Solving the Ramsey Model

NumEcon

Jeppe Druedahl

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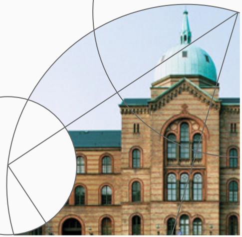


Plan

1. Introduction
2. Model
3. Solution algorithm
4. Example

Introduction

Introduction

- **Subject:** Solve the Ramsey model numerically (using Python)
- **NumEcon module** (under construction)
  1. **Source files:** GitHub.com
  2. **Interactive version:** MyBinder.org
- **Today:**
  1. **Notebook:** course_macro3\Ramsey.ipynb
  2. **Code:** numecon\course_macro3\Ramsey.py
- **Python introduction:** misc\Python in 15 Minutes.ipynb

Model

Model

- **Households** (of measure 1):
  Own capital, supply labor and consume.
- **Firms:** Rent capital and hire labor to produce.
- **Variables:**
  1. Capital: K_t
  2. Labor supply: L_t
  3. Output: Y_t
  4. Consumption: C_t
- **Per worker:** $k_t \equiv K_t/L_t$, $y_t \equiv Y_t/L_t$ and $c_t \equiv C_t/L_t$
- **Prices** are taken as given by households and firms
  1. r_t, rental rate on capital
  2. w_t, wage rate
- **Net return factor on capital:** $R_t \equiv 1 + r_t - \delta$
  where $\delta > 0$ is the depreciation rate

Households I

- **Inealastically supply labour**, $L_t = 1$
- **Maximize the discounted sum of utility from consumption**

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad u' > 0, u'' < 0$$

under the constraints

$$k_{t+1} = R_t k_t + w_t - c_t$$

$$\lim_{t \to \infty} \mathcal{R}_t^{-1} k_t \geq 0$$

$$\mathcal{R}_t = \prod_{j=0}^{t} R_j$$

$$k_0 \text{ givet}$$

and given time paths for $\{R_t\}_{t=0}^{\infty}$ and $\{w_t\}_{t=0}^{\infty}$

Households II

- **Optimal behavior** imply the Euler-equation $\frac{u'(c_t)}{u'(c_{t+1})} = \beta R_{t+1}$
- **CRRA utility:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \sigma > 0$

1. **Euler-equation**

$$c_{t+1} = \left(\beta R_{\mathbf{t+1}}\right)^{1/\sigma} c_t$$

2. **Consumption function**

$$c_0 = \frac{1}{\theta}[R_0 k_0 + h_0]$$

$$h_0 \equiv \sum_{t=0}^{\infty} \mathcal{R}_t^{-1} w_t$$

$$\theta \equiv \sum_{t=0}^{\infty} (\beta^t \mathcal{R}_t)^{1/\sigma} \mathcal{R}_t^{-1}$$

Firms

- **Production function:** $Y_t = F(K_t, L_t) = f(k_t)L_t$
  where F is neoclassical
- **Maximize profits**

$$\max_{K_t, L_t} f(k_t)L_t - r_t K_t - w_t L_t =$$

- The **first order conditions** imply

$$\begin{aligned} f'(k_t) &= r_t \\ f(k_t) - f'(k_t)k_t &= w_t \end{aligned}$$

Low-of-motion

- The **law-of-motion**, $(k_{t+1}, c_{t+1}) = \Gamma(k_t, c_t)$, is given by the solution to the equation system

$$\begin{aligned} k_{t+1} &= k_t(1-\delta) + f(k_t) - c_t \\ \frac{u'(c_t)}{u'(c_{t+1})} &= \beta(1 + f'(k_{t+1}) - \delta) \end{aligned}$$

- **Curves (loci)** where k or c is constant

  1. k: $\{(k, c) \,|\, c = f(k) - \delta k\}$
  2. c: $\{(k, c) \,|\, 1 = \beta(1 + f'(k) - \delta)\}$

Solution algorithm

Find steady state

- In **steady state**:

  1. capital, k^*, solves
  $$\beta(1 + f'(k^*) - \delta) = 1$$
  2. consumption, c^*, then equals
  $$c^* = f(k^*) - \delta k^*$$

- **Cobb-Douglas:**
$$k^* = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha - 1}}$$

Find initial consumption, c_0

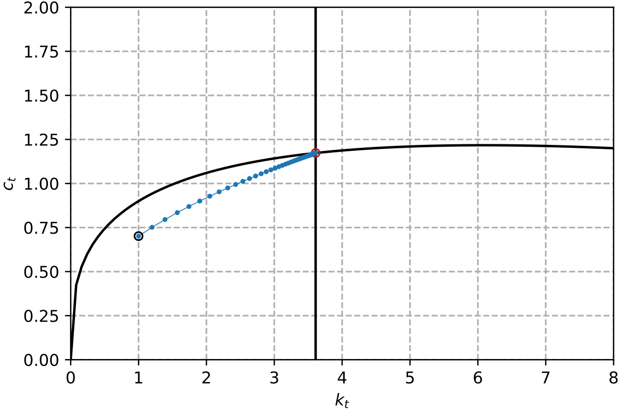
1. Choose **initial capital** $k_0 > 0$ and **tolerance level** $\tau > 0$
2. Set **initial bounds** as $(\underline{c}_0, \overline{c}_0) =$
$$\begin{cases} (0, f(k_0) - \delta k_0) & \text{if } k_0 < k^* \\ (f(k_0) - \delta k_0, f(k_0) + k_0 - \delta k_0) & \text{if } k_0 \geq k^* \end{cases}$$
3. Set $c_0 = (\underline{c}_0 + \overline{c}_0)/2$ and $c = c_0$ and $k = k_0$
4. **Update** using the law-of-motion: $(k, c) = \Gamma(k, c)$
   a. If $\sqrt{(c^* - c)^2 + (k^* - k)^2} < \tau$ **stop**
   b. If $k_0 \leq k^*$ then
      If $c \leq c^* \wedge k \leq k^*$ go to step 4
      If $k > k^*$ set $\underline{c}_0 = c_0$ and go to step 3
      If $c > c^*$ set $\overline{c}_0 = c_0$ and go to step 3
   c. If $k_0 > k^*$ then
      If $c \geq c^* \wedge k \geq k^*$ go to step 4
      If $k < k^*$ set $\overline{c}_0 = c_0$ and go to step 3
      If $c < c^*$ set $\underline{c}_0 = c_0$ and go to step 3

Example

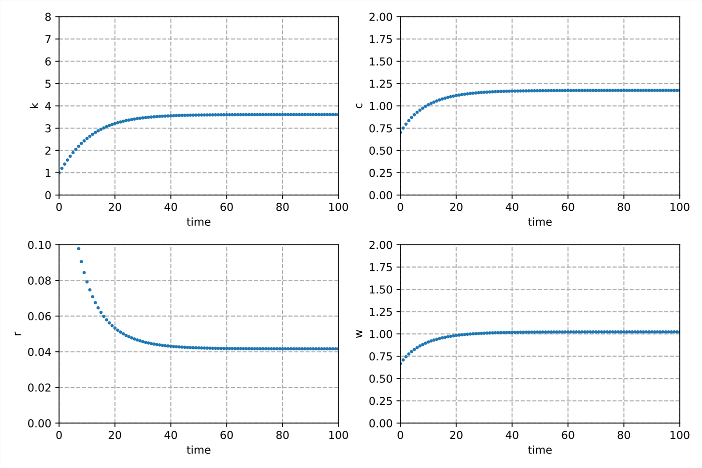
Calibration

1. $f(k_t) = k_t^\alpha$ (Cobb-Douglas)
2. $\beta = 0.96$
3. $\sigma = 2$
4. $\alpha = 1/3$
5. $\delta = 0.10$
6. $k_0 = 1$
7. $\tau = 10^{-6}$

Phase diagram



Time profiles



Diverging paths