



Vortex and spiral instabilities at gap edges in three-dimensional self-gravitating disc–satellite simulations

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ABSTRACT

Numerical simulations of global three-dimensional (3D), self-gravitating discs with a gap opened by an embedded planet are presented. The simulations are customized to examine planetary gap stability. Previous results, obtained by Lin & Papaloizou from 2D disc models, are reproduced in 3D. These include (i) the development of vortices associated with local vortensity minima at gap edges and their merging on dynamical time-scales in weakly self-gravitating discs, (ii) the increased number of vortices as the strength of self-gravity is increased and their resisted merging, and (iii) suppression of the vortex instability and development of global spiral arms associated with local vortensity maxima in massive discs. The vertical structure of these disturbances is examined. In terms of the relative density perturbation, the vortex disturbance has weak vertical dependence when self-gravity is neglected. Vortices become more vertically stratified with increasing self-gravity. This effect is seen even when the unperturbed region around the planet’s orbital radius has a Toomre stability parameter ~ 10 . The spiral modes display significant vertical structure at the gap edge, with the mid-plane density enhancement being several times larger than that near the upper disc boundary. However, for both instabilities the vertical Mach number is typically a few per cent, and on average vertical motions near the gap edge do not dominate horizontal motions.

Key words: hydrodynamics – instabilities – methods: numerical – planet–disc interactions – protoplanetary discs – planetary systems.

1 INTRODUCTION

Astrophysical discs may develop radial structure for several reasons (Armitage 2011). It has been suggested that protoplanetary discs contain ‘dead zones’ in which the magnetorotational instability is inefficient, leading to a reduced accretion rate in this region (Gammie 1996). Matter then accumulates at the radial boundary of a dead zone and the actively accreting region, leading to a local density bump.

Structure can also be induced by an external potential such as an embedded satellite. A sufficiently massive planet opens a gap in the disc (Lin & Papaloizou 1986), and the gap edges involve radial structure with characteristic length-scales of the local disc scale-height.

It is well established that localized radial structure in thin discs can be dynamically unstable (Lovelace et al. 1999; Li et al. 2000, 2001). The evolution of radially structured discs may then be affected by such instabilities. More specifically, a necessary condition for instability is the existence of an extremum in the ratio of vor-

ticity to surface density, or vortensity.¹ Indeed, instabilities have been demonstrated explicitly for dead-zone boundaries (Varnière & Tagger 2006; Lyra et al. 2008, 2009b; Crespe, Gonzalez & Arena 2011) as well as planetary gap edges (Koller, Li & Lin 2003; Li et al. 2005; de Val-Borro et al. 2007; Lyra et al. 2009a; Lin & Papaloizou 2010).

These studies consider non-self-gravitating or weakly self-gravitating discs. In fact, instabilities in structured astrophysical discs can be traced back to Lovelace & Hohlfeld (1978), who considered self-gravitating particle discs. Sellwood & Kahn (1991) studied a similar system, while self-gravitating gaseous discs were examined by Papaloizou & Lin (1989), Papaloizou & Savonije (1991) and Meschiari & Laughlin (2008, who adopted disc profiles to mimic planetary gaps).

Lin & Papaloizou (2011a,b) explored in more detail the role of self-gravity on the stability of gaps self-consistently opened by a planet. They performed a series of linear and non-linear calculations for a range of disc masses. They found vortex formation in weakly self-gravitating discs and global spiral arms in massive discs, but both are associated with the gap edge.

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¹This is modified by a factor involving the entropy for non-barotropic discs.

The above studies have employed the razor-thin or two-dimensional (2D) disc approximation. It is natural to extend these models to 3D. However, note that non-axisymmetric instabilities in pressure-supported thick discs, i.e. the Papaloizou–Pringle instability (PPI), was originally studied in 3D (Papaloizou & Pringle 1984, 1985, 1987; Goldreich, Goodman & Narayan 1986). The vortex-forming instability mentioned above is essentially the PPI operating in a thin disc with the density bump being analogous to a torus. Some early studies of slender tori also included self-gravity (e.g. Goodman & Narayan 1988; Christodoulou & Narayan 1992).

Recently, 3D non-self-gravitating, rotationally supported global discs have been simulated with a local density bump, either set as an initial condition (Meheut et al. 2010; Meheut, Yu & Lai 2012b; Meheut et al. 2012a) or self-consistently generated by a resistivity jump in magnetic discs (Lyra & Mac Low 2012). The latter models the dead-zone scenario. These simulations display vortex formation similar to 2D discs. On the other hand, instabilities at planetary gap edges in 3D self-gravitating discs have not yet been simulated.

In this work we extend the 2D self-gravitating disc models of Lin & Papaloizou (2011a,b) to 3D. Because the instabilities are associated with radial structure with comparable size to the disc thickness, it is not obvious at first that 2D is a good approximation. Thus, our priority in this first study is to verify results obtained in Lin & Papaloizou (2011a,b) by simulating equivalent systems in 3D. We also identify some 3D effects that set the direction for future investigations.

This paper is organized as follows. After reviewing the main 2D results in the next subsection, we describe our 3D disc–planet models in Section 2. Numerical methods are stated in Section 3. We go through our simulations in Section 4 with additional result analyses presented in Section 5. We summarize and conclude in Section 6 with a discussion of several limitations of our simulations.

1.1 Gap stability in 2D

For discussion purposes here we consider a barotropic disc.² For a radially structured disc the quantity governing stability is the vortensity profile

$$\eta \equiv \frac{\kappa^2}{2\Omega\Sigma}, \quad (1)$$

where $\kappa^2 \equiv R^{-3} d(R^4 \Omega^2)/dR$ is the square of the epicycle frequency, Ω is the disc angular velocity and Σ is its surface density. Here R is the cylindrical radius. If the disc is self-gravitating, the Toomre parameter Q is also important,

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma} = \frac{c_s}{\pi G} \left(\frac{2\Omega\eta}{\Sigma} \right)^{1/2}, \quad (2)$$

where c_s is the sound speed. $Q < 1$ signifies local axisymmetric gravitational instability (Toomre 1964). We remark that some studies of instabilities in structured discs employ values of κ^2 marginally above zero or even negative (e.g. Li et al. 2000, 2001), implying that the classic Toomre instability may operate. Of course, if the disc is strictly non-self-gravitating and $\kappa^2 > 0$, then Σ can be rescaled so that $Q \gg 1$, giving a self-consistent model.

The connection between Q and η results in the vortensity profile of a planetary gap to resemble its Toomre parameter profile because of

² Our locally isothermal numerical models are not strictly barotropic. However, the instabilities of interest are associated with localized structure and we adopt sound-speed profiles that vary slowly in these regions. Hence we can consider it to be isothermal and barotropic.

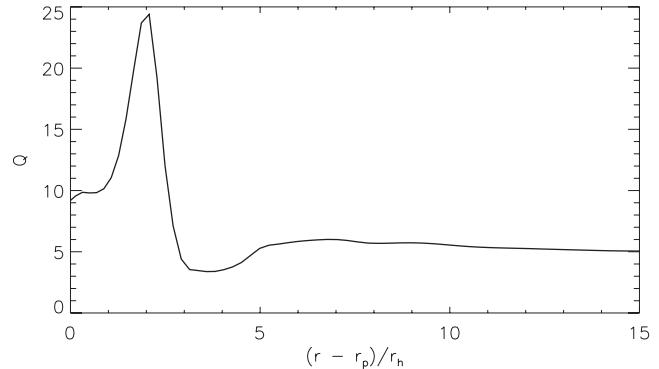


Figure 1. Typical structure of the outer edge of a planetary gap in terms of the Toomre parameter Q . The horizontal axis is the displacement away from the planet in units of its Hill radius (defined in Section 2.4).

vortensity generation and destruction across planet-induced shocks (Lin & Papaloizou 2010). Extrema in Q and η nearly coincide at the same radius. Fig. 1 shows a typical Toomre profile. Here we focus on the outer disc where we find instabilities strongest in the numerical simulations.

As mentioned in Section 1, disc profiles with stationary points in η (and therefore in Q for planetary gaps) can be dynamically unstable. The gap profile fulfils this requirement and has the following stability properties (taken from Lin & Papaloizou 2011a,b).

(i) In weakly or non-self-gravitating discs, instability is associated with the vortensity minimum or $\min(Q)$, leading to local vortex formation.

(ii) As the strength of self-gravity is increased, the vortex mode shifts to higher azimuthal wavenumber m . This is partly due to the stabilization effect of self-gravity on low m vortex modes.

(iii) The time-scale for vortex merging increases with the strength of self-gravity. In non-self-gravitating discs, vortices merge on dynamical time-scales and the result is a single, azimuthally extended vortex. Multi-vortex configurations can last much longer with increased disc gravity. Merging eventually takes place but the resulting vortex is azimuthally localized. In the moderately self-gravitating case discussed in Lin & Papaloizou (2011a), a vortex pair persists until the end of the simulation.

(iv) The vortex mode is suppressed with sufficiently strong self-gravity and replaced by a global spiral instability associated with the local vortensity maxima or $\max(Q)$. The instability can be physically understood as gravitational coupling between the gap edge and the wider disc exterior to it.

We shall confirm the above in 3D disc models, except for the azimuthally localized, post-merger vortices in (iii). This requires very long simulations with self-gravity (~ 200 orbits in Lin & Papaloizou 2011a), which are currently impractical in 3D.

These instabilities also affect planetary migration, leading to vortex–planet and spiral–planet interactions (Lin & Papaloizou 2010, 2011a,b, 2012). In order to focus on gap stability we will not consider migration (and because of resolution limits in a 3D simulation), but we can still measure disc–planet torques. In particular, we will confirm that spiral modes make the disc-on-planet torques more positive with increasing instability strength.

2 DISC-PLANET MODELS

We consider a 3D gas disc of mass M_d with an embedded planet of mass M_p , both rotating about a central star of mass M_* . To describe

the system, we use both spherical polar coordinates $\mathbf{r} = (r, \theta, \phi)$ and cylindrical coordinates $\mathbf{r} = (R, \phi, z)$ centred on the star. The frame is non-rotating. The disc is governed by the standard fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi_{\text{eff}}, \quad (4)$$

$$p = c_s^2 \rho, \quad (5)$$

where ρ is the mass density, \mathbf{u} is the velocity field, p is the pressure and Φ_{eff} is an effective potential. Physical viscosity is not included in this study (but artificial viscosity is employed in the numerical simulations to treat shocks). The effect of viscosity on vortex modes and spiral modes has been investigated previously (de Val-Borro et al. 2007; Lin & Papaloizou 2011b). We do not expect its effect to differ in 3D.

2.1 Equation of state

We adopt a modified isothermal equation of state (EOS) where the sound speed c_s depends on R and the planet position if present. Without a planet, we set

$$c_s = c_{\text{iso}} \equiv H \Omega_k \quad \text{no planet,} \quad (6)$$

where $H = hR$ is the disc scale-height with constant aspect ratio h and $\Omega_k = \sqrt{GM_*/R^3}$ is the Keplerian frequency. When a planet is present, we set

$$c_s = \frac{HH_p \sqrt{\Omega_k^2 + \Omega_{\text{kp}}^2}}{\left(H^{7/2} + H_p^{7/2}\right)^{2/7}} \quad \text{with planet,} \quad (7)$$

where $H_p = h_p d_p$, $\Omega_{\text{kp}}^2 = GM_p/d_p^3$ and $d_p = \sqrt{|\mathbf{r} - \mathbf{r}_p|^2 + \epsilon_p^2}$ is the softened distance to the planet at position \mathbf{r}_p , where ϵ_p is the softening length defined later. This EOS is taken from Pepliński, Artynowicz & Mellema (2008) and is used here to increase the temperature near to planet in order to reduce mass accumulation in this region. The dimensionless parameter h_p controls the temperature increase at \mathbf{r}_p relative to that given by c_{iso} .

2.2 Effective potential

The effective potential is as follows:

$$\Phi_{\text{eff}} = \Phi_* + \Phi_p + \Phi + \Phi_i, \quad (8)$$

where

$$\Phi_* = -\frac{GM_*}{r} \quad (9)$$

is the stellar potential and

$$\Phi_p = -\frac{GM_p}{d_p} \quad (10)$$

is the softened planet potential. Φ is the gravitational potential due to the disc material and is given via the Poisson equation:

$$\nabla^2 \Phi = 4\pi G \rho. \quad (11)$$

In equation (8), Φ_i is the indirect potential due to the disc and the planet,

$$\Phi_i(\mathbf{r}) = \int \frac{G\rho(\mathbf{r}')}{r'^3} \mathbf{r} \cdot \mathbf{r}' d^3 r' + \frac{GM_p}{|\mathbf{r}_p|^3} \mathbf{r} \cdot \mathbf{r}_p. \quad (12)$$

The indirect potential accounts for the acceleration of the coordinate origin relative to the inertial frame. This term is included for consistency but is unimportant for the instabilities of interest.

2.3 Initial disc

The physical disc occupies $r \in [r_i, r_o]$, $\theta \in [\theta_{\min}, \pi - \theta_{\min}]$ and $\phi \in [0, 2\pi]$. The vertical domain is such that $\tan(\pi/2 - \theta_{\min})/h = n$ scale-heights. The density field is initialized to

$$\rho(t=0) = \beta \rho_0, \quad (13)$$

where ρ_0 is the density profile corresponding to a non-self-gravitating disc,

$$\rho_0(R, z) = \frac{\Sigma_0}{\sqrt{2\pi}H} \left(\frac{R}{r_i}\right)^{-\sigma} \left[1 - \sqrt{\frac{r_i}{R + hr_i}}\right] \times \exp\left(-\frac{\Phi_*}{c_{\text{iso}}^2} - \frac{1}{h^2}\right), \quad (14)$$

with fixed power-law index $\sigma = 3/2$. β is a function to account for vertical self-gravity, such that the surface density of the initial disc is the same as that corresponding to ρ_0 . We calculate β in Appendix A with some approximations. Because of this, we always first evolve the disc without a planet.

The constant Σ_0 is chosen via the Keplerian Toomre parameter Q_0 at the outer boundary:

$$Q_0 \equiv \left. \frac{c_{\text{iso}} \Omega_k}{\pi G \Sigma} \right|_{R=r_o}, \quad (15)$$

$$\Sigma \equiv \int_{z_{\min}}^{z_{\max}} \rho_0 \, dz. \quad (16)$$

Note that the integration for surface density Σ is taken over the finite vertical domain being considered.

The disc is initialized with zero meridional velocity ($u_r = u_\theta = 0$). The azimuthal velocity is set by centrifugal balance with stellar gravity, pressure and self-gravity, but for the disc models being considered, which are thin and not very massive, the initial azimuthal velocity is essentially Keplerian.

Our disc models are labelled by $Q_0 \propto M_d^{-1}$. This gives an indication of the strength of self-gravity. Specifically it measures gravitational stability against local axisymmetric perturbations at the outer disc boundary. All of our discs satisfy the Toomre criterion for stability.

2.4 Planet configuration

In this work the planet is treated as a fixed external potential. Its purpose is to create and maintain a structured disc. The planet is held on a circular orbit, $\mathbf{r}_p = (r_p, \pi/2, \phi_p)$ with $\phi_p(t) = \Omega_k(r_p)t$ in spherical coordinates. The softening length of the planet potential is fixed to $\epsilon_p = 0.1r_h$, where $r_h = (q/3)^{1/3}r_p$ is the Hill radius and $q \equiv M_p/M_*$. The EOS parameter is set to $h_p = 0.5$.

3 NUMERICAL METHOD

We evolve the disc–planet system using the ZEUS-MP finite-difference code in spherical coordinates (Hayes et al. 2006). The computational domain is divided into (N_r, N_θ, N_ϕ) zones, logarithmically spaced in radius and uniformly spaced in the angular coordinates. We assume symmetry about the mid-plane, so the computational domain only covers the upper plane ($z > 0$). Hydrodynamic boundary conditions are outflow at r_i, r_o , reflecting at θ_{\min} and periodic in ϕ .

ZEUS-MP was chosen for its ability to treat self-gravity on a spherical grid with parallelization. It solves the discretized Poisson equation using a conjugate gradient method (for details, see Hayes et al. 2006). To supply boundary conditions to the solver, we approximate the boundary potential using spherical harmonic expansion as described in Boss (1980). The expansion in spherical harmonics Y_{lm} is truncated at l_{\max}, m_{\max} . We assume negligible contributions to the disc potential beyond the physical disc boundaries.

3.1 Simulation set-up

Computational units are such that $G = M_* = 1$. The radial range of the disc is $(r_i, r_o) = (1, 25)$. The vertical extent is $n_H = 2$ scale-heights. The grid resolution is $(N_r, N_\theta, N_\phi) = (256, 32, 512)$. We quote time in units of $P_0 = 2\pi/\Omega_k(r_p)$. Between $0 \leq t < 10P_0$ the disc is evolved without a planet and $(l_{\max}, m_{\max}) = (48, 0)$. The planet is introduced at $t = 10P_0$ and its mass smoothly increased from zero to its full value between $10P_0 \leq t \leq 20P_0$. For $t > 20P_0$ we set $(l_{\max}, m_{\max}) = (16, 10)$.

The unstable modes of interest are associated with vortensity extrema at gap edges, so these features need to be resolved. Numerical diffusivity, e.g. due to low resolution or grid choice, may inhibit such instabilities. For example, in their 2D studies of vortex generation at gap edges, de Val-Borro et al. (2007) did not find instability in Cartesian simulations.

Test runs with half the resolution in each coordinate also produce vortex and spiral modes, but with reduced growth rate. While the resolution adopted here can confirm these instabilities operate in 3D, it may be inadequate to probe secondary instabilities on longer time-scales (see Section 6.3).

We set $m_{\max} = 10$ for the boundary potential as a compromise between accounting for the non-axisymmetry of unstable modes and computation time. We typically find vortex modes with azimuthal wavenumber $m = 3-6$. We tested a run (Case 3) with $m_{\max} = 12$ and obtained similar results. However with $m_{\max} = 0$, vortex merging proceeds soon after their formation, whereas it is resisted with $m_{\max} = 10$ (consistent with high-resolution 2D results). The global spiral modes have $m = 2$, so large m_{\max} is not crucial. Indeed, tests with $m_{\max} = 4$ also yield the spiral instability.

4 RESULTS

Our simulations are summarized in Table 1. If the stellar mass is taken to be $M_* = M_\odot$ then $q = 10^{-3}$ corresponds to a Jupiter-mass planet. The planetary masses considered here are larger than our previous 2D investigations (Lin & Papaloizou 2011a,b) in order to achieve higher instability growth rates and shorten the computation time. We will examine gap stability as a function of Q_0 .

Table 1. Simulation parameters. Q_p is the Keplerian Toomre parameter evaluated at r_p . Case 0 was ran without self-gravity.

Case	h	$10^3 q$	Q_0, Q_p	M_d/M_*	Mode
0	0.07	2	∞	0.021	Vortex
1	0.07	2	8.0, 14.8	0.021	Vortex
2	0.07	2	4.0, 7.40	0.042	Vortex
3	0.07	2	3.0, 5.54	0.056	Vortex
4	0.05	1	4.0, 7.39	0.030	Vortex
5	0.05	1	3.0, 5.54	0.040	Vortex
6	0.05	1	1.7, 3.14	0.070	Spiral
7	0.05	1	1.5, 2.77	0.080	Spiral

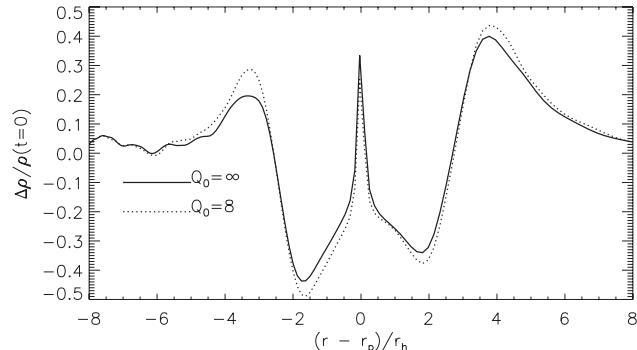


Figure 2. Gap profile opened by a giant planet in a non-self-gravitating disc (solid, Case 0) and a self-gravitating disc (dotted, Case 1). The azimuthally averaged relative density perturbation in the mid-plane is shown. Profiles for other cases are similar.

4.1 Vortex modes in weakly self-gravitating discs

We first compare Cases 0 and 1. The set-up for these runs is identical except that the disc potential is neglected in Case 0, which corresponds to the standard approach to model disc–planet systems (e.g. D’Angelo & Lubow 2010). Case 1 is the self-gravitating version of Case 0. We show below that, even with a minimum Toomre parameter of $Q_0 = 8$, disc self-gravity affects the evolution of the vortex instability.

Fig. 2 shows the mid-plane gap profile at $t = 25P_0$ in terms of the relative density perturbation. The snapshot is taken before instabilities develop. Case 1 has a slightly deeper gap and steeper gap edges than Case 0. This is because in a self-gravitating calculation such as Case 1, fluid bound to the planet adds to its mass. For both runs, the mass in the planet’s Hill sphere is $\sim 0.02M_p$.

4.1.1 Development of instability

Fig. 3 shows the evolution of the gap. It is clear that the vortex instability can develop in 3D with or without self-gravity. At $t = 30P_0$, two vortices are visible in Case 0, while three vortices are seen in Case 1 (in both cases there may be another vortex coinciding with the outer planetary wake).

For the weakly self-gravitating discs considered here, vortex modes with the same m have been excited, but without self-gravity vortices merge soon after formation. (In Case 0, the overdensity at the outer gap edge just ahead of the planet appears to be a merging vortex pair, rather than a single vortex from the instability.) At $t = 40P_0$, only a vortex pair remains in Case 0 while merging is delayed in Case 1 with a three-vortex configuration.

The differences between self-gravitating and non-self-gravitating simulations shown in Fig. 3 are similar to those seen in 2D simulations (Lyra et al. 2009b; Lin & Papaloizou 2011a). Lin & Papaloizou demonstrated that self gravity sets a minimal inter vortex distance so that merging is resisted in the $r\varphi$ plane. However, at this level of self-gravity merging is only delayed. The final state for both cases is a single vortex circulating the gap edge.

4.1.2 Effect of self-gravity on vortex vertical structure

Here we compare the final vortex in Cases 0 and 1. Fig. 4 shows their vertical structure in the Rz plane. The snapshots correspond to

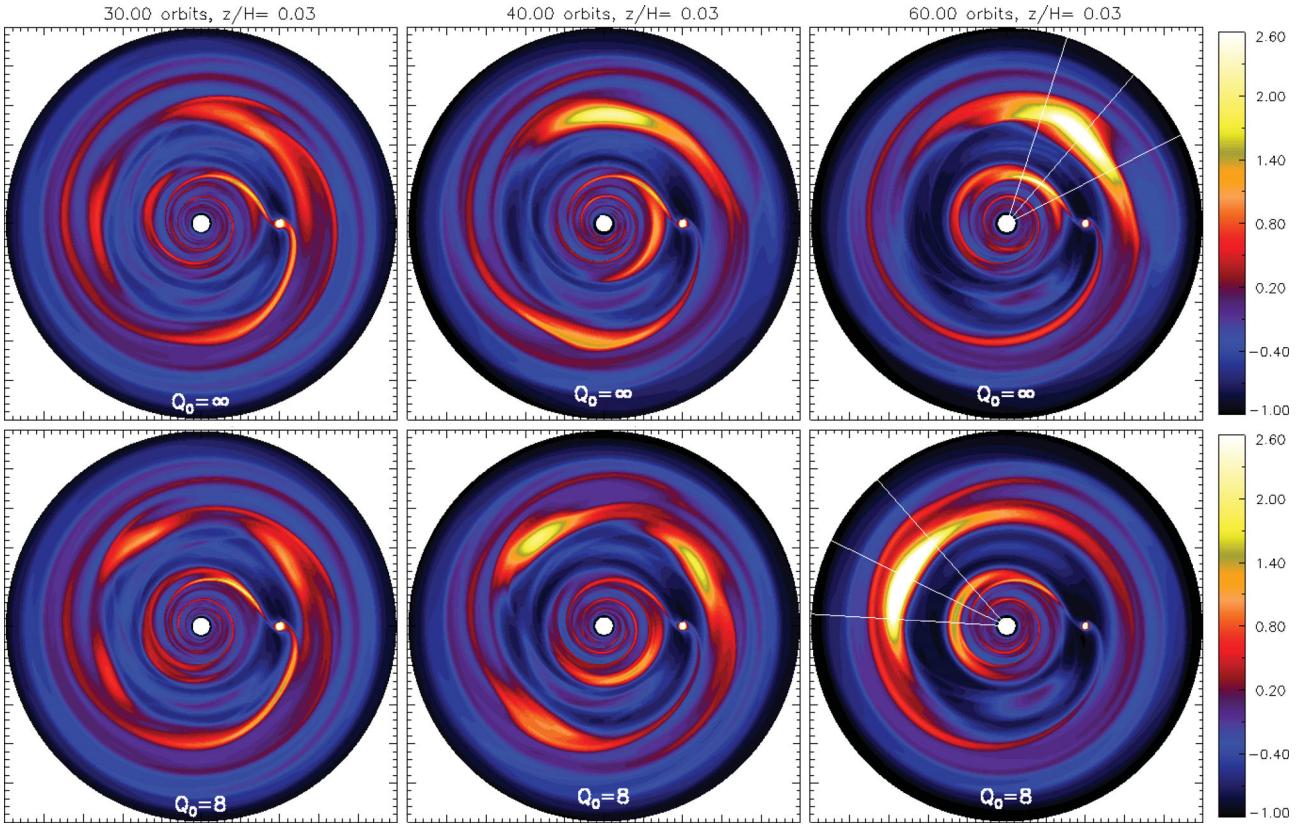


Figure 3. Development of the vortex instability in the non-self-gravitating Case 0 (top) and the weakly self-gravitating Case 1 (bottom). The mid-plane relative density perturbation is shown. White lines at intermediate azimuths correspond to the vortex centroid in Fig. 4, while the azimuthal range marked by the lines was set for averaging in Fig. 5.

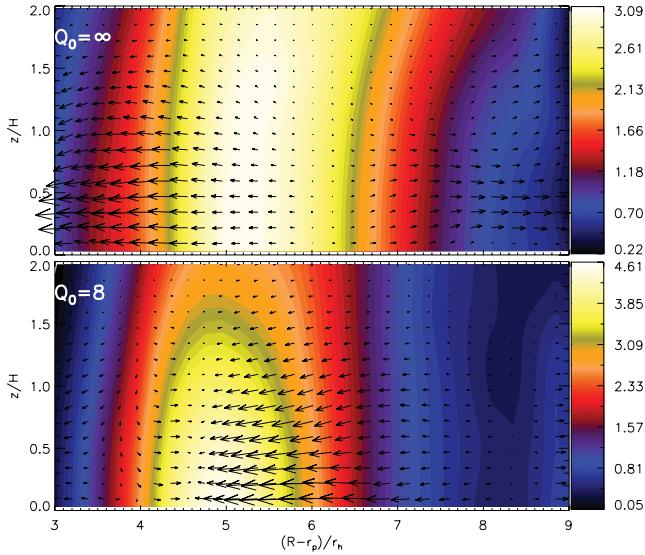


Figure 4. Vertical structure of a vortex in a non-self-gravitating disc (top, Case 0) and a weakly self-gravitating disc (bottom, Case 1). Contours show the relative density perturbation and the arrows are mass flux vectors ρu projected on to this plane. The snapshots correspond to the vortex centroids marked by white lines at intermediate azimuths in Fig. 3 (right-hand panel).

the vortex centroid, marked by white lines at intermediate azimuths in the right-hand panel of Fig. 3. Without self-gravity the vortex instability produces predominantly columnar disturbances in the relative density perturbation W .

Case 0 is consistent with recent linear calculations of the vortex instability in non-self-gravitating 3D discs, which show that W has essentially no vertical dependence at the radius where vortex-formation is expected (Lin 2012; Meheut et al. 2012b). In Fig. 4 the Case 0 vortex does show very weak vertical dependence near the upper boundary. This is likely due to the finite vertical domain adopted in our model.³

By contrast, the self-gravitating vortex in Case 1 clearly display stratification in the relative density perturbation. The vortex is more concentrated towards the mid-plane. With self-gravity, density enhancement in the vortex can be ~ 50 per cent higher than without.

In Fig. 5 we plot vertical velocities averaged over the vortices. The non-self-gravitating case typically involves positive vertical velocity (also seen in Meheut et al. 2012b). Perhaps not surprisingly, the self-gravitating case has as strong overdensity near the mid-plane to provide vertical acceleration, so on average the vertical velocity is negative. The precise values in Fig. 5 depend on the averaging procedure but the contrast in $\text{sgn}(u_z)$ between the two cases is robust (even when we consider the point in the $R\phi$ plane where density perturbation is largest and do not perform an average). The quantity $\langle u_z^2 / c_{\text{iso}}^2 \rangle^{1/2}$ behaves similarly in both cases.

Although there is vertical motion, it is worth noting that the vertical Mach number is only a few per cent. Fig. 3 also indicates that the flow is horizontal on average. This suggests approximate vertical hydrostatic balance. If self-gravity is included, then just

³ Linear calculations of vertically isothermal discs usually assume an atmosphere of infinite extent.

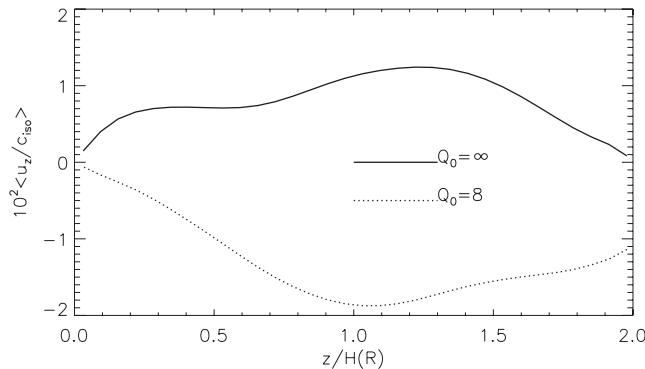


Figure 5. Average vertical velocity u_z , normalized by the sound speed, in a non-self-gravitating vortex (Case 0, solid) and a self-gravitating vortex (Case 1, dotted). The radial range for the average is taken over $r - r_p \in [4, 6.5]r_h$ for Case 0 and $r - r_p \in [4, 6]r_h$ for Case 1, since the latter vortex is slightly smaller (see Fig. 4). The azimuthal range is that marked by white lines in Fig. 3 (right-hand panel).

like in the set-up of the initial disc, the additional vertical force will enhance the mid-plane density. Hence, we observe a more stratified vortex in Case 1.

4.2 Vortex modes with moderate self-gravity

Cases 2–5 are all self-gravitating and develop the vortex instability. Cases 2 and 3 are continuations of Case 1 to more massive discs. Cases 4 and 5 are identical runs except a Jupiter-mass planet ($q = 10^{-3}$) and a thinner disc ($h = 0.05$) are adopted.

We compare the relative density perturbation between Cases 2 and 3 in Fig. 6. The snapshots are taken at $z = H$ but are very similar to razor-thin disc simulations (Li et al. 2009; Yu et al. 2010; Lin & Papaloizou 2011a). In both Cases the $m = 5$ vortex mode is excited (cf. $m = 3–4$ in Case 1). At $t = 30P_0$, vortices are just beginning to emerge in Case 3, whereas distinct blobs can already be identified in Case 2, with larger overdensities than vortices in Case 3. This indicates a stronger instability with decreasing strength

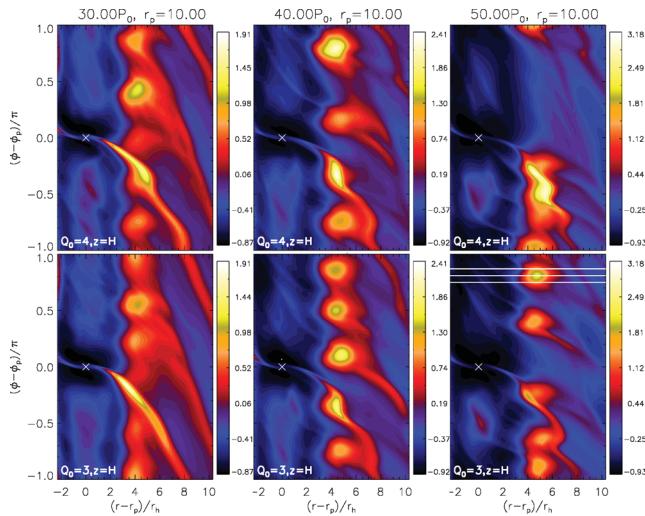


Figure 6. Development of the vortex instability at gap edges in moderately self-gravitating discs. The relative density perturbation is shown for Case 2 (top) and Case 3 (bottom) at $z = H$. The planet position is marked by the cross. Horizontal lines in the plot for $Q_0 = 3$ and $t = 50P_0$ indicate azimuths taken in Fig. 7.

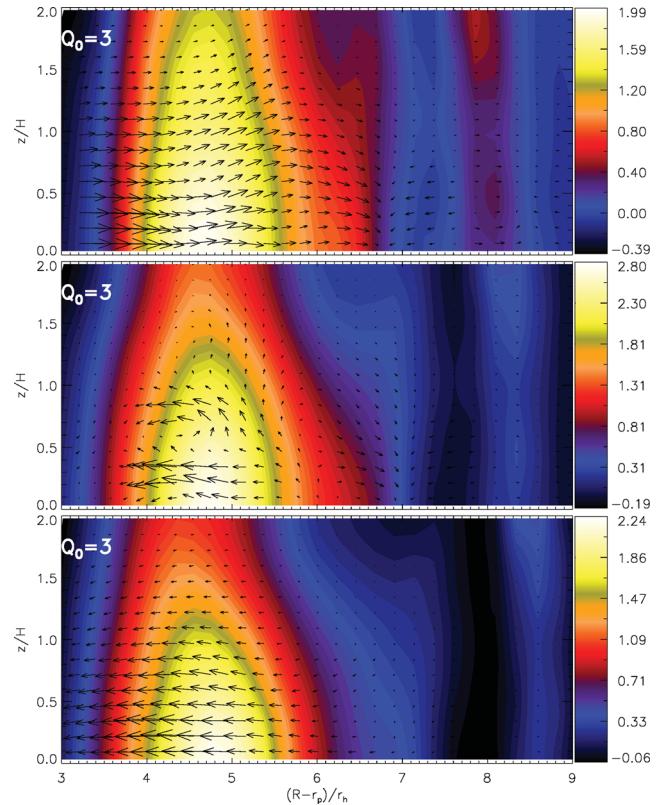


Figure 7. Vertical structure of a vortex in the moderately self-gravitating Case 3. The relative density perturbation are overlaid by the mass flux vectors $\rho \mathbf{u}$ projected on to this plane. The slices are taken at $t = 50P_0$ and azimuths $(\phi - \phi_p)/\pi = 0.86$ (top), 0.81 (middle) and 0.76 (bottom). These correspond to the horizontal lines marked in Fig. 6.

of self-gravity. Vortex merging ensues in Case 2 but does not occur in Case 3 within the simulation time-scale (unlike Case 1).

For razor-thin discs, Lin & Papaloizou (2011a) have shown that self-gravity has a stabilizing effect against vortex modes with low m . This contributes to favouring higher m modes and hence more vortices with increasing self-gravity. Fig. 6, together with results for Case 1, shows that this effect persists in 3D. Resisted merging, seen in 2D models, also occurs in 3D. Because this is due to vortices executing mutual horseshoe turns, only horizontal self-gravitational forces are important. We do not expect the vertical dimension to significantly affect vortex–vortex gravitational interaction.

4.2.1 A 3D vortex in Case 3

Here we examine the vortex in Case 3 marked by horizontal white lines in Fig. 6. Note that no merging has occurred. The vortex is radially located about $r_v \sim r_p + 5r_h$. It has azimuthal and radial sizes $\Delta\phi_v \sim 9h$ and $\Delta r_v \sim 2.5H$, respectively. Its mass is $M_v \sim 8.4 \times 10^{-4} M_*$.⁴ The corresponding Hill radius of the vortex $r_{hv} \sim 0.9H$ is smaller than its horizontal size but is comparable to its vertical size at the vortex centroid.

Fig. 7 shows the vertical structure of the vortex described above. As expected it is more stratified than in weakly self-gravitating discs

⁴ Since no merging has occurred, M_v can also be estimated by the mass contained in an annulus about r_v in the unperturbed disc divided by $m = 5$. This gives $8 \times 10^{-4} M_*$ if the annulus width is $2r_h$.

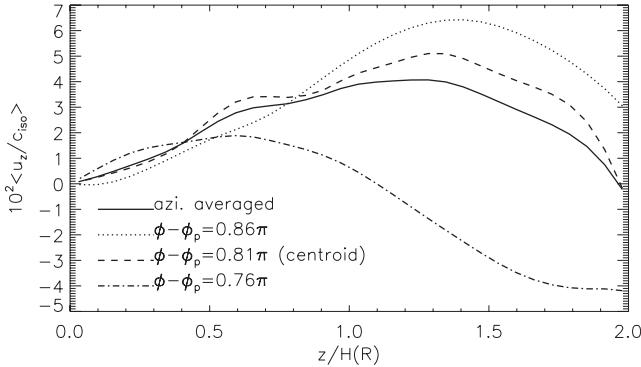


Figure 8. Vertical velocities u_z , normalized by the sound speed, for the vortex in the self-gravitating Case 3 shown in Fig. 6. The velocities are all radially averaged over $r - r_p \in [4.0, 5.5]r_h$. The solid line is also averaged over the vortex azimuthal extent. Azimuthal slices ahead of the vortex (dotted), at the centroid (dashed) and behind the vortex (dash-dotted) are also shown.

(Fig. 4), especially at the vortex centroid where the overdensity is maximum. The initial Keplerian Toomre parameter at $r = r_v$ is $Q = 4.3$. A density enhancement by a factor of $\gtrsim 2$ easily gives $Q \lesssim 2$, so that self-gravity in the perturbed state is dynamically important.

Ahead and behind the vortex centroid, the flow field is mostly horizontal (top and bottom panels in Fig. 7). It corresponds to the motion of an anticyclonic disc vortex common in 2D simulations (Li et al. 2001). In this plane, vertical motions only become significant compared to u_R close to the vortex centroid (middle panel). We plot the average u_z for the vortex in Fig. 8. Azimuthal slices at the vortex centroid, behind it and ahead of it, are also shown. The average vertical velocity is positive (solid line). This is qualitatively different to the merged vortex in the weakly self-gravitating Case 1 (Fig. 4). We have examined other vortices but could not identify a ‘typical’ vertical flow structure (cf. anticyclonic horizontal flow is generic). Although the vertical velocities appear to display a range of behaviour, the vertical Mach number is still very small.

We also find that the vortices have height-dependent azimuthal structure. Fig. 9 shows several slices in the ϕz plane. The choice of radii for these plots was based on the vortex described above (corresponding to the right vortex in the figure). The vortices are more columnar closer to the gap edge ($r - r_p = 4r_h$, top slice). Moving away from the gap edge, the vortices develop front-back asymmetry and are thinner with increasing height ($r - r_p = 5.5r_h$, bottom panel).

The increased 3D structure away from the gap edge could be related to density waves emitted by the vortex (Paardekooper, Lesur & Papaloizou 2010). Back-reaction of these waves on the vortex vertical structure, if any, is expected to be weaker on the side of the vortex adjacent to the low-density gap edge (top slice in Fig. 9).

The perturbed azimuthal velocities away from the vortex centroids again follow the expected anticyclonic motion in the horizontal plane (being positive interior and negative exterior to the centroid). However, unlike the previous Rz plots, here the vertical velocities can often be comparable or larger than the perturbed azimuthal velocities (e.g. $r - r_p = 5.5r_h$, bottom panel). Of course, the total azimuthal velocity is supersonic and therefore much larger than vertical velocities. As remarked above, the vortices do not share the same structure. For example, the left vortex centroid involves negative vertical velocity while $u_z > 0$ for the right vortex ($r - r_p = 4.75r_h$, middle panel).

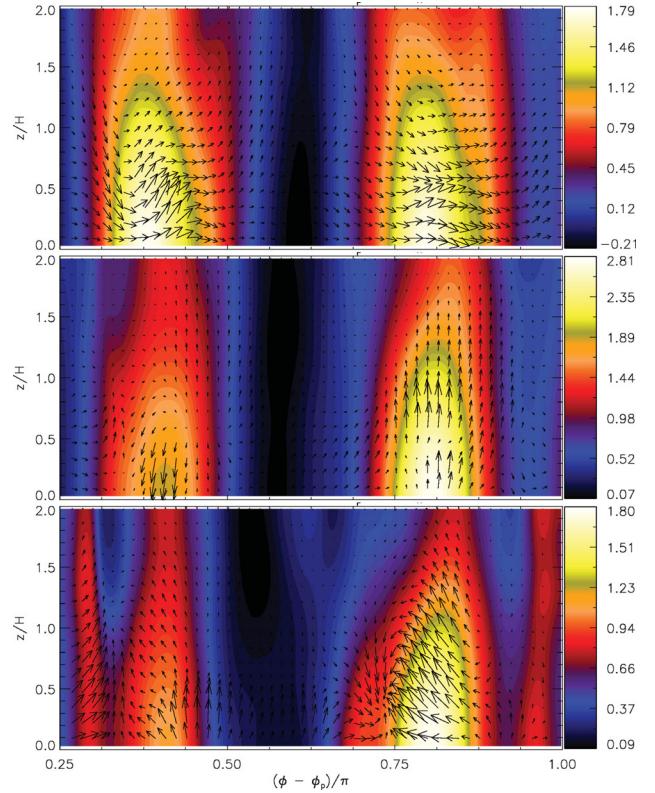


Figure 9. Azimuthal vortex structures in the self-gravitating Case 3 at $t = 50P_0$. The background flow is from left to right. The relative density perturbations are overlaid by the perturbed mass flux vectors $\rho(\mathbf{u} - R\Omega_k \hat{\phi})$ projected on to this plane. The slices are taken at radii $(r - r_p)/r_h = 4.0$ (top), 4.75 (middle) and 5.5 (bottom). These correspond to the flow just interior, at, and just exterior to the vortex centroid on the right.

4.2.2 Cases 4 and 5

Cases 4 and 5 are additional examples of the vortex instability in 3D discs with smaller h ($=0.05$) than the above runs (with $h = 0.07$). However, initial Toomre profile is nearly independent of h , so the strength of self-gravity is unchanged. We also used a smaller planetary mass, $q = 10^{-3}$ so the ratio $q/h^3 = 8$ is not significantly larger than Cases 2 and 3 ($q/h^3 = 5.8$). We checked that prior to instability, the outer gap edge profiles of these cases are similar.

We found that the instability grows slower with $h = 0.05$ as vortices become identifiable at $t = 35P_0$, compared to $t = 30P_0$ for Cases 2 and 3. A decrease in growth rate with sound speed (which is proportional to h) was already noted in 2D (Li et al. 2000). Thus, the effect of sound speed on the vortex instability remains unchanged by the 3D geometry.

Fig. 10 shows several snapshots of Cases 4 and 5 at $t = 50P_0$ near the upper disc boundary. As before, high in the atmosphere the density enhancement is weaker with increasing self-gravity. Consistent with 2D simulations, the more self-gravitating Case 5 develops the $m = 6$ vortex mode, whereas $m = 5$ vortices develop in Case 4 with weaker self-gravity. Merging is strongly resisted in these runs. In fact, Case 5 was extended to $t = 135P_0$ and only one vortex pair merged.

4.3 Edge-spiral modes in massive discs

Lin & Papaloizou (2011a,b) found that as the strength of self-gravity is increased, instability eventually shifts from localized vortices to

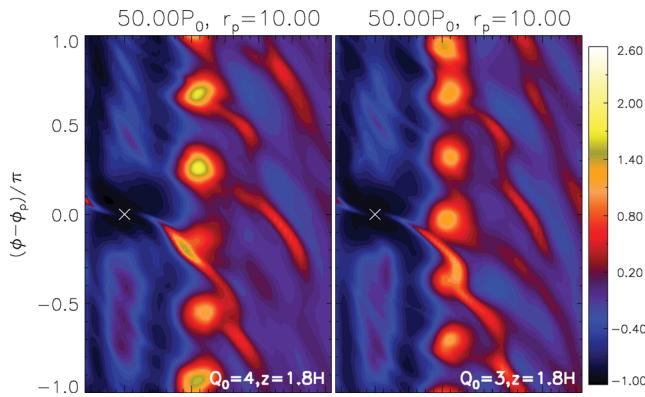


Figure 10. Vortex instability in Case 4 (left) and Case 5 (right). The relative density perturbation near the upper vertical boundary. These runs are similar to Cases 2 and 3, except a colder disc is employed ($h = 0.05$) and a smaller planetary mass is used to open the gap $q = 10^{-3}$.

low- m global spirals extending from the outer gap edge to the outer disc. The vortex instability is suppressed because they are associated with vortensity minima (which we will check in Section 5.2). Lin & Papaloizou (2011a) give a simple energy argument which shows that such association is not possible for sufficiently strong self-gravity. Association with vortensity maximum is favoured instead. These are the spiral instabilities. Lin & Papaloizou called them edge modes since they can still be considered associated with the gap edge.

We begin to observe edge modes in our 3D models when $Q_0 = 1.7$ (Case 6).⁵ Its evolution is depicted in Fig. 11. A $m = 2-3$ disturbance develops at the outer gap edge at $t = 35P_0$ and induces spiral density waves in the exterior disc through self-gravity. Interaction between the edge disturbance and the wider disc leads to the global spiral pattern seen at $t = 40P_0$ and $45P_0$. This coupling is necessary for instability. The edge mode is associated with a local $\max(Q)$ just inside the unperturbed outer gap edge. In Case 6, $\max(Q) \sim 8$ so a *local* gravitational instability is not possible. However, we can still consider the global edge mode as being composed of two parts: an edge disturbance where density perturbation is largest and the spiral arm it induces.

Comparison between the two heights in Fig. 11 shows that edge modes are significantly vertically stratified. Most of the density perturbation is confined near the mid-plane. Unlike vortex modes the spirals appear transient. Their amplitudes are much reduced by $t = 50P_0$, but are still visible. This is likely a radial boundary effect. The transition to a low-density annulus towards the outer disc edge is a result of the standard outflow boundary condition applied there. While the linear edge mode instability is insensitive to boundary conditions, its long-term evolution is affected. The outer disc edge can reflect waves back to gap edge to stabilize it, causing saturation (Lin & Papaloizou 2011b).

4.3.1 Vertical structure of an edge mode

Several vertical cuts of the edge mode in Case 6 are shown in Fig. 12. The slices are taken at azimuths marked by white lines in Fig. 11. The top three plots are associated with the edge disturbance,

⁵ Note that the transition from vortex to spiral modes with decreasing Q_0 is not abrupt. It is possible to have a mixture.

while the bottom plot is taken at the transition between the edge disturbance and its spiral arm extending to the outer disc.

Fig. 12 shows that the horizontal flow in the edge disturbance differs significantly from the vortex mode. If the second plot is considered the ‘centroid’, then the inward (outward) flow ahead (behind) it is in the opposite sense to anticyclonic motion associated with a vortex mode. The centroid is the most stratified region. Its mid-plane overdensity ~ 3.37 corresponds to a Toomre $Q \sim 1$ in the centroid, but we do not observe fragmentation.

As we move away from the centroid in the decreasing ϕ direction, the edge mode decreases in amplitude but occupies more of the vertical domain. The bottom plot in Fig. 12 shows a radial split: the columnar disturbance in $R - r_p \in [5.8, 6.5]r_h$ is the beginning of the spiral wave excited by the edge disturbance. The spiral arm is fully three dimensional.

In Fig. 13 we plot the average vertical velocity inside an edge disturbance of the spiral mode. A simulation with $Q_0 = 1.5$ (Case 7) is also plotted for comparison (its mid-plane density perturbation is shown in Fig. 14). The flow is typically downwards towards the mid-plane. This is expected because of strong vertical self-gravity, as these are massive discs and the mid-plane density enhancement is large. Note that u_z approaches zero again beyond $z/H \sim 1.5$ because of the imposed reflective upper boundary.

5 ADDITIONAL RESULTS ANALYSIS

In this section we examine some secondary quantities derived from the hydrodynamic simulations above. To keep this discussion concise, we will use selected simulations from above for illustration.

5.1 Three-dimensionality

A simple measure of three-dimensionality of the flow is to compare vertical to horizontal motion. Since we are interested in non-axisymmetric perturbations to the gap edge, we first Fourier transform the meridional momentum densities

$$(v_{Rm}, v_{zm}) \equiv \int_0^{2\pi} \rho \times (u_R, u_z) \exp(-im\phi) d\phi. \quad (17)$$

We define the three-dimensionality as $\Theta_m(z/H)$, where

$$\Theta_m^2 \equiv \frac{\langle |v_{zm}|^2 \rangle}{\langle |v_{zm}|^2 \rangle + \langle |v_{Rm}|^2 \rangle}, \quad (18)$$

and $\langle \cdot \rangle$ denotes a radial average. Admittedly, this is a crude measure, and exact values of Θ_m vary somewhat with details of the average. However, we have experimented with different averaging domains and found that the features described below are robust.

The top panel in Fig. 15 shows Θ_m for Cases 1–3 at $t = 40P_0$. The radial average is taken over $r - r_p \in [3, 7]r_h$. These are all vortex modes (see Figs 3 and 6). The flow becomes increasingly three dimensional away from the mid-plane but $\Theta_m = O(10^{-1})$ is small. In an averaged sense the flow is mostly horizontal. At the end of the simulation for Case 1, an azimuthally extended vortex dominates the flow, for which we measured $\Theta_1 \sim 0.2-0.3$. Thus, although vertical motion can become an appreciable fraction of horizontal motion, the former never dominates.

Θ_m for Cases 4–7 are shown in the bottom panel of Fig. 15. The radial average is performed over $r - r_p \in [2, 6]r_h$ because the global spirals in Cases 6 and 7 significantly protrude the gap edge. The snapshot is taken at $t = 50P_0$ for Cases 4 and 5, at $t = 40P_0$ for Case 6 and at $t = 30P_0$, so that the vortices and spirals have comparable overdensities at the gap edge. It also reflects the fact that spiral

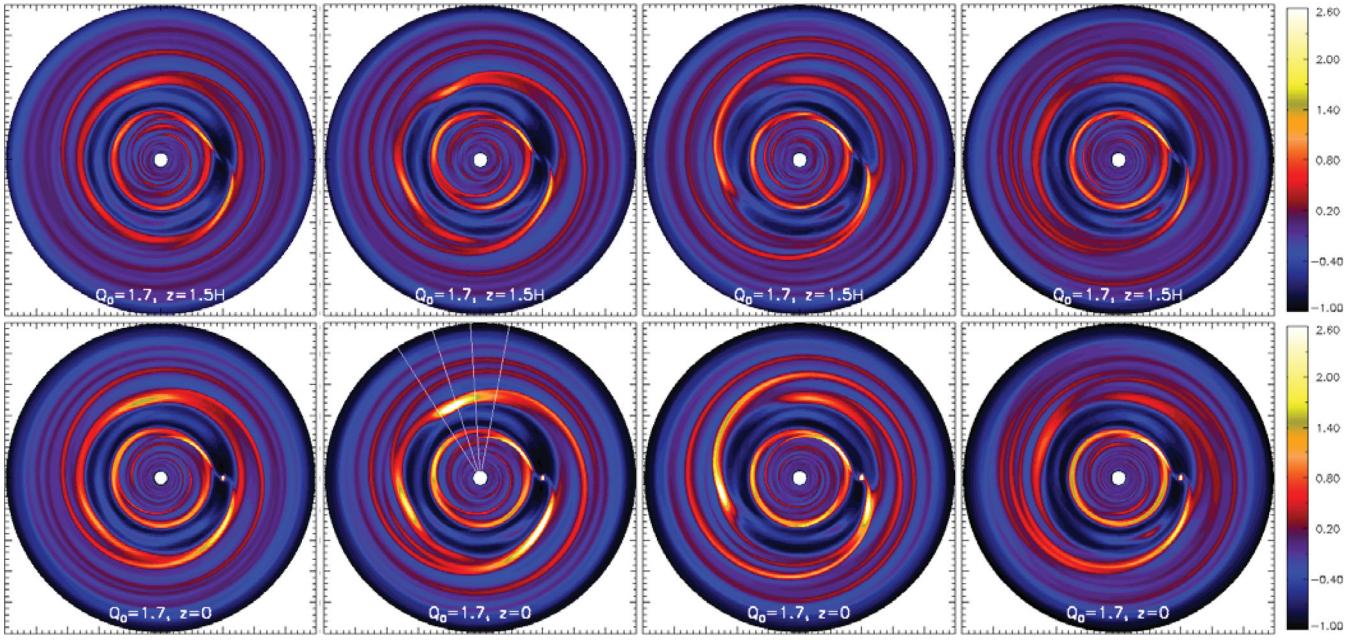


Figure 11. Case 6: development of the edge mode instability at the gap edge of a Jovian mass planet. The relative density perturbation in the mid-plane (bottom) and in the atmosphere (top) is shown at times (left to right) $t = 35P_0, 40P_0, 45P_0$ and $50P_0$. White lines indicate azimuthal cuts taken in Fig. 12.

modes are more unstable than vortex modes and develop earlier (Lin & Papaloizou 2011b). $\Theta_m \sim 0.2$ is again not particularly large, but the spiral modes are distinctly more three dimensional than vortex modes. This is likely due to additional vertical acceleration provided by the strong self-gravity in those cases.

5.2 Vortensity field

A fundamental distinction between the linear vortex and edge mode instability is their association with local vortensity minimum and maximum, respectively. In this section we compare vortensity fields of discs with vortex modes (Case 2) and edge modes (Case 7). More specifically, we examine the relative perturbation to the vertical component of vortensity:

$$\Delta\eta_z \equiv \frac{\eta_z - \eta_z(t=0)}{\eta_z(t=0)}, \quad (19)$$

where

$$\eta_z \equiv \frac{\hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}}{\rho}. \quad (20)$$

Fig. 16 compares $\Delta\eta_z$ in the mid-plane when vortices and spirals develop. For planetary gaps, vortensity maxima and minima are both located near the gap edges with characteristic separation of the local scale-height. The vortensity ring at the inner gap edge ($r - r_p \simeq -2r_h$) remains well defined. The vortex instability is associated with the local vortensity minimum near the outer gap edge – seen as localized closed contour lines centred about $r - r_p \sim 4r_h$. The vortensity ring at $r - r_p \sim +2r_h$ becomes distorted as a consequence of large-scale vortex formation just exterior to it. By contrast, the edge-spiral mode is associated with the local vortensity maximum. Their development inherently disrupts the vortensity rings. This is seen in the right-hand panel as the outer ring is broken up.

The vertical structures also differ. Fig. 17 compares $\Delta\eta_z$ at azimuths coinciding with a vortex or the edge disturbance of the spiral mode. Both instabilities involve $\Delta\eta_z < 0$. It is clear that the spiral mode has stronger vertical dependence. Its region of $\Delta\eta_z < 0$

becomes thinner away from the mid-plane. In the vortex case this region remains about the same width and $\Delta\eta_z$ is approximately uniform within it. While $\min(\Delta\eta_z)$ is of comparable magnitude, the vortensity ring at $r - r_p = 2r_h$ is much weaker and thinner in the spiral case ($\Delta\eta_z$ being a factor of ~ 4 smaller than the vortex case).

5.3 Disc–planet torques

The presence of non-axisymmetric disturbances at gap edges is expected to significantly affect disc–planet torques. It has been confirmed in 2D simulations that both vortex and spiral modes lead to oscillatory torques of either sign (Li et al. 2005; Lin & Papaloizou 2011b). In this section we measure the disc-on-planet torques in several of the above simulations to confirm the main features found in 2D.

We calculate the specific torque acting on the planet due to a mass element as

$$dT(\mathbf{r}) = \frac{\mathbf{r}_p \times \mathbf{r} G \rho(\mathbf{r}) d^3 r}{d_p^3} f(\mathbf{r}, \mathbf{r}_p), \quad (21)$$

$$f(\mathbf{r}, \mathbf{r}_p) \equiv 1 - \exp\left(-\frac{1}{2} \left|\frac{\mathbf{r} - \mathbf{r}_p}{\epsilon_c r_h}\right|^2\right). \quad (22)$$

The tapering function f reduces contributions from close to the planet, thereby reducing numerical artefacts arising from this region because of the diverging potential and limited resolution. We set the parameter $\epsilon_c = 1$ so that tapering does not significantly reduce contributions from the instabilities, since they develop at $\gtrsim 2r_h$ away from the planet’s orbital radius.

Fig. 18 shows the disc-on-planet torques in Cases 3 and 5, which develop the $m = 5$ and 6 vortex modes, respectively. These plots are qualitatively similar to 2D simulations (e.g. Li et al. 2005). The torques oscillate on orbital time-scales and its instantaneous values can be of either sign. However, upon averaging over the simulation we find that the total torques are negative in both cases. This means that inward migration is still favoured.

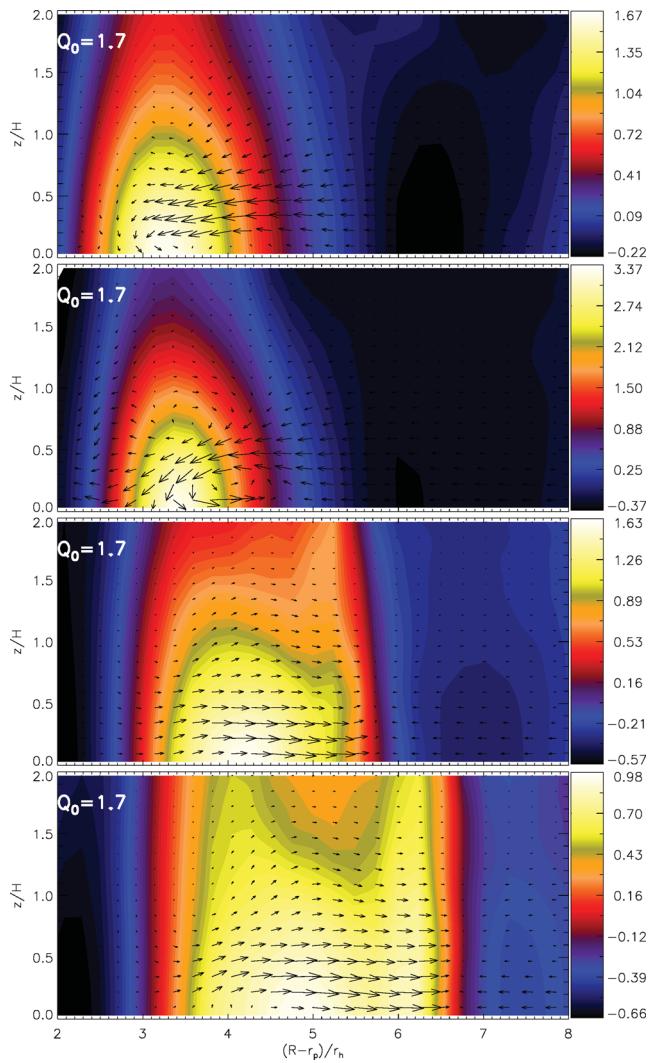


Figure 12. Vertical structure of the spiral instability at the gap edge in Case 6. The relative density perturbations are overlaid by the mass flux vectors $\rho\mathbf{u}$ projected on to this plane. The slices are taken at $t = 40P_0$ and azimuths marked by white lines in Fig. 11 (with decreasing ϕ from top to bottom).

We extended Case 5 to $t = 135P_0$ and found that the vortices have similar overdensities as at $t = 50P_0$. However, Fig. 18 shows that the torque oscillation amplitudes decrease towards the end of Case 5 compared to $t \in [40, 80]P_0$. At $t = 50P_0$ the vortices are located in $r - r_p \in [3.5, 5.5]r_h$ but by $t = 135P_0$ they are located in $r - r_p \in [4, 6]r_h$. Given that $t \in [40, 80]P_0$ is only 20 P_0 to 50 P_0 after the planet potential has been fully introduced, gap formation is probably ongoing during this time. We expect torque amplitudes to be larger during gap formation since the vortices lie closer to the planet.

Next we examine disc–planet torques in the presence of the spiral modes. The top panel of Fig. 19 shows the instantaneous disc-on-planet torques. Contributions from the inner disc ($r < r_p$) and outer disc ($r > r_p$) are plotted separately for comparison with Fig. 18(b) in Lin & Papaloizou (2011b), which is similar to the present plot. Large oscillations in the outer torque due to edge mode spirals cause the total torque to be positive or negative at a given instant.

Unlike the vortex modes, Fig. 19 shows that spiral modes can lead to a positive running-time-averaged torques (bottom panel).

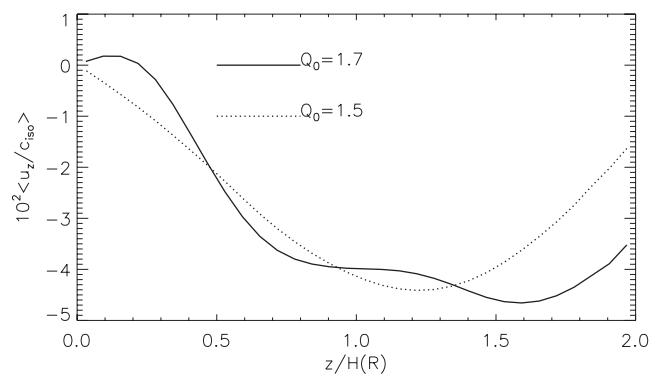


Figure 13. Average vertical velocity of the edge disturbance in Case 6 (solid) and Case 7 (dotted). The average is taken over $r - r_p \in [2.5, 5.5]r_h$. The azimuthal range is indicated by white lines in Fig. 12 for Case 6 (first and third azimuth in the clockwise direction) and in Fig. 14 for Case 7.

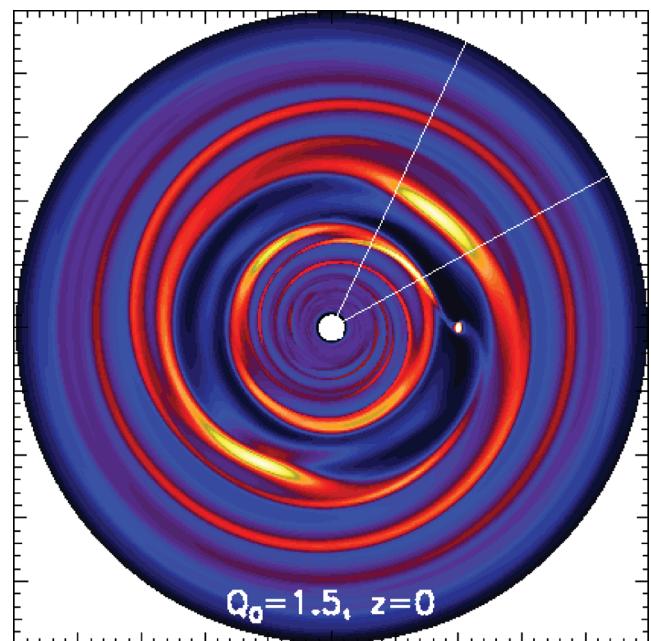


Figure 14. The $m = 2$ edge mode in Case 7. The colour bar range is the same as in Fig. 11. White lines indicate the azimuthal range taken for averaging in Fig. 13 (dotted line). The relative density perturbation of the edge disturbance in the Rz plane is very similar to Case 6 (Fig. 12).

The average torques become more positive with time after spiral modes develop, and with increasing self-gravity (which increases the instability strength). This was also observed in high-resolution 2D simulations in Lin & Papaloizou (2011b). There it was suggested that the creation of large ‘voids’ in between spiral arms decreases the time-averaged density in the planet-induced wakes, thereby reducing associated torque magnitudes. Since the outer planetary wake normally provides a negative torque, the spiral modes make the total torque more positive. The similarity between 2D and 3D results indicates that outward migration induced by spiral modes, which was seen in 2D by Lin & Papaloizou (2011b, 2012), will also operate in 3D.

6 SUMMARY AND DISCUSSION

We have performed customized numerical simulations of 3D self-gravitating discs, in which an embedded satellite or planet has

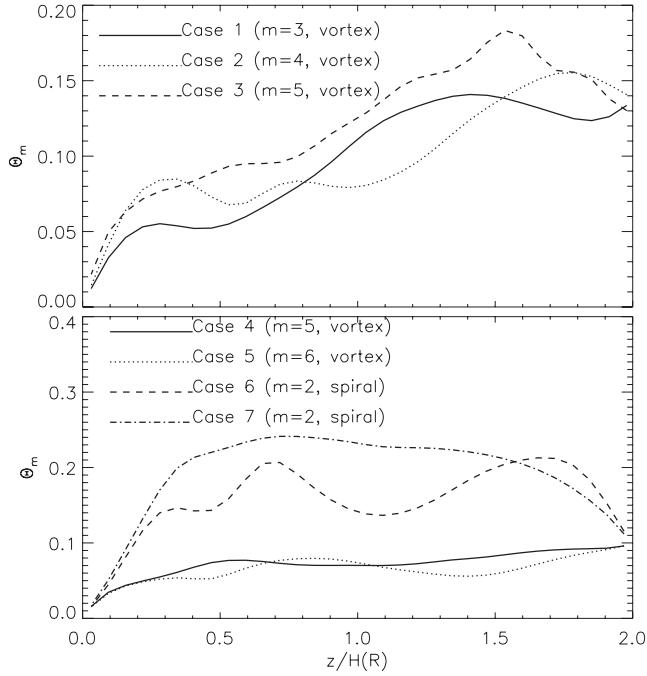


Figure 15. Three-dimensionality of the non-axisymmetric flow near the outer gap edge. Top: Cases 1–3 (vortex modes). Bottom: Cases 4 and 5 (vortex modes) and Cases 6 and 7 (spiral modes). The azimuthal wavenumber m is chosen to match the number of vortices or large-scale spirals observed.

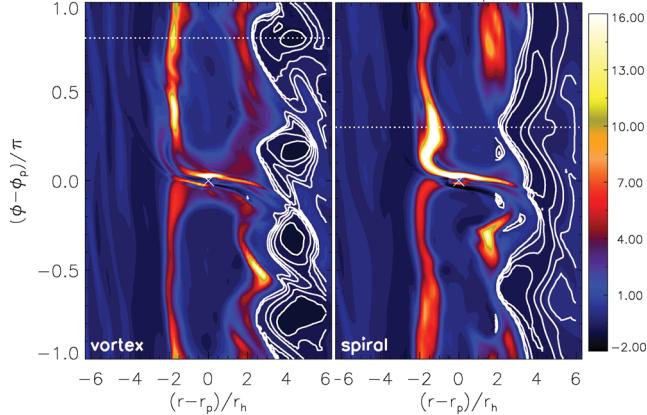


Figure 16. Relative perturbation to the vertical component of vortensity at the mid-plane in a disc with the vortex instability (left, Case 2 at $t = 40P_0$) and the spiral instability (right, Case 7 at $t = 30P_0$). Negative perturbations in the region $r - r_p \in [2, 6]r_h$ are outlined by white lines. Dotted horizontal lines indicate azimuthal cuts taken in Fig. 17.

opened a gap. We explicitly verified in 3D the main results on gap stability previously obtained from 2D calculations (Koller et al. 2003; Li et al. 2005; de Val-Borro et al. 2007; Meschiari & Laughlin 2008), in particular those by Lin & Papaloizou (2011a,b).

Planetary gaps are potentially unstable because of the existence of vortensity extrema generated by planet-induced shocks. Disc-satellite interaction also occurs in other systems such as stars in black hole accretion discs (Baruteau, Cuadra & Lin 2011; Kocsis, Yunes & Loeb 2011; McKernan et al. 2011). Furthermore, gaps opened by satellites are just one type of structured disc. Other examples include dead-zone boundaries mentioned in Section 1 and transition discs (Regály et al. 2012).

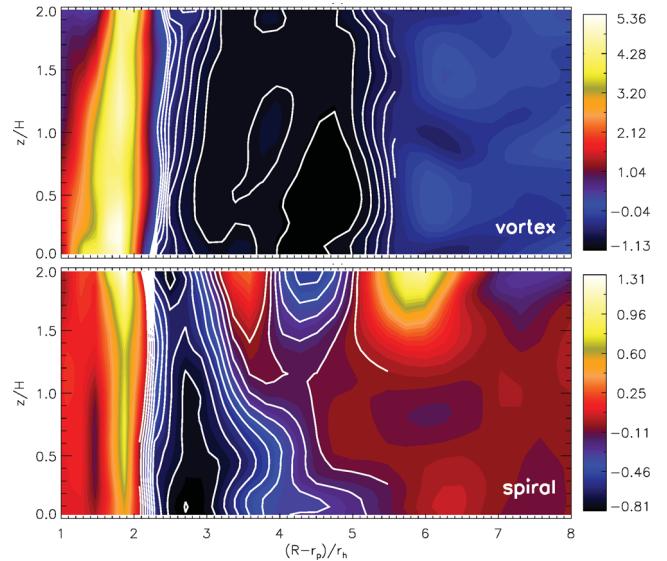


Figure 17. Relative vertical vortensity perturbation associated with the vortex instability (top) and spiral instability (bottom). The slices are taken at azimuths shown by white dotted lines in Fig. 16. Negative perturbations in the region $r - r_p \in [2, 5.5]r_h$ are outlined by white lines.

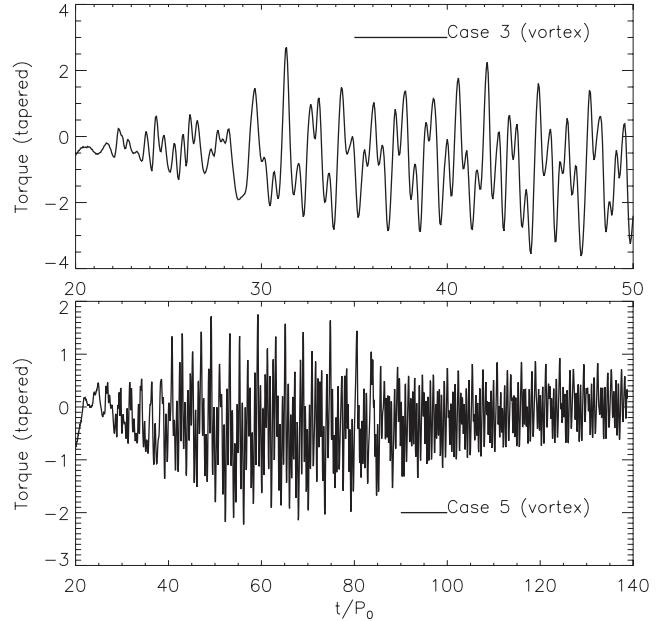


Figure 18. Instantaneous disc-on-planet torques in simulations where the vortex mode develops. Top: Case 3. Bottom: Case 5. Note that Case 5 has been extended to $t = 135P_0$.

Thus, although we were motivated by previous works on planetary gap stability, and hence considered disc–planet systems, we expect our results to be applicable to discs with radial structure of other origin, provided that the vortensity profiles involve stationary points and therefore prone to the same instabilities.

6.1 Confirmed 2D results

We demonstrated the development of the vortex instability at the outer gap edge opened by a giant planet in 3D discs. We first considered a non-self-gravitating disc, in which a few vortices develop then quickly merge. The quasi-steady state is a single azimuthally

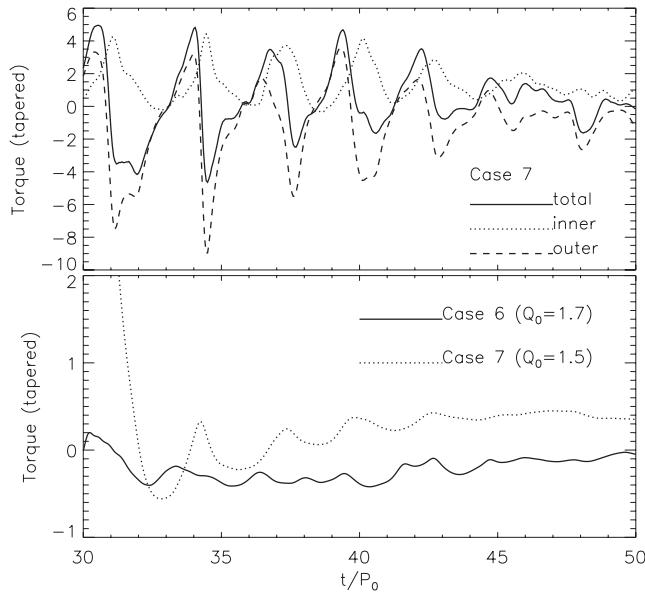


Figure 19. Disc-on-planet torques in the presence of spiral modes associated with the outer gap edge. Top: total torque (solid), torque from the inner disc (dotted) and from the outer disc (dashed) in Case 7. Bottom: time-averaged torques in Case 6 (solid) and Case 7 (dotted).

extended vortex at the gap edge. This evolution is similar to the 2D simulation in de Val-Borro et al. (2007).

We also showed that the effect of self-gravity on the vortex instability observed in 2D (Lyra et al. 2009b; Lin & Papaloizou 2011a) persists in 3D. We observe more vortices as the strength of self-gravity is increased by increasing the density scale. These vortices resist merging on dynamical time-scales. In our disc models with $Q_0 = 3$, the multi-vortex configuration lasts until the end of the simulations.

As expected from 2D linear theory (Lin & Papaloizou 2011b), vortex modes are suppressed in our massive disc models. Instead, a global spiral instability develops which is associated with the local vortensity maximum just inside the outer gap edge. These are distinct from vortex modes since self-gravity is essential.

Our limited numerical resolution does not permit accurate disc–planet torque measurements, but the qualitative effect of the vortex and spiral instabilities, previously studied in 2D, has been reproduced in 3D – oscillatory torques of either sign and the tendency for spiral modes to provide on average a positive torque. The similarity to 2D results is not surprising since for giant planets $r_h \gtrsim H$, so the razor-thin disc approximation is expected to be valid as far as disc–planet interaction is concerned. Furthermore, vertical self-gravity increases the mid-plane density while reducing that in the atmosphere (Appendix A), so that given a fixed temperature profile the 2D approximation is even better for self-gravitating discs.

It is worth mentioning here that in shearing sheet simulations, Mamatsashvili & Rice (2009) found self-gravity to favour vortices of smaller scale. They initialized a local patch of an unstructured disc with random velocity perturbations. Their gravito-turbulent discs are dominated by small vortices limited by the local Jeans length (which is smaller than the scale-height). Without self-gravity, they merge to form larger vortices. These observations are similar to the above results for the vortex mode. However, the set-ups are quite different as we consider radially structured, laminar global discs. Our large-scale vortices develop from a linear instability and have horizontal sizes of a few scale-heights. Thus, confirmation of the

above results in 3D is only valid for the edge instabilities considered in this paper.

6.2 3D effects

In our simulations the dominant 3D effect is vertical self-gravity on the density field. In the non-self-gravitating limit, the relative density perturbation associated with a vortex is columnar with weak vertical dependence. This is consistent with 3D linear and non-linear simulations (Meheut et al. 2010, 2012a,b; Lin 2012).

As the strength of self-gravity is increased, vortices become more vertically stratified – they are condensed towards the mid-plane. For moderately self-gravitating discs, the vortex mid-plane density enhancement can be twice that near the upper boundary. The spiral modes display significant vertical structure near the gap edge, while the density waves they launch in the outer disc are columnar. The latter is probably due to the chosen EOS (see below).

The effect of vertical self-gravity on the vortex mode is seen even in our least massive disc model with $Q_0 = 8$. One can consider an initially smooth disc which is justified to be non-self-gravitating. However, this approximation may become less good with the creation of a vortensity minimum, because it can also be a local minimum in the Toomre Q (equation 2). That is, the Toomre parameter is decreased with the development of local radial structure (such as a density bump). The non-self-gravitating approximation worsens further when the vortex instability associated with $\min(\eta)$ develops, because the vortices are regions of enhanced density, especially if they merge into a single large vortex. Thus, the non-self-gravitating approximation is not guaranteed to hold in the perturbed state even if it does in the initial disc.

It is interesting to note that linear calculations of the vortex instability in vertically isothermal, non-self-gravitating discs show that the vertical velocity vanishes at the vortex centroid (Lin 2012; Meheut et al. 2012b), but this is not observed in non-linear calculations (Meheut et al. 2012b, and in the present simulations). This contrasts to their anticyclonic horizontal flow, which can be computed in linear theory and seen in hydrodynamic simulations (Li et al. 2000, 2001). This suggests that vertical motions in this case may be associated with secondary processes (e.g. Goodman, Narayan & Goldreich 1987).

Although we observe somewhat complicated vertical flow for both vortex and spiral instabilities, the vertical Mach number is at most a few per cent in magnitude. Also, the magnitude of vertical motion is at most ~ 20 per cent of the radial motion on average. This suggests that the disturbances at the gap edge are roughly two dimensional in the present disc models. This would be consistent with early studies which find that instabilities associated with corotation singularities are two dimensional (Papaloizou & Pringle 1985; Goldreich et al. 1986). Recent 3D linear calculations also find that, even with a vertical temperature gradient, the vortex mode (without self-gravity) is largely 2D near the vortensity minimum (Lin 2012).

6.3 Caveats and outlooks

In order to provide 3D examples of previous 2D results, a range of disc models had to be simulated, each for many dynamical time-scales with full self-gravity. To maintain reasonable computational cost, numerical resolution in the $r\phi$ plane is much reduced compared to razor-thin disc simulations ($\sim 4\text{--}6$ cells per H compared to ~ 16 in 2D). Despite this, our plots in the $r\phi$ plane closely resemble those obtained from 2D simulations.

On the other hand, the low resolutions adopted here are unlikely to capture elliptic instabilities which may destroy vortices in 3D (Lesur & Papaloizou 2009, 2010). This might not be a serious issue when the condition for the vortex instability is maintained (by a planet in the present context). Also, the vortex grows on dynamical time-scales whereas the elliptic instability takes much longer (Lesur & Papaloizou 2009). The initial development of vortices is not expected to be suppressed by the elliptic instability (as found by Meheut et al. 2012b).

The vortex instability produces smaller vortices with increasing self-gravity. According to Lesur & Papaloizou (2009), vortices with small aspect ratios ($\lesssim 4$) are strongly unstable in 3D, but note that their model is a local patch of a smooth disc without self-gravity, whereas we considered a gap edge in a global self-gravitating disc.

Inclusion of self-gravity may change the stability properties of vortices. In particular, we found that a vortex can flatten somewhat under its own weight. Lithwick (2009) suggested that vertical gravity helps to stabilize vortices in 3D. Vertical self-gravity can enhance this effect. Lithwick found in local 3D simulations that ‘tall’ vortices are unstable, whereas ‘short’ vortices survive as in 2D (Godon & Livio 1999). We may expect the more stratified vortices formed in self-gravitating discs to be more stable than those in non-self-gravitating discs (Barranco & Marcus 2005). The elliptic instability is also weakened by stratification (Lesur & Papaloizou 2009), so vertical self-gravity may also be stabilizing in this respect.

We have adopted the locally isothermal EOS for simplicity and direct comparison with previous 2D results. This EOS limits the 3D structure of density waves compared to thermally stratified discs (e.g. Lin, Papaloizou & Savonije 1990; Lubow & Ogilvie 1998; Ogilvie & Lubow 1999), which can cause refraction of waves out of the mid-plane. The locally isothermal EOS represents the limit of efficient cooling (Boss 1998). This might apply in optically thin regions of a disc,⁶ but is violated if high densities develop, such as self-gravitating clumps (Pickett et al. 2000). Our edge instabilities only reach moderate overdensities. Nevertheless, enhanced vertical stratification of the edge disturbances observed in our simulations is likely exaggerated by the EOS. Clearly, it is necessary to extend models of edge instabilities in self-gravitating discs to include an appropriate energy equation.

Another important issue is vertical boundary conditions. We simulated a thin disc and imposed a reflecting upper boundary to prevent mass loss from above. This set-up may enhance the two-dimensionality of the problem. The vortex instability tends to involve the entire column of fluid, especially in the weak self-gravity regime. It is therefore a global instability in z . We suspect that the spiral mode is less affected by vertical boundary conditions because the instability tends to concentrate material at the mid-plane. Future simulations will consider varying the vertical domain size and upper disc boundary conditions.

Our torque measurements indicate that migration will be significantly affected by the vortex and spiral instabilities. Preliminary 3D simulations with a freely migrating planet have been performed. We recover the vortex–planet scattering and the spiral-induced outward migration described by Lin & Papaloizou (2010, 2012). The disc–planet torque is determined by the density field, and the above instabilities have density perturbations that either have weak vertical dependence or concentrated at the mid-plane. Thus, we believe that, at present, 2D simulations are more advantageous for stud-

ies focusing on migration, because high resolution is feasible and needed.

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REFERENCES

- Armitage P. J., 2011, *ARA&A*, 49, 195
 Barranco J. A., Marcus P. S., 2005, *ApJ*, 623, 1157
 Baruteau C., Cuadra J., Lin D. N. C., 2011, *ApJ*, 726, 28
 Boss A. P., 1980, *ApJ*, 236, 619
 Boss A. P., 1998, *ApJ*, 503, 923
 Christodoulou D. M., Narayan R., 1992, *ApJ*, 388, 451
 Cossins P., Lodato G., Clarke C., 2010, *MNRAS*, 401, 2587
 Crespe E., Gonzalez J.-F., Arena S. E., 2011, in Alecian G., Belkacem K., Samadi R., Valls-Gabaud D., eds, SF2A-2011: Proc. Annual meeting of the French Society of Astronomy and Astrophysics, p. 469 (available at: <http://sf2a.cesr.fr/php/spip/spip.php?article350>)
 D’Angelo G., Lubow S. H., 2010, *ApJ*, 724, 730
 de Val-Borro M., Artymowicz P., D’Angelo G., Peplinski A., 2007, *A&A*, 471, 1043
 Gammie C. F., 1996, *ApJ*, 457, 355
 Godon P., Livio M., 1999, *ApJ*, 523, 350
 Goldreich P., Goodman J., Narayan R., 1986, *MNRAS*, 221, 339
 Goodman J., Narayan R., 1988, *MNRAS*, 231, 97
 Goodman J., Narayan R., Goldreich P., 1987, *MNRAS*, 225, 695
 Hayes J. C., Norman M. L., Fiedler R. A., Bordner J. O., Li P. S., Clark S. E., ud-Doula A., Mac Low M., 2006, *ApJS*, 165, 188
 Kocsis B., Yunes N., Loeb A., 2011, *Phys. Rev. D*, 84, 024032
 Koller J., Li H., Lin D. N. C., 2003, *ApJ*, 596, L91
 Lesur G., Papaloizou J. C. B., 2009, *A&A*, 498, 1
 Lesur G., Papaloizou J. C. B., 2010, *A&A*, 513, A60
 Li H., Finn J. M., Lovelace R. V. E., Colgate S. A., 2000, *ApJ*, 533, 1023
 Li H., Colgate S. A., Wendroff B., Liska R., 2001, *ApJ*, 551, 874
 Li H., Li S., Koller J., Wendroff B. B., Liska R., Orban C. M., Liang E. P. T., Lin D. N. C., 2005, *ApJ*, 624, 1003
 Li H., Lubow S. H., Li S., Lin D. N. C., 2009, *ApJ*, 690, L52
 Lin D. N. C., Papaloizou J., 1986, *ApJ*, 309, 846
 Lin D. N. C., Papaloizou J. C. B., Savonije G. J., 1990, *ApJ*, 364, 326
 Lin M.-K., 2012, *ApJ*, 754, 21
 Lin M.-K., Papaloizou J. C. B., 2010, *MNRAS*, 405, 1473
 Lin M.-K., Papaloizou J. C. B., 2011a, *MNRAS*, 415, 1426
 Lin M.-K., Papaloizou J. C. B., 2011b, *MNRAS*, 415, 1445
 Lin M.-K., Papaloizou J. C. B., 2012, *MNRAS*, 421, 780
 Lithwick Y., 2009, *ApJ*, 693, 85
 Lovelace R. V. E., Hohlfeld R. G., 1978, *ApJ*, 221, 51
 Lovelace R. V. E., Li H., Colgate S. A., Nelson A. F., 1999, *ApJ*, 513, 805
 Lubow S. H., Ogilvie G. I., 1998, *ApJ*, 504, 983
 Lyra W., Mac Low M.-M., 2012, *ApJ*, 756, 62
 Lyra W., Johansen A., Klahr H., Piskunov N., 2008, *A&A*, 491, L41
 Lyra W., Johansen A., Klahr H., Piskunov N., 2009a, *A&A*, 493, 1125
 Lyra W., Johansen A., Zsom A., Klahr H., Piskunov N., 2009b, *A&A*, 497, 869
 Mamatsashvili G. R., Rice W. K. M., 2009, *MNRAS*, 394, 2153
 McKernan B., Ford K. E. S., Lyra W., Perets H. B., Winter L. M., Yaqoob T., 2011, *MNRAS*, 417, L103
 Meheut H., Casse F., Varniere P., Tagger M., 2010, *A&A*, 516, A31
 Meheut H., Keppens R., Casse F., Benz W., 2012a, *A&A*, 542, A9
 Meheut H., Yu C., Lai D., 2012b, *MNRAS*, 422, 2399

⁶ For example, Cossins, Lodato & Clarke (2010) find optical depths $\tau < 0.2$ beyond ~ 100 au in their models of protoplanetary discs.

- Meschiari S., Laughlin G., 2008, *ApJ*, 679, L135
 Ogilvie G. I., Lubow S. H., 1999, *ApJ*, 515, 767
 Paardekooper S., Lesur G., Papaloizou J. C. B., 2010, *ApJ*, 725, 146
 Papaloizou J. C. B., Lin D. N. C., 1989, *ApJ*, 344, 645
 Papaloizou J. C. B., Pringle J. E., 1984, *MNRAS*, 208, 721
 Papaloizou J. C. B., Pringle J. E., 1985, *MNRAS*, 213, 799
 Papaloizou J. C. B., Pringle J. E., 1987, *MNRAS*, 225, 267
 Papaloizou J. C., Savonije G. J., 1991, *MNRAS*, 248, 353
 Pepliński A., Artymowicz P., Meléma G., 2008, *MNRAS*, 386, 164
 Pickett B. K., Durisen R. H., Cassen P., Mejía A. C., 2000, *ApJ*, 540, L95
 Regály Z., Juhász A., Sándor Z., Dullemond C. P., 2012, *MNRAS*, 419, 1701
 Sellwood J. A., Kahn F. D., 1991, *MNRAS*, 250, 278
 Toomre A., 1964, *ApJ*, 139, 1217
 Varnière P., Tagger M., 2006, *A&A*, 446, L13
 Yu C., Li H., Li S., Lubow S. H., Lin D. N. C., 2010, *ApJ*, 712, 198

APPENDIX A: MODIFICATION TO VERTICAL STRUCTURE BY SELF-GRAVITY

We describe a simple procedure to set up the vertical structure of a locally isothermal, self-gravitating disc. We imagine that setting up a non-self-gravitating disc then slowly switches on the vertical force due to self-gravity. We expect the mid-plane density to increase at the expense of gas density higher in the atmosphere. It is assumed that the temperature profile remains unchanged.

Vertical hydrostatic equilibrium between gas pressure, stellar gravity and self-gravity reads

$$c_{\text{iso}}^2(R) \frac{\partial \ln \rho}{\partial z} = -\frac{\partial \Phi_*}{\partial z} - \frac{\partial \Phi}{\partial z}. \quad (\text{A1})$$

Assuming a smooth radial density profile, we use the plane-parallel atmosphere approximation for the disc potential, i.e.

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho. \quad (\text{A2})$$

Next, we write the density field as

$$\rho(R, z) = \rho_N(R, z) \times \beta(z; R), \quad (\text{A3})$$

where ρ_N is the density field corresponding to the non-self-gravitating disc:

$$c_{\text{iso}}^2(R) \frac{\partial \ln \rho_N}{\partial z} = -\frac{\partial \Phi_*}{\partial z}, \quad (\text{A4})$$

$$\begin{aligned} \rho_N &= \frac{\Sigma(R)}{\sqrt{2\pi} H(R)} \exp(-z^2/2H^2) \\ &\equiv \rho_{N0}(R) \exp(-z^2/2H^2), \end{aligned} \quad (\text{A5})$$

where $\rho_{N0} = \rho_N(R, z = 0)$ is the mid-plane density. The explicit expression for ρ_N above assumes a thin disc. The function β de-

scribes the modification to the local density in order to be consistent with self-gravity. By construction, its governing equation is

$$c_{\text{iso}}^2(R) \frac{\partial^2 \ln \beta}{\partial z^2} = -4\pi G \rho_N \beta. \quad (\text{A6})$$

Let

$$\chi = \ln \beta - z^2/2H^2, \quad (\text{A7})$$

$$\xi = \left(\frac{4\pi G \rho_{N0}}{c_{\text{iso}}^2} \right)^{1/2} z, \quad (\text{A8})$$

then the governing equation can be written in dimensionless form

$$\begin{aligned} \frac{\partial^2 \chi}{\partial \xi^2} &= -K - \exp \chi, \\ K &\equiv \frac{c_{\text{iso}}^2}{4\pi G \rho_{N0} H^2}. \end{aligned} \quad (\text{A9})$$

Note that $K(R)$ is proportional to the local Keplerian Toomre parameter. Equation (A9) can be further reduced to a first-order differential equation, but this is unnecessary because we pursue a numerical solution at the end. Appropriate boundary conditions are

$$\chi(z = 0) = \ln \beta_0, \quad (\text{A10})$$

$$\left. \frac{\partial \chi}{\partial \xi} \right|_{z=0} = 0. \quad (\text{A11})$$

$\beta_0(R)$ is the mid-plane density enhancement. To determine its value, we impose the surface density before and after modification by self-gravity to remain the same. Then we require

$$\begin{aligned} F(\beta_0) &\equiv \sqrt{\frac{2K}{\pi}} \int_0^{n/\sqrt{K}} \exp \chi(\xi; \beta_0) d\xi - \text{erf} \left(\frac{n}{\sqrt{2}} \right) \\ &= 0, \end{aligned} \quad (\text{A12})$$

where n is the number of scale-heights of the non-self-gravitating disc we originally considered. At a given cylindrical radius R , we solve equation (A12) using Newton–Raphson iteration. Each iteration involves integrating the governing ordinary differential equation for χ (equation A9). At the end of the iteration, we have $\beta(z; R)$ and the mid-plane enhancement $\beta_0(R)$.

We comment that the procedure outlined above can be extended to polytropic discs. In this case, there is an additional unknown – the new disc thickness after adjustment by self-gravity and an additional constraint – the density should vanish at the new disc surface.

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