Handout 1: Calculus Review

STA 131B

For more details, please see a calculus text.

1. ***Series***

   a) $1 + a + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ if $a \neq 1$.

   b) $1 + a + a^2 + \cdots = \sum_{i=0}^{\infty} a^i = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}$ if $|a| < 1$.

   c) $\sum_{i=1}^{\infty} ia^{i-1} = \sum_{i=1}^{\infty} \frac{d}{da} a^i = \frac{d}{da} \frac{a}{1-a} = \frac{1}{(1-a)^2}$,

      $\sum_{i=2}^{\infty} i(i-1)a^{i-2} = \sum_{i=2}^{\infty} \frac{d^2}{d^2a} a^i = \frac{d^2}{d^2a} \sum_{i=2}^{\infty} a^i = \frac{d^2}{d^2a}(\frac{a^2}{1-a}) = \frac{2}{(1-a)^3}$ for $|a| < 1$.

   d) $\sum_{i=0}^{\infty} \frac{a^i}{i!} = e^a = \exp(a)$.

   e) $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$ (Binomial Theorem).

2. ***Limits***

   a) $\lim_{n\to\infty}(1 + \frac{a}{n})^n = e^a$.

   b) $\lim_{x\to\infty} e^{-x}x^a$ for all $a > 0$.

3. ***Differentiation of an inverse function***

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

4. ***Integration by parts***

$$\int_a^b u(x)dv(x) = -\int_a^b v(x)du(x) + \lim_{x\to b} u(x)v(x) - \lim_{x\to a} u(x)v(x),$$

   or for differentiable functions f and g,

$$\int_a^b f(x)g'(x)dx = -\int_a^b f'(x)g(x)dx + f(x)g(x)|_a^b.$$

5. ***Gamma function***
   Definition: $\Gamma(\alpha) = \int_0^{\infty} e^{-x}x^{\alpha-1}dx$ for $\alpha > 0$.

   a) For $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.

   b) For $\alpha > 0$, let $[\alpha]$ = greatest integer not exceeding α. Then by (a),

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)\cdots(\alpha-[\alpha])\Gamma(\alpha-[\alpha]).$$

      For example $\Gamma(5.3) = (4.3)(3.3)(2.3)(1.3)(0.3)\Gamma(0.3)$. Thus if we know $\Gamma(p)$ for all $0 < p \leq 1$, then we can calculate $\Gamma(\alpha)$ for all $\alpha > 0$.

c) $\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$

$\Gamma(n+1) = \int_0^\infty e^{-x} x^n = n!.$

d) For any nonnegative integer n,

$$\frac{1}{n!} \int_a^\infty e^{-x} x^n dx = \sum_{l=0}^{n} e^{-a} \frac{a^l}{l!}.$$

e) $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

6. ***Beta function***
Definition: $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$ for $a > 0, b > 0$.

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

7. ***Multiple integrals***
Let $g_i : \mathbb{R}^n \to \mathbb{R}$ be functions, $i = 1, \dots, n$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be the function $x = (x_1, \dots, x_n) \to y = (y_1, \dots, y_n), y_i = g_i(x_1, \dots, x_n), i = 1, \dots, n$. Denote the inverse function of g by h, $x_i = h_i(y_1, \dots, y_n), i = 1, \dots, n$ for functions $h_i : \mathbb{R}^n \to \mathbb{R}$.
Assume: Partial derivatives $\frac{\partial g_i}{\partial x_j}$ exist for all i, j and the matrix $[(\frac{\partial g_i}{\partial x_j})]_{i,j}$ has a non-zero determinant. Then

$$J_g = \det \left[\left(\frac{\partial g_i}{\partial x_j} \right) \right]_{i,j}$$

is called the Jacobian of the function (transformation) g. J_g is a function $\mathbb{R}^n \to \mathbb{R}$ (in x). It holds that

$$J_g = \det \left[\left(\frac{\partial g_i}{\partial x_j} \right) \right]_{i,j} = \frac{1}{\det \left[\left(\frac{\partial h_i}{\partial y_j} \right) \right]_{i,j}} = \frac{1}{J_h},$$

where J_h is a function of y.

a) Change of variables formula:
For $B \subset \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}$,

$$\begin{aligned}
&\int \cdots \int_B f(x_1, \dots, x_n) dx_1 \dots dx_n \\
&= \int \cdots \int_{g(B)} f(h_1(y_1, \dots, y_n), \dots, h_n(y_1, \dots, y_n)) \frac{1}{|J_g|} dy_1 \dots dy_n \\
&= \int \cdots \int_{g(B)} f(h_1(y_1, \dots, y_n), \dots, h_n(y_1, \dots, y_n)) |J_h| dy_1 \dots dy_n.
\end{aligned}$$

b) Application - Transformation to poloar coordinates:
Problem: Evaluate $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-u}u^{-1/2}du$.
Let $u = x^2$. Then $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-x^2}dx$. Now write

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-x_1^2}e^{-x_2^2}dx_1dx_2$$

and transform (x_1, x_2) to polar coordinates (r, θ):

$$x_1 = r\cos(\theta), \qquad x_2 = r\sin(\theta), \qquad r^2 = x_1^2 + x_2^2, \qquad \cos(\theta) = \frac{x_1}{r}$$

$$g : (x_1, x_2) \in (0, \infty)^2 \to (r, \theta) \in (0, \infty) \times \left(0, \frac{\pi}{2}\right)$$

$$J_h = \det\begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} = r.$$

Hence $(\Gamma(\frac{1}{2}))^2 = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}rd\theta dr = 2\pi\int_0^\infty e^{-r^2}rdr = \pi\int_0^\infty e^{-t}dt = \pi$, substituting $t = r^2$. Thus $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.