Appendix for Tropical Calculus (v.2)

Minho Jeong

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Abstract

This material is a supplement to M-AXIS's presentation on tropical calculus and contains proofs, definitions, and theorems not included in the presentation. It is best read in conjunction with the accompanying PowerPoint file when listening to the presentation to aid in understanding. You can access to the most recent Appendix file via mino0112.github.io/archive/mathematics.

Contents

1 Definition of Semiring

A **semiring** is a set R equipped with two binary operations $+$ and $\cdot$, called addition and multiplication, such that:

- $(R, +)$ is a commutative monoid with identity element called 0:
  - (Associative) $(a+b)+c = a+(b+c)$,
  - (Commutative) $a+b = b+a$,
  - (Identity) $0+a = a$.
- $(R, \cdot)$ is a monoid with identity element called 1:
  - (Associative) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
  - (Identity) $1 \cdot a = a \cdot 1 = a$.
- (Distributive) $r \cdot (a+b) = r \cdot a + r \cdot b$ and $(a+b) \cdot r = a \cdot r + b \cdot r$.
- $0 \cdot a = a \cdot 0 = 0$.

2 Proof of Law of Exponents

Tropical exponentiation is defined as iterated tropical multiplication. For natural number n and m, these properties are established as follows:

(E1) $x^{\otimes n} \otimes x^{\otimes m} = nx \otimes mx = nx + mx = (n+m)\,x = x^{\otimes (n+m)} = x^{\otimes (n \otimes m)}$.

(E2) $\left(x^{\otimes n}\right)^{\otimes m} = (nx)^{\otimes m} = nmx = x^{\otimes nm}$.

(E3) (freshman's dream) $(x \otimes y)^{\otimes m} = n\,(x+y) = nx + ny = x^{\otimes n} \otimes y^{\otimes n}$.

In the same way that we extended the exponential law to real numbers, we can extend the tropical exponential law to real numbers.

3 Proof of Properties of Differentiation and Integration

Let $f(x) = a_0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x^1 + b_2x^2 + \cdots + b_mx^m$, where $n \leq m$ (without any loss of generality), $a_i \in \mathbb{R}$ for $i \in \mathbb{Z}_{n+1}$, and $b_j \in \mathbb{R}$ for $j \in \mathbb{Z}_{m+1}$. Here and below, $T(\cdot)$ denotes tropical transformation.

3.1 Proof of Sum Law of Differentiation

We want to show that $T(f+g)' = T(f)' \oplus T(g)'$ for some prime p. For the left-hand side("LHS"), we get:

$$\begin{aligned}
T(f+g)' &= T\left((a_0+b_0) + \cdots + (a_n+b_n)x^n + b_{n+1}x^{n+1} + \cdots + b_mx^m\right)' \\
&= \left(v(a_0+b_0) \oplus \cdots \oplus v(a_n+b_n) \otimes x^{\otimes n} \oplus v(b_{n+1}) \otimes x^{\otimes(n+1)} \oplus \cdots \oplus v(b_m) \otimes x^{\otimes m}\right)' \\
&= v(a_1+b_1) \oplus \cdots \oplus v(n(a_n+b_n)) \otimes x^{\otimes(n-1)} \oplus v((n+1)b_{n+1}) \otimes x^{\otimes n} \oplus \cdots \oplus v(mb_m) \otimes x^{\otimes(m-1)}.
\end{aligned}$$

Otherwise, for the right-hand side("RHS"), we have:

$$\begin{aligned}
T(f)' \oplus T(g)' &= T(a_0 + \cdots + a_nx^n)' \oplus T\left(b_0 + \cdots + b_nx^n + b_{n+1}x^{n+1} + \cdots + b_mx^m\right)' \\
&= \left(v(a_0) \oplus \cdots \oplus v(a_n) \otimes x^{\otimes n}\right)' \oplus \left(v(b_0) \oplus \cdots \oplus v(b_m) \otimes x^{\otimes m}\right)' \\
&= \left(v(a_1) \oplus \cdots \oplus v(na_n) \otimes x^{\otimes(n-1)}\right) \oplus \left(v(b_1) \oplus \cdots \oplus v(mb_m) \otimes x^{\otimes(m-1)}\right) \\
&= (v(a_1) \oplus v(b_1)) \oplus \cdots \oplus (v(na_n) \oplus v(nb_n)) \otimes x^{\otimes(n-1)} \oplus v((n+1)b_{n+1}) \otimes x^{\otimes n} \oplus \cdots \oplus v(mb_m) \otimes x^{\otimes(m-1)}.
\end{aligned}$$

Not all prime p satiesfies $v(a_k+b_k) = v(a_k) \oplus v(b_k)$, however, it is trivial that there exists at least one p such that $v(a_k+b_k) = v(a_k) \oplus v(b_k)$ for every $k \in \mathbb{Z}_{n+1}$. Thus, $T(f+g)' = T(f)' \oplus T(g)'$ for some prime p. □

3.2 Proof of Product Law of Differentiation

We want to show that $T(f \cdot g)' = T(f)' \otimes T(g) \oplus T(f) \otimes T(g)'$ for some prime p. For LHS, we can derive:

$$\begin{aligned}
T(f \cdot g)' &= T\left(\sum_{k=0}^{n}\left(\sum_{i=0}^{k} a_ib_{k-i}x^k\right) + \sum_{k=n+1}^{m}\left(\sum_{i=0}^{n} a_ib_{k-i}x^k\right) + \sum_{k=m+1}^{m+n}\left(\sum_{i=k-m}^{n} a_ib_{k-i}x^k\right)\right)' \\
&= \left(v(a_0b_0) \oplus \cdots \oplus v\left(\sum_{i=0}^{n} a_ib_{n-i}\right) \otimes x^{\otimes n} \oplus \cdots \oplus v\left(\sum_{i=0}^{n} a_ib_{m-i}\right) \otimes x^{\otimes m} \oplus \cdots \oplus v(a_nb_m) \otimes x^{\otimes(m+n)}\right)' \\
&= v(a_0b_1 + a_1b_0) \oplus \cdots \oplus v\left(n\sum_{i=0}^{n} a_ib_{n-i}\right) \otimes x^{\otimes(n-1)} \oplus \cdots \oplus v((m+n)a_nb_m) \otimes x^{\otimes(m+n-1)}.
\end{aligned}$$

While, for RHS, we have:

$$\begin{aligned}
T(f)' \otimes T(g) \oplus T(f) \otimes T(g)' &= \left(\bigoplus_{k=1}^{n} v(k) \otimes v(a_k) \otimes x^{\otimes(k-1)}\right) \otimes \left(\bigoplus_{k=1}^{m} v(b_k) \otimes x^{\otimes k}\right) \\
&\quad \oplus \left(\bigoplus_{k=1}^{n} v(a_k) \otimes x^{\otimes k}\right) \otimes \left(\bigoplus_{k=1}^{m} v(k) \otimes v(b_k) \otimes x^{\otimes(k-1)}\right) \\
&= \left(\bigoplus_{k=1}^{n-1}\left(\bigoplus_{i=1}^{k} v(a_i) \otimes v(b_{k-i})\right) \otimes x^{\otimes k}\right) \oplus \left(\bigoplus_{k=n}^{m}\left(\bigoplus_{i=1}^{n} v(a_i) \otimes v(b_{k-i})\right) \otimes x^{\otimes k}\right) \\
&\quad \oplus \left(\bigoplus_{k=m+1}^{m+n-1}\left(\bigoplus_{i=k-m}^{n-1} v(a_i) \otimes v(b_{k-i})\right) \otimes x^{\otimes k}\right) \oplus \left(\bigoplus_{k=1}^{n}\left(\bigoplus_{i=0}^{k-1} v(a_i) \otimes v(b_{k-i})\right) \otimes x^{\otimes k}\right) \\
&\quad \oplus \left(\bigoplus_{k=n+1}^{m-1}\left(\bigoplus_{i=0}^{n} v(a_i) \otimes v(b_{k-i})\right) \otimes x^{\otimes k}\right) \oplus \left(\bigoplus_{k=m}^{m+n-1}\left(\bigoplus_{i=k-m}^{n} v(a_i) \otimes v(b_{k-i})\right) \otimes x^{\otimes k}\right) \\
&= (v(a_1b_0) \oplus v(a_0b_1)) \oplus \cdots \oplus (v(n) \oplus v(m)) \otimes v(a_nb_m) \otimes x^{\otimes(m+n-1)}.
\end{aligned}$$

From above, we can easily recognize that, for some prime p, $T(f \cdot g)' = T(f)' \otimes T(g) \oplus T(f) \otimes T(g)'$. □

3.3 Proof of Composite Law of Differentiation

We want to show that $T(f \circ g)' = T(f(T(g)))' \otimes T(g)'$ for some prime p. We can easily derive this formula as in other properties, however, we leave this for readers to try themselves.

3.4 Proof of Sum Law of Integration

We want to show that $\int T(f+g) = \int T(f) \oplus \int T(g)$ for some prime p. For the LHS, we have:

$$\begin{aligned}
\int T(f+g) &= \int T\left((a_0+b_0)+\cdots+(a_n+b_n)x^n+b_{n+1}x^{n+1}+\cdots+b_m x^m\right) \\
&= \int \left(v(a_0+b_0)\oplus\cdots\oplus v(a_n+b_n)\otimes x^{\otimes n}\oplus v(b_{n+1})\otimes x^{\otimes(n+1)}\oplus\cdots\oplus v(b_m)\otimes x^{\otimes m}\right) \\
&= C\oplus v(a_0+b_0)\otimes x\oplus\cdots\oplus v\left(\frac{a_n+b_n}{n+1}\right)\otimes x^{\otimes(n+1)}\oplus v\left(\frac{b_{n+1}}{n+2}\right)\otimes x^{\otimes(n+2)}\oplus\cdots\oplus v\left(\frac{b_m}{m+1}\right)\otimes x^{\otimes(m+1)}.
\end{aligned}$$

On the other hand, for the RHS, we can derive:

$$\begin{aligned}
\int T(f)\oplus\int T(g) &= \int T(a_0+\cdots+a_n x^n)\oplus\int T\left(b_0+\cdots+b_n x^n+b_{n+1}x^{n+1}+\cdots+b_m x^m\right) \\
&= \int \left(v(a_0)\oplus\cdots\oplus v(a_n)\otimes x^{\otimes n}\right)\oplus\int\left(v(b_0)\oplus\cdots\oplus v(b_m)\otimes x^{\otimes m}\right) \\
&= \left(C_1\oplus v(a_0)\otimes x\oplus\cdots\oplus v\left(\frac{a_n}{n+1}\right)\otimes x^{\otimes(n+1)}\right)\oplus\left(C_2\oplus v(b_0)\otimes x\oplus\cdots\oplus v\left(\frac{b_m}{m+1}\right)\otimes x^{\otimes(m+1)}\right) \\
&= C\oplus(v(a_0)\oplus v(b_0))\otimes x\oplus\cdots\oplus\left(v\left(\frac{a_n}{n+1}\right)\oplus v\left(\frac{b_n}{n+1}\right)\right)\otimes x^{\otimes(n+1)}\oplus\cdots\oplus v\left(\frac{b_m}{m+1}\right)\otimes x^{\otimes(m+1)}
\end{aligned}$$

where $C=\min\{C_1,C_2\}$. For some prime p, we get the same coefficients, thus, $\int T(f+g)=\int T(f)\oplus\int T(g)$. $\square$