Set 2.2)

Problem 9

As the coefficient matrix is tridiagonal, it is very unlikely to benefit from pivoting. Hence we use the non-pivoting LU decomposition functions written for tridiagonal matrices.

```
% problem2_2_9
n = 10;
c = ones(n-1,1)*(-1.0); e = c;
d = ones(n,1)*4.0;
b = ones(n,1)*5.0; b(1) = 9.0;
[c,d,e] = LUdec3(c,d,e);
x = LUsol3(c,d,e,b)
```

```
>> x =
    2.9019
    2.6077
    2.5288
    2.5075
    2.5011
    2.4971
    2.4873
    2.4519
    2.3205
    1.8301
```

Problem 20

We apply the conservation equation

$$\Sigma\,(Qc)_{\text{in}} + \Sigma\,(Qc)_{\text{out}} = 0$$

to each vessel, where Q is the flow rate of water, and c is the concentration. The results are

$$\begin{array}{lr}
1 & -8c_1 + 4c_2 + 4(20) = 0 \\
2 & 8c_1 - 10c_2 + 2c_3 = 0 \\
3 & 6c_2 - 11c_3 + 5c_4 = 0 \\
4 & 3c_3 - 7c_4 + 4c_5 = 0 \\
5 & 2c_4 - 4c_5 + 2(15) = 0
\end{array}$$

Since these equations are tridiagonal, we solve them with `LUdec3` and `LUsol3`:

```
% problem2_2_20
d1 = [8 6 3 2]';
d2 = [-8 -10 -11 -7 -4]';
d3 = [4 2 5 4]';
rhs = [-80 0 0 0 -30]';
[d1,d2,d3] = LUdec3(d1,d2,d3);
c = LUsol3(d1,d2,d3,rhs)
```

The solution for the concentrations is (units are mg/m^3):

```
c =

   19.7222
   19.4444
   18.3333
   17.0000
   16.0000
```

Set 8.2)

Problem 6

$$y'' = xy \qquad y(1) = 1.5 \qquad y(2) = 3$$

The finite difference equations are

$$\begin{aligned} y_1 &= 1.5 \\ y_{i-1} - 2y_i + y_{i+1} - h^2 x_i y_i &= 0, \quad i = 2, 3, \ldots, n-1 \\ y_n &= 3 \end{aligned}$$

or

$$\begin{aligned} y_0 &= 1.5 \\ y_{i-1} - \left(2 + h^2 x_i\right) y_i + y_{i+1} &= 0 \quad i = 2, 3, \ldots, n-1 \\ y_n &= \quad y_1 = 1.5 \end{aligned}$$

The following program is based on the function `fDiff6` in Example 8.6.

```
function p8_2_6
% Finite difference method for the second-order,
% linear boundary value problem in Problem 6,
% Problem Set 8.2.

xStart = 1; xStop = 2;   % Range of integration.
n = 21;                  % Number of mesh points.
freq = 2;                % Printout frequency.

h = (xStop - xStart)/(n-1);
x = linspace(xStart,xStop,n)';
[c,d,e,b] = fDiffEqs(x,h,n);
[c,d,e] = LUdec3(c,d,e);
printSol(x,LUsol3(c,d,e,b),freq)

function [c,d,e,b] = fDiffEqs(x,h,n)
% Sets up the tridiagonal coefficient matrix and the
% constant vector of the finite difference equations.
h2 = h*h;
d = -h2.*x - 2;
c = ones(n-1,1);
e = ones(n-1,1);
b = zeros(n,1);
d(1) = 1; e(1) = 0; b(1) = 1.5;
d(n) = 1; c(n-1) = 0; b(n) = 3;
```

The program prints every second point of the solution:

```
 x           y1
1.0000e+00   1.5000e+00
1.1000e+00   1.5372e+00
1.2000e+00   1.5914e+00
1.3000e+00   1.6647e+00
1.4000e+00   1.7597e+00
1.5000e+00   1.8793e+00
1.6000e+00   2.0272e+00
1.7000e+00   2.2075e+00
1.8000e+00   2.4255e+00
1.9000e+00   2.6872e+00
2.0000e+00   3.0000e+00
```

Problem 8

$$y'' = -\frac{1}{x}y' - \frac{1}{x^2}y \qquad y(1) = 0 \qquad y(2) = 0.638961$$

The finite difference equations are

$$\begin{aligned} y_1 &= 0 \\ y_{i-1} - 2y_i + y_{i+1} - h^2\left(-\frac{y_{i+1} - y_{i-1}}{2hx_i} - \frac{y_i}{x_i^2}\right) &= 0, \quad i = 2, 3, \ldots, n-1 \\ y_n &= 0.638961 \end{aligned}$$

or

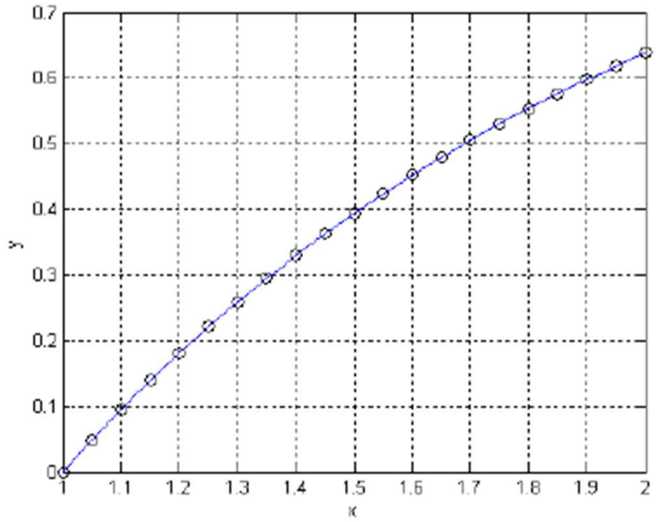
$$\begin{aligned} y_1 &= 0 \\ \left(1 - \frac{h}{2x_i}\right) y_{i-1} - \left(2 - \frac{h^2}{x_i^2}\right) y_i + \left(1 + \frac{h}{2x_i}\right) y_{i+1} &= 0, \quad i = 2, 3, \ldots, n-1 \\ y_n &= 0.638961 \end{aligned}$$

Here are the finite difference equations:

```
function p8_2_8
% Finite difference method for the second-order,
% linear boundary value problem in Problem 8,
% Problem Set 8.2.

xStart = 1; xStop = 2;      % Range of integration.
n = 21;                     % Number of mesh points.
freq = 2;                   % Printout frequency.

h = (xStop - xStart)/(n-1);
x = linspace(xStart,xStop,n)';
[c,d,e,b] = fDiffEqs(x,h,n);
[c,d,e] = LUdec3(c,d,e);
y = LUsol3(c,d,e,b);
printSol(x,y,freq)
plot(x,y,'ko'); hold on
fplot('sin(log(x))',[1,2]); grid on
xlabel('x'); ylabel('y')

function [c,d,e,b] = fDiffEqs(x,h,n)
% Sets up the tridiagonal coefficient matrix and the
% constant vector of the finite difference equations.
h2 = h*h;
d = h2./x./x - 2;
c = -0.5*h./x(2:n) + 1;
e = 0.5*h./x(1:n-1) + 1;
b = zeros(n,1);
d(1) = 1; e(1) = 0;
d(n) = 1; c(n-1) = 0; b(n) = 0.638961;
end
end
```

x	y1
1.0000e+00	0.0000e+00
1.1000e+00	9.5170e-02
1.2000e+00	1.8132e-01
1.3000e+00	2.5937e-01
1.4000e+00	3.3016e-01
1.5000e+00	3.9445e-01
1.6000e+00	4.5289e-01
1.7000e+00	5.0608e-01
1.8000e+00	5.5452e-01
1.9000e+00	5.9868e-01
2.0000e+00	6.3896e-01

The plot shows the numerical solution (open circles) together with the analytical solution (solid line).



Problem 9

$$y'' = y^2 \sin y \qquad y'(0) = 0 \qquad y(\pi) = 1$$

The finite difference equations are

$$\begin{aligned} -2y_1 + 2y_2 - h^2 F(x_1, y_1, y_1') &= 0 \\ y_{i-1} - 2y_i + y_{i+1} - h^2 F(x_i, y_i, y_i') &= 0, \quad i = 2, 3, \ldots, n-1 \\ y_n &= 1 \end{aligned}$$

In arriving at the first equation, we utilize the equivalent boundary condition $y_0 = y_2$. The quadratic $y = (x/\pi)^2$ was chosen for the starting solution (note that its satisfies the prescribed boundary conditions). The following program is based on the function `fDiff7` in Example 8.7.

```
function p8_2_9
% Finite difference method for the second-order,
% nonlinear boundary value problem in Problem 9,
% Problem Set 8.2.

xStart = 0; xStop = pi;       % Range of integration.
n = 21;                       % Number of mesh points.
freq = 2;                     % Printout frequency.
x = linspace(xStart,xStop,n)';
y = x.*x/pi^2;                % Starting values of y.

h = (xStop - xStart)/(n-1);
y = newtonRaphson2(@residual,y,1.0e-5);
printSol(x,y,freq)

    function r = residual(y)
    % Residuals of finite difference equations: left-hand
    % sides of Eqs (8.11).
    r = zeros(n,1);
    r(1) = -2*y(1) + 2*y(2) - h*h*(y(1)^2)*sin(y(1));
    r(n) = y(n) - 1;
    for i = 2:n-1
        r(i) = y(i-1) - 2*y(i) + y(i+1) - h*h*(y(i)^2)*sin(y(i));
    end
    end
end
```

```
 x          y1
0.0000e+00  4.1338e-01
3.1416e-01  4.1678e-01
6.2832e-01  4.2714e-01
9.4248e-01  4.4498e-01
1.2566e+00  4.7127e-01
1.5708e+00  5.0757e-01
1.8850e+00  5.5631e-01
2.1991e+00  6.2132e-01
2.5133e+00  7.0875e-01
2.8274e+00  8.2892e-01
3.1416e+00  1.0000e+00
```

Problem 10

$$y'' = -2y(2xy' + y) \qquad y(0) = \frac{1}{2} \qquad y'(1) = -\frac{2}{9}$$

The finite difference equations are

$$\begin{aligned} y_1 &= 0.5 \\ y_{i-1} - 2y_i + y_{i+1} - h^2 F(x_i, y_i, y_i') &= 0, \quad i = 2, 3, \ldots, n-1 \\ y_{n-1} - 2y_n + y_{n+1} - h^2 F(x_n, y_n, y_n') &= 0 \end{aligned}$$

The bounday condition $y_n' = -2/9$ is equivalent to

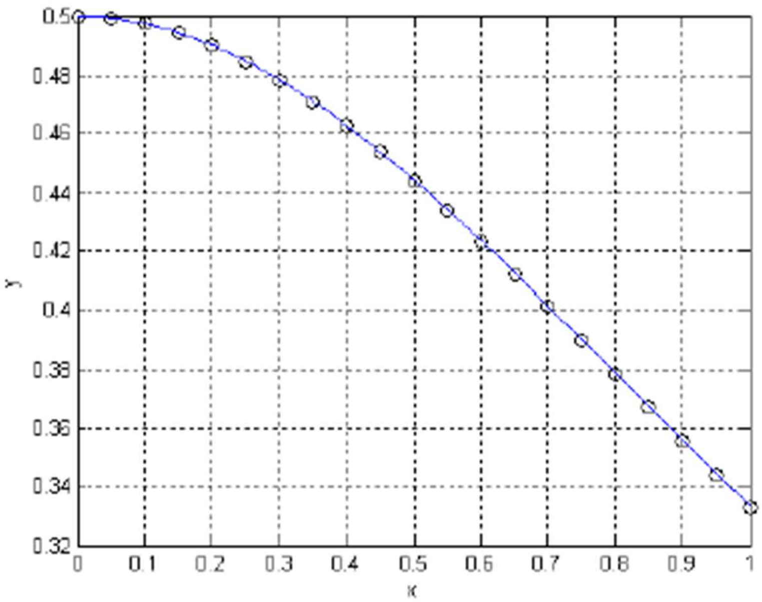
$$\frac{y_{n+1} - y_{n-1}}{2h} = -\frac{2}{9} \qquad y_{n+1} = y_{n-1} - \frac{4}{9}h$$

so that the last finite difference equation becomes

$$2y_{n-1} - 2y_n - \frac{4}{9}h - h^2 F\left(x_n, y_n, -\frac{2}{9}\right) = 0$$

```
function p8_2_10
% Finite difference method for the second-order,
% nonlinear boundary value problem in Problem 10,
% Problem Set 8.2.

xStart = 0; xStop = 1;      % Range of integration.
n = 21;                     % Number of mesh points.
freq = 2;                   % Printout frequency.
x = linspace(xStart,xStop,n)';
y = -2/9*x + 0.5;           % Starting values of y.
h = (xStop - xStart)/(n-1);
y = newtonRaphson2(@residual,y,1.0e-5);
printSol(x,y,freq)
plot(x,y,'ko'); hold on
fplot('1/(2+x.*x)',[0 1]); grid on
xlabel('x'); ylabel('y')

function r = residual(y)
% Residuals of finite difference equations: left-hand
% sides of Eqs (8.11).
r = zeros(n,1);
r(1) = y(1) - 0.5;
r(n) = 2*y(n-1) - 2*y(n) - 4/9*h...
    - h*h*y2Prime(x(n),y(n),-2/9);
for i = 2:n-1
    r(i) = y(i-1) - 2*y(i) + y(i+1)...
        - h*h*y2Prime(x(i),y(i),(y(i+1) - y(i-1))/(2*h));
end
end

function F = y2Prime(x,y,yPrime)
% Second-order differential equation F = y".
F = -2*y*(2*x*yPrime + y);
end
end
```

The plot shows the numerical solution (open circles) together with the analytical solution (solid line).



Problem 18

It is convenient to introduce he variable $x = r/a$. The differential equation and the boundary conditions then become

$$\frac{d^2T}{dx^2} = -\frac{1}{x}\frac{dT}{dx} \qquad T|_{x=0.5} = 0 \qquad T|_{x=1} = 200^\circ\ \mathrm{C}$$

Using 11 mesh points, the finite difference equations, Eqs. (8.11), are

$$\begin{aligned} T_1 &= 0 \\ T_{i-1} - 2T_i + T_{i+1} - h^2\left(-\frac{1}{x_i}\frac{T_{i+1} - T_{i-1}}{2h}\right) &= 0, \quad i = 2, 3, \ldots 10 \\ T_{11} &= 200 \end{aligned}$$

or

$$\begin{aligned} T_1 &= 0 \\ \left(1 - \frac{h}{2x_i}\right)T_{i-1} - 2T_i + \left(1 + \frac{h}{2x_i}\right)T_{i+1} &= 0, \quad i = 2, 3, \cdots 10 \\ T_{11} &= 200 \end{aligned}$$

The following program is based on Example 8.6. It utilizes the tridiagonal structure of the equations.

```
function problem8_2_18
xStart = 0.5; xStop = 1;
n = 11;
h = (xStop - xStart)/(n-1);
x = zeros(n,1); y = zeros(n,2);
x(1) = xStart;
for i = 2:n
    x(i) = x(i-1) + h;
    y(i,2) = 200*(1 - log(x(i))/log(0.5)); % Analytical soln.
end
[c,d,e,b] = fdEqs(x,h,n);
[c,d,e] = LUdec3(c,d,e);
y(:,1) = LUsol3(c,d,e,b);       % Numerical soln.
printSol(x,y,1)

function[c,d,e,b] = fdEqs(x,h,n)
% Sets ub finite difference (tridiagonal) equations
h2 = h*h;
d = ones(n,1)*(-2);
c = zeros(n-1,1);
e = zeros(n-1,1);
for i = 1:n-1
    c(i) = 1 - h/2/x(i+1);
    e(i) = 1 + h/2/x(i);
end
b = zeros(n,1);
e(1) = 0; d(1) = 1;
b(n) = 200; d(n) = 1; c(n-1) = 0;
```

In the printout y1 is the numerical solution and y2 is the analytical solution. The two are in good agreement.

x	y1	y2
5.0000e-01	0.0000e+00	0.0000e+00
5.5000e-01	2.7492e+01	2.7501e+01
6.0000e-01	5.2594e+01	5.2607e+01
6.5000e-01	7.5687e+01	7.5702e+01
7.0000e-01	9.7070e+01	9.7085e+01
7.5000e-01	1.1698e+02	1.1699e+02
8.0000e-01	1.3560e+02	1.3561e+02
8.5000e-01	1.5310e+02	1.5311e+02
9.0000e-01	1.6959e+02	1.6960e+02
9.5000e-01	1.8520e+02	1.8520e+02
1.0000e+00	2.0000e+02	2.0000e+02