Growth of Functions and Asymptotic Notations

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Overview:

- Study a way to describe behavior of functions in the limit ... *asymptotic efficiency*
- Describe growth of functions
- Focus on what's important by abstracting lower-order terms and constant factors
- Indicate running times of algorithms
- A way to compare "sizes" of functions

  $O \approx \leq$
  $\Omega \approx \geq$
  $\Theta \approx =$

  In addition, $o \approx <$ and $\omega \approx >$

O-notation

- $g(n)$ is an asymptotic upper bound for $f(n)$:

  $$f(n) = O(g(n))$$

  if there exists constants c and n_0 such that

  $$0 \leq f(n) \leq c \cdot g(n) \quad \text{for } n \geq n_0$$

- Example:
  - $2n + 10 = O(n^2)$, pick $c = 1$ and $n_0 = 5$

More on O-notation

- $O(g(n))$ is a set of functions

  $$O(g(n)) = \{f(n) : \exists\, c, n_0 \text{ such that } 0 \le f(n) \le c \cdot g(n) \text{ for } n \ge n_0\}$$

- Examples of functions in $O(n^2)$:
  $n^2 + n$
  $n^2 + 1000n$
  $1000n^2 + 1000n$
  $n/1000$
  $n^2/\lg n$

Ω-notation

- $g(n)$ is an asymptotic lower bound for $f(n)$.

$$f(n) = \Omega(g(n))$$

  if there exists constants c and n_0 such that

$$0 \leq c \cdot g(n) \leq f(n) \quad \text{for } n \geq n_0$$

- Example:
  - $\sqrt{n} = \Omega(\lg n)$, pick $c = 1$ and $n_0 = 16$

More on Ω-notation

- $\Omega(g(n))$ is a set of functions

  $$\Omega(g(n)) = \{f(n) : \exists\, c, n_0 \text{ such that } 0 \leq c \cdot g(n) \leq f(n) \text{ for } n \geq n_0\}$$

- Examples of functions in $\Omega(n^2)$:
  n^2
  $n^2 + n$
  $n^2 - n$
  $1000n^2 + 1000n$
  $1000n^2 - 1000n$
  $n^{2.00001}$
  $n^2 \lg n$
  n^3

Θ-notation

- $g(n)$ is an asymptotic tight bound for $f(n)$.

$$f(n) = \Theta(g(n))$$

  if there exists constants c_1, c_2 and n_0 such that

$$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 g(n) \quad \text{for } n \geq n_0$$

- Example:
  - $\frac{1}{2}n^2 - 2n = \Theta(n^2)$, pick $c_1 = \frac{1}{4}$ $c_2 = \frac{1}{2}$ and $n_0 = 8$.
  - If $p(n) = \sum_{i=1}^{d} a_i n^i$ and $a_d > 0$, then $p(n) = \Theta(n^d)$

More on Θ-notation

- $\Theta(g(n))$ is a set of functions

  $\Omega(g(n)) =$
  $\{f(n) : \ \exists\ c_1, c_2, n_0$ such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 g(n)$ for $n \geq n_0\}$

- Examples of functions in $\Theta(n^2)$:
  n^2
  $n^2 + n$
  $n^2 - n$
  $1000n^2 + 1000n$
  $1000n^2 - 1000n$

Theorem

Theorem. O and Ω iff Θ.

Using limits for comparing orders of growth

In order to determine the relationship between $f(n)$ and $g(n)$, it is often usefuly to examine

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$$

The possible outcomes:

1. $L = 0$: $f(n) = O(g(n))$
2. $L = \infty$: $f(n) = \Omega(g(n))$
3. $L \neq 0$ is finite: $f(n) = \Theta(g(n))$
4. There is no limit: this technique cannot be used to determine the asymptotic relationship between $f(n)$ and $g(n)$.

Examples

1. $f(n) = n^2$ and $g(n) = n \lg n$

$$n^2 = \Omega(n \lg n)$$

2. $f(n) = n^{100}$ and $g(n) = 2^n$

$$n^{100} = O(2^n)$$

3. $f(n) = 10n(n+1)$ and $g(n) = n^2$

$$10n(n+1) = \Theta(n^2)$$