OLG model in DGE

Ch 10.1. Overlapping Generation Models with Idiosyncratic Income Uncertainty

Dynamic General Equilibrium

3/20/2022

Table of Contents

Table of Contents

Demographics

- Born: $s = 1$ (real-age 21)
  Retire: $s = 45$ (real-age 65)
  Max-live: $s = T = 70$ (real-age 90)
- $N_t(s)$: the number of agents of age s at t
  N_t: total population at t
- ϕ_t^s: probability that at t, all agents of age s survive until age $s+1$ where $\phi_t^0 = 1$ and $\phi_t^T = 0$
- n_t: the rate at which the newborn cohort grows in period t

$$N_t(1) = (1 + n_t)N_{t-1}(1)$$

- $\phi_t^s = \phi^s$ and $n_t = n$: constant

Table of Contents

Households

- Each household comprises one (possibly retired) worker
- Maximize expected utility at the beginning of age 1 in period t

$$\max \sum_{s=1}^{J} \beta^{s-1}(\Pi_{j=1}^{s}\phi^{J-1})E_t[u(c_{t+s-1}^{s}, l_{t+s-1}^{s})]$$

- $\beta > 0$: discount factor
- Utility function: consumption c, leisure $1 - l$

$$u(c, 1-l) = \frac{(c^{\gamma}(1-l)^{1-\gamma})^{1-\eta}}{1-\eta}$$

- γ: share of consumption in utility, $1/\eta$: IES

Households

- During working life, $0 \leq l^s \leq l^{max}$ for $s = 1, .., T^w$
  During retirement, $l^s = 0$ for $s = T^w + 1, .., T$
- $\epsilon(s, \theta, e) A_t w_t l_t^s$ = labor income of the s-year-old worker in period t
- A_t: aggregate productivity, grows at the exogenous rate g_A = economic growth rate of per capita income in steady-state
- w_t: wage per efficiency unit, l_t^s: working time
- $\epsilon(s, \theta, e) = \theta e \overline{y}^s$: idiosyncratic productivity
  1. θ : stochastic and follows finite Markov process
     n_θ possible values
     $prob(\theta'|\theta)$: transition probability
     The individual θ's are assumed to be independent across agents and the law of large numbers holds $\rightarrow$ there is no aggregate uncertainty
  2. e: education type (high school/college), $e \in \{e_1, e_2\}$
  3. $\overline{y}^s$: age component, hump-shaped over the life-cycle

Households

- tr_t: All households receive transfers from the government
- τ^l: rate of labor income tax
- τ^P: pays social security contribution proportional to labor income
- pen_t: retired worker receives a pension (independent of individuals' contributions)
- net non-capital income

$$y_t^s = \begin{cases} (1 - \tau_t^l - \tau_t^p)\epsilon(s, \theta, e)A_t w_t l_t^s & s = 1, ..., T^w \\ pen_t & s = T^w + 1, ..., T \end{cases}$$

Households

To express the budget constraint in stationary variables, divide non-stationary individual variables by A_t ($\tilde{y}_t^s = y_t^s / A_t$)

$$(1+\tau_t^c)\tilde{c}_t{}^s =$$
$$\tilde{y}_t^s + (1-\tau_t^k)(r_t-\delta)\tilde{k}_t^s + \tilde{k}_t^s + R_t^b \tilde{b}_t^s + \tilde{tr}_t - (1+g_A)\tilde{k}_{t+1}^{s+1} - (1+g_A)\tilde{b}_{t+1}^{s+1}$$

- k_t^s, b_t^s: capital stock, government bonds of s-year-old agent at the beginning of period t
- Household is born without assets and leave no bequests $k_t^1 = k_t^{T+1} = 0, b_t^1 = b_t^{T+1} = 0$
- Interest income r_t: $R_t^b = 1 + r_t$ on capital and govt. bonds
- labor income tax, capital income tax, consumption tax at the rate of $\tau_t^l, \tau_t^k, \tau_t^c$
- capital depreciation δk_t^s is tax exempt
- $b_t^s \geq 0, k_t^s \geq 0$

Table of Contents

Technology

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

$$\Pi_t = Y_t - r_t K_t - w_t A_t L_t$$

$$w_t = (1-\alpha) K_t^{\alpha}(A_t L_t)^{-\alpha}$$

$$r_t = \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha}$$

Table of Contents

Government and Social Security

- Labor income taxes τ^l, capital income tax τ^k, consumption tax τ^c
- Confiscates all accidental bequests Beq_t and pays aggregate transfers Tr_t
- G_t: total public expenditure
- Pays interest r_t^b on the accumulated debt B_t
- In each period, the government budget is financed by issuing government debt:

$$Tr_t + G_t + r_t^b B_t - Tax_t - Beq_t = B_{t+1} - Bt$$

where

$$Tax_t = \tau_t^l A_t L_t w_t + \tau_t^k (r_t - \delta) K_t + \tau_t^c C_t$$

Ct : Aggregate consumption

Government and Social Security

- Pay-as-you-go pensions to the retirees which it finances with the contributions of the workers
- Pen_t: Aggregate pension payments
- The social security budget is assumed to balance

$$Pen_t = \tau_t^p A_t L_t w_t$$

Table of Contents

Households' Optimization Problem

- Define households assets $a_t^s = k_t^s + b_t^s$
- State vector: $z = (s, \theta, e, a)$, or $z_s = (\theta, e, a)$
- Choice variables: c, l, a'
- $V_t(z_s)$: value function of the s-year-old household in period t

$$V_t(z_s) = max_{c,l,a'}\{u(c,l) + \beta\phi_t^s \sum_{\theta'} prob(\theta'|\theta)V_{t+1}(z_{s+1})\}$$

subject to

$$(1+\tau_t^c)c = y + [1 + (1-\tau_t^k)(r_t - \delta)]a + tr - a', a \geq 0$$

with the terminal condition $V_t(z_{T+1}) = 0$

Households' Optimization Problem

- To formulate this optimization problem in stationary variables, define

$$\tilde{V}_t = \frac{V_t}{A_t^{\gamma(1-\eta)}}, \;\; \tilde{z}_s = (\theta, e, \tilde{a})$$

$$\text{and } \; \tilde{u}(\tilde{c}, 1-l) = \frac{u(c,l)}{A_t^{\gamma(1-\eta)}} = \frac{(\tilde{c}^{\gamma}(1-l)^{1-\gamma})^{1-\eta}}{1-\eta}$$

Households' Optimization Problem

- The Bellman equation can be rewritten in stationary form

$$\tilde{V}_t(\tilde{z}_s) = \max_{\tilde{c},l,\tilde{a}'}\{\tilde{u}(\tilde{c},l) + (1+g_A)^{\gamma(1-\eta)}\beta\phi_t^s \sum_{\theta'} prob(\theta'|\theta)\tilde{V}_{t+1}(\tilde{z}_{s+1})\}$$

with the terminal condition $\tilde{V}_t(\tilde{z}_{T+1}) = 0$

- The budget constraint of the household in stationary variables:

$$(1+\tau_t^c)\tilde{c} = \tilde{y} + [1+(1-\tau_t^k)(r_t-\delta)]\tilde{a} + \tilde{tr} - (1+g_A)\tilde{a}'$$

with

$$\tilde{y} = \begin{cases} (1-\tau_t^l-\tau_t^p)\epsilon(s,\theta,e)w_t l & s=1,...,T^w \\ \tilde{pen}_t & s=T^w+1,...,T \end{cases}$$

Table of Contents

Stationary Equilibrium

- Assume that we have already discretized the state space
- Asset space $\overline{a} \in A = \{\overline{a}_1, \overline{a}_2, .., \overline{a}_{n_a}\}$
- Let $f(s, e, \theta, \overline{a})$: discretized density function associated with the state $\overline{z} = (s, e, \theta, \overline{a})$
- In order to express the equilibrium in terms of stationary variables,

$$\tilde{X}_t \equiv \frac{X_t}{A_t N_t}$$

for the aggregate variables $X \in \{Pen, Tr, G, B, Beq, Tax, Y, K, C\}$
In particular, $\tilde{Tr}_t = \tilde{tr}_t, \overline{L}_t = L_t / N_t$

Stationary Equilibrium

In equilibrium,

$$\tilde{L}_t = \sum_{s=1}^{T^w} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \epsilon(s, \theta_{i_\theta}, e_j,) l(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}) f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})$$

$$\tilde{\Omega}_t = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \tilde{a}_{i_a} f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})$$

$$\Omega_t = B_t + K_t$$

1. Aggregate effective labor supply is equal to the sum of the individual effective labor supplies
2. Aggregate wealth is equal to the sum of individual wealth levels
3. In capital market equilibrium

Stationary Equilibrium

At the beginning of period t+1, the government collects accidental bequests from the s-year old households who do not survive from period t till period t+1

$$Beq_{t+1} = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a}$$
$$(1-\Phi_t^s)[1+(1-\tau_{t+1}^k)(r_{t+1}-\delta)]\overline{a}'(s,\theta_{i_\theta},e_j,\overline{a}_{i_a})f(s,\theta_{i_\theta},e_j,\overline{a}_{i_a})$$

where we have used the condition that the after-tax returns on the two assets are equal

$$r_t^b = (1-\tau_r^k)(r_t-\delta)$$

Stationary Equilibrium

In equilibrium, the goods markets clear:

$$\tilde{Y}_t = \tilde{C}_t + \tilde{G}_t + (1+g_A)(1+n)\tilde{K}_{t+1} - (1-\delta)\tilde{K}_t$$

where aggregate consumption is the sum of individual consumptions:

$$C_t = \sum_{s=1}^{T} \sum_{i_\theta=1}^{n_\theta} \sum_{j=1}^{2} \sum_{i_a=1}^{n_a} \tilde{c}(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a}) f(s, \theta_{i_\theta}, e_j, \tilde{a}_{i_a})$$

Table of Contents

Calibration

Parameter	Value
α	0.35
δ	8.3%
g_A	2.0%
n	0.75%
$1/\eta$	1/2
γ	0.33
β	1.011
$\tau^l + \tau^p$	28%
τ^k	36%
τ^c	5%
G/Y	18%
B/Y	63%
$repl$	35.2%
$\{e_1, e_2\}$	$\{0.57, 1.43\}$

Calibration

- Idiosyncratic productivity shock θ follows a Markov process,

$$\theta' = \rho\theta + \xi \text{ where } \xi \sim N(0, \sigma_\xi)$$

  $\rho = 0.96, \sigma_\xi^2 = 0.045, \sigma_{y_1} = 0.38$ and mean $\overline{y}^1$
- Discretize the state space $\Theta = (\theta_1, .., \theta_{n_\theta})$ using $n_\theta = 5$
- The states are equally spaced and range from $-\sigma_{y_1}$ to σ_{y_1} $\rightarrow \Theta = (-0.7576, -0.3788, 0.0000, 0.3788, 0.7576)$
- Probability $v(\theta) = (0.1783, 0.2010, 0.2413, 0.2010, 0.1783)$ respectively for each $e_i, i = 1, 2$
- Each $e_i, i = 1, 2$ has a share of 50% in each cohort

Calibration

- Transition probabilities are computed using Tauchen's method (Ch 12.2.1 in DGE textbook)

$$Prob(\theta'|\theta) = \begin{pmatrix} 0.7734 & 0.2210 & 0.0056 & 0.0000 & 0.0000 \\ 0.1675 & 0.6268 & 0.2011 & 0.0046 & 0.0000 \\ 0.0037 & 0.1823 & 0.6281 & 0.1823 & 0.0033 \\ 0.0000 & 0.0046 & 0.2011 & 0.6268 & 0.1675 \\ 0.0000 & 0.0000 & 0.0056 & 0.2210 & 0.7734 \end{pmatrix}$$

Table of Contents

Computation - Demographic

- Define the measure of the s-year old in period t

$$\mu_t^s \equiv \frac{N_t(s)}{N_t}, \ \sum_{s=1}^{70} \mu_t^s = 1.0$$

- To compute the stationary measure $\{\mu^s\}_{s=1}^{70}$, initialize the mass of the 1-year old, μ^1 equal to one(mass[0])

$$\mu^{s+1} = \frac{\phi^s}{1+n}\mu^s$$

- The above formula is from dividing both $N_t(1) = (1+n)N_{t-1}(1)$ and $N_t(s) = \phi^{s-1}N_{t-1}(s-1)$ by N_t
- Normalize the measures so that their sum is equal to one

Computation - Labor supply

The first-order condition of the worker respect to his labor supply:

$$\frac{(1-\tau^l-\tau^p)\epsilon(s,\theta,e)w}{1+\tau^c} = \frac{1-\gamma}{\gamma}\frac{\tilde{c}}{1-l}$$

After the substitution of $\tilde{c}$ from the budget constraint of the worker,

$$l = \gamma - \frac{1-\gamma}{(1-\tau^l-\tau^p)\epsilon(s,\theta,e)w}([1+(1-\tau^k)(r-\delta)]\tilde{a} + \tilde{tr} - (1+g_A)\tilde{a}')$$

with

$$0 \leq l \leq 0.60$$

Euler equation residual

- Assessing the accuracy of the optimization:
- To evaluate the accuracy of our policy function approximation, we study the so-called 'Euler residual' which is defined as the percentage deviation of the Euler equation
- For the s-year old worker with wealth level $\tilde{a}$ and productivity type $\epsilon(s, \theta, e)$, the Euler equation residual is defined by:

$$R(\tilde{a}) = 1 - \frac{\tilde{u}_c(\tilde{c}, l)}{\beta \phi^s E\{\tilde{u}_c(\tilde{c}', l')\}}$$

- If $R(\tilde{a} = 0)$, the Euler equation error is zero and we have a perfect fit

Euler equation residual

- Euler equation residual for the s-year old worker with wealth level $\tilde{a}$ and productivity type $\epsilon(s, \theta, e)$:

$$R(\tilde{a}) = 1 - \frac{\tilde{u}_c(\tilde{c}, 1-l)}{\beta(1+r^b)(1+g_A)^{\gamma(1-\eta)-1}\phi^s E\{\tilde{u}_c(\tilde{c}', 1-l')\}}$$

derived from

$$\frac{\partial \tilde{u}(\tilde{c}, 1-l)}{\partial \tilde{c}} = (1+g_A)^{\gamma(1-\eta)-1}\beta(1+r^b)\phi^s E\{\frac{\partial \tilde{u}(\tilde{c}, 1-l)}{\partial \tilde{c}'}\}$$

Aggregate

Compute $\tilde{\Omega}, \tilde{K}, \tilde{L}, \bar{l}, \tau^p, \tau^l, \tilde{tr}$

- For given $\tilde{K}$ and $\tilde{L}$, we can compute $\tilde{Y}$ and, hence, $\tilde{B} = 0.63\tilde{Y}$
- Aggregate wealth $\tilde{Omega}$: sum of $\tilde{a}$ weighted by $f(s, e, \theta, \tilde{a})$
- Capital market equilibrium: $\tilde{K} = \tilde{\Omega} - \tilde{B}$
- Pension contribution rate τ^p: computed with the help of the social security budget

Aggregate

Computation of government transfers:

- government consumption $\tilde{G} = 0.18\tilde{Y}$
- accidental bequests $\tilde{Beq}$
- aggregate consumption $\tilde{C}$
- total taxes: $\tilde{Tax} = \tau^l w\tilde{L} + \tau^k(r-\delta)\tilde{K} + \tau^c\tilde{C}$
- government transfers: With the help of the steady-state government budget constraint,

$$\tilde{tr} = \tilde{Tr} = \tilde{Tax} + [(1+g_A)(1+n) - (1+r^b)]\tilde{B} - \tilde{G}$$

- $\tilde{pen} = repl * w * \bar{l}$
- $\tau^p = \frac{\tilde{Pen}}{w\tilde{L}}$