1 (50 points).

a)

①

$\frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -u\delta_x\phi^n - v\delta_y\phi^n + O(\Delta t, \Delta x^2, \Delta y^2)$ (FTCS)

$\frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -u\delta_x\phi^{n+1} - v\delta_y\phi^{n+1} + O(\Delta t, \Delta x^2, \Delta y^2)$ (BTCS)

Crank-Nicholson method is the average of FTCS and BTCS,

$$\therefore \frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -\frac{u}{2}\delta_x(\phi^{n+1}+\phi^n) - \frac{v}{2}\delta_y(\phi^{n+1}+\phi^n) + O(\Delta t^2, \Delta x^2, \Delta y^2)$$ +3

②

Answer in ① can be rewritten as

$$\left(\mathbf{I} + \frac{u\Delta t}{2}\delta_x + \frac{v\Delta t}{2}\delta_y\right)\phi^{n+1} = \left(\mathbf{I} - \frac{u\Delta t}{2}\delta_x - \frac{v\Delta t}{2}\delta_y\right)\phi^{n} + \Delta t O(\Delta t^2, \Delta x^2, \Delta y^2)$$ +3

Factoring each side

$$\left(\mathbf{I} + \frac{u\Delta t}{2}\delta_x\right)\left(\mathbf{I} + \frac{v\Delta t}{2}\delta_y\right)\phi^{n+1} - \frac{uv\Delta t^2}{4}\delta_x\delta_y\phi^{n+1}$$
$$= \left(\mathbf{I} - \frac{u\Delta t}{2}\delta_x\right)\left(\mathbf{I} - \frac{v\Delta t}{2}\delta_y\right)\phi^{n} - \frac{uv\Delta t^2}{4}\delta_x\delta_y\phi^{n} + \Delta t O(\Delta t^2, \Delta x^2, \Delta y^2)$$

or

$$\therefore \left(\mathbf{I} + \frac{u\Delta t}{2}\delta_x\right)\left(\mathbf{I} + \frac{v\Delta t}{2}\delta_y\right)\phi^{n+1} = \left(\mathbf{I} - \frac{u\Delta t}{2}\delta_x\right)\left(\mathbf{I} - \frac{v\Delta t}{2}\delta_y\right)\phi^{n} + \Delta t O(\Delta t^2, \Delta x^2, \Delta y^2)$$

where $\frac{uv\Delta t^2}{4}\delta_x\delta_y\underbrace{(\phi^{n+1}-\phi^n)}_{O(\Delta t)}$ is included in error term, $\Delta t O(\Delta t^2, \Delta x^2, \Delta y^2)$ +4

③

For $j = 2 \sim J-1$,

$$\therefore g_{i,j}^n = \phi_{i,j}^n - \frac{v\Delta t}{2}\left(\frac{\phi_{i,j+1}^n - \phi_{i,j-1}^n}{2\Delta y}\right)$$ +3

④

$$R_{i,j}^n = \left(\mathbf{I} - \frac{u\Delta t}{2}\delta_x\right)g_{i,j}^n$$

For $i = 2 \sim I-1$,

$$\therefore R_{i,j}^n = g_{i,j}^n - \frac{u\Delta t}{2}\left(\frac{g_{i+1,j}^n - g_{i-1,j}^n}{2\Delta x}\right)$$ +3

⑤

$$\left(\mathbf{I}+\frac{u\Delta t}{2}\delta_x\right)\phi_{i,j}^{*}=R_{i,j}^{n}$$

For $j=2\sim J-1$,

$$\phi_{i,j}^{*}+\frac{u\Delta t}{2}\left(\frac{\phi_{i+1,j}^{*}-\phi_{i-1,j}^{*}}{2\Delta x}\right)=R_{i,j}^{n}$$

$$\rightarrow -\frac{u\Delta t}{4\Delta x}\phi_{i-1,j}^{*}+\phi_{i,j}^{*}+\frac{u\Delta t}{4\Delta x}\phi_{i+1,j}^{*}=R_{i,j}^{n}$$

or

$$\therefore -r_x\phi_{i-1,j}^{*}+\phi_{i,j}^{*}+r_x\phi_{i+1,j}^{*}=R_{i,j}^{n} \qquad +3$$

where

$$r_x=\frac{u\Delta t}{4\Delta x}$$

⑥

For $j=2\sim J-1$,

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -r_x & 1 & r_x & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -r_x & 1 & r_x \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \phi_{1,j}^{*} \\ \phi_{2,j}^{*} \\ \vdots \\ \phi_{I-1,j}^{*} \\ \phi_{I,j}^{*} \end{bmatrix}=\begin{bmatrix} 0 \\ R_{2,j}^{n} \\ \vdots \\ R_{I-1,j}^{n} \\ 0 \end{bmatrix} \qquad +4$$

⑦

For $i=2\sim I-1$,

$$\left(\mathbf{I}+\frac{v\Delta t}{2}\delta_y\right)\phi_{i,j}^{n+1}=\phi_{i,j}^{*}$$

$$\phi_{i,j}^{n+1}+\frac{v\Delta t}{2}\left(\frac{\phi_{i,j+1}^{n+1}-\phi_{i,j-1}^{n+1}}{2\Delta y}\right)=\phi_{i,j}^{*}$$

$$\rightarrow -\frac{v\Delta t}{4\Delta y}\phi_{i,j-1}^{n+1}+\phi_{i,j}^{n+1}+\frac{v\Delta t}{4\Delta y}\phi_{i,j+1}^{n+1}=\phi_{i,j}^{*}$$

or

$$\therefore -r_y\phi_{i,j-1}^{n+1}+\phi_{i,j}^{n+1}+r_y\phi_{i,j+1}^{n+1}=\phi_{i,j}^{*} \qquad +3$$

where

$$r_y=\frac{v\Delta t}{4\Delta y}$$

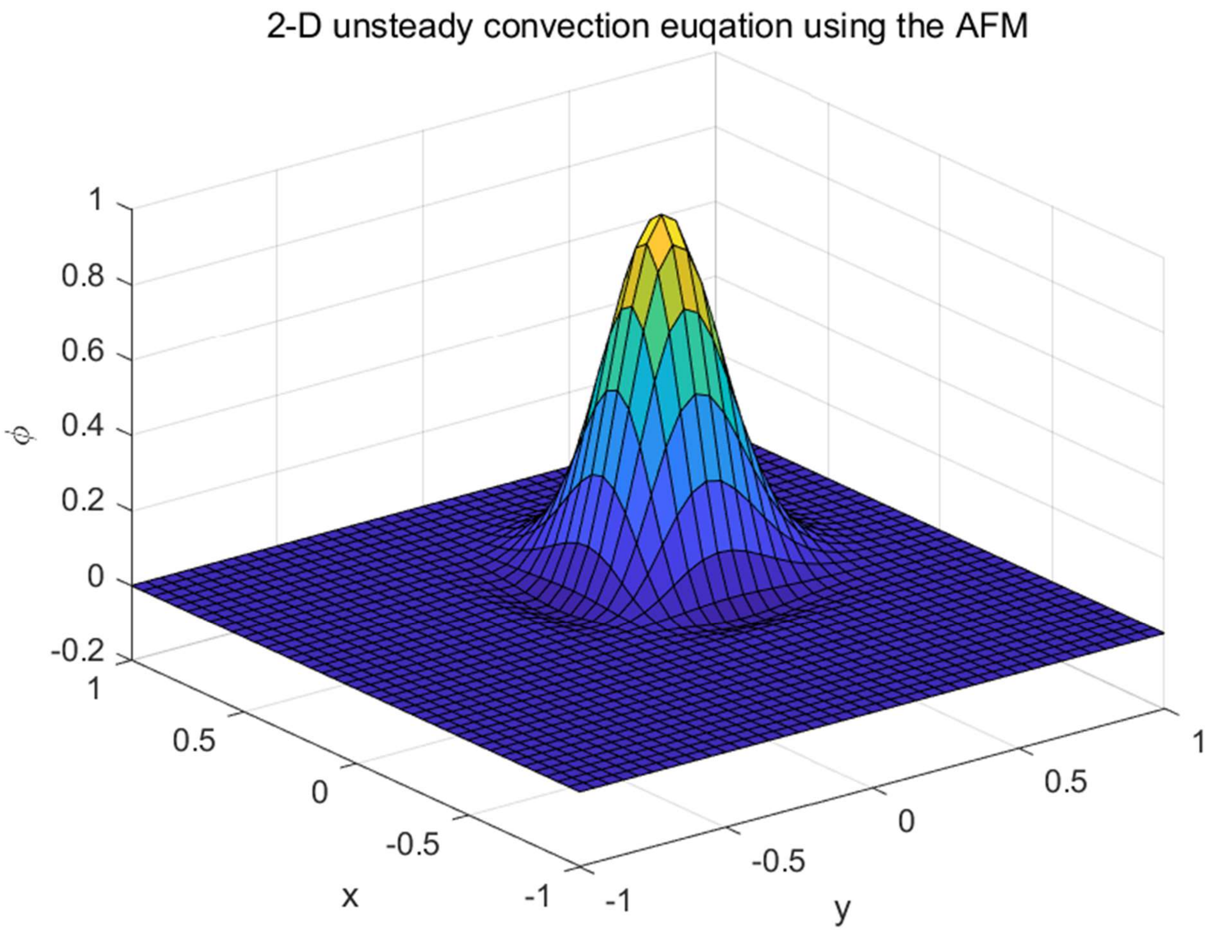
⑧

For $i = 2 \sim I - 1$,

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -r_y & 1 & r_y & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -r_y & 1 & r_y \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{i,1}^{n+1} \\ \phi_{i,2}^{n+1} \\ \vdots \\ \phi_{i,J-1}^{n+1} \\ \phi_{i,J}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_{i,2}^{*} \\ \vdots \\ \phi_{i,J-1}^{*} \\ 0 \end{bmatrix}$$

+4

b) (Programming) +20



2 (25 points).

a)

For $i = 2 \sim I - 1$,

$$\frac{df_i}{dt} = v\frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} - f_i\frac{f_{i+1} - f_{i-1}}{2h} \qquad +5$$

and

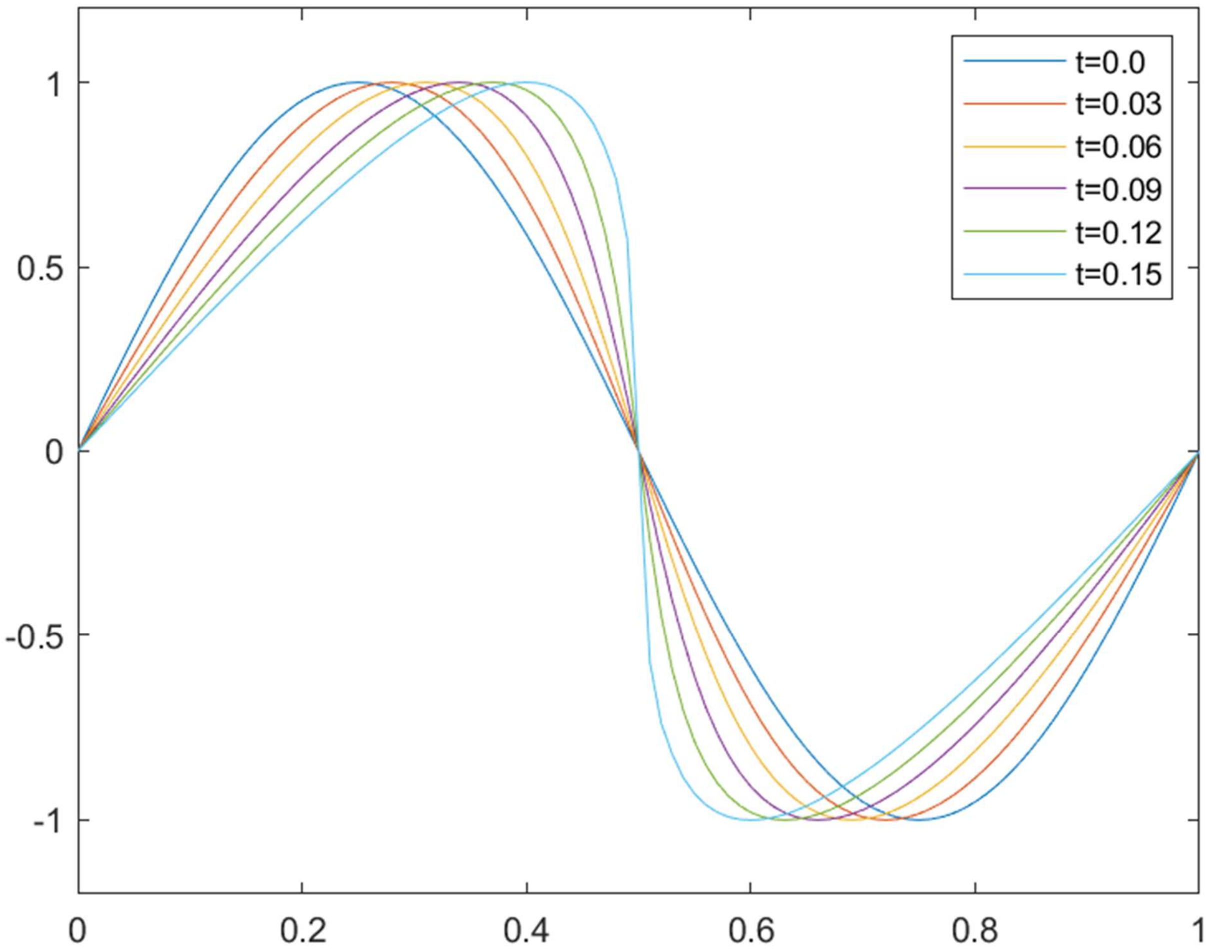
$$\frac{df_1}{dt} = 0 \quad / \quad \frac{df_I}{dt} = 0 \qquad +2.5 \,/\, +2.5$$

or

or

$$\vec{F}(t, \vec{f}) = \frac{d\vec{f}}{dt} = \begin{bmatrix} \frac{df_1}{dt} \\ \vdots \\ \frac{df_i}{dt} \\ \vdots \\ \frac{df_I}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ v\frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} - f_i\frac{f_{i+1} - f_{i-1}}{2h} \\ \vdots \\ 0 \end{bmatrix} \qquad +10$$

b) (Programming) +15



3 (30 points).

a)

Discretized Poisson equation

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{h^2} = 0$$

Rearranging for $T_{i,j}$

$$T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right)$$

SOR iteration is

$$\therefore T_{i,j}^{n+1} = \frac{\beta}{4}\left(T_{i+1,j}^{n} + T_{i-1,j}^{n+1} + T_{i,j+1}^{n} + T_{i,j-1}^{n+1}\right) + (1-\beta)T_{i,j}^{n}$$ +5

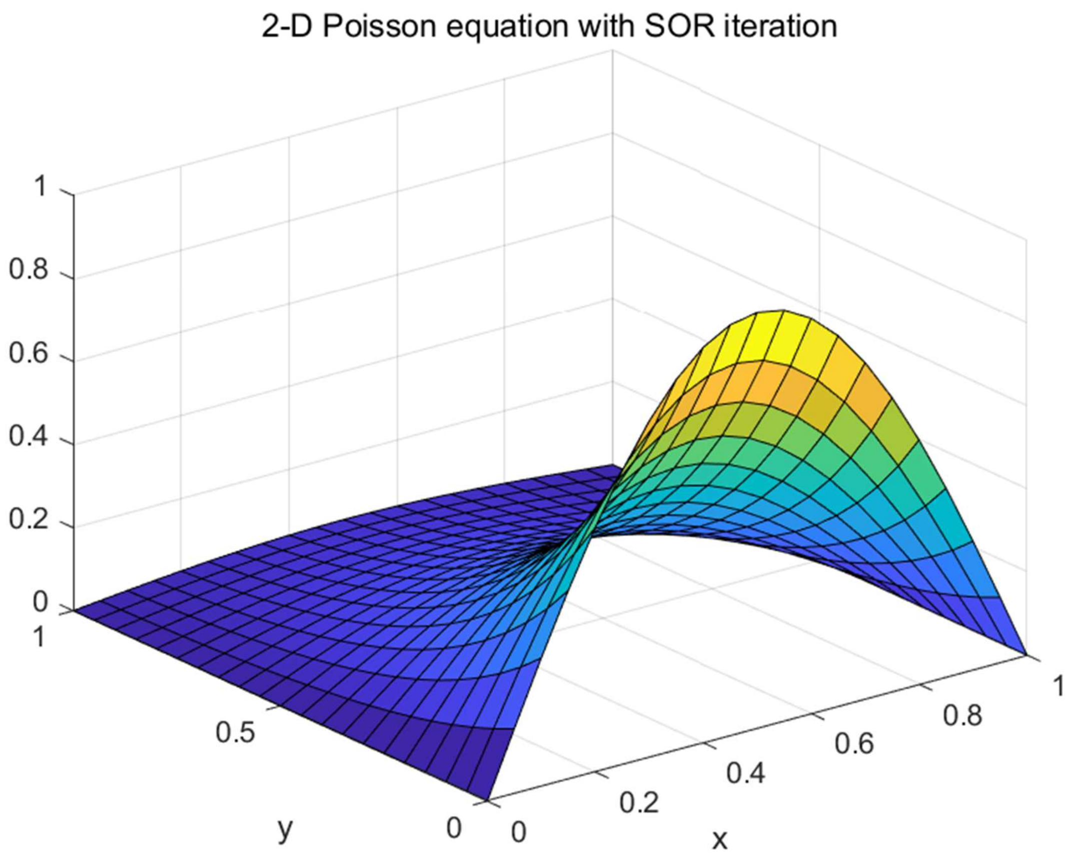
b)

$$-\frac{1}{4}\tilde{T}_{i-1,j} + \tilde{T}_{i,j} - \frac{1}{4}\tilde{T}_{i+1,j} = \frac{1}{4}\left(T_{i,j-1}^{n+1} + T_{i,j+1}^{n}\right)$$ +2.5

And then over-relax

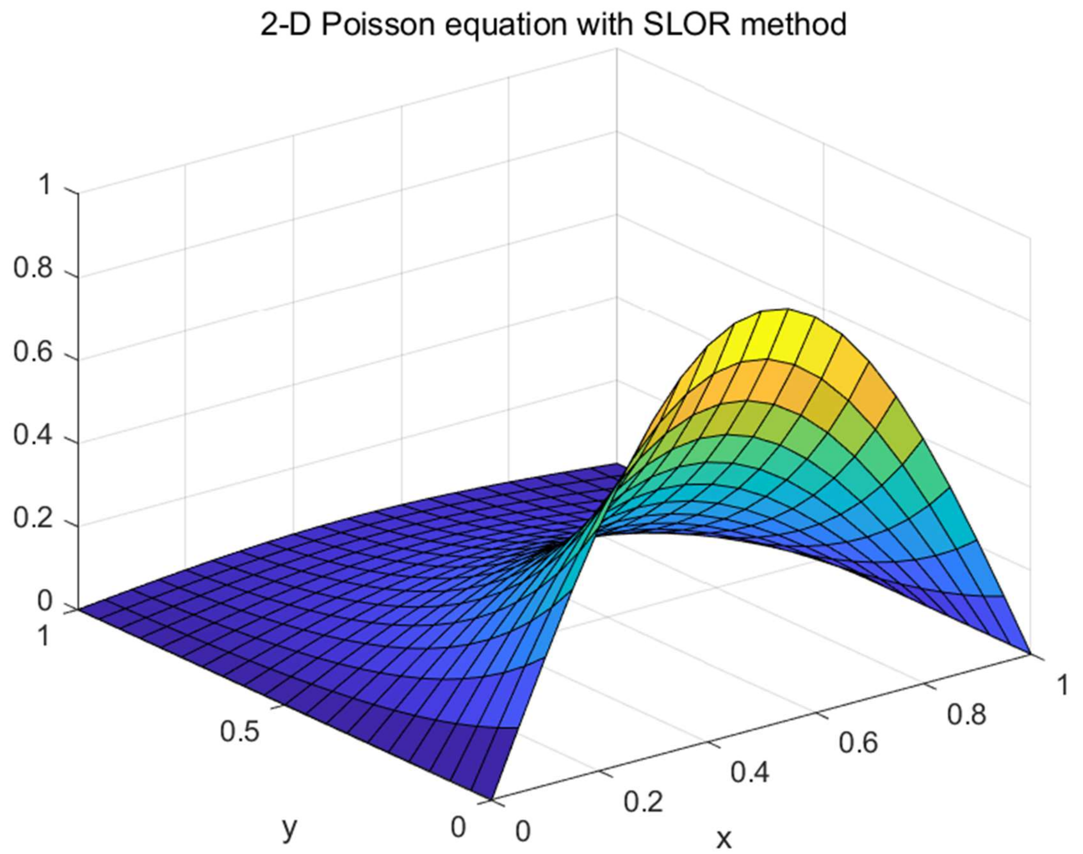
$$\therefore T_{i,j}^{n+1} = \beta\tilde{T}_{i,j} + (1-\beta)T_{i,j}^{n}$$ +2.5

c) (Programming)



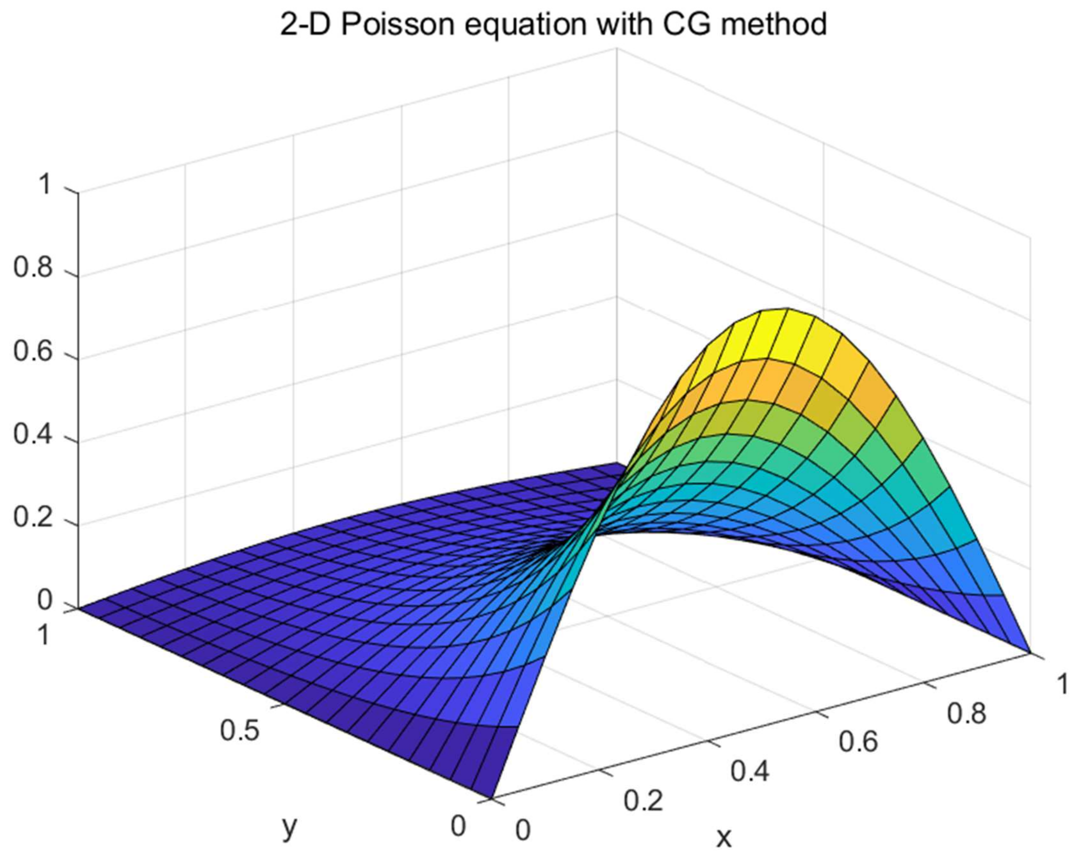
+5

d) (Programming)



+10

e) (Programming)



+5

4 (25 points).

a)

$$k_{i-1}(x_{i-1}-x_i)+2k_i(x_{i-1}-x_{i+1})+k_{i+1}(x_i-x_{i+1})=6\left(\frac{y_{i-1}-y_i}{x_{i-1}-x_i}-\frac{y_i-y_{i+1}}{x_i-x_{i+1}}\right), i=1,2,3$$

or

+4

If the data points are evenly spaced at intervals *h*, the above equation simplify to

$$k_{i-1}+4k_i+k_{i+1}=6(y_{i-1}-2y_i+y_{i+1}), i=1,2,3$$

b)

$$k_0+4k_1+k_2=5k_1+k_2=6(1-2(1)+0.5)=-3$$ +1

$$k_1+4k_2+k_3=6(1-2(0.5)+0)=0$$ +1

$$k_2+4k_3+k_4=k_2+5k_3=6(0.5-2(0)+(-0.5))=0$$ +1

$$\rightarrow \begin{bmatrix}5 & 1 & 0\\1 & 4 & 1\\0 & 1 & 5\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}-3\\0\\0\end{bmatrix}$$ +4

Using Doolittle's decomposition,

$$\text{row 2} \leftarrow \text{row 2} - \frac{1}{5}\times \text{row 1 (eliminates } A_{21}) \Rightarrow \text{Storing the mulipliers } L_{21}=\frac{1}{5}$$

$$\Rightarrow \mathbf{A}'=\begin{bmatrix}5 & 1 & 0\\ [1/5] & {}^{19}/_{5} & 1\\ [0] & 1 & 5\end{bmatrix}$$

$$\text{row 3} \leftarrow \text{row 3} - \frac{5}{19}\times \text{row 2 (eliminates } A_{32}) \Rightarrow \text{Storing the mulipliers } L_{32}=\frac{5}{19}$$

$$\Rightarrow \mathbf{A}''=[\mathbf{L}\setminus\mathbf{U}]=\begin{bmatrix}5 & 1 & 0\\ \left[{}^{1}/_{5}\right] & {}^{19}/_{5} & 1\\ [0] & \left[{}^{5}/_{19}\right] & {}^{90}/_{19}\end{bmatrix}$$

$$\Rightarrow \mathbf{L}=\begin{bmatrix}1 & 0 & 0\\ {}^{1}/_{5} & 1 & 0\\ 0 & {}^{5}/_{19} & 1\end{bmatrix} \quad \mathbf{U}=\begin{bmatrix}5 & 1 & 0\\ 0 & {}^{19}/_{5} & 1\\ 0 & 0 & {}^{90}/_{19}\end{bmatrix}$$ +4

c)

Solving $\mathbf{Ly}=\mathbf{b}$ by forward substitution

$$[\mathbf{L}|\mathbf{b}]=\left[\begin{array}{ccc|c}1 & 0 & 0 & -3\\ {}^{1}/_{5} & 1 & 0 & 0\\ 0 & {}^{5}/_{19} & 1 & 0\end{array}\right] \Rightarrow$$

$$y_1=-3$$ +1

$$y_2=-\frac{y_1}{5}=\frac{3}{5}$$ +1

$$y_3=-\frac{5}{19}y_2=-\frac{5}{19}\left(\frac{3}{5}\right)=-\frac{3}{19}$$ +1

Solving $\mathbf{Uk} = \mathbf{y}$ by forward substitution

$$k_3 = -\frac{1}{30} \qquad +1$$

$$[\mathbf{U}|\mathbf{y}] = \left[\begin{array}{ccc|c} 5 & 1 & 0 & -3 \\ 0 & {}^{19}\!/_{5} & 1 & \frac{3}{5} \\ 0 & 0 & {}^{90}\!/_{19} & -\frac{3}{19} \end{array}\right] \Rightarrow k_2 = \frac{\frac{3}{5} - k_3}{\frac{19}{5}} = \frac{3 - 5\left(-\frac{1}{30}\right)}{19} = \frac{1}{6} \qquad +1$$

$$k_1 = \frac{-3 - k_2}{5} = \frac{-3 - \left(\frac{1}{6}\right)}{5} = -\frac{19}{30} \qquad +1$$

d)

$$f_{i,i+1}(x) = \frac{k_i}{6}\left[\frac{(x - x_{i+1})^3}{x_i - x_{i+1}} - (x - x_{i+1})(x_i - x_{i+1})\right] - \frac{k_{i+1}}{6}\left[\frac{(x - x_i)^3}{x_i - x_{i+1}} - (x - x_i)(x_i - x_{i+1})\right] + \frac{y_i(x - x_{i+1}) - y_{i+1}(x - x_i)}{x_i - x_{i+1}}$$

$$\rightarrow f_{3,4}(x) = -\frac{k_3}{6}[(x - x_4)^3 - (x - x_4)] + \frac{k_4}{6}[(x - x_3)^3 - (x - x_3)] - [y_3(x - x_4) - y_4(x - x_3)] \qquad +3$$

$$\therefore f_{3,4}(3) = 0 \qquad +1$$