Handout 2

Let $\{Z_n\}$ be a sequence of random variables.

Definition: Modes of Convergence

1. $Z_n \longrightarrow Z$ in probability $\Longleftrightarrow Z_n - Z \longrightarrow 0$ in probability.

   $\Longleftrightarrow$ For any $\epsilon > 0, P(|Z_n - Z| > \epsilon) \longrightarrow 0$, as $n \longrightarrow \infty$.

2. $Z_n \longrightarrow Z$ in distribution $\Longleftrightarrow Z_n - Z \longrightarrow 0$ in distribution.

   $\Longleftrightarrow F_{Z_n}(t) \longrightarrow F_Z(t), \forall t$ that is a point of continuity of $F_Z(\cdot)$.

3. $Z_n \longrightarrow Z$ in quadratic mean $\Longleftrightarrow E\{(Z_n - Z)^2\} \longrightarrow 0$.

Properties:

1. $Z_n \longrightarrow Z$ in quadratic mean $\Longrightarrow Z_n \longrightarrow Z$ in probability.

2. $Z_n \longrightarrow Z$ in probability $\Longrightarrow Z_n \longrightarrow Z$ in distribution.

   * So convergence in quadratic mean and convergence in distribution are, respectively, the strongest and weakest mode of convergence among the three modes of convergence.

3. $Z_n \longrightarrow c$ (a constant) in probability $\Longleftrightarrow Z_n \longrightarrow c$ in distribution.

4. $Z_n \longrightarrow Z$ in probability, and g is a continuous function $\Longrightarrow g(Z_n) \longrightarrow g(Z)$ in probability.

   $Z_n \longrightarrow Z$ in distribution, and g is a continuous function $\Longrightarrow g(Z_n) \Longrightarrow g(Z)$ in distribution.

5. $X_n \longrightarrow X$ in probability, $Y_n \longrightarrow Y$ in probability $\Longrightarrow X_n \pm Y_n \longrightarrow X \pm Y$ in probability;

   $X_n Y_n \longrightarrow XY$ in probability; and $X_n/Y_n \longrightarrow X/Y$, if $P(Y = 0) = 0$.

6. *Slutsky's Theorem*: $X_n \longrightarrow X$ in distribution, $Y_n \longrightarrow c$ (a constant)

   $\Longrightarrow X_n \pm Y_n \longrightarrow X \pm c$ in distribution; $X_n Y_n \longrightarrow cX$ in distribution;

   and $X_n/Y_n \longrightarrow X/c$, if $c \neq 0$.

   * This is a very important theorem to find the asymptotic distribution of several estimators.