PROBLEM SET 5.1

Problem 1

$$\begin{aligned}
f(x-h_1) &= f(x) - f'(x)h_1 + \frac{1}{2}f''(x)h_1^2 - \frac{1}{6}f'''(x)h_1^3 + \cdots \\
f(x+h_2) &= f(x) + f'(x)h_2 + \frac{1}{2}f''(x)h_2^2 + \frac{1}{6}f'''(x)h_2^3 + \cdots
\end{aligned}$$

Multiplying the first expression by h_2/h_1 and adding it to the second expression yields

$$\begin{aligned}
&\frac{h_2}{h_1}f(x-h_1) + f(x+h_2) \\
= \quad &\left(\frac{h_2}{h_1}+1\right)f(x) + \frac{1}{2}f''(x)\left(\frac{h_2}{h_1}h_1^2 + h_2^2\right) + \frac{1}{6}f'''(x)\left(\frac{h_2}{h_1}h_2^3 - h_1^3\right) + \cdots
\end{aligned}$$

$$f''(x) = \frac{\frac{h_2}{h_1}f(x-h_1) - \left(\frac{h_2}{h_1}+1\right)f(x) + f(x+h_2)}{\frac{h_2}{h_1}\left(1+\frac{h_2}{h_1}\right)\frac{h_1^2}{2}} + \mathcal{O}(h) \quad \blacktriangleleft$$

Problem 2

$$\begin{aligned}
f'''(x) &= [f''(x)]' = \left[\frac{f(x-2h) - 2f(x-h) + f(x)}{h^2}\right]' \\
&= \frac{1}{h^2}\left[f'(x-2h) - 2f'(x-h) + f'(x)\right] \\
&= \frac{1}{h^2}\left[\frac{f(x-2h)-f(x-3h)}{h} - 2\frac{f(x-h)-f(x-2h)}{h} + \frac{f(x)-f(x-h)}{h}\right] \\
&= \frac{-f(x-3h) + 3f(x-2h) - 3f(x-h) + f(x)}{h^3} \quad \blacktriangleleft
\end{aligned}$$

Problem 3

Central difference approximations for $f''(x)$ of $\mathcal{O}(h^2)$ are

$$\begin{aligned} g(h) &= \frac{f(x+h)-2f(x)+f(x-h)}{h^2} \\ g(2h) &= \frac{f(x+2h)-2f(x)+f(x-2h)}{(2h)^2} \end{aligned}$$

Richardson's extrapolation gives us an approximation of $\mathcal{O}(h^4)$:

$$\begin{aligned} f''(x) &\approx \frac{4g(h)-g(2h)}{4-1} \\ &= \frac{16}{12}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} - \frac{1}{12}\frac{f(x+2h)-2f(x)+f(x-2h)}{h^2} \\ &= \frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2} \quad \blacktriangleleft \end{aligned}$$

Problem 4

Taylor series:

$$\begin{aligned} f(x+h) &= f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(x)+\cdots && \text{(a)} \\ f(x+2h) &= f(x)+2hf'(x)+2h^2f''(x)+\frac{4h^3}{3}f'''(x)+\cdots) && \text{(b)} \\ f(x+3x) &= f(x)+3hf'(x)+\frac{9h^2}{2}f''(x)+\frac{9h^3}{2}f'''(x)+\cdots && \text{(c)} \end{aligned}$$

Eq. (b) $-2\times$ Eq. (a):

$$f(x+2h)-2f(x+h) = -f(x)+h^2f''(x)+h^3f'''(x)+\cdots \qquad \text{(d)}$$

Eq. (c) $-3\times$ Eq. (a):

$$f(x+3h)-3f(x+h) = -2f(x)+3h^2f''(x)+4h^3f'''(x)+\cdots \qquad \text{(e)}$$

Eq. (e) $-3\times$ Eq. (d):

$$f(x+3h)-3(x+2h)+3f(x+h) = f(x)+h^3f'''(x)+\cdots$$

$$f'''(x) \approx \frac{-f(x)+3f(x+h)-3f(x+2h)+f(x+3h)}{h^3} \quad \blacktriangleleft$$

Problem 5

Taylor series:

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + \cdots \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) - \cdots
\end{aligned}
$$

Adding the expressions yields

$$
f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12}f^{(4)}(x) + \cdots \tag{a}
$$

Taylor series:

$$
\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4h^3}{3}f'''(x) + \frac{2h^4}{3}f^{(4)}(x) + \cdots \\
f(x-2h) &= f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4h^3}{3}f'''(x) + \frac{2h^4}{3}f^{(4)}(x) - \cdots
\end{aligned}
$$

Adding the expressions gives us

$$
f(x+2h) + f(x-2h) = 2f(x) + 4h^2 f''(x) + \frac{4h^4}{3}f^{(4)}(x) + \cdots \tag{b}
$$

$4\times$ Eq. (a) $-$ Eq. (b):

$$
-f(x+2h) + 4f(x+h) + 4f(x-h) - f(x-2h) = 6f(x) - h^4 f^{(4)}(x) + \cdots
$$

$$
f^{(4)}(x) \approx \frac{f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x-2h)}{h^4} \quad ◀
$$

Problem 6

x	2.36	2.37	2.38	2.39
$f(x)$	0.85866	0.86289	0.86710	0.87129

Use forward differences of $\mathcal{O}(h^2)$:

$$
\begin{aligned}
f'(x) &\approx \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} \\
f'(2.36) &\approx \frac{-0.86710 + 4(0.86289) - 3(0.85866)}{0.02} = 0.424 \quad ◀
\end{aligned}
$$

$$
\begin{aligned}
f''(x) &\approx \frac{-f(x+3h) + 4f(x+2h) - 5f(x+h) + 2f(x)}{h^2} \\
f''(2.36) &\approx \frac{-0.87129 + 4(0.86710) - 5(0.86289) + 2(0.85866)}{0.01^2} \\
&= -0.200 \quad ◀
\end{aligned}
$$

Problem 7

x	0.97	1.00	1.05
$y = f(x)$	0.85040	0.84147	0.82612

As the spacing of data points is uneven, use a polynomial interpolant:

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

where the coefficients are given by the equations

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \\ \sum y_i x_i^2 \end{bmatrix}$$

$$\begin{bmatrix} 3.0000 & 3.0200 & 3.0434 \\ 3.0200 & 3.0434 & 3.0703 \\ 3.0434 & 0.3070 & 3.1008 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.5180 \\ 2.5338 \\ 2.5524 \end{bmatrix}$$

The solution is

$$a_0 = 1.0260 \qquad a_1 = -0.0678 \qquad a_2 = -0.1167$$

$$\begin{aligned} f'(1) &\approx P_2'(1) = a_1 + 2a_2 = -0.0678 - 2(0.1167) = -0.301 \blacktriangleleft \\ f''(1) &\approx P_2''(1) = 2a_2 = 2(-0.1167) = -0.233 \blacktriangleleft \end{aligned}$$

Problem 8

x	0.84	0.92	1.00	1.08	1.16
$f(x)$	0.431 711	0.398 519	0.367 879	0.339 596	0.313 486

Central difference approximations for $f''(x)$ of $\mathcal{O}(h^2)$ are

$$\begin{aligned} g(h) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \\ g(2h) &= \frac{f(x+2h) - 2f(x) + f(x-2h)}{(2h)^2} \end{aligned}$$

At $x = 1.0$ we get

$$\begin{aligned} g(0.8) &= \frac{0.339\,596 - 2(0.367\,879) + 0.398\,519}{0.8^2} = 3.682\,81 \times 10^{-3} \\ g(1.6) &= \frac{0.313\,486 - 2(0.367\,879) + 0.431\,711}{1.6^2} = 3.687\,11 \times 10^{-3} \end{aligned}$$

Richardson's extrapolation gives us an approximation of $\mathcal{O}(h^4)$:

$$f''(1) \approx \frac{4g(0.8) - g(1.6)}{4-1} = \frac{4(3.682\,81) - 3.687\,11}{3} \times 10^{-3} = 3.6814 \times 10^{-3} \blacktriangleleft$$

Problem 9

x	0	0.1	0.2	0.3	0.4
$y = f(x)$	0.000 000	0.078 348	0.138 910	0.192 916	0.244 981

Central difference approximations for $f'(x)$ of $\mathcal{O}(h^2)$ are

$$\begin{aligned} g(h) &= \frac{f(x+h) - f(x-h)}{2h} \\ g(2h) &= \frac{f(x+2h) - f(x-2h)}{4h} \end{aligned}$$

At $x = 0.2$:

$$\begin{aligned} g(0.1) &= \frac{0.192\,916 - 0.078\,348}{0.2} = 0.57\,284 \\ g(0.2) &= \frac{0.244\,981 - 0}{0.4} = 0.612\,45\,3 \end{aligned}$$

We obtain an approximation of $\mathcal{O}(h^3)$ by Richardson's extrapolation:

$$f'(0.2) \approx \frac{4g(0.1) - g(0.2)}{4-1} = \frac{4(0.57\,284) - 0.612\,45\,3}{3} = 0.\,559\,64 \blacktriangleleft$$

Problem 10

The true result is

$$f'(0.8) = \cos(0.8) = 0.696\,707$$

(a)

First forward difference approximation:

$$f'(0.8) \approx \frac{\sin(0.8+h) - \sin(0.8)}{h} = \frac{\sin(0.8+h) - 0.71736}{h}$$

h	$\sin(0.8+h)$	$f'(0.8)$
0.001	0.71805	0.69
0.0025	0.71910	0.696 ◀
0.005	0.72083	0.694

Note that two significant figures were lost in the computations.

(b)

First central difference approximation:

$$f'(0.8) \approx \frac{\sin(0.8+h) - \sin(0.8-h)}{2h}$$

h	$\sin(0.8+h)$	$\sin(0.8-h)$	$f'(0.8)$
0.01	0.72429	0.71035	0.697
0.025	0.73455	0.69972	0.6966 ◀
0.05	0.75128	0.68164	0.6964

Here one significant figure was lost in the computation.

Problem 11

x	-2.2	-0.3	0.8	1.9
$f(x)$	15.180	10.962	1.920	-2.040

Since there are four data points, the interpolant intersecting all these points is a cubic:

$$f(x) \approx a_1x^3 + a_2x^2 + a_3x + a_4$$

so that

$$f'(0) \approx a_3 \qquad f''(0) \approx 2a_2 \quad \blacktriangleleft$$

The following program computes the coefficients of the polynomial with the function `polynFit` described in Art. 3.4.

```
% problem5_1_11
xData = [-2.2 -0.3 0.8 1.9];
yData = [15.180 10.962 1.920 -2.040];
a = polynFit(xData,yData,4);
first_derivative = a(3)
second_derivative = 2*a(2)

>> first_derivative =
   -8.5600
second_derivative =
   -0.6000
```

Problem 12

$$x = R\left(\cos\theta + \sqrt{2.5^2 - \sin^2\theta}\right)$$

Letting $\omega = d\theta/dt$ (constant), we have

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\theta}\omega \qquad \ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2x}{d\theta^2}\omega^2$$

Using central differences of $O(h^2)$, the finite difference approximation for the acceleration is

$$\ddot{x} \approx R\omega^2 \frac{f(\theta+h) - 2f(\theta) + f(\theta-h)}{h^2}$$

where

$$f(\theta) = \cos\theta + \sqrt{2.5^2 - \sin^2\theta}$$

```
% problem5_1_12
func = inline('cos(x) + sqrt(6.25 - sin(x)^2)','x');
h = 0.1*pi/180; R = 0.09; omega = 5000*(2*pi)/60;
c = R*(omega^2)/h^2;
fprintf('Angle (deg)  Acceleration (m/s/s)\n')
for angle = 0:5:180
    x = angle*pi/180;
    accel = c*(func(x-h) - 2*func(x) + func(x+h));
    fprintf('%7.1f %19.4e\n',angle,accel)
end
```

Here is a partial printout of the output:

```
>> Angle (deg)  Acceleration (m/s/s)
    0.0        -3.4544e+004
    5.0        -3.4318e+004
   10.0        -3.3643e+004
   15.0        -3.2527e+004
   20.0        -3.0986e+004
   25.0        -2.9041e+004
   30.0        -2.6720e+004
```

Problem 13

t (s)	9	10	11
α	54.80°	54.06°	53.34°
β	65.59°	64.59°	63.62°

$$x = a\frac{\tan\beta}{\tan\beta - \tan\alpha} \qquad y = a\frac{\tan\alpha\tan\beta}{\tan\beta - \tan\alpha}$$

Using central differences of $O(h^2)$, we have for the velocities at $t = 10$ s

$$v_x = \dot{x} \approx \frac{x(11\text{ s}) - x(9\text{ s})}{2} \qquad v_y = \dot{y} \approx \frac{y(11\text{ s}) - y(9\text{ s})}{2}$$

The speed and the climb angle are

$$v = \sqrt{v_x^2 + v_y^2} \qquad \gamma = \tan^{-1}(v_y/v_x)$$

```
% problem5_1_13
alpha_9 = 54.80*pi/180; alpha_11 = 53.34*pi/180;
beta_9  = 65.59*pi/180; beta_11  = 63.62*pi/180;
x9 =  500*tan(beta_9) /(tan(beta_9)  - tan(alpha_9));
x11 = 500*tan(beta_11)/(tan(beta_11) - tan(alpha_11));
y9 = x9*tan(alpha_9); y11 = x11*tan(alpha_11);
vx = (x11 - x9)/2; vy = (y11 - y9)/2;
v = sqrt(vx^2 + vy^2)
gamma = atan(vy/vx)*180/pi

>> v =
   50.0994
gamma =
   15.1380
```

Thus $v = 50.10$ m/s and $\gamma = 15.14°$ at $t = 10s$ ◀

Problem 14

θ (deg)	0	30	60	90	120	150
β (deg)	59.96	56.42	44.10	25.72	-0.27	-34.29

$$\dot{\beta} = \frac{d\beta}{d\theta}\frac{d\theta}{dt} = \frac{d\beta}{d\theta}(1 \text{ rad/s})$$

We compute $d\beta/d\theta$ using Eq. (5.10):

$$\begin{aligned} f'_{i,i+1}(x) &= \frac{k_i}{6}\left[\frac{3(x - x_{i+1})^2}{x_i - x_{i+1}} - (x_i - x_{i+1})\right] \\ &\quad -\frac{k_{i+1}}{6}\left[\frac{3(x - x_i)^2}{x_i - x_{i+1}} - (x_i - x_{i+1})\right] + \frac{y_i - y_{i+1}}{x_i - x_{i+1}} \end{aligned} \tag{a}$$

Evaluating at $x = x_i$ yields

$$\begin{aligned} f'_{i,i+1}(x_i) &= \frac{k_i}{6}\left[\frac{3(x_i - x_{i+1})^2}{x_i - x_{i+1}} - (x_i - x_{i+1})\right] \\ &\quad -\frac{k_{i+1}}{6}\left[\frac{3(x_i - x_i)^2}{x_i - x_{i+1}} - (x_i - x_{i+1})\right] + \frac{y_i - y_{i+1}}{x_i - x_{i+1}} \end{aligned}$$

Letting $h = x_{i+1} - x_i$, the last expression reduces to

$$f'_{i,i+1}(x_i) = -\left(\frac{k_i}{3} + \frac{k_{i+1}}{6}\right)h - \frac{y_i - y_{i+1}}{h}$$

At the last knot we need to evaluate Eq. (a) at $x = x_{i+1}$:

$$\begin{aligned} f'_{i,i+1}(x_{i+1}) &= \frac{k_i}{6}\left[\frac{3(x_{i+1}-x_{i+1})^2}{x_i - x_{i+1}} - (x_i - x_{i+1})\right] \\ &\quad -\frac{k_{i+1}}{6}\left[\frac{3(x_{i+1}-x_i)^2}{x_i - x_{i+1}} - (x_i - x_{i+1})\right] + \frac{y_i - y_{i+1}}{x_i - x_{i+1}} \\ &= \left(\frac{k_i}{6} + \frac{k_{i+1}}{3}\right)h - \frac{y_i - y_{i+1}}{h} \end{aligned}$$

There is no need to convert angles into radians, since $d\beta/d\theta$ is dimensionless.

```
% problem5_1_14
x = [0 30 60 90 120 150];
y = [59.96 56.42 44.10 25.72 -0.27 -34.29];
h = 30; n = length(x) - 1;
k = splineCurv(x,y);
fprintf('Theta (deg)  Beta_dot (rad/s)\n')
for i = 1:n
    betaDot = -(k(i)/3 + k(i+1)/6)*h - (y(i) - y(i+1))/h;
    fprintf('%7.1f %15.4f\n',x(i),betaDot)
end
betaDot = (k(n)/6 + k(n+1)/3)*h - (y(n) - y(n+1))/h;
fprintf('%7.1f %15.4f\n',x(n),betaDot)
```

```
>> Theta (deg)  Beta_dot (rad/s)
    0.0         -0.0505
   30.0         -0.2530
   60.0         -0.5235
   90.0         -0.7229
  120.0         -1.0220
  120.0         -1.1900
```

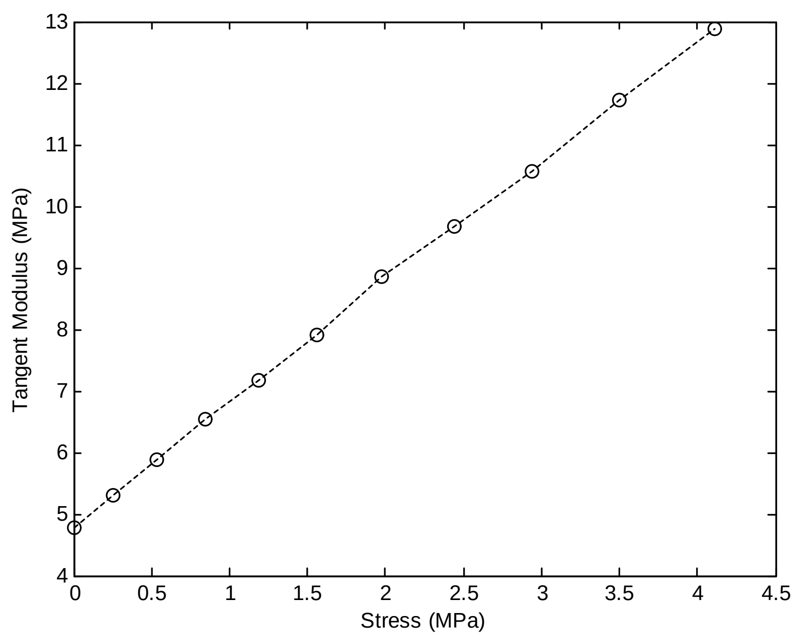
Problem 15

```
% problem5_1_15
stress = [0 0.252 0.531 0.840 1.184 1.558 1.975...
          2.444 2.943 3.500 4.115]';
strain = 0:0.05:0.5;
n = length(stress); dstrain = 0.05;

% Compute modulus = d(stress)/d(strain) by finite
% differences of O(h^2) and plot modulus vs. stress
```

```
modulus = zeros(n,1);
modulus(1) =-3*stress(1) + 4*stress(2) - stress(3);
for i = 2:n-1
    modulus(i) = -stress(i-1) + stress(i+1);
end
modulus(n) = stress(n-2) - 4*stress(n-1) + 3*stress(n);
modulus = modulus./(2*dstrain);
plot(stress,modulus,'k:o');
xlabel('Stress (MPa)'); ylabel('Tangent Modulus (MPa)')

% Use regression formulas in Eq. (3.19) to find a and b
avStress = mean(stress);
z = stress - avStress;
b = dot(modulus,z)/dot(stress,z)
a = mean(modulus) - avStress*b
```




```
b =
    1.9648

a =
    4.8433
```

The plot shows that the relationship between the tangent modulus ($d\sigma/d\varepsilon$) and the stress is indeed very close to linear. The straight line that best fits the data is

$$\frac{d\sigma}{d\varepsilon} = 4.84 + 1.96\sigma \text{ MPa} \blacktriangleleft$$

where σ is measured in MPa.