Homework #3*

4.1 (a) We are given $p = 2$, $\mu = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & -.8 \times \sqrt{2} \\ -.8 \times \sqrt{2} & 1 \end{bmatrix}$ so $|\Sigma| = .72$ and

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{.72} & \frac{\sqrt{2}}{.9} \\ \frac{\sqrt{2}}{.9} & \frac{2}{.72} \end{bmatrix}$$

$$f(x) = \frac{1}{(2\pi)\sqrt{.72}} \exp\left(-\frac{1}{2}\left[\frac{1}{.72}(x_1-1)^2 + \frac{2\sqrt{2}}{.9}(x_1-1)(x_2-3) + \frac{2}{.72}(x_2-3)^2\right]\right)$$

(b)

$$\frac{1}{.72}(x_1-1)^2 + \frac{2\sqrt{2}}{.9}(x_1-1)(x_2-3) + \frac{2}{.72}(x_2-3)^2$$

4.3 We apply Result 4.5 that relates zero covariance to statistical independence

a) No, $\sigma_{12} \neq 0$

b) Yes, $\sigma_{23} = 0$

c) Yes, $\sigma_{13} = \sigma_{23} = 0$

d) Yes, by Result 4.3, $(X_1+X_2)/2$ and X_3 are jointly normal and their covariance is $\frac{1}{2}\sigma_{13} + \frac{1}{2}\sigma_{23} = 0$.

e) No, by Result 4.3 with $A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$, form $A \Sigma A'$ to see that the covariance is 10 and not 0.

4.4 a) $3X_1 - 2X_2 + X_3$ is $N(13,9)$

b) Require $\text{Cov}(X_2, X_2 - a_1X_1 - a_3X_3) = 3 - a_1 - 2a_3 = 0$. Thus any $\underset{\sim}{a}' = [a_1, a_3]$ of the form $\underset{\sim}{a}' = [3-2a_3, a_3]$ will meet the requirement. As an example, $\underset{\sim}{a}' = [1,1]$.

4.5 a) $X_1 | x_2$ is $N(\frac{1}{\sqrt{2}}(x_2-2), \frac{3}{2})$

b) $X_2 | x_1, x_3$ is $N(-2x_1-5, 1)$

c) $X_3 | x_1, x_2$ is $N(\frac{1}{2}(x_1+x_2+3), \frac{1}{2})$

4.6 (a) X_1 and X_2 are independent since they have a bivariate normal distribution with covariance $\sigma_{12} = 0$.

(b) X_1 and X_3 are dependent since they have nonzero covariance $\sigma_{13} = -1$.

(c) X_2 and X_3 are independent since they have a bivariate normal distribution with covariance $\sigma_{23} = 0$.

(d) X_1, X_3 and X_2 are independent since they have a trivariate normal distribution where $\sigma_{12} = 0$ and $\sigma_{32} = 0$.

(e) X_1 and $X_1 + 2X_2 - 3X_3$ are dependent since they have nonzero covariance

$$\sigma_{11} + 2\sigma_{12} - 3\sigma_{13} = 4 + 2(0) - 3(-1) = 7$$

4.7 (a) $X_1|x_3$ is $N(1 + .5(x_3 - 2)\ , 3.5)$

(b) $X_1|x_2, x_3$ is $N(1 + .5(x_3 - 2)\ , 3.5)$. Since X_2 is independent of X_1, conditioning further on x_2 does not change the answer from Part a).

4.15 First,

$$\sum_{j=1}^{n} (\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})' = (\bar{\mathbf{x}}-\boldsymbol{\mu})[\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})']$$
$$= (\bar{\mathbf{x}}-\boldsymbol{\mu})(\sum_{j=1}^{n}\mathbf{x}_j-n\bar{\mathbf{x}})'$$
$$= (\bar{\mathbf{x}}-\boldsymbol{\mu})(n\bar{\mathbf{x}}-n\bar{\mathbf{x}})' = \mathbf{0}$$

Also,

$$\sum_{j=1}^{n} (\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})' = [\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})']' = \mathbf{0}' = \mathbf{0}.$$

4.16 (a) By Result 4.8, with $c_1 = c_3 = 1/4$, $c_2 = c_4 = -1/4$ and $\boldsymbol{\mu}_j = \boldsymbol{\mu}$ for $j = 1, ..., 4$ we have $\sum_{j=1}^{4} c_j\boldsymbol{\mu}_j = \mathbf{0}$ and $(\ \sum_{j=1}^{4} c_j^2\ )\ \Sigma = \frac{1}{4}\Sigma$. Consequently, V_1 is $N(0,\ \frac{1}{4}\Sigma\ )$. Similarly, setting $b_1 = b_2 = 1/4$ and $b_3 = b_4 = -1/4$, we find that V_2 is $N(0,\ \frac{1}{4}\Sigma\ )$.

(b) Again by Result 4.8, we know that V_1 and V_2 are jointly multivariate normal with covariance

$$(\ \sum_{j=1}^{4} b_jc_j\ )\ \Sigma = \left(\ \frac{1}{4}(\ \frac{1}{4}\ ) + \frac{-1}{4}(\ \frac{1}{4}\ ) + \frac{1}{4}(\ \frac{-1}{4}\ ) + \frac{-1}{4}(\ \frac{-1}{4}\ )\ \right)\ \Sigma = \mathbf{0}$$

That is,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \text{is distributed } N_{2p}\left(\mathbf{0}, \begin{bmatrix} \frac{1}{4}\Sigma & \mathbf{0} \\ \mathbf{0} & \frac{1}{4}\Sigma \end{bmatrix}\right)$$

so the joint density of the $2p$ variables is

$$f(\mathbf{v}_1,\ \mathbf{v}_2) = \frac{1}{(2\pi)^p|\ \frac{1}{4}\Sigma\ |}\exp\left(\ -\frac{1}{2}[\ \mathbf{v}_1',\ \mathbf{v}_2'\ ]\begin{bmatrix} \frac{1}{4}\Sigma & \mathbf{0} \\ \mathbf{0} & \frac{1}{4}\Sigma \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_1 \end{bmatrix}\ \right)$$

$$= \frac{1}{(2\pi)^p|\ \frac{1}{4}\Sigma\ |}\exp\left(\ -\frac{1}{8}(\ \mathbf{v}_1'\ \Sigma^{-1}\mathbf{v}_1\ +\ \mathbf{v}_2'\ \Sigma^{-1}\mathbf{v}_2\ )\right)$$