



# Solving a Ramsey Model

NumEcon

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# Plan

1. Introduction
2. Model
3. Solution algorithm
4. Example
5. Extensions

# Introduction

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# Introduction

- **Subject:** Solve a Ramsey model numerically (using Python)
- **NumEcon module** (under construction)
  1. **Source files:** GitHub.com
  2. **Interactive version:** MyBinder.org
- **Today:**
  1. **Notebook:** course\_macro3\Ramsey.ipynb
  2. **Code:** numecon\course\_macro3\Ramsey.py
- **Python introduction:** misc\Python in 15 Minutes.ipynb

# Model



- **Households** (of measure 1):  
Own capital, supply labor and consume.
- **Firms:** Rent capital and hire labor to produce.
- **Variables:**
  1. Capital:  $K_t$
  2. Labor supply:  $L_t$
  3. Output:  $Y_t$
  4. Consumption:  $C_t$
- **Per worker:**  $k_t \equiv K_t/L_t$ ,  $y_t \equiv Y_t/L_t$  and  $c_t \equiv C_t/L_t$
- **Prices** are taken as given by households and firms
  1.  $r_t$ , rental rate on capital
  2.  $w_t$ , wage rate
- **Net return factor on capital:**  $R_t \equiv 1 + r_t - \delta$   
where  $\delta > 0$  is the depreciation rate

# Households I

- Inelastically supply labour,  $L_t = 1$
- Maximize the discounted sum of utility from consumption

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad u' > 0, u'' < 0$$

under the constraints

$$k_{t+1} = R_t k_t + w_t - c_t$$

$$\lim_{t \rightarrow \infty} \mathcal{R}_t^{-1} k_t \geq 0$$

$$\mathcal{R}_t = \prod_{j=0}^t R_j$$

$k_0$  given

and given time paths for  $\{R_t\}_{t=0}^{\infty}$  and  $\{w_t\}_{t=0}^{\infty}$

- **Optimal behavior** imply the Euler-equation  $\frac{u'(c_t)}{u'(c_{t+1})} = \beta R_{t+1}$
- **CRRA utility:**  $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \sigma > 0$

1. **Euler-equation**

$$c_{t+1} = (\beta R_{t+1})^{1/\sigma} c_t$$

2. **Consumption function**

$$\begin{aligned} c_0 &= \frac{1}{\theta} [R_0 k_0 + h_0] \\ h_0 &\equiv \sum_{t=0}^{\infty} \mathcal{R}_t^{-1} w_t \\ \theta &\equiv \sum_{t=0}^{\infty} (\beta^t \mathcal{R}_t)^{1/\sigma} \mathcal{R}_t^{-1} \end{aligned}$$



- **Production function:**  $Y_t = F(K_t, L_t) = f(k_t)L_t$   
where  $F$  is neoclassical
- **Maximize profits**

$$\max_{K_t, L_t} f(k_t)L_t - r_t K_t - w_t L_t =$$

- The **first order conditions** imply

$$\begin{aligned} f'(k_t) &= r_t \\ f(k_t) - f'(k_t)k_t &= w_t \end{aligned}$$

- The **law-of-motion**,  $(k_{t+1}, c_{t+1}) = \Gamma(k_t, c_t)$ , is given by the solution to the equation system

$$k_{t+1} = k_t(1 - \delta) + f(k_t) - c_t$$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + f'(k_{t+1}) - \delta)$$

- **Curves (loci)** where  $k$  or  $c$  is constant
  1.  $k$ :  $\{(k, c) \mid c = f(k) - \delta k\}$
  2.  $c$ :  $\{(k, c) \mid 1 = \beta(1 + f'(k) - \delta)\}$

## **Solution algorithm**

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# Find steady state

- In steady state:

1. capital,  $k^*$ , solves

$$\beta(1 + f'(k^*) - \delta) = 1$$

2. consumption,  $c^*$ , then equals

$$c^* = f(k^*) - \delta k^*$$

- Cobb-Douglas:

$$k^* = \left( \frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}}$$

## Find initial consumption, $c_0$

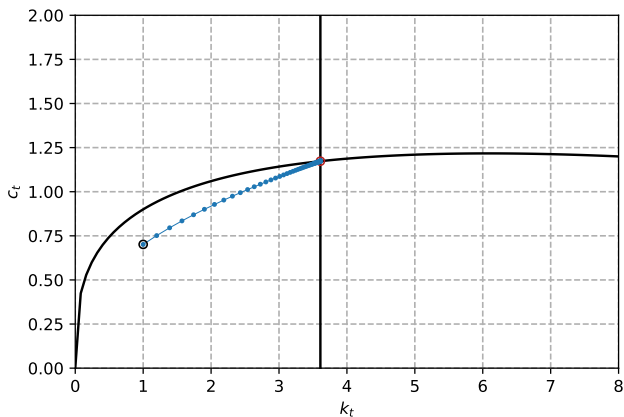
1. Choose **initial capital**  $k_0 > 0$  and **tolerance level**  $\tau > 0$
2. Set **initial bounds** as  $(\underline{c}_0, \bar{c}_0) =$ 
$$\begin{cases} (0, f(k_0) - \delta k_0) & \text{if } k_0 < k^* \\ (f(k_0) - \delta k_0, f(k_0) + k_0 - \delta k_0) & \text{if } k_0 \geq k^* \end{cases}$$
3. Set  $c_0 = (\underline{c}_0 + \bar{c}_0)/2$  and  $c = c_0$  and  $k = k_0$
4. **Update** using the law-of-motion:  $(k, c) = \Gamma(k, c)$ 
  - a. If  $\sqrt{(c^* - c)^2 + (k^* - k)^2} < \tau$  **stop**
  - b. If  $k_0 \leq k^*$  then
    - If  $c \leq c^* \wedge k \leq k^*$  go to step 4
    - If  $k > k^*$  set  $\underline{c}_0 = c_0$  and go to step 3
    - If  $c > c^*$  set  $\bar{c}_0 = c_0$  and go to step 3
  - c. If  $k_0 > k^*$  then
    - If  $c \geq c^* \wedge k \geq k^*$  go to step 4
    - If  $k < k^*$  set  $\bar{c}_0 = c_0$  and go to step 3
    - If  $c < c^*$  set  $\underline{c}_0 = c_0$  and go to step 3

## Example

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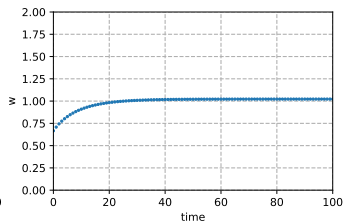
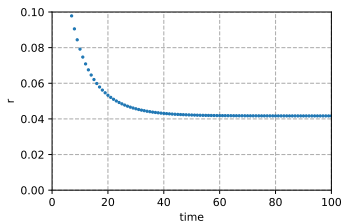
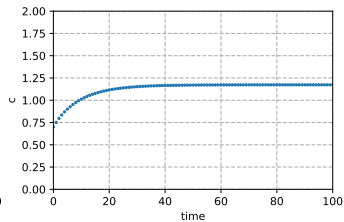
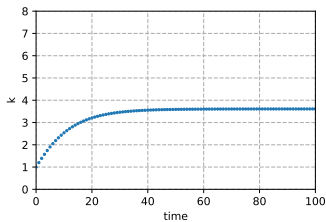
1.  $f(k_t) = k_t^\alpha$  (Cobb-Douglas)
2.  $\beta = 0.96$
3.  $\sigma = 2$
4.  $\alpha = 1/3$
5.  $\delta = 0.10$
6.  $k_0 = 1$
7.  $\tau = 10^{-6}$

# Phase diagram

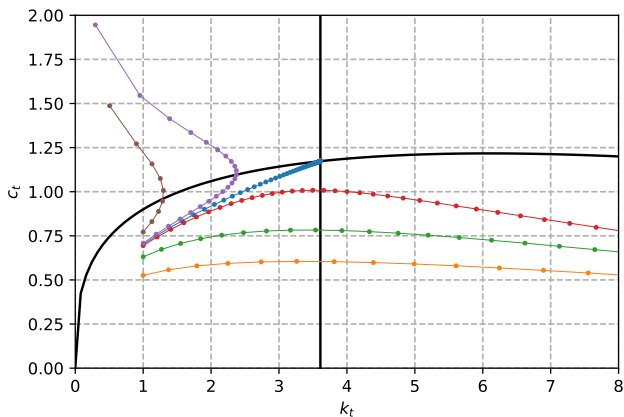




# Time profiles



# Diverging paths



# Extensions

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# Potential extensions

1. **Government** (taxes and spending)
2. **Temporary shock(s)**
3. **Announced shock(s)**
4. **Endogenous labor supply**