Homework #7

6.6 a) Treatment 2: Sample mean vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$; sample covariance matrix $\begin{bmatrix} 1 & -3/2 \\ -3/2 & 3 \end{bmatrix}$

Treatment 3: Sample mean vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$; sample covariance matrix $\begin{bmatrix} 2 & -4/3 \\ -4/3 & 4/3 \end{bmatrix}$

$$S_{pooled} = \begin{bmatrix} 1.6 & \\ -1.4 & \end{bmatrix}$$

b) $T^2 = [2-3,\ 4-2] \left[\left(\frac{1}{3} + \frac{1}{4}\right) \begin{bmatrix} 1.6 & -1.4 \\ -1.4 & 2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 2-3 \\ 4-2 \end{bmatrix} = 3.88$

$$\frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,n_1+n_2-p-1}(.01) = \frac{(5)2}{4}(18) = 45$$

Since $T^2 = 3.88 < 45$ do not reject $H_0: \underline{\mu}_2 - \underline{\mu}_3 = \underline{0}$ at the $\alpha = .01$ level.

c). 99% simultaneous confidence intervals:

$$\mu_{21} - \mu_{31}: (2-3) \pm \sqrt{45} \sqrt{\left(\frac{1}{3}+\frac{1}{4}\right)1.6} = -1 \pm 6.5$$

$$\mu_{22} - \mu_{32}: 2 \pm 7.2$$

6.7 $T^2 = [74.4 \quad 201.6] \left[\left(\frac{1}{45} + \frac{1}{55}\right) \begin{bmatrix} 10963.7 & 21505.5 \\ 21505.5 & 63661.3 \end{bmatrix} \right]^{-1} \begin{bmatrix} 74.4 \\ 201.6 \end{bmatrix} = 16.1$

$$\frac{(n_1+n_2-2)p}{n_1+n_2-p-1} F_{p,n_1+n_2-p-1}(.05) = 6.26$$

Since $T^2 = 16.1 > 6.26$ reject $H_0: \underline{\mu}_1 - \underline{\mu}_2 = \underline{0}$ at the $\alpha = .05$ level.

$$\hat{\mathbf{a}} = S_{pooled}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = \begin{bmatrix} .0017 \\ .0026 \end{bmatrix}$$

6.8 a) For first variable:

observation = mean + treatment effect + residual

$$\begin{bmatrix} 6 & 5 & 8 & 4 & 7 \\ 3 & 1 & 2 & & \\ 2 & 5 & 3 & 2 & \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & & \\ 4 & 4 & 4 & 4 & \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ -2 & -2 & -2 & & \\ -1 & -1 & -1 & -1 & \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 & -2 & 1 \\ 1 & -1 & 0 & & \\ -1 & 2 & 0 & -1 & \end{bmatrix}$$

$SS_{obs} = 246$ $\quad SS_{mean} = 192$ $\quad SS_{tr} = 36$ $\quad SS_{res} = 18$

For second variable:

$$\begin{bmatrix} 7 & 9 & 6 & 9 & 9 \\ 3 & 6 & 3 & & \\ 3 & 1 & 1 & 3 & \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & & \\ 5 & 5 & 5 & 5 & \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & & \\ -3 & -3 & -3 & -3 & \end{bmatrix} + \begin{bmatrix} -1 & 1 & -2 & 1 & 1 \\ -1 & 2 & -1 & & \\ 1 & -1 & -1 & 1 & \end{bmatrix}$$

$SS_{obs} = 402$ $\quad SS_{mean} = 300$ $\quad SS_{tr} = 84$ $\quad SS_{res} = 18$

Cross product contributions:

275 $\quad$ 240 $\quad$ 48 $\quad$ -13

b) MANOVA table:

Source of Variation	SSP	d.f.
Treatment	$B = \begin{bmatrix} 36 & 48 \\ 48 & 84 \end{bmatrix}$	$3 - 1 = 2$
Residual	$W = \begin{bmatrix} 18 & -13 \\ -13 & 18 \end{bmatrix}$	$5 + 3 + 4 - 3 = 9$
Total (corrected)	$\begin{bmatrix} 54 & 35 \\ 35 & 102 \end{bmatrix}$	11

c) $\Lambda^* = \frac{|W|}{|B+W|} = \frac{155}{4283} = .0362$

Using Table 6.3 with $p = 2$ and $g = 3$

$$\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right)\left(\frac{\Sigma n_\ell - g - 1}{g - 1}\right) = 17.02\,.$$

Since $F_{4,16}(.01) = 4.77$ we conclude that treatment differences exist at $\alpha = .01$ level.

Alternatively, using Bartlett's procedure,

$$-\left(n-1-\frac{(p+g)}{2}\right)\ln \Lambda^* = -(12-1-\tfrac{5}{2})\ln(.0362) = 28.209$$

Since $\chi^2_4(.01) = 13.28$ we again conclude treatment differences exist at $\alpha = .01$ level.

6.9 For <u>any</u> matrix C

$$\underset{\sim}{\bar{d}} = \frac{1}{n}\Sigma\, \underset{\sim}{d}_j = C(\frac{1}{n}\Sigma\, \underset{\sim}{x}_j) = C\underset{\sim}{\bar{x}}$$

and $\quad \underset{\sim}{d}_j - \underset{\sim}{\bar{d}} = C(\underset{\sim}{x}_j - \underset{\sim}{\bar{x}})$

so $\quad S_d = \frac{1}{n-1}\Sigma(\underset{\sim}{d}_j-\underset{\sim}{\bar{d}})(\underset{\sim}{d}_j-\underset{\sim}{\bar{d}})' = C(\frac{1}{n-1}\Sigma(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}})(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}})')C' = CSC'$

6.10 $(\bar{x}\,\underset{\sim}{1})'[(\bar{x}_1-\bar{x})\underset{\sim}{u}_1 + \ldots + (\bar{x}_g-\bar{x})\underset{\sim}{u}_g]$

$$= \bar{x}[(\bar{x}_1-\bar{x})n_1 + \ldots + (\bar{x}_g-\bar{x})n_g]$$

$$= \bar{x}[n_1\bar{x}_1 + \ldots + n_g\bar{x}_g - \bar{x}(n_1 + \ldots + n_g)]$$

$$= \bar{x}[(n_1 + \ldots + n_g)\bar{x} - \bar{x}(n_1 + \ldots + n_g)] = 0$$

6.19

a) $\bar{\mathbf{x}}_1 = \begin{bmatrix} 12.219 \\ 8.113 \\ 9.590 \end{bmatrix}; \quad \bar{\mathbf{x}}_2 = \begin{bmatrix} 10.106 \\ 10.762 \\ 18.168 \end{bmatrix};$

$$S_1 = \begin{bmatrix} 223.0134 & 12.3664 & 2.9066 \\ & 17.5441 & 4.7731 \\ & & 13.9633 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 4.3623 & .7599 & 2.3621 \\ & 25.8512 & 7.6857 \\ & & 46.6543 \end{bmatrix};$$

$$S_{pooled} = \begin{bmatrix} 15.8112 & 7.8550 & 2.6959 \\ & 20.7458 & 5.8960 \\ & & 26.5750 \end{bmatrix}$$

$$[(\frac{1}{n_1} + \frac{1}{n_2})S_{pooled}]^{-1} = \begin{bmatrix} 1.0939 & -.4084 & -.0203 \\ & .8745 & -.1525 \\ & & .5640 \end{bmatrix}$$

H_0: $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \mathbf{0}$

Since $T^2 = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'[(\frac{1}{n_1} + \frac{1}{n_2})S_{pooled}]^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = 50.92$

$$> \frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,n_1+n_2-p-1}(.01) = \frac{(57)(3)}{55} F_{3,55}(.01) = 13,$$

we reject H_0 at the $\alpha = .01$ level. There is a difference in the (mean) cost vectors between gasoline trucks and diesel trucks.

b) $$\hat{\mathbf{a}} = S_{pooled}^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = \begin{bmatrix} 3.58 \\ -1.88 \\ -4.48 \end{bmatrix}$$

c) 99% simultaneous confidence intervals are:

$$\mu_{11} - \mu_{21}: \ 2.113 \pm 3.790$$

$$\mu_{12} - \mu_{22}: \ -2.650 \pm 4.341$$

$$\mu_{13} - \mu_{23}: \ -8.578 \pm 4.913$$

d) Assumption $\Sigma_1 = \Sigma_2$.

Since S_1 and S_2 are quite different, it may not be reasonable to pool. However, using "large sample" theory ($n_1 = 36$, $n_2 = 23$) we have, by Result 6.4,

$$(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2))'[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2]^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)) \sim \chi_p^2$$

Since

$$(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2]^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = 43.15 > \chi_3^2(.01) = 11.34$$

we reject $H_0: \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \mathbf{0}$ at the $\alpha = .01$ level. This is consistent with the result in part (a).