Solution: Homework 4

25.7. Here $r = 6, n = 8$, and

$$\begin{aligned} SSTR &= 854.5292, df = r-1 = 5, MSTR = SSTR/5 = 170.9058 \\ SSE &= 30.0700, df = (n-1)r = 42, MSE = SSE/42 = 0.71595. \\ SSTO &= 884.5992, df = n-1 = 47. \end{aligned}$$

(a) $H_0 : \sigma_\mu^2 = 0$, $H_1 : \sigma_\mu^2 \neq 0$.
$F^* = MSTR/MSE = 238.7119$.
Decision rule: reject H_0 if $F^* > F(0.99; 5, 42) = 3.4882$.
Since $F^* > F(0.99; 5, 42)$, we reject H_0. P-value≈ 0.0000.
(b) Here

$$\hat{\mu} = \bar{Y}_{..} = 17.6292, s(\hat{\mu}) = \sqrt{MSTR/n_T} = \sqrt{170.9058/48} = 1.8869.$$

99% confidence interval for μ is

$$\begin{aligned} &\hat{\mu} \pm t(0.995; 5)s(\hat{\mu}), \text{ i.e.,} \\ &17.6292 \pm (4.0321)(1.8869), \text{ i.e., } (10.021, 25.237). \end{aligned}$$

25.8
(a) 99% confidence interval for $\sigma_\mu^2/(\sigma_\mu^2 + \sigma^2)$ is given by (L^*, U^*) where

$$\begin{aligned} L^* &= L/(1+L), U^* = U/(1+U), \\ L &= \frac{1}{n}\left[F^*/F(0.995; , 5, 42) - 1\right], \\ U &= \frac{1}{n}\left[F^*/F(0.005; 5, 42 - 1\right]. \end{aligned}$$

Note that F^* is as given in (25.7a). Noting that $F(0.005; 5, 42) = 1/F(0.995; 42, 5) = 12.51164$,we have

$$\begin{aligned} L &= \frac{1}{8}[238.7119/3.9528 - 1] = 7.4238, \\ L^* &= L/(1+L) = 0.8813, \\ U &= \frac{1}{8}[(238.7119)(12.5116) - 1] = 44.7790, \\ U^* &= U/(1+U) = 0.9782. \end{aligned}$$

Thus a 99% confidence interval for $\sigma_\mu^2/(\sigma_\mu^2 + \sigma^2)$ is $(0.881, 0.978)$.
(b) Estimates of σ^2 and σ_μ^2 are

$$\begin{aligned} \hat{\sigma}^2 &= MSE = 0.71595, \\ s_\mu^2 &= \frac{MSTR - MSE}{n} = 21.2737. \end{aligned}$$

(c) 99% confidence interval for σ^2 is given by

$$\begin{aligned} & (42MSE/\chi^2(0.995;42), 42MSE/\chi^2(0.005;42) \\ = & ((42)(0.71595)/69.3360, (42)(0.71595)/22.1385) \\ = & (0.4337, 1.3583). \end{aligned}$$

(d) Want to test $H_0 : \sigma_\mu^2/\sigma^2 = 2$ against $H_1 : \sigma_\mu^2/\sigma^2 > 2$.
(e) An example of MLS procedure is given in 26.16g. It will not be repeated here.

25.11. As has been discussed in the class if $\{\delta_{ij} : j = 1, \ldots, b, i = 1, \ldots, a\}$ are iid $N(0, \sigma_{\alpha\beta}^2)$, then $(\alpha\beta)_{ij}$ is taken to be $(\alpha\beta)_{ij} = \delta_{ij} - \bar{\delta}_{.j}$. Thus $\sum_i (\alpha\beta)_{ij} = 0$. However, $\sum_j (\alpha\beta)_{ij} = b[\bar{\delta}_{i.} - \bar{\delta}_{..}] \sim N(0, b(1-1/a)\sigma_{\alpha\beta}^2)$. Thus $P[\sum_j (\alpha\beta)_{ij} = 0] = 0$.

25.16.

Here, $a = 3, b = 3, n = 5$, factor A is random and factor B is fixed. The following summary can be obtained.

$$\begin{aligned} SSA &= 24.5778, df = a-1 = 2, MSA = 12.2889, \\ SSB &= 28.3111, df = b-1 = 2, MSB = 14.1556, \\ SSAB &= 1215.2889, df = (a-1)(b-1) = 4, MSAB = 303.8222, \\ SSE &= 1872.4000, df = (n-1)ab = 36, MSE = 52.0111, \\ \bar{Y}_{.1.} &= 56.13333, \bar{Y}_{.2.} = 56.60000 \bar{Y}_{.3.} = 54.73333. \end{aligned}$$

(a) $H_0 : \sigma_{\alpha\beta}^2 = 0$, $H_1 : \sigma_{\alpha\beta}^2 \neq 0$, $F^* = MSAB/MSE = 5.8415$.
Decision rule; reject H_0 if $F^* > F(0.99; 4, 36) = 3.8903$.
Since $F^* > F(0.99; 4, 36)$, we reject H_0. P-value$\approx$ 0.0010.
(b) Estimate of $\sigma_{\alpha\beta}^2$ is

$$s_{\alpha\beta}^2 = \frac{MSAB - MSE}{n} = 50.3622.$$

An approximate estimate of $\sigma_{\alpha\beta}^2/\sigma^2$ is $s_{\alpha\beta}^2/MSE = 0.9683$. Thus it appears $\sigma_{\alpha\beta}^2$ and σ^2 may not be much different.
(c) $H_0 : \sigma_\alpha^2 = 0$, $H_1 : \sigma_\alpha^2 \neq 0$, $F^* = MSA/MSE = 0.2363$.
Decision rule: reject H_0 if $F^* > F(0.99; 2, 36) = 5.2479$.
Here $F^* < F(0.99; 2, 36)$, we cannot reject H_0. P-value$\approx$ 0.7908
(d) $H_0 : \sigma_\beta^2 = 0$, $H_1 : \sigma_\beta^2 \neq 0$, $F^* = MSB/MSAB = 0.0466$.
Decision rule: reject H_0 if $F^* > F(0.99; 2, 36) = 5.2479$.
Here $F^* < F(0.99; 2, 36)$, we cannot reject H_0. P-value$\approx$ 0.9545.
(e) Factor B means and their estimates are $\mu_{.j} = \mu_{..} + \beta_j$ and $\hat{\mu}_{.j} = \bar{Y}_{.j.}$ Thus an estimate of $\mu_{.j} - \mu_{.j'} = \beta_j - \beta_{j'}, j \neq j'$, is $\bar{Y}_{.j.} - \bar{Y}_{.j'.}$ For the mixed model

here (with A random and B fixed), we have

$$\begin{aligned}
Var(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot j'\cdot}) &= \frac{2}{a}(n\sigma^2_{\alpha\beta} + \sigma^2), \text{ and hence} \\
s^2(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot j'\cdot}) &= \frac{2}{an}MSAB = \frac{2}{(3)(5)}(303.8222) = 40.5096, \\
s(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot j'\cdot}) &= 6.3647.
\end{aligned}$$

Tukey's multiplier is

$$T = \frac{1}{\sqrt{2}}q(0.95; 3, 4) = \frac{1}{\sqrt{2}}5.0402 = 3.5640.$$

Simultaneous confidence intervals for the pairwise differences are $\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot j'} \pm Ts(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot j'\cdot}), j \neq j'$, i.e.,

$$\begin{aligned}
\mu_{\cdot 1} - \mu_{\cdot 2} &: \bar{Y}_{\cdot 1\cdot} - \bar{Y}_{\cdot 2\cdot} \pm (3.5640)(6.3647), \text{ i.e., } (-23.150, 22.217), \\
\mu_{\cdot 1} - \mu_{\cdot 3} &: \bar{Y}_{\cdot 1\cdot} - \bar{Y}_{\cdot 3\cdot} \pm (3.5640)(6.3647), \text{ i.e., } (-21.284, 24.084), \\
\mu_{\cdot 2} - \mu_{\cdot 3} &: \bar{Y}_{\cdot 2\cdot} - \bar{Y}_{\cdot 3\cdot} \pm (3.5640)(6.3647), \text{ i.e., } (-20.817, 24.550).
\end{aligned}$$

All the intervals contain zero. Thus there is no evidence that means for the different disk drive makes are different.
(f) Estimate of $\mu_{\cdot 1}$ is $\bar{Y}_{\cdot 1\cdot}$. We know that

$$\begin{aligned}
E(\bar{Y}_{\cdot 1\cdot}) &= \mu_{\cdot 1}, \text{ and} \\
Var(\bar{Y}_{\cdot 1\cdot}) &= (1/a)\sigma^2_\alpha + (1/a)(1 - 1/b)\sigma^2_{\alpha\beta} + \sigma^2/(an) \\
&= \frac{b-1}{nab}(n\sigma^2_{\alpha\beta} + \sigma^2) + \frac{1}{nab}(nb\sigma^2_\alpha + \sigma^2) \\
&= c_1 E(MSAB) + c_2 E(MSA), \text{ where} \\
c_1 &= \frac{b-1}{nab} = 2/45, c_2 = \frac{1}{nab} = 1/45.
\end{aligned}$$

Thus an estimate of $Var(\bar{Y}_{\cdot 1\cdot})$ is

$$\begin{aligned}
s^2(\bar{Y}_{\cdot 1\cdot}) &= c_1 MSAB + c_2 MSA \\
&= (2/45)(303.8222) + (1/45)(12.2889) = 13.77630.
\end{aligned}$$

If we write $L = c_1E(MSAB) + c_2E(MSA)$ and $\hat{L} = c_1MSAB + c_2MSA = s^2(\bar{Y}_{\cdot 1\cdot})$, then, noting that $MSAB$ and MSA are independent, we can cay that $(df)\hat{L}/L \overset{approx}{\sim} \chi^2_{df}$ where

$$df = \frac{\hat{L}^2}{\frac{(c_1MSAB)^2}{df(MSAB)} + \frac{(c_2MSA)^2}{df(MSA)}} = \frac{(13.77630)^2}{\frac{((2/45)(303.8222))^2}{4} + \frac{((1/45)(12.2889))^2}{2}} = 4.16.$$

Even though interpolation can be used, we will take the $df = 4$.

Thus an approximate 99% confidence interval for $\mu_{.1}$ is

$$\bar{Y}_{.1.} \pm t(0.99; df)s(\bar{Y}_{.1.}), \text{ i.e., } \bar{Y}_{.1.} \pm t(0.995; 4)s(\bar{Y}_{.1.}), \text{i.e.,}$$
$$56.13333 \pm (4.6041)\sqrt{13.77630}, \text{ i.e., } (39.045, 73.222).$$

(g) Estimate of σ_α^2 is

$$s_\alpha^2 = \frac{MSA - MSE}{nb} = \frac{12.2889 - 52.0111}{(5)(3)} = -2.64815.$$

Since σ_α^2 cannot be negative, its estimate is $\max(s_\alpha^2, 0) = 0$.

In order to use the MLS procedure, we will need to find quite a few constants. We may write $s_\alpha^2 = c_1 MSA + c_2 MSE$, where $c_1 = 1/15$, $c_2 = -1/15$, $df_1 = df(MSA) = 2, df_2 = df(MSE) = 36$.

Approximate 99% confidence interval for σ_α^2 is $(s_\alpha^2 - H_L, s_\alpha^2 + H_U)$, where the constants H_L and H_U are obtained after a series of complicated formulas.

$$\begin{aligned}
F_1 &= F(0.995; df_1, \infty) = F(0.995; 2, \infty) = 5.2983, \\
F_2 &= F(0.995; df_2, \infty) = F(0.995; 36, \infty) = 1.7106, \\
F_3 &= F(0.995; \infty, df_1) = F(0.995; \infty, 2) = 199.4916, \\
F_4 &= F(0.995; \infty, df_2) = F(0.995; \infty, 36) = 2.0127, \\
F_5 &= F(0.995; df_1 df_2) = F(0.995; 2, 36) = 6.1606, \\
F_6 &= F(0.995; df_2, df_1) = F(0.995; 36, 2) = 199.4718, \\
G_1 &= 1 - 1/F_1 = 0.8113, \\
G_2 &= 1 - 1/F_2 = 0.4154, \\
G_3 &= \frac{(F_5 - 1)^2 - (G_1 F_5)^2 - (F_4 - 1)^2}{F_5} = 0.1015, \\
G_4 &= F_6 \left[\left(\frac{F_6 - 1}{F_6} \right)^2 - \left(\frac{F_3 - 1}{F_3} \right)^2 - G_2^2 \right] = -34.4205, \\
H_L &= \left\{ [G_1 c_1 MS_1]^2 + [(F_4 - 1)c_2 MS_2]^2 - G_3 c_1 c_2 MS_1 MS_2 \right\}^{1/2} = 3.6139, \\
H_U &= \left\{ [(F_3 - 1)c_1 MS_1]^2 + (G_2 c_2 MS_2)^2 - G_4 c_1 c_2 MS_1 MS_2 \right\}^{1/2} = 162.3217.
\end{aligned}$$

Thus $(s_\alpha^2 - H_L, s_\alpha^2 + H_U) = (-6.2621, 159.6735)$. Since σ_α^2 is nonnegative, the 99% confidence interval is $[0, 159.67)$.

25.30. The randomized block design when the treatment effects are random is given by

$$Y_{ij} = \mu_{..} + \rho_i + \tau_j + \varepsilon_{ij}, i = 1, \ldots, n_b, j = 1, \ldots, r,$$

where $\mu_{..}$ is the overall mean, $\{\rho_i\}$ are the block effects with $\sum \rho_i = 0$, $\{\tau_j\}$ are iid $N(0, \sigma_1^2)$, $\{\varepsilon_{ij}\}$ are iid $N(0, \sigma^2)$, and $\{\tau_j\}$ and $\{\varepsilon_{ij}\}$ are independent. For this model

$$Var(Y_{ij}) = Var(\tau_j) + Var(\varepsilon_{ij}) = \sigma_1^2 + \sigma^2.$$

Since $\bar{Y}_{.j} = \mu_{..} + \tau_j + \bar{\varepsilon}_{.j}$, we have

$$Var(\bar{Y}_{.j}) = Var(\tau_j) + Var(\bar{\varepsilon}_{.j}) = \sigma_1^2 + \sigma^2/n_b.$$

25.34. Here the blocks are random and the treatment effects are fixed. Thus we have for $j \neq j'$,

$$\begin{aligned} Cov(Y_{ij}, Y_{ij'}) &= Cov(\mu_{..} + \rho_i + \tau_j + \varepsilon_{ij}, \mu_{..} + \rho_i + \tau_{j'} + \varepsilon_{ij'}) \\ &= Cov(\rho_i + \varepsilon_{ij}, \rho_i + \varepsilon_{ij'}) = Cov(\rho_i, \rho_i) = \sigma_\rho^2, \end{aligned}$$

using the facts that $\{\rho_i\}$ and $\{\varepsilon_{ij}\}$ are independent, and ε_{ij} and $\varepsilon_{ij'}$ are independent.

25.30. A confidence interval for σ_μ^2/σ^2 with confidence coefficient $1-\alpha$ is given by (L, U). Thus $P(L \leq \sigma_\mu^2/\sigma^2 \leq U) = 1-\alpha$. Hence

$$\begin{aligned} P(1/U &\leq \sigma^2/\sigma_\mu^2 \leq 1/L) \\ &= P(1/U + 1 \leq \sigma^2/\sigma_\mu^2 + 1 \leq 1/L + 1) \\ &= P\left[(1+U)/U \leq (\sigma_\mu^2 + \sigma^2)/\sigma_\mu^2 \leq (1+L)/L\right] \\ &= P\left[L/(1+L) \leq \sigma_\mu^2/(\sigma_\mu^2 + \sigma^2) \leq U/(1+U)\right] \\ &= P\left[L^* \leq \sigma_\mu^2/(\sigma_\mu^2 + \sigma^2) \leq U^*\right]. \end{aligned}$$