131B HW#4 solution

7.7 Sufficient Statistics

10. The p.d.f. for this uniform distribution is

$$f(x|a) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise,} \end{cases}$$

which leads to the joint p.d.f.

$$f_n(\mathbf{x}|a) = \begin{cases} \frac{1}{(b-a)^n} & \text{if } a \le x_i \le b \\ 0 & \text{otherwise.} \end{cases}$$

This can be re-written as

$$f_n(\mathbf{x}|a) = \frac{1}{(b-a)^n} 1_{\{T \ge a\}} 1_{\{\max(x_i) \le b\}},$$

where $1_{\{\}}$ denotes the indicator function. By the factorization criterion, T is a sufficient statistic for a.

12. The likelihood function is

$$f(\mathbf{x}|a, x_0) = \frac{\alpha^n x_0^{\alpha n}}{[\prod_{i=1}^n x_i]^{\alpha+1}},$$

for all $x_i \ge x_0$, which can be expressed as

$$\frac{\alpha^n x_0^{\alpha n}}{[\prod_{i=1}^n x_i]^{\alpha+1}} 1_{\{\min(x_i) \ge x_0\}}.$$

a) If x_0 is known, write

$$\left[\frac{\alpha^n x_0^{\alpha n}}{[\prod_{i=1}^n x_i]^{\alpha+1}}\right] \times [1_{\{\min(x_i) \ge x_0\}}].$$

By the factorization criterion, $T = \prod_{i=1}^n x_i$ is sufficient for α.

b) If α is known, write

$$\left[\frac{\alpha^n}{[\prod_{i=1}^n x_i]^{\alpha+1}}\right] \times [x_0^{\alpha n} 1_{\{\min(x_i)\geq x_0\}}].$$

By the factorization criterion, $T = \min(x_i)$ is sufficient for x_0.

14. The likelihood for the gamma distribution is

$$f(\mathbf{x}|\alpha,\beta) = \frac{\beta^{n\alpha}}{[\Gamma(\alpha)]^n}\left(\prod_{i=1}^n x_i\right)^{\alpha-1} e^{-\beta\sum_{i=1}^n x_i} 1_{\{\min(x_i)>0\}},$$

which can be written

$$\frac{\beta^{n\alpha}}{[\Gamma(\alpha)]^n}\left(\prod_{i=1}^n e^{\log(x_i)}\right)^{\alpha-1} e^{-\beta\sum_{i=1}^n x_i} 1_{\{\min(x_i)>0\}},$$

or

$$\left[\frac{\beta^{n\alpha}}{[\Gamma(\alpha)]^n} e^{(\alpha-1)\sum_{i=1}^n \log(x_i)}\right] \times \left[e^{-\beta\sum_{i=1}^n x_i} 1_{\{\min(x_i)>0\}}\right].$$

By the factorization criterion, $T = \sum_{i=1}^n \log(x_i)$ is sufficient for α.

16. Let $\xi(\theta)$ be a prior p.d.f. for θ. The posterior p.d.f. of θ is, according to Bayes' Theorem,

$$\xi(\theta|\mathbf{x}) = \frac{f_n(\mathbf{x}|\theta)\xi(\theta)}{\int f_n(\mathbf{x}|\theta)\xi(\theta)d\theta} = \frac{u(\mathbf{x})v[r(\mathbf{x}),\theta]\xi(\theta)}{\int u(\mathbf{x})v[r(\mathbf{x}),\theta]\xi(\theta)d\theta} = \frac{v[r(\mathbf{x}),\theta]\xi(\theta)}{\int v[r(\mathbf{x}),\theta]\xi(\theta)d\theta},$$

where the second equality uses the factorization criterion. One can see that this last expression depends on $\mathbf{x}$ only through $r(\mathbf{x})$. Notice that $u(\mathbf{x})$ is constant with respect to the integral in the denominator, and therefore cancels with the numerator.

7.8 Jointly Sufficient Statistics

2. The joint p.d.f. is

$$f_n(\mathbf{x}|\alpha,\beta) = \left[\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n [\prod_{i=1}^n x_i]^{\alpha-1}[\prod_{i=1}^n (1-x_i)]^{\beta-1}\right] 1_{\{\min(x_i)\geq 0\}} 1_{\{\max(x_i)\leq 1\}}.$$

By the factorization criterion $T_1 = \prod_{i=1}^n X_i$ and $T_2 = \prod_{i=1}^n (1-X_i)$ are jointly sufficient for α and β.

4. The joint p.d.f. is

$$f_n(\mathbf{x}|\theta) = \frac{1}{3^n} 1_{\{\min(x_i)\geq\theta\}} 1_{\{\max(x_i)\leq\theta+3\}}.$$

By the factorization criterion $T_1 = \min(X_i)$ and $T_2 = \max(X_i)$ are jointly sufficient for θ.

6. The joint p.d.f. or joint p.f. is

$$f_n(\mathbf{x}|\theta) = \left[\prod_{j=1}^{n} b(x_j)\right] \times \left[[a(\theta)]^n \exp\left[\sum_{i=1}^{k} c_i(\theta) \sum_{j=1}^{n} d_i(x_j)\right]\right].$$

It follows from the factorization criterion that $T_1, \ldots, T_k$ are jointly sufficient statistics for θ.

7. For each part, we will identify a, b, c_1, d_1, c_2, and d_2 as parts of the p.d.f. corresponding to exercise 6. Note that in each part, we are showing that the p.d.f.s take the form of a 2-parameter exponential family ($k = 2$).

a) Let $\theta = (\mu, \sigma^2)$. Then

$$\begin{aligned} a(\theta) &= \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \\ b(x) &= 1 \\ c_1(\theta) &= -\frac{1}{2\sigma^2} \\ d_1(x) &= x^2 \\ c_2(\theta) &= \frac{\mu}{\sigma^2} \\ d_2(x) &= x \end{aligned}$$

b) Let $\theta = (\alpha, \beta)$. Then

$$\begin{aligned} a(\theta) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \\ b(x) &= 1 \\ c_1(\theta) &= \alpha - 1 \\ d_1(x) &= \log x \\ c_2(\theta) &= -\beta \\ d_2(x) &= x \end{aligned}$$

c) Let $\theta = (\alpha, \beta)$. Then

$$\begin{aligned} a(\theta) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\ b(x) &= 1 \\ c_1(\theta) &= \alpha - 1 \\ d_1(x) &= \log x \\ c_2(\theta) &= \beta - 1 \\ d_2(x) &= \log(1-x) \end{aligned}$$

Supplementary Exercises

14. The joint p.d.f. is

$$f_n(\mathbf{x}|\beta, \theta) = \beta^n \exp\left(n\beta\theta - \beta \sum_{i=1}^n x_i\right) 1_{\{\min(x_i) \geq \theta\}}.$$

Hence, by the factorization criterion, $\sum_{i=1}^n X_i$ and $\min(X_i)$ is a pair of jointly sufficient statistics for (β, θ).