1 (30 points).

①

$$\frac{1}{x_i}\frac{\theta_{i+1}-\theta_{i-1}}{2h}+\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2}+S(x_i)=0$$ +1

②

Applying L'Hôpital's rule to the term at $x=0$,

$$\lim_{x\to 0}\frac{\frac{d\theta}{dx}}{x}=\lim_{x\to 0}\frac{\frac{d^2\theta}{dx^2}}{1}=\left.\frac{d^2\theta}{dx^2}\right|_{x=0}$$ +1

Then, the governing equation at $x=0$ becomes,

$$\frac{1}{x}\frac{d\theta}{dx}+\frac{d^2\theta}{dx^2}+S(x)=0$$

$$\rightarrow 2\frac{d^2\theta}{dx^2}+S(x)=0 \ \text{(at } x=0\text{)}$$ +1

2nd order finite difference approximation of the ODE at $x=0$ is

$$2\frac{\theta_2-2\theta_1+\theta_0}{h^2}+S(x_1)=0$$ +1

θ_0 can be eliminated with the boundary condition of $\theta'(0)=0$

$$\frac{\theta_2-\theta_0}{2h}=0 \rightarrow \theta_2=\theta_0$$ +1

Then, the finite difference approximation becomes

$$\therefore 4\frac{\theta_2-\theta_1}{h^2}+S(x_1)=0$$ +1

③

the governing equation at $x=1$ is

$$\frac{1}{x}\frac{d\theta}{dx}+\frac{d^2\theta}{dx^2}+S(x)=0$$

2nd order finite difference approximation of the ODE at $x=1$ is

$$\frac{1}{x_n}\frac{\theta_{n+1}-\theta_{n-1}}{2h}+\frac{\theta_{n+1}-2\theta_n+\theta_{n-1}}{h^2}+S(x_n)=0$$ +1

θ_{n+1} can be eliminated with the boundary condition of $\theta'(1)+\theta(1)=0$

$$\frac{\theta_{n+1}-\theta_{n-1}}{2h}+\theta_n=0 \rightarrow \theta_{n+1}=\theta_{n-1}-2h\cdot\theta_n$$ +1

Then, the finite difference approximation becomes

$$\therefore -\frac{\theta_n}{x_n}+\frac{2\theta_{n-1}-(2h+2)\theta_n}{h^2}+S(x_n)=0$$ +1

④

First, we rewrite the finite differential approximation as follows:

1) For $i = 1$,

$$4\frac{\theta_2 - \theta_1}{h^2} + S(x_1) = 0$$

$$\Rightarrow -4\theta_1 + 4\theta_2 = -h^2 S(x_1) \qquad +1$$

2) For $i = 2,3,\dots,n-1$,

$$\frac{1}{x_i}\frac{\theta_{i+1} - \theta_{i-1}}{2h} + \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + S(x_i) = 0$$

$$\Rightarrow \frac{h}{2x_i}(\theta_{i+1} - \theta_{i-1}) + \theta_{i+1} - 2\theta_i + \theta_{i-1} + h^2 S(x_i) = 0$$

$$\Rightarrow \left(1 - \frac{h}{2x_i}\right)\theta_{i-1} - 2\theta_i + \left(1 + \frac{h}{2x_i}\right)\theta_{i+1} = -h^2 S(x_i) \qquad +2$$

3) For $i = n$,

$$-\frac{\theta_n}{x_n} + \frac{2\theta_{n-1} - (2h+2)\theta_n}{h^2} + S(x_n) = 0$$

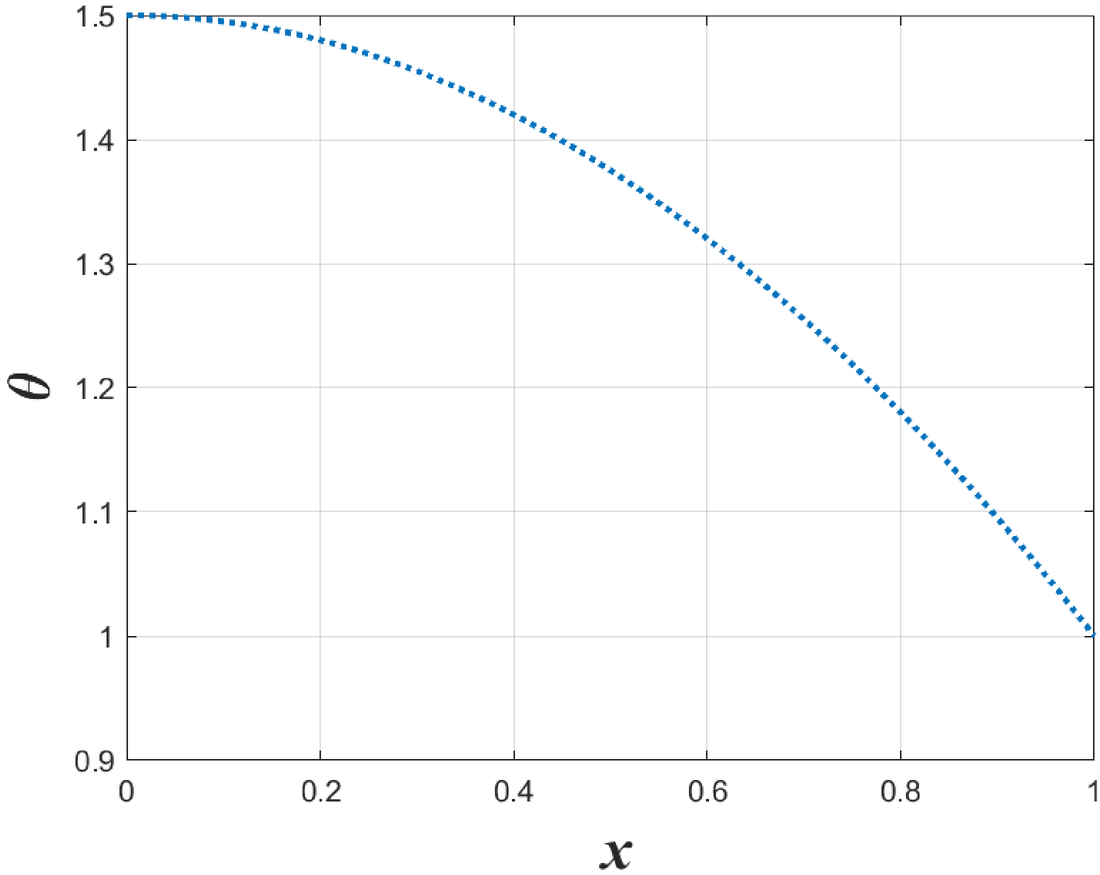
$$\Rightarrow -\frac{h^2}{x_n}\theta_n - (2h+2)\theta_n + 2\theta_{n-1} + h^2 S(x_n) = 0$$

$$\Rightarrow 2\theta_{n-1} - \left(\frac{h^2}{x_n} + 2h + 2\right)\theta_n = -h^2 S(x_n) \qquad +1$$

$$\begin{bmatrix} -4 & 4 & & & & \\ 1 - \frac{h}{2x_i} & -2 & 1 + \frac{h}{2x_i} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 - \frac{h}{2x_i} & -2 & 1 + \frac{h}{2x_i} \\ & & & & 2 & -\left(\frac{h^2}{x_n} + 2h + 2\right) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{n-1} \\ \theta_n \end{bmatrix} = \begin{bmatrix} -h^2 S(x_1) \\ -h^2 S(x_2) \\ \vdots \\ -h^2 S(x_{n-1}) \\ -h^2 S(x_n) \end{bmatrix} \qquad +2$$

The coefficient matrix is tridiagonal, so that the equations can be solved efficiently by using LU decomposition method.

⑤ Programming



+15

2 (25 points).

①

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ z \\ z' \end{bmatrix} \quad +1.5 \qquad \text{and} \qquad \boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \\ z' \\ z'' \end{bmatrix} = \begin{bmatrix} y_2 \\ \alpha^2 y_3 \sin(x + y_1) \\ y_4 \\ \cos(x + y_1) \end{bmatrix} \quad +1.5$$

②

```
function y=inCond(u)
    y = [0 u(1) u(2) 0];
end
```

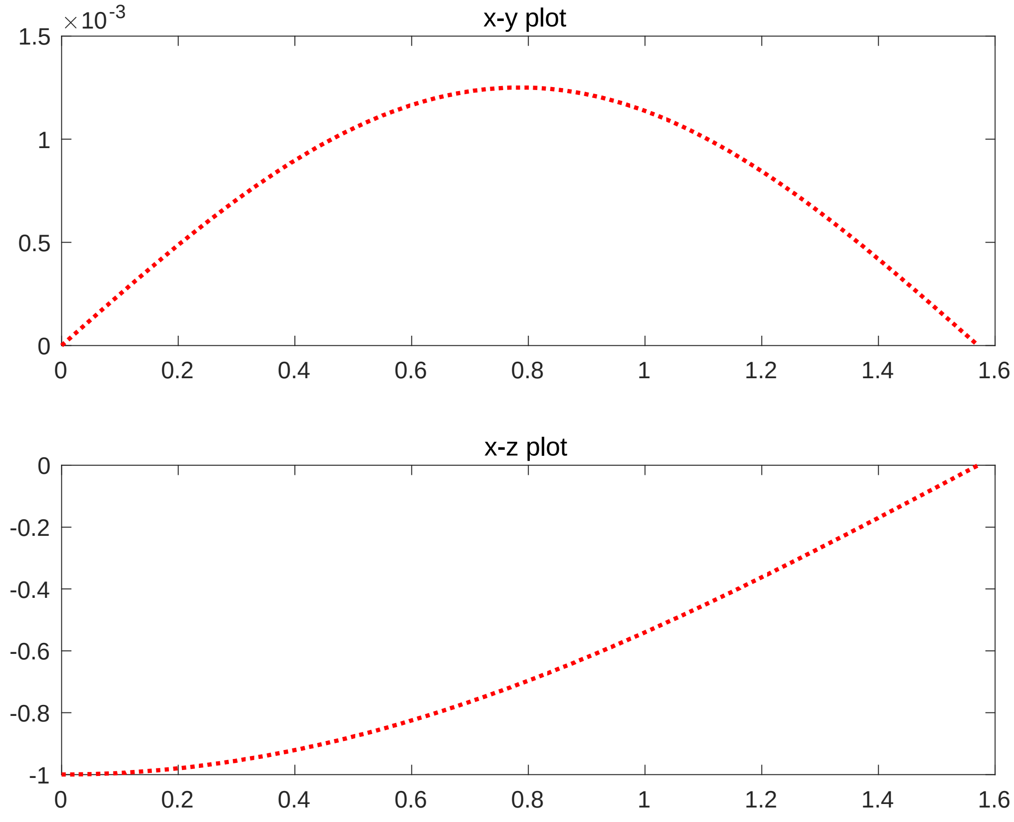
+3.5

③

```
r(1)=ySol(lastRow,1);
r(2)=ySol(lastRow,3);
```

+3.5

④ Programming



+15

3 (25 points).

①

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \alpha^2 z_i \sin(x_i + y_i) \qquad +1$$

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} = \cos(x_i + y_i) \qquad +1$$

②

$$r(1) = y(1) \ \ / \ \ r(n) = y(n) \ \ / \ \ r(2n) = y(2n) \qquad \text{+0.5 for each}$$

2nd order finite difference approximation of the ODE at $z = 0$ is

$$\frac{z_2 - 2z_1 + z_0}{h^2} = \cos(x_1 + y_1) \qquad +0.5$$

z_0 can be eliminated with the boundary condition of $z'(0) = 0$

$$\frac{z_2 - z_0}{2h} = 0 \ \rightarrow \ z_2 = z_0 \qquad +1$$

Then, the finite difference approximation becomes

$$\therefore \frac{2z_2 - 2z_1}{h^2} = \cos(x_1 + y_1) \qquad +1$$

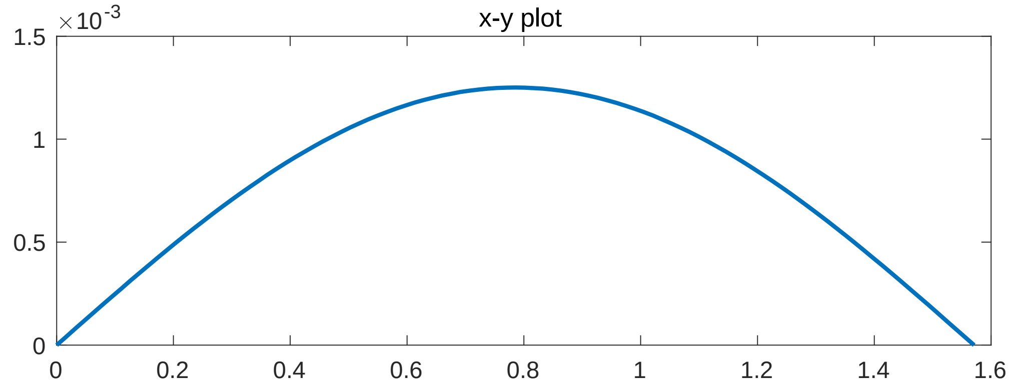
$$\rightarrow r(n+1) = \frac{2y(n+2) - 2y(n+1)}{h^2} - \cos[x(1) + y(1)] \qquad +1$$

For $i = 2:n-1$

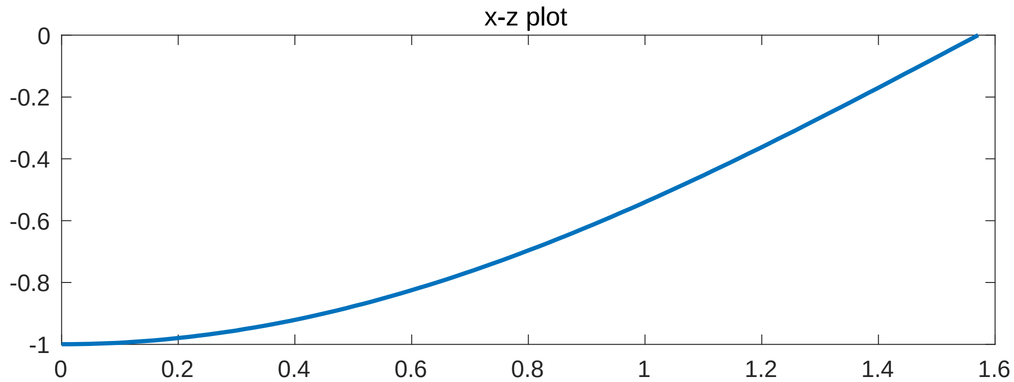
$$r(i) = \frac{y(i+1) - 2y(i) + y(i-1)}{h^2} - \alpha^2 y(n+i)\sin[x(i) + y(i)] \qquad +1.5$$

$$r(n+i) = \frac{y(n+i+1) - 2y(n+i) + y(n+i-1)}{h^2} - \cos[x(i) + y(i)] \qquad +1.5$$

③ Programming



+15



4 (20 points).

$$\mathbf{A} = \begin{bmatrix} 3 & -3 & 3 \\ -3 & 5 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 9 \\ -7 \\ 12 \end{bmatrix}$$

$$\begin{array}{l} \text{row } 2 \leftarrow \text{row } 2 - (-1) \times \text{row } 1 \text{ (eliminates } A_{21}) \\ \text{row } 3 \leftarrow \text{row } 3 - 1 \times \text{row } 1 \text{ (eliminates } A_{21}) \end{array} \Rightarrow \text{Storing the mulipliers } L_{21} = -1 \text{ and } L_{21} = 1$$

$$\Rightarrow \mathbf{A}' = \begin{bmatrix} 3 & -3 & 3 \\ [-1] & 2 & 4 \\ [1] & 4 & 2 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - 2 \times \text{row } 2 \text{ (eliminates } A_{32}) \Rightarrow \text{Storing the mulipliers } L_{32} = 2$$

$$\Rightarrow \mathbf{A}'' = [\mathbf{L} \setminus \mathbf{U}] = \begin{bmatrix} 3 & -3 & 3 \\ [-1] & 2 & 4 \\ [1] & [2] & -6 \end{bmatrix} \Rightarrow \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 3 & -3 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{bmatrix}$$

+5.5 +5.5

Solving $\mathbf{Ly} = \mathbf{b}$ by forward substitution

$$[\mathbf{L}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ -1 & 1 & 0 & -7 \\ 1 & 2 & 1 & 12 \end{array}\right] \Rightarrow \begin{array}{c} y_1 = 9 \\ y_2 = -7 + y_1 = -7 + 9 = 2 \\ y_3 = 12 - 2y_2 - y_1 = 12 - 2(2) - 9 = -1 \end{array}$$

+1.5 for each

Solving $\mathbf{Ux} = \mathbf{y}$ by forward substitution

$$[\mathbf{U}|\mathbf{y}] = \left[\begin{array}{ccc|c} 3 & -3 & 3 & 9 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -6 & -1 \end{array}\right] \Rightarrow \begin{array}{c} x_3 = \dfrac{1}{6} \\ x_2 = \dfrac{2 - 4x_3}{2} = \dfrac{2 - 4\left(\frac{1}{6}\right)}{2} = \dfrac{2}{3} \\ x_1 = \dfrac{9 + 3x_2 - 3x_3}{3} = \dfrac{9 + 3\left(\frac{2}{3}\right) - 3\left(\frac{1}{6}\right)}{3} = \dfrac{7}{2} \end{array}$$

+1.5 for each