Solutions for Practice Midterm I

STA 131B
Winter 2016
University of California, Davis

Exam Rules: This exam is closed book and closed notes. Use of calculators, cell phones or other communication devices is not allowed. You must show all of your work to receive credit. You will have 50 minutes to complete the exam.

Note: The practice exam is longer than the actual exam so you will have more to practice.

Name : ____________________________

ID : ____________________________

1. Suppose that the number of defects in a 1200-foot roll of magnetic recording tape has a Poisson distribution for which the value of the mean (θ) is unknown.

   a) Suppose five rolls of this tape are selected at random. Determine the joint p.d.f. $f(x_1, \ldots, x_5|\theta)$ of the five rolls.

   Answer: The p.d.f. of this Poisson distribution is
   $$f(x|\theta) = \frac{\theta^x}{x!}e^{-\theta}.$$
   So the joint p.d.f. is
   $$f(x_1, \ldots, f_5|\theta) = \prod_{i=1}^{5} f(x_i|\theta) = \prod_{i=1}^{5} \frac{\theta^{x_i}}{x_i!}e^{-\theta} = \frac{\theta^{\sum_{i=1}^{5} x_i}}{\prod_{i=1}^{5}(x_i!)}e^{-5\theta}.$$

   b) Suppose the number of defects found on the rolls are $2, 2, 6, 0$ and 3. If the prior distribution of θ is the gamma distribution with parameters $\alpha = 3$ and $\beta = 1$, find the posterior distribution of θ.

   Answer: The posterior p.d.f. of θ is
   $$\xi(\theta|\mathbf{x}) \propto f(x_1, \ldots, x_5|\theta)\xi(\theta) \propto \theta^{\sum_{i=1}^{5} x_i}e^{-5\theta}\theta^{\alpha-1}e^{-\beta\theta} = \theta^{\sum_{i=1}^{5} x_i+\alpha-1}e^{-(5+\beta)\theta}$$
   It follows that the posterior distribution is a gamma distribution with parameters $\sum_{i=1}^{5} x_i + \alpha = (2 + 2 + 6 + 0 + 3) + 3 = 16$ and $5 + \beta = 5 + 1 = 6$.

   c) Find the Bayes estimator with respect to the squared error loss function.

   Answer: The Bayes estimator is the mean of the posterior gamma distribution, which is 16/6=2.667.

2. Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution for which the p.d.f. $f(x|\theta)$ is as follows:
   $$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$
   Also, suppose the value of θ is unknown, and is a fixed (non-random) value $\theta > 0$.

a) Determine the joint p.d.f. of $X_1, \ldots, X_n$.

Answer: The joint p.d.f. is

$$f(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n (\prod_{i=1}^{n} x_i)^{\theta-1}.$$

b) Find the M.L.E. of θ.

Answer: The log-likelihood function is

$$L(\theta) = \log f(x_1, \ldots, x_n|\theta) = n \log \theta + (\theta - 1) \log(\prod_{i=1}^{n} x_i) = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log x_i.$$

Let $\dfrac{dL(\theta)}{d\theta} = \dfrac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0$, we get $\hat{\theta} = -\dfrac{n}{\sum_{i=1}^{n} \log x_i}$.
Furthermore, $\dfrac{d^2 L(\theta)}{d\theta^2} = -\dfrac{n}{\theta^2} < 0$.
Thus, $\hat{\theta}$ is the M.L.E. of θ.

c) Find the M.L.E. of θ^2.

Answer: By the invariance property, the the M.L.E. of θ^2 is $\hat{\theta}^2 = \dfrac{n^2}{(\sum_{i=1}^{n} \log x_i)^2}$.

3. Suppose that $X_1, \ldots, X_n$ form a random sample from the Poisson distribution with parameter θ.

a) Determine the method of moments estimator of θ based on the first moment.

Answer: Note that $E(X_1) = \theta$, the MoM estimator is $\tilde{\theta} = \dfrac{\sum_{i=1}^{n} x_i}{n} = \bar{x}_n$.

b) Show that your estimator in part (a) is the same as the M.L.E.

Answer: The joint p.d.f. is

$$f(x_1, \ldots, x_n|\theta) = \frac{\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} (x_i!)} e^{-n\theta}$$

The log-likelihood function is

$$L(\theta) = \log f(x_1, \ldots, x_n|\theta) = \log(\theta) \sum_{i=1}^{n} x_i - n\theta - \log(\prod_{i=1}^{n} (x_i!)).$$

Let $\dfrac{dL(\theta)}{d\theta} = \dfrac{\sum_{i=1}^{n} x_i}{\theta} - n = 0$, we get $\hat{\theta} = \dfrac{\sum_{i=1}^{n} x_i}{n} = \bar{x}_n$.
In addition, $\dfrac{d^2 L(\theta)}{d\theta^2} = -\dfrac{\sum_{i=1}^{n} x_i}{\theta^2} < 0$, so $\hat{\theta}$ is the M.L.E. of θ and is the same as $\tilde{\theta}$.

c) Find a method of moment estimator for the standard deviation of this Poisson distribution.

Answer: Note that $\text{var}(X_1) = \theta$, so a MoM estimator of the standard deviation is $\sqrt{\tilde{\theta}} = \sqrt{\bar{x}_n}$.

4. Suppose that the proportion θ of defective items in large shipment is unknown and that the prior distribution of θ is the beta distribution with parameters 1 and 10. Assume in a random sample of 20 items one find that 1 item is defective.

   a) What is the expected value and variance of the prior distribution?

      Answer: The expected value equals $\frac{1}{1+10} = 0.091$.
      The variance of the prior distribution equals $\frac{1 \cdot 10}{(1+10)^2(1+10+1)} = 0.007$.

   b) What is the posterior distribution?

      Answer: The prior p.d.f. of θ is $\xi(\theta) \propto \theta^{1-1}(1-\theta)^{10-1} = (1-\theta)^9$.
      Let $X_1, \ldots, X_{20}$ be the random samples. Then $X_i|\theta$ follows iid Bernoulli(θ) for $i = 1, \ldots, 20$. The likelihood function is $f(\mathbf{x}|\theta) = \theta(1-\theta)^{19}$.
      The posterior p.d.f. of θ is

      $$\xi(\theta|\mathbf{x}) \propto f(\mathbf{x}|\theta)\xi(\theta) \propto \theta(1-\theta)^{19}(1-\theta)^9 = \theta(1-\theta)^{28}.$$

      Thus, the posterior distribution of θ is the beta distribution with parameters 2 and 29.

   c) What is the Bayes estimator for θ if one uses the quadratic loss function?

      Answer: The Bayes estimator for θ is the mean of the posterior distribution, which is $\frac{2}{2+29} = 0.065$.

   d) Find the M.L.E. for θ. Is it the same as the Bayes estimator?

      Answer: The log-likelihood function is

      $$L(\theta) = \log f(\mathbf{x}|\theta) = \log\theta + 19\log(1-\theta).$$

      Let $\dfrac{dL(\theta)}{d\theta} = \dfrac{1}{\theta} - \dfrac{19}{1-\theta} = 0$, we get the M.L.E. $\hat{\theta} = 1/20 = 0.05$.
      Thus, the M.L.E. is not the same as the Bayes estimator.

   e) Suppose that you change the sampling plan and will keep on sampling until you find 3 defective items. Let X be the number of non-defective items until this happens. Derive the M.L.E. and Bayes estimator again.

      Answer: X has the negative binomial distribution with parameters 3 and θ. So the likelihood function is

      $$g(x|\theta) = \binom{x+2}{2}\theta^3(1-\theta)^x$$

      The log-likelihood function is

      $$L_1(\theta) = \log g(x|\theta) = 3\log\theta + x\log(1-\theta) + \log\binom{x+2}{2}.$$

      Let $\dfrac{dL_1(\theta)}{d\theta} = \dfrac{3}{\theta} - \dfrac{x}{1-\theta} = 0$, we get the M.L.E is $3/(3+x)$.
      The posterior p.d.f. of θ is

      $$\xi(\theta|x) \propto g(x|\theta)\xi(\theta) \propto \theta^3(1-\theta)^x(1-\theta)^9 = \theta^3(1-\theta)^{x+9}$$

      It corresponds to the p.d.f. of the beta distribution with parameters 4 and $x+10$.
      Thus, the Bayes estimator is the mean of this distribution, which is $\frac{4}{4+(x+10)} = \frac{4}{x+14}$.