Handout 8

Random and Mixed Effects Models.

Two factor models (both factors random, ANOVA model II).

For this model we have two factors A and B, both random. An example (data on "miles per gallon") is given later where both the factors diver (factor A) and driver (factor B) are random. The model is

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},\ k = 1, \ldots, n, j = 1, \ldots, b, i = 1, \ldots, a,$$

where $\mu_{\cdot\cdot}$ is an unknown constant where

(i) α_i's are iid $N(0, \sigma_\alpha^2)$, (ii) β_j's are iid $N(0, \sigma_\beta^2)$, (iii) $(\alpha\beta)_{ij}$'s iid $N(0, \sigma_{\alpha\beta}^2)$,

(iv) ε_{ijk}'s iid $N(0, \sigma^2)$, (v) $\{\alpha_i\}$, $\{\beta_j\}$, $\{(\alpha\beta)_{ij}\}$ and $\{\varepsilon_{ijk}\}$ are all pairwise independent.

For this model, we have

$$\begin{aligned}
E(Y_{ijk}) = \mu_{\cdot\cdot},\ Var(Y_{ijk}) &= \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_{\alpha\beta}^2 + \sigma^2, \\
Cov(Y_{ijk}, Y_{ijk'}) &= \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_{\alpha\beta}^2,\ k \neq k', \\
Cov(Y_{ijk}, Y_{ij'k'}) &= \sigma_\alpha^2, \text{ whenever } j \neq j', \\
Cov(Y_{ijk}, Y_{i'jk'}) &= \sigma_\beta^2, \text{ whenever } i \neq i', \\
Cov(Y_{ijk}, Y_{i'j'k'}) &= 0, \text{ whenever } i \neq i' \text{ and } j \neq j'.
\end{aligned}$$

Note that $Var(Y_{ijk})$ consists of four terms $\sigma_\alpha^2, \sigma_\beta^2, \sigma_{\alpha\beta}^2, \sigma^2$: these are called the components of variance (of $Var(Y_{ijk})$) or simply "variance components".

Two factors (A fixed, B random, ANOVA III, mixed model)

For this model we have two factors: A fixed and B random. An example (data on "imitation pearls") is given below in which number of coats is factor A (fixed) and bath is factor B (random). The model is

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},\ k = 1, \ldots, n, j = 1, \ldots, b, i = 1, \ldots, a,$$

where $\mu_{\cdot\cdot}$ is an unknown constant, α_i's are constants, where

(i) $\sum \alpha_i = 0$, (ii) β_j's are iid $N(0, \sigma_\beta^2)$, (iii) $\sum_i (\alpha\beta)_{ij} = 0$ for all j, (iv) $(\alpha\beta)_{ij} \sim N(0, (1 - 1/a)\sigma_{\alpha\beta}^2)$,

(v) $Cov((\alpha\beta)_{ij}, (\alpha\beta)_{i'j}) = -\sigma_{\alpha\beta}^2/a,\ i \neq i'$,

(vi) $Cov((\alpha\beta)_{ij}, (\alpha\beta)_{i'j'}) = 0$ if $i \neq i'$ and $j \neq j'$, (vii) ε_{ijk}'s iid $N(0, \sigma^2)$,

(ix) $\{\beta_j\}$, $\{(\alpha\beta)_{ij}\}$ and $\{\varepsilon_{ijk}\}$ are all pairwise independent.

In order to understand these variances and covariance for the mixed effects model, look at the Appendix where some technical details are given on the definition of $(\alpha\beta)_{ij}$.

For this model, we have

$$
\begin{aligned}
E(Y_{ijk}) &= \mu_{\cdot\cdot} + \alpha_i,\ Var(Y_{ijk}) = \sigma^2_\beta + (1-1/a)\sigma^2_{\alpha\beta} + \sigma^2,\\
Cov(Y_{ijk}, Y_{ijk'}) &= \sigma^2_\beta + (1-1/a)\sigma^2_{\alpha\beta},\ k \neq k',\\
Cov(Y_{ijk}, Y_{ijk'}) &= \sigma^2_\beta - (1/a)\sigma^2_{\alpha\beta},\ i \neq i',\\
Cov(Y_{ijk}, Y_{ij'k'}) &= 0, \text{ whenever } j \neq j',\\
Cov(Y_{ijk}, Y_{i'j'k'}) &= \sigma^2_\beta, \text{ whenever } j \neq j'.
\end{aligned}
$$

Analysis of Variance:

Calculations of SSTO, SSA, SSB, SSAB and SSE as in the two factor case when both factors are fixed. It also holds that SSTO can be written as the sum of SSA, SSB, SSAB and SSE. The question is: what do theses sum of squares and mean squares estimate? What statistics can be used to carry out various tests? The following tables (next page) provide the answers.

Estimation of variance components:

ANOVA II: We begin by noting that MSE estimates σ^2. In order to estimate the other variance components, note that

$$E(MSAB) - E(MSE) = n\sigma^2_{\alpha\beta},\ E(MSA) - E(MSAB) = nb\sigma^2_\alpha, E(MSB) - E(MSAB) = na\sigma^2_\beta.$$

So unbiased estimates of the variance components are

$$s^2_{\alpha\beta} = \frac{MSAB - MSE}{n},\ \ s^2_\alpha = \frac{MSA - MSAB}{nb},\ \ s^2_\beta = \frac{MSB - MSAB}{na}.$$

Ideally we we would like the estimates of $\sigma^2_{\alpha\beta}, \sigma^2_\alpha$ and σ^2_β to be non-negative, but $s^2_{\alpha\beta}, s^2_\alpha,$ and s^2_β are not guaranteed to be nonnegative. One way to modify them is to set the to zero whenever they are negative, i.e.,

$$\hat{\sigma}^2_{\alpha\beta} = \max(s^2_{\alpha\beta}, 0),\ \ \hat{\sigma}^2_\alpha = \max(s^2_\alpha, 0) \text{ and } \hat{\sigma}^2_\beta = \max(s^2_\beta, 0).$$

For the "Miles per gallon" data, we have

$$
\begin{aligned}
MSE &= 0.17575,\\
s^2_{\alpha\beta} &= \frac{MSAB - MSE}{n} = \frac{0.20308 - 0.17575}{2} = 0.01367,\\
s^2_\alpha &= \frac{MSA - MSAB}{nb} = \frac{93.4282 - 0.20308}{(2)(5)} = 9.3225,\\
s^2_\beta &= \frac{MSB - MSAB}{na} = \frac{23.6784 - 0.20308}{(2)(4)} = 2.9344.
\end{aligned}
$$

ANOVA III. Estimate of σ^2 is MSE as before. Now note that

$$E(MSAB) - E(MSE) = n\sigma^2_{\alpha\beta}, (MSB) - E(MSE) = na\sigma^2_\beta.$$

Table 1: Expected Values of Mean Squares

		Expected value		
Mean Square	df	A fixed, B fixed	A random, B random	A fixed, B random
MSA	$a-1$	$\sigma^2 + \frac{nb}{a-1}\sum \alpha_i^2$	$\sigma^2 + nb\sigma_\alpha^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + \frac{nb}{a-1}\sum \alpha_i^2 + n\sigma_{\alpha\beta}^2$
MSB	$b-1$	$\sigma^2 + \frac{na}{b-1}\sum \alpha_i^2$	$\sigma^2 + na\sigma_\beta^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + na\sigma_\beta^2$
MSAB	$(a-1)(b-1)$	$\sigma^2 + \frac{n}{(a-1)(b-1)}\sum\sum(\alpha\beta)_{ij}^2$	$\sigma^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + n\sigma_{\alpha\beta}^2$
MSE	$(n-1)ab$	σ^2	σ^2	σ^2

Table 2: Various F tests

	Some tests	
ANOVA I	ANOVA II	ANOVA III
$H_0 : \alpha_i = 0$ for all i, $F^* = \frac{MSA}{MSE}$	$H_0 : \sigma_\alpha^2 = 0$, $F^* = \frac{MSA}{MSAB}$	$H_0 : \alpha_i = 0$ for all i, $F^* = \frac{MSA}{MSAB}$
$H_0 : \beta_j = 0$ for all j, $F^* = \frac{MSB}{MSE}$	$H_0 : \sigma_\beta^2 = 0$, $F^* = \frac{MSB}{MSAB}$	$H_0 : \sigma_\beta^2 = 0$, $F^* = \frac{MSB}{MSE}$
$H_0 : (\alpha\beta)_{ij} = 0$ for all i,j, $F^* = \frac{MSAB}{MSE}$	$H_0 : \sigma_{\alpha\beta}^2 = \frac{MSAB}{MSE}$	$H_0 : \sigma_{\alpha\beta}^2 = 0$, $F^* = \frac{MSAB}{MSE}$

So unbiased estimates of $\sigma_{\alpha\beta}^2$ and σ_β^2 are

$$s_{\alpha\beta}^2 = \frac{MSAB - MSE}{n}, \; s_\beta^2 = \frac{MSB - MSE}{na}.$$

As in the ANOVA III case, $s_{\alpha\beta}^2$ and s_β^2 are not guaranteed to be non-negative and, they can be modified making them zero whenever they are negative.

For the "Imitation pearls" data, we have

$$\begin{aligned} MSE &= 4.8229, \\ s_{\alpha\beta}^2 &= \frac{MSAB - MSE}{n} = \frac{0.30863 - 4.8229}{4} < 0, \\ s_\beta^2 &= \frac{MSB - MSE}{na} = \frac{50.9506 - 4.8229}{(4)(3)} = 3.8440. \end{aligned}$$

Clearly estimate of $\sigma_{\alpha\beta}^2$ here is $\hat{\sigma}_{\alpha\beta}^2 = \max(s_{\alpha\beta}^2, 0) = 0$.

Estimation of means:

ANOVA II. Suppose we would like to estimate the overall mean $\mu_{\cdot\cdot}$. An unbiased estimate is $\bar{Y}_{\cdots}$. We can verify that

$$\begin{aligned} E(\bar{Y}_{\cdots}) &= \mu_{\cdot\cdot}, \; Var(\bar{Y}_{\cdots}) = \frac{nb\sigma_\alpha^2 + na\sigma_\beta^2 + n\sigma_{\alpha\beta}^2 + \sigma^2}{nab}, \\ s^2(\bar{Y}_{\cdots}) &= \frac{MSA + MSB - MSAB}{nab}. \end{aligned}$$

For the "Miles per gallon" data,

$$\bar{Y}_{\cdots} = 30.0475,\ s^2(\bar{Y}_{\cdots}) = \frac{93.4282 + 23.6784 - 0.20388}{(2)(4)(5)} = 2.9226,$$
$$s(\bar{Y}_{\cdots}) = 1.7096.$$

ANOVA III (mixed effects, A fixed, B random)

Let $\mu_i = E(Y_{ijk}) = \mu_{\cdot\cdot} + \alpha_i$. Often we are interested in estimating a linear contrast of $\mu_{i\cdot}$'s, i.e., $L = \sum c_i \mu_{i\cdot}$ (linear contrast means $\sum c_i = 0$). Estimate of L, its variance and estimated variance are

$$\hat{L} = \sum c_i \bar{Y}_{i\cdot\cdot},\ Var(\hat{L}) = \frac{n\sigma^2_{\alpha\beta} + \sigma^2}{nb} \sum c_i^2,\ s^2(\hat{L}) = \frac{MSAB}{nb} \sum c_i^2.$$

So a $(1-\alpha)100\%$ confidence interval for L is given by $\hat{L} \pm t(1-\alpha/2, (a-1)(b-1))s(\hat{L})$.

If we want to estimate $L = \mu_{3\cdot} - \mu_{1\cdot}$ (which compares levels 1 and 3 of factor A), then $c_1 = -1, c_2 = 0$ and $c_3 = 1$. Estimates of L and its SE are given below

$$\hat{L} = \bar{Y}_{3\cdot\cdot} - \bar{Y}_{1\cdot\cdot} = 76.9250 - 73.1063 = 3.8187,$$
$$s(\hat{L}) = \sqrt{\frac{MSAB}{nb} \sum c_i^2} = \sqrt{\frac{0.30863}{(4)(4)}(2)} = 0.1964.$$

Unbalanced Case:

As in the case of fixed effects, the unbalanced case requires coding of variables for factors. Let us review the fixed effects case. For the one-factor case, we can define $a-1$ independent variables and rewrite the one-factor ANOVA model as $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{X}$ is a $n_T \times a$ matrix whose first column consists of 1's (n_T is the total number of observations). Then the Gauss-Markov theorem is applied to obtain the normal equations $\mathbf{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{Y}$ and then solve for $\hat{\boldsymbol{\beta}}$. If an additive model is fitted for the two-factor model we can still write the model as $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{X}$ is a $n_T \times (a+b-1)$ matrix with the first column of 1's. In case we fit a model with interactions, then the two-factor model can be written as $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$ with $\boldsymbol{X}$ as a $n_T \times (ab)$ matrix and the first column consists of 1's.

The same strategy can be used for two and higher factor ANOVA models. The approach taken in chapter 24 resembles those for the fixed effects cases. This approach is called the **Henderson's** approach. This implictly assumes that the random factor effects satisfy the constraints as in the fixed efects case. Thus Henderson's approach is not the most appropriate one. A more appropriate approach uses the maximum likelihood method or its variants and its coding can be done as described below.

In the random and mixed models, one can carry out the same way of coding the factors that are fixed. However, it is easier if the coding of the random factors are done a bit differently using a 0-1 coding instead of {1,0,-1} coding. In the case we have two factors and we have an additive model, then we may write $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} + \boldsymbol{\varepsilon}$, where $\boldsymbol{X\beta}$ is the fixed effects part and $\boldsymbol{Z\gamma}$ is the random effects part, by making a slight change in notation by denoting the vector of main effects of factor B (random) by $\boldsymbol{\gamma}$. Here $\boldsymbol{Z}$ is $n_T \times b$ with all the factor levels coded coded as 0-1. **Note that unlike in the fixed factor case where b levels lead to $b-1$ variables, here all of them are coded leading to b variables.**

Higher order and other models.

If we have three factors, then conceptually it is not a problem to carry out a random effects model (with all random factors) or a mixed effects model. The expressions are quite complicated. However, all the packages have options to carry out random or mixed effects models.

It is also possible to have randomized design with the blocking factor random. The analysis are the same as in ANOVA models II and III.

Analysis of covariance with random factors.

Analysis of covariance with random factors can be easily carried out and all packages have the options of doing them. For instance, when investigating the quantitative skills of 10-12 olds, we may want to give them a standardized test and measure the scores on the test. So we may choose 10 schools at random (factor A, random), and then choose 3 boys and 3 girls at random (factor B, gender, fixed) and give these children the test. Also measured is each child's family income. If Y_{ijk} is the score of the k^{th} child of the j^{th} gender in the i^{th} school and X_{ijk} (covariate) is the corresponding family income, then a reasonable model may be

$$Y_{ijk} = \mu_{\cdots} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \gamma X_{ijk} + \varepsilon_{ijk}, k = 1, \ldots, 3, j = 1, 2, i = 1, \ldots 10,$$

where a's (random), β's (fixed) and $(\alpha\beta)$'s (random) are the same as in the ANOVA model III.

Structure of mixed effects models.

In the literature, the mixed effects models are often written as

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}_1\boldsymbol{\gamma}_1 + \cdots + \boldsymbol{Z}_q\boldsymbol{\gamma}_q + \boldsymbol{\varepsilon},$$

where $\boldsymbol{X}\boldsymbol{\beta}$ contains the fixed effects part or the combinations of the fixed effects and the covariates as in ANCOVA, $\boldsymbol{Z}_1\boldsymbol{\gamma}_1, \ldots, \boldsymbol{Z}_q\boldsymbol{\gamma}_q$ are the random effects parts, with $\boldsymbol{Z}_1, .., \boldsymbol{Z}_q$ as known design matrices, where $\boldsymbol{Z}_i$ is $n_T \times r_i, i = 1, ...q$, $\boldsymbol{\gamma}_1, ..., \boldsymbol{\gamma}_q$ are independent with $\boldsymbol{\gamma}_i$ is a r_i dimensional vector of iid $N(0, \sigma_i^2)$ vectors. In this setting, and $\boldsymbol{\varepsilon}$ consists of n_T iid $N(0, \sigma^2)$ variables. In this setting,

$$E(\boldsymbol{Y}) = \boldsymbol{X}\boldsymbol{\beta},\ Cov(\boldsymbol{Y}) = \sigma_1^2 \boldsymbol{Z}_1\boldsymbol{Z}_1' + \cdots + \sigma_q^2 \boldsymbol{Z}_q\boldsymbol{Z}_q' + \sigma^2\boldsymbol{I}.$$

Here $\sigma_1^2, ..., \sigma_q^2$ and σ^2 are the variance components. Two well known methods are employed to estimate the parameters: maximum likelihood and restricted maximum likelihood (REML).

Data Sets.

Example 1. (Miles per gallon, Exercise 25.15 in the text) An automobile manufacturer wished to study the effects of differences between drivers (factor A) and differences between cars (factor B) on gasoline consumption. Four drivers were selected at random; also five cars of the same model with manual transmission were randomly selected from the assembly line. Each driver drove each car twice over a 40-mile test course and miles per gallon were recorded. The data follow.

Factor A			Factor B		
	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	25.3	28.9	24.8	28.4	27.1
	25.2	30.0	25.1	27.9	26.6
$i=2$	33.6	36.7	31.7	35.6	33.7
	32.9	36.5	31.9	35.0	33.9
$i=3$	27.7	30.7	26.9	29.7	29.2
	28.5	30.4	26.3	30.2	28.9
$i=4$	29.2	32.4	27.7	31.8	30.3
	29.3	32.4	28.9	30.7	29.9

Here $\bar{Y}_{\dots} = 30.0475$.

ANOVA Table:

Source	df	SS	MS	F	p-value
Driver	$a-1=3$	280.2847	93.4282	$MSA/MSAB = 458.262$	0.000
Car	$b-1=4$	94.7135	23.6784	$MSB/MSAB = 116.142$	0.000
Interaction	$(a-1)(b-1)=12$	2.4465	0.20388	$MSAB/MSE = 1.160$	0.372
Error	$(n-1)ab=20$	3.5150	0.17575		
Total	$nab-1=39$	380.9597			

Example 2. (Imitation pearls, Exercise 25.17 in the text)

Preliminary research on the production of imitation pearls entailed studying the effects of the number of coats a special lacquer (factor A) applied to an opalescent plastic bead used as the base of the pearl on the market value of the pearl. Four batches of 12 beads (factor B) were used in this study, and it is desired to also consider their effect on the market value. The three levels of factor A (6, 8, and 10 coats) were fixed in advance, while the four batches can be regarded as a random sample of batches from the bead production process. The market value of each pearl was determined by a panel of experts. The market value data (coded) follow.

Factor A			Factor B	
	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	72.0	72.1	75.2	70.4
	$\dots$	$\dots$	$\dots$	$\dots$
	72.8	73.3	77.8	72.4
$i=2$	76.9	80.3	80.2	74.3
	$\dots$	$\dots$	$\dots$	$\dots$
	74.2	77.2	79.9	72.9
$i=3$	76.3	80.9	79.2	71.6
	$\dots$	$\dots$	$\dots$	$\dots$
	75.0	80.2	81.2	74.4

Here, $\bar{Y}_{\cdot\cdot\cdot} = 75.6083$, $\bar{Y}_{1\cdot\cdot} = 73.1063$, $\bar{Y}_{2\cdot\cdot} = 76.7938$, $\bar{Y}_{3\cdot\cdot} = 76.9250$.

ANOVA Table:

Source	df	SS	MS	F	p-value
Coat	$a-1=2$	150.3879	75.1940	$MSA/MSAB = 243.638$	0.000
Batch	$b-1=3$	152.8517	50.9506	$MSB/MSE = 10.564$	0.000
Interaction	$(a-1)(b-1)=6$	1.8521	0.30863	$MSAB/MSE = 0.064$	0.999
Error	$(n-1)ab=36$	173.6250	4.8229		
Total	$nab=47$				

Appendix.

Here is an important fact that is always useful in obtaining results in ANOVA or other models.

Fact. (a) If $\{Z_i : i = 1, \dots, a\}$ are **mean zero** variables which are mutually uncorrelated with common variance γ^2, then $E\left[\sum (Z_i - \bar{Z})^2\right] = (a-1)\gamma^2$.

(b) If $\{W_{ij} : i = 1, \dots, a, j = 1, \dots, b\}$ are mutually uncorrelated **mean zero** variables with common variance τ^2, then

$$E\left[\sum_i \sum_j (W_{ij} - \bar{W}_{i\cdot} - \bar{W}_{\cdot j} + \bar{W}_{\cdot\cdot})^2\right] = (a-1)(b-1)\tau^2.$$

Proof. (a) This part is clear since $\sum (Z_i - \bar{Z})^2 = \sum Z_i^2 - n\bar{Z}^2$ and $E(\bar{Z}) = Var(\bar{Z}) = \gamma^2/a$.

(b) Fix j, and denoting $D_i = W_{ij} - \bar{W}_{i\cdot}$, then $\bar{D} = \bar{W}_{\cdot j} - \bar{W}_{\cdot\cdot}$. Note that D_i's have zero mean, are mutually uncorrelated and the common variance of D_i is $\tau^2(b-1)/b = \gamma^2$. Hence using part (a), we have

$$\begin{aligned}
&\sum_i E(W_{ij} - \bar{W}_{i\cdot} - \bar{W}_{\cdot j} + \bar{W}_{\cdot\cdot})^2 \\
&= \sum_i E(D_i - \bar{D})^2 = (a-1)\gamma^2 = [(a-1)(b-1)/b]\tau^2.
\end{aligned}$$

Hence,

$$\sum_i \sum_j E(W_{ij} - \bar{W}_{i\cdot} - \bar{W}_{\cdot j} + \bar{W}_{\cdot\cdot})^2 = \sum_j \left[\sum_i E(W_{ij} - \bar{W}_{i\cdot} - \bar{W}_{\cdot j} + \bar{W}_{\cdot\cdot})^2 \right]$$
$$= \sum_j [(a-1)(b-1)/b]\tau^2 = (a-1)(b-1)\tau^2.$$

For ANOVA II and III models, the proof for $E(MSE) = \sigma^2$ are the same. Important part to note that the quantity $Y_{ijk} - \bar{Y}_{ij\cdot} = \varepsilon_{ijk} - \bar{\varepsilon}_{ij\cdot}$ does not depend α's, β's and $(\alpha\beta)$'s. Now for any given i and j

$$\sum_k E(Y_{ijk} - \bar{Y}_{ij\cdot})^2 = \sum_k E(\varepsilon_{ijk} - \bar{\varepsilon}_{ij\cdot})^2 = (n-1)\sigma^2.$$

Hence

$$E(SSE) = \sum_i \sum_j \sum_k E(Y_{ijk} - \bar{Y}_{ij\cdot})^2 = \sum_i \sum_j (n-1)\sigma^2 = (n-1)ab\sigma^2.$$

This shows that $E(MSE) = \sigma^2$.

ANOVA II models.

(i) Variance of $\bar{Y}_{\cdots}$: Note that we can write $\bar{Y}_{\cdots} = \mu_{\cdot\cdot} + \bar{\alpha} + \bar{\beta} + \overline{(\alpha\beta)}_{\cdot\cdot} + \bar{\varepsilon}_{\cdots}$, and since α's, β's, $(\alpha\beta)$'s and ε's are independent, we have

$$\begin{aligned} Var(\bar{Y}_{\cdots}) &= Var(\bar{\alpha}) + Var(\bar{\beta}) + Var(\overline{(\alpha\beta)}) + Var(\bar{\varepsilon}) \\ &= \sigma_\alpha^2/a + \sigma_\beta^2/b + \sigma_{\alpha\beta}^2/(ab) + \sigma^2/(nab) \\ &= \frac{nb\sigma_\alpha^2 + na\sigma_\beta^2 + n\sigma_{\alpha\beta}^2 + \sigma^2}{nab}. \end{aligned}$$

(ii) Expectation of mean squares: $\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdots}$, $\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdots}$ and $\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdots}$ as $\hat{\alpha}_i, \hat{\beta}_j$ and $\widehat{(\alpha\beta)_{ij}}$, respectively, we have

$$\begin{aligned} SSA &= \sum\sum\sum \hat{\alpha}_i^2 = nb\sum_i \hat{\alpha}_i^2, \\ SSB &= \sum\sum\sum \hat{\alpha}_i^2 = na\sum_j \hat{\beta}_j^2, \\ SSAB &= \sum\sum\sum \widehat{(\alpha\beta)_{ij}}^2 = n\sum_i\sum_j \widehat{(\alpha\beta)_{ij}}^2. \end{aligned}$$

Note that

$$\begin{aligned} \hat{\alpha}_i = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdots} &= \left\{ \mu_{\cdot\cdot} + \alpha_i + \bar{\beta} + \overline{(\alpha\beta)}_{i\cdot} + \bar{\varepsilon}_{i\cdot\cdot} \right\} - \{ \mu_{\cdot\cdot} + \bar{\alpha} + \bar{\beta} + \overline{(\alpha\beta)}_{\cdot\cdot} + \bar{\varepsilon}_{\cdots} \} \\ &= [\alpha_i - \bar{\alpha}] + [\overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{\cdot\cdot}] + [\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdots}]. \end{aligned}$$

Verify that (using part (a) of the Fact given above)

$$\sum_i E(\alpha_i - \bar{\alpha})^2 = (a-1)\sigma^2_\alpha,$$

$$\sum_i E(\overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{\cdot\cdot}))^2 = (a-1)(\sigma^2_{\alpha\beta}/b), \text{[Part (a) with } Z_i = \overline{(\alpha\beta)}_{i\cdot}]$$

$$\sum_i E(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot\cdot\cdot})^2 = (a-1)\{\sigma^2/(nb)\}, \text{ [Part (a) with } Z_i = \bar{\varepsilon}_{i\cdot\cdot}].$$

Independence of α's, β's, $(\alpha\beta)$'s and ε's then allows us

$$\begin{aligned}
E(SSA) &= nb\sum E(\hat{\alpha}_i^2) \\
&= nb\sum Var(\hat{\alpha}_i^2) \text{ [since } E(\hat{\alpha}_i) = 0] \\
&= nb\sum \Big[Var(\alpha_i - \bar{\alpha}) + Var(\overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{\cdot\cdot})) + Var(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot\cdot\cdot})\Big] \\
&= nb\left[(a-1)\sigma^2_\alpha + (a-1)(\sigma^2_{\alpha\beta}/b) + (a-1)\{\sigma^2/(nb)\}\right] \\
&= nb(a-1)\sigma^2_\alpha + n(a-1)\sigma^2_{\alpha\beta} + (a-1)\sigma^2, \text{ and hence} \\
E(MSE) &= E(SSA)/(a-1) = nb\sigma^2_\alpha + n\sigma^2_{\alpha\beta} + \sigma^2.
\end{aligned}$$

Basically the same type of argument will show

$$E(MSB) = na\sigma^2_\beta + n\sigma^2_{\alpha\beta} + \sigma^2.$$

In order to check the result for $MSAB$, note that

$$\begin{aligned}
\widehat{(\alpha\beta)_{ij}} &= \bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot} \\
&= [\mu_{\cdot\cdot} + \alpha_i + \beta_j + \overline{(\alpha\beta)}_{ij} + \bar{\varepsilon}_{ij\cdot}] - [\mu_{\cdot\cdot} + \alpha_i + \bar{\beta} + \overline{(\alpha\beta)}_{i\cdot} + \bar{\varepsilon}_{i\cdot\cdot}] \\
&\quad - [\mu_{\cdot\cdot} + \bar{\alpha} + \beta_j + \overline{(\alpha\beta)}_{\cdot j} + \bar{\varepsilon}_{\cdot j\cdot}] + [\mu_{\cdot\cdot} + \bar{\alpha} + \bar{\beta} + \overline{(\alpha\beta)}_{\cdot\cdot} + \bar{\varepsilon}_{\cdot\cdot\cdot}] \\
&= [\overline{(\alpha\beta)}_{ij} - \overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{\cdot j} + \overline{(\alpha\beta)}_{\cdot\cdot}] + [\bar{\varepsilon}_{ij\cdot} - \bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot j\cdot} + \bar{\varepsilon}_{\cdot\cdot\cdot}].
\end{aligned}$$

Now part (b) of the above Fact tells us

$$\sum\sum E(\overline{(\alpha\beta)}_{ij} - \overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{\cdot j} + \overline{(\alpha\beta)}_{\cdot\cdot})^2 = (a-1)(b-1)\sigma^2_{\alpha\beta} \text{ [with } W_{ij} = \overline{(\alpha\beta)}_{ij} \text{ in part (b)]}$$

$$\sum\sum E(\bar{\varepsilon}_{ij\cdot} - \bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot j\cdot} + \bar{\varepsilon}_{\cdot\cdot\cdot})^2 = (a-1)(b-1)[\sigma^2/n] \text{ [with } W_{ij} = \bar{\varepsilon}_{ij\cdot} \text{ in part (b)]}.$$

Hence

$$\begin{aligned}
E(SSAB) &= n[(a-1)(b-1)\sigma^2_{\alpha\beta} + (a-1)(b-1)\{\sigma^2/n\}] \\
&= (a-1)(b-1)[n\sigma^2_{\alpha\beta} + \sigma^2], \\
E(MSAB) &= n\sigma^2_{\alpha\beta} + \sigma^2.
\end{aligned}$$

ANOVA III (A fixed, B random).

The following technical point is important. Let γ_{ij} be iid $N(0, \sigma^2_{\alpha\beta})$, then you text defines $(\alpha\beta)_{ij} = \gamma_{ij} - \bar{\gamma}_{\cdot j}$ (actually the notation for γ_{ij} in the text is $(\alpha\beta)^*_{ij}$. Following this definition, $\sum_i (\alpha\beta)_{ij} = 0$, $Var(\alpha\beta)_{ij}) = [(a-1)/a]\sigma^2_{\alpha\beta}$. Moreover, $\overline{(\alpha\beta)}_{i\cdot} = \bar{\gamma}_{i\cdot} - \bar{\gamma}_{\cdot\cdot}$. Hence,

$$Var(\overline{(\alpha\beta)}_{i\cdot}) = [(a-1)/(ab)]\sigma^2_{\alpha\beta}, \text{ and}$$
$$Cov(\overline{(\alpha\beta)}_{i\cdot}, \overline{(\alpha\beta)}_{i'\cdot}) = -\sigma^2_{\alpha\beta}/(ab),\ i \neq i'.$$

(i) Let $\hat{L} = \sum c_i \bar{Y}_{i\cdot\cdot}$. Then it is easy to check that $E(\hat{L}) = L$. Denote $\mu_{i\cdot} = \mu_{\cdot\cdot} + \alpha_i$ and noting that $\sum c_i = 0$, we have

$$\hat{L} = \sum c_i \{\mu_{i\cdot} + \bar{\beta} + \overline{(\alpha\beta)}_{i\cdot} + \bar{\varepsilon}_{i\cdot\cdot} = L + \sum c_i \overline{(\alpha\beta)}_{i\cdot} + \sum c_i \bar{\varepsilon}_{i\cdot\cdot}.$$

Since $\sum c_i \overline{(\alpha\beta)}_{i\cdot} = \sum c_i (\bar{\gamma}_{i\cdot} - \bar{\gamma}_{\cdot\cdot}) = \sum c_i \bar{\gamma}_{i\cdot}$ and that $\bar{\gamma}_{i\cdot}$'s are mutually uncorrelated with mean 0 and common variance $\sigma^2_{\alpha\beta}/b$, we have

$$\begin{aligned}
Var(\hat{L}) &= Var\left(\sum c_i \overline{(\alpha\beta)}_{i\cdot}\right) + Var\left(\sum c_i \bar{\varepsilon}_{i\cdot\cdot}\right) \\
&= Var\left(\sum c_i \bar{\gamma}_{i\cdot}\right) + \sum c_i^2 \{\sigma^2/(nb)\} \\
&= \sum c_i^2 \sigma^2_{\alpha\beta}/b + \sum c_i^2\{\sigma^2/(nb)\} = \frac{n\sigma^2_{\alpha\beta} + \sigma^2}{nb}.
\end{aligned}$$

(ii) Note that

$$\begin{aligned}
\hat{\alpha}_i &= [\alpha_i - \bar{\alpha}] + [\overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{\cdot\cdot}] + [\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot\cdot\cdot}] \\
&= \alpha_i + \overline{(\alpha\beta)}_{i\cdot} + [\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot\cdot\cdot}] \text{ [since } \sum \alpha_i = 0 \text{ and } \overline{(\alpha\beta)}_{\cdot\cdot} = 0] \\
&= \alpha_i + [\bar{\gamma}_{i\cdot} - \bar{\gamma}_{\cdot\cdot}] + [\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot\cdot\cdot}].
\end{aligned}$$

We will only write down the outlines since many of the details are similar to the ANOVA II case. Note that

$$\begin{aligned}
E(SSA) &= nb \sum \hat{\alpha}_i^2 \\
&= nb \sum \alpha_i^2 + nb \sum E(\bar{\gamma}_{i\cdot} - \bar{\gamma}_{\cdot\cdot})^2 + nb \sum E(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot\cdot\cdot})^2 \\
&= nb \sum \alpha_i^2 + nb(a-1)\sigma^2_{\alpha\beta}/b + nb(a-1)\sigma^2/(nb) \\
&= nb \sum \alpha_i^2 + (a-1)[n\sigma^2_{\alpha\beta} + \sigma^2].
\end{aligned}$$

Rest of the details for obtaining $E(MSA)$ is skipped. Similar argument will show the results for $E(MSA)$ and $E(MSAB)$.