Practice Midterm II

STA 131B
Winter 2016
University of California, Davis

Exam Rules: This exam is closed book and closed notes. Use of calculators, cell phones or other communication devices (including smartwatch) is not allowed. You must show all of your work to receive credit. You will have 50 minutes to complete the exam.

Note: The practice exam is longer than the actual exam so you will have more to practice.

Name : ______________________

ID : ______________________

Signature : ______________________

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x|\theta_1, \theta_2) = \frac{1}{\theta_2} e^{-(x-\theta_1)/\theta_2},$$

for $x \geq \theta_1$, $-\infty < \theta_1 < \infty$, and $\theta_2 > 0$.

a) Find jointly sufficient statistics (T_1, T_2) where θ_1 and θ_2 are both unknown.

b) If θ_2 is known, find a sufficient statistic for θ_1.

c) Is the M.L.E. for θ_1 minimal sufficient (assuming that θ_2 is known)? Justify your answer.

d) Does the Fisher information for θ_1 exist? Explain briefly.

2. Suppose $X_1, \ldots, X_n$ form a random sample from a distribution with known mean μ and unknown variance σ^2.

   (a) Show that the following estimator is an unbiased estimator of σ^2:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

   (b) Show that the Fisher information for $\theta = \sigma^2$ based on a random sample from $N(\mu, \sigma^2)$ is $\frac{n}{2\sigma^4}$.

   (c) Based on the results in (b), what can you say about the efficiency of the estimator in (a)? Explain clearly but you may use the fact that the fourth central moment $E(X-\mu)^4$ of $N(\mu, \sigma^2)$ is $3\sigma^4$.

3. Suppose we draw a sample $X_1, \ldots, X_n$ of size n from the distribution $N(\mu_1, \sigma_1^2)$ and a sample $Y_1, \ldots, Y_m$ of size m from the distribution $N(\mu_2, \sigma_2^2)$. Assume $\sigma_1^2 = 4\sigma_2^2$ and $\mu_1 = \mu_2 =: \mu$. We aim to estimate $\theta = \mu$ and use the estimator $\theta_\alpha = \alpha \bar{X}_n + (1-\alpha)\bar{Y}_m$, where $\bar{X}$ denotes the sample means.

   (a) Obtain the bias of θ_α for all α.

   (b) Obtain the MSE of θ_α for all α.

   (c) For what value of α is the MSE minimized? What is the value of the MSE at the minimum?

(d) How does this MSE compare to that of the estimator that is obtained when you pool the two samples into one and take the sample average as estimator?

(e) Now assume $\mu_1 = 2\mu_2$, $n = 2m$ and that the target is the parameter $\theta = \frac{\mu_1+\mu_2}{2}$. Redo (b) and (c) under these assumptions, for the estimator θ_α of θ.

4. Suppose $X_1, \ldots, X_n$ form a random sample from a distribution with p.d.f. $f(x|\theta) = \theta x^{\theta-1}$, for $0 < x < 1$ and $\theta > 0$.

   a) Find the M.L.E. of θ.

   b) Is the M.L.E. in (a) minimal sufficient? Justify your answer.

c) Is the sample mean admissible for estimating θ. Explain briefly.

d) Use that fact that $E(\log X_i) = \frac{1}{\theta}$ to find an UMVUE of $\frac{1}{\theta}$.