131B HW#7 solution

8.1 The Sampling Distribution of a Statistic

2. It is known that $\bar{X}_n$ has the normal distribution with mean θ and variance $4/n$. Therefore,

$$E_\theta(|\bar{X}_n - \theta|^2) = \mathrm{var}(\bar{X}_n) = 4/n,$$

and $4/n \le 0.1$ if and only if $n \ge 40$.

8.2 The Chi-Square Distributions

6. We need to calculate $P(\sqrt{X^2+Y^2+Z^2} \le 4\sigma)$. At time 2, each of the independent variables X, Y and Z has a normal distribution with mean 0 and variance $2\sigma^2$. Therefore, each of the variables $X/(\sqrt{2}\sigma)$, $Y/(\sqrt{2}\sigma)$ and $Z/(\sqrt{2}\sigma)$ has a standard normal distribution. Hence, $V = (X^2+Y^2+Z^2)/(2\sigma^2)$ has a χ^2 distribution with three degrees of freedom. It follows that

$$P(\sqrt{X^2+Y^2+Z^2} \le 4\sigma) = P(X^2+Y^2+Z^2 \le 16\sigma^2) = P(V < 8) = 0.954.$$

9. It is known that $\bar{X}_n$ has the normal distribution with mean μ and variance σ^2/n. Therefore, $(\bar{X}_n - \mu)/(\sigma/\sqrt{n})$ has a standard normal distribution and the square of this variable has the χ^2 distribution with one degree of freedom.

10. Each of the variables $X_1+X_2+X_3$ and $X_4+X_5+X_6$ has the normal distribution with mean 0 and variance 3. Then $(X_1+X_2+X_3)/\sqrt{3}$ and $(X_4+X_5+X_6)/\sqrt{3}$ have the standard normal distribution. So $(X_1+X_2+X_3)^2/3$ and $(X_4+X_5+X_6)^2/3$ have the χ^2 distribution with one degree of freedom. Because of the independence, $(X_1+X_2+X_3)^2/3+(X_4+X_5+X_6)^2/3 = Y/3$ has the χ^2 distribution with two degrees of freedom. Thus, $c = 1/3$.

8.3 Joint Distribution of the Sample Mean and Sample Variance

4. The 3×3 matrix of the transformation from (X_1, X_2, X_3) to (Y_1, Y_2, Y_3) is

$$A = \begin{bmatrix} 0.8 & 0.6 & 0 \\ (0.3)\sqrt{2} & -(0.4)\sqrt{2} & -(0.5)\sqrt{2} \\ (0.3)\sqrt{2} & -(0.4)\sqrt{2} & (0.5)\sqrt{2} \end{bmatrix}$$

Since the matrix A is orthogonal, it follows from Theorem 8.3.4 that Y_1, Y_2 and Y_3 are independent and each has a standard normal distribution.

5. Let $Y_i = (X_i - \mu)/\sigma$ for $i = 1, 2$. Then Y_1 and Y_2 are independent and each has a standard normal distribution. Since $X_i = \mu + \sigma Y_i$, it follows that

$$X_1 + X_2 = 2\mu + \sigma(Y_1 + Y_2) = 2\mu + \sqrt{2}\sigma Z_1$$
$$X_1 - X_2 = \sigma(Y_1 - Y_2) = \sqrt{2}\sigma Z_2$$

where $Z_1 = (Y_1 + Y_2)/\sqrt{2}$ and $Z_2 = (Y_1 - Y_2)/\sqrt{2}$.

That is,

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

The 2×2 matrix A of the transformation is orthogonal, it follows from Theorem 8.3.4 that Z_1 and Z_2 are independent. Therefore, $X_1 + X_2$ and $X_1 - X_2$ are also independent.

8.4 The t Distributions

3. $X_1 + X_2$ has the normal distribution with mean 0 and variance 2. Therefore, $Y = (X_1 + X_2)/\sqrt{2}$ has a standard normal distribution. Also, $Z = X_3^2 + X_4^2 + X_5^2$ has the χ^2 distribution with 3 degrees of freedom, and Y and Z are independent. Therefore, $\frac{Y}{(Z/3)^{1/2}}$ has the t distribution with 3 degrees of freedom. Thus, $c = \sqrt{3/2}$.

5. Let $\bar{X}_2 = (X_1 + X_2)/2$ and $S_2^2 = \sum_{i=1}^{2}(X_i - \bar{X}_2)^2 = (X_1 - X_2)^2/2$. Then

$$W = \frac{(X_1 + X_2)^2}{(X_1 - X_2)^2} = \frac{2\bar{X}_2^2}{S_2^2}.$$

It follows from Eq. (8.4.4) that $U = \sqrt{2}\bar{X}_2/\sqrt{S_2^2}$ has the t distribution with one degree of freedom. Since $W = U^2$, we have

$$P(W < 4) = P(-2 < U < 2) = 2P(U < 2) - 1 = 2(0.852) - 1 = 0.704.$$

8.9 Supplementary Exercises

5. Since X_i has the exponential distribution with parameter β, it follows that $2\beta X_i$ has the exponential distribution with parameter 1/2. But this exponential distribution is the χ^2 distribution with 2 degrees of freedom. Therefore, the sum of the i.i.d. χ^2 random variables $\sum_{i=1}^{n} 2\beta X_i = 2\beta \sum_{i=1}^{n} X_i$ has a χ^2 distribution with $2n$ degrees of freedom.

8. $X_{n+1} - \bar{X}_n$ has the normal distribution with mean 0 and variance $(1+1/n)\sigma^2$. Hence, the distribution of $[n/(n+1)]^{1/2}(X_{n+1} - \bar{X}_n)/\sigma$ is a standard normal distribution. Also, nT_n^2/σ^2 has an independent χ^2 distribution with $n-1$ degrees of freedom. Thus, the following ratio has the t distribution with $n-1$ degrees of freedom. That is,

$$\frac{\left(\frac{n}{n+1}\right)^{1/2}(X_{n+1} - \bar{X}_n)/\sigma}{\left[\frac{nT_n^2}{(n-1)\sigma^2}\right]^{1/2}} = \left(\frac{n-1}{n+1}\right)^{1/2} \frac{X_{n+1} - \bar{X}_n}{T_n}$$

It can be seen that $k = [(n-1)/(n+1)]^{1/2}$.