131B HW#3 solution

7.2 Prior and Posterior Distributions

2. The joint p.f. of the eight observations is

$$f_n(\mathbf{x}|\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} = \theta^2(1-\theta)^6.$$

Therefore,

$$\begin{aligned}
\xi(0.1|\mathbf{x}) = P(\theta = 0.1|\mathbf{x}) &= \frac{\xi(0.1)f_n(\mathbf{x}|0.1)}{\xi(0.1)f_n(\mathbf{x}|0.1) + \xi(0.2)f_n(\mathbf{x}|0.2)} \\
&= \frac{(0.7)(0.1)^2(0.9)^6}{(0.7)(0.1)^2(0.9)^6 + (0.3)(0.2)^2(0.8)^6} \\
&= 0.5418.
\end{aligned}$$

And $\xi(0.2|\mathbf{x}) = 1 - \xi(0.1|\mathbf{x}) = 0.4582$.

10. The p.d.f. of X is

$$f(x|\theta) = \begin{cases} 1 & \text{for } \theta - 1/2 \le x \le \theta + 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

and the prior p.d.f. of θ is

$$\xi(\theta) = \begin{cases} \frac{1}{10} & \text{for } 10 \le \theta \le 20, \\ 0 & \text{otherwise.} \end{cases}$$

The condition that $\theta - 1/2 \le x \le \theta + 1/2$ is the same as the condition that $x - 1/2 \le \theta \le x + 1/2$. Therefore, $f(x|\theta)\xi(\theta)$ is a positive constant only for values of θ which satisfy both conditions that $x - 1/2 \le \theta \le x + 1/2$ and $10 \le \theta \le 20$. Since $X = 12$, $\xi(\theta|x) \propto f(x|\theta)\xi(\theta)$ is a positive constant only for $11.5 < \theta < 12.5$. In other words, the posterior distribution of θ is a uniform distribution on the interval [11.5,12.5].

7.3 Conjugate Prior Distributions

4. Let α_1 and β_1 denote the parameters of the posterior beta distribution, and let $\gamma = \alpha_1/(\alpha_1+\beta_1)$. Then γ is the mean of the posterior distribution and we are told that $\gamma = 2/51$. The variance of the posterior distribution is

$$\begin{aligned}
\frac{\alpha_1\beta_1}{(\alpha_1+\beta_1)^2(\alpha_1+\beta_1+1)} &= \frac{\alpha_1}{\alpha_1+\beta_1}\cdot\frac{\beta_1}{\alpha_1+\beta_1}\cdot\frac{1}{\alpha_1+\beta_1+1} \\
&= \gamma(1-\gamma)\frac{1}{\alpha_1+\beta_1+1} \\
&= \frac{2}{51}\cdot\frac{49}{51}\cdot\frac{1}{\alpha_1+\beta_1+1} \\
&= \frac{98}{(51)^2}\cdot\frac{1}{\alpha_1+\beta_1+1}.
\end{aligned}$$

From the value of this variance given it is now evident that $\alpha_1+\beta_1+1 = 103$. Hence, $\alpha_1+\beta_1 = 102$ and $\alpha_1 = \gamma(\alpha_1+\beta_1) = 2(102)/51 = 4$. In turn, it follows that $\beta_1 = 102-4 = 98$. Since the posterior distribution is a beta distribution, it follows from Theorem 7.3.1 that the prior distribution must have been a beta distribution with parameters α and β such that $\alpha+3 = \alpha_1$ and $\beta+97 = \beta_1$. Therefore, $\alpha = \beta = 1$. Note that the beta distribution for which $\alpha = \beta = 1$ is the uniform distribution on the interval [0,1].

18. Suppose that the prior distribution of θ is the Pareto distribution with parameters x_0 and α ($x_0 > 0$ and $\alpha > 0$). Then the prior p.d.f. $\xi(\theta)$ has the form

$$\xi(\theta) \propto 1/\theta^{\alpha+1} \quad \text{for } \theta \geq x_0.$$

If $X_1,\ldots,X_n$ form a random sample from a uniform distribution on $[0,\theta]$, then the likelihood function

$$f_n(\mathbf{x}|\theta) \propto 1/\theta^n \quad \text{for } \theta \geq \max\{x_1,\ldots,x_n\}.$$

Hence, the posterior p.d.f. of θ has the form

$$\xi(\theta|\mathbf{x}) \propto \xi(\theta)f_n(\mathbf{x}|\theta) \propto 1/\theta^{\alpha+n+1} \quad \text{for } \theta \geq \max\{x_0,x_1,\ldots,x_n\}$$

Hence, the posterior distribution of θ is recognized as the Pareto distribution with parameters $\max\{x_0,x_1,\ldots,x_n\}$ and $\alpha+n$.

7.4 Bayes Estimators

2. Based on Theorem 7.3.1, the posterior distribution of θ is the beta distribution with parameters 5+1=6 and 10+19=29. Therefore, the Bayes estimate of θ is the mean of this posterior distribution, which is 6/(6+29)=6/35.

10. Let α and β denote the parameters of the prior gamma distribution of θ. Then $\alpha/\beta = 0.2$ and $\alpha/\beta^2 = 1$. Therefore, $\beta = 0.2$ and $\alpha = 0.04$. Furthermore, the total time required to serve the sample of 20 customers is $y = 20(3.8) = 76$. Therefore, by Theorem 7.3.4, the posterior distribution of θ is the gamma distribution with parameters 0.04+20=20.04 and 0.2+76=76.2. The Bayes estimate is the mean of this posterior distribution and is equal to 20.04/76.2=0.263.

12. a. A's prior distribution for β is the beta distribution with parameters $\alpha = 2$ and $\beta = 1$. Based on Theorem 7.3.1, A's posterior distribution for θ is the beta distribution with parameters 2+710=712 and 1+290=291. B's prior distribution for θ is a beta distribution with parameters $\alpha = 4$ and $\beta = 1$. Similarly, B's posterior distribution for θ is the beta distribution with parameters 4+710=714 and 1+290=291.

b. A's Bayes estimate of θ is 712/(712+291)=712/1003. B's Bayes estimate of θ is 714/(714+291)=714/1005.

c. If y denotes the number in the sample who were in favor of the proposition, then A's posterior distribution for θ will be the beta distribution with parameters $2 + y$ and $1 + 1000 - y = 1001 - y$, and B's posterior distribution will be a beta distribution with parameters $4 + y$ and $1 + 1000 - y = 1001 - y$. Therefore, A's Bayes estimate of θ will be (2+y)/1003 and B's Bayes estimate of θ will be $(4 + y)/1005$. Then

$$\left|\frac{4+y}{1005} - \frac{2+y}{1003}\right| = \frac{2(1001-y)}{(1005)(1003)}.$$

This difference is a maximum when $y = 0$, but even then its value is only

$$\frac{2(1001)}{(1005)(1003)} < \frac{2}{1000}.$$

Additional Problems

7.3 Conjugate Prior Distributions

7. In the notation of Theorem 7.3.3, we have $\sigma^2 = 4$, $\mu = 68$, $\upsilon^2 = 1$, $n = 10$, and $\bar{x}_n = 69.5$. Therefore, the posterior distribution of θ is the normal distribution with mean $\mu_1 = 967/14$ and variance $\upsilon_1^2 = 2/7$.

8. Since the p.d.f. of a normal distribution attains its maximum value at the mean of the distribution and then drops off on each side of the mean, among all intervals of length 1 unit, the interval that is centered at the mean will contain the most probability. Therefore,

the answer in part (a) is the interval centered at the mean of the prior distribution of θ and the answer in part (b) is the interval centered at the mean of the posterior distribution of θ. In part (c), if the distribution of θ is specified by its prior distribution, then $Z = \theta - 68$ will have a standard normal distribution. Therefore,

$$P(67.5 \leq \theta \leq 68.5) = P(-0.5 \leq Z \leq 0.5) = \Phi(0.5) - \Phi(-0.5) = 0.3830.$$

Similarly, if the distribution of θ is specified by its posterior distribution, then $Z = (\theta - \mu_1)/\upsilon_1 = (\theta - 69.07)/0.5345$ will have a standard normal distribution. Therefore,

$$P(68.57 \leq \theta \leq 69.57|\theta) = P(-0.9355 \leq Z \leq 0.9355) = \Phi(0.9355) - \Phi(-0.9355) = 0.6506.$$

7.10 Supplementary Exercises

12. The prior distribution of θ is the Pareto distribution with parameters $x_0 = 1$ and $\alpha = 1$. Therefore, it follows from Exercise 18 of Sec. 7.3 that the posterior distribution of θ will be a Pareto distribution with parameters $\max\{x_0, x_1, \ldots, x_n\}$ and $\alpha + n$. In this exercise $n = 4$ and $\max\{x_0, x_1, \ldots, x_n\} = \max\{1, 0.6, 0.4, 0.8, 0.9\} = 1$. Hence, the posterior Pareto distribution has parameters 1 and 5. The Bayes estimate of θ will be the mean of this posterior distribution, namely

$$\hat{\theta} = \int_1^\infty \theta \frac{5}{\theta^6} d\theta = \frac{5}{4}.$$

13. The Bayes estimate of θ will be the median of the posterior Pareto distribution. This will be the value m such that

$$\frac{1}{2} = \int_m^\infty \frac{5}{\theta^6} d\theta = \frac{1}{m^5}.$$

Hence, $\hat{\theta} = m = 2^{1/5}$.