1. (20 points)

a)

$$y_1 = F \qquad y_1' = y_4$$
$$y_2 = G \qquad y_2' = y_5$$
$$y_3 = H \quad \rightarrow \quad y_3' = -2y_1$$
$$y_4 = F' \qquad y_4' = y_1^2 - y_2^2 + y_4 y_3$$
$$y_5 = G' \qquad y_5' = 2y_1 y_2 + y_3 y_5$$

+5

b)

$$y_1(0) = 0$$
$$y_2(0) = 1$$
$$y_3(0) = 0$$
$$y_4(0) = u(1)$$
$$y_5(0) = u(2)$$

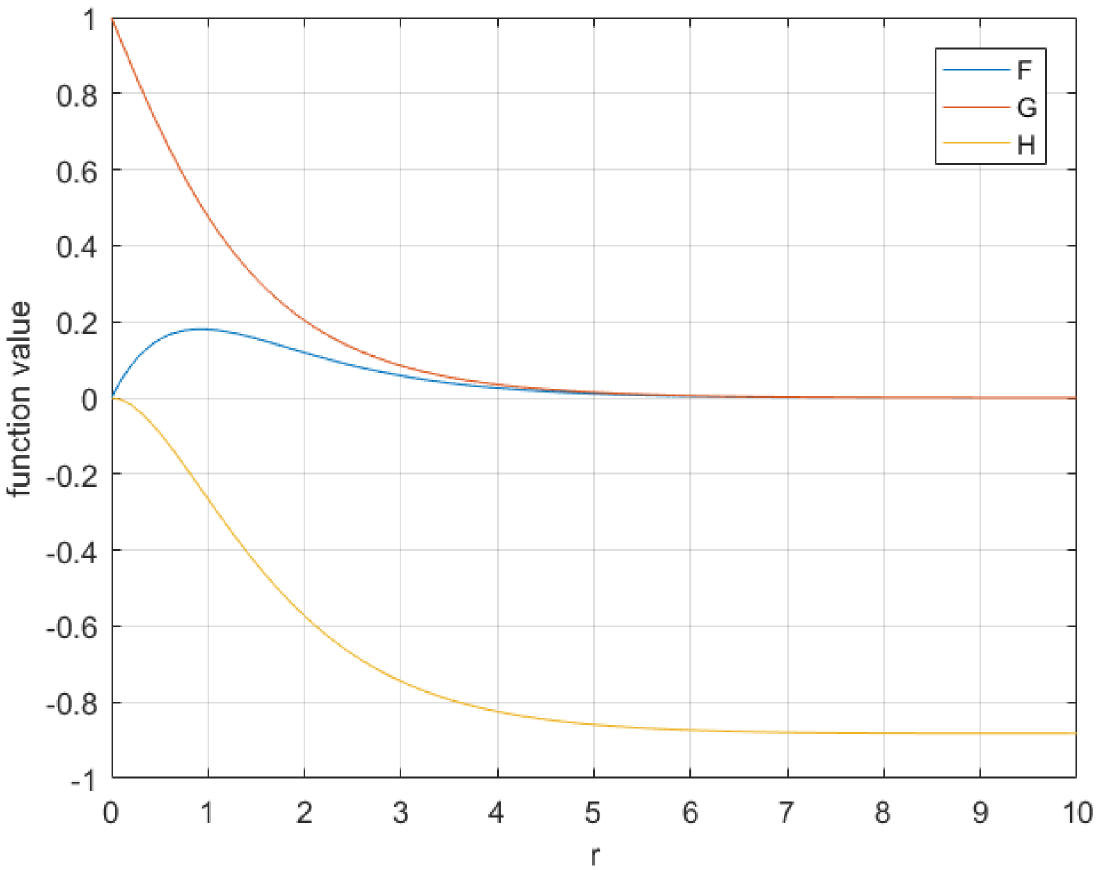
+3

c)

$$r(1) = y_{1'}(10) - 0$$
$$r(2) = y_2(10) - 0$$

+2

d) (Programming)



2. (30 points)

a)

$$\frac{T_{i+1}-2T_i+T_{i-1}}{\Delta x^2}+x_i\frac{T_{i+1}-T_{i-1}}{2\Delta x}=-qDX_iY_i\exp(-\frac{T_a}{T_i})$$

$$\frac{1}{Le_x}\frac{X_{i+1}-2X_i+X_{i-1}}{\Delta x^2}+x_i\frac{X_{i+1}-X_{i-1}}{2\Delta x}=\alpha_x DX_iY_i\exp(-\frac{T_a}{T_i})$$

$$\frac{1}{Le_Y}\frac{Y_{i+1}-2Y_i+Y_{i-1}}{\Delta x^2}+x_i\frac{Y_{i+1}-Y_{i-1}}{2\Delta x}=\alpha_Y DX_iY_i\exp(-\frac{T_a}{T_i})$$

+6

b)

$$r(1)=y(1)-0.2$$
$$r(n)=y(n)-0.2$$
$$r(n+1)=y(n+1)-1$$
$$r(2n)=y(2n)-0$$
$$r(2n+1)=y(2n+1)-0$$
$$r(3n)=y(3n)-1$$

+3

for $i=2:n-1$

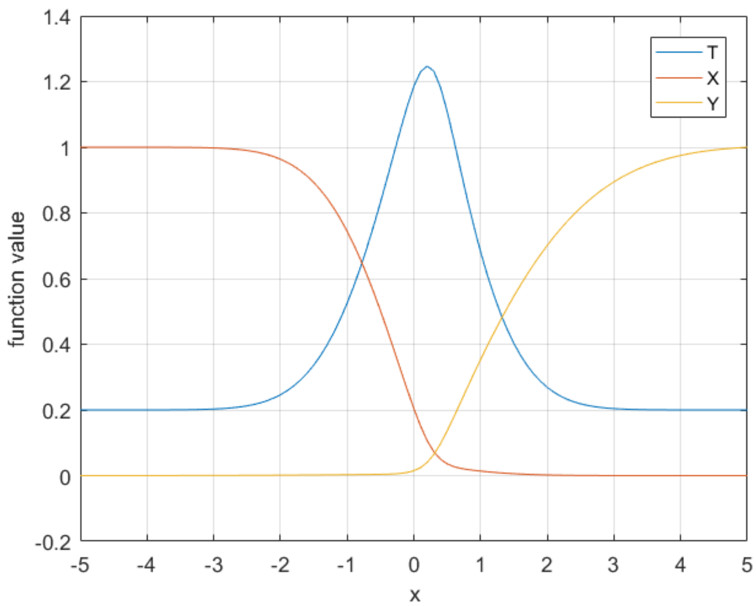
$$r(i)=\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta x^2}+x_i\frac{y_{i+1}-y_{i-1}}{2\Delta x}+qDy_{(i+n)}y_{(i+2n)}\exp(-\frac{T_a}{y_i})$$

$$r(i+n)=\frac{1}{Le_x}\frac{y_{i+n+1}-2y_{i+n}+y_{i+n-1}}{\Delta x^2}+x_i\frac{y_{i+n+1}-y_{i+n-1}}{2\Delta x}-\alpha_x Dy_{(i+n)}y_{(i+2n)}\exp(-\frac{T_a}{y_i})$$

$$r(i+2n)=\frac{1}{Le_Y}\frac{y_{i+2n+1}-2Y_{i+2n}+Y_{i+2n-1}}{\Delta x^2}+x_i\frac{y_{i+2n+1}-y_{i+2n-1}}{2\Delta x}-\alpha_Y Dy_{(i+n)}y_{(i+2n)}\exp(-\frac{T_a}{y_i})$$

+6

c) (Programming)



3. (40 points)

①

$$\frac{T_{i,j}^{n+1/2}-T_{i,j}^{n}}{\Delta t/2}=\alpha(\frac{T_{i+1,j}^{n+1/2}-2T_{i,j}^{n+1/2}+T_{i-1,j}^{n+1/2}}{h^2}+\frac{T_{i,j+1}^{n}-2T_{i,j}^{n}+T_{i,j-1}^{n}}{h^2})+S_{i,j}$$

$$\Rightarrow\ -rT_{i+1,j}^{n+1/2}+(1+2r)T_{i,j}^{n+1/2}-rT_{i-1,j}^{n+1/2}=rT_{i,j+1}^{n}+(1-2r)T_{i,j}^{n}+rT_{i,j-1}^{n}+\frac{\Delta t}{2}S_{i,j}\quad\left(r=\frac{\alpha\Delta t}{2h^2}\right)\quad +2$$

②

For i = 1

$$\Rightarrow\ T_{1,j}=1\quad +1$$

For i = I

$$\Rightarrow\ \left.\frac{\partial T}{\partial x}\right|_{1,y,t}\approx\frac{T_{I+1,j}-T_{I-1,j}}{2h}=0\quad\Rightarrow\quad T_{I+1,j}=T_{I-1,j}\quad +1$$

$$\Rightarrow\ -rT_{I+1,j}^{n+1/2}+(1+2r)T_{I,j}^{n+1/2}-rT_{I-1,j}^{n+1/2}=rT_{I,j+1}^{n}+(1-2r)T_{I,j}^{n}+rT_{I,j-1}^{n}+\frac{\Delta t}{2}S_{I,j}$$

$$\Rightarrow\ -rT_{I-1,j}^{n+1/2}+(1+2r)T_{I,j}^{n+1/2}-rT_{I-1,j}^{n+1/2}=rT_{I,j+1}^{n}+(1-2r)T_{I,j}^{n}+rT_{I,j-1}^{n}+\frac{\Delta t}{2}S_{I,j}$$

$$\Rightarrow\ -2rT_{I-11,j}^{n+1/2}+(1+2r)T_{I,j}^{n+1/2}=rT_{I,j+1}^{n}+(1-2r)T_{I,j}^{n}+rT_{I,j-1}^{n}+\frac{\Delta t}{2}S_{I,j}\quad +2$$

③

$$\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ -r & 1+2r & -r & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & -r & 1+2r & -r\\ 0 & \cdots & 0 & -2r & 1+2r\end{bmatrix}\begin{bmatrix}T_{1,,j}^{n+1/2}\\ T_{2,j}^{n+1/2}\\ \vdots\\ T_{I-1,j}^{n+1/2}\\ T_{I,j}^{n+1/2}\end{bmatrix}=\begin{bmatrix}1\\ rT_{2,j+1}^{n}+(1-2r)T_{2,j}^{n}+rT_{2,j-1}^{n}+\frac{\Delta t}{2}S_{2,j}\\ \vdots\\ rT_{I-1,j+1}^{n}+(1-2r)T_{I-1,j}^{n}+rT_{I-1,j-1}^{n}+\frac{\Delta t}{2}S_{I-1,j}\\ rT_{I,j+1}^{n}+(1-2r)T_{I,j}^{n}+rT_{I,j-1}^{n}+\frac{\Delta t}{2}S_{I,j}\end{bmatrix}\quad +3$$

④

$$\frac{T_{i,j}^{n+1}-T_{i,j}^{n+1/2}}{\Delta t/2}=\alpha(\frac{T_{i+1,j}^{n+1/2}-2T_{i,j}^{n+1/2}+T_{i-1,j}^{n+1/2}}{h^2}+\frac{T_{i,j+1}^{n+1}-2T_{i,j}^{n+1}+T_{i,j-1}^{n+1}}{h^2})+S_{i,j}$$

$$=> \quad -rT_{i,j+1}^{n+1}+(1+2r)T_{i,j}^{n+1}-rT_{i,j-1}^{n+1}=rT_{i+1,j}^{n+1/2}+(1-2r)T_{i,j}^{n+1/2}+rT_{i-1,j}^{n+1/2}+\frac{\Delta t}{2}S_{i,j} \qquad \left(r=\frac{\alpha\Delta t}{2h^2}\right)$$ +2

⑤

For j = 1

=> $T_{i,1}=0$ +1

For j = J

=> $T_{i,J}=0$ +1

⑥

For i = 2 : I-1

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -r & 1+2r & -r & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -r & 1+2r & -r \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} T_{i,,1}^{n+1} \\ T_{i,2}^{n+1} \\ \vdots \\ T_{i,J-1}^{n+1} \\ T_{i,J}^{n+1} \end{bmatrix}=\begin{bmatrix} 0 \\ rT_{i+1,2}^{n}+(1-2r)T_{i,2}^{n}+rT_{i-1,2}^{n}+\frac{\Delta t}{2}s_{i,2} \\ \vdots \\ rT_{i-1,J-1}^{n}+(1-2r)T_{i,J-1}^{n}+rT_{i,J-1}^{n}+\frac{\Delta t}{2}s_{i,J-1} \\ 0 \end{bmatrix} \qquad \left(r=\frac{\alpha\Delta t}{2h^2}\right)$$ +2

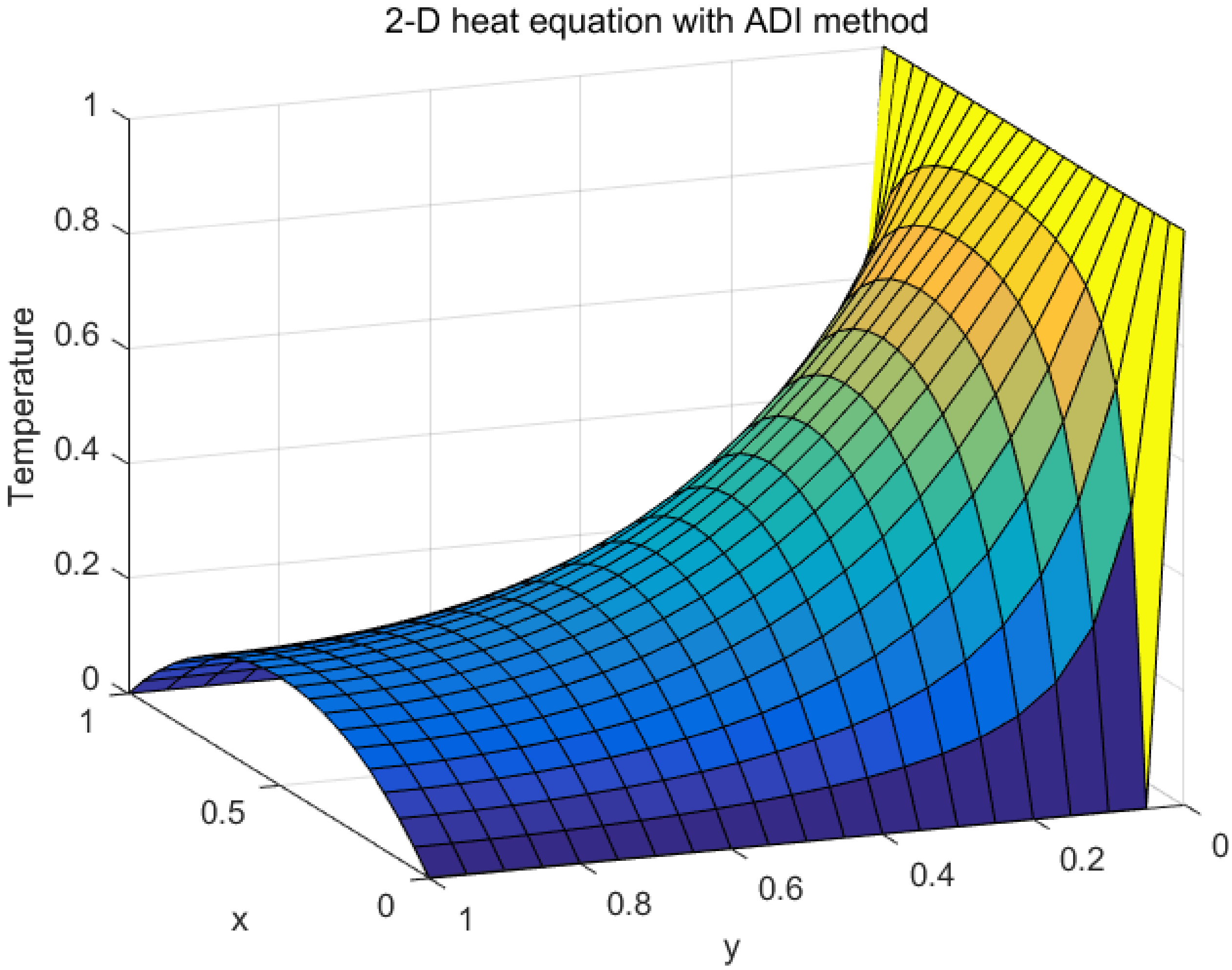
For i = I

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -r & 1+2r & -r & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -r & 1+2r & -r \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} T_{I,,1}^{n+1} \\ T_{I,2}^{n+1} \\ \vdots \\ T_{I,J-1}^{n+1} \\ T_{I,J}^{n+1} \end{bmatrix}=\begin{bmatrix} 0 \\ 2rT_{I-1,2}^{n}+(1-2r)T_{I,2}^{n}+\frac{\Delta t}{2}s_{I,2} \\ \vdots \\ 2rT_{I-1,J-1}^{n}+(1-2r)T_{I,J-1}^{n}+\frac{\Delta t}{2}s_{I,J-1} \\ 0 \end{bmatrix}$$ +2

⑦ +3

By solving matrix equation ③ using LU decomposition for j = 2 ~ j = J-1, we can get solutions of all grid at n+1/2 step.

Then solve the matrix equation ⑥ in the same way for i = 2 ~ i = I to find solutions at n+1



2-D heat equation with ADI method
Temperature
1
0.8
0.6
0.4
0.2
0
x
1
0.5
0
y
1
0.8
0.6
0.4
0.2
0

4. **(40 points)**

$$\frac{T_{i+1,j}-2T_{i,j}-T_{i-1,j}}{h^2}+\frac{T_{i,j+1}-2T_{i,j}-T_{i,j-1}}{h^2}=0$$

=> $T_{i+1,j}+T_{i-1,j}+T_{i,j-1}+T_{i,j+1}-4T_{i,j}=0$

=> $T_{i,j}=\frac{1}{4}(T_{i+1,j}+T_{i-1,j}+T_{i,j-1}+T_{i,j+1})$

(a)

$$T_{i,j}^{n+1}=\beta\frac{1}{4}(T_{i+1,j}^{n}+T_{i-1,j}^{n+1}+T_{i,j-1}^{n+1}+T_{i,j+1}^{n})+(1-\beta)T_{i,j}^{n} \quad +5$$

(b)

$$-\frac{1}{4}\overline{T}_{i+1,j}^{n+1}+\overline{T}_{i,j}^{n+1}-\frac{1}{4}\overline{T}_{i-1,j}^{n+1}=\frac{1}{4}(T_{i,j+1}^{n}+T_{i1,j-1}^{n+1})$$

$$T_{i,j}^{n+1}=\beta\overline{T_{i,j}^{n+1}}+(1-\beta)T_{i,j}^{n} \quad +5$$