Linear vertical shear instability in protoplanetary disks

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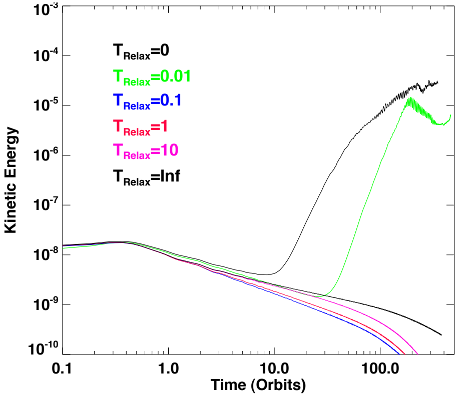
Outline

Thermodynamic condition for the VSI

- Astrophysical disks generally have $\partial_z \Omega \neq 0$ — neccessary for VSI, but also need rapid cooling .



(Nelson et al., 2013)

Can we quantify this requirement?

Thermodynamic condition for the VSI

Lin-Youdin VSI condition

$$t_{\mathrm{cool}}\Omega_{\mathrm{K}} < \frac{h|q|}{\gamma - 1} \equiv \beta_{\mathrm{crit}}$$

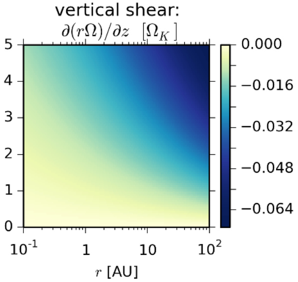
(Vertically isothermal disk with $T \propto r^q$, $h \equiv H/r$, and $t_{\mathrm{cool}} = \beta/\Omega_{\mathrm{K}}$.)

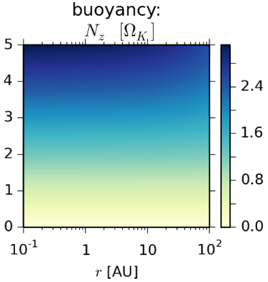
(Lin & Youdin, 2015)

Rapid cooling needed because of buoyancy

Vertical motion associated with VSI is opposed by buoyancy forces

$$\underbrace{r\partial_z\Omega}_{\text{destabilizing vert. shear}} \quad \text{v.s.} \quad \underbrace{N_z}_{\text{stabilizing vert. buoyancy}}$$





- Vertical shear is *weak*, $r\partial_z ln\Omega \sim O(h) \ll 1$, so need $l_z/l_r \gg 1$
- Vertical buoyancy is *strong*, $N_z/\Omega \sim O(1)$

Linear theory: previous analyses and our contribution

- Vertically and radially local, with energy equation
  (Urpin & Brandenburg, 1998; Urpin, 2003, G. Mohandas)
- Vertically global, radially local, no buoyancy
  (Nelson et al., 2013; McNally & Pessah, 2014; Barker & Latter, 2015)
- Vertically and radially global, no buoyancy
  (Barker & Latter, 2015; Umurhan et al., 2015)

Lin & Youdin (2015)

- Vertically global, radially local, including energy equation (i.e. with buoyancy)
- Both constant cooling and realistic cooling functions

Isothermal limit (instantaneous cooling)

Linearized fluid equations $\rightarrow$

$$0 = W'' + \left[\ln \rho' - \frac{\mathrm{i}K}{(1-\nu^2)\,\Omega_{\mathrm{K}}^2 h}\frac{d\Omega^2}{dz}\right] W' + \nu^2\left(1 + \frac{K^2}{1-\nu^2}\right) W.$$

$K = k_x H$, $\nu = \omega/\Omega_{\mathrm{K}}$.

- Formal[1] limit on the growth rate of low-frequency modes

$$\boxed{\sigma < \max\left|r\frac{d\Omega}{dz}\right|}$$

(Unbound if approximating vertical shear $\propto z$)

- General frequency waves in a thin-disk without a surface

$$\nu^4 - \left(L + 1 + K^2\right)\nu^2 + L\left(1 + \mathrm{i}hqK\right) = 0, \quad L = 1, 2\cdots$$

VSI is the low-frequency (inertial) branch.

[1] Via Cauchy-Schwarz inequality...etc.

Linear theory with finite cooling

- Parameterized cooling: $t_{\mathrm{cool}}\Omega_{\mathrm{K}} \equiv \beta = \mathrm{const.}$

Single ODE reduced model (low-freq., thin-disk, no explicit $\partial_r P$)

$$\boxed{0 = \delta v_z''(z) - zA\delta v_z'(z) + \left(B - Cz^2\right)\delta v_z(z).}$$

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- Transformation $\rightarrow$ Hermite ODE (as before)
- As in Lubow & Pringle (1993) but A, B, C now complex because $\partial_z\Omega \neq 0$
- Important: reduced model is only valid for $t_{\rm cool}\Omega_{\rm K} \lesssim O(1)$ (OK for VSI)

Linear theory with finite cooling

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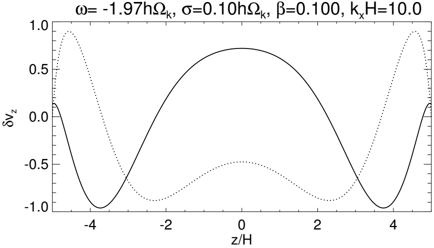
- Finite K.E. density as $|z| \to \infty \Rightarrow$ dispersion relation $\omega = \omega(k_x; \beta, M)$
- Mode number $M = 0, 1, 2 \cdots$
- Fundamental mode $M = 0$ has special importance



Critical cooling time

- Assume $\beta = \beta_{\mathrm{c}}$ at marginal stability ($\sigma = 0$) and large k_x

Find

$$\frac{\partial \beta_{\mathrm{c}}}{\partial M} < 0$$

(if the disk is sufficiently thin). Then $M = 0$ has the longest critical cooling time.

The fundamental mode is the most difficult to stabilize with increasing t_{cool}.

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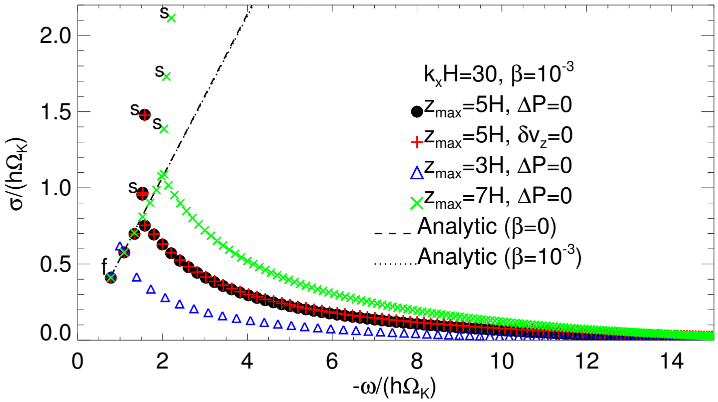
So condition for VSI is

$$t_\mathrm{cool}\Omega_\mathrm{K} < \beta_\mathrm{c}(M = 0) = \frac{h|q|}{\gamma - 1}$$

- $h|q|$: vertical shear (destabilizing)
- $\gamma - 1$: vertical buoyancy (stabilizing)

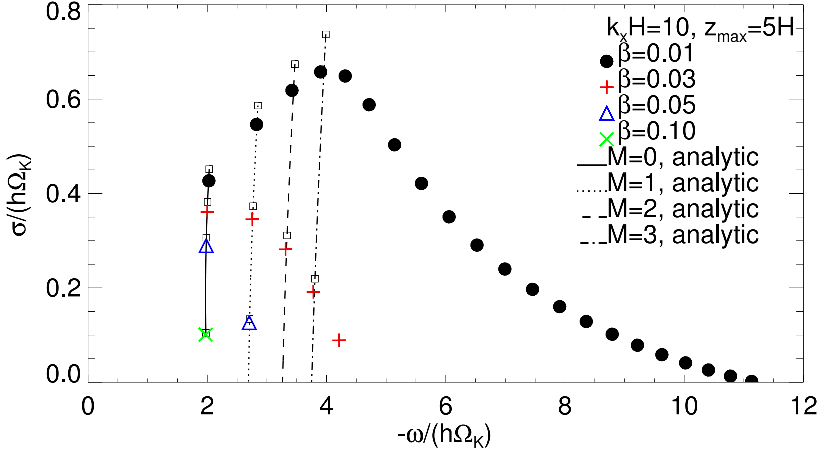
Numerical calculations

- Solve linearized equation in the radially local approx.
- Relax all other assumptions in reduced model



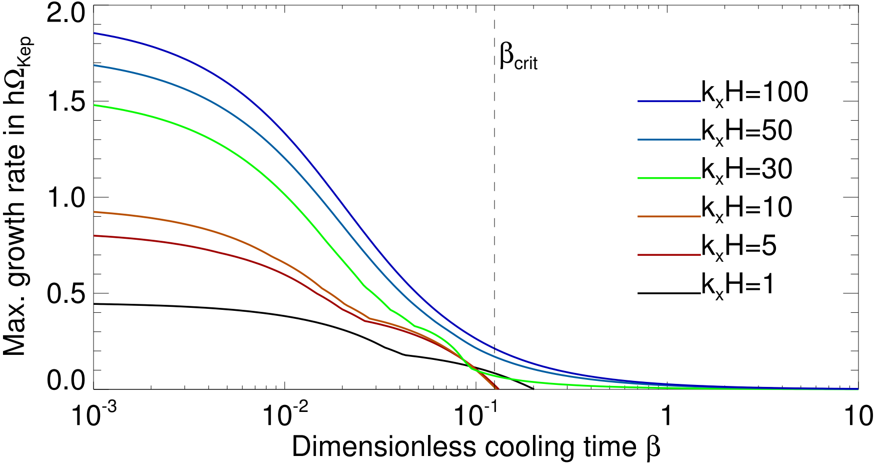
- Theory describes the lowest order modes inc. fundamental mode
- 'Surface modes' are entirely due to disk surface (imposed or physical)

Effect of increasing the cooling time



Testing the critical cooling timescale

$$t_{\mathrm{cool}}\Omega_{\mathrm{K}} < \frac{h|q|}{\gamma - 1}$$



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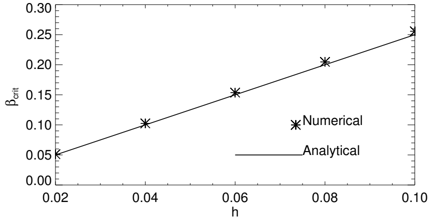

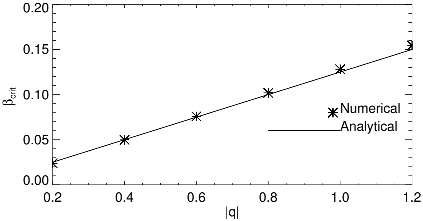

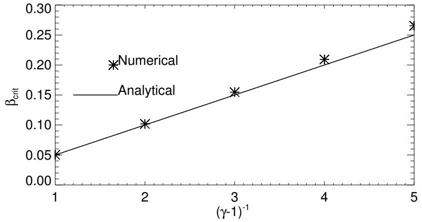

Application to protoplanetary disks

Estimate cooling times in the Minimum Mass Solar Nebula (Chiang & Youdin, 2010) based on dust opacity ($\propto T^2$):

$$t_{\rm cool}\Omega_{\rm K} \equiv$$

$$\boxed{\beta(z; r, K) = 3.9\times10^{-3}\frac{r_{\rm AU}^{9/14}}{\kappa_{\rm d}}\left[1 + \frac{1.9\times10^7\kappa_{\rm d}^2}{r_{\rm AU}^{33/7}K^2}\exp\left(-\frac{z^2}{2H^2}\right)\right]}$$

- $\kappa_{\rm d}$: opacity scale relative to MMSN
- Optically thin/Newtonian cooling for very small scales, $\boxed{\text{fast for large } \kappa_{\rm d}}$
- Radiative diffusion for longer scales, $\boxed{\text{fast for small } \kappa_{\rm d}}$
- Vert. dependence through ρ

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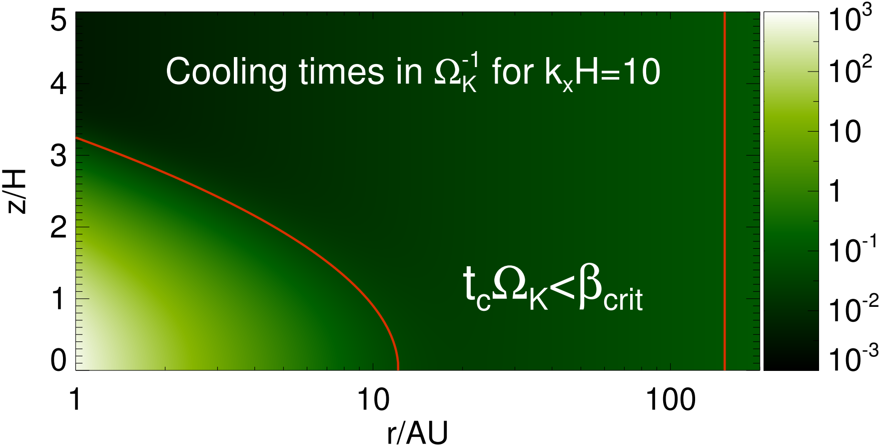
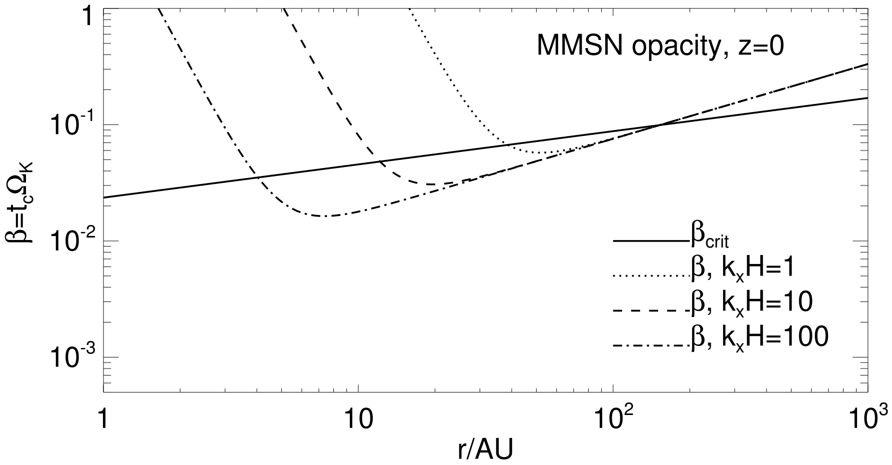


β versus $\beta_{\rm crit}$



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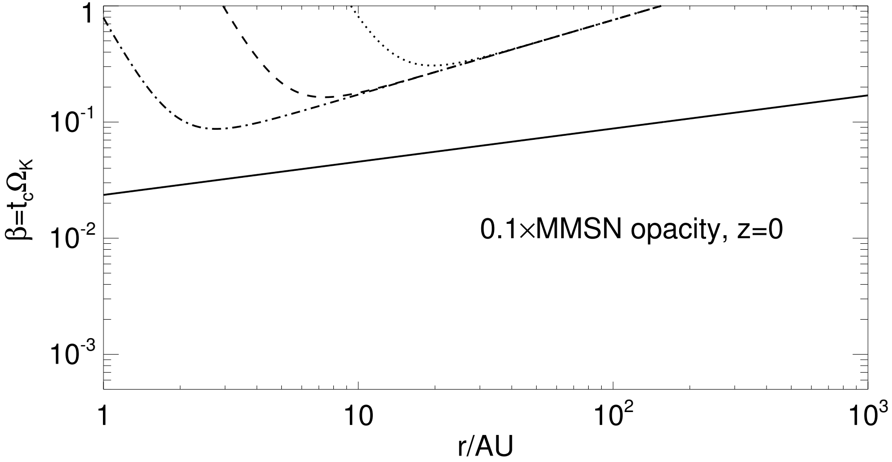

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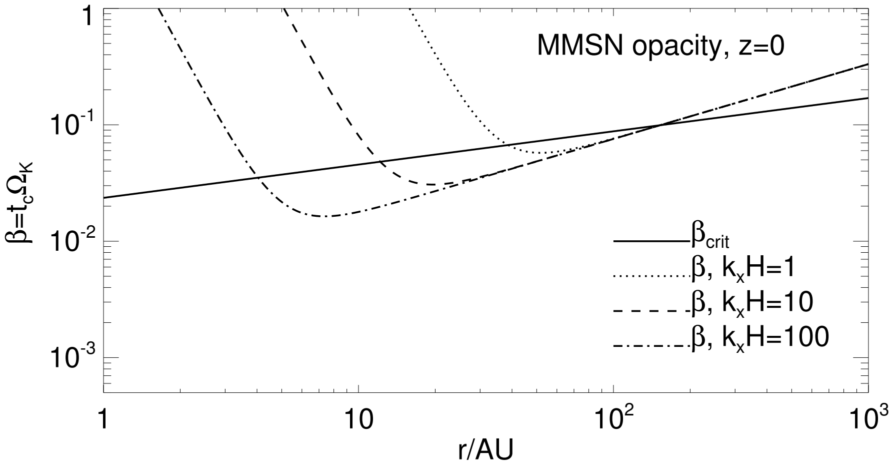
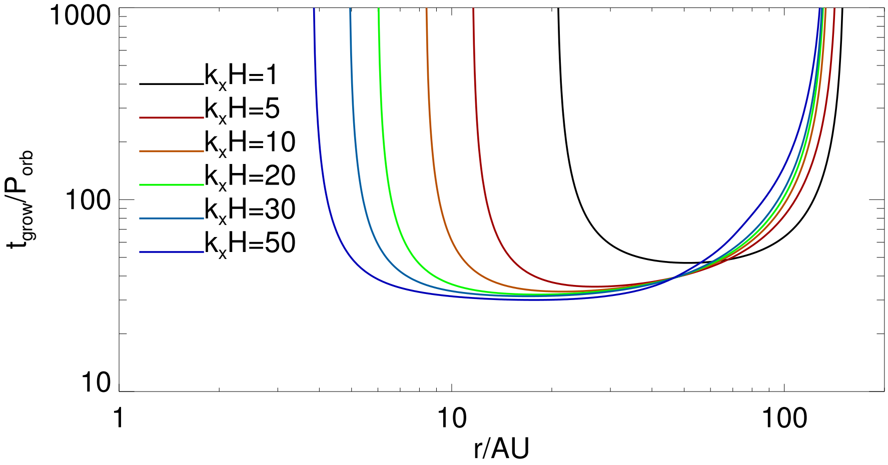


VSI in the solar nebula



With $\beta = \beta(z; r, K)$

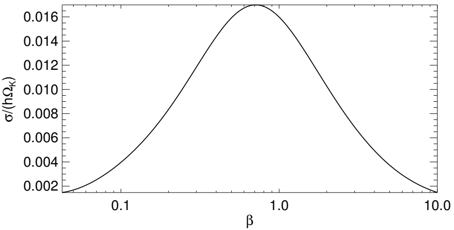
Further applications and extensions

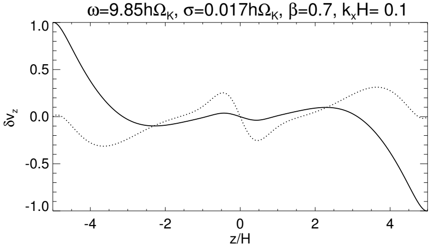
- Use β_{crit} as a simple, first 'go to' criteria to assess stability against VSI
- Enroll β_{crit} in 1D accretion models, e.g. $\alpha_{\mathrm{VSI}}(t_{\mathrm{cool}}, \beta_{\mathrm{crit}})$. (Cf. GI stress from Toomre parameter.)

Further applications and extensions

- Use β_{crit} as a simple, first 'go to' criteria to assess stability against VSI
- Enroll β_{crit} in 1D accretion models, e.g. $\alpha_{\mathrm{VSI}}(t_{\mathrm{cool}}, \beta_{\mathrm{crit}})$. (Cf. GI stress from Toomre parameter.)

- Radially-global problem (with O. Umurhan)
- Non-axisymmetric problem
- Other instabilities are supported in the current model, e.g. convective overstability (Klahr & Hubbard, 2014; Lyra, 2014)





Conclusions

Lin-Youdin criterion

$$\boxed{t_{\rm cool}\Omega_{\rm K} < \frac{h|q|}{\gamma - 1}}$$

- Astrophysical disks generally have $\partial_z \Omega \neq 0$
- Thin PPDs are unstable if buoyancy ineffective:

  $N_z = 0$ and/or $\boxed{t_{\rm cool}\Omega_{\rm K} \ll 1}$
- Fast cooling needed because vertical shear is weak but buoyancy is strong
- Thermodynamic requirement satisfied at 10s of AU in typical PPDs

References

Barker A. J., Latter H. N., 2015, MNRAS, 450, 21

Chiang E., Youdin A. N., 2010, Annual Review of Earth and Planetary Sciences, 38, 493

Klahr H., Hubbard A., 2014, ApJ, 788, 21

Lin M.-K., 2014, MNRAS, 437, 575 [GI + resistive MRI: linear calc.]

Lin M.-K., Youdin A. N., 2015, ApJ, 811, 17

Lubow S. H., Pringle J. E., 1993, ApJ, 409, 360

Lyra W., 2014, ApJ, 789, 77

Mamatsashvili G. R., Rice W. K. M., 2010, MNRAS, 406, 2050

McNally C. P., Pessah M. E., 2014, ArXiv e-prints

Nelson R. P., Gressel O., Umurhan O. M., 2013, MNRAS, 435, 2610

Umurhan O. M., Nelson R. P., Gressel O., 2015, ArXiv e-prints

Urpin V., 2003, A&A, 404, 397

Urpin V., Brandenburg A., 1998, MNRAS, 294, 399