NP-Completeness (part 1 of 3)

Outline

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II P and NP

III. NP-complete (NPC): formal definition

IV. How to prove a problem is NPC

V. How to solve a NPC problem: approximate algorithms

I. Introduction

Outline

1. Tractable and intractable problems
2. NP-complete problems – informal definition
3. P vs NP
4. Optimization problems and decision problems

I. Introduction

Tractable and intractable problems

- Problems that are solvable by polynomial-time algorithms are tractable
- Problems that require superpolynomial time are intractable.

Almost all the algorithms we have studied thus far have been polynomial-time algorithms on inputs of size n, their worst-case running time is $O(n^k)$ for some constant k.

I. Introduction

NP-complete (NPC) problems: an informal definition

A class of very diverse problems share the following properties:

1. We *only know* how to solve those problems in time much larger than polynomial, namely exponential time.

2. If we could *solve one NPC porblem* in polynomial time, then there is a way to *solve every NPC problem* in polynomial time.

I. Introduction

Reasons to study NPC porblems: practical one

- you can use a known algorithm for it, and accept that it will take a long long time to solve;
- you can settle for approximating the solution, e.g., finding a nearly best solution rather than the optimum; or
- you can change your problem formulation so that it is solvable in polynomial time.

I. Introduction

Reasons to study NPC porblems: theoretical one

- We stated above that "*We only know*" how to solve those problems in time much larger than polynomial, *Not that we have proven* that these problems require exponential time.
- Indeed, this is one of the most famous problems in computer science:

$$\mathrm{P} \stackrel{?}{=} \mathrm{NP}$$

  or

  Whether NPC problems have polynomial solutions?

- First posed in 1971
  http://www.claymath.org/millennium-problems

I. Introduction

P-vs-NP Example 1.

- Shortest path:[1]
  finding the shortest path from a single source in a directed graph.
- Longest path:
  finding the longest *simple* path between two vertices in a directed graph.

The first one is solvable in polynomial time, and the second is NPC, but the difference appears to be slight.

[1] The Bellman-Ford algorithm

I. Introduction

P-vs-NP Example 2.

- Euler tour:[2]
  given a connected, directed graph G, is there a cycle that visits each edge exactly once (although it is allowed to visit each vertex more than once)?
- Hamiltonian cycle:
  given a connected directed graph G, is there a simple cycle that visits each vertex exactly once?

The first one is solvable in polynomial time, and the second is NPC, but the difference appears to be slight

[2] Euler cycle of $G = (V, E)$ **iff** in-degree(v) = out-degree(v) for $\forall v \in V$.

I. Introduction

P-vs-NP Example 3.

- Minimum spanning tree (MST):[3]
  Given a weighted graph and an integer k, is there a spanning tree whose total weight is k or less?

- Traveling salesperson problem (TSP):
  given a weighted graph and an integer k, is there a cycle that visits all vertices exactly once whose total weight is k or less?

The first one is solvable in polynomial time, and the second is NPC, but the difference appears to be slight

[3] Prim's and Kruskal's algorithms

I. Introduction

P-vs-NP Example 4.

- Circuit value:
  given a Boolean formula and its input, is the output True?
- Circuit satisfiability (SAT):
  given a Boolean formula, is there a way to set the inputs so that the output is True?

The first one is solvable in polynomial time, and the second is NPC, but the difference appears to be slight.

I. Introduction

Optimization problems and Decision problems

- Most of problems occur naturally as optimization problems,
- but they can also be formulated as decision problems, that is, problems for which the output is a simple *Yes* or *No* answer for each input.

Remarks:

- *To simplify discussion, we can consider only decision problems, rather than optimization problems.*
- The optimization problems are at least as hard to solve as the related decision problems, we have not lost anything essential by doing so.

I. Introduction

Optimization-vs-Decision Example 1.

Graph coloring: A coloring of a graph $G = (V, E)$ is a mapping

$$C : V \to S$$

where S is a finite set of "colors", such that

$$(u, v) \in E \Rightarrow C(u) \neq C(v)$$

- optimization problem: given G, determine the smallest number of colors needed.
- decision problem: given G and a positive integer k, is there a coloring of G using at most k colors?

I. Introduction

Optimization-vs-Decision Example 2.

Hamiltonian cycle: A Hamiltonian cycle is cycle that passes through every vertex exactly once.

- decision problem: Does a given graph have a Hamiltonian cycle?
- optimization problem: Give a list of vertices of a Hamiltonian cycle.

I. Introduction

Optimization-vs-Decision Example 3.

TSP (Traveling Salesperson Problem): given a weighted graph and an integer k, is there a cycle that visits all vertices exactly once (Hamiltonian cycle) whose total weight is k or less?

- optimization problem: given a weighted graph, find a minimum Hamiltonian cycle.
- decision problem: given a weighted graph and an integer k, is there a Hamiltonian cycle with total weight at most k?

I. Introduction – recap

1. Tractable and intractable problems
   polynomial-boundness: $O(n^k)$
2. NP-complete problems – informal definition
3. P vs NP
   difference may appear "*only slightly*"
4. Optimization problems and decision problems