Handout 3: Method of Moments

STA 131B

Definition of Method of Moments Estimator

This is another quick method for getting estimates that relies on a substitution idea. Let $X_1, \ldots, X_n$ be a random sample from a population with parameter indexed by θ. Suppose that $\mu_1(\theta), \ldots, \mu_r(\theta)$ are the first r moments of the population we are sampling from. Thus,

$$\mu_j(\theta) = E(X_1^j|\theta).$$

Define the jth sample moment m_j

$$m_j = \frac{1}{n}\sum_{i=1}^{n} X_i^j \text{ for } j \geq 1.$$

To apply the method of moments to the problem of estimating $q(\theta)$, a function of θ, we need to be able to express q as a function g of the first r moments. Thus, suppose

$$q(\theta) = g(\mu_1(\theta), \ldots, \mu_r(\theta)).$$

The *method of moments* prescribes that we estimate $q(\theta)$ by $g(m_1, \ldots, m_r)$.

For instance, suppose that

$$q(\theta) = Var(X) = \sigma^2 = \mu_2(\theta) - \mu_1^2(\theta).$$

Our moment estimator of σ^2 would be

$$\hat{\sigma}^2 = m_2 - m_1^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\overline{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

The last equality is a consequence of $2\overline{X}\sum_{i=1}^{n} X_i = 2n\overline{X}^2$.

Example 3.1

Suppose that $X_1, \ldots, X_n$ is a $N(\mu, \sigma^2)$ sample. The method of moments estimates of μ and σ^2 are $\overline{X}$ and $\hat{\sigma}^2$.

Example 3.2

Suppose that $X_1, \ldots, X_n$ are indicators of a set of binomial trials with probability of success θ. Since $\mu_1(\theta) = \theta$ the method of moments leads to the natural estimate of θ, $\overline{X}$, the frequency of successes. To estimate the population variance $\theta(1-\theta)$ we are led by the first moment to the estimate $\overline{X}(1-\overline{X})$. Since we are dealing with (unrestricted) binomial trials, these are the frequency substitution estimates.

There are often several methods of moment estimates for the same $q(\theta)$. For example, if we are sampling from a Poisson population with parameter θ, then θ is both the population mean and the population variance. The method of moments can lead to either the sample mean or the sample variance. Other method of moments estimates for θ can also be constructed. The issue of which moment estimator is preferred will be addressed in a later topic.

What are the good points of the method of moments and frequency substitution?

a) The generally lead to procedures that are easy to compute and are therefore valuable as preliminary estimates.

b) If the sample size is large, these estimates are likely to be close to the value estimated (consistency).

This minimal property and the large sample behavior of the estimates will be discusses later on. The main difficulty with these methods is that they do not provide a unique estimate. A drawback of the method of moments is that unlike the MLE they often do not lead to efficient estimator. The issue of efficiency will be addressed later but below we provide an example where the method of moment estimators may be unreasonable.

Example 3.3

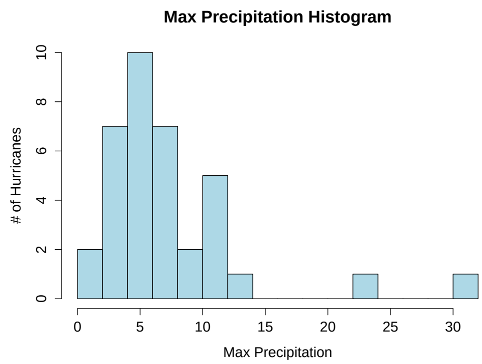
Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $[0, \theta]$. Since $\mu = \mu_1 = \theta/2$, the method of moment estimator for $\theta = 2\mu_1$ is $2\overline{X}$. This is clearly a foolish estimator if $X_{(n)} = maxX_i$, the largest order statistics, is greater than $2\overline{X}$, since in this model θ is always at least as large as $X_{(n)}$.

Case Study

Although hurricanes generally strike only the eastern and southern coastal regions of the United States, they do occasionally sweep inland before completely dissipating. The U.S. Weather Bureau confirms that in the period from 1900 to 1969 a total of 36 hurricanes moved as far as the Appalachians. The table below lists a subset of the 36 maximum 24-

hour precipitation levels recorded.

Year	Name	Location	Max Precipitation (in)
1969	*Camille*	Tye River, Va.	31.00
1968	*Candy*	Hickley, N.Y.	2.82
1965	*Betsy*	Haywood Gap, N.C.	3.98
1960	*Brenda*	Cairo, N.Y.	4.02
⋮	⋮	⋮	⋮
1900		St. Johnsbury, Vt.	0.67



The histogram of the data shows that because of its skewed shape, Y (the maximum number of 24-hour precipitation) might be well approximated by a member of the gamma family,

$$f_Y(y;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \text{ for } y > 0.$$

Here α and β are the parameters to be estimated. The complexity of $f_Y(y;\alpha,\beta)$, though, makes the method of maximum likelihood unwieldy. As an alternative, we will find the method-of-moments estimates.

Recall that the first two moments of a gamma distribution are given by

$$\mu_1(\theta) = E(Y) = \frac{\alpha}{\beta}$$

and

$$\mu_2(\theta) = E(Y^2) = \frac{\alpha(\alpha+1)}{\beta^2}.$$

For the given data, the first two sample moments are

$$m_1 = \frac{1}{36}\sum_{i=1}^{36} y_i = \frac{1}{36}(262.35) = 7.29$$

and

$$m_2 = \frac{1}{36}\sum_{i=1}^{36} y_i^2 = \frac{1}{36}(3081.2177) = 85.89.$$

The system of equation that derives from setting the μ_js equal to their corresponding m_js reduces to

$$\frac{\alpha}{\beta} = 7.29$$

$$\frac{\alpha(\alpha+1)}{\beta^2} = 85.89$$

Solving for β first and substituting the result into the second gives

$$\alpha^2\left(\frac{7.29}{\alpha}\right)\left(\frac{7.29}{\alpha}+1\right) = 7.29\alpha + 53.14 = 85.89.$$

Therefore,

$$\hat{\alpha} = \frac{85.89 - 53.14}{7.29} = 4.45$$

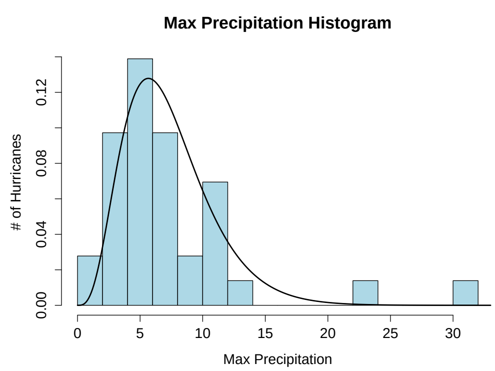
and

$$\hat{\beta} = \frac{7.29}{\hat{\alpha}} = 1.64.$$

The figure below shows the fitted model

$$f_Y(y; 4.45, 1.64) = \frac{1}{4.45^{1.64}\Gamma(1.64)} y^{0.64} e^{-y/4.45}$$

superimposed over the original data.



Considering the relatively small number of observations in the sample, the agreement is quite good. This would come as no surprise to a meteorologist; the gamma distribution is a very frequently used model for describing precipitation levels.