ECS122A Homework Assignment #2 **Due: 4:00pm, January 22, 2016**

1. (a) Prove that $(n+3)^3 = \Theta(n^3)$

   (b) Prove that for any real constants a and b, where $b > 0$,
   $$(n+a)^b = \Theta(n^b)$$
   Note: to establish the relationship $f(n) = \Theta(g(n))$, we need to find the proper constants c_1, c_2 and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ whenever $n \ge n_0$.

2. (a) Is $2^{n+1} = O(2^n)$? why? (b) Is $2^{2n} = O(2^n)$? why?

3. Order the following functions into a list such that if $f(n)$ comes before $g(n)$ in the list then $f(n) = O(g(n))$. If any two (or more) of the same asymptotic order, indicate which.

   (a) Start with these basic functions
   $$n,\ 2^n,\ n\lg n,\ n^3,\ \lg n,\ n - n^3 + 7n^5,\ n^2 + \lg n$$
   (b) Combine the following functions into your answer for part (a). Assume that $0 < \epsilon < 1$.
   $$e^n,\ \sqrt{n},\ 2^{n-1},\ \lg\lg n,\ (\sqrt{2})^{\lg n},\ \ln n,\ (\lg n)^2,\ n!,\ n^{1+\epsilon},\ 1$$

4. A method to solve the recurrence relations is to expand out the recurrence a few times, until a pattern emerges. For instance, let us start with the recurrence
   $$T(n) = 2T(n/2) + O(n).$$
   Think of $O(n)$ as being $\le cn$ for some constant c so $T(n) \le 2T(n/2) + cn$. By repeatedly applying this rule, we can bound $T(n)$ in terms of $T(n/2)$, then $T(n/2^2)$, then $T(n/2^3)$, and so on, at each step getting closer to the basis value of $T(\cdot)$ we do know, namely $T(1) = O(1)$:
   $$\begin{aligned} T(n) &\le 2T(n/2) + cn \le 2[2T(n/2^2) + cn/2] + cn = 2^2 T(n/2^2) + 2cn \\ &\le 2^2[2T(n/2^3) + cn/2^2] + 2cn = 2^3 T(n/2^3) + 3cn \le \ldots \end{aligned}$$
   A pattern is emerging. The general term is $T(n) \le 2^k T(n/2^k) + kcn$. Plugging in $k = \lg n$, we get $T(n) \le nT(1) + cn\lg n = O(n\lg n)$.

   Do the same thing for the recurrence
   $$T(n) = 3T(n/2) + O(n).$$
   (a) What is the general kth term in this case?

   (b) What value of k should be plugged in to get the answer?

5. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for sufficient small n, and c is a constant. Make your bounds as tight as possible, and justify your answers.

   (a) $T(n) = T(n-1) + 1/n$
   (b) $T(n) = T(n-1) + c^n$, where $c > 1$ is some constant
   (c) $T(n) = 2T(n-1) + 1$
   (d) $T(n) = 2 \cdot T(\frac{n}{2}) + \sqrt{n}$
   (e) $T(n) = 2T(n/4) + 1$
   (f) $T(n) = 2T(n/4) + n$
   (g) $T(n) = 3 \cdot T(\frac{n}{2}) + cn$
   (h) $T(n) = 27 \cdot T(\frac{n}{3}) + cn^3$
   (i) $T(n) = 5 \cdot T(\frac{n}{4}) + cn^2$