# An operator-based approach to upscaling the pressure equation

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## Abstract

Permeability and porosity parameters of a porous medium are known only in a statistical sense. For risk assessment, one must perform multiple flow simulations of a single site, varying these input parameters. Because multiple simulations of large sites are computationally prohibitive, upscaling from fine to coarse scales is necessary. Traditional upscaling techniques determine a new effective or upscaled permeability field defined on a coarser scale, which is then used in a standard coarse grid discretization operator. We develop here a method of determining a new coarse grid discretization operator that provides an upscaled solution but bypasses the determination of effective permeability and porosity fields. The method has two steps. We first solve for fine scale flow information internal to each coarse grid cell. Because the problems are small, this step is relatively fast. Then we determine a modified coarse grid operator for solving the upscaled problem that includes the fine scale flow information from the first step. The method is developed for single-phase flow in the context of the mixed finite element method; therefore, the method is locally mass conservative. Unlike traditional upscaling methods (such as homogenization) we do not impose arbitrary boundary conditions on the coarse grid. We present comparisons of our method with the harmonic average permeability upscaling technique.

# 1 Introduction

Modeling of physical systems requires input data which may have significance on many different spatial scales, such as permeability and porosity. The engineer obtains limited core sample and other measurements, generates geostatistical extrapolation, and ends up with a very finely resolved description of very uncertain parameters.

For risk assessment, the engineer runs a number of flow simulations varying the input parameter distribution. Running a large number of detailed simulations is not generally computationally feasible in reasonable time; thus, the engineer is forced to work with a coarser description of the input parameters. *Upscaling* is the process of redefining (averaging) the physical system's parameters *up to* a coarser grid, forming *effective or equivalent* parameters, see [1].

Upscaling for both single and multiphase flow is currently an active area of research (e.g., [1, 2, 3, 4, 5, 6]). For single-phase, the simplest and cheapest method is harmonic averaging of the fine grid permeability field. Even for this simple method, little is known about the accuracy of the resulting coarse-grid flow simulation [7]. Simple averaging seems most reliable when the directions of preferential flow are parallel to the coordinate directions, and when the correlation lengths of the permeability heterogeneity are small compared with the size of the coarse grid cells [1]. Harmonic averaging, although incorrect in some situations, is the most common upscaling technique.

Our method differs fundamentally from other upscaling methods because no new effective coefficients (permeability and porosity) are defined on the coarse grid. The method extends to multiphase flow, but will be presented only for single-phase flow.

# 2 The Mixed Finite Element Method

Combining Darcy's Law with conservation of mass gives the pressure equation for single phase flow. Let $\Omega$ be the medium domain with boundary $\Gamma$. A single time step of the equation for the pressure or head $p$ and the Darcy velocity $u$ is the first order system

$$\begin{cases} u = -K\nabla p & \text{in } \Omega, \\ \phi p + \nabla \cdot u = q & \text{in } \Omega, \end{cases} \tag{1}$$

where $K$ depends on the permeability, $\phi$ depends on the rock compressibility, porosity, and time step, and $q$ depends on the source

terms, porosity, and time step. We assume that on each fine grid cell $K$ is a diagonal tensor and both $K$ and $\phi$ are constant. Neumann boundary conditions are naturally imposed on the solution, so we illustrate the essential Dirichlet conditions only: $p = p^D$ on $\Gamma$.

In the mixed finite element method (see [8]), we use a rectangular grid and approximate the pressure in a finite element space $W$ and the velocity in $V$. The nodes of $W$ are at the centers of the cells; that is, they are the piece-wise discontinuous constant functions. The velocity space $V$ consists of vectors in which the nodes are at the faces of the cells. The first component function is a piecewise continuous linear function in the $x_1$ direction, and it is discontinuous constant in the $x_2$ and $x_3$ directions. Similar statements apply to the other directions. Thus the velocity and consequent mass flux is continuous normal to each cell face. On any cell, a function in $V$ has the form $(\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2, \alpha_3 x_3 + \beta_3)$ for some constants $\alpha_i$ and $\beta_i$.

It will be convenient to use the notation $(f, g)$ and $\langle f, g \rangle$ for the integrals $\int_\Omega f(\mathbf{x}) \, g(\mathbf{x}) \, dx$ and $\int_\Gamma f(\mathbf{x}) \, g(\mathbf{x}) \, dA(x)$. We write the system in variational form by multiplying by appropriate test functions, integrating over $\Omega$, and invoking the Divergence Theorem for one of the terms. Then (1) reduces to finding $u \in V$ and $p \in W$ satisfying

$$\begin{cases} (K^{-1}u, \, v) - (p, \nabla \cdot v) = -\langle p^D, v \cdot \nu \rangle, \\ (\phi p, \, w) + (\nabla \cdot u, \, w) = (q, \, w), \end{cases} \tag{2}$$

for all basis functions $v \in V$ and $w \in W$, where $\nu$ is the unit outer normal vector. Since this discrete problem is posed on the fine scale, we consider its solution as the true solution.

## 3   Our Operator-Based Upscaling Technique

The idea is to decompose the fine scale solution into a coarse piece plus a remainder. Recall that $W$ and $V$ are the function spaces for pressure and velocity on the full fine grid. Let $W_c$ and $V_c$ be the corresponding spaces on the coarse grid. Our goal is to capture some of the fine grid flow internal to each coarse cell without solving the full fine grid problem. Call the fine grid unknowns internal to each coarse grid cell the $\delta$-problem unknowns, denoted $\delta W$ and $\delta V$. Then,

$$\begin{cases} W = W_c \oplus \delta W, \\ \delta W = \{\delta w \in W : \int \delta w \, dx = 0 \text{ on each coarse cell}\}, \end{cases} \tag{3}$$

$$\begin{cases} V \approx V_c + \delta V, \\ \delta V = \{\delta v \in V : \delta v \cdot \nu = 0 \text{ on the boundary} \\ \qquad\qquad\qquad\qquad \text{of each coarse cell}\}. \end{cases} \quad (4)$$

We will solve a $\delta$-problem on each coarse grid cell for internal fine grid unknowns. Line 2 of (4) is the only restrictive simplifying assumption in the definition of our method. This *closure assumption* states that we impose homogeneous Neumann boundary conditions on the $\delta$-problems. It allows us to decouple the $\delta$-problems from coarse grid cell to coarse grid cell and would make a parallel implementation of the method straight-forward. Note, however, that we are *not* imposing arbitrary boundary conditions on the final upscaled problem. We do not discuss other possible closure assumptions here. We also note that in one spatial dimension, $V = V_c + \delta V$, so no simplifying closure assumption is imposed and our upscaling scheme reproduces exactly the fine scale solution.

Substitute $u = u_c + \delta u$, $p = p_c + \delta p$, $v = v_c + \delta v$, and $w = w_c + \delta w$ in (2) and note that several integrals vanish. Then separate the system into two subproblems corresponding to the $\delta$-scale (for test functions $v = \delta v$, $w = \delta w$ only) and the coarse scale ($v = v_c$, $w = w_c$ only).

## 3.1 The $\delta$-Problems

On each coarse cell, $\delta u = \delta u(p_c, u_c, q) \in \delta V$ and $\delta p = \delta p(p_c, u_c, q) \in \delta W$ satisfy

$$\begin{cases} (K^{-1}(u_c + \delta u), \delta v) - (\delta p, \nabla \cdot \delta v) = 0, \\ (\phi(p_c + \delta p), \delta w) + (\nabla \cdot \delta u, \delta w) = (q, \delta w), \end{cases} \quad (5)$$

for all $\delta v \in \delta V$ and $\delta w \in \delta W$. We do not know the values of $u_c$ and $p_c$ at this stage. We discuss the solution of these $\delta$-problems in Subsection 3.3, where we show how to represent these functions in terms of $q$ and the nodal values of $p_c$ and $u_c$.

## 3.2 The Coarse Problem

The second step of the upscaling process is to use $\delta u$ and $\delta p$ to find $u_c \in V_c$ and $p_c \in W_c$ solving the upscaled coarse problem

$$\begin{cases} (K^{-1}(u_c + \delta u(p_c, u_c, q)), v_c) - (p_c, \nabla \cdot v_c) = -\langle p^D, v_c \cdot \nu \rangle, \\ (\phi(p_c + \delta p(p_c, u_c, q)), w_c) + (\nabla \cdot u_c, w_c) = (q, w_c), \end{cases} \quad (6)$$

for all $v_c \in V_c$ and $w_c \in W_c$. We have found a new coarse-grid linear operator $\mathcal{L}(u_c, p_c) = K^{-1}(u_c + \delta u(u_c, p_c))$, rather than a new value for $K$. Hence the designation *operator-based* upscaling. The effect of the $\delta u$ and $\delta p$ terms is to modify the entries in the matrix of the linear system. If no upscaling were performed, this would correspond simply to taking $\delta u = 0$ and $\delta p = 0$.

### 3.3 Decoupling the Coarse and $\delta$-Scales

For simplicity, we illustrate the solution of (5) in one spatial dimension on the coarse cell from 0 to $h$. On this cell, $p_c$ has 1 nodal value $N_p$, so $p_c = N_p$. Let $N_l$ and $N_r$ be the 2 nodal values of $u_c$ (there are 4 in 2-D and 6 in 3-D). Then $u_c = N_l (h - x)/h + N_r x/h$.

Given any $U$, $P$, and $Q$, we can find $\delta U = \delta U(P, U, Q) \in \delta V$ and $\delta P = \delta P(P, U, Q) \in \delta W$ such that

$$\begin{cases} (K^{-1}(U + \delta U), \delta v) - (\delta P, \nabla \cdot \delta v) = 0, \\ (\phi(P + \delta P), \delta w) + (\nabla \cdot \delta U, \delta w) = (Q, \delta w), \end{cases} \qquad (7)$$

for all $\delta v \in \delta V$ and $\delta w \in \delta W$. Let us solve for

$$\begin{cases} (\delta u_p, \delta p_p) = (\delta U(1, 0, 0), \delta P(1, 0, 0)), \\ (\delta u_l, \delta p_l) = (\delta U(0, (h - x)/h, 0), \delta P(0, (h - x)/h, 0)), \\ (\delta u_r, \delta p_r) = (\delta U(0, x/h, 0), \delta P(0, x/h, 0)), \\ (\delta u_q, \delta p_q) = (\delta U(0, 0, q), \delta P(0, 0, q)). \end{cases} \qquad (8)$$

Each of these is readily solved (and in parallel) over the relatively small coarse grid cell. The desired $\delta$-solutions are then clearly

$$\begin{cases} \delta u = N_p \delta u_p + N_l \delta u_l + N_r \delta u_r + \delta u_q, \\ \delta p = N_p \delta p_p + N_l \delta p_l + N_r \delta p_r + \delta p_q. \end{cases} \qquad (9)$$

## 4 Numerical Experiments

We present the results of 3 numerical experiments. We consider a 2-D unit square domain, take $K$ as a scalar, and use a fine grid of $40 \times 40$, upscaled to $10 \times 10$. In Exp. 1, we set $\phi = .1$ and $q = 0$, take a heterogeneous $K$, and impose no flow boundary conditions on two adjacent edges and a unit pressure drop across the other two edges. In Exp. 2, we set $\phi$ and $q$ as above, take a streaked heterogeneous $K$, and impose a unit pressure drop across opposite faces so that channeling

of the flow can occur. In Exp. 3, we set $\phi = .001$ and $K = 1$, and impose no flow boundary conditions on all edges. The source term represents 2 wells of opposite strength, each with $q = \pm 10$ and acting over the region of a single fine grid cell, located at the corner of a coarse cell: injector at $7/40 \le x \le 8/40$, $7/40 \le y \le 8/40$ and extractor at $35/40 \le x \le 36/40$, $32/40 \le y \le 33/40$.

We compute five solutions. Three are the full *fine scale* solution, the *coarse scale* solution (computed with no upscaling), and the *upscaled* solution. When $K$ is heterogeneous, we can find its harmonic average $\bar{K}$ over the coarse scale. The *harmonic coarse scale* solution is computed on the coarse scale using $\bar{K}$. Finally, (6) is posed on the coarse scale. We replace $K$ by $\bar{K}$ in that equation only (i.e., not in (5) or (7)), giving the *harmonic upscaled* solution.

Table 1: The coarse scale relative errors.

| Exp. | Coarse | Upscaled | Harmonic Coarse | Harmonic Upscaled |
|------|--------|----------|-----------------|-------------------|
| $1-p$ | 0.2139 | 0.1855 | 0.0236 | 0.0208 |
| $1-u$ | 0.3826 | 0.3518 | 0.2310 | 0.2397 |
| $2-p$ | 0.1027 | 0.0886 | 0.0264 | 0.0256 |
| $2-u$ | 0.6984 | 0.6553 | 0.1063 | 0.1664 |
| $3-p$ | 0.1002 | 0.0162 | — | — |
| $3-u$ | 0.1519 | 0.0672 | — | — |

Table 2: The fine scale relative errors.

| Exp. | Coarse | Upscaled | Harmonic Coarse | Harmonic Upscaled |
|------|--------|----------|-----------------|-------------------|
| $1-p$ | 0.2245 | 0.1862 | 2.8775 | 0.0331 |
| $1-u$ | 0.5433 | 0.4512 | 1.1683 | 0.4200 |
| $2-p$ | 0.1167 | 0.0951 | 0.5527 | 0.0901 |
| $2-u$ | 0.8022 | 0.7248 | 0.9347 | 0.5779 |
| $3-p$ | 0.1458 | 0.0380 | — | — |
| $3-u$ | 0.2389 | 0.1563 | — | — |

We report in Tables 1–2 relative errors of $p$ in $L^2$ and $u$ in $L^1$, by comparing the coarse, upscaled, harmonic coarse, and harmonic upscaled solutions with the fine scale solution. These errors are computed on either the coarse or fine scale. On the coarse scale, we actually compare with the projection of the fine scale solution to the coarse level; that is, with the best possible coarse scale representation

of the fine scale solution in $W_c \times V_c$. Thus, we see only $(p_c, u_c)$ of the upscaled solutions. On the fine scale, we compare with the full fine scale solution in $W \times V$, and we include the $\delta$-problem corrections to the upscaled solutions. It is important to note that Table 2 is more germane to nonlinear problems, since we need accurate fine scale solutions to predict relative permeabilities and capillary pressures.
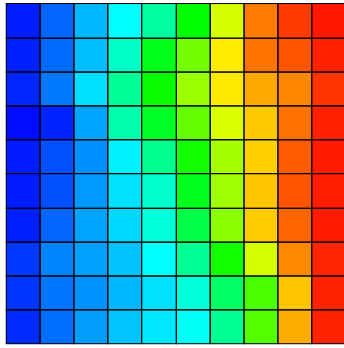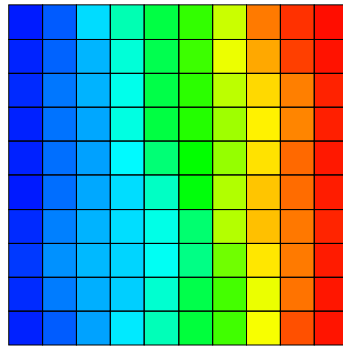


Figure 1: Harmonic coarse scale solution (Exp. 2)



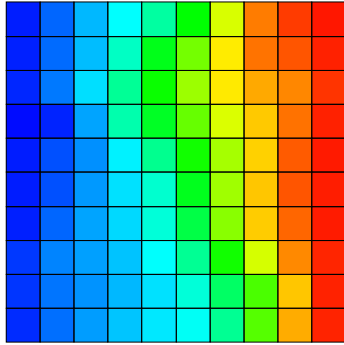Figure 2: Upscaled solution on the coarse scale $p_c$



Figure 3: Harmonic upscaled solution on the coarse scale $p_c$
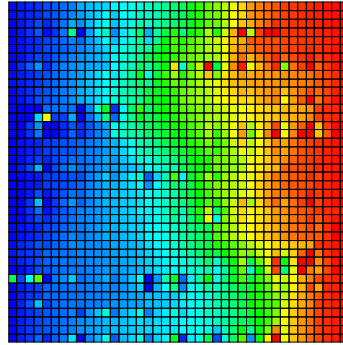


Figure 4: Harmonic upscaled solution on the fine scale $p_c + \delta p$

One can prove from the theory of finite elements that the upscaled solution error will be no worse than the coarse solution error. Such an estimate is not known for harmonic averaging; it is possible for the harmonic solution to be worse than the coarse solution. On the coarse scale, Exps. 1 and 2 show that for heterogeneous $K$, the solution improves with harmonic averaging, and that our upscaling is at best of marginal benefit. However, on the fine scale, the combination of

harmonic averaging and our upscaling produces superior results, and upscaling alone is better than harmonic averaging. We illustrate the pressure of Exp. 2 in Figs. 1–4. Exp. 3 shows that it is possible and important to resolve the location of wells in a coarse mesh.

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## References

[1] Durlofsky, L.J., Numerical calculation of equivalent grid block permeability tensors for heterogeneous porous media, *Water Resources Research*, 27, pp. 699–708, 1991.

[2] Amaziane, B., Bourgeat, A. & Koebbe, J., Numerical simulation and homogenization of two-phase flow in heterogeneous porous media, *Transport in Porous Media*, 9, pp. 519–547, 1991.

[3] Espedal, M.S. & Saevareid, O., Upscaling of permeability based on wavelet representation, *4th European Conf. on the Math. of Oil Recovery*, Norway, 1994.

[4] King, M.J., King, P.R., McGill, C.A. & Williams, J.K., Effective properties for flow calculations, *Trans. Por. Med.*, 20, pp. 169–96, 1995.

[5] Christie, M.A., Mansfield, M., King, P.R., Barker, J.W. & Culverwell, I.D., A renormalization-based upscaling technique for WAG floods in heterogeneous reservoirs, *13th Symp. on Res. Sim.*, SPE 29127, pp. 353–361, 1995.

[6] Hou, T.Y. & Wu, X.H., A multiscale finite element method for elliptic problems in composite materials and porous media, *J. Comput. Phys.*, 134, pp. 169–189, 1997.

[7] King, P.R., & Williams, J., Upscaling permeability: mathematics of renormalization, *4th European Conf. on the Math. of Oil Recovery*, Norway, 1994.

[8] Brenner, S.C. & Scott, L.R., *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.