Statistics 206

Homework 4 Solution

Due : October 26, 2015, In Class

1. Tell true or false of the following statements.

   (a) The multiple coefficient of determination R^2 is always larger/not-smaller for models with more X variables.

   False. When the X variables in a smaller model are not entirely included in the larger model, then it is possible that the R^2 of the smaller model is larger than the R^2 of the larger model.

   (b) If all the regression coefficients associated with the X variables are estimated to be zero, then $R^2 = 0$.

   True. The fitted regression surface is horizontal so $\widehat{Y}_i = \overline{Y}$ for all i and thus $SSR = 0$, then $R^2 = \frac{SSR}{SSTO} = 0$.

   (c) The adjusted multiple coefficient of determination R_a^2 may decrease when adding additional X variables into the model.

   True. $R_a^2 = 1 - \frac{n-1}{n-p}\frac{SSE}{SSTO}$, the decrease in SSE may be more than offset by the loss of degrees of freedom in the denominator $n - p$.

   (d) Models with larger R^2 is always preferred.

   False. Models with larger R^2 may have a lot of X variables that are unrelated to the response variable or are highly correlated with each other which just over-fit the data and make prediction and interpretation difficult.

   (e) If the response vector is a linear combination of the columns of the design matrix $\mathbf{X}$, then the coefficient of multiple determination $R^2 = 1$.

   TRUE. Sine now $Y_i = \widehat{Y}_i$ for all $i = 1, \cdots, n$. Thus all residuals $e_i \equiv 0$ and then $SSE = 0$.

2. Under the multiple regression model (with X variables $X_1, \cdots, X_{p-1}$), show the following.

   (a) The LS estimator of the regression intercept is:

   $$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}_1 - \cdots - \hat{\beta}_{p-1}\overline{X}_{p-1},$$

   where $\hat{\beta}_k$ is the LS estimator of β_k, and $\overline{X}_k = \frac{1}{n}\sum_{i=1}^n X_{ik}$ $(k = 1, \cdots, p-1)$.

   (Hint: Plug in $\hat{\beta}_1, \cdots, \hat{\beta}_{p-1}$ to the least squares criterion function $Q(\cdot)$ and solve for b_0 that minimizes that function.)

   By the hint, taking partial derivative of $Q(\cdot)$ w.r.t. b_0 we have

   $$-\sum_{i=1}^n (Y_i - b_0 - \hat{\beta}_1 X_{i1} - \cdots - \hat{\beta}_{p-1}X_{p1}) = 0$$

   $$\Rightarrow \hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}_1 - \cdots - \hat{\beta}_{p-1}\overline{X}_{p-1}.$$

(b) SSE and the coefficient of multiple determination R^2 remain the same if we first center all the variables and then fit the regression model.

(Hint: Use part (a) and the fact that SSE is the minimal value achieved by the least squares criterion function.)

By part (a), if we center all the variables, the only change to the least squares criterion function is adding a constant term and taking out a constant term which is equal to $\hat{\beta}_0$, but now $\hat{\beta}_0^*$ for the new setup of centered data is 0 i.e. nothing really changed in the criterion function. Hence the minimal value achieved by the least squares criterion function SSE remains unchanged. And also the $SSTO$ is unchanged, therefore the coefficient of multiple determination would remain the same as well.

3. **Multiple regression.** The following data set has 30 cases, one response variable Y and two predictor variables X_1, X_2.

```
case Y    X1      X2
1   2.86  0.36  2.14
2  -0.50  0.66  0.74
3   3.24  0.66  1.91
4   0.44 -0.52 -0.41
5   0.04 -0.68  0.45
...  ...   ...   ...
29  2.60  0.84 -0.49
30  0.98 -0.11  2.41
```

Consider fitting the nonadditive model with interaction between X_1 and X_2. (R output is given at the end.)

(a) Write down the first 4 rows of the design matrix $\mathbf{X}$.

$$\begin{vmatrix} 1 & 0.36 & 2.14 & 0.7704 \\ 1 & 0.66 & 0.74 & 0.4884 \\ 1 & 0.66 & 1.91 & 1.2606 \\ 1 & -0.52 & -0.41 & 0.2132 \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

(b) We want to conduct prediction at $X_1 = 0, X_2 = 0$ and it is given that

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 0.087 & -0.014 & -0.035 & -0.004 \\ -0.014 & 0.115 & -0.012 & -0.052 \\ -0.035 & -0.012 & 0.057 & -0.014 \\ -0.004 & -0.052 & -0.014 & 0.050 \end{bmatrix}.$$

What is the predicted value? What is the prediction standard error? Construct a 95% prediction interval.

The predicted value when $X_1 = 0, X_2 = 0$ is
$\hat{Y_h} = X_h'\hat{\boldsymbol{\beta}} = 0.9918 + 0 + 0 + 0 = 0.9918$ where

$$X_h' = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{\beta}}' = \begin{bmatrix} 0.9918 & 1.5424 & 0.5799 & -0.1491 \end{bmatrix}.$$

$s(pred) = \sqrt{MSE\left[1 + X_h'(X'X)^{-1}X_h\right]} = 1.02 \times \sqrt{(1 + 0.087)} = 1.063.$
A 95% confidence prediction interval is $0.9918 \pm t(0.975; 30 - 4) \times 1.063 = 0.9918 \pm 2.056 \times (1.063) = (-1.19, 3.18)$.

(c) What are the regression sum of squares and error sum of squares of this model? What is SSTO?
$SSR = 58.232+5.490+0.448 = 64.17$. $SSE = 27.048$. $SSTO = 64.17 + 27.048 = 91.218$.

(d) Derive the following sum of squares:

$$SSR(X_1), \quad SSE(X_1), \quad SSR(X_2|X_1), \quad SSR(X_2, X_1 \cdot X_2|X_1),$$

$$SSR(X_1 \cdot X_2|X_1, X_2), \quad SSR(X_1, X_2), \quad SSE(X_1, X_2).$$

$SSR(X_1) = 58.232$. $SSE(X_1) = SSTO - SSR(X_1) = 32.986$. $SSR(X_2, X_1 \cdot X_2|X_1) = 5.490 + 0.448 = 5.938$.
$SSR(X_1 \cdot X_2|X_1, X_2) = 0.448$. $SSR(X_1, X_2) = 58.232+5.490 = 63.722$. $SSE(X_1, X_2) = 27.048 + 0.448 = 27.496$.

(e) Test whether both X_2 and the interaction term X_1X_2 can be dropped out of the model at level 0.01. Write down the full model and the reduced model. State the null and alternative hypotheses, test statistic and its null distribution, decision rule and the conclusion.
Full model: $Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i1} X_{i2} + \epsilon_i, \ i = 1, \cdots, n$.
Reduced model: $Y_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i, i = 1, \cdots, n$.
$H_0 : \beta_2 = \beta_3 = 0$.
H_a : Not both β_2 and β_3 are 0.

Test Statistic:

$$F^* = \frac{\frac{SSE(R) - SSE(F)}{df_R - df_F}}{SSE(F)/df_F} = \frac{\frac{SSR(X_2, X_1X_2|X_1)}{2}}{SSE(F)/df_F} = \frac{(5.49 + 0.448)/2}{27.048/26} = 2.85.$$

Under null hypothesis, the test statistic follows $F_{2,26}$ distribution.

Decision rule: Reject the null if $F^* > F(0.99; 2, 26) = 5.53$.

Since $F^* < F(0.99; 2, 26)$ we can not reject the null. Thus we conclude that both X_2 and the interaction X_1X_2 can be dropped from the model when X_1 is retained.

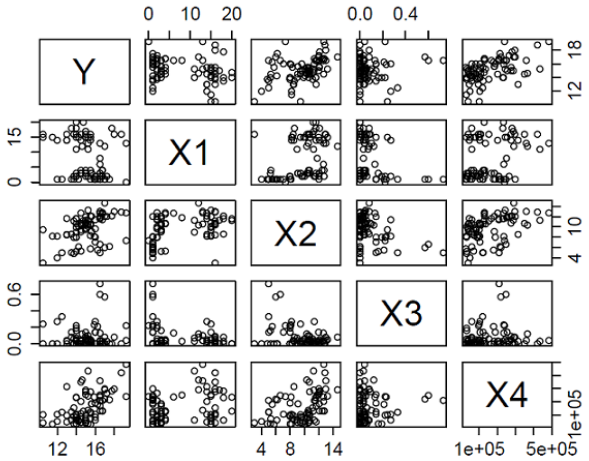
```
Call:
lm(formula = Y ~ X1 + X2 + X1:X2, data = data)

Residuals:
Min      1Q  Median      3Q     Max
-2.8660 -0.2055  0.1754  0.5436  2.0143

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.9918     0.3006   3.299 0.002817 **
X1            1.5424     0.3455   4.464 0.000138 ***
X2            0.5799     0.2427   2.389 0.024433 *
X1:X2        -0.1491     0.2271  -0.657 0.517215
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

Residual standard error: 1.02 on 26 degrees of freedom
Multiple R-squared: 0.7035,     Adjusted R-squared: 0.6693
F-statistic: 20.56 on 3 and 26 DF,  p-value: 4.879e-07


Analysis of Variance Table

Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 58.232  58.232 55.9752 6.067e-08 ***
X2         1  5.490   5.490  5.2775    0.0299 *
X1:X2      1  0.448   0.448  0.4311    0.5172
Residuals 26 27.048   1.040
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1
```

4. **A multiple linear regression case study by R**. You should use R and the *lm()* function and its associated functions (e.g., *summary(), anova(), confint(), predict.lm()*) to do this problem. Please also attach your R codes and plots.

   A commercial real estate company evaluates age (X_1), operating expenses (X_2, in thousand dollar), vacancy rate (X_3), total square footage (X_4) and rental rates (Y, in thousand dollar) for commercial properties in a large metropolitan area in order to provide clients with quantitative information upon which to make rental decisions. The data are taken from 81 *suburban commercial properties.* (The data is on smartsite under Resources/Homework/property.txt; The first column is Y, followed by X_1, X_2, X_3, X_4.)

   (a) Read data into R. Draw the scatter plot matrix and obtain the correlation matrix. What do you observe?

```
> property=read.table('property.txt',header=FALSE)
> names(property)=c('Y','X1','X2','X3','X4')
> pairs(property)
```



No obvious nonlinearity.

```
> cor(property)
```

```
Y          X1          X2          X3          X4
Y   1.00000000 -0.2502846  0.4137872  0.06652647 0.53526237
X1 -0.25028456  1.0000000  0.3888264 -0.25266347 0.28858350
X2  0.41378716  0.3888264  1.0000000 -0.37976174 0.44069713
X3  0.06652647 -0.2526635 -0.3797617  1.00000000 0.08061073
X4  0.53526237  0.2885835  0.4406971  0.08061073 1.00000000
```

X_1 and X_3, X_2 and X_3, X_1 and Y are negatively correlated, X_3 and X_4, X_3 and Y are not much correlated, other pairs are moderately positively correlated.

(b) Perform regression of the rental rates Y on the four predictors X_1, X_2, X_3, X_4 (Model 1). What are the Least-squares estimators? Write down the fitted regression function. What are MSE, R^2 and R_a^2?

```
> fit1=lm(Y~X1+X2+X3+X4,data=property)
> summary(fit1)

Call:
lm(formula = Y ~ X1 + X2 + X3 + X4, data = property)

Residuals:
Min      1Q  Median      3Q     Max
```

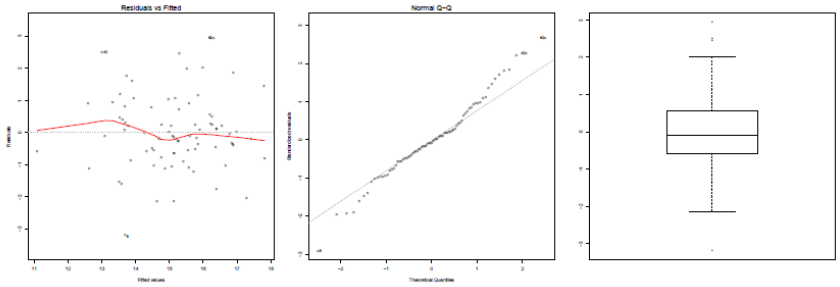
```
-3.1872 -0.5911 -0.0910  0.5579  2.9441

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.220e+01  5.780e-01  21.110  < 2e-16 ***
X1          -1.420e-01  2.134e-02  -6.655 3.89e-09 ***
X2           2.820e-01  6.317e-02   4.464 2.75e-05 ***
X3           6.193e-01  1.087e+00   0.570     0.57
X4           7.924e-06  1.385e-06   5.722 1.98e-07 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.137 on 76 degrees of freedom
Multiple R-squared: 0.5847,Adjusted R-squared: 0.5629
F-statistic: 26.76 on 4 and 76 DF,  p-value: 7.272e-14
```

Fitted regression function:

$$Y = 12.2 - 0.142X_1 + 0.282X_2 + 0.619X_3 + 7.92 \times 10^{-6}X_4$$

$MSE = 1.137^2 = 1.293$, $R^2 = 0.5847$, $R_a^2 = 0.5629$.

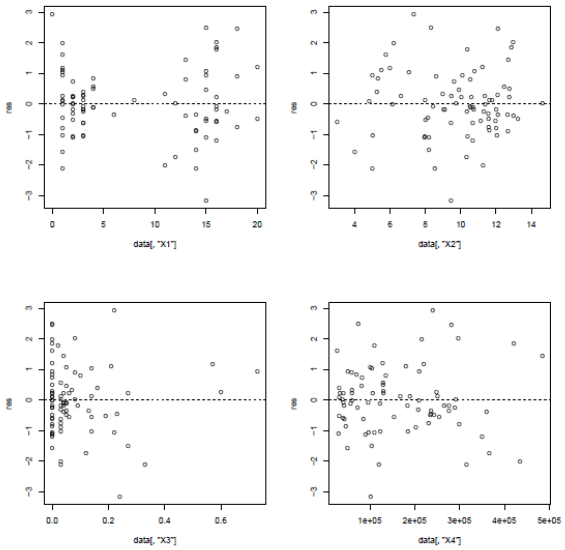
(c) Draw residuals vs. fitted values plot, residuals Normal Q-Q plot and residuals boxplot. Comment on the model assumptions based on these plots. (Hint: for a compact report, please use *par(mfrow)* to create one multiple paneled plot).



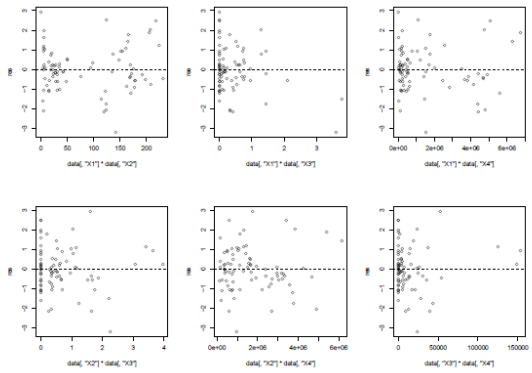
Residuals vs. fitted values plot shows no obvious nonlinearity. Residuals Q-Q plot shows slightly heavy tails. Residuals boxplot shows that most of the residuals are in between -2 and 2 and residual distribution is nearly symmetric.

(d) Draw residuals vs. each predictor variable plots, and residuals vs. each two-way interaction term plots. How many two-way interaction terms are there? Analyze your plots and summarize your findings. (Hint: again, use the *par(mfrow)* to create a few multiple paneled plots; Otherwise, there will be too many pages of plots!).

Residuals vs. each predictor:



No obvious pattern.
Residuals vs. each two-way interaction (6 in total):



No obvious pattern.

(e) For each regression coefficient, test whether it is zero or not (under the Normal error model) at level 0.01. State the null and alternative hypotheses, the test statistic, its null distribution and the pvalue. Which regression coefficient(s) is (are) significant, which is/are not? What is the implication? (Hint: you can find relevant information from the R *summary()* output.)

- $H_0 : \beta_0 = 0 \quad vs. \quad H_a : \beta_0 \neq 0$, $T^* = 21.11$, Under H_0, $T^* \sim t_{(76)}$, $pvalue < 2 \times 10^{-16}$
- $H_0 : \beta_1 = 0 \quad vs. \quad H_a : \beta_1 \neq 0$, $T^* = -6.655$, Under H_0, $T^* \sim t_{(76)}$, $pvalue = 3.89 \times 10^{-9}$

- $H_0 : \beta_2 = 0$ *vs.* $H_a : \beta_2 \neq 0$, $T^* = -4.464$, Under H_0, $T^* \sim t_{(76)}$, $pvalue = 2.75 \times 10^{-5}$
- $H_0 : \beta_3 = 0$ *vs.* $H_a : \beta_3 \neq 0$, $T^* = 0.57$, Under H_0, $T^* \sim t_{(76)}$, $pvalue = 0.57$
- $H_0 : \beta_4 = 0$ *vs.* $H_a : \beta_4 \neq 0$, $T^* = 5.722$, Under H_0, $T^* \sim t_{(76)}$, $pvalue = 1.98 \times 10^{-7}$

$\beta_0, \beta_1, \beta_2$ and β_4 are significant and β_3 is not significant. This implies that we could consider dropping X_3 from the model.

(f) Obtain SSTO, SSR, SSE and their degrees of freedom. Summarize these into an ANOVA table. Test whether there is a regression relation at $\alpha = 0.01$. State the null and alternative hypotheses, the test statistic, its null distribution, the decision rule and your conclusion. (Hint: you can find relevant information from the R *anova()* output.)

```
> anova(fit1)
Analysis of Variance Table

Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 14.819  14.819 11.4649  0.001125 **
X2         1 72.802  72.802 56.3262 9.699e-11 ***
X3         1  8.381   8.381  6.4846  0.012904 *
X4         1 42.325  42.325 32.7464 1.976e-07 ***
Residuals 76 98.231   1.293
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Source of Variation	SS	d.f.	MS	F^*
Regression	$SSR = 138.327$	4	$MSR = 34.58175$	$F^* = 26.75543$
Error	$SSE = 98.231$	76	$MSE = 1.292513$	
Total	$SSTO = 236.558$	80		

Note $SSR = 14.819 + 72.802 + 8.381 + 42.325 = 138.27$, and $SSTO = SSR + SSE = 236.558$.

$$H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0 \quad vs.$$

$$H_a : \text{ not all } \beta_k \ (k = 1, 2, 3, 4) \text{ equal zero.}$$

$$F^* = \frac{MSR}{MSE} = 26.75543$$

Under H_0, $F^* \sim F_{4,76}$.

```
> qf(0.99,4,76)
[1] 3.57652
```

Since $F^* = 26.75543 > 3.58 = F(0.99; 4, 76)$, reject H_0 and conclude that there is regression relation between Y and the set of X variables $\{X_1, X_2, X_3, X_4\}$.

(g) You now decide to fit a different model by regressing the rental rates Y on three predictors X_1, X_2, X_4 (Model 2). Why would you make such a decision? Get the Least-squares estimators and write down the fitted regression function. What are MSE, R^2 and R_a^2? How do these numbers compare with those from Model 1?

We consider Model 2 because β_3 is not significant (from part (e)).

```
> fit2=lm(Y~X1+X2+X4,data=property)
> summary(fit2)

Call:
lm(formula = Y ~ X1 + X2 + X4, data = property)

Residuals:
Min      1Q  Median      3Q     Max
-3.0620 -0.6437 -0.1013  0.5672  2.9583

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.237e+01  4.928e-01  25.100  < 2e-16 ***
X1          -1.442e-01  2.092e-02  -6.891 1.33e-09 ***
X2           2.672e-01  5.729e-02   4.663 1.29e-05 ***
X4           8.178e-06  1.305e-06   6.265 1.97e-08 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.132 on 77 degrees of freedom
Multiple R-squared: 0.583,Adjusted R-squared: 0.5667
F-statistic: 35.88 on 3 and 77 DF,  p-value: 1.295e-14
```

Fitted regression function:

$$Y = 12.37 - 0.1442X_1 + 0.2672X_2 + 8.178 \times 10^{-6}X_4$$

$MSE = 1.132^2 = 1.281$, $R^2 = 0.583$, $R_a^2 = 0.5667$. Compared to Model 1, MSE is a little bit smaller(1.281 vs. 1.293), R^2 is a little bit smaller(0.583 vs. 0.5847) but R_a^2 is a little bit larger(0.5667 vs. 0.5629).

(h) Compare the standard errors of the regression coefficient estimates for X_1, X_2, X_4 under Model 2 with those under Model 1. What do you find? Construct 95% confidence intervals for regression coefficients for X_1, X_2, X_4 under Model 2. If these intervals were constructed under Model 1, how would their widths compare with the widths of the intervals you just constructed, i.e., being wider or narrower? Justify your answer.

The standard errors of the regression coefficient estimates are smaller under Model 2. For $\hat{\beta}_1$, 2.134×10^{-2}(Model 1)$> 2.092 \times 10^{-2}$(Model 2); For $\hat{\beta}_2$, 6.317×10^{-2}(Model

1)$> 5.729 \times 10^{-2}$(Model 2); For $\hat{\beta}_4$, 1.385×10^{-6}(Model1)$> 1.305 \times 10^{-6}$(Model 2)
95% confidence interval under Model 2:

```
> cf2=confint(fit2,parm=c('X1','X2','X4'),level=.95)
> cf2
2.5 %        97.5 %
X1 -1.858219e-01 -1.025074e-01
X2  1.530784e-01  3.812557e-01
X4  5.578873e-06  1.077755e-05
```

If these intervals were constructed under Model 1, their width would be wider since the standard errors of the regression coefficient estimates are larger in Model 1, as well as the multipliers (t-percentiles) due to less degrees of freedom under Model 1. We can check this in R:

```
> cf1=confint(fit1,parm=c('X1','X2','X4'),level=.95)
> cf1          # confidence interval under Model 1
2.5 %        97.5 %
X1 -1.845411e-01 -9.952615e-02
X2  1.561979e-01  4.078352e-01
X4  5.166283e-06  1.068232e-05
> cf2[,2]-cf2[,1]          # width under Model 2
X1           X2           X4
8.331458e-02 2.281773e-01 5.198675e-06
> cf1[,2]-cf1[,1]          # width under Model 1
X1           X2           X4
8.501498e-02 2.516373e-01 5.516038e-06
```

(i) Consider a property with the following characteristics: $X_1 = 4, X_2 = 10, X_3 = 0.1, X_4 = 80,000$. Construct 99% prediction intervals under Model 1 and Model 2, respectively. Compare these two sets of intervals, what do you find?

```
> newX=data.frame(X1=4,X2=10,X3=0.1,X4=80000)
```

99% prediction interval under Model 1:

```
> predict.lm(fit1, newX, interval="prediction", level=0.99, se.fit=TRUE)
$fit
fit     lwr      upr
1 15.1485 12.1027 18.19429

$se.fit
[1] 0.1908982

$df
[1] 76
```

```
$residual.scale
[1] 1.136885
```

$$s(pred) = \sqrt{0.1908982^2 + 1.293} = 1.153$$

99% prediction interval under Model 2:

```
> predict.lm(fit2, newX, interval="prediction", level=0.99, se.fit=TRUE)
$fit
fit      lwr      upr
1 15.11985 12.09134 18.14836

$se.fit
[1] 0.1833524

$df
[1] 77

$residual.scale
[1] 1.131889
```

$$s(pred) = \sqrt{0.1833524^2 + 1.281} = 1.147$$

The width of the interval is narrower and the standard error is smaller under Model 2.

(j) Which of the two Models you would prefer and why?

We prefer Model 2 because it is simpler (less X variables) and essentially the same goodness of fit (R^2 similar to that of Model 1). It also has smaller standard errors and more degrees of freemdom, resulting in narrower confidence intervals and prediction intervals.

5. **(Optional Problem.)** Show the following with regard to the multiple coefficient of determination R^2.

(a)

$$R^2 = \widehat{Corr}^2(Y, \hat{\beta}_1 X_1 + \cdots + \hat{\beta}_{p-1} X_{p-1}) = \widehat{Corr}^2(Y, \widehat{Y}).$$

(Hint: By 2 (b), we can assume that all the variables are centered and then show $R^2 = \frac{(\mathbf{Y}^T \mathbf{X}\hat{\boldsymbol{\beta}})^2}{(\mathbf{Y}^T\mathbf{Y})(\mathbf{X}\hat{\boldsymbol{\beta}})^T(\mathbf{X}\hat{\boldsymbol{\beta}})}$*.)*

Since the data is centered, we have $\bar{Y} = 0$ so

$$\begin{aligned} R^2 &= \frac{SSR}{SSTO} = \frac{\hat{Y}'\hat{Y}}{Y'Y} \\ &= \frac{(\hat{Y}'\hat{Y})^2}{(Y'Y)(\hat{Y}'\hat{Y})} = \frac{[Y'X(X'X)^{-1}X'X(X'X)^{-1}X'Y][Y'X(X'X)^{-1}X'X(X'X)^{-1}X'Y]}{(Y'Y)[Y'X(X'X)^{-1}X'X(X'X)^{-1}X'Y]} \\ &= \frac{(Y'X\hat{\beta})^2}{(Y'Y)(X\hat{\beta})'(X\hat{\beta})}, \end{aligned}$$

which is $R^2 = \widehat{Corr}^2(Y, \hat{\beta}_1 X_1 + \cdots + \hat{\beta}_{p-1} X_{p-1})$.

(b)

$$R^2 = \max_{b_1, \cdots, b_{p-1}} \widehat{Corr}^2(Y, b_1 X_1 + \cdots + b_{p-1} X_{p-1}).$$

(Hint: Show that $(b_1, \cdots, b_{p-1})$ *that maximize the function* $\widehat{Corr}^2(Y, b_1 X_1 + \cdots + b_{p-1} X_{p-1})$ *are the LS estimators* $(\hat{\beta}_1, \cdots, \hat{\beta}_{p-1})$ *and then use part (a).)*
Take the derivative with respect to b, and look at the numerator

$$Y'Yb'X'XbY'XbY'X - Y'XbY'XbY'Yb'X'X.$$

If we substitute b by the least squares estimates $\hat{\beta}$ which satisfy $X'X\hat{\beta} = X'Y$, then it will make the numerator zero.