Set 7.1)

Problem 7

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta$$

With the notation $\theta = y_1$, $\dot{\theta} = y_2$ the equivalent first-order differential equations are

$$\mathbf{F} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -\sin y_1 \end{bmatrix}$$

We release the pendulum from rest at $\theta = 1$, $\tau = 0$ and determine the time it takes for it to return to the starting point for the first time. Hence the initial conditions are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To assure that the integration covers one period, we stop at $\tau = 2.2\pi$ (this is 10% larger than the period for small amplitudes). We use the 4th-order Runge-Kutta method with $h = 0.25$.

```
% problem7_1_7
clc; clear;
x = 0; y = [1 0]; xStop = 2.2*pi; h = 0.25;
[xSol,ySol] = runKut4(@dEqs,x,y,xStop,h);
printSol(xSol,ySol,1)

   function F = dEqs(x,y)
   F = zeros(1,2);
   F(1) = y(2);
   F(2) = -sin(y(1));
   end % function dEqs
```

The part of the printout that spans the return of the pendulum to the release position is (note the change in the sign of the velocity y_2):

```
>>        x              y1             y2
   6.5000e+000    9.8315e-001    1.6776e-001
   6.7500e+000    9.9892e-001   -4.1972e-002
```

The value of τ at the instant when $d\theta/d\tau = 0$ can be estimated from two-term Taylor series expansion

$$\begin{aligned}
\left.\frac{d\theta}{d\tau}\right|_{6.75+\Delta\tau} &= \left.\frac{d\theta}{d\tau}\right|_{6.75} + \left.\frac{d^2\theta}{d\tau^2}\right|_{6.75} \Delta\tau \\
0 &= -0.041\,972 + (-\sin 0.99892)\,\Delta\tau \\
\Delta\tau &= -0.04991 \\
\tau &= 6.75 - 0.04991 = 6.700 \quad \blacktriangleleft
\end{aligned}$$

Thus the period is $6.700\sqrt{L/g}$ ◄

Problem 18

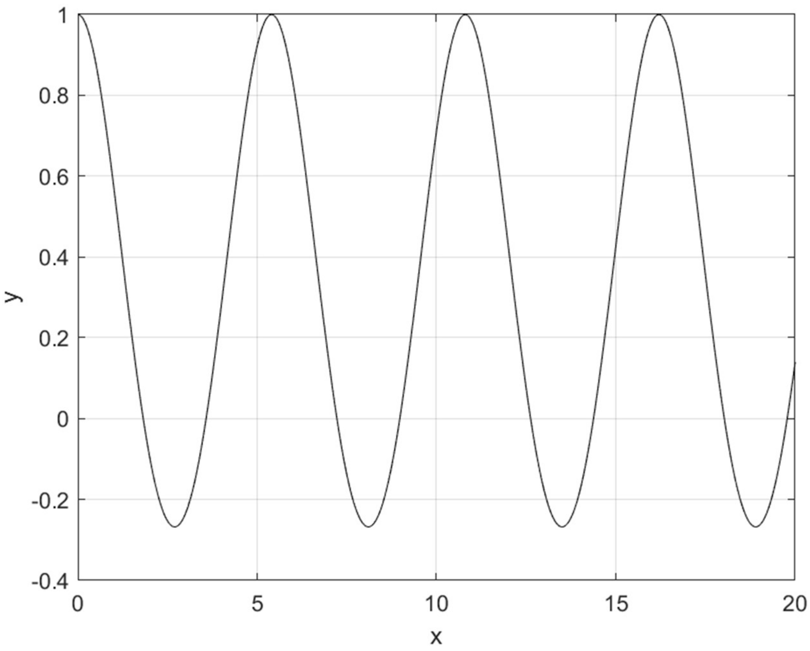
We use the notation $y = y_1$, $y' = y_2$ in both problems. Being unable to determine a suitable time increment h beforehand, we let the program do it for us. Starting with an initial guess for h, the program integrates the differential equations with h, $h/2$, $h/4$, etc. until the results of two successive integrations agree within a prescribed tolerance.

(a)

$$y'' + 0.5(y^2 - 1) + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

$$\mathbf{F} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} - \begin{bmatrix} y_2 \\ -0.5(y_1^2 - 1) - y_1 \end{bmatrix}$$

```
% problem7_1_18a
clc; clear;
x = 0; xStop = 20; h = 0.2; y = [1 0];
yOld = 0;

while 1
    [xSol,ySol] = runKut4(@dEqs,x,y,xStop,h);
    yNew = ySol(size(ySol,1));
    if abs(yNew - yOld) < 1.0e-4; break
    else; h = h/2; yOld = yNew; end
end
h
plot(xSol,ySol(:,1),'k-'); grid on
xlabel('x'); ylabel('y')

   function F = dEqs(x,y)
   F = zeros(1,2);
   F(1) = y(2);
   F(2) = -0.5*(y(1)^2-1)-y(1);
   end % function dEqs
```



The output is

```
>> h = 0.050000
```

Note that the initial increment $h = 0.2$ was reduced to 0.05 in the last run of the program.

(b)

$$y'' = y\cos 2x \qquad y(0) = 0 \qquad y'(0) = 1$$

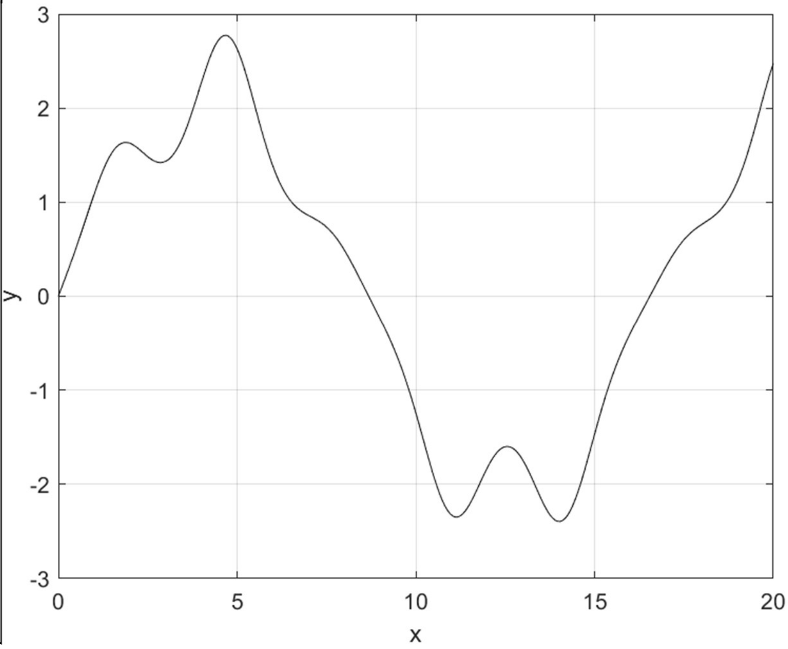
This differential equation is called Mathieu's equation. The equivalent first-order equations are

$$\mathbf{F} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} - \begin{bmatrix} y_2 \\ y_1 \cos 2x \end{bmatrix}$$

We used the program listed in Part (a); only $\mathbf{F}$ and the initial conditions were changed.

```
>> h = 0.050000
```

```
% problem7_1_18b
clc; clear;
x = 0; xStop = 20; h = 0.2; y = [0 1];
yOld = 0;

while 1
    [xSol,ySol] = runKut4(@dEqs,x,y,xStop,h);
    yNew = ySol(size(ySol,1));
    if abs(yNew - yOld) < 1.0e-4; break
    else; h = h/2; yOld = yNew; end
end
h
plot(xSol,ySol(:,1),'k-'); grid on
xlabel('x'); ylabel('y')

   function F = dEqs(x,y)
   F = zeros(1,2);
   F(1) = y(2);
   F(2) = y(1)*cos(2*x);
   end % function dEqs
```



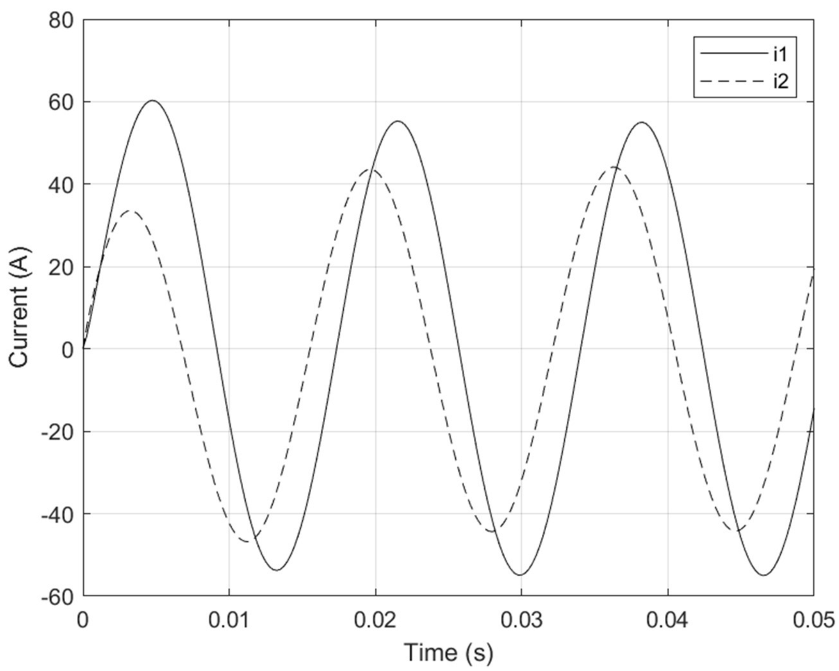
Problem 21

$$\begin{aligned}\frac{di_1}{dt} &= \frac{-3Ri_1 - 2Ri_2 + E}{L}\\ \frac{di_2}{dt} &= -\frac{2}{3}\frac{di_1}{dt} - \frac{i_2}{3RC} + \frac{\dot{E}}{3R}\\ i_1(0) &= i_2(0) = 0\end{aligned}$$

Using the notation $i_1 = y_1$, $i_2 = y_2$, the differential equations are

$$F = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} (-3Ry_1 - 2Ry_2 + E)/L \\ \left[-2\dot{y}_1 - y_2/(RC) + \dot{E}/R\right]/3 \end{bmatrix}$$

```
% problem7_1_21
clc; clear;
x = 0; xStop = 0.05; h = 0.00025; y = [0 0];
[xSol,ySol] = runKut4(@dEqs,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol(:,1),'k-'); hold on
plot(xSol,ySol(:,2),'k--'); grid on
xlabel('Time (s)'); ylabel('Current (A)')
legend('i1','i2')

    function F = dEqs(x,y)
    % Differential eqs. used in Problem 21, Problem Set 7.1.
    R = 1; L = 0.2e-3; C = 3.5e-3;
    E = 240*sin(120*pi*x);
    dE = 240*120*pi*cos(120*pi*x);
    F = zeros(1,2);
    F(1) = (-3*R*y(1) - 2*R*y(2) + E)/L;
    F(2) = (-2*F(1) -y(2)/R/C + dE/R)/3;
    end
```



Set 7.2)

Problem 3

The analytical solution is

$$y(5) = 0.1(5) - 0.01 + 10.01e^{-10(5)} = 0.4900$$

```
% problem7_2_3
clear; clc;
for h = [0.1 0.25 0.5]
x = 0; xStop = 5; y = [10];
[xSol,ySol] = runKut4(@dEqs,x,y,xStop,h);
fprintf('\nh = %6.3f\n',h)
printSol(xSol,ySol,0)
end

   function F = dEqs(x,y)
   F = x-10*y(1);
   end % function dEqs
```

```
h =   0.100
        x              y1
   0.0000e+000    1.0000e+001
   5.0000e+000    4.9000e-001
h =   0.250
        x              y1
   0.0000e+000    1.0000e+001
   5.0000e+000    4.9173e-001
h =   0.500
        x              y1
   0.0000e+000    1.0000e+001
   5.0000e+000    2.3457e+012
```

In Problem 2 the stable range of h was estimated as $h < 0.2$. Thus $h = 0.1$ is stable and $h = 0.5$ is unstable, as verified by the numerical results. On the other hand. $h = 0.25$ is close to the borderline—it is stable in the specified range of integration, but not accurate.

Problem 4

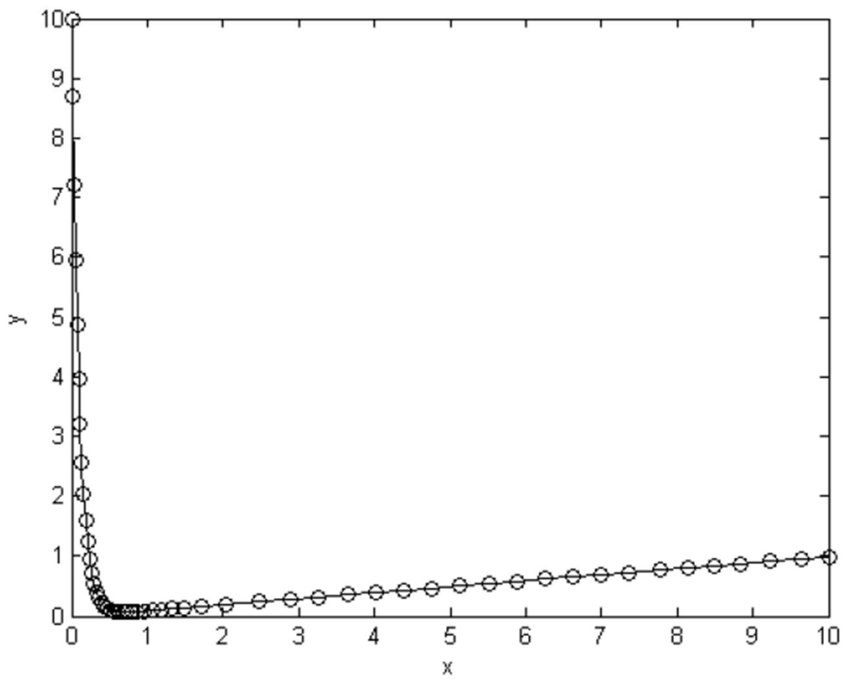
The analytical solution is

$$y(10) = 0.1(10) - 0.01 + 10.01e^{-10(10)} = 0.9900$$

```
% problem7_2_4
clear; clc;
x = 0; xStop = 10; y = [10]; h = 0.1;
[xSol,ySol] = runKut5(@dEqs,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol,'k-o')
xlabel('x');ylabel('y')

   function F = dEqs(x,y)
   F = x-10*y(1);
   end % function dEqs
```

```
>>       x              y1
   0.0000e+000    1.0000e+001
   1.0000e+001    9.9000e-001
```

The plot of the solution shows the integration points as circles. Note the greater density points where y varies rapidly.



Problem 5

$$\ddot{y} = -\frac{c}{m}\dot{y} - \frac{k}{m}y \qquad y(0) = 0.01 \text{ m} \qquad \dot{y}(0) = 0$$

(a)

With $y = y_1$, $\dot{y} = y_2$ the equivalent first-order differential equations are

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

These equations are of the form $\dot{\mathbf{y}} = -\mathbf{\Lambda}\mathbf{y}$, where

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & -1 \\ k/m & c/m \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 450/2 & 460/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 225 & 230 \end{bmatrix}$$

The eigenvalues of $\mathbf{\Lambda}$ are the roots of

$$\begin{vmatrix} 0 - \lambda & -1 \\ 225 & 230 - \lambda \end{vmatrix} = 0 \qquad \lambda^2 - 230\lambda + 225 = 0$$

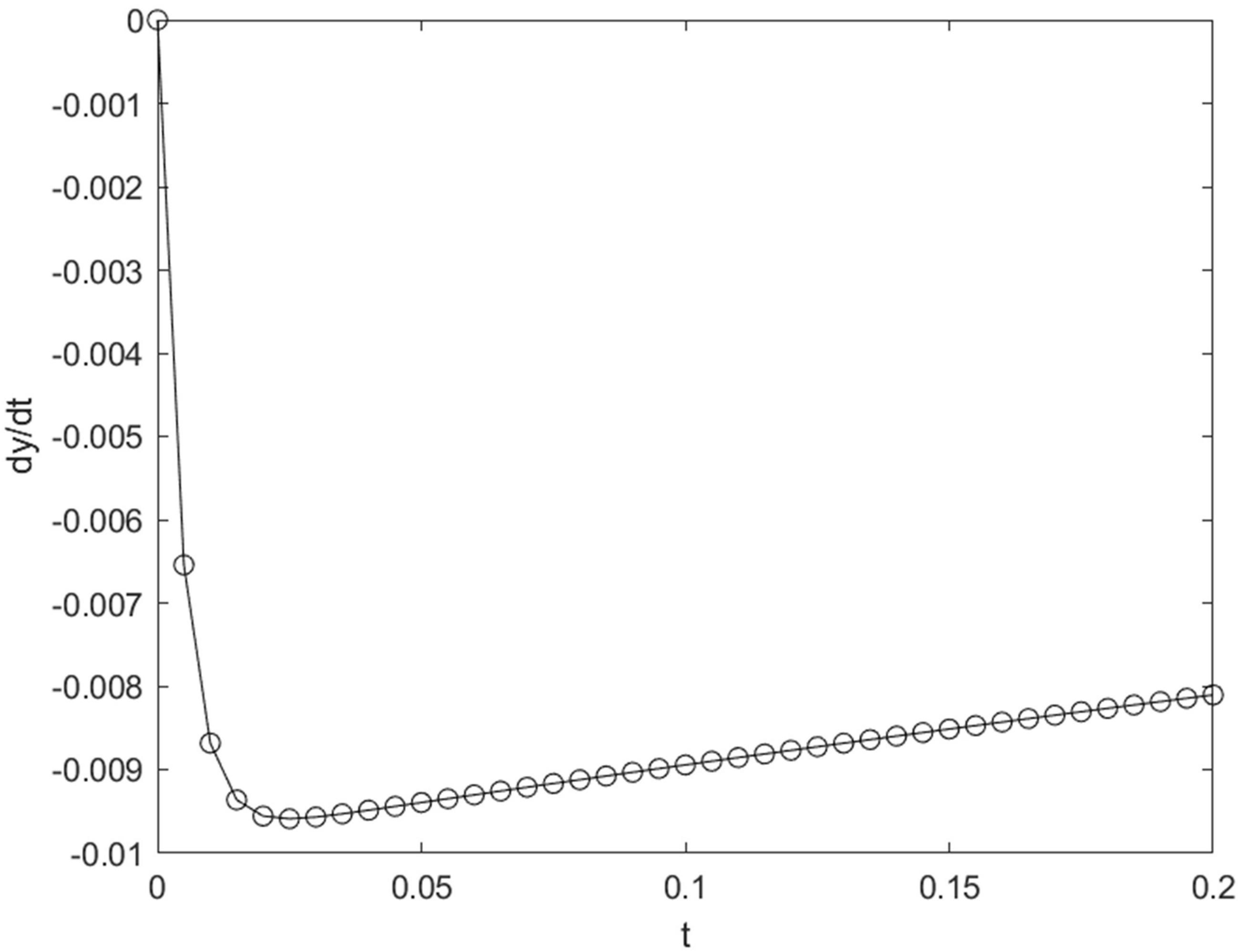
The solution is $\lambda_1 = 0.982\,458$, $\lambda_2 = 229.0175$. Since there is a large disparity in the eigenvalues, the problem is stiff. Numerical integration requires

$$h < \frac{2}{\lambda_2} = \frac{2}{229.0175} = 0.008\,733$$

A reasonable choice would be $h = 0.005$ ◄

(b)

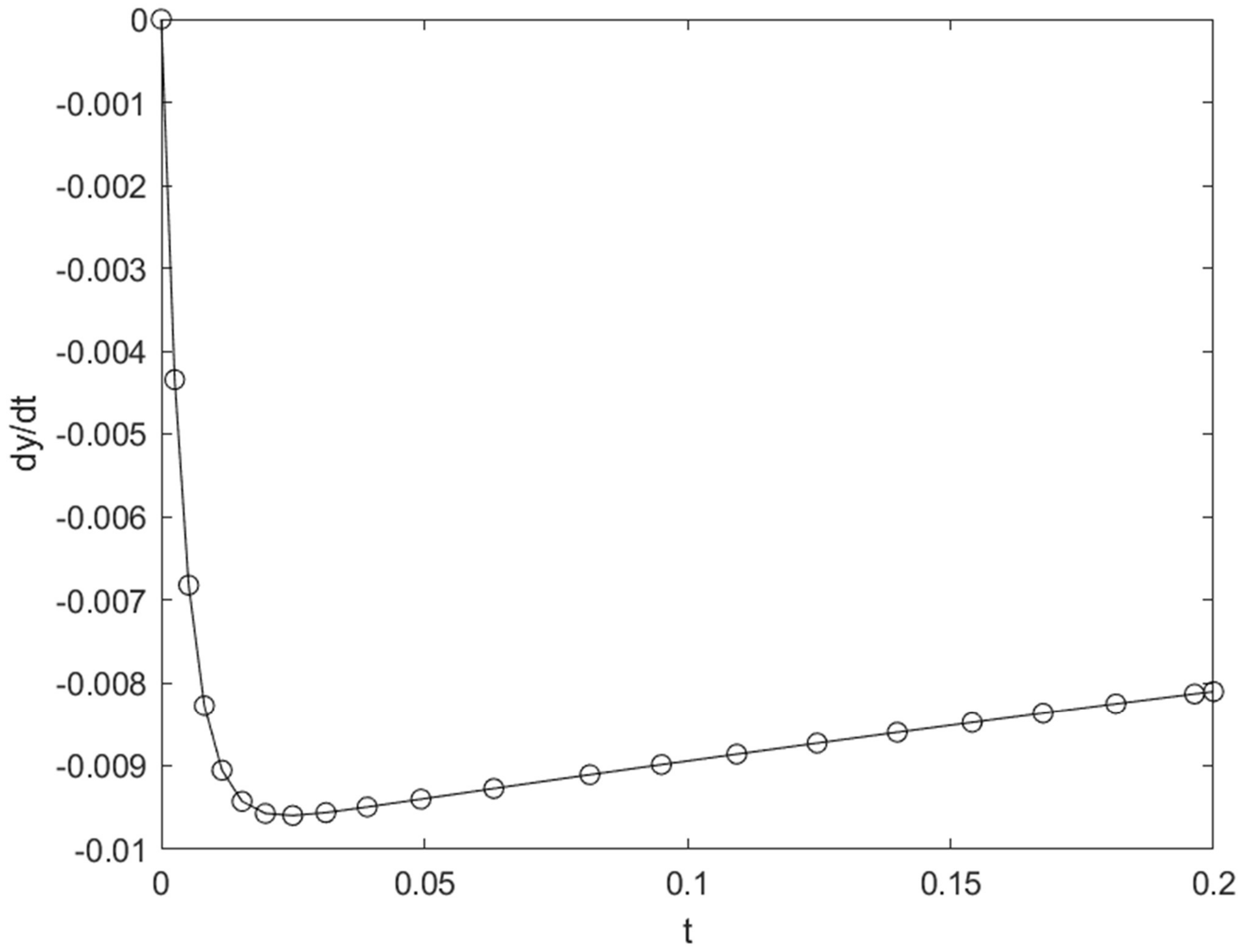
```
% problem7_2_5
clear; clc;
x = 0; xStop = 0.2; y = [0.01 0];
h = 0.005;
[xSol,ySol] = runKut4(@dEqs,x,y,xStop,h);
printSol(xSol,ySol,0)
plot(xSol,ySol(:,2),'k-o')
xlabel('t');ylabel('dy/dt')

   function F = dEqs(x,y)
   F = zeros(1,2);
   F(1) = y(2);
   F(2) = -225*y(1)-230*y(2);
   end % function dEqs
```



Problem 6

The program in Problem 5 was used with the function call runKut4 replaced by runKut5.

```
>>      x              y1              y2
   0.0000e+000   1.0000e-002   0.0000e+000
   2.0000e-001   8.2515e-003  -8.1065e-003
```

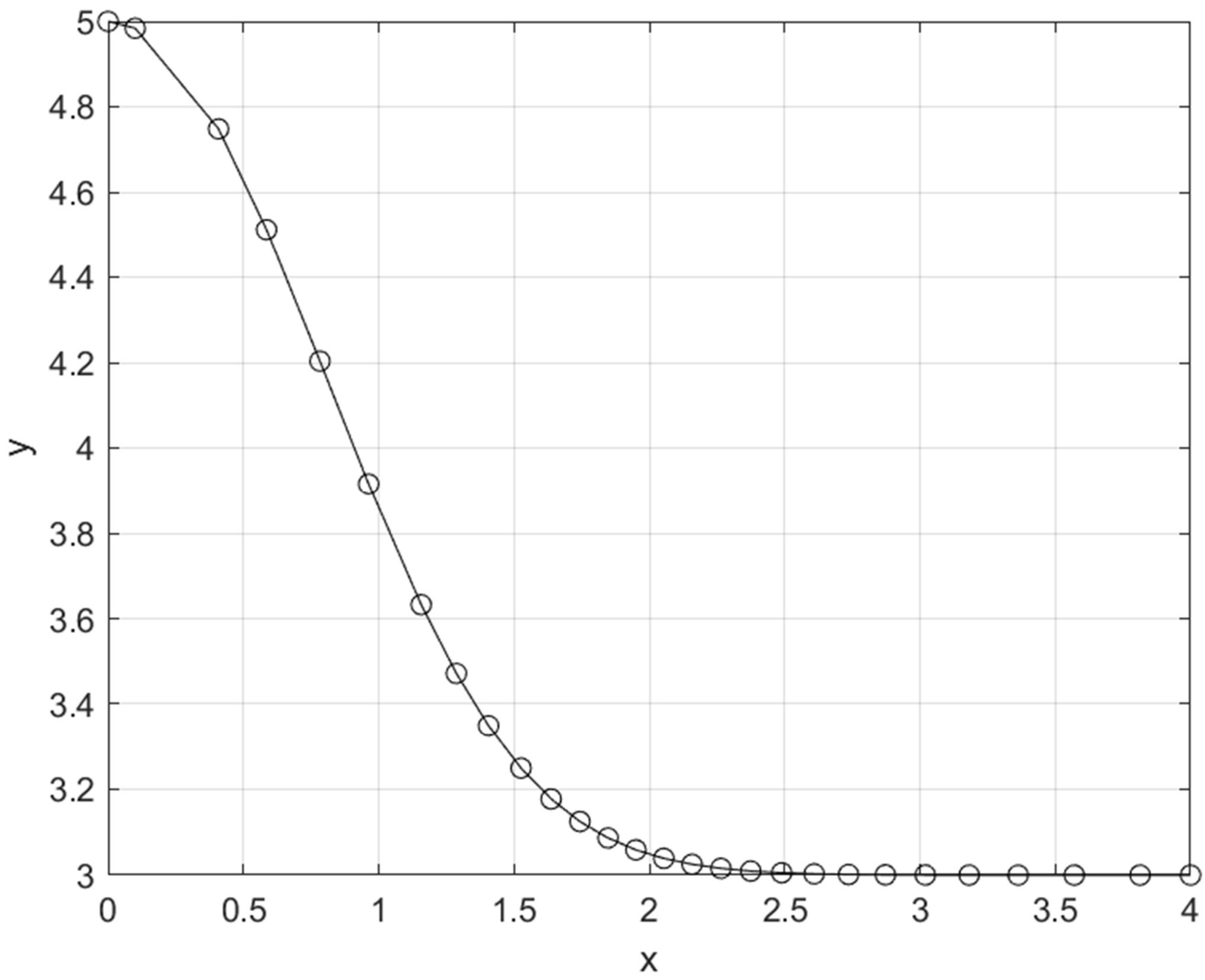


Problem 13

$$y' = \left(\frac{9}{y} - y\right)x \qquad y(0) = 5$$

```
% problem7_2_13
clear; clc;
x = 0; xStop = 4; y = [5]; h = 0.1;
[xSol,ySol] = runKut5(@dEqs,x,y,xStop,h);
printSol(xSol,ySol,2)
plot(xSol,ySol,'k-o')
xlabel('x');ylabel('y')
grid on

   function F = dEqs(x,y)
   F = (9/y(1)-y(1))*x;
   end % function dEqs
```

```
>>       x              y1
   0.0000e+000    5.0000e+000
   3.4132e-001    4.8208e+000
   8.1895e-001    4.1451e+000
   1.1506e+000    3.6412e+000
   1.4276e+000    3.3294e+000
   1.6806e+000    3.1543e+000
   1.9219e+000    3.0656e+000
   2.1668e+000    3.0243e+000
   2.4268e+000    3.0074e+000
   2.7155e+000    3.0017e+000
   3.0525e+000    3.0002e+000
   3.4747e+000    3.0000e+000
   4.0000e+000    3.0000e+000
```



Problem 15

$$y'' = -\frac{1}{x}y' - \frac{1}{x^2}y \qquad y(1) = 0 \qquad y'(1) = -2$$

```
% problem7_2_15
clear; clc;
x = 1; xStop = 20; y = [0 -2];
H = 1;
[xSol,ySol] = bulStoer(@dEqs,x,y,xStop,H);
printSol(xSol,ySol,2)
plot(xSol,ySol(:,1),'k-o'); hold on
plot(xSol,ySol(:,2),'k-s'); grid on
xlabel('x'); legend('y',"y'")

   function F = dEqs(x,y)
   F = zeros(1,2);
   F(1) = y(2);
   F(2) = -y(2)/x-y(1)/x^2;
   end % function dEqs
```

x	y1	y2
1.0000e+00	0.0000e+00	-2.0000e+00
3.0000e+00	-1.7812e+00	-3.0322e-01
5.0000e+00	-1.9985e+00	1.5451e-02
7.0000e+00	-1.8609e+00	1.0468e-01
9.0000e+00	-1.6203e+00	1.3028e-01
1.1000e+01	-1.3540e+00	1.3381e-01
1.3000e+01	-1.0904e+00	1.2897e-01
1.5000e+01	-8.4019e-01	1.2100e-01
1.7000e+01	-6.0704e-01	1.1210e-01
1.9000e+01	-3.9177e-01	1.0322e-01
2.0000e+01	-2.9070e-01	9.8938e-02>>