Divide-and-Conquer recurrences and Master Theorem

Divide-and-Conquer recurrences

- Divide-and-Conquer (DC) recurrence

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

  where constants $a \geq 1$ and $b > 1$, function $f(n)$ is nonnegative.

- Example: the cost function of Merge Sort

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$$

  where

  - $a = 2$ (the number of subproblems),
  - $b = 2$ ($n/2$ is the size of subproblems),
  - $f(n) = n$ is the cost to divide and combine.

Solving DC recurrences by explicit substitution

- We illustrate by the following example

$$T(n) = 4T(\frac{n}{2}) + n, \qquad n = 2^k$$

By iterating the recurrence (i.e. explicit substitution), we have

$$\begin{aligned} T(n) &= 4T(\frac{n}{2}) + n = 4^2T(\frac{n}{2^2}) + 2n + n \\ &= 4^3T(\frac{n}{2^3}) + 2^2n + 2n + n = \cdots \\ &= 4^kT(\frac{n}{2^k}) + 2^{k-1}n + \cdots + 2n + n \\ &= n^2T(1) + (2^{k-1} + \cdots + 2 + 1)n \\ &= n^2T(1) + n(n-1) = \Theta(n^2) \end{aligned}$$

- For the general DC recurrence, let $n = b^k$, then we have

$$T(n) = n^{\log_b a}T(1) + \sum_{j=0}^{k-1} a^j f(\frac{n}{b^j})$$

The master theorem/method to solve DC recurrences

Case 1: *If* $n^{\log_b a}$ *is polynomially larger than* $f(n)$, *i.e,*

$$\frac{n^{\log_b a}}{f(n)} = \Omega(n^{\epsilon}) \quad \textit{for some constant } \epsilon > 0$$

Then

$$T(n) = \Theta(n^{\log_b a}).$$

Example: $T(n) = 7 \cdot T(\frac{n}{2}) + \Theta(n^2)$

The master theorem/method to solve DC recurrences

Case 2: *If* $n^{\log_b a}$ *and* $f(n)$ *are on the same order, i.e.,*

$$f(n) = \Theta(n^{\log_b a})$$

Then

$$T(n) = \Theta(n^{\log_b a} \lg n)$$

Example: $T(n) = 2 \cdot T(\frac{n}{2}) + \Theta(n)$

The master theorem/method to solve DC recurrences

Case 3: *If $f(n)$ is polynomially greater than $n^{\log_b a}$, i.e.,*

$$\frac{f(n)}{n^{\log_b a}} = \Omega(n^{\epsilon}) \quad \textit{for some constant } \epsilon > 0$$

*and $f(n)$ satisfies the regularity condition (*see next slide*).*
Then

$$T(n) = \Theta(f(n))$$

Example: $T(n) = 4 \cdot T(\frac{n}{2}) + n^3$

Remarks

1. $f(n)$ satisfies the *regularity condition* if

$$af\left(\frac{n}{b}\right) \leq cf(n)$$

   for some constant $c < 1$ and for all sufficient large n.
2. The proof of the master theorem is involved, shown in section 4.6, which we can safely skip for now.
3. The master method cannot solve every DC recurrences.