Handout 6

Three-factor ANOVA

We will deal with three factor models here. Note that once we know how to analyze the two-factor case, this knowledge basically allows us to tackle three or higher order studies, except that notations become increasingly messy. This note is concerned with three-factor studies. Suppose that we have three factors: factor A has a levels, factor B has b levels, factor C has c levels, and there are n observations for each of the abc combinations of the factors. Let Y_{ijkm} be the m^{th} observation when factor A is at level i, factor B is at level j, and factor C is at level k. The cell means model is

$$Y_{ijkm} = \mu_{ijk} + \varepsilon_{ijkm},\ m = 1, ..., n, k = 1, ..., c, j = 1, ..., b, i = 1, ..., a,$$

where ε_{ijkm} are independent $N(0, \sigma^2)$, and μ_{ijk}'s are the means. Note that the total number of observations is equal to $n_T = nabc$. As before we can write this as a factor effects model. Here are some notations

$$\mu_{\cdots} = \sum\sum\sum \mu_{ijk}/(abc),\ \mu_{ij\cdot} = \sum_k \mu_{ijk}/c, \mu_{i\cdot k} = \sum_j \mu_{ijk}/b, \mu_{\cdot jk} = \sum_i \mu_{ijk}/a,$$
$$\mu_{i\cdot\cdot} = \sum_j\sum_k \mu_{ijk}/(bc), \mu_{\cdot j\cdot} = \sum_i\sum_k \mu_{ijk}/(ac), \mu_{\cdot\cdot k} = \sum_i\sum_j \mu_{ijk}/(ab).$$

The main effects and the interactions are

$$\begin{aligned}
\text{Main effect of factor A}: &\ \alpha_i = \mu_{i\cdot\cdot} - \mu_{\cdots},\\
\text{Main effect of factor B}: &\ \beta_j = \mu_{\cdot j\cdot} - \mu_{\cdots},\\
\text{Main effect of factor C}: &\ \gamma_k = \mu_{\cdot\cdot k} - \mu_{\cdots},\\
\text{Interaction between A and B}: &\ (\alpha\beta)_{ij} = \mu_{ij\cdot} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu_{\cdots},\\
\text{Interaction between A and C}: &\ (\alpha\gamma)_{ik} = \mu_{i\cdot k} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu_{\cdots},\\
\text{Interaction between B and C}: &\ (\beta\gamma)_{jk} = \mu_{\cdot jk} - \mu_{\cdot j\cdot} - \mu_{\cdot\cdot k} + \mu_{\cdots},\\
\text{Three factor interaction between A, B and C}: &\ (\alpha\beta\gamma)_{ijk}\\
&= \mu_{ijk} - \mu_{ij\cdot} - \mu_{i\cdot k} - \mu_{\cdot jk} + \mu_{i\cdot\cdot} + \mu_{\cdot j\cdot} + \mu_{\cdot\cdot k} - \mu_{\cdots}\\
&= \mu_{ijk} - \{\mu_{\cdots} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}\}.
\end{aligned}$$

One can show that

$$\mu_{ijk} = \mu_{\cdots} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk}.$$

So the factor effects model is

$$Y_{ijkm} = \mu_{\cdots} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkm}.$$

The mains effects and the interactions satisfy the following constrains

$$\begin{aligned}
\sum \alpha_i = 0, \sum \beta_j &= 0, \sum \gamma_k = 0, \\
\sum_j (\alpha\beta)_{ij} &= 0 \text{ for all } i, \ \sum_i (\alpha\beta)_{ij} = 0 \text{ for all } j, \\
\sum_k (\alpha\gamma)_{ik} &= 0 \text{ for all } i, \ \sum_i (\alpha\gamma)_{ik} = 0 \text{ for all } k, \\
\sum_k (\beta\gamma)_{ij} &= 0 \text{ for all } j, \ \sum_j (\beta\gamma)_{ij} = 0 \text{ for all } k, \\
\sum_j (\alpha\beta\gamma)_{ijk} &= 0 \text{ for all } i \text{ and } k, \sum_k (\alpha\beta\gamma)_{ijk} = 0 \text{ for all } i \text{ and } j, \text{ and} \\
\sum_i (\alpha\beta\gamma)_{ijk} &= 0 \text{ for all } j \text{ and } k.
\end{aligned}$$

Let us fix some notations

$$\begin{aligned}
\bar{Y}_{\cdots\cdot} &= \sum\sum\sum\sum Y_{ijkm}/(nabc), \bar{Y}_{ijk\cdot} = \sum_m Y_{ijkm}/n, \\
\bar{Y}_{ij\cdot\cdot} &= \sum_k\sum_m Y_{ijkm}/(nc), \bar{Y}_{i\cdot k\cdot} = \sum_j\sum_m Y_{ijkm}/(nb), \bar{Y}_{\cdot jk\cdot} = \sum_i\sum_m Y_{ijkm}/(na), \\
\bar{Y}_{i\cdots} &= \sum_j\sum_k\sum_m Y_{ijkm}/(nbc), \bar{Y}_{\cdot j\cdot\cdot} = \sum_i\sum_k\sum_m Y_{ijkm}/(nac), \bar{Y}_{\cdot\cdot k\cdot} = \sum_i\sum_j\sum_m Y_{ijkm}/(nab).
\end{aligned}$$

Estimates are

$$\begin{aligned}
\hat{\mu}_{\cdots} &= \bar{Y}_{\cdots\cdot}, \hat{\mu}_{ijk} = \bar{Y}_{ijk\cdot}, \hat{\mu}_{ij\cdot} = \bar{Y}_{ij\cdot\cdot}, \hat{\mu}_{i\cdot k} = \bar{Y}_{i\cdot k\cdot}, \hat{\mu}_{\cdot jk} = \bar{Y}_{\cdot jk\cdot}, \\
\hat{\mu}_{i\cdot\cdot} &= \bar{Y}_{i\cdots}, \hat{\mu}_{\cdot j\cdot} = \bar{Y}_{\cdot j\cdot\cdot}, \hat{\mu}_{\cdot\cdot k} = \bar{Y}_{\cdot\cdot k\cdot}, \\
\hat{\alpha}_i &= \hat{\mu}_{i\cdot\cdot} - \hat{\mu}_{\cdots}, \hat{\beta}_j = \hat{\mu}_{\cdot j\cdot} - \hat{\mu}_{\cdots}, \hat{\gamma}_k = \hat{\mu}_{\cdot\cdot k} - \hat{\mu}_{\cdots}, \\
\widehat{(\alpha\beta)}_{ij} &= \hat{\mu}_{ij\cdot} - \hat{\mu}_{i\cdot\cdot} - \hat{\mu}_{\cdot j\cdot} + \hat{\mu}_{\cdots}, \widehat{(\alpha\gamma)}_{ik} = \hat{\mu}_{i\cdot k} - \hat{\mu}_{i\cdot\cdot} - \hat{\mu}_{\cdot\cdot k} + \hat{\mu}_{\cdots}, \widehat{(\beta\gamma)}_{jk} = \hat{\mu}_{\cdot jk} - \hat{\mu}_{\cdot j\cdot} - \hat{\mu}_{\cdot\cdot k} + \hat{\mu}_{\cdots}, \\
\widehat{(\alpha\beta\gamma)}_{ijk} &= \hat{\mu}_{ijk} - \hat{\mu}_{ij\cdot} - \hat{\mu}_{i\cdot k} - \hat{\mu}_{\cdot jk} + \hat{\mu}_{i\cdot\cdot} + \hat{\mu}_{\cdot j\cdot} + \hat{\mu}_{\cdot\cdot k} - \hat{\mu}_{\cdots} \\
&= \hat{\mu}_{ijk} - \{\hat{\mu}_{\cdots} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_k + \widehat{(\alpha\beta)}_{ij} + \widehat{(\alpha\gamma)}_{ik} + \widehat{(\beta\gamma)}_{jk}\}.
\end{aligned}$$

The fitted values and the residuals are

$$\hat{Y}_{ijkm} = \hat{\mu}_{ijk},\ e_{ijkm} = Y_{ijkm} - \hat{Y}_{ijkm}.$$

Sums of squares and the mean squares are

$$SSTO = \sum\sum\sum\sum (Y_{ijkm} - \bar{Y}_{\dots})^2,\ df = nabc - 1,$$

$$SSA = \sum\sum\sum\sum \hat{\alpha}_i^2 = nbc \sum_i \hat{\alpha}_i^2,\ df = a - 1,\ MSA = \frac{SSA}{a-1},$$

$$SSB = \sum\sum\sum\sum \hat{\beta}_j^2 = nac \sum_j \hat{\beta}_j^2,\ df = b - 1,\ MSB = \frac{SSB}{b-1},$$

$$SSC = \sum\sum\sum\sum \hat{\gamma}_k^2 = nab \sum_k \hat{\gamma}_k^2,\ df = c - 1,\ MSC = \frac{SSC}{c-1},$$

$$SSAB = \sum\sum\sum\sum \widehat{(\alpha\beta)}_{ij}^2 = nc \sum_i \sum_j \widehat{(\alpha\beta)}_{ij}^2,\ df = (a-1)(b-1),\ MSAB = \frac{SSAB}{(a-1)(b-1)},$$

$$SSAC = \sum\sum\sum\sum \widehat{(\alpha\gamma)}_{ik}^2 = nb \sum_i \sum_k \widehat{(\alpha\gamma)}_{ik}^2,\ df = (a-1)(c-1),\ MSAC = \frac{SSAC}{(a-1)(c-1)},$$

$$SSBC = \sum\sum\sum\sum \widehat{(\beta\gamma)}_{jk}^2 = na \sum_j \sum_k \widehat{(\beta\gamma)}_{jk}^2,\ df = (b-1)(c-1),\ MSBC = \frac{SSBC}{(b-1)(c-1)},$$

$$SSABC = \sum\sum\sum\sum \widehat{(\alpha\beta\gamma)}_{ijk}^2 = n \sum_i \sum_j \sum_k \widehat{(\alpha\beta\gamma)}_{ijk}^2,\ df = (a-1)(b-1)(c-1),$$

$$MSABC = \frac{SSABC}{(a-1)(b-1)(c-1)},$$

$$SSE = \sum\sum\sum\sum (Y_{ijkm} - \hat{\mu}_{ijk})^2,\ df = nabc - abc = (n-1)abc,\ MSE = \frac{SSE}{(n-1)abc}.$$

Note that MSE estimates σ^2.

The following identities hold

$$SSTO = SSA + SSB + SSC + SSAB + SSAC + SSBC + SSABC + SSE,$$

$$df(SSTO) = df(SSA) + df(SSB) + df(SSC) + df(SSAB) + df(SSAC) + df(SSBC) + df(SSABC) + df(SSE).$$

F-tests:

The following table presents the null and alternative hypotheses for various tests, the corresponding F-statistics and the degrees of freedom associated with these F-tests.

Null	Alternative	F-statistic
$\alpha_i = 0$ for all i	not all α_i's are zero	$F^* = \frac{MSA}{MSE}$
$\beta_j = 0$ for all j	not all β_j's are zero	$F^* = \frac{MSB}{MSE}$
$(\alpha\beta)_{ij} = 0$ for all i and j	not all $(\alpha\beta)_{ij}$'s are zero	$F^* = \frac{MSAB}{MSE}$
$(\alpha\gamma)_{ik} = 0$ for all i and k	not all $(\alpha\gamma)_{ij}$'s are zero	$F^* = \frac{MSAC}{MSE}$
$(\beta\gamma)_{jk} = 0$ for all j and k	not all $(\beta\gamma)_{jk}$'s are zero	$F^* = \frac{MSBC}{MSE}$
$(\alpha\beta\gamma)_{jk} = 0$ for all i, j and k	not all $(\alpha\beta\gamma)_{ijk}$'s are zero	$F^* = \frac{MSABC}{MSE}$

Case Hardening:

An experiment involving the case hardening of lightweight shafts machined from bars of an allow was run to study the effects of the amount of a chemical agent added to the alloy in a molten state (factor A), the temperature of the hardening process (factor B), and the time duration of the hardening process (factor C) on the outside hardness of the shaft. All the factors were at two levels (1: low, 2:high), and the number of rods tested for each treatment was $n = 3$. The data on hardness (in Brinell units) follow.

	$k=1$		$k=2$	
	$j=1$	$j=2$	$j=1$	$j=2$
	39.9	53.5	56.0	70.9
$i=1$	32.2	50.7	56.9	73.3
	36.3	52.8	56.6	71.6
	45.2	63.3	69.4	82.9
$i=2$	48.0	65.5	66.6	85.2
	47.5	63.6	68.8	82.3

The following tables give the sample means $\bar{Y}_{ijk\cdot}$'s.

	$k=1$	
	$j=1$	$j=2$
$i=1$	36.13	52.33
$i=2$	46.90	64.13

	$k=2$	
	$j=1$	$j=2$
$i=1$	56.50	71.93
$i=2$	68.27	83.47

ANOVA table

Source	df	SS	MS	F	p-value
A	$a-1=1$	788.91	788.91	234.88	0.000
B	$b-1=1$	1539.20	1539, 20	458.27	0.000
C	$c-1=1$	2440.17	2440.17	726.51	0.000
AB	$(a-1)(b-1)=1$	0.24	0.24	0.07	0.793
AC	$(a-1)(c-1)=1$	0.20	0.20	0.06	0.810
BC	$(b-1)(c-1)=1$	2.94	2.94	0.88	0.363
ABC	$(a-1)(b-1)(c-1)=1$	0.60	0.60	0.18	0.678
Error	$(n-1)abc=16$	53.74	3.35875		
Total	$nabc-1=23$	4826.00			

From this ANOVA table it seems that the all the main effects of all the three factors are present, but all the two-factor and three factor interaction terms can be ignored.

Let us now construct simultaneous confidence intervals dor $D_1 = \alpha_2 - \alpha_1$, Effects Plot for Y.emf

$D_2 = \beta_2 - \beta_1$ and $D_3 = \gamma_2 - \gamma_1$, using Bonferroni method. Note that

$$\begin{aligned}
\hat{D}_1 &= \hat{\mu}_{2..} - \hat{\mu}_{1..} = \bar{Y}_{2..} - \bar{Y}_{1..} = 65.69 - 54.23 = 11.46, \\
\hat{D}_2 &= \hat{\mu}_{.2.} - \hat{\mu}_{.1.} = \bar{Y}_{.2.} - \bar{Y}_{.1.} = 67.97 - 51.95 = 16.02, \\
\hat{D}_3 &= \hat{\mu}_{..2} - \hat{\mu}_{..1} = \bar{Y}_{..2} - \bar{Y}_{..1} = 70.04 - 49.88 = 20.16, \\
s(\hat{D}_1) &= \sqrt{\frac{2}{nbc} MSE} = \sqrt{\frac{2}{(3)(2)(2)}(3.3588)} = 0.7482, \\
s(\hat{D}_2) &= \sqrt{\frac{2}{nac} MSE} = \sqrt{\frac{2}{(3)(2)(2)}(3.3588)} = 0.7482, \\
s(\hat{D}_3) &= \sqrt{\frac{2}{nab} MSE} = \sqrt{\frac{2}{(3)(2)(2)}(3.3588)} = 0.7482.
\end{aligned}$$

From the t-table we have $t(1-(0.5)/(2)(3), (n-1)abc) = t(0.9917, 16) = 2.675$.
So simultaneous 95% confidence intervals are

$$\begin{aligned}
&D_1 : \hat{D}_1 \pm t(0.9917, 16)s(\hat{D}_1), \text{ i.e., } 11.46 \pm (2.675)(0.7482), \text{ i.e., } 11.46 \pm 2.001, \text{ i.e., } (9.46, 13.46), \\
&D_2 : \hat{D}_2 \pm t(0.9917, 16)s(\hat{D}_2), \text{ i.e., } 16.02 \pm (2.675)(0.7482), \text{ i.e., } 16.02 \pm 2.001, \text{ i.e., } (14.02, 18.02), \\
&D_3 : \hat{D}_3 \pm t(0.9917, 16)s(\hat{D}_3), \text{ i.e., } 20.16 \pm (2.675)(0.7482), \text{ i.e., } 20.16 \pm 2.001, \text{ i.e., } (18.16, 22.16).
\end{aligned}$$

This clearly pointed out what we already know from the F-table, i.e., the main effects of the factors A, B and C are significant.

Unbalanced case:

When a three factor model is unbalanced then it is of the form

$$Y_{ijkm} = \mu_{...} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkm},$$

where $m = 1, \ldots, n_{ijk}$, $k = 1, \ldots, c, j = 1, \ldots, b, i = 1, \ldots, a$, and $\{\varepsilon_{ijkm}\}$ are iid $N(0, \sigma^2)$. Here α's, β's, γ's and the interactions sastify the same constraints as in the balanced case. In this case, it is customary to use code the factors so that a regression model can be used as described in STA 206. As a matter of fact computers use coding to cfreate a model of the form $Y = X\beta + \varepsilon$ in order to estimate the parameters and carry out inferece. We will not say more this as these are along the some lines as done in STA 206.