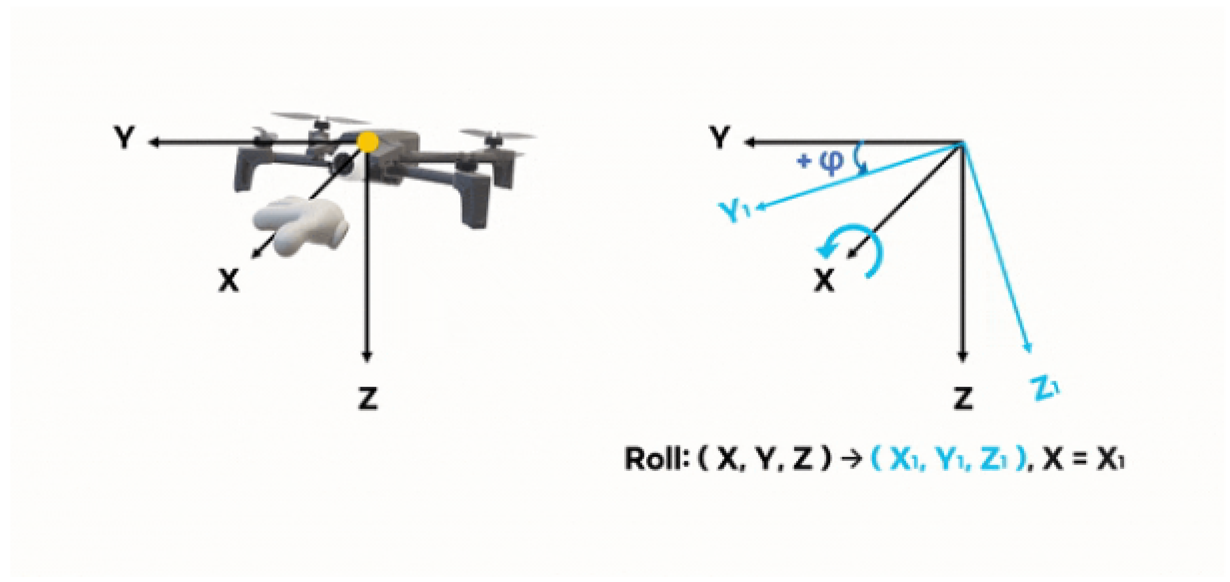
Euler rotations theorem

3D 공간에서 물체가 세 가지 독립적인 회전 축을 따라 회전하는 개념
이러한 회전 축은 롤 (Roll), 피치 (Pitch) 및 요 (Yaw) 축
회전을 적용하는 순서가 중요하다.
즉 오일러 각의 회전은 교환법칙이 성립하지 않는다는 뜻이다



〈 x축을 기준으로 회전하는 Roll 회전의 양의 방향과 좌표계, 오일러 1 회전(Euler 1 rotation) 〉

X축 기준으로 회전하면 X는 변하지 않고 Y,Z만 변화

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{R}_{xyz} = \mathbf{R}_x(\phi)\mathbf{R}_y(\theta)\mathbf{R}_z(\psi)$$
$$= \begin{bmatrix} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\theta \\ \sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi & \sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

Z->Y->X 순대로 회전

XYZ(Yaw, Pitch, Roll)순 곱

$$\mathbf{R}_{zyx} = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)$$
$$= \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

ZYX(Roll-Pitch-Yaw)순 곱

QuanutmCircuits
single qubit problems

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목 차

1. **Why are they called “rotations”**

2 **. Given a unitary matrix (operator) U, can you decompose it into a product of rotations?**

\+ what is the Euler rotations?

1. Why are they called “rotations”

Rotations

=> A useful advantage of the Bloch sphere representation is that the evolution of the qubit state is describable by rotations of the Bloch sphere.

Unitary Operator U can be written as $U = e^{i\alpha}R_{\hat{n}}(\theta)$

Applying rotational Operator about an arbitrary direction, $U = e^{-i(\frac{\theta}{2})\vec{\sigma}\,\hat{n}}$

According to Euler’s Formula, $e^{-i(\frac{\theta}{2})} = \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})$

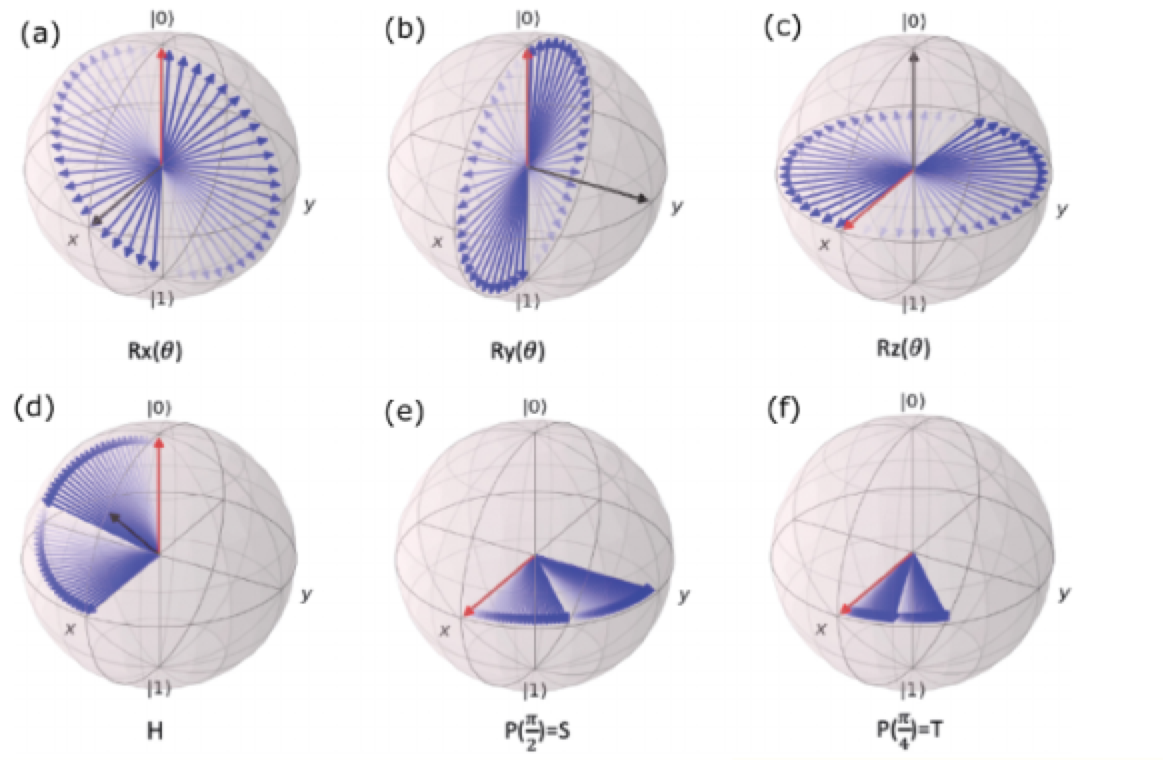
Therefore, $U = e^{i\alpha}\,[I\,\cos(\frac{\theta}{2}) - i\,(\vec{\sigma}\,\hat{n})\sin(\frac{\theta}{2})]$

For $\alpha = \frac{\pi}{2}$ $and\ \theta = \pi$,

$U = e^{i\frac{\pi}{2}}\,[I\,\cos(\frac{\pi}{2}) - i\,(\vec{\sigma}\,\hat{n})\sin(\frac{\pi}{2})]$

$U = i[-i\,(\vec{\sigma}\,\hat{n})] = \vec{\sigma}\,\hat{n}$

Therefore, for X, Y and Z axes, $U = \vec{\sigma}\,\hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$



Rotation about X axis	Rotation about Y axis	Rotation about Z axis
$R_x(\emptyset) = e^{-i(\frac{\emptyset}{2})\sigma_x}$	$R_y(\emptyset) = e^{-i(\frac{\emptyset}{2})\sigma_y}$	$R_z(\emptyset) = e^{-i(\frac{\emptyset}{2})\sigma_z}$
$R_x(\emptyset) = [I\,\cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2})\sigma_x]$	$R_y(\emptyset) = [I\,\cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2})\sigma_y]$	$R_z(\emptyset) = [I\,\cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2})\sigma_z]$
$R_x(\emptyset) = \begin{bmatrix}\cos(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2})\end{bmatrix} - \begin{bmatrix}0 & i\sin(\frac{\emptyset}{2}) \\ i\sin(\frac{\emptyset}{2}) & 0\end{bmatrix}$	$R_y(\emptyset) = \begin{bmatrix}\cos(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2})\end{bmatrix} - \begin{bmatrix}0 & \sin(\frac{\emptyset}{2}) \\ \sin(\frac{\emptyset}{2}) & 0\end{bmatrix}$	$R_z(\emptyset) = \begin{bmatrix}\cos(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2})\end{bmatrix} - \begin{bmatrix}i\sin(\frac{\emptyset}{2}) & 0 \\ 0 & -i\sin(\frac{\emptyset}{2})\end{bmatrix}$
$R_x(\emptyset) = \begin{bmatrix}\cos(\frac{\emptyset}{2}) & -i\sin(\frac{\emptyset}{2}) \\ -i\sin(\frac{\emptyset}{2}) & \cos(\frac{\emptyset}{2})\end{bmatrix}$	$R_y(\emptyset) = \begin{bmatrix}\cos(\frac{\emptyset}{2}) & -\sin(\frac{\emptyset}{2}) \\ -\sin(\frac{\emptyset}{2}) & \cos(\frac{\emptyset}{2})\end{bmatrix}$	$R_z(\emptyset) = \begin{bmatrix}\cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2}) + i\sin(\frac{\emptyset}{2})\end{bmatrix}$
		$R_z(\emptyset) = \begin{bmatrix}e^{-i(\frac{\emptyset}{2})} & 0 \\ 0 & e^{i(\frac{\emptyset}{2})}\end{bmatrix}$
		Adding a global phase $e^{i(\frac{\emptyset}{2})}$ results in $R_z(\emptyset) = \begin{bmatrix}1 & 0 \\ 0 & e^{i\emptyset}\end{bmatrix}$

2 . Given a unitary matrix (operator) U, can you decompose it into a product of rotations?

Unitary Matrix

- $U^\dagger = U^{-1}$
- U is diagonizable
- $|det(U)| = e^{i\theta}$ for any θ [10]
- For $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $\sqrt{A^2 + B^2} = 1$.

If A is a (n x n) matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$A = A^{-1} = A^\dagger$

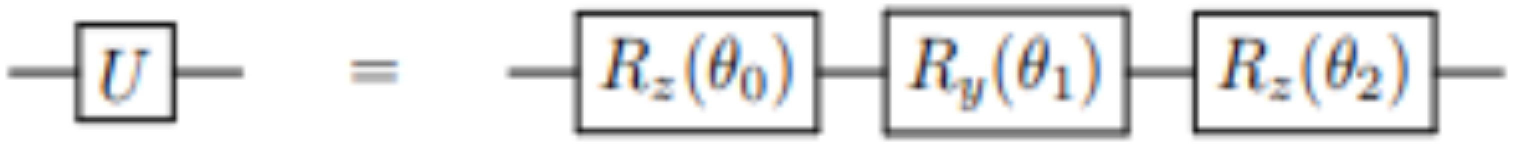
Where, $A^\dagger = (A^T)^*$ and $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

det(A) is given by det(A) = ad-bc

두개의 회전 게이트 RZ, 한개의 회전 게이트 RY
각도: Φ, α, β, γ

$$U(2) = e^{-i\Phi} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = e^{-i\Phi} R_z(\alpha) R_y(\beta) R_z(\gamma)$$

$$SU(2) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = R_z(\alpha) R_y(\beta) R_z(\gamma)$$



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