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1. (a)

① $f = y_1$ $\quad\quad$ $T = y_4$

$f' = y_2 = y_1'$ $\quad\quad$ $T' = y_5 = y_4'$

$f'' = y_3 = y_2'$ $\quad\quad$ $T'' = -3Pr\,y_1 y_5$

$f''' = -3y_1 y_3 + 2\cdot y_2^2 - y_4$

$$\bar{F} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ -3y_1y_3 + 2y_2^2 - y_4 \\ y_5 \\ -3Pr\,y_1y_5 \end{bmatrix}$$

+5

② function y = InCond(u)

```
y = [0 0 u(1) 1 u(2)];
end
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+2

③ function r = residual(u)

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r = zeros(length(u),1);
x = xStart;
[xSol,ySol] = runkut4(@dEqs, x, InCond(u), xStop, h);
lastlow = size(ySol,1);
r(1) = ySol(lastlow,2);
r(2) = ySol(lastlow,4);
end
```

+2

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2.

① $$\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta x^2} = D(y_i - x_i)(y_i + x_i)e^{-y_i}$$ ← ans

+4

② $$y'(-5) = y_1' = \frac{y_2 - y_0}{2\Delta x} = -1 \Rightarrow y_0 = y_2 + 2\Delta x$$ +2

$i=1$

$$\frac{y_2 - 2y_1 + y_0}{\Delta x^2} = D(y_1 - x_1)(y_1 + x_1)e^{-y_1}$$

$$2\frac{y_2 - y_1 + \Delta x}{\Delta x^2} = D(y_1 - x_1)(y_1 + x_1)e^{-y_1}$$ +2 ← ans

③ $$y'(5) = y_n' = \frac{y_{n+1} - y_{n-1}}{2\Delta x} = 1 \Rightarrow y_{n+1} = y_{n-1} + 2\Delta x$$ +2

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} = D(y_n - x_n)(y_n + x_n)e^{-y_n}$$

$$2\frac{y_{n-1} - y_n + \Delta x}{\Delta x^2} = D(y_n - x_n)(y_n + x_n)e^{-y_n}$$ ← ans

+2

④
$$r_1 = 2\left(\frac{y_2 - y_1 + \Delta x}{\Delta x^2}\right) - D(y_1 - x_1)(y_1 + x_1)e^{-y_1}$$

$$r_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} - D(y_i - x_i)(y_i + x_i)e^{-y_i} \quad (i = 2, 3, \cdots, n-1)$$

$$r_n = 2\left(\frac{y_{n-1} - y_n + \Delta x}{\Delta x^2}\right) - D(y_n - x_n)(y_n + x_n)e^{-y_n}$$

+3

3. $\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + S(x,y) \qquad r = \frac{\alpha \Delta t}{h^2}$

①

$$T_{i,j}^{n+1/2} - T_{i,j}^{n} = \frac{\alpha \Delta t}{2h^2}\left((T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}) + (T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n})\right) + \Delta t/2\, S_{i,j}$$

$$-r/2\, T_{i-1,j}^{n+1/2} + (1+r) T_{i,j}^{n+1/2} - r/2\, T_{i+1,j}^{n+1/2} = r/2\, T_{i,j-1}^{n} + (1-r) T_{i,j}^{n} + r/2\, T_{i,j+1}^{n} + \Delta t/2\, S_{i,j}$$

② $i=1$

$T_{1,j} = 1 \qquad (\because T(0,y,t) = 1)$

+3

$i = I$

$$-r/2\, T_{I-1,j}^{n+1/2} + (1+r) T_{I,j}^{n+1/2} - r/2\, T_{I+1,j}^{n+1/2} = r/2\, T_{I,j-1}^{n} + (1-r) T_{I,j}^{n} + r/2\, T_{I,j+1}^{n} + \Delta t/2\, S_{I,j}$$

$$\Rightarrow -r\, T_{I-1,j}^{n+1/2} + (1+r) T_{I,j}^{n+1/2} = r/2\, T_{I,j-1}^{n} + (1-r) T_{I,j}^{n} + r/2\, T_{I,j+1}^{n} + \Delta t/2\, S_{I,j}$$

+3

③

$$\begin{bmatrix} 1 & 0 & & & \\ -r/2 & 1+r & -r/2 & & \\ & \ddots & \ddots & \ddots & \\ & & & -r & 1+r \end{bmatrix} \begin{bmatrix} T_{1,j}^{n+1/2} \\ T_{2,j}^{n+1/2} \\ \vdots \\ T_{I,j}^{n+1/2} \end{bmatrix} = \begin{bmatrix} 0 \\ r/2\, T_{2,j-1}^{n} + (1+r) T_{2,j}^{n} + r/2\, T_{2,j+1}^{n} + \Delta t/2\, S_{2,j} \\ \vdots \\ r/2\, T_{i,j-1}^{n} + (1-r) T_{i,j}^{n} + r/2\, T_{i,j+1}^{n} + \Delta t/2\, S_{i,j} \\ \vdots \\ r/2\, T_{I,j-1}^{n} + (1-r) T_{I,j}^{n} + r/2\, T_{I,j+1}^{n} + \Delta t/2\, S_{I,j} \end{bmatrix}$$

+2

④

$$T_{i,j}^{n+1} - T_{i,j}^{n+1/2} = \frac{\alpha \Delta t}{2h^2}\left((T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2}) + (T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1})\right) + \Delta t/2\, S_{i,j}$$

$$-r/2\, T_{i,j-1}^{n+1} + (1+r) T_{i,j}^{n+1} - r/2\, T_{i,j+1} = r/2\, T_{i-1,j}^{n+1/2} + (1-r) T_{i,j}^{n+1/2} + r/2\, T_{i+1,j}^{n+1/2} + \Delta t/2\, S_{i,j}$$

+3

⑤ $j=1$ $\qquad j=J$

$T_{i,1} = 0 \quad (\because T(x,0,t) = 0) \qquad T_{i,J} = 0 \quad (\because T(x,1,t) = 0)$

+3

⑥

$$\begin{bmatrix} 1 & 0 & & & \\ -r/2 & 1+r & -r/2 & & \\ & \ddots & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{i,1}^{n+1} \\ T_{i,2}^{n+1} \\ \vdots \\ T_{i,J}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ r/2\, T_{i-1,2}^{n+1/2} + (1-r) T_{i,2}^{n+1/2} + r/2\, T_{i+1,2}^{n+1/2} + \Delta t/2\, S_{i,2} \\ \vdots \\ r/2\, T_{i-1,j}^{n+1/2} + (1-r) T_{i,j}^{n+1/2} + r/2\, T_{i+1,j}^{n+1/2} + \Delta t/2\, S_{i,j} \\ \vdots \\ 0 \end{bmatrix}$$

+2

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(7) Instead of solving one set of linear equations for the 2-D heat equation, solve 1D equations for each grid line. It is ADI method, and it consists of first treating one row implicitly with backward Euler then reversing roles and treating the other by backward Euler.

+4

4.

(a) let $-\frac{\dot{Q}}{k} = S$. then,

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{h^2} = S_{i,j}$$

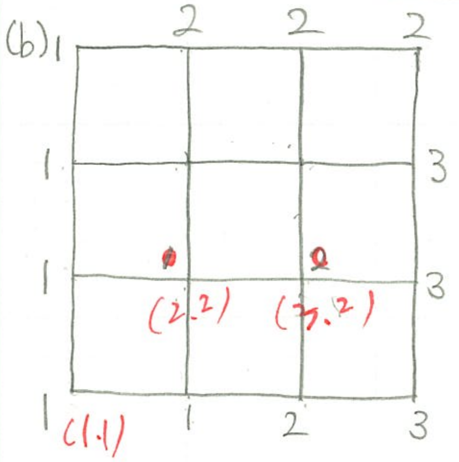
$$\therefore T_{i,j} = \frac{1}{4}(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}) - h^2 \cdot S_{i,j}$$

G-S

$$\Rightarrow T_{i,j}^{n+1} = \frac{1}{4}(T_{i+1,j}^{n} + T_{i-1,j}^{n+1} + T_{i,j+1}^{n} + T_{i,j-1}^{n+1} - h^2 S_{i,j})$$

+4

(b)



$$T_{2,2}^{n+1} = \frac{1}{4}(1+1+1+1+1) = \frac{5}{4} = 1.25$$

$$T_{2,3}^{n+1} = \frac{1}{4}(1+1+2+\frac{5}{4}+1) = 1.5625$$

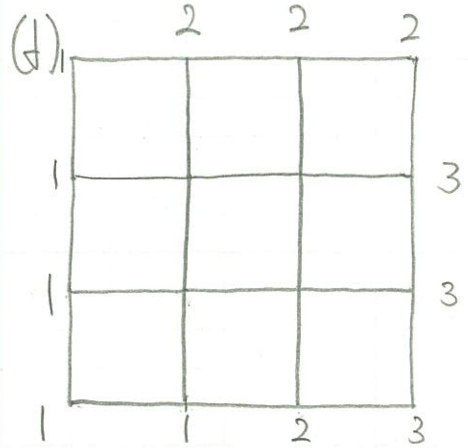
$$T_{3,2}^{n+1} = \frac{1}{4}(3+1.25+1+2+1) = 2.0625$$

$$T_{3,3}^{n+1} = \frac{1}{4}(3+1.5625+2+2.0625+1) = 2.4063$$

+4

(c) $T_{i,j}^{n+1} = \frac{1}{4}(T_{i-1,j}^{n+1} + T_{i,j-1}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j+1}^{n} - h^2 S_{i,j})$ +4

(d)



$$\begin{cases} T_{2,2}^{n+1} = \frac{1}{4}(1+1+1+T_{3,2}^{n+1}+1) \\ T_{3,2}^{n+1} = \frac{1}{4}(T_{2,2}^{n+1}+2+3+1+1) \end{cases}$$

$\therefore T_{2,2}^{n+1} = 1.533 \quad T_{3,2}^{n+1} = 2.1332$

$$\begin{cases} T_{2,3}^{n+1} = \frac{1}{4}(1+1.533+2+T_{3,3}^{n+1}+1) \\ T_{3,3}^{n+1} = \frac{1}{4}(2+3+1+T_{2,3}^{n+1}+2.1332) \end{cases}$$

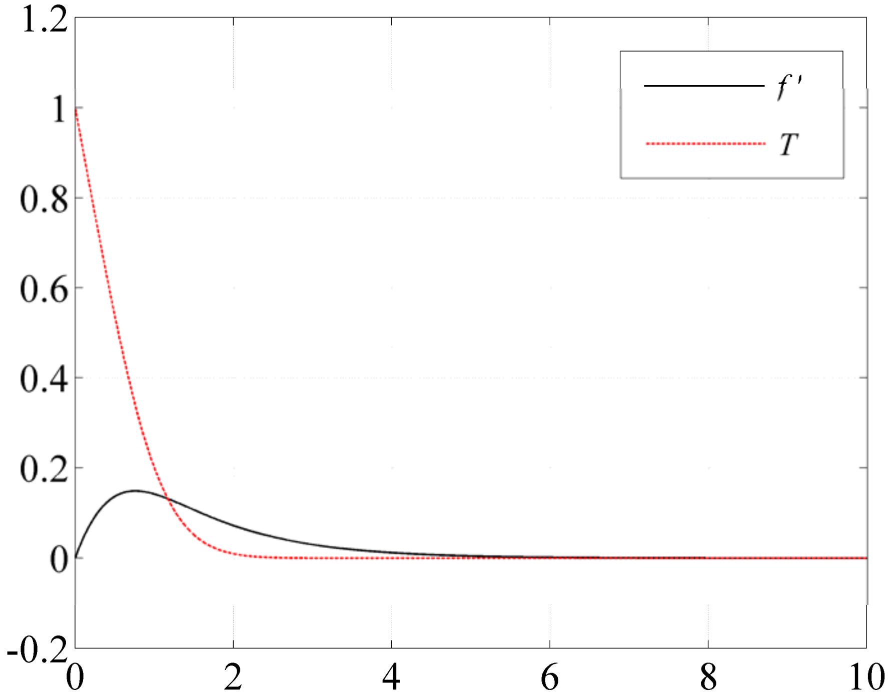
$\therefore T_{2,3}^{n+1} = 2.0174 \quad , T_{3,3}^{n+1} = 2.5376$

+4

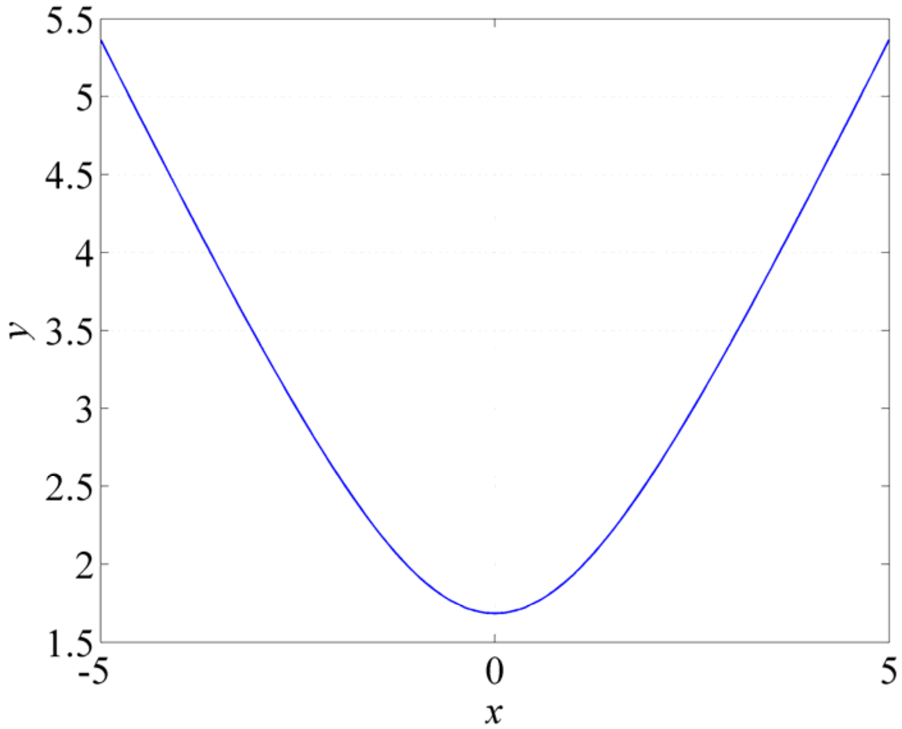
(e) The LGS method is more faster than GS method +4.

Because the boundary condition can affect the interior point. (more easily)

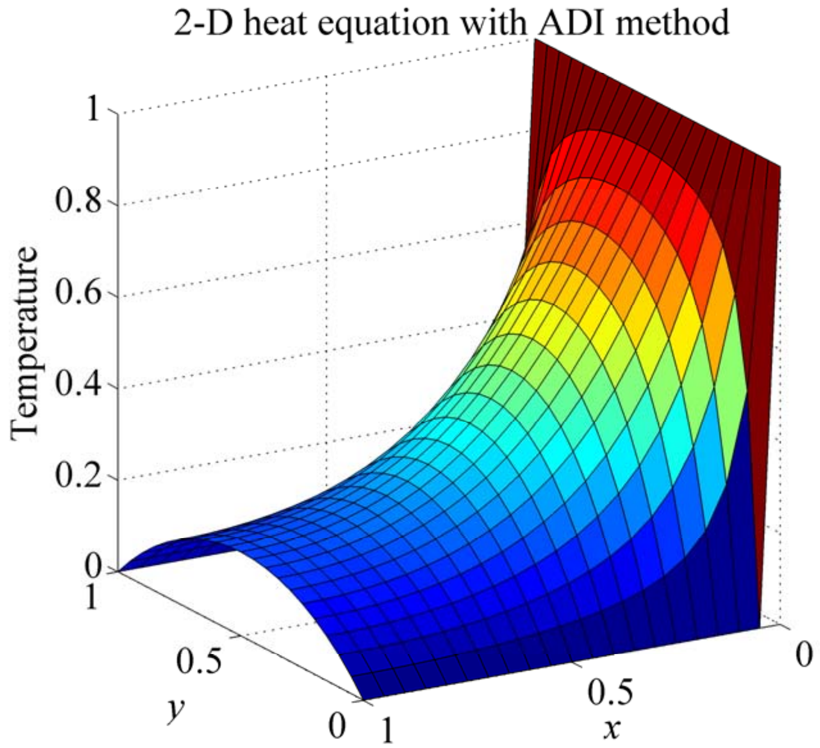
1-b



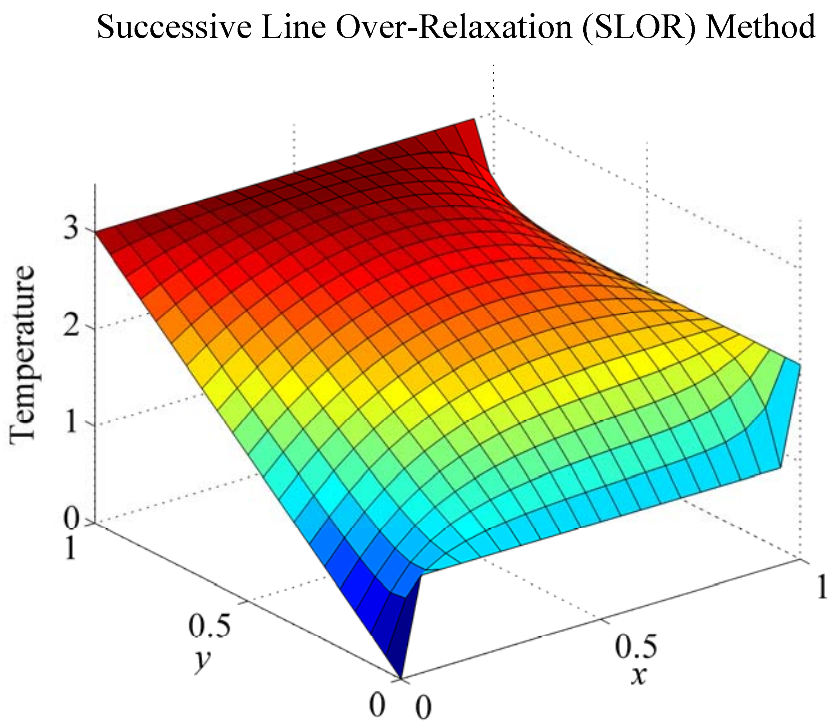
2-b



3-b


2-D heat equation with ADI method
Temperature
1
0.8
0.6
0.4
0.2
0
1
0.5
y
0
1
0.5
x
0

4-f


Successive Line Over-Relaxation (SLOR) Method
Temperature
3
2
1
0
1
0.5
y
0
0
0.5
x
1