Proof of Euler's Formula

MINO Present
(feat. IS)

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof #1 (Taylor Series)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Proof #1 (Taylor Series)

$$
\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= \cos x + i \sin x \quad \blacksquare
\end{aligned}
$$

* 재배열 급수에 관한 내용은 생략하기로 한다.

Proof #2 (Differentiation)

$$f(x) \coloneqq e^{-ix}(\cos x + i \sin x)$$

$$\begin{aligned} f'(x) &= -ie^{-ix}(\cos x + i \sin x) + e^{-ix}(-\sin x + i \cos x) \\ &= e^{-ix}(-i \cos x + \sin x - \sin x + i \cos x) \\ &= 0 \end{aligned}$$

$$\therefore f(x) = \text{const.}$$

Proof #2 (Differentiation)

Since $f(0) = 1$, we get

$$\forall x \in \mathbb{R}, \qquad f(x) = e^{-ix}(\cos x + i\sin x) = 1$$

$$\therefore e^{ix} = \cos x + i\sin x \quad \blacksquare$$

Proof #3 (Calculus)

$$y \coloneqq \cos x + i \sin x$$

$$\begin{aligned} y' &= -\sin x + i \cos x \\ &= i^2 \sin x + i \cos x \\ &= i(\cos x + i \sin x) \\ &= iy \end{aligned}$$

Proof #3 (Calculus)

$$\int \frac{1}{y} dy = \int i dx$$

$$\log y = ix + \text{const.}$$

$$\log y = ix$$

$$y = e^{ix}$$

$$e^{ix} = \cos x + i \sin x \quad \blacksquare$$

Proof #4 (Differential Equation)

$$f(x) := e^{ix}$$

$$f'(x) = ie^{ix}$$

$$f''(x) = i^2 e^{ix} = -e^{ix}$$

$$f(x) + f''(x) = 0$$

$$\therefore f(x) = A\cos x + B\sin x$$

Proof #4 (Differential Equation)

$$f(0) = e^{i0} = 1 = A$$
$$f'(0) = ie^{i0} = i = B$$

$$\therefore f(x) = e^{ix} = \cos x + i \sin x \ \blacksquare$$

Proof #5 (approximation)

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$ (de Moivre's formula)

$$\theta := \frac{x}{n}$$

$$\left(\cos\frac{x}{n} + i\sin\frac{x}{n}\right)^n = \cos x + i\sin x$$

Proof #5 (approximation)

$$
\begin{aligned}
\cos x + i \sin x &= \lim_{n\to\infty} \left(\cos\frac{x}{n} + i \sin\frac{x}{n}\right)^{n} \\
&= \lim_{n\to\infty} \left(1 + \frac{ix}{n}\right)^{n} \\
&= e^{ix} \ \blacksquare
\end{aligned}
$$

Proof #6 (trigonometry)

$$e^{ix} = \lim_{n\to\infty}\left(1+\frac{ix}{n}\right)^n$$

$$1+\frac{ix}{n} = \sqrt{1+\frac{x^2}{n^2}}\left(\cos\left(\tan^{-1}\frac{x}{n}\right)\right) + i\left(\sin\left(\tan^{-1}\frac{x}{n}\right)\right)$$

$$\left(1+\frac{ix}{n}\right)^n = \left(1+\frac{x^2}{n^2}\right)^{\frac{n}{2}}\left(\cos\left(n\tan^{-1}\frac{x}{n}\right)\right) + i\left(\sin\left(n\tan^{-1}\frac{x}{n}\right)\right)$$

Proof #6 (trigonometry)

$$\lim_{n\to\infty}\left(1+\frac{x^2}{n^2}\right)^{\frac{n}{2}} = 1$$

$$\lim_{n\to\infty} n\tan^{-1}\left(\frac{x}{n}\right) = x$$

$$\therefore e^{ix} = \lim_{n\to\infty}\left(1+\frac{ix}{n}\right)^{n} = \cos x + i\sin x \ \blacksquare$$

Proof #7 (matrix)

$$J \coloneqq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \coloneqq e^{Jx} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Proof #7 (matrix)

$$e^{Jx}\begin{bmatrix}1\\0\end{bmatrix} = \left(\frac{x^0}{0!}J^0 + \frac{x^1}{1!}J^1 + \frac{x^2}{2}J^2 + \frac{x^3}{3!}J^3 + \frac{x^4}{4!}J^4 + \cdots\right)\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + x\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} - \frac{x^2}{2!}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \frac{x^3}{3!}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} + \cdots\right)\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \begin{bmatrix}1 + 0 - \frac{x^2}{2!} - 0 + \frac{x^4}{4!} + \cdots\\ 0 - x - 0 + \frac{x^3}{3!} + 0 + \cdots\end{bmatrix}$$

$$= \begin{bmatrix}\cos x\\ -\sin x\end{bmatrix}$$

Proof #7

$$\therefore e^{Jx}\begin{bmatrix}1\\0\end{bmatrix} = (\cos x + J\sin x)\begin{bmatrix}1\\0\end{bmatrix} \blacksquare$$

References

- Jichao, Mathematics Stack Exchange,
  "How to prove Euler's formula: $e^{i\varphi} = \cos\varphi + i\sin\varphi$?,"
  https://math.stackexchange.com/q/3510, Aug. 18, 2023

Thank you
for your listening