University of California, Davis
Department of Statistics

Name ....Solution.........
P R I N T P L E A S E

STA 135

Sample
Midterm I

Instructions: 1. **WORK ALL PROBLEMS.** Please, give details and explanations and **SHOW ALL YOUR WORK** so that partial credits can be given.
2. You may use **two** sheets of **notes** and a **calculator** but **no** other reference materials.

Points

1. Let A be an nxn matrix and Q an nxn orthogonal matrix.

(a) Show that tr (Q' A Q) = tr (A).
(b) Find the determinant of Q.
(c) Show that Q' A Q and A have the same eigenvalues.

(25)

(a) $tr(Q'AQ) = tr(AQQ')$

$= tr(AI)$

$= tr(A)$

(b) Since $Q'Q = I$

$|Q'Q| = |Q'||Q| = |Q|^2 = 1$

Then $|Q| = \pm 1$

(c) Let λ be an eigenvalue of A.

$|A - \lambda I| = 0$

By part (b), $|Q'||Q| = 1$. Then

$|Q'||A - \lambda I||Q| = 0$

$|Q'(A - \lambda I)Q| = 0$

$|Q'AQ - \lambda Q'Q| = 0$

$|Q'AQ - \lambda I| = 0$

and λ is an eigenvalue of A.

2. A is a 2x2 symmetric matrix with $a_{11} = 1$, $a_{12} = 2$ and $a_{22} = -2$.

(a) Find A^{-1}.
(b) Compute eigenvalues and normalized eigenvectors of A^{-1}.
(c) Write down the spectral decomposition of A^{-1}.

(25)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) $$A^{-1} = \frac{1}{(1)(-2)-(2)(2)} \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/6 \end{bmatrix}$$

(b) $|A^{-1} - \lambda I| = 0$

$$\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6}-\lambda \end{vmatrix} = 0$$

$\lambda^2 - \frac{1}{6}\lambda - \frac{1}{6} = 0 \Rightarrow \lambda_1 = \frac{1}{2}, \ \lambda_2 = -\frac{1}{3}$

Let $\underline{e}_k$ be the eigenvector corresponding to λ_k.

$A^{-1}\underline{e}_1 = \lambda_1 \underline{e}_1$

$$\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/6 \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix} \Rightarrow \begin{matrix} \frac{1}{3}e_{11} + \frac{1}{3}e_{21} = \frac{1}{2}e_{11} \\ \frac{1}{3}e_{11} - \frac{1}{6}e_{21} = \frac{1}{2}e_{21} \end{matrix} \Rightarrow$$

Select one of the equations (first one) and let $e_{11} = 1$

$\frac{1}{3}e_{21} = \frac{1}{6} \Rightarrow e_{21} = \frac{1}{2}$

The normalized first eigenvector is $\underline{e}_1 = \frac{1}{\sqrt{1^2 + (\frac{1}{2})^2}}\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

Similarly, the second normalized eigenvector is $\underline{e}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$.

(c) $$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2/\sqrt{5} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 2/\sqrt{5}, & 1/\sqrt{5} \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1/\sqrt{5}, & -2/\sqrt{5} \end{bmatrix}$$

3. The following data matrix is observed for a three-dimensional random vector $\underline{X}$.

$$\mathbf{X} = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

(a) Compute the sample mean vector and sample covariance matrix of $\underline{X}$.

(b) Let $\underline{b}'$ = [1 2 -3] and compute sample mean vector and sample covariance matrix of $\underline{b}'\,\underline{X}$.

(20)

$n=3 \quad p=3$

(a) $\bar{x}_1 = \frac{1+6+8}{3} = 5$, $\bar{x}_2 = \frac{4+2+3}{3} = 3$, $\bar{x}_3 = \frac{3+6+3}{3} = 4 \Rightarrow \bar{x} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

$\underline{d}_1 = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix} \quad \underline{d}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \underline{d}_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

$S_{11} = \frac{1}{n-1} \underline{d}_1' \underline{d}_1 = \frac{1}{2}(26) = 13$

$S_{22} = \frac{1}{n-1} \underline{d}_2' \underline{d}_2 = \frac{1}{2}(2) = 1$

$S_{33} = \frac{1}{n-1} \underline{d}_3' \underline{d}_3 = \frac{1}{2}(6) = 3$

$S_{12} = \frac{1}{n-1} \underline{d}_1' \underline{d}_2 = \frac{1}{2}(-5) = -2.5$

$S_{13} = \frac{1}{n-1} \underline{d}'_1 \underline{d}_3 = \frac{1}{2}(3) = -1.5$

$d_{23} = \frac{1}{n-1} \underline{d}'_2 d_3 = \frac{1}{2}(-3) = -1.5$

Then

$$S = \begin{bmatrix} 13 & -2.5 & -1.5 \\ -2.5 & 1 & -1.5 \\ -1.5 & -1.5 & 3 \end{bmatrix}$$

(b)

Sample Mean $\underline{b}'\underline{x} = \underline{b}'\underline{\bar{x}} = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} = -1$

Sample Variance $\underline{b}'x = \underline{b}' S \underline{b} = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 13 & -2.5 & -1.5 \\ -2.5 & 1 & -1.5 \\ -1.5 & -1.5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 12.5 & 4 & -13.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 61$

4. The three-dimensional random vector $\underline{X}$ has a multivariate normal distribution. The means of X_1, and X_2 are 1 , and the mean of X_3 is 0. Variance of X_1 is 6, variance of X_2 is 4 and variance of X_3 is 2. X_2 and X_3 are uncorrelated, but the correlation between X_1 and X_2 is $1/\sqrt{24}$, and the correlation between X_1 and X_3 is $-1/\sqrt{12}$.

(a) Find the distribution of X_2 given X_3.
(b) Find the distribution of $X_1 + 2X_2 + 3X_3$.
(c) Partition $\underline{X}$ into $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$, where $\underline{X}^{(2)}$ is a 2x1 vector. Find the distribution of $\underline{X}^{(2)}$.
(d) Find the conditional distribution of $\underline{X}^{(2)}$ given $X_1 = 2$.

(30)

$$\underline{\mu} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \sigma_{11} = 6, \ \sigma_{22} = 4, \ \sigma_{33} = 2, \ \sigma_{12} = \sqrt{6}\sqrt{4}\,\frac{1}{\sqrt{24}} = 1$$

$$\sigma_{13} = \sqrt{6}\sqrt{2}\left(-\frac{1}{\sqrt{12}}\right) = -1, \quad \sigma_{23} = 0$$

$$\Sigma = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(a) $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\right)$, X_2 and X_3 are independent

$X_2 \mid X_3 \sim N(1, 4)$

(b) Let $Y = X_1 + 2X_2 + 3X_3 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

$$E(Y) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 3$$

$$Cov(Y) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix} = 38$$

$Y \sim N(3, 38)$

(c) $X^{(2)} = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\right)$

(d) $X^{(2)} \mid X_1 \sim N\left(\mu^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(x^{(1)} - \mu^{(1)}),\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$

$$\text{Mean Vector} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{6}(2-1) = \begin{bmatrix} 7/6 \\ -1/6 \end{bmatrix}$$

$$\text{Cov. matrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\frac{1}{6}\right) \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 23 & 1 \\ 1 & 11 \end{bmatrix}$$