Practice Final Exam

STA 131B
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University of California, Davis

Exam Rules: This exam is closed book and closed notes. Use of calculators, cell phones or other communication devices is not allowed. You must show all of your work to receive credit.

Name : ______________________________

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1. Let $X_1, \ldots, X_{25}$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. Suppose you observe $\sum_{i=1}^{25} X_i = 100$ and $\sum_{i=1}^{25} X_i^2 = 900$.

   a) Derive a 99% confidence interval for μ with the shortest length and interpret the meaning of this confidence interval.

   Answer. $\bar{X} = 4$, $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1}(\sum_{i=1}^n X_i^2 - n\bar{X}^2) = \frac{900-25\cdot 4^2}{24} = 20.83$.

   Since $\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \sim t_{n-1}$, a 99% confidence interval for μ with the shortest length is

   $$\bar{X} \pm T_{n-1}^{-1}(0.995)\hat{\sigma}/\sqrt{n}$$

   hereafter $T_m^{-1}(\alpha)$ is the α quantile of the t distribution with m degrees of freedom.
   That is, $4 \pm T_{24}^{-1}(0.995) \cdot 4.56/\sqrt{25} = 4 \pm (2.797)(4.56)/5 = (1.449, 6.551)$.
   It means that (1.449, 6.551) covers μ with 99% confidence.

   b) Derive a 99% confidence interval for σ^2.

   Answer. Since $\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$, a 99% confidence interval for σ^2 can be derived from

   $$\begin{aligned} 0.99 &= P\left(\chi^2_{24}(0.005) \le \frac{\sum_{i=1}^{25}(X_i - \bar{X})^2}{\sigma^2} \le \chi^2_{24}(0.995)\right) \\ &= P\left(\frac{\sum_{i=1}^{25}(X_i - \bar{X})^2}{\chi^2_{24}(0.995)} \le \sigma^2 \le \frac{\sum_{i=1}^{25}(X_i - \bar{X})^2}{\chi^2_{24}(0.005)}\right) \end{aligned}$$

   hereafter $\chi^2_m(\alpha)$ is the α quantile of the χ^2 distribution with m degrees of freedom.
   It follows that a 99% confidence interval for σ^2 is

   $$\left(\frac{\sum_{i=1}^{25}(X_i - \bar{X})^2}{\chi^2_{24}(0.995)}, \frac{\sum_{i=1}^{25}(X_i - \bar{X})^2}{\chi^2_{24}(0.005)}\right) = \left(\frac{500}{45.56}, \frac{500}{9.886}\right) = (10.97, 50.58).$$

   c) Test at the 5% level the null hypothesis $H_0 : \sigma = 1$ against the alternative hypothesis $H_1 : \sigma < 1$.

Answer. A 95% confidence upper limit for σ can be derived from

$$0.95 = P\left(\frac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\sigma^2} \geq \chi^2_{24}(0.05)\right) = P\left(\frac{\sqrt{\sum_{i=1}^{25}(X_i-\bar{X})^2}}{\sigma} \geq \sqrt{\chi^2_{24}(0.05)}\right) = P\left(\sqrt{\frac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\chi^2_{24}(0.05)}} \geq \sigma\right)$$

Thus, the 95% confidence upper limit for σ is $\sqrt{\frac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\chi^2_{24}(0.05)}} = \sqrt{\frac{500}{13.85}} = 6.01$. And the one-sided confidence interval is $(-\infty, 6.01)$. Since this interval covers 1, we do not reject H_0.

d) Test at the 5% level the null hypothesis $H_0: \sigma = 1$ against the alternative hypothesis $H_1: \sigma \neq 1$.
Answer. By the similar argument as in (b), a 95% confidence interval for σ is

$$\left(\sqrt{\frac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\chi^2_{24}(0.975)}}, \sqrt{\frac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\chi^2_{24}(0.025)}}\right) = \left(\sqrt{\frac{500}{39.36}}, \sqrt{\frac{500}{12.40}}\right) = (3.56, 6.35)\,.$$

Since this interval does not cover 1, we reject H_0.

e) True or false:

i) There is a 99% chance that the interval you found in (a) covers the true mean.
Answer. False. For a found confidence interval, the probability that it covers the true mean is either 0 or 1.

ii) The confidence interval in (b) is unique.
Answer. False. As long as the confidence interval (a, b) satisfies $P(a < \sigma^2 < b) = 0.99$.

iii) We are 99% confident that the interval in (a) and (b) together covers both true parameters μ and σ.
Answer. False. Using the inequality that $P(A \cap B) \geq P(A) + P(B) - 1$, we have

$$\begin{aligned} & P\left(\bar{X} - T_{24}^{-1}(0.995)\frac{\hat{\sigma}}{\sqrt{25}} \leq \mu \leq \bar{X} + T_{24}^{-1}(0.995)\frac{\hat{\sigma}}{\sqrt{25}}, \sqrt{\frac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\chi^2_{24}(0.995)}} \leq \sigma \leq \sqrt{\frac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\chi^2_{24}(0.005)}}\right) \\ \geq\ & 0.99 + 0.99 - 1 = 0.98. \end{aligned}$$

f) Derive jointly sufficient statistics for μ and σ.
Answer. The joint p.d.f. is

$$f_n(\mathbf{x}|\mu,\sigma) = \frac{1}{(2\pi)^{n/2}\sigma^n}\exp\left\{-\frac{\sum_{i=1}^n(x_i-\mu)^2}{2\sigma^2}\right\} = \frac{1}{(2\pi)^{n/2}\sigma^n}\exp\left\{-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2} + \frac{\mu\sum_{i=1}^n x_i}{\sigma^2} - \frac{n\mu^2}{2\sigma^2}\right\}$$

By the factorization criterion, $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ are jointly sufficient statistics.

2. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the exponential distribution with parameter θ. Suppose we wish to test the hypotheses

$$H_0: \theta \geq \theta_0$$
$$H_1: \theta < \theta_0.$$

Let $Y = \sum_{i=1}^n X_i$ and let δ_c be the test that rejects H_0 if $Y \geq c$.

a) Show that the power function is an decreasing function of θ.
Answer. Note that $2\theta Y$ has the χ^2 distribution with $2n$ degrees of freedom. The power function

$$\pi(\theta|\delta) = P(Y \geq c) = P(2\theta Y \geq 2c\theta) = 1 - G(2c\theta)$$

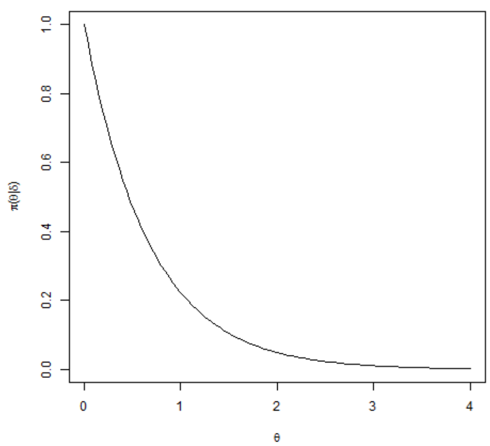
where $G(\cdot)$ is the c.d.f. of the χ^2 distribution with $2n$ degrees of freedom.
Since G is a c.d.f, $G(2c\theta)$ is an increasing function in θ. Therefore, $1 - G(c\theta)$, namely, $\pi(\theta|\delta)$ is an decreasing function of θ.

b) Find c to make δ_c have size .05.
Answer. Using the conclusion in (a), we have

$$0.05 = \sup_{\theta \geq \theta_0} \pi(\theta|\delta) = \pi(\theta_0|\delta) = 1 - G(2c\theta_0).$$

It follows that $c = \chi^2_{2n}(0.95)/(2\theta_0)$.

c) Let $\theta_0 = 2, n = 1$ and suppose the test has size .05. Find the precise form of the test δ_c and sketch its power function.
Answer. $c = \chi^2_2(0.95)/(2 \times 2) = 1.50$. That is, we reject H_0 if $\sum_{i=1}^n X_i \geq 1.5$.
And the power function $\pi(\theta|\delta) = e^{-1.5\theta}$ with the graph below.



3. Let X be a random variable with a Poisson distribution for which the mean λ is unknown.

a) Show that the only unbiased estimator of $e^{-\lambda}$ based on a sample of size 1 is:

$$\delta(X) = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Answer. The unbiased estimator $\delta(X)$ satisfy $E(\delta(X)) = e^{-\lambda}$ for any λ. That is,

$$\sum_{x=0}^{\infty} \delta(x) \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \quad \Rightarrow \quad \sum_{x=0}^{\infty} \frac{\delta(x)}{x!} \lambda^x = 1$$

The desired δ is obtained by comparing the coefficients of polynomial of λ on both sides. The uniqueness of the estimator is also justified by the same argument.

b) Derive an unbiased estimator of $e^{-\lambda}$ based on a random sample $X_1, \ldots, X_n$.
Answer. An unbiased estimator of $e^{-\lambda}$ is $\frac{1}{n} \sum_{i=1}^n \delta(X_i)$ because

$$E\left(\frac{1}{n} \sum_{i=1}^n \delta(X_i)\right) = E(\delta(X_1)) = e^{-\lambda}.$$

c) Is the estimator in (b) admissible for the squared error loss function? If not show how to improve it (you need not derive the improved estimator).

Answer. The joint p.d.f. is

$$f_n(\mathbf{x}|\lambda) = e^{n\lambda}\lambda^{\sum_{i=1}^n x_i}\prod_{i=1}^n \frac{1}{x_i!}$$

It follows from the factorization criterion that $\sum_{i=1}^n X_i$ is a sufficient statistic.
Since $\frac{1}{n}\sum_{i=1}^n \delta(X_i) = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i=0\}}$ is not a function of $\sum_{i=1}^n X_i$, it is inadmissible. And it can be improved by $E[\frac{1}{n}\sum_{i=1}^n \delta(X_i)|\sum_{i=1}^n X_i] = E[\delta(X_1)|\sum_{i=1}^n X_i] = E[1_{\{X_1=0\}}|\sum_{i=1}^n X_i]$.

d) Is the estimator in (b) efficient for $e^{-\lambda}$? Prove or disprove.

Answer. No.

$$\begin{aligned}
\text{var}\left[\frac{1}{n}\sum_{i=1}^n 1_{\{X_i=0\}}\right] &= \frac{1}{n}\text{var}[1_{\{X_1=0\}}] = \frac{1}{n}\left[E(1^2_{\{X_1=0\}}) - E(1_{\{X_1=0\}})^2\right] \\
&= \frac{1}{n}\left[P(X_1=0) - (P(X_1=0))^2\right] = \frac{1}{n}(e^{-\lambda} - e^{-2\lambda}).
\end{aligned}$$

The Fisher information is

$$I(\lambda) = -E\left[\frac{d^2\log f(X|\lambda)}{d\lambda^2}\right] = -E\left[\frac{d^2}{d\lambda^2}(X\log\lambda - \lambda - \log(X!))\right] = -E\left[-\frac{X}{\lambda^2}\right] = \frac{1}{\lambda}.$$

So the C-R lower bound is $\dfrac{[\frac{d}{d\lambda}(e^{-\lambda})]^2}{nI(\lambda)} = \dfrac{e^{-2\lambda}\lambda}{n}$.

Using the inequality that $e^\lambda > 1+\lambda$ for any λ, we have

$$\frac{1}{n}(e^{-\lambda} - e^{-2\lambda}) = \frac{e^{-2\lambda}(e^\lambda - 1)}{n} > \frac{e^{-2\lambda}\lambda}{n}.$$

Thus, the C-R lower bound is not achieved. Namely, $\frac{1}{n}\sum_{i=1}^n \delta(X_i)$ is not efficient.

e) Derive the UMVUE of λ using a random sample of size n.

Answer. The log-likelihood function

$$\log f_n(\mathbf{x}|\lambda) = \log\lambda\sum_{i=1}^n x_i - n\lambda - \sum_{i=1}^n \log(x_i!)$$

And

$$\frac{d}{d\lambda}(\log f_n(\mathbf{x}|\lambda)) = \frac{\sum_{i=1}^n x_i}{\lambda} - n = \frac{n\bar{x}_n}{\lambda} - n$$

It implies that

$$\bar{x}_n = \frac{\lambda}{n}\left[\frac{d}{d\lambda}(\log f_n(\mathbf{x}|\lambda))\right] + \lambda.$$

So $\bar{X}_n$ is efficient for $E(\bar{X}_n) = \lambda$. Thus, it is the UMVUE of λ.

4. Suppose X_1 and X_2 are independent standard normal random variables. Let

$$\begin{aligned}
Y_1 &= X_1 + 2X_2 \\
Y_2 &= 2X_1 - X_2.
\end{aligned}$$

a) Prove that Y_1 and Y_2 are i.i.d. and find their distributions.
Answer. Let $Z_i = Y_i/\sqrt{5}$ for $i = 1, 2$. Then

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Since the matrix A of the transformation is orthogonal, it follows that Z_1 and Z_2 are i.i.d. and have the standard normal distribution. Thus, Y_1 and Y_2 are i.i.d. and have the normal distribution with mean 0 and variance 5.

b) Let X_3 be another standard normal random variable that is independent of X_1 and X_2. Derive the distribution of

$$\frac{\overline{X}_2}{\sqrt{(X_1 - X_2)^2 + 2X_3^2}},$$

where $\overline{X}_2$ is the average of X_1 and X_2.
Answer. Note that $(X_1 - X_2)^2 = 2[(X_1 - \overline{X}_2)^2 + (X_2 - \overline{X}_2)^2]$, so $\overline{X}_2$ and $(X_1 - X_2)^2$ are independent.
Clearly,

$$\frac{\overline{X}_2}{\sqrt{(X_1 - X_2)^2 + 2X_3^2}} = \frac{\sqrt{2}\bar{X}_2}{2\sqrt{(X_1 - \overline{X}_2)^2 + (X_2 - \overline{X}_2)^2 + X_3^2}}$$

It is easy to know that $\sqrt{2}\bar{X}_2 \sim N(0, 1)$ and $(X_1 - \overline{X}_2)^2 + (X_2 - \overline{X}_2)^2 \sim \chi_1^2$. Moreover, $X_3^2 \sim \chi_1^2$ and is independent of $\overline{X}_2$ and $(X_1 - \overline{X}_2)^2 + (X_2 - \overline{X}_2)^2$. So $(X_1 - \overline{X}_2)^2 + (X_2 - \overline{X}_2)^2 + X_3^2 \sim \chi_2^2$ and is independent of $\overline{X}_2$.
It implies that

$$\frac{\sqrt{2}\bar{X}_2}{\sqrt{((X_1 - \overline{X}_2)^2 + (X_2 - \overline{X}_2)^2 + X_3^2)/2}} \sim t_2$$

Thus, $\frac{\overline{X}_2}{\sqrt{(X_1 - X_2)^2 + 2X_3^2}} \sim \frac{1}{2\sqrt{2}} \cdot t_2$.

5. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $[0, \theta]$.

a) What is the parameter space?
Answer. The parameter space $\Omega = \{\theta : \theta > 0\}$.

b) Find the MLE of θ.
Answer. The likelihood function

$$f_n(\mathbf{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \frac{1}{\theta^n} 1_{\{x_{(n)} \le \theta\}} 1_{\{x_{(1)} \ge 0\}}.$$

where $x_{(1)} = \min\{x_1, \ldots, x_n\}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\}$.
Since $\frac{1}{\theta^n}$ is decreasing in θ, $\hat{\theta} = x_{(n)}$ is the maximizer of the likelihood. Namely, $\hat{\theta} = x_{(n)}$ is the MLE of θ.

c) Find a method of moment estimator for θ.
Answer. Note that $E(X_1) = \theta/2$, so a method of moment estimator for θ is $2\bar{X}_n$.

d) Are either of the estimators in (b) or (c) minimal sufficient? Prove or disprove.
Answer. By the factorization criterion, $X_{(n)}$ is sufficient. It is also the MLE of θ. Thus, it is minimal sufficient. $\bar{X}_n$ is not minimal sufficient as it is even not sufficient nor a function of $X_{(n)}$.

e) Are $(X_{(1)}, X_{(n)})$ jointly sufficient statistics for θ? Are they minimal sufficient? Prove or disprove.
Answer. $(X_{(1)}, X_{(n)})$ are jointly sufficient statistics as $X_{(n)}$ can be written as a function of them. But they are not minimal sufficient, because $X_{(n)}$ is already sufficient, adding anything else makes it a waste.

Assume in parts (f) and (g) that θ has a prior density $h(\theta) = \theta e^{-\theta}$, for $\theta > 0$.

f) Find the Bayes estimator of θ for the squared error loss function. Just provide an expression but do not work out the final answer.
Answer. The posterior p.d.f. of θ

$$\xi(\theta|\mathbf{x}) \propto h(\theta) f_n(\mathbf{x}|\theta) \propto \frac{e^{-\theta}}{\theta^{n-1}} 1_{\{\theta \geq x_{(n)}\}}$$

So the posterior p.d.f. of θ is

$$\xi(\theta|\mathbf{x}) = \frac{e^{-\theta}/\theta^{n-1}}{\int_{x_{(n)}}^{\infty} e^{-\theta}/\theta^{n-1} d\theta} \quad \text{for} \quad \theta \geq x_{(n)}$$

The Bayes estimator is the posterior mean $E(\theta|\mathbf{x})$, that is,

$$E(\theta|\mathbf{x}) = \frac{\int_{x_{(n)}}^{\infty} \theta e^{-\theta}/\theta^{n-1} d\theta}{\int_{x_{(n)}}^{\infty} e^{-\theta}/\theta^{n-1} d\theta} = \frac{\int_{x_{(n)}}^{\infty} e^{-\theta}/\theta^{n-2} d\theta}{\int_{x_{(n)}}^{\infty} e^{-\theta}/\theta^{n-1} d\theta}$$

g) Derive a 95% Bayes confidence interval for θ. Identify the upper and lower confidence bounds clearly.
Answer. Take the 2.5% quantile of the posterior distribution as the lower bound $A(\mathbf{x})$ and the 97.5% quantile as the upper bound $B(\mathbf{x})$. Then it satisfies $P(A(\mathbf{x}) \leq \theta \leq B(\mathbf{x})|\mathbf{x}) = 0.95$.

6. Let $X_{(1)}$ and $X_{(n)}$ be the smallest and largest order statistics corresponding to a random sample $X_1, \ldots, X_n$ from an exponential distribution with rate λ.

a) Find the expected value of $X_{(1)}$ and use this to construct an unbiased estimator of $\frac{1}{\lambda}$.
Answer. The c.d.f. of $X_{(1)}$ is

$$\begin{aligned} F(x) &= P(X_{(1)} \leq x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x, \ldots, X_n > x) = 1 - \prod_{i=1}^{n} P(X_i > x) \\ &= 1 - (e^{-\lambda x})^n = 1 - e^{-n\lambda x} \end{aligned}$$

So the p.d.f. of $X_{(1)}$ is

$$f(x) = \frac{dF(x)}{dx} = n\lambda e^{-n\lambda x}$$

It follows that $E(X_{(1)}) = \int_0^{\infty} x \cdot n\lambda e^{-n\lambda x} dx = \frac{1}{n\lambda}$. Then $E(nX_{(1)}) = \frac{1}{\lambda}$.
Thus, $nX_{(1)}$ is an unbiased estimator of $\frac{1}{\lambda}$.

b) Find the MLE of the mean.
Answer. The log-likelihood function is

$$L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i$$

Letting $\frac{dL(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$, we have $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}_n}$. Moreover, $\frac{d^2 L(\lambda)}{d\lambda^2} = -\frac{n}{\lambda^2} < 0$, so $\hat{\lambda}$ is the MLE of λ.
The mean of the distribution is $1/\lambda$. By the invariance property, the MLE of $1/\lambda$ is $1/\hat{\lambda} = \bar{X}_n$.

c) Is the estimator in (b) an efficient estimator of the mean? Explain why.
Answer. The Fisher information is

$$I(\lambda) = -E\left[\frac{d^2}{d\lambda^2}(\log\lambda - \lambda X)\right] = \frac{1}{\lambda^2}.$$

So the C-R lower bound is

$$\frac{[\frac{d}{d\lambda}(\frac{1}{\lambda})]^2}{nI(\lambda)} = \frac{1}{n\lambda^2}$$

Since $\mathrm{var}(\bar{X}_n) = \frac{1}{n}\mathrm{var}(X_1) = \frac{1}{n\lambda^2}$, the lower bound is achieved. Thus, $\bar{X}_n$ is efficient.

d) Show that $e^{\overline{X}_n}$ is a sufficient statistic for λ.
Answer. It is a sufficient statistic because $\overline{X}_n$ is sufficient by the factorization criterion and $g(x) = e^x$ is a one-to-one function.

7. Suppose $X_1, \ldots, X_n$ are a random sample from the normal distribution with unknown mean μ and known variance σ^2. Consider the likelihood ratio test for the hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0.$$

a) Show that the likelihood ratio test for these hypotheses rejects H_0 when $\overline{X} \geq c$ for some constant c.
Answer. The likelihood function $f_n(\mathbf{x}|\mu) = \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left\{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}\right\}$.
Under H_0, $f_n(\mathbf{x}|\mu_0) = \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left\{-\frac{\sum_{i=1}^n (x_i-\mu_0)^2}{2\sigma^2}\right\}$.
For $\mu \geq \mu_0$, using the equality that $\sum_{i=1}^n (x_i-\mu)^2 = \sum_{i=1}^n (x_i-\bar{x})^2 + n(\bar{x}-\mu)^2$, we obtain the MLE of μ

$$\hat{\mu} = \begin{cases} \bar{x} & \text{if } \bar{x} \geq \mu_0 \\ \mu_0 & \text{if } \bar{x} < \mu_0 \end{cases}$$

It implies that

$$\sup_{\mu\geq\mu_0} f_n(\mathbf{x}|\mu) = \begin{cases} \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left\{-\frac{\sum_{i=1}^n (x_i-\bar{x})^2}{2\sigma^2}\right\} & \text{if } \bar{x} \geq \mu_0 \\ \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left\{-\frac{\sum_{i=1}^n (x_i-\mu_0)^2}{2\sigma^2}\right\} & \text{if } \bar{x} < \mu_0 \end{cases}$$

It follows that

$$\Lambda(\mathbf{X}) = \frac{\sup_{\mu=\mu_0} f_n(\mathbf{X}|\theta)}{\sup_{\mu\geq\mu_0} f_n(\mathbf{X}|\theta)} = \begin{cases} 1 & \text{if } \bar{X} < \mu_0 \\ \exp\{-\frac{n(\bar{X}-\mu_0)^2}{2\sigma^2}\} < 1 & \text{if } \bar{X} \geq \mu_0 \end{cases}$$

We reject $H_0 \Leftrightarrow \Lambda(\mathbf{X})$ is small, e.g. $\Lambda(\mathbf{X}) \leq D$.
If $D = 1$, it indicates that we always reject H_0, so D has to be less than 1.

$$\begin{aligned} \text{This means that we reject} H_0 \quad &\Leftrightarrow \quad \Lambda(\mathbf{X}) \leq D < 1 \\ &\Leftrightarrow \quad (\bar{X}-\mu_0)^2 \geq D^* \text{ for some } D^* > 0 \ \ \& \ \ \bar{X} \geq \mu_0 \\ &\Leftrightarrow \quad |\bar{X}-\mu_0| \geq \sqrt{D^*} \ \ \& \ \ \bar{X} \geq \mu_0 \\ &\Leftrightarrow \quad \bar{X}-\mu_0 \geq \sqrt{D^*} (\text{The case } \bar{X}-\mu_0 \leq -\sqrt{D^*} \text{ is excluded because } \bar{X} \geq \mu_0) \\ &\Leftrightarrow \quad \bar{X} \geq \mu_0 - \sqrt{D^*} \end{aligned}$$

The desired result is obtained by taking $c = \mu_0 - \sqrt{D^*}$.

b) If the test is to be conducted at level .02, find c.

Answer. Under H_0, we have

$$0.02 = P(\bar{X} \geq c) = P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{c - \mu_0}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(\frac{c - \mu_0}{\sigma/\sqrt{n}}\right).$$

It follows that $c = \mu_0 + \Phi^{-1}(0.98)\sigma/\sqrt{n}$.