STA 135 Multivariate Data Analysis

Homework #2

2.24

a) $\Sigma^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\lambda_1 = 4, \quad \mathbf{e}_1 = [1,0,0]'$
$\lambda_2 = 9, \quad \mathbf{e}_2 = [0,1,0]'$
$\lambda_3 = 1, \quad \mathbf{e}_3 = [0,0,1]'$

c) For Σ^{-1}: $\lambda_1 = 1/4, \quad \mathbf{e}_1' = [1,0,0]'$
$\lambda_2 = 1/9, \quad \mathbf{e}_2' = [0,1,0]'$
$\lambda_3 = 1, \quad \mathbf{e}_3' = [0,0,1]'$

2.25

a) $V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \quad \boldsymbol{\rho} = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -.2 & .267 \\ -.2 & 1 & .167 \\ .267 & .167 & 1 \end{bmatrix}$

b) $V^{1/2} \boldsymbol{\rho} V^{1/2} =$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 4/5 & 1/2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \Sigma$$

2.27

a) $\mu_1 - 2\mu_2, \quad \sigma_{11} + 4\sigma_{22} - 4\sigma_{12}$

b) $-\mu_1 + 3\mu_2, \quad \sigma_{11} + 9\sigma_{22} - 6\sigma_{12}$

c) $\mu_1 + \mu_2 + \mu_3, \quad \sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23}$

d) $\mu_1 + 2\mu_2 - \mu_3, \quad \sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} - 2\sigma_{13} - 4\sigma_{23}$

e) $3\mu_1 - 4\mu_2, \quad 9\sigma_{11} + 16\sigma_{22}$ since $\sigma_{12} = 0$.

2.31 (a)

$$E[\boldsymbol{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{(b)} \quad \mathbf{A}\boldsymbol{\mu}^{(1)} = [\,1 \quad -1\,] \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1$$

(c)

$$\text{Cov}(\boldsymbol{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}' = [\,1 \quad -1\,] \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4$$

(e)

$$E[\boldsymbol{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{(f)} \quad \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(g)

$$\text{Cov}(\boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}' = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 48 & -8 \\ -8 & 4 \end{bmatrix}$$

(i)

$$\text{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(j)

$$\text{Cov}(\mathbf{A}\boldsymbol{X}^{(1)}, \mathbf{B}\boldsymbol{X}^{(2)}) = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}' = [\,1 \quad -1\,] \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = [\,0 \quad 2\,]$$

2.41 (a) $E(\mathbf{AX}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\mu_X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

(b) $Cov(\mathbf{AX}) = \mathbf{A}Cov(\mathbf{X})\mathbf{A}' = \mathbf{A}\Sigma_X\mathbf{A}' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix}$

(c) All pairs of linear combinations have zero covariances.

3.1

a) $\bar{\mathbf{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

b) $\mathbf{e}_1 = \mathbf{y}_1 - \bar{x}_1 \mathbf{1} = [4, 0, -4]'$

$\mathbf{e}_2 = \mathbf{y}_2 - \bar{x}_2 \mathbf{1} = [-1, 1, 0]'$

c) $L_{\mathbf{e}_1} = \sqrt{32}$; $L_{\mathbf{e}_2} = \sqrt{2}$

Let θ be the angle between $\mathbf{e}_1$ and $\mathbf{e}_2$, then $\cos(\theta) = -4/\sqrt{32 \times 2} = -.5$

Therefore $n\, s_{11} = L^2_{\mathbf{e}_1}$ or $s_{11} = 32/3$; $n\, s_{22} = L^2_{\mathbf{e}_2}$ or $s_{22} = 2/3$; $n\, s_{12} = \mathbf{e}_1'\mathbf{e}_2$ or $s_{12} = -4/3$. Also, $r_{12} = \cos(\theta) = -.5$. Consequently $S_n = \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & -.5 \\ -.5 & 1 \end{bmatrix}$.

3.5 a) $\mathbf{X}' = \begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}$; $\bar{\mathbf{x}}\,\mathbf{1}' = \begin{bmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \end{bmatrix}$

$$2\,S = (\mathbf{X} - \bar{\mathbf{x}}\,\mathbf{1}')(\mathbf{X} - \bar{\mathbf{x}}\,\mathbf{1}')' = \begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix}$$

so $S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$ and $|S| = 12$

b) $X' = \begin{bmatrix} 3 & 6 & 3 \\ 4 & -2 & 1 \end{bmatrix}$; $\bar{x}\,1' = \begin{bmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$$2S = (X - 1\bar{x}')'(X - 1\bar{x}') = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 18 \end{bmatrix}$$

so $S = \begin{bmatrix} 3 & -9/2 \\ -9/2 & 9 \end{bmatrix}$ and $|S| = 27/4$

3.10 (a) We calculate $\bar{x} = [5, 2, 3\,]'$ and

$$X_c = \begin{bmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and we notice } \mathrm{col}_1(\,X_c) + \mathrm{col}_2(\,X_c) = \mathrm{col}_1(\,X_c)$$

so $a = [1, 1, -1\,]'$ gives $X_c a = 0$.

(b)

$$S = \begin{bmatrix} 2.5 & 0 & 2.5 \\ 0 & 2.5 & 2.5 \\ 2.5 & 2.5 & 5 \end{bmatrix} \quad \text{so } |\,S\,| = \begin{array}{ccccc} 5(2.5)^2 & + & 0 & + & 0 \\ -(2.5)^3 & - & 0 & - & (2.5)^3 \end{array} = 0$$

Using the save coefficient vector a as in Part a) $Sa = 0$.

(c) Setting $Xa = 0$,

$$\begin{array}{rcl} 3a_1 + a_2 & = 0 \\ 7a_1 + 3a_3 & = 0 \\ 5a_1 + 3a_2 + 4a_3 & = 0 \end{array} \quad \text{so} \quad \begin{array}{rcl} a_1 & = & -\tfrac{3}{7}a_3 \\ 5a_1 & - & 3(3a_1) + 4a_3 = 0 \end{array}$$

so we must have $a_1 = a_3 = 0$ but then, by the first equation in the first set, $a_2 = 0$. The columns of the data matrix are linearly independent.

3.14 a) From first principles we have

$$\underset{\sim}{b}'\,\underset{\sim}{x}_1 = [2 \quad 3]\begin{bmatrix} 9 \\ 1 \end{bmatrix} = 21$$

Similarly $\underset{\sim}{b}'\,\underset{\sim}{x}_2 = 19$ and $\underset{\sim}{b}'\,\underset{\sim}{x}_3 = 8$ so

$$\text{sample mean} = \frac{21+19+8}{3} = 16$$

$$\text{sample variance} = \frac{(21-16)^2+(19-16)^2+(8-16)^2}{2} = 49$$

Also $\underset{\sim}{c}'\,\underset{\sim}{x}_1 = [-1 \quad 2]\begin{bmatrix} 9 \\ 1 \end{bmatrix} = -7$; $\underset{\sim}{c}'\,\underset{\sim}{x}_2 = 1$ and $\underset{\sim}{c}'\,\underset{\sim}{x}_3 = 3$

so

$$\text{sample mean} = -1$$

$$\text{sample variance} = 28$$

Finally sample covariance $= \dfrac{(21-16)(-7+1)+(19-16)(1+1)+(8-16)(3+1)}{2} =$ -28.

b) $\bar{\underset{\sim}{x}}' = [5 \quad 2]$ and $S = \begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}$

Using (3-36)

$$\text{sample mean of } \mathbf{b}'\mathbf{X} = \mathbf{b}'\bar{\mathbf{x}} = [2 \quad 3]\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 16$$

$$\text{sample mean of } \mathbf{c}'\mathbf{X} = [-1 \quad 2]\begin{bmatrix} 5 \\ 2 \end{bmatrix} = -1$$

$$\text{sample variance of } \mathbf{b}'\mathbf{X} = \mathbf{b}'S\mathbf{b} = [2 \quad 3]\begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 49$$

$$\text{sample variance of } \mathbf{c}'\mathbf{X} = \mathbf{c}'S\mathbf{c} = [-1 \quad 2]\begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 28$$

sample covariance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$

$$= \mathbf{b}'S\mathbf{c} = [2 \quad 3]\begin{bmatrix} 16 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -28$$

Results same as those in part (a).

3.16 Since

$$\begin{aligned}
\Sigma_V &= E(\mathbf{V}-\boldsymbol{\mu}_V)(\mathbf{V}-\boldsymbol{\mu}_V)' \\
&= E(\mathbf{V}\mathbf{V}' - \mathbf{V}\boldsymbol{\mu}_V' - \boldsymbol{\mu}_V\mathbf{V}' + \boldsymbol{\mu}_V\boldsymbol{\mu}_V') \\
&= E(\mathbf{V}\mathbf{V}') - E(\mathbf{V})\boldsymbol{\mu}_V' - \boldsymbol{\mu}_V E(\mathbf{V}') + \boldsymbol{\mu}_V\boldsymbol{\mu}_V' \\
&= E(\mathbf{V}\mathbf{V}') - \boldsymbol{\mu}_V\boldsymbol{\mu}_V' - \boldsymbol{\mu}_V\boldsymbol{\mu}_V' + \boldsymbol{\mu}_V\boldsymbol{\mu}_V' \\
&= E(\mathbf{V}\mathbf{V}') - \boldsymbol{\mu}_V\boldsymbol{\mu}_V' ,
\end{aligned}$$

we have $E(\mathbf{V}\mathbf{V}') = \Sigma + \boldsymbol{\mu}_V\boldsymbol{\mu}_V'$.