Cross-covariance Functions for Tangent Vector Fields on the Sphere

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Outline

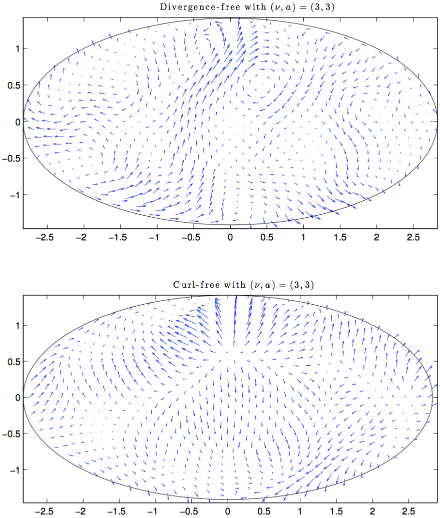
Tangent Vector Fields on the Sphere

- Suppose $\boldsymbol{Y}(\boldsymbol{s}) \in \mathbb{R}^3, \boldsymbol{s} \in \mathbb{S}^2$ is a random tangent vector field defined on the sphere.
- Let $\boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{\mu}(\boldsymbol{s}) + \boldsymbol{e}(\boldsymbol{s})$,
  $\boldsymbol{\mu}(\boldsymbol{s})$: non-random mean; $\boldsymbol{e}(\boldsymbol{s})$: zero-mean random process.
- Cross-covariance function: $\boldsymbol{C}(\boldsymbol{s}, \boldsymbol{t}) = \mathrm{Cov}(\boldsymbol{e}(\boldsymbol{s}), \boldsymbol{e}(\boldsymbol{t}))$, $\boldsymbol{s}, \boldsymbol{t} \in \mathbb{S}^2$.

The Helmholtz-Hodge Decomposition

- Any tangent vector field can be uniquely decomposed into the sum of a divergence-free component and a curl-free component.
- Suppose the fields are velocity fields of fluids.
- Divergence-free: incompressible flow.
- Curl-free: irrotational flow.

An Example


Divergence-free with $(\nu, a) = (3, 3)$
1
0.5
0
−0.5
−1
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
Curl-free with $(\nu, a) = (3, 3)$
1
0.5
0
−0.5
−1
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5

The Surface Gradient and the Surface Curl

- Suppose f is a scalar function defined on the sphere.
- The surface gradient of f at location $\boldsymbol{s}$,

$$\nabla_{\boldsymbol{s}}^{*} f =: \boldsymbol{P}_{\boldsymbol{s}} \nabla_{\boldsymbol{s}} f,$$

  where $\boldsymbol{P}_{\boldsymbol{s}} = \boldsymbol{I}_3 - \boldsymbol{s}\boldsymbol{s}^T$, and $\nabla_{\boldsymbol{s}}$ is the usual gradient on $\mathbb{R}^3$.
- The surface curl of f at location $\boldsymbol{s}$,

$$L_{\boldsymbol{s}}^{*} f =: \boldsymbol{Q}_{\boldsymbol{s}} \nabla_{\boldsymbol{s}} f,$$

  where

$$\boldsymbol{Q}_{\boldsymbol{s}} = \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix}.$$

- $\nabla_{\boldsymbol{s}}^{*} f$ is curl-free and $L_{\boldsymbol{s}}^{*} f$ is divergence-free.

Outline

Main Idea

- Suppose $Z(\boldsymbol{s})$ is a sufficiently smooth univariate random field defined on the sphere. Besides, $Z(\boldsymbol{s})$ is stationary with mean zero and covariance function

  $$C(\boldsymbol{h}) = \mathrm{Cov}(Z(\boldsymbol{s}), Z(\boldsymbol{t})),$$

  where $\boldsymbol{h} = \boldsymbol{s} - \boldsymbol{t}$.
- Apply the surface gradient (or the surface curl) operator to the sample paths of $Z(\boldsymbol{s})$. The resulting random tangent vector field is curl-free (or divergence-free) in terms of its sample paths.

Main Idea (cont.)

- The cross-covariance functions of the tangent vector fields are
$$\boldsymbol{C}_{\mathrm{curl},Z}(\boldsymbol{s},\boldsymbol{t}) = -\boldsymbol{P}_{\boldsymbol{s}} \nabla_{\boldsymbol{h}} \nabla_{\boldsymbol{h}}^T C(\boldsymbol{h}) \Big|_{\boldsymbol{h}=\boldsymbol{s}-\boldsymbol{t}} \boldsymbol{P}_{\boldsymbol{t}}^T,$$
and
$$\boldsymbol{C}_{\mathrm{div},Z}(\boldsymbol{s},\boldsymbol{t}) = -\boldsymbol{Q}_{\boldsymbol{s}} \nabla_{\boldsymbol{h}} \nabla_{\boldsymbol{h}}^T C(\boldsymbol{h}) \Big|_{\boldsymbol{h}=\boldsymbol{s}-\boldsymbol{t}} \boldsymbol{Q}_{\boldsymbol{t}}^T.$$

Outline

Tangent Mixed Matérn

- Suppose $\boldsymbol{Z}(\boldsymbol{s}) = (Z_1(\boldsymbol{s}), Z_2(\boldsymbol{s}))^T$ is an isotropic bivariate Gaussian random field defined on the sphere with mean zero and cross-covariance function $\boldsymbol{C}(\|\boldsymbol{h}\|) = [C_{ij}(\|\boldsymbol{h}\|)]_{1\le i,j\le 2}$, where $\boldsymbol{h} = \boldsymbol{s} - \boldsymbol{t}$.
- The cross-covariance function satisfies a parsimonious bivaraite Matérn model [Gneiting et al., 2010].
- $C_{ii}(\|\boldsymbol{h}\|) = \sigma_i^2 M(\|\boldsymbol{h}\|; \nu_i, a), i = 1, 2,$
- $C_{12}(\|\boldsymbol{h}\|) = C_{21}(\|\boldsymbol{h}\|) = \rho_{12}\sigma_1\sigma_2 M(\|\boldsymbol{h}\|; (\nu_1 + \nu_2)/2, a)$.
- ρ_{12}, ν_1 and ν_2 satisfy a sufficient and necessary condition for non-negative definiteness.

Tangent Mixed Matérn (cont.)

- Based on the Helmholtz-Hodge decomposition, we construct a random tangent vector field on the sphere as

$$\underbrace{\nabla_{\boldsymbol{s}}^{*} Z_1(\boldsymbol{s})}_{\text{curl-free}} + \underbrace{L_{\boldsymbol{s}}^{*} Z_2(\boldsymbol{s})}_{\text{divergence-free}} .$$

- The cross-covariance function $\boldsymbol{C}_{\text{mix},\boldsymbol{Z}}(\boldsymbol{s},\boldsymbol{t})$ can be computed correspondingly.
- The smoothness parameters ν_1 and ν_2 are required to be larger than 1.

Propositions

- In climatology, tangent vector fields are often represented in terms of zonal and meridional components (i.e., u and v components).
- $$u(\theta, \phi) = \frac{1}{\sin\theta}\frac{\partial Z_1}{\partial \phi} + \frac{\partial Z_2}{\partial \theta} \quad \text{P-a.e.,}$$
  and
  $$v(\theta, \phi) = \frac{1}{\sin\theta}\frac{\partial Z_2}{\partial \phi} - \frac{\partial Z_1}{\partial \theta} \quad \text{P-a.e.}$$
- Axial symmetry: $\mathrm{Cov}(X(\theta_s, \phi_s), X(\theta_t, \phi_t)) = C(\theta_s, \theta_t, \phi_s - \phi_t)$.
- The u and v components are axially symmetric both marginally and jointly.

Propositions (cont.)

- Allows for negative covariances for the u and v components.
- This characteristic is very common in meteorological variables, such as wind fields [Daley, 1991].

Fast Computation

- The evaluation of the likelihood function requires $O(n^3)$ operations, where n is the number of sampling locations.
- If the observations are on a regular grid on the sphere, the discrete Fourier transform (DFT) can be used to speed up the computation [Jun, 2011].
- When $n_{\text{lat}} \sim n_{\text{lon}}$, the time complexity can be reduced to $O(n^2)$, where $n = n_{\text{lat}} n_{\text{lon}}$.
- $n_{\text{lat}} = 25$, $n_{\text{lon}} = 50$.
- Without using the DFT: 34.430 seconds.
- Using the DFT: 3.025 seconds.

Data Example

- Ocean surface wind dataset called QuikSCAT.
- Level 3 dataset.
- Monthly mean ocean surface winds from January 2000 to December 2008.
- The large-scale non-stationary variation is modeled by the method of empirical orthogonal function (EOF) analysis.
- Tangent Mixed Matérn is applied to the residual wind fields in the region of the Indian Ocean with a number of leading EOFs subtracted.

Residual Wind Fields

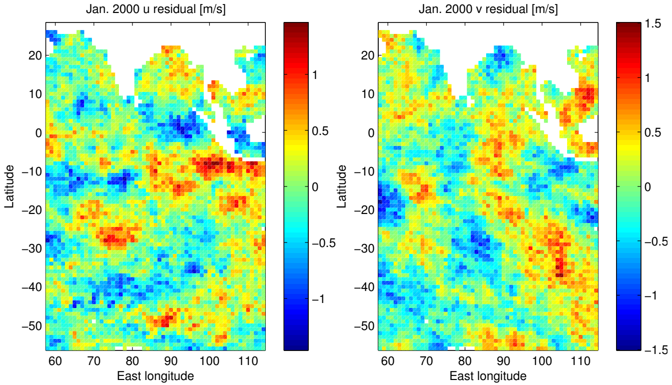


Figure : The u and v residual wind fields of January, 2000 in the region of the Indian Ocean.

List of Cross-covariance Models

- PARS-BM: parsimonious bivariate Matérn model.
- TMM: tangent Mixed Matérn model.

Table : Maximum likelihood estimates of parameters

Model	PARS-BM	TMM
σ_1^2	0.184 (2.85e-3)	-
σ_1	-	0.029 (4.25e-4)
σ_2^2	0.157 (2.24e-3)	-
σ_2	-	0.055 (8.05e-4)
ρ_{12}	-0.080 (6.58e-3)	0.281 (7.05e-3)
ν_1	1.239 (0.033)	1.758 (0.022)
ν_2	1.132 (0.032)	2.034 (0.020)
$1/a$	0.058 (1.24e-3)	-
a	-	9.472 (0.16)
τ_1	0.218 (1.3e-3)	0.210 (1.48e-3)
τ_2	0.203 (1.5e-3)	0.196 (1.48e-3)
Log-likelihood	-46995	**-45126**
# of parameters	8	8

Cokriging

- We randomly hold out the data at 170 locations for evaluation and estimate the parameters using the data at the remaining 900 locations.
- The cross-validation procedure is repeated 20 times.
- Scoring rules: the mean squared prediction error (MSPE) and the mean absolute error (MAE).

Table : Cokriging cross-validation results (20 times) in terms of the mean squared prediction error (MSPE) and the mean absolute error (MAE)

Model	Scoring Rule	U Residual Field [m/s]			V Residual Field [m/s]		
		Median	Max	Min	Median	Max	Min
PARS-BM	MSPE	0.0753	0.0803	0.0725	0.0680	0.0731	0.0651
	MAE	0.2157	0.2228	0.2117	0.2064	0.2120	0.2020
TMM	MSPE	0.0748	0.0803	0.0718	0.0671	0.0721	0.0642
	MAE	0.2150	0.2226	0.2107	0.2051	0.2106	0.2009

References

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