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1. $F''' + \theta = 0 \qquad 0 \le x \le 5$

$\theta'' + F\theta' = 0 \qquad F(0) = F'(0) = 0,\ \theta(0) = 1$

$F''(5) = 0,\ \theta(5) = 0$

a) ① $F = y_1 \qquad \theta = y_4$

$F' = y_1' = y_2 \qquad \theta' = y_4' = y_5$

$F'' = y_2' = y_3 \qquad \theta'' = y_5' = -y_1 y_5$

$F''' = y_3' = -y_4$

$$\therefore y' = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}' = \begin{bmatrix} y_2 \\ y_3 \\ -y_4 \\ y_5 \\ -y_1 y_5 \end{bmatrix}$$

+4

②
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function y = InCond(u)
y = [0 0 u(1) 1 u(2)];
end
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+3

③
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function r = residual(u)
r = zeros(length(u),1);
x = xStart;
[xSol, ySol] = runkut4(@dEqs, x, InCond(u), xStop, h);
lastrow = size(ySol,1);
r(1) = ySol(lastrow,3);
r(2) = ySol(lastrow,4);
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$\Big\}$ +3

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2. $\frac{d^2\theta}{dx^2} = \frac{1}{2}\theta e^{-(\theta+\mu x)}$ $-6 \le x \le 6$ $\theta'(-6) = -1$

$\theta'(6) = 0$

a) ① $\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} = \frac{1}{2}\theta_i e^{-(\theta_i + \mu x_i)}$ +4

② BC $\theta'(-6) = -1$

$\frac{\theta_2 - \theta_0}{2h} = -1$ $\therefore \theta_0 = \theta_2 + 2h$ +2

$\frac{\theta_2 - 2\theta_1 + \theta_0}{h^2} = \frac{1}{2}\theta_1 e^{-(\theta_1 + \mu x_1)}$

$\Rightarrow \frac{2\theta_2 - 2\theta_1 + 2h}{h^2} = \frac{1}{2}\theta_1 e^{-(\theta_1 + \mu x_1)}$ +2

③ BC $\theta'(6) = 0$

$\frac{\theta_{n+1} - \theta_{n-1}}{2h} = 0$ $\therefore \theta_{n+1} = \theta_{n-1}$ +2

$\Rightarrow \frac{2\theta_{n-1} - 2\theta_n}{h^2} = \frac{1}{2}\theta_n e^{-(\theta_n + \mu x_n)}$ +2

④
$$\left.\begin{aligned} r(1) &= \frac{2\theta_2 - 2\theta_1 + 2h}{h^2} - \frac{1}{2}\theta_1 e^{-(\theta_1 + \mu x_1)} \\ r(i) &= \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} - \frac{1}{2}\theta_i e^{-(\theta_i + \mu x_i)} \\ r(n) &= \frac{2\theta_{n-1} - 2\theta_n}{h^2} - \frac{1}{2}\theta_n e^{-(\theta_n + \mu x_n)} \end{aligned}\right\} +3$$

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3. $\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + S(x,y)$

$T(\pm 1, y, t) = 0$, $T(x, \pm 1, t) = 0$, $T(x, y, 0) = 0$, $S(x,y) = 2(2 - x^2 - y^2)$

a)

① $$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{\alpha}{2}\left(\frac{T_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{h^2} + \frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{h^2}\right) + \frac{\alpha}{2}\left(\frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{h^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{h^2}\right)$$

$$\Rightarrow \frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{\alpha}{2}\delta_{xx}(T^{n+1} + T^{n}) + \frac{\alpha}{2}\delta_{yy}(T^{n+1} + T^{n}) + S + O(\Delta t^2, h^2)$$

+3

② $$\Rightarrow \left(I - \frac{\alpha\Delta t}{2}\delta_{xx} - \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n+1} = \left(I + \frac{\alpha\Delta t}{2}\delta_{xx} + \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n} + \Delta t S + \Delta t\, O(\Delta t^2, h^2)$$

$$\Rightarrow \left(I - \frac{\alpha\Delta t}{2}\delta_{xx}\right)\left(I - \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n+1} - \frac{\alpha^2\Delta t^2}{4}\delta_{xx}\delta_{yy}T^{n+1} =$$
$$\left(I + \frac{\alpha\Delta t}{2}\delta_{xx}\right)\left(I + \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n} - \frac{\alpha^2\Delta t^2}{4}\delta_{xx}\delta_{yy}T^{n} + \Delta t S + \Delta t\, O(\Delta t^2, h^2)$$

+2

$$\rightarrow \left(I - \frac{\alpha\Delta t}{2}\delta_{xx}\right)\left(I - \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n+1} = \left(I + \frac{\alpha\Delta t}{2}\delta_{xx}\right)\left(I + \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n} + \Delta t S + \Delta t\, O(\Delta t, h^2)$$

+2

$\left(\because \frac{\alpha^2\Delta t^2}{4}\delta_{xx}\delta_{yy}(T^{n+1} - T^{n}) = O(\Delta t^3)\right)$ $\therefore$ we can neglect

+2

③ $$\left(I - \frac{\alpha\Delta t}{2}\delta_{xx}\right)T^{*} = \left(I + \frac{\alpha\Delta t}{2}\delta_{xx}\right)\left(I + \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n} + \Delta t S \quad \cdots (5)$$

$$\left(I - \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n+1} = T^{*}$$

(5) RHS

$$r_{ij}^{*} = \left(I + \frac{\alpha\Delta t}{2}\delta_{yy}\right)T^{n} = T_{i,j}^{n} + \frac{\alpha\Delta t}{2h^2}\left(T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}\right)$$ +2

$$r_{i,j}^{n} = \left(I + \frac{\alpha\Delta t}{2}\delta_{xx}\right)r^{*} = r_{i,j}^{*} + \frac{\alpha\Delta t}{2h^2}\left(r_{i+1,j}^{*} - 2r_{i,j}^{*} + r_{i-1,j}^{*}\right) + \Delta t S_{i,j}$$ +2

④ $T^{*}_{i,j} - \dfrac{\alpha \Delta t}{2h^2}\left(T^{*}_{i+1,j} - 2T^{*}_{i,j} + T^{*}_{i-1,j}\right) = r^{n}_{i,j}$ +2

⑤ $T^{*}_{1,j} = T^{n+1}_{1,j} - \dfrac{\alpha \Delta t}{2h^2}\left(T^{n+1}_{1,j+1} - 2T^{n+1}_{1,j} + T^{n+1}_{1,j-1}\right)$

$T^{n+1}_{1,j+1} = T^{n+1}_{1,j} = T^{n+1}_{1,j-1} = 0 \quad \therefore T^{*}_{1,j} = 0.$

In the same way,

$T^{*}_{I,j} = 0$

~~5~~ +3

⑥ $C = \dfrac{\alpha \Delta t}{2h^2}$

$$\begin{pmatrix} 1 & 0 & 0 & & \\ -C & 1+2C & -C & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T^{*}_{1,j} \\ T^{*}_{2,j} \\ \vdots \\ T^{*}_{i,j} \\ \vdots \\ T^{*}_{I,j} \end{pmatrix} = \begin{pmatrix} 0 \\ r^{n}_{2,j} \\ \vdots \\ r^{n}_{i,j} \\ \vdots \\ 0 \end{pmatrix}$$

+3

⑦ $T^{n+1}_{i,j} - \dfrac{\alpha \Delta t}{2h^2}\left(T^{n+1}_{i,j+1} - 2T^{n+1}_{i,j} + T^{n+1}_{i,j-1}\right) = T^{*}_{i,j}$ +2

⑧ $T^{n+1}_{i,1} = 0$

$T^{n+1}_{i,J} = 0$ +2 from B.C. $T(x, \pm 1, t) = 0$

⑨

$$\begin{pmatrix} 1 & 0 & 0 & & \\ -C & 1+2C & -C & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T^{n+1}_{i,1} \\ T^{n+1}_{i,2} \\ \vdots \\ T^{n+1}_{i,j} \\ \vdots \\ T^{n+1}_{i,J} \end{pmatrix} = \begin{pmatrix} 0 \\ T^{*}_{i,2} \\ \vdots \\ T^{*}_{i,j} \\ \vdots \\ 0 \end{pmatrix}$$

+3

⑩ T^{*} and T^{n+1} are coupled, and these are expressed by tridiagonal matrices, respectively. Therefore, LUdec3 & LUsol3 can be used to get T^{*} and T^{n+1}.

+2

4.

(a) $$\cancel{\frac{\partial T}{\partial t}} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + S(x,y)$$

$$\therefore \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{1}{\alpha} S(x,y),$$ elliptic equation

+5

(b)

$$\frac{T^{n}_{i+1,j} - 2T^{n+1}_{i,j} + T^{n+1}_{i-1,j}}{\Delta x^2} + \frac{T^{n}_{i,j+1} - 2T^{n+1}_{i,j} + T^{n+1}_{i,j-1}}{\Delta y^2} = -\frac{1}{\alpha} S_{i,j}$$

$$\Rightarrow T^{n+1}_{i,j} = \frac{\beta}{4}\left(T^{n}_{i+1,j} + T^{n+1}_{i-1,j} + T^{n}_{i,j+1} + T^{n+1}_{i,j-1} + \frac{h^2}{\alpha} S_{i,j}\right) + (1-\beta)T^{n}_{i,j}$$

+5

(c)

$$-\frac{1}{4}T^{n+1}_{i-1,j} + T^{n+1}_{i,j} - \frac{1}{4}T^{n+1}_{i+1,j} = \frac{1}{4}\left(T^{n+1}_{i,j-1} + T^{n}_{i,j+1} + \frac{h^2}{\alpha}S_{i,j}\right)$$

$$\Rightarrow -\frac{1}{4}\tilde{T}_{i-1,j} + \tilde{T}_{i,j} - \frac{1}{4}\tilde{T}_{i+1,j} = \frac{1}{4}\left(T^{n+1}_{i,j-1} + T^{n}_{i,j+1} + \frac{h^2}{\alpha}S_{i,j}\right)$$

and then

+5

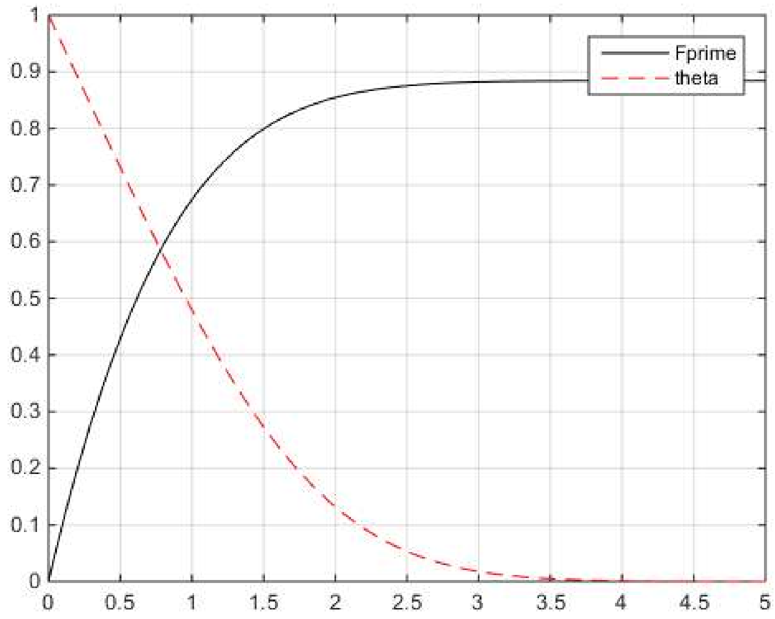
$$T^{n+1}_{i,j} = \beta\tilde{T}_{i,j} + (1-\beta)T^{n}_{i,j}$$

(d) Convergence rate : SLOR > SOR.

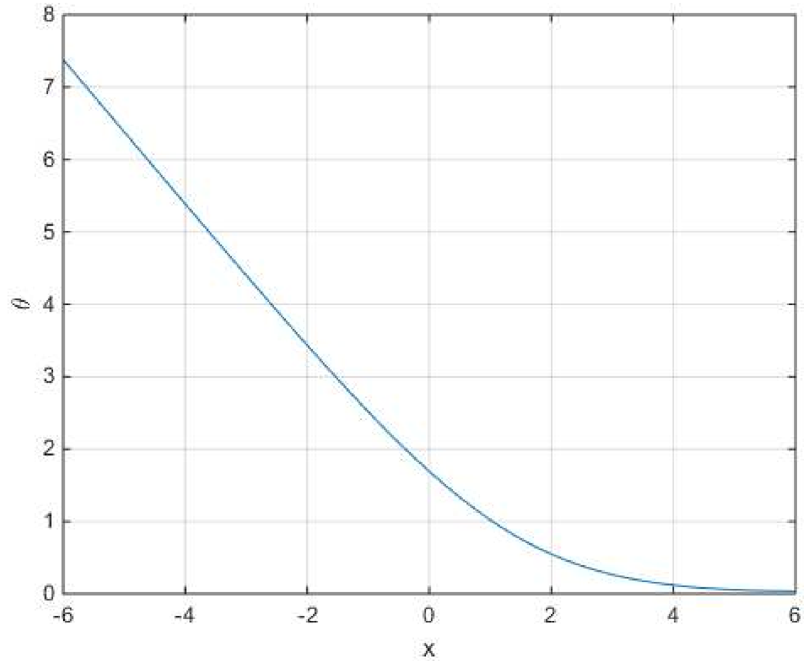
Because the boundary condition affect more faster when using SLOR.

+5

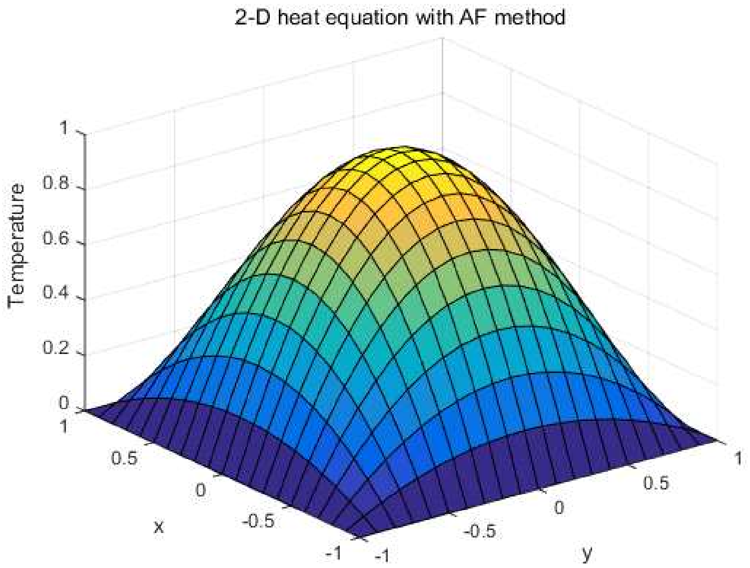
Problem 1



Problem 2



Problem 3



Problem 4 (The source term, S, should be correct.)