Solution: Homework 7

Statistics 207
Winter Quarter, 2016

14.9
(a) $b_0 = -10.3089, b_1 = 0.01892$.
Fitted response function is

$$\begin{aligned}\hat{\pi} &= \exp(-10.3089 + 0.01892X)/[1 + \exp(-10.3089 + 0.01892X)] \\ &= 1/[1 + \exp(10.3089 - 0.01892X)].\end{aligned}$$

(b) I am leaving the plots to you.
(c) $\exp(b_1) = 1.019$. If θ_0 and θ_1 are the odds of performance at X and $X + 1$, then the odds ratio is $\theta_1/\theta_0 = \exp(\beta_1)$ and $\exp(b_1)$ is an estimate of this odds ratio.
(d) Estimated probability at $X = 550$ can be obtained using the fitted response in part (a) and it equals 0.5243.
(e) Need to solve for X when $\hat{\pi} = 0.7$, i.e., solve

$$0.7 = 1/\exp[10.3089 - 0.01892X].$$

The value of X is 589.65.

14.11
(a) Note that

j	1	2	3	4	5	6
p_j	0.144	0.206	0.340	0.592	0.812	0.898

Plot of $\{p_j\}$ against $\{X_j\}$ indicates an S-shape. Thus a logistic linear model may be used.
(b) $b_0 = -2.07656, b_1 = 0.13585$.
The fitted response function is

$$\hat{\pi} = 1/[1 + \exp(2.07656 - 0.13585X)].$$

(c) The following table gives the observed proportions and the fitted proportions

j	1	2	3	4	5	6
p_j	0.144	0.206	0.340	0.592	0.812	0.898
$\hat{\pi}_j$	0.141	0.198	0.328	0.655	0.789	0.881

The fit is reasonably good except perhaps at $X_j = 4$.
(d) $\exp(b_1) = 1.1455$. If θ_0 and θ_1 are the odds of return at X and $X + 1$, then the odds ratio is $\theta_1/\theta_0 = \exp(\beta_1)$ and $\exp(b_1)$ is an estimate of this odds ratio.
(e) Probability at $X = 15$ can be obtained by using the formula in part (b) and this estimate is 0.4903.
(f) A solution of the equation

$$0.75 = \hat{\pi} = 1/[1 + \exp(2.07656 - 0.13585X)]$$

leads to $X = 23.3726$.

14.14.

(a) $b_0 = -1.17717, b_1 = 0.07279, b_2 = -0.09899, b_3 = 0.43397.$.

Fitted response function is

$$\hat{\pi} = 1/[1 + \exp(1.17717 - 0.07279X_1 + 0.09899X_2 - 0.43397X_3)].$$

(b) Here

$$\exp(b_1) = 1.0755, \exp(b_2) = 0.9058, \exp(b_3) = 1.5434.$$

If θ_0 and θ_1 are the odds receiving flue shots at (X_1, X_2, X_3) and $(X_1 + 1, X_2, X_3)$, then the odds ratio $\theta_1/\theta_0 = \exp(\beta_1)$ and $\exp(b_1)$ is an estimate of this odds ratio. Similar explanations hold for $\exp(b_2)$ and $\exp(b_3)$.

(c) When $(X_1, X_2, X_3) = (55, 60, 1)$, the probability of getting a flu shot can obtained from the formula in part (a) and it is 0.0642.

14.20

(a) Bonferroni multiplier is $B = z\,(1 - 0.05/(2)(2)) = z(0.975) = 1.96$.

Simultaneous confidence intervals for β_1 and β_2 are

$$\begin{aligned} \beta_1 &: \quad b_1 \pm 1.96s(b_1), i.e., 0.07270 \pm 0.059506, \\ \beta_2 &: \quad b_2 \pm 1.96s(b_2), i.e., -0.09899 \pm 0.065523. \end{aligned}$$

Thus simultaneous confidence intervals for $\exp(30\beta_1)$ and $\exp(25\beta_2)$ are

$$\begin{aligned} \exp(30\beta_1) &: \quad [\exp(30(b_1 - 1.96s(b_1)), 30(b_1 + 1.96s(b_1))], i.e.[1.49, 52.92], \\ \exp(25\beta_2) &: \quad [\exp(25(b_2 - 1.96s(b_2)), 25(b_2 + 1.96s(b_2))], i.e.[0.016, 0.433]. \end{aligned}$$

(b) $H_0 : \beta_3 = 0$ vs $H_1 : \beta_3 \neq 0$.

$z^* = b_3/s(b_3) = 0.8324$.

Decision rule: reject H_0 if $|z^*| > z(0.975) = 1.96$.

Since $|z^*| < 1.96$, we cannot reject H_0. p-val≈ 0.405.

(c) $H_0 : \beta_3 = 0$ vs $H_1 : \beta_3 \neq 0$.

$G^2 = 0.702$.

Decision rule: if $G^2 > \chi^2(0.95; 1) = 3.8415$.

Since $G^2 < 3.8415$, we cannot reject H_0.

Conclusion: we may drop variable X_3.

p-val$\approx$0.4021

(d) $H_0 : \beta_3 = \beta_4 = \beta_5 = 0$ vs H_1 : not all $\beta_k = 0$, $k = 3, 4, 5$.

$G^2 = 1.534$.

Decision rule: reject H_0 if $G^2 > \chi^2(0.95; 3) = 7.81$.

Since $G^2 < 7.81$, we cannot reject H_0.

Conclusion: the three second order terms should not be included in the model.

p-val$\approx$0.6744.

14.40. Multiply the numerator and denominator of

$$\pi_i = \exp(\beta_0 + \beta_1 X_i)/[1 + \exp(\beta_0 + \beta_1 X_i)]$$

by $\exp(-\beta_0 - \beta_1 X_i)$ to get the result.

14.41. Formula (14.26) holds for given observations $Y_1, \ldots, Y_n$. Assembling all terms with a given X value, X_j, we obtain

$$y_{\cdot j}(\beta_0 + \beta_1 X_j) - n_j \ln[1 + \exp(\beta_0 + \beta_1 X_j)]$$

since there are n_j cases with X value X_j, of which $y_{\cdot j}$ have value $Y_i = 1$. There are $\binom{n_j}{y_{\cdot j}}$ ways of choosing these $y_{\cdot j}$ 1's out of n_j, all of which are equally likely. Hence is the likelihood function of the $y_{\cdot j}$, we must add $\ln \binom{n_j}{y_{\cdot j}}$ to the above term for given X_j:

$$\ln \binom{n_j}{y_{\cdot j}} + y_{\cdot j}(\beta_0 + \beta_1 X_j) - n_j \ln[1 + \exp(\beta_0 + \beta_1 X_j)].$$

Assembling the terms for all X_j, we obtain (14.34).

Problem 9.
(a) The conditional pdf of Y given β is $N(X\beta, \sigma^2 I)$. Thus the joint pdf of (Y, β) is

$$\begin{aligned}
& f_{Y|\beta}(y|\beta) f_\beta(\beta) \\
= \; & \frac{1}{(2\pi)^{n/2} |\Sigma_{Y|\beta}|^{1/2}} \exp\left\{-(1/2)(y - X\beta)^T \Sigma_{Y|\beta}^{-1} (y - X\beta)\right\} \times \\
& \frac{1}{(2\pi)^{p/2} |\Sigma_\beta|^{1/2}} \exp\left\{-(1/2)\beta^T \Sigma_\beta \beta\right\} \\
= \; & (2\pi)^{-(n+p)/2} \sigma^{-n} \tau^{-p} \times \exp\left\{-||y - X\beta||^2/(2\sigma^2)\right\} \exp\left\{-||\beta||^2/(2\tau^2)\right\}.
\end{aligned}$$

We have used the facts that $\Sigma_{Y|\beta} = \sigma^2 I_{n \times n}$ and $\Sigma_\beta = \tau^2 I_{p \times p}$.
(b) Note that

$$l(\beta) = ||y - X\beta||^2/\sigma^2 + ||\beta||^2/\tau^2 + c,$$

where c does not depend on β. Re-expressing $\tau^2 = \sigma^2/k$, we have

$$l(\beta) = \sigma^{-2}\left[||y - X\beta||^2 + k||\beta||^2\right] + c.$$

Note that

$$\frac{\partial l(\beta)}{\partial \beta} = 0 \Longrightarrow (X^T X + kI)\beta = X^T Y.$$

Since

$$\frac{\partial^2 l(\beta)}{\partial\beta\partial\beta} = 2\sigma^{-2}\left[X^T X + kI\right]$$

is positive definite, the minimum of $l(\beta)$ is attained at $\hat{\beta} = (X^T X + kI)^{-1} X^T Y$.

Problem 10.

You need to use the following formulas given in the handout on Ridge Regression.

If $\beta(k) = E[\hat{\beta}(k)]$, then

$$\begin{aligned} D(k) &= E\left[||\hat{\beta}(k) - \beta||^2\right] = E[||\hat{\beta}(k) - \beta(k)||^2] + ||\beta(k) - \beta||^2 \\ &= \sigma^2 \sum \frac{\lambda_j}{(\lambda_j + k)^2} + k^2 \sum \frac{1}{(\lambda_j + k)^2} (e_j^T \beta)^2. \end{aligned}$$

Also

$$\begin{aligned} L(k) &= E\left[||X\hat{\beta}(k) - X\beta||^2\right] = E[||X\hat{\beta}(k) - X\beta(k)||^2] + ||X\beta(k) - X\beta||^2 \\ &= \sigma^2 \sum \frac{\lambda_j^2}{(\lambda_j + k)^2} + k^2 \sum \frac{\lambda_j}{(\lambda_j + k)^2} (e_j^T \beta)^2. \end{aligned}$$