STA 243: Homework 4

Homework due in Canvas: 06/04/2019 at 11:59PM. Please follow the instructions provided in Canvas about homeworks, carefully. **Please be aware of the end of the quarter time period and plan your time accordingly.**

1. (**7.5 points**) Consider the following probability density function:

$$p(x) \propto \frac{e^{-x}}{1+x^2}, \quad x > 0.$$

Use rejection sampling to sample from $p(x)$ with the following envelope density functions:

$$q_1(x) = e^{-x}, \quad q_2(x) = \frac{2}{\pi(1+x^2)}, \quad x > 0.$$

   (a) For each envelope density function ($q_1(x)$ and $q_2(x)$), draw 5000 samples and plot the histogram (or the estimated density function) for $0 < x < 5$, along with the true density function $p(x)$.

   (b) Comment on the speeds of sampling and the results using $q_1(x)$ and $q_2(x)$.

2. (**5 points**) Use Monte Carlo to evaluate each of the following integrals:

   (a) $\int_0^1 x^2 dx$

   (b) $\int_0^1 \int_{-2}^2 x^2 \cos(xy) dx dy$

   (c) $\int_0^\infty \frac{3}{4} x^4 e^{-x^3/4} dx$

Provide confidence interval for your estimates and compare your estimate to the true integral value (you don't have to show your integral calculation).

3. (**5 points**) Let

$$I = \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx.$$

Estimate I using importance sampling. Take $q(x)$ to be $N(1.5, \nu^2)$ with $\nu = 0.1, 1$ and 10. Plot a histogram of the values you are averaging to see if there are any extreme values.

4. (**7.5 points**) A random variable Z has a inverse Gaussian distribution if it has density

$$p(z) \propto z^{-3/2} \exp\left\{-\theta_1 z - \frac{\theta_2}{z} + 2\sqrt{\theta_1\theta_2} + \log\sqrt{2\theta_2}\right\}, z > 0,$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are parameters. It can be shown that

$$E(Z) = \sqrt{\frac{\theta_2}{\theta_1}} \quad \text{and} \quad E\left(\frac{1}{Z}\right) = \sqrt{\frac{\theta_1}{\theta_2}} + \frac{1}{2\theta_2}.$$

Let $\theta_1 = 1.5$ and $\theta_2 = 2$. Draw a sample of size 1,000 using the *independence-Metropolis-Hastings algorithm*. Here *Independence-Metropolis-Hastings Algorithm* is a version of MCMC where we draw the proposal from a fixed distribution q, instead of a conditional distribution. Specifically, in page (5) of the notes, we have $\mathbf{x}' \sim q(\mathbf{x})$ instead of $q(\mathbf{x}|\mathbf{x}^{(t)})$. The acceptance probability becomes

$$r(\mathbf{x}, \mathbf{x}') = \min\left\{1, \frac{p(\mathbf{x}')}{p(\mathbf{x})} \frac{q(\mathbf{x})}{q(\mathbf{x}')}\right\}.$$

Use a Gamma distribution for the proposal density q. To assess the accuracy, compare the samples mean of Z and $1/Z$ using the samples you generated, to the theoretical expectation values. Try different parameters for the Gamma distributions to see if you can get accurate estimates of the true expectations. Record your results in a table.