Handout 2

Minjie Fan

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1 Neyman-Scott problem

Model:

$$y_{ij} = \mu_i + \epsilon_{ij},$$

where

$$\epsilon_{ij} \overset{i.i.d.}{\sim} N(0, \sigma^2), i = 1, \cdots, m; j = 1, 2.$$

This is a Gaussian mixed model. By (1.7) and (1.8) in the textbook, we can obtain the MLEs of β and σ^2.

$$\hat{\beta}_{MLE} = \frac{1}{2} X' y,$$

$$\hat{\sigma}^2_{MLE} = \frac{1}{4m} \sum_{i=1}^{m} (y_{i1} - y_{i2})^2.$$

To show that $\hat{\sigma}^2_{MLE}$ is inconsistent, i.e.,

$$\hat{\sigma}^2_{MLE} \xrightarrow{P} \frac{\sigma^2}{2},$$

where σ^2 is the true value.

Since $y_{i1} - y_{i2} \overset{i.i.d.}{\sim} N(0, 2\sigma^2)$,

$$\frac{(y_{i1} - y_{i2})^2}{2\sigma^2} \overset{i.i.d.}{\sim} \chi^2_1,$$

By the Weak Law of Large Number, we have

$$\frac{\sum_{i=1}^{m} (y_{i1} - y_{i2})^2}{2m\sigma^2} \xrightarrow{P} E(\chi^2_1) = 1.$$

Thus,

$$\hat{\sigma}^2_{MLE} \xrightarrow{P} \frac{\sigma^2}{2}.$$

Similarly, we can show that the MLE of σ^2 based on $z_i = y_{i1} - y_{i2}$ is consistent, i.e.,

$$\hat{\sigma}^2_{REML} \xrightarrow{P} \sigma^2.$$

2 Verifying Equation (1.21)

For the REML estimator, the restricted Fisher Information matrix has the following form:

$$Var\left(\frac{\partial l_R}{\partial \theta}\right) = -E\left(\frac{\partial^2 l_R}{\partial \theta \partial \theta'}\right).$$

Show that

$$E\left(\frac{\partial^2 l_R}{\partial \theta_i \partial \theta_j}\right) = -\frac{1}{2} tr\left(P\frac{\partial V}{\partial \theta_i} P \frac{\partial V}{\partial \theta_j}\right).$$

Hint: show

$$\frac{\partial P}{\partial \theta_j} = -P\frac{\partial V}{\partial \theta_j} P,$$

and make use of the first formula in Appendix C.2.

3 Problem 1.14

What is a multivariate t-distribution? Check the definition on Page 17 of the textbook and Wikipedia. Its pdf and the following way of constructing a multivariate t-distribution may be useful: If $y \sim t_n(X\beta, V, d)$, then there exist $u \sim N(0, V)$ and $w \sim \chi^2_d$ such that

$$y = X\beta + u\sqrt{d/w},$$

where n is the dimension of y and d is the degree of freedom.

Suppose the ANOVA model is defined as

$$y = X\beta + Z\alpha + \epsilon,$$

where

$$Z\alpha = Z_1\alpha_1 + \cdots + Z_s\alpha_s,$$

the components of α_i are independent and distributed as $N(0, \sigma_i^2)$ and the components of ϵ are independent and distributed as $N(0, \tau^2)$, X is an n by p matrix.

We can show that the REML equations derived under the multivariate t-distribution are

$$\frac{n-p+d}{2} \cdot \frac{1}{d + y'Py} y'P\frac{\partial V}{\partial \theta_i} Py - \frac{1}{2} tr\left(P\frac{\partial V}{\partial \theta_i}\right) = 0, \tag{1}$$

where

$$V = \tau^2 I_n + \sum_{i=1}^{s} \sigma_i^2 Z_i Z_i',$$

P is the projection matrix defined as $A(A'VA)^{-1}A'$ and

$$\theta_1 = \tau^2, \theta_2 = \sigma_1^2, \cdots, \theta_{s+1} = \sigma_s^2.$$

We have already known that the REML equations derived under the multivariate normal distribution are (see textbook P16 (1.23))

$$y'P^2y = tr(P),$$

$$y'PZ_iZ_i'Py = tr(Z_i'PZ_i). \tag{2}$$

or in another equivalent form are (see textbook P13 (1.18))

$$\frac{1}{2}\left\{y'P\frac{\partial V}{\partial \theta_i}Py - tr\left(P\frac{\partial V}{\partial \theta_i}\right)\right\} = 0 \tag{3}$$

Thus, the remaining problem is to show that (1) and (2) are equivalent.

Here the key is that if (2) holds true, then you can show that

$$y'Py = n - p. \tag{4}$$

Now let us show that from (2) and (3) we can get (1): Suppose (2) and (3) hold true, then (4) holds true as well. Plug in (4) to the LHS of (1), we have

$$LHS = \frac{1}{2}\left\{y'P\frac{\partial V}{\partial \theta_i}Py - tr\left(P\frac{\partial V}{\partial \theta_i}\right)\right\} = 0.$$

Thus, we get (1).

The other direction (from (1) to get (2)) is left as exercise.