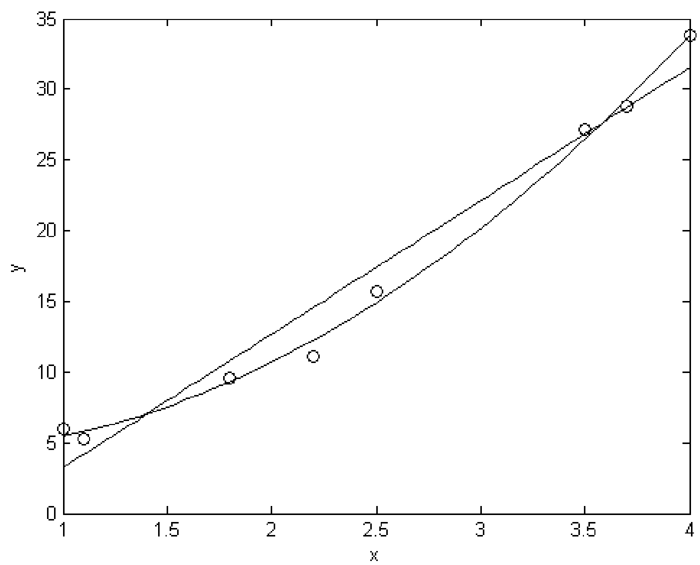


Problem 9

The program below prompts for the degree of the fitting polynomial until it receives an empty input (by pressing 'return').

```
% problem3_2_9
xData = [1 2.5 3.5 4 1.1 1.8 2.2 3.7];
yData = [6.008 15.722 27.130 33.772 5.257 9.549 ...
        11.098 28.828];
format short e
while 1
    m = input('Degree of polynomial ==> ');
    if isempty(m); fprintf('Done'); break; end
    c = polynFit(xData,yData,m+1)
    std_deviation = stdDev(c,xData,yData)
    % Plotting the data and the fitting function
    x1 = min(xData); x2 = max(xData); dx = (x2 - x1)/50;
    x = x1:dx:x2;
    y = zeros(1,length(x));
    for i = 1:m+1
        y = y + c(i)*x.^(m-i+1);
    end
    plot(xData,yData,'ko'); hold on; plot(x,y,'k-')
    label('x'); ylabel('y')
end
```

```
Degree of polynomial ==> 1
c =
  9.4385e+000
 -6.1899e+000
std_deviation =
  2.2436e+000
Degree of polynomial ==> 2
c =
  2.1081e+000
 -1.0689e+000
  4.4057e+000
std_deviation =
  8.1293e-001
Degree of polynomial ==>
Done
```



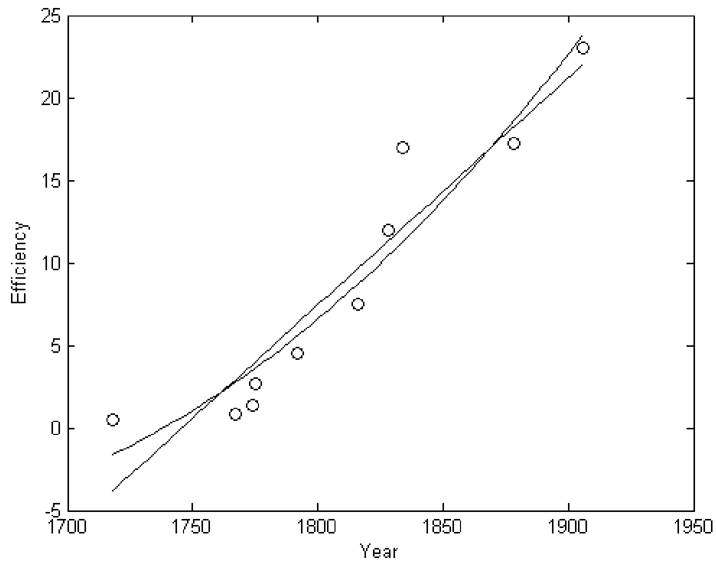
The *quadratic* is a much better fit.

Problem 10

Since we have to do extrapolation, the fitting function should be either straight line or a quadratic. Let us try both:

```
% problem3_2_10
xData = [1718 1767 1774 1775 1792 1816 1828 1834 1878 1906];
yData = [0.5 0.8 1.4 2.7 4.5 7.5 12.0 17.0 17.2 23.0];
format short e
while 1
    m = input('Degree of polynomial ==> ');
    if isempty(m); fprintf('Done'); break; end
    c = polynFit(xData,yData,m+1)
    std_deviation = stdDev(c,xData,yData)
    % Plotting the data and the fitting function
    x1 = min(xData); x2 = max(xData); dx = (x2 - x1)/50;
    x = x1:dx:x2;
    y = zeros(1,length(x));
    for i = 1:m+1
        y = y + c(i)*x.^(m-i+1);
    end
    plot(xData,yData,'ko'); hold on; plot(x,y,'k-')
    xlabel('Year'); ylabel('Efficiency')
end
```

```
Degree of polynomial ==> 1
c =
  1.3769e-001
 -2.4039e+002
std_deviation =
  2.8552e+000
Degree of polynomial ==> 2
c =
  3.3374e-004
 -1.0740e+000
  8.5852e+002
std_deviation =
  2.7684e+000
Degree of polynomial ==>
Done
```



As the standard devations are about equal, we choose the straight line (always the safer of the two for extrapolation). Thus the predicted efficiency in year 2000 is $-240.39 + 0.13769(2000) = 35.0\%$ ◄

Problem 11

	T	$T^2 \times 10^{-3}$	$T^3 \times 10^{-6}$	$T^4 \times 10^{-9}$	k	kT	$kT^2 \times 10^{-3}$
	79	6.24	0.49	0.04	1.000	79.00	6.24
	190	36.10	6.86	1.30	0.932	177.08	33.65
	357	127.45	45.50	16.24	0.839	299.52	106.93
	524	274.58	143.88	75.39	0.759	397.72	208.40
	690	476.10	328.51	226.67	0.693	478.17	329.94
$\sum$	1840	920.47	525.24	319.64	4.223	1431.49	685.16

The coefficients $\mathbf{a}$ of the quadratic are given by the solution of the equation

$$\begin{bmatrix} 5 & \sum T & \sum T^2 \\ \sum T & \sum T^2 & \sum T^3 \\ \sum T^2 & \sum T^3 & \sum T^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum k \\ \sum kT \\ \sum kT^2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1840 & 920.47 \times 10^3 \\ 1840 & 920.47 \times 10^3 & 525.24 \times 10^6 \\ 920.47 \times 10^3 & 525.24 \times 10^6 & 319.64 \times 10^9 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4.223 \\ 1431.49 \\ 685.16 \times 10^3 \end{bmatrix}$$

The solution is

$$a_1 = 1.0526 \qquad a_2 = -0.6807 \times 10^{-3} \qquad a_3 = 0.2310 \times 10^{-6}$$

Thus the quadratic approximation is

$$k = 1.0526 - 0.6807 \times 10^{-3}T + 0.2310 \times 10^{-6}T^2 \quad \blacktriangleleft$$

Problem 12

$$\begin{aligned} f(x) &= ax^b \\ F(x) &= \ln f(x) = \ln a + b \ln x \end{aligned}$$

The residuals are

$$\begin{aligned} r_i &= y_i - f(x_i) = y_i - ax_i^b && \text{(a)} \\ R_i &= \ln y_i - F(x_i) = \ln y_i - \ln a - b\ln x_i = \ln \frac{y_i}{a} - b \ln x_i && \text{(b)} \end{aligned}$$

But from Eq. (a)

$$y_i - r_i = ax_i^b$$

so that

$$\begin{aligned} \ln(y_i - r_i) &= \ln a + b \ln x_i \\ b \ln x_i &= \ln \frac{y_i - r_i}{a} \end{aligned}$$

Substitution into Eq. (b) yields

$$R_i = \ln \frac{y_i}{a} - \ln \frac{y_i - r_i}{a} = \ln \frac{y_i}{y_i - r_i} = \ln \frac{1}{1 - r_i/y_i}$$

For small r_i/y_i we can approximate

$$R_i \approx \ln\left(1 + \frac{r_i}{y_i}\right) \approx \frac{r_i}{y_i} \quad \text{Q.E.D.}$$

Problem 13

i	1	2	3	4	5	6
x_i	-0.5	-0.19	0.02	0.20	0.35	0.50
y_i	-3.558	-2.874	-1.995	-1.040	-0.068	0.677

The fitting function is a linear form with

$$f_1(x) = \sin \frac{\pi t}{2} \qquad f_2(x) = \cos \frac{\pi t}{2}$$

The coefficients a and b are given by the solution of the equations

$$\begin{bmatrix} \sum_i f_1^2(x_i) & \sum_i f_1(x_i)f_2(x_i) \\ \sum_i f_1(x_i)f_2(x_i) & \sum_i f_2^2(x_{i)} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_i f_1(x_i)y_i \\ \sum_i f_2(x_i)y_i \end{bmatrix} \tag{a}$$

i	$f_1(x_i)$	$f_2(x_i)$	$f_1^2(x_i)$	$f_1(x_i)f_2(x_i)$	$f_2^2(x_{i)}$	$f_1(x_i)y_2$	$f_2(x_i)y_i$
1	−0.7071	0.7071	0.5000	−0.5000	0.5000	2.5159	−2.5159
2	−0.2940	0.9558	0.0865	−0.2810	0.9135	0.8451	−2.7469
3	0.0314	0.9995	0.0010	0.0314	0.9990	−0.0627	−1.9940
4	0.3090	0.9511	0.0955	0.2939	0.9045	−0.3214	−0.9891
5	0.5225	0.8526	0.2730	0.4455	0.7270	−0.0355	−0.0580
6	0.7071	0.7071	0.5000	0.5000	0.5000	0.4716	0.4716
$\sum$			1.4560	0.4898	4.5440	3.4130	−7.8323

Equations (a) are

$$\begin{bmatrix} 1.4560 & 0.4898 \\ 0.4898 & 4.5440 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3.4130 \\ -7.8323 \end{bmatrix}$$

The solution is

$$a = 3.034 \qquad b = -2.051 \blacktriangleleft$$

Problem 14

i	1	2	3	4	5
x_i	0.5	1.0	1.5	2.0	2.5
y_i	0.49	1.60	3.36	6.44	10.16

Rather than fitting $y = ae^{bx}$, we use linear regression to fit $\ln y = \ln a + bx$ with the weights $W_i = y_i$.

i	$z_i = \ln y_i$	$y_i^2 x_i$	$y_i^2 z_i$	$x_i - \hat{x}$	$y_i^2 x_i(x_i - x)$	$y_i^2 z_i(x_i - x)$
1	−0.7133	0.12	−0.17	−1.7711	−0.213	0.303
2	0.4700	2.56	1.20	−1.2711	−3.254	−1.529
3	1.2119	16.93	13.68	−0.7711	−13.058	−10.551
4	1.8625	82.95	77.25	−0.2711	−22.487	−20.941
5	2.3185	258.06	239.32	0.2289	59.071	54.781
$\sum$		360.63	331.28		20.059	22.063

$$\sum_i y_i^2 = 0.49^2 + 1.60^2 + 3.36^2 + 6.44^2 + 10.16^2 = 158.79$$

$$\begin{aligned} \hat{x} &= \frac{\sum_i y_i^2 x_i}{\sum_i y_i^2} = \frac{360.63}{158.79} = 2.2711 \\ \hat{z} &= \frac{\sum_i y_i^2 z_i}{\sum_i y_i^2} = \frac{331.28}{158.79} = 2.0863 \end{aligned}$$

$$b = \frac{\sum_i y_i^2 z_i(x_i - x)}{\sum_i y_i^2 x_i(x_i - x)} = \frac{22.063}{20.059} = 1.0999 \quad \blacktriangleleft$$

$$\begin{aligned} \ln a &= \hat{z} - b\hat{x} = 2.0863 - 1.0999(2.2711) = -0.41168 \\ a &= e^{-0.41168} = 0.6625 \quad \blacktriangleleft \end{aligned}$$

Problem 15

i	1	2	3	4	5
x	0.5	1.0	1.5	2.0	2.5
y	0.541	0.398	0.232	0.106	0.052

The fitting function is $y = axe^{bx}$, or $y/x = ae^{bx}$. We linearize the problem by fitting $\ln(y/x) = \ln a + bx$ with the weights $W_i = y_i$.

i	$z_i = \ln y_i/x_i$	$y_i^2 x_i$	$y_i^2 z_i$	$x_i - \hat{x}$	$y_i^2 x_i(x_i - x)$	$y_i^2 z_i(x_i - x)$
1	0.0788	0.1463	0.0231	−0.2993	−0.04380	−0.00690
2	−0.9213	0.1584	−0.1459	0.2007	0.03179	−0.02929
3	−1.8665	0.0807	−0.1005	0.7007	0.05657	−0.07039
4	−2.9375	0.0225	−0.0330	1.2007	0.02698	−0.03963
5	−3.8728	0.0068	−0.0105	1.7007	0.01150	−0.01781
$\sum$		0.4147	−0.2668		0.08304	−0.16402

$$\sum_i y_i^2 = 0.541^2 + 0.398^2 + 0.232^2 + 0.106^2 + 0.052^2 = 0.5188$$

$$\begin{aligned} \hat{x} &= \frac{\sum_i y_i^2 x_i}{\sum_i y_i^2} = \frac{0.4147}{0.5188} = 0.7993 \\ \hat{z} &= \frac{\sum_i y_i^2 z_i}{\sum_i y_i^2} = \frac{-0.2668}{0.5188} = -0.5143 \end{aligned}$$

$$b = \frac{\sum_i y_i^2 z_i(x_i - x)}{\sum_i y_i^2 x_i(x_i - x)} = \frac{-0.16402}{0.08304} = -1.9752 \quad \blacktriangleleft$$

$$\begin{aligned} \ln a &= \hat{z} - b\hat{x} = -0.5143 - (-1.9752)(0.7993) = 1.0645 \\ a &= e^{\ln a} = e^{1.0645} = 2.899 \quad \blacktriangleleft \end{aligned}$$

Computation of standard deviation:

$$f(x) = axe^{bx} = 2.899xe^{-1.9752x}$$

i	y_i	$f(x_i)$	$[y_i - f(x_i)]^2 \times 10^6$
1	0.541	0.53998	1.047
2	0.398	0.40226	18.122
3	0.232	0.22475	52.609
4	0.106	0.11162	21.552
5	0.052	0.05197	0.000
$\sum$			103.330

$$\sigma = \sqrt{\frac{S}{5-2}} = \sqrt{\frac{103.330 \times 10^{-6}}{3}} = 5.87 \times 10^3 \blacktriangleleft$$

Problem 16

The fitting function is $\gamma = ae^{bt}$ (the value of b should come out to be negative). We linearize the problem by fitting

$$\ln \gamma = \ln a + bt \tag{a}$$

with the weights $W_i = \gamma_i$. The half-life $t_{1/2}$ is obtained by substituting $\gamma = 0.5$ into Eq. (a) and solving for t, obtaining

$$t_{1/2} = \frac{\ln 0.5 - \ln a}{b}$$

```
% problem3_2_16
x = [0:0.5:5.5];
y = [1 0.994 0.990 0.985 0.979 0.977 ...
     0.972 0.969 0.967 0.960 0.956 0.952];
format short e
z = log(y);
xBar = dot(x,y.^2)/dot(y,y);
zBar = dot(z,y.^2)/dot(y,y);
b = dot(y.^2,z.*(x - xBar))/dot(y.^2,x.*(x - xBar));
ln_a = zBar -b*xBar;
half_life = (log(0.5) - ln_a)/b

>> half_life =
  7.9996e+001
```

Problem 17

The function to be minimized is

$$S(a,b,c) = \sum_{i=1}^{n}(z_i - a - bx_i - cy_i)^2$$

which yields

$$\begin{aligned}
\frac{\partial S}{\partial a} &= -2\sum_{i=1}^{n}(z_i - a - bx_i - cy_i) = 0 \\
\frac{\partial S}{\partial b} &= -2\sum_{i=1}^{n}x_i(z_i - a - bx_i - cy_i) = 0 \\
\frac{\partial S}{\partial c} &= -2\sum_{i=1}^{n}y_i(z_i - a - bx_i - cy_i) = 0
\end{aligned}$$

This can be written as

$$\begin{aligned}
na + b\sum_{i=1}^{n}x_i + c\sum_{i=1}^{n}y_i &= \sum_{i=1}^{n}z_i \\
a\sum_{i=1}^{n}x_i + b\sum_{i=1}^{n}x_i^2 + c\sum_{i=1}^{n}x_iy_i &= \sum_{i=1}^{n}x_iz_i \\
a\sum_{i=1}^{n}y_i + b\sum_{i=1}^{n}x_iy_i + c\sum_{i=1}^{n}y_i^2 &= \sum_{i=1}^{n}y_iz_i
\end{aligned}$$

Q.E.D.

Problem 18

The normal equations to be solved are

$$\begin{bmatrix} n & \Sigma x_i & \Sigma y_i \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_iy_i \\ \Sigma y_i & \Sigma x_iy_i & \Sigma y_i^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \Sigma z_i \\ \Sigma x_iz_i \\ \Sigma y_iz_i \end{bmatrix}$$

From the given data we have

$$\begin{aligned}
n &= 6 \qquad \Sigma x_i = 7 \qquad \Sigma y_i = 4 \qquad \Sigma z_i = 5.88 \\
\Sigma x_i^2 &= 13 \qquad \Sigma y_i^2 = 6 \qquad \Sigma x_iy_i = 6 \\
\Sigma x_iz_i &= 4.44 \qquad \Sigma y_iz_i = 4.55
\end{aligned}$$

Thus the normal equations are

$$\begin{bmatrix} 6 & 7 & 4 \\ 7 & 13 & 6 \\ 4 & 6 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5.88 \\ 4.44 \\ 4.55 \end{bmatrix}$$

The solution is

$$a = 1.413 \qquad b = -0.621 \qquad c = 0.438$$

so that the fitting function becomes

$$f(x, y) = 1.413 - 0.621x + 0.438y \quad \blacktriangleleft$$