STA 135 Multivariate Data Analysis

Homework #1

2.2

a) $5A = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix}$

b) $BA = \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$

c) $A'B' = \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix}$

d) $C'B = [12, -7]$

e) No.

2.3

a) $A' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A$ so $(A')' = A' = A$

b) $C' = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$; $(C')^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix}$

$C^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{4}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$; $(C^{-1})' = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix} = (C')^{-1}$

c)

$(AB)' = \begin{bmatrix} 7 & 8 & 7 \\ 16 & 4 & 11 \end{bmatrix}' = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix} = (AB)'$

d) AB has (i,j)th entry

$$a_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{\ell=1}^{k} a_{i\ell}b_{\ell j}$$

Consequently, (AB)' has $(i,j)^{th}$ entry

$$c_{ji} = \sum_{\ell=1}^{k} a_{j\ell}b_{\ell i} .$$

Next B' has i^{th} row $[b_{1i}, b_{2i}, \cdots, b_{ki}]$ and A' has j^{th} column $[a_{j1}, a_{j2}, \cdots, a_{jk}]'$ so B'A' has $(i,j)^{th}$ entry

$$b_{1i}a_{j1} + b_{2i}b_{j2} + \cdots + b_{ki}a_{jk} = \sum_{\ell=1}^{k} a_{j\ell}b_{\ell i} = c_{ji}$$

Since i and j were arbitrary choices, (AB)' = B'A'.

2.4 a) $I = I'$ and $AA^{-1} = I = A^{-1}A$. Thus $I' = I = (AA^{-1})' = (A^{-1})'A'$ and $I = (A^{-1}A)' = A'(A^{-1})'$. Consequently, $(A^{-1})'$ is the inverse of A' or $(A')^{-1} = (A^{-1})'$.

b) $(B^{-1}A^{-1})AB = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I$ so AB has inverse $(AB)^{-1} = B^{-1}A^{-1}$. It was sufficient to check for a left inverse but we may also verify $AB(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{I}A^{-1} = AA^{-1} = I$.

2.6 a) Since $A = A'$, A is symmetric.

b) Since the quadratic form

$$\underset{\sim}{x}'A\underset{\sim}{x} = [x_1, x_2]\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 - 4x_1x_2 + 6x_2^2$$

$$= (2x_1 - x_2)^2 + 5(x_1^2 + x_2^2) > 0 \quad \text{for} \quad [x_1, x_2] \neq [0,0]$$

we conclude that A is positive definite.

2.7 a) Eigenvalues: $\lambda_1 = 10$, $\lambda_2 = 5$.

Normalized eigenvectors: $\underset{\sim}{e}_1' = [2/\sqrt{5}, -1/\sqrt{5}] = [.894, -.447]$

$\underset{\sim}{e}_2' = [1/\sqrt{5}, 2/\sqrt{5}] = [.447, .894]$

b) $A = \begin{bmatrix} 9 & -2 \\ -2 & 9 \end{bmatrix} = 10 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ -1/\sqrt{5}] + 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, \ 2/\sqrt{5}]$

c) $A^{-1} = \dfrac{1}{9(6)-(-2)(-2)} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} .12 & .04 \\ .04 & .18 \end{bmatrix}$

d) Eigenvalues: $\lambda_1 = .2, \ \lambda_2 = .1$

Normalized eigenvectors: $\underset{\sim}{e}_1' = [1/\sqrt{5}, \ 2/\sqrt{5}]$

$\underset{\sim}{e}_2' = [2/\sqrt{5}, \ -1/\sqrt{5}]$

2.12 By (2-20), $A = P\Lambda P'$ with $PP' = P'P = I$. From Result 2A.11(e) $|A| = |P| \ |\Lambda| \ |P'| = |\Lambda|$. Since Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_p$, we can apply Exercise 2.11 to get $|A| = |\Lambda| = \prod_{i=1}^{p} \lambda_i$.

2.16 $(A'A)' = A'(A')' = A'A$ showing $A'A$ is symmetric.

$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = A\underset{\sim}{x}$. Then $0 \le y_1^2 + y_2^2 + \cdots + y_p^2 = \underset{\sim}{y}'\underset{\sim}{y} = \underset{\sim}{x}'A'A\underset{\sim}{x}$

and $A'A$ is non-negative definite by definition.

2.20 Using matrix A in Exercise 2.3, we determine

$\lambda_1 = 1.382, \quad \underset{\sim}{e}_1 = [.8507, \quad -.5257]'$

$\lambda_2 = 3.618, \quad \underset{\sim}{e}_2 = [.5257, \quad .8507]'$

We know

$$A^{1/2} = \sqrt{\lambda_1}\, \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \sqrt{\lambda_2}\, \underset{\sim}{e}_2 \underset{\sim}{e}_2' = \begin{bmatrix} 1.376 & .325 \\ .325 & 1.701 \end{bmatrix}$$

$$A^{-1/2} = \frac{1}{\sqrt{\lambda_1}} \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \frac{1}{\sqrt{\lambda_2}} \underset{\sim}{e}_2 \underset{\sim}{e}_2' = \begin{bmatrix} .7608 & -.1453 \\ -.1453 & .6155 \end{bmatrix}$$

We check

$$A^{1/2}\,A^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1/2}\,A^{1/2}$$

2.21 (a)

$$\mathbf{A'A} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

$0 = |\mathbf{A'A} - \lambda\,\mathbf{I}\,| = (9-\lambda)^2 - 1 = (10-\lambda)(8-\lambda)$, so $\lambda_1 = 10$ and $\lambda_2 = 8$. Next,

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(b)

$$\mathbf{AA'} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

$$0 = |\mathbf{AA'} - \lambda\mathbf{I}\,| = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 8-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{vmatrix}$$

$= (2-\lambda)(8-\lambda)^2 - 4^2(8-\lambda) = (8-\lambda)(\lambda-10)\lambda$ so $\lambda_1 = 10, \lambda_2 = 8$, and $\lambda_3 = 0$.

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{aligned} 4e_3 &= 8e_1 \\ 8e_2 &= 10e_2 \end{aligned} \quad \text{so} \quad e_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{aligned} 4e_3 &= 6e_1 \\ 4e_1 &= 0 \end{aligned} \quad \text{so} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Also, $e_3 = [-2/\sqrt{5}, 0, 1/\sqrt{5}\,]'$.

(c)

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \sqrt{10}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}\left[\ \tfrac{1}{\sqrt{2}},\ \tfrac{1}{\sqrt{2}}\ \right] + \sqrt{8}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\left[\ \tfrac{1}{\sqrt{2}},\ -\tfrac{1}{\sqrt{2}}\ \right]$$

2.22 (a)

$$\mathbf{AA}' = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} = \begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix}$$

$0 = |\mathbf{AA}' - \lambda\,\mathbf{I}\,| = (144-\lambda)(126-\lambda) - (12)^2 = (150-\lambda)(120-\lambda)$, so $\lambda_1 = 150$ and $\lambda_2 = 120$. Next,

$$\begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 150\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives} \quad e_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

and $\lambda_2 = 120$ gives $e_2 = [1/\sqrt{5}, 2/\sqrt{5}\,]'$.

(b)

$$\mathbf{A}'\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix}\begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}$$

$$0 = |\mathbf{A}'\mathbf{A} - \lambda\mathbf{I}\,| = \begin{vmatrix} 25-\lambda & 50 & 5 \\ 50 & 100-\lambda & 10 \\ 5 & 10 & 145-\lambda \end{vmatrix} = (150-\lambda)(\lambda-120)\lambda$$

so $\lambda_1 = 150$, $\lambda_2 = 120$, and $\lambda_3 = 0$. Next,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 150\begin{bmatrix} e_1 \\ e_2 \\ e_2 \end{bmatrix}$$

$$\text{gives} \quad \begin{matrix} -120e_1 & + & 60e_2 & = 0 \\ -25e_1 & + & 5e_3 & = 0 \end{matrix} \quad \text{or } e_1 = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 120\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\text{gives} \quad \begin{array}{rcrl} 60e_1 & + & 60e_3 & = 0 \\ -120e_2 & + & -240e_3 & = 0 \end{array} \quad \text{or} \quad e_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Also, $e_3 = [2/\sqrt{5}, -1/\sqrt{5}, 0]'$.

(c)

$$\begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

$$= \sqrt{150} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \left[\begin{array}{ccc} \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{array} \right] + \sqrt{120} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \left[\begin{array}{ccc} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{array} \right]$$