131B HW#5 solution

7.8 Jointly Sufficient Statistics

10. The p.d.f. of the uniform distribution is $f(x|\theta) = \frac{1}{\theta} 1_{\{0 \le x \le \theta\}}$, so the likelihood is

$$f_n(\mathbf{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \frac{1}{\theta^n} 1_{\{\max\{x_1,\ldots,x_n\} \le \theta\}} 1_{\{0 \le \min\{x_1,\ldots,x_n\}\}}$$

It implies that the M.L.E. $\hat{\theta} = \max\{X_1, \ldots, X_n\}$.

By the factorization criterion, $\hat{\theta}$ is the sufficient statistic, so it is a minimal sufficient statistic.

12. The likelihood is

$$f_n(\mathbf{x}|\theta) = \prod_{i=1}^{n} f(x|\theta) = \frac{2^n \prod_i x_i}{\theta^{2n}} 1_{\{\max\{x_1,\ldots,x_n\} \le \theta\}} 1_{\{0 \le \min\{x_1,\ldots,x_n\}\}}$$

The M.L.E. of θ is $\hat{\theta} = \max\{X_1, \ldots, X_n\}$. $\hat{\theta}$ is also a sufficient statistic by the factorization criterion.

The median of the distribution is the value m such that $\int_0^m f(x|\theta)dx = 1/2$, it implies that $m = \theta/\sqrt{2}$.

By the invariance property, $\hat{m} = \hat{\theta}/\sqrt{2}$ is the M.L.E. of m. Note that $\hat{m}$ is also a sufficient statistic, so it is a minimal sufficient statistic.

16. It follows from Theorem 7.3.2 that the Bayes estimator of λ is $(\alpha + \sum_{i=1}^{n} X_i)/(\beta + n)$. Since $\sum_{i=1}^{n} X_i$ is a sufficient statistic for λ, the Bayes estimator is also a sufficient statistic for λ. Hence, this estimator is a minimal sufficient statistic.

7.9 Improving an Estimator

2. It is derived that $\max\{X_1, \ldots, X_n\}$ is a sufficient statistic. Since $2\bar{X}_n$ is not a function of the sufficient statistic, it is inadmissible.

6. The likelihood is

$$f(\mathbf{x}|\alpha) = \frac{\beta^{n\alpha}}{[\Gamma(\alpha)]^n} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} e^{-\beta \sum_{i=1}^{n} x_i} 1_{\{\min(x_i)>0\}}$$

By the factorization criterion, $\prod_{i=1}^{n} X_i$ is a sufficient statistic. Since $\bar{X}_n$ is not a function of the sufficient statistic, it is inadmissible.

7.10 Supplementary Exercises

18. The p.d.f. is $f(x|x_0, \alpha) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}} 1_{\{x \geq x_0\}}$, so the likelihood is

$$f_n(\mathbf{x}|x_0, \alpha) = \frac{\alpha^n x_0^{n\alpha}}{(\prod_{i=1}^{n} x_i)^{\alpha+1}} 1_{\{\min\{x_1,\ldots,x_n\} \geq x_0\}}.$$

It implies that $\hat{x}_0 = \min\{X_1, \ldots, X_n\}$ is the M.L.E. of x_0, since this value of x_0 maximizes the likelihood function regardless of the value of α. If we substitute $\hat{x}_0$ for x_0 and let $L(\alpha)$ denote the log-likelihood, then

$$L(\alpha) = n \log \alpha + n\alpha \log \hat{x}_0 - (\alpha + 1) \sum_{i=1}^{n} \log x_i$$

and let

$$\frac{dL(\alpha)}{d\alpha} = \frac{n}{\alpha} + n \log \hat{x}_0 - \sum_{i=1}^{n} \log x_i = 0.$$

We obtain the M.L.E. of α

$$\hat{\alpha} = \left(\frac{1}{n} \sum_{i=1}^{n} \log x_i - \log \hat{x}_0 \right)^{-1}$$

It follows from the factorization criterion that $T_1 = \min\{X_1, \ldots, X_n\}$ and $T_2 = \prod_{i=1}^{n} x_i$ are jointly sufficient statistics for x_0 and α. $\hat{x}_0$ and $\hat{\alpha}$ form a one-to-one transform of T_1 and T_2, so $\hat{x}_0$ and $\hat{\alpha}$ are also jointly sufficient statistics. Note that they are M.L.E. for x_0 and α, hence, they are minimal jointly sufficient statistics.

Additional Problems

8.7 Unbiased Estimators

4. It follows from Eq. (5.5.7) that $E(X) = (1-p)/p = 1/p - 1$. Therefore, $E(X+1) = 1/p$, which implies that $\delta(X) = X + 1$ is an unbiased estimator of $1/p$.

11. (a) $E(\hat{\theta}) = \alpha E(\bar{X}_m) + (1-\alpha)E(\bar{Y}_n) = \alpha\theta + (1-\alpha)\theta = \theta$. Hence, $\hat{\theta}$ is an unbiased estimator of θ for all values of α, m and n.

(b) Since the two samples are taken independently, $\bar{X}_m$ and $\bar{Y}_n$ are independent. Hence,

$$\mathrm{var}(\hat{\theta}) = \alpha^2 \mathrm{var}(\bar{X}_m) + (1-\alpha)^2 \mathrm{var}(\bar{Y}_n) = \alpha^2 \frac{\sigma_A^2}{m} + (1-\alpha)^2 \frac{\sigma_B^2}{n}$$

Since $\sigma_A^2 = 4\sigma_B^2$, it follows that

$$\mathrm{var}(\hat{\theta}) = \left[\frac{4\alpha^2}{m} + \frac{(1-\alpha)^2}{n}\right]\sigma_B^2.$$

By differentiating the coefficient of σ_B^2, it is found that $\mathrm{var}(\hat{\theta})$ is a minimum when $\alpha = m/(m+4n)$.

2. Based on Theorem 7.3.3, the posterior distribution of θ is the normal distribution with mean $\mu_1 = \frac{68\sigma^2 + 4n\bar{X}_n}{\sigma^2+4n}$.

(a). The Bayes estimator is the mean of the posterior distribution. That is,

$$\theta_2 = \frac{68\sigma^2 + 4n\bar{X}_n}{\sigma^2 + 4n}.$$

Then

$$E(\theta_2) = \frac{68\sigma^2 + 4nE(\bar{X}_n)}{\sigma^2 + 4n} = \frac{68\sigma^2 + 4n\theta}{\sigma^2 + 4n}$$

And $E(\theta_2) - \theta = \frac{(68-\theta)\sigma^2}{\sigma^2+4n}$. So if $\theta = 68$, then θ_2 is unbiased, otherwise, it is biased and the bias is $\frac{(68-\theta)\sigma^2}{\sigma^2+4n}$.

(b). It is easy to know that the M.L.E. $\theta_1 = \bar{X}_n$. Then the mean squared error of θ_1 is

$$R(\theta, \theta_1) = \mathrm{var}(\theta_1) = \sigma^2/n.$$

The mean squared error of θ_2 is

$$R(\theta, \theta_2) = (\mathrm{bias}(\theta_2))^2 + \mathrm{var}(\theta_2) = \left(\frac{(68-\theta)\sigma^2}{\sigma^2+4n}\right)^2 + \frac{16n\sigma^2}{(\sigma^2+4n)^2} = \frac{(68-\theta)^2\sigma^4 + 16n\sigma^2}{(\sigma^2+4n)^2}$$

Then

$$\frac{R(\theta,\theta_2)}{R(\theta,\theta_1)} = \frac{(68-\theta)^2 n\sigma^2 + 16n^2}{(\sigma^2+4n)^2} = \frac{(68-\theta)^2 n\sigma^2 + 16n^2}{\sigma^4 + 8n\sigma^2 + 16n^2}.$$

Thus, if $\sigma^2 \le ((68-\theta)^2 - 8)n$, then $R(\theta,\theta_1) \le R(\theta,\theta_2)$, otherwise, $R(\theta,\theta_1) > R(\theta,\theta_2)$.