Matrix-matrix multiplication

- Problem:
  Given $n \times n$ matrices A and B, compute the product $C = AB$.
- Traditional method: triple-loop
  Complexity: $T(n) = \Theta(n^3)$
- Divide-and-conquer
  1. Naive implementation – partition and then direct block multiplication

     $$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

     Complexity: $T(n) = 8 \cdot T(\frac{n}{2}) + \Theta(n^2) = \Theta(n^3)$. (*no improvement*)
  2. Strassen's method reduces the complexity to

     $$T(n) = 7 \cdot T(\frac{n}{2}) + \Theta(n^2) = \Theta(n^{\lg 7}).$$

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Strassen's method – Step 1: Divide

$$A = \begin{array}{cc} & \begin{array}{cc} \frac{n}{2} & \frac{n}{2} \end{array} \\ \begin{array}{c} \frac{n}{2} \\ \frac{n}{2} \end{array} & \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \end{array} \quad \text{and} \quad B = \begin{array}{cc} & \begin{array}{cc} \frac{n}{2} & \frac{n}{2} \end{array} \\ \begin{array}{c} \frac{n}{2} \\ \frac{n}{2} \end{array} & \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right] \end{array}$$

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Strassen's method – Step 2: Compute 10 matrices by $\pm$ only:

$$
\begin{aligned}
S_1 &= B_{12} - B_{22} \\
S_2 &= A_{11} + A_{12} \\
S_3 &= A_{21} + A_{22} \\
S_4 &= B_{21} - B_{11} \\
S_5 &= A_{11} + A_{22} \\
S_6 &= B_{11} + B_{22} \\
S_7 &= A_{12} - A_{22} \\
S_8 &= B_{21} + B_{22} \\
S_9 &= A_{11} - A_{21} \\
S_{10} &= B_{11} + B_{12}
\end{aligned}
$$

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Strassen's method – Step 3: Compute 7 matrices by multiplication:

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 \\
P_2 &= S_2 \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 \\
P_5 &= S_5 \cdot S_6 \\
P_6 &= S_7 \cdot S_8 \\
P_7 &= S_9 \cdot S_{10}
\end{aligned}
$$

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Strassen's method – Step 4: Add and subtract the P_i to construct submatrices C_{ij} of the product C

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$