Supplementary Note 1

Equality of Variances in ANOVA

The data on Winding Speeds is taken from Appl. Lin. Stat. Models (Problem 18.17). There were four speeds ($i = 1$: slow, $i = 2$: normal, $i = 3$: fast, $i = 4$: maximum) at which threads were wound and the response variable is the number of thread breaks. Thus the factor is speed and Y_{ij} is the number of breaks for the j^{th} spool at the i^{th} speed. You will find below a plot of the data as well as a table of the sample means and SD's. There is a clear indication that the SD tends to increase with speed. Thus the the usual assumption of equal variance of the four population is not justifiable in this case. And the formal test justifies this observation. For each i, the mean $\bar{Y}_i$ and SD s_i of the n_i observations are given in the table below.

n_i	$\bar{Y}_{i\cdot}$	s_i
16	3.563	1.094
16	5.875	1.996
16	10.687	3.240
16	16.562	5.379

Consider the problem of testing the equality of variances of k populations, i.e., $H_0 : \sigma_1 = \cdots = \sigma_k$ vs H_1 : not all σ_i are not the same

Bartlett's test statistic= 32.555, df=$k - 1 = 3$, p-value=0.000 $[\chi^2(0.99, 3) = 11.34]$,

Hartley statistic: $H = 24.175$, p-value=0.000 $[H(0.99, 4, 15) = 5.5]$,

Levene's (Brown-Forsythe) statistic=9.018, df=(3,59), p-value=0.000 $[F(0.99, 3, 59) = 4.13]$.

All the three statistics indicate that assumption of equal variance is untenable here.

Box-Cox Transformation

We wish to find the appropriate Box-Cox transformation so that the assumption of equal variance is reasonable for the transformed variable. The following table indicates that the assumption of equal variance is reasonable for the transformed varaible $Y' = \ln(Y)$.

n_i	$\bar{Y}_{i\cdot}$	s_i		$\bar{Y}'_{i\cdot}$	s'_i
16	3.563	1.094		1.224	0.323
16	5.875	1.996		1.704	0.403
16	10.687	3.240		2.321	0.330
16	16.562	5.379		2.750	0.365

How do we choose the appropriate transformation? Note that for any of the three statistics, H_0 is retained if the value of test statistic is small. Thus we may choose the transformation which minimizes the statistic.

The table given below lists the values of Bartlett, Levene (Brown-Forsythe) and Hartley statistics. It seems that logarithmic transformation may be the most appropriate transformation which makes the variance of the groups as similar as possible.

Transformation	Bartlett	Levene	Hartley
$Y' = Y$	34.580	9.542	24.192
$Y' = \sqrt{Y}$	10.292	2.888	5.549
$Y' = \ln(Y)$	0.937	0.098	1.562
$Y' = 1/\sqrt{Y}$	9.900	1.036	3.826
$Y' = 1/Y$	33.040	2.652	14.092

Some Tests for Equality of Variances.

The set up is as follows. We have k normal populations and we have n_i iid observations from the i^{th} population. The goal is to test if the population variances are equal i.e., $H_0 : \sigma_1 = \cdots = \sigma_k$ vs H_1 : not all σ_i are not the same.

We write down three tests here, though there are many others. Let s_i^2 denote the sample variance based on observations $\{Y_{ij} : j = 1, \ldots, n_i\}$ from the i^{th} population and let s^2_{pooled} be the pooled variance, i.e.,

$$\begin{aligned} s_i^2 &= \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\cdot})^2, i = 1, \ldots, k, \\ s^2_{pooled} &= \sum (n_i - 1) s_i^2 / (n_T - k) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\cdot})^2 / (n_T - k) = MSE. \end{aligned}$$

Bartlett and Hartley test statistics are

$$\begin{aligned} \text{Bartlett} &: \quad LRT = \sum (n_i - 1) \log(s^2_{pooled}/s_i^2), \\ \text{Hartley} &: \quad H = \max_i s_i^2 / \min_i s_i^2. \end{aligned}$$

It is clear that $H \geq 1$ and it can be shown that $LRT \geq 0$. Bartlett's statistic is named LRT since it the likelihood-ratio test statistic.

Under H_0, $LRT \overset{approx}{\sim} \chi^2_{k-1}$ and H_0 is rejected if $LRT > \chi^2_{k-1}(1 - \alpha)$, where α is the given level of significance.

There is a table for the distribution $H_{k,n-1}$ (where $n_1 = \cdots = n_k = n$) of the Hartley statistic under H_0 for the balanced case (i.e., $n_1 = \cdots = n_k$) and one can use this distribution to obtain the critical value for the test.

Levene's test (also known as Brown-Forsythe) is somewhat different from the other two tests given above. This test is constructed as follows. Let $\tilde{Y}_i$ be the median of the observations from the i^{th} population and consider the deviations

$$d_{ij} = |Y_{ij} - \tilde{Y}_i|, j = 1, \ldots, n_i, i = 1, \ldots, k.$$

Now carry out a regular test for the equality of means (an in the one-factor ANOVA) with $\{d_{ij}\}$. Let $MSTR_d$ and MSE_d be the treatment means square and error mean-squares based on $\{d_{ij}\}$ and let the F-statistic be

$$F^* = MSTR_d / MSE_d.$$

Under H_0, $F^* \overset{approx}{\sim} F_{k-1,n_T-k}$, and we reject H_0 if $F^* > F(1-\alpha; k-1, n_T-k)$.

Figure 1: Winding Speeds Data