Practice Final Exam

STA 131B
Winter 2016
University of California, Davis

Exam Rules: This exam is closed book and closed notes. Use of calculators, cell phones or other communication devices is not allowed. You must show all of your work to receive credit.

Name : ______________________

ID : ______________________

Signature : ______________________

1. Let $X_1, \dots, X_{25}$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. Suppose you observe $\sum_{i=1}^{25} X_i = 100$ and $\sum_{i=1}^{25} X_i^2 = 900$.

   a) Derive a 99% confidence interval for μ with the shortest length and interpret the meaning of this confidence interval.

   b) Derive a 99% confidence interval for σ^2.

   c) Test at the 5% level the null hypothesis $H_0 : \sigma = 1$ against the alternative hypothesis $H_1 : \sigma < 1$.

   d) Test at the 5% level the null hypothesis $H_0 : \sigma = 1$ against the alternative hypothesis $H_1 : \sigma \neq 1$.

   e) True or false:

      i) There is a 99% chance that the interval you found in (a) covers the true mean.

      ii) The confidence interval in (b) is unique.

      iii) We are 99% confident that the interval in (a) and (b) together covers both true parameters μ and σ.

   f) Derive jointly sufficient statistics for μ and σ.

2. Let $X_1, \dots, X_n$ be i.i.d. random variables with the exponential distribution with parameter θ. Suppose we wish to test the hypotheses

$$H_0 : \theta \geq \theta_0$$
$$H_1 : \theta < \theta_0.$$

   Let $Y = \sum_{i=1}^{n} X_i$ and let δ_c be the test that rejects H_0 if $Y \geq c$.

   a) Show that the power function is an decreasing function of θ.

   b) Find c to make δ_c have size .05.

   c) Let $\theta_0 = 2, n = 1$ and suppose the test has size .05. Find the precise form of the test δ_c and sketch its power function.

3. Let X be a random variable with a Poisson distribution for which the mean λ is unknown.

   a) Show that the only unbiased estimator of $e^{-\lambda}$ based on a sample of size 1 is:

$$\delta(X) = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise.} \end{cases}$$

b) Derive an unbiased estimator of $e^{-\lambda}$ based on a random sample $X_1, \ldots, X_n$.

c) Is the estimator in (b) admissible for the squared error loss function? If not show how to improve it (you need not derive the improved estimator).

d) Is the estimator in (b) efficient for $e^{-\lambda}$? Prove or disprove.

e) Derive the UMVUE of λ using a random sample of size n.

4. Suppose X_1 and X_2 are independent standard normal random variables. Let

$$Y_1 = X_1 + 2X_2$$
$$Y_2 = 2X_1 - X_2.$$

a) Prove that Y_1 and Y_2 are i.i.d. and find their distributions.

b) Let X_3 be another standard normal random variable that is independent of X_1 and X_2. Derive the distribution of

$$\frac{\overline{X}_2}{\sqrt{(X_1 - X_2)^2 + 2X_3^2}},$$

where $\overline{X}_2$ is the average of X_1 and X_2.

5. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $[0, \theta]$.

a) What is the parameter space?

b) Find the MLE of θ.

c) Find a method of moment estimator for θ.

d) Are either of the estimators in (b) or (c) minimal sufficient? Prove or disprove.

e) Are $(X_{(1)}, X_{(n)})$ jointly sufficient statistics for θ? Are they minimal sufficient? Prove or disprove.

Assume in parts (f) and (g) that θ has a prior density $h(\theta) = \theta e^{-\theta}$, for $\theta > 0$.

f) Find the Bayes estimator of θ for the squared error loss function. Just provide an expression but do not work out the final answer.

g) Derive a 95% Bayes confidence interval for θ. Identify the upper and lower confidence bounds clearly.

6. Let $X_{(1)}$ and $X_{(n)}$ be the smallest and largest order statistics corresponding to a random sample $X_1, \ldots, X_n$ from an exponential distribution with rate λ.

a) Find the expected value of $X_{(1)}$ and use this to construct an unbiased estimator of $\frac{1}{\lambda}$.

b) Find the MLE of the mean.

c) Is the estimator in (b) an efficient estimator of the mean? Explain why.

d) Show that $e^{\overline{X}_n}$ is a sufficient statistic for λ.

7. Suppose $X_1, \ldots, X_n$ are a random sample from the normal distribution with unknown mean μ and known variance σ^2. Consider the likelihood ratio test for the hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0.$$

a) Show that the likelihood ratio test for these hypotheses rejects H_0 when $\overline{X} \geq c$ for some constant c.

b) If the test is to be conducted at level .02, find c.