ECS122A Homework Assignment #2 **Solution**

1. (a) We need to show that there are constants $c_1, c_2, n_0 > 0$ such that

$$0 \leq c_1 n^3 \leq (n+3)^3 \leq c_2 n^3 \quad \text{for all } n \geq n_0.$$

Note that

$$\frac{1}{2}n \leq n+3 \leq 2n \quad \text{when } n \geq 3$$

When all parts are raised to the power 3.

$$\frac{1}{2^3}n^3 \leq (n+3)^3 \leq 2^3 n^3 \quad \text{when } n \geq 3$$

Thus $c_1 = \frac{1}{2^3}$, $c_2 = 2^3$ and $n_0 = 3$.

(b) We still need to show that there are constants $c_1, c_2, n_0 > 0$, which depend on a and b, such that

$$0 \leq c_1 n^b \leq (n+a)^b \leq c_2 n^b \quad \text{for all } n \geq n_0.$$

Note that

$$n + a \leq n + |a| \leq 2n \quad \text{when } \; n \geq |a|.$$

and

$$n + a \geq n - |a| \geq \frac{1}{2}n \quad \text{when } \; n \geq 2|a|.$$

Therfore, we have

$$\frac{1}{2}n \leq n + a \leq 2n \quad \text{when } n \geq 2|a|$$

Since $b > 0$, the inequality still holds when all parts are raised to the power b:

$$\frac{1}{2^b}n^b \leq (n+a)^b \leq 2^b n^b.$$

Thus, $c_1 = 1/2^b$, $c_2 = 2^b$, and $n_0 = \lceil 2|a| \rceil$.

2. (a) This is true. Since $2^{n+1} = 2 \cdot 2^n$ for all n, the definition of the Big-O notation is satisfied with $c = 2$ and $n_0 = 1$.

(b) This is false. Assume that there are constatns c, n_0 such that

$$2^{2n} \leq c \cdot 2^n \quad \text{for all } n \geq n_0$$

Then

$$2^{2n} = 2^n \cdot 2^n \leq c \cdot 2^n \quad \text{for all } n \geq n_0$$

This implies that

$$2^n \leq c \quad \text{for all } n \geq n_0$$

But there is no constant c is greater than 2^n for all $n \geq n_0$. So the assumption leads to a contradiction.

3. (a) There is the ordering, where functions on the same line are the same order

$$\begin{aligned}
&\lg n,\\
&n,\\
&n\lg n,\\
&n^2+\lg n,\\
&n^3,\\
&n-n^3+7n^5,\\
&2^n
\end{aligned}$$

(b) Here is the ordering

$$\begin{aligned}
&1\\
&\lg\lg n,\\
&\lg n, \quad \ln n,\\
&(\lg n)^2,\\
&\sqrt{n}=(\sqrt{2})^{\lg n},\\
&n,\\
&n\lg n,\\
&n^{1+\epsilon},\\
&n^2+\lg n,\\
&n^3,\\
&n-n^3+7n^5,\\
&2^n, \quad 2^{n-1},\\
&e^n,\\
&n!
\end{aligned}$$

Note that the problem does not ask you to justify these relations (otherwise, it will be too long!). But here is an example. Why $n\lg n = O(n^{1+\epsilon})$? We can compute $\lim_{n\to\infty}[n\lg n/n^{1+\epsilon}] = 0$ (using L'Hopital's rule).

4. For the recurrence $T(n) = 3T(n/2) + O(n)$, we have

$$\begin{aligned}
T(n) &\le 3T(n/2)+cn\\
&\le 3[3T(n/2^2)+cn/2]+cn = 3^2T(n/2^2)+(3/2)cn+cn\\
&\le 3^2[3T(n/2^3)+cn/2^2]+(3/2)cn+cn = 3^3T(n/2^3)+(3/2)^2cn+(3/2)cn+cn\\
&\dots
\end{aligned}$$

A pattern is emerging. The general term is

$$T(n) \le 3^kT(n/2^k) + \left(\sum_{i=0}^{k-1}(3/2)^i\right)cn = 3^kT(n/2^k)+2(3^k/2^k-1)cn$$

Plugging in $k=\lg n$, we get

$$T(n) \le 3^{\lg n}T(1)+2(3^{\lg n}/n-1)cn = n^{\lg 3}T(1)+2(n^{\lg 3}/n-1)cn = \Theta(n^{\lg 3}).$$

5. (a) $T(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + O(1) = \Theta(\lg n)$ (the nth Harmonic number)
Note: the master method does not apply here.

(b) $T(n) = T(n-1) + c^n = T(n-2) + c^{n-1} + c^n = \cdots = T(0) + c^0 + \cdots + c^{n-1} + c^n$
$= T(0) + \frac{c^{n+1}-1}{c-1} = \Theta(c^n)$
Note: the master method does not apply here.

(c) $T(n) = 2T(n-1)+1 = 2(2T(n-2)+1)+1 = \cdots = 2^n T(0) + 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = \Theta(2^n)$
Note: the master method does not apply here.

(d) We have $a = 2$, $b = 2$ and $f(n) = \sqrt{n}$, then

$$n^{\log_b a} = n^{\log_2 2} = n$$

The quotient

$$\frac{f(n)}{n^{\log_b a}} = \frac{\sqrt{n}}{n} = n^{-1/2}.$$

Case 1 of the master theorem applies, and $T(n) = \Theta(n^{\log_b a}) = \Theta(n)$.

(e) We have $a = 2$, $b = 4$ and $f(n) = 1$, then

$$n^{\log_b a} = n^{\log_4 2} = n^{1/2}$$

The quotient

$$\frac{f(n)}{n^{\log_b a}} = \frac{1}{n^{1/2}} = n^{-1/2}.$$

Case 1 of the master theorem applies, and $T(n) = \Theta(n^{\log_b a}) = \Theta(\sqrt{n})$.

(f) We have $a = 2$, $b = 4$ and $f(n) = n$, then

$$n^{\log_b a} = n^{\log_4 2} = n^{1/2}$$

The quotient

$$\frac{f(n)}{n^{\log_b a}} = \frac{n}{\sqrt{n}} = n^{1/2}.$$

Case 3 of the master theorem applies, and $T(n) = \Theta(f(n)) = \Theta(n)$.
Note: strictly speaking, in order to apply case 3 of the master theorem we have to also prove the regularity of $f(n)$. In fact, since $af(n/b) = 2(n/4) = n/2$, the ineqaulity the regularity condition $af(n/b) = n/2 \leq cn$ holds for all any c such that $\frac{1}{2} \leq c < 1$.

(g) We have $a = 3$, $b = 2$ and $f(n) = cn$, then

$$n^{\log_b a} = n^{\log_2 3}$$

The quotient

$$\frac{f(n)}{n^{\log_b a}} = \frac{cn}{n^{\lg 3}} = cn^{-0.585}.$$

Case 1 of the master theorem applies, and $T(n) = \Theta(n^{\lg 3})$.

(h) We have $a = 27$, $b = 3$ and $f(n) = cn^3$, then

$$n^{\log_b a} = n^{\log_3 27} = n^3$$

The quotient

$$\frac{f(n)}{n^{\log_b a}} = \frac{cn^3}{n^3} = c.$$

Case 2 of the master theorem applies, and $T(n) = \Theta(n^3 \lg n)$.

(i) We have $a = 5$, $b = 4$ and $f(n) = cn^2$, then

$$n^{\log_b a} = n^{\log_4 5}$$

The quotient

$$\frac{f(n)}{n^{\log_b a}} = \frac{cn^2}{n^{\log_4 5}} = cn^{2-1.161} = cn^{0.839}$$

Case 3 of the master theorem applies, and $T(n) = \Theta(n^2)$.

Note: in order to apply case 3 of the master theorem we have to also prove the regularity of $f(n) = cn^2$. In fact, since $af(n/b) = 5c\frac{n^2}{16}$, the regularity condition $af(n/b) = 5c\frac{n^2}{16} \leq \hat{c}f(n)$. holds for any $\hat{c}$ such that $\frac{5}{16} \leq \hat{c} < 1$.