STA 131B HW 8

8.5

4. Since X_i are iid normal, $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a standard normal distribution. So

$$P\left(-1.96 < \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} < 1.96\right) = 0.95, \quad \text{rewritten as}$$
$$P\left(\bar{X}_n - \frac{1.96\sigma}{\sqrt{n}} < \mu < \bar{X}_n + \frac{1.96\sigma}{\sqrt{n}}\right) = 0.95.$$

Therefore $(\bar{X}_n - 1.96\sigma/\sqrt{n}, \bar{X}_n + 1.96\sigma/\sqrt{n})$ will be a confidence interval for μ with confidence coefficient 0.95. The length of this interval is $3.92\sigma/\sqrt{n}$ which should be less than 0.01σ. This means $n > 153664$ (or $n \geq 153665$).

6. The exponential distribution with mean μ is the same as the gamma distribution with $\alpha = 1$ and $\beta = 1/\mu$. Therefore, by Theorem 5.7.7, $\sum_{i=1}^{n} X_i$ follows the gamma distribution with parameters $\alpha = n$ and $\beta = 1/\mu$. Then it follows from Exercise 5.7.1 that $2\sum_{i=1}^{n} X_i/\mu$ has the gamma distribution with $\alpha = n$ and $\beta = 1/2$. By Definition 8.2.1, $2\sum_{i=1}^{n} X_1/\mu$ has the χ^2 distribution with $2n$ degrees of freedom. There are infinite number of pairs of numbers (q_1, q_2) such that $P(q_1 < \chi^2_{2n} < q_2) = \gamma$, we can choose for example $q_1 = \chi^2_{2n}((1-\gamma)/2)$ and $q_2 = \chi^2_{2n}((1+\gamma)/2)$, where $\chi^2_{2n}(\alpha)$ is the α-quantile of a χ^2_{2n} random variable, the value of which can be found in the Table on page 858. Let $c_1 = q_1/2$ and $c_2 = q_2/2$ and we have

$$P\left(q_1 < \frac{2}{\mu}\sum_{i=1}^{n} X_i < q_2\right) = \gamma, \quad \text{so}$$
$$P\left(c_1 < \frac{1}{\mu}\sum_{i=1}^{n} X_i < c_2\right) = \gamma.$$

Reorder the last display we have

$$P\left(\frac{1}{c_2}\sum_{i=1}^{n} X_i < \mu < \frac{1}{c_1}\sum_{i=1}^{n} X_i\right) = \gamma.$$

Then $(\sum_{i=1}^{n} X_i/c_2, \sum_{i=1}^{n} X_i/c_1)$ is a confidence interval for μ with confidence coefficient γ.

7. The average of the $n = 20$ values is $\bar{x}_n = 156.85$, and $\hat{\sigma} = \sqrt{(n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2} = 22.64$. The appropriate t distribution quantile is $T_{19}^{-1}(0.95) = 1.729$. The endpoints of the confidence interval are then $156.85 \pm 22.64 \times 1.729/20^{1/2}$. Completing the calculation we have the interval $(148.1, 165.6)$.

Supplementary Exercises 8.9

11. Let $c = T_{n-1}^{-1}(0.99)$ denote the 0.99 quantile of the t distribution. Then $P(\sqrt{n}(\bar{X}_n - \mu)/\hat{\sigma} < c) = 0.99$, or equivalently, $P(\mu > \bar{X}_n - c\hat{\sigma}/\sqrt{n}) = 0.99$. Hence, $L = \bar{X}_n - c\hat{\sigma}/\sqrt{n}$.

13.

(a) The posterior distribution of θ is the normal distribution as given by equations (7.3.1) and (7.3.2), which has mean and variance

$$\mu_1 = \frac{\sigma^2\mu + n\nu^2\bar{x}_n}{\sigma^2 + n\nu^2}, \quad \nu_1^2 = \frac{\sigma^2\nu^2}{\sigma^2 + n\nu^2}. \tag{1}$$

Therefore under this distribution,

$$P(\mu_1 - 1.96\nu_1 < \theta < \mu_1 + 1.96\nu_1) = 0.95,$$

so $I = (\mu_1 - 1.96\nu_1, \mu_1 + 1.96\nu_1)$. This interval I is the shortest one that has the required probability because it is symmetrically placed around the mean μ_1 of the posterior normal distribution.

(b) It follows from (1) that $\mu_1 \to \bar{x}_n$ and that $\nu_1^2 \to \sigma^2/n$ as $\nu^2 \to \infty$. Hence, the interval I converges to

$$(\bar{x}_n - \frac{1.96\sigma}{\sqrt{n}}, \bar{x}_n + \frac{1.96\sigma}{\sqrt{n}}).$$

It was shown in Exercise 8.5.4 that this interval is a confidence interval for θ with confidence coefficient 0.95.

22.

(a) The pdf of Y_n can be found by using the method in Example 3.9.6, which is

$$f(y|\theta) = \begin{cases} ny^{n-1}/\theta^n & \text{if } 0 \le y \le \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = Y_n/\theta$. The pdf of Z is found by Jacobian method to be

$$g(z|\theta) = f(z\theta|\theta)\theta = \begin{cases} nz^{n-1} & \text{if } 0 \le z \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

The cdf of Z is then $G(z) = z^n$ for $0 < z < 1$. The quantile function if $G^{-1}(p) = p^{1/n}$.

(b) The bias of Y_n as an estimator of θ is

$$E(Y_n) - \theta = \int_0^\theta y\frac{ny^{n-1}}{\theta^n}dy - \theta = -\frac{\theta}{n+1}.$$

(c) We found the distribution of $Z = Y_n/\theta$ in (a), which does not depend on θ (so is the same for all θ). So it is a pivotal.

(d) Using the pdf of the pivotal Y_n/θ, for any positive constants $a < b \le 1$,

$$P(a < \frac{Y_n}{\theta} < b) = b^n - a^n.$$

So choose a and b such that $b^n - a^n = \gamma$ (for example $b = ((1+\gamma)/2)^{1/n}$ and $a = ((1-\gamma)/2)^{1/n}$), we have

$$P(a < \frac{Y_n}{\theta} < b) = \gamma \quad \text{which means}$$
$$P(\frac{Y_n}{b} < \theta < \frac{Y_n}{a}) = \gamma.$$

Then $(Y_n/b, Y_n/a)$ is a confidence interval for θ with coefficient γ.

9.1

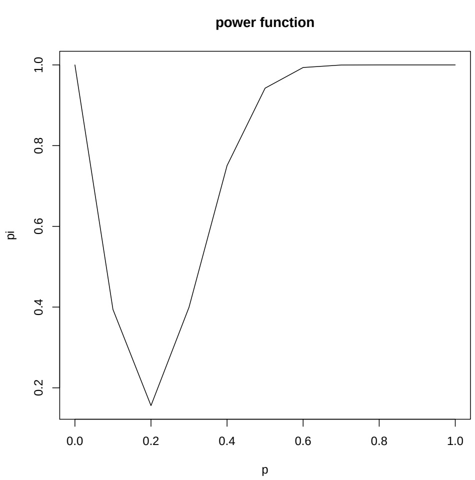
3.

(a) For any given value of p, $\pi(p|\delta) = P(Y \ge 7) + P(Y \le 1)$, where Y has a binomial distribution with parameters $n = 20$ and p. For $p = 0$, $P(Y \ge 7) = 0$ and $P(Y \le 1) = 1$. Therefore, $\pi(0|\delta) = 1$. For $p = 0.2$, it is found from the table of the binomial distribution (p854) that

$$P(Y \ge 7) = .545 + .0222 + .0074 + .0020 + .0005 + .0001 = .0867.$$

and $P(Y \le 1) = .0115 + .0576 = .0691$. Hence, $\pi(0.2|\delta) = 0.1558$. By continuing to use the tables in this way, we can find the values of $\pi(0.1|\delta)$, $\pi(0.3|\delta)$, $\pi(0.4|\delta)$, and $\pi(0.5|\delta)$. For $p = 0.6$, we must use the fact that if Y has a binomial distribution with *parameters* 20 and 0.6, then $Z = 20 - Y$ has a binomial distribution with parameters 20 and 0.4. Also, $P(Y \ge 7) = P(Z \le 13)$ and $P(Y \le 1) = P(Z \ge 19)$. It is found from the tables that $P(Z \le 13) = .9935$ and $P(Z \ge 19) = .0000$. Hence $\pi(0.6|\delta) = .9935$. Similarly, if $p = 0.7$, then $Z = 20 - Y$ will have a binomial distribution with parameter 20 and 0.3. In this case it is found that $P(Z \le 13) = 0.9998$ and $P(Z \ge 19) = .0000$. Hence $\pi(0.7|\delta) = .9998$. Similarly we can find $\pi(0.8|\delta)$, $\pi(0.9|\delta)$, and $\pi(1|\delta) = 1$.

p	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000
$\pi(p\|\delta)$	1.0000	0.3941	0.1558	0.3996	0.7505	0.9424	0.9935	0.9998	1.0000	1.0000	1.0000



(b) Since H_0 is a simple hypothesis, the size α of the test is just the value of the power function at the point specified by H_0. Thus, $\alpha = \pi(0.2|\delta) = 0.1558$.

5. A hypothesis is simple if and only if it specifies a single value of both μ and σ. Therefore, only the hypothesis in (a) is simple. All others are composite. In particular, although the hypothesis in (d) specifies a single value for μ, it leaves the value of σ arbitrary.

19. We want our test to reject H_0 if $\bar{X}_n \leq Y$, where Y is some quantity. We need the test to have level α_0, so

$$P(\bar{X}_n \leq Y|\mu = \mu_0, \sigma^2) = \alpha_0 \tag{2}$$

is necessary. We know that $\sqrt{n}(\bar{X}_n - \mu_0)/\hat{\sigma}$ has the t distribution with $n-1$ degrees of freedom if $\mu = \mu_0$, hence (2) will hold if $Y = \mu_0 - \hat{\sigma}T_{n-1}^{-1}(1-\alpha_0)/\sqrt{n}$. Now, $\bar{X}_n \leq Y$ (reject H_0) if and only if $\mu_0 \geq \bar{X}_n + \hat{\sigma}T_{n-1}^{-1}(1-\alpha_0)/\sqrt{n}$. This is equivalent to μ_0 is not in the interval

$$(-\infty, \bar{X}_n + \hat{\sigma}T_{n-1}^{-1}(1-\alpha_0)/\sqrt{n}).$$

Additional Problem

The log-likelihood function is

$$l(\mu, \sigma^2) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{\sum_{i=1}^n (X_i - \mu)^2}{2\sigma^2}.$$

The first derivative is

$$\begin{cases} \frac{\partial l}{\partial \mu} = \frac{\sum_{i=1}^n (X_i - \mu)}{\sigma^2} \\ \frac{\partial l}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{\sum_{i=1}^n (X_i - \mu)^2}{2\sigma^4} \end{cases}$$

The MLE $\hat{\sigma}_0^2$ under $H_0 : \mu = \mu_0$ is obtained by setting $\mu = \mu_0$ and $\frac{\partial l}{\partial \sigma^2} = 0$ and is $\sigma_0^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu_0)^2$.
The MLE $(\hat{\mu}, \hat{\sigma}_1^2)$ under H_1 is obtained by setting $\frac{\partial l}{\partial \mu} = 0$ and $\frac{\partial l}{\partial \sigma^2} = 0$ and is $\hat{\mu} = \bar{X}$ and $\hat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$. So the likelihood ratio statistic is

$$\begin{aligned}
\Lambda(\mathbf{X}) &= \frac{\sup_{\mu=\mu_0} f_n(\mathbf{X}|\theta)}{\sup_{\mu,\sigma^2} f_n(\mathbf{X}|\theta)} \\
&= \frac{f_n(\mathbf{X}|\mu = \mu_0, \sigma^2 = \hat{\sigma}_0^2)}{f_n(\mathbf{X}|\mu = \bar{X}, \sigma^2 = \hat{\sigma}_1^2)} \quad \text{since the estimates are MLE} \\
&= \frac{\left(\frac{1}{\sqrt{2\pi}\hat{\sigma}_0}\right)^n e^{-n/2}}{\left(\frac{1}{\sqrt{2\pi}\hat{\sigma}_1}\right)^n e^{-n/2}} \quad \text{by plugging } \hat{\mu}, \hat{\sigma}_0^2, \hat{\sigma}_1^2 \text{ into normal density} \\
&= \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2}\right)^{n/2} \\
&= \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (X_i - \mu_0)^2}\right]^{n/2}.
\end{aligned}$$

Since

$$\frac{1}{\Lambda(\mathbf{X})} = \left[\frac{\sum_{i=1}^n (X_i - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right]^{n/2}.$$

and

$$\sum_{i=1}^n (X_i - \mu_0)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu_0)^2,$$

we have

$$\frac{1}{\Lambda(\mathbf{X})} = \left[1 + \frac{n(\bar{X} - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right]^{n/2}.$$

Therefore, reject H_0 for small value of $\Lambda(\mathbf{X})$ is equivalent to reject H_0 for large values of $\frac{n(\bar{X}-\mu_0)^2}{\sum_{i=1}^n (X_i-\bar{X})^2}$. Define $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$, the LRT is equivalent to reject H_0 when $\left|\frac{\sqrt{n}(\bar{X}-\mu_0)}{\hat{\sigma}}\right|$ is large. Hence, the lRT reject H_0 when $\left|\frac{\sqrt{n}(\bar{X}-\mu_0)}{\hat{\sigma}}\right| > C$, for some constant $C > 0$. It now remains to find the critical value C.

We know $\frac{\sqrt{n}(\bar{X}-\mu_0)}{\hat{\sigma}}$ follows t_{n-1} distribution, and we want a α level test which means the rejection probability is α, we should choose $C = T_{n-1}^{-1}(1 - \alpha/2)$. Therefore, the likelihood ratio test rejects the null hypothesis if and only if

$$\left|\frac{\sqrt{n}(\bar{X} - \mu_0)}{\hat{\sigma}}\right| \geq T_{n-1}^{-1}(1 - \alpha/2).$$

(this shows the likelihood ratio test for a normal $\mu = \mu_0$ is the same as the t-test.)