A Study on the Solution of the Wave Equation Simulating the Lateral Vibration of Elastic Strings

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Abstract

In this study, we focus on the properties of solutions to the wave equation from a theoretical perspective to investigate the characteristics of solutions describing the lateral vibration of elastic strings. Particularly, using integral inequality and Lagrange's multiplier method, we approach solutions of the wave equation describing linear or polynomially decaying motion of elastic strings in the form of

$$\rho u_{tt}(x,t) - T u_{xx}(x,t) + a|u_t(x,t)|^{p-2} u_t(x,t) = 0$$

from an energy standpoint to examine their properties.

Keywords: Elastic strings, Wave equation, Damping, Energy decay rate

Contents

I. Introduction

Elastic strings play a crucial role in various fields of modern science and engineering such as materials science, geoscience, and seismology. Accurately describing and understanding the motion of elastic strings is essential for research and applications in this domain, ultimately contributing to advancements in science and technology. In seismology, understanding subsurface elastic phenomena is crucial for predicting and responding to earthquakes. For instance, accurately modeling the motion of elastic strings can be utilized to minimize human and property damages by improving prediction models for subsurface elastic phenomena.

Furthermore, in materials engineering and structural engineering, it plays a significant role in evaluating material properties and assessing the stability of structures. Accurate description of elastic string motion contributes to designing safe and efficient structures and facilities. Additionally, obtaining accurate solutions to the wave equation can lead to improved outcomes in various applications and industrial sectors.

In this study, we aim to focus on the properties of solutions to the wave equation from a theoretical perspective, investigating the characteristics of solutions describing the lateral vibration of elastic strings. Utilizing the theoretical foundations of elastic phenomena and wave theory, we aim to model appropriately and theoretically investigate the energy of waves. According to previous research, solutions to wave equations without damping terms can be clearly obtained using Fourier series. However, in cases where the wave equation contains damping terms, deriving clear solutions becomes challenging. Hence, in this study, we employ various mathematical techniques such as integral inequality and Lagrange's multiplier method to investigate the energy decay rate of solutions, thereby understanding the properties of solutions.

II. Theoretical Background

1. Fourier Series (Kreyszig 2011, 476)

For a real function f defined on the interval $[-\pi, \pi]$ with period 2π and Riemann integrable on its domain $\mathbb{R}$, the Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where the Fourier coefficients a_n and b_n are defined as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad n \in \{0\} \cup \mathbb{N},$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \quad n \in \mathbb{N}.$$

2. 1-D Wave Equation (Kreyszig 2011, 543-550)

2.1. 1-D Wave Equation

The 1-dimensional wave equation describing the form of a wave is given as

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

Therefore, functions satisfying the wave equation also satisfy

$$f(x,t) = f(x - vt, 0) = g(x - vt).$$

2.2. Modeling of 1-D Wave Equation

The assumptions regarding the physical behavior of waves required for this process are as follows

1. The linear density of the string is constant. The string is a perfectly elastic body and experiences no resistance when vibrating.
2. Before fixing the ends of the string, the tension applied to the string is extremely high, allowing us to ignore gravity.
3. The motion of the string is a small lateral vibration within a vertical plane.

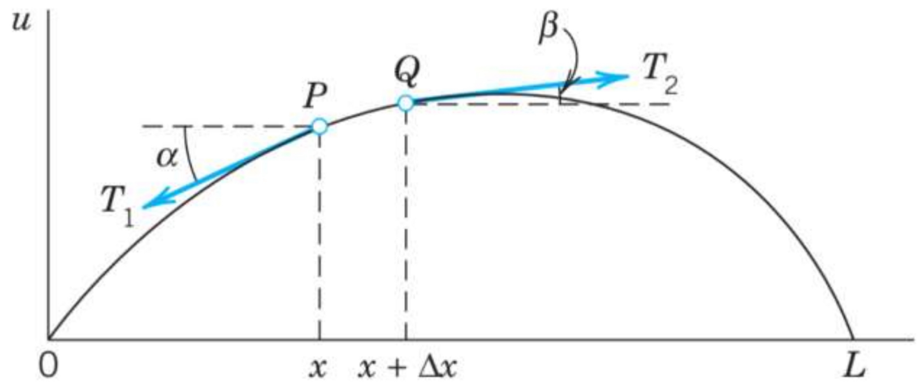

Figure 1: Modeling of Vibrating String

Let L be the length of the string, ρ be the linear density of the string, and $u(x,t)$ represent the displacement of the string. According to the aforementioned assumptions, the string's motion is lateral vibration, so the net force in the horizontal direction is 0, leading to

$$T_1 \cos\alpha + T_2 \cos\beta = 0.$$

Thus, the following holds

$$T_1 \cos\alpha = T_2 \cos\beta = T.$$

Here, T represents the horizontal tension. Applying Newton's second law to the vertical force gives

$$T_1 \sin\alpha + T_2 \sin\beta = \rho \Delta x u_{tt}.$$

Dividing both sides by T results in

$$\tan\beta - \tan\alpha = \frac{\rho}{T} \Delta x u_{tt}.$$

Where tan represents the slope at a point, thus,

$$\tan\alpha = u_x(x,t), \quad \tan\beta = u_x(x+\Delta x, t).$$

Therefore,

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}[u_x(x+\Delta x,t) - u_x(x,t)] = \frac{\rho}{T} u_{tt}(x,t).$$

Manipulating this leads to

$$u_{xx} = c^2 u_{tt}.$$

Here, $c^2 = \frac{\rho}{T}$.

2.3. Solutions of the 1-D Wave Equation

The form of the Fourier series solution to the wave equation with the following Dirichlet boundary conditions and initial values is well known as

$$\rho u_{tt}(x,t) - Tu_{xx}(x,t) = 0, \quad (x,t) \in (0,L) \times (0,\infty),$$
$$u(0,t) = u(L,t) = 0, \quad t \in (0,\infty),$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,L).$$

The form of the Fourier series solution is given as follows:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{L}.$$

Where $\lambda_n = \frac{cn\pi}{L}$, and rewriting this expression gives

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}.$$

Substituting with Fourier sine functions yields

$$u(x,t) = \frac{1}{2}\left[f^{'}(x-ct) + f^{'}(x+ct)\right].$$

3. Linear Integral Inequality (Komornik 1994, 103-104)

Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function, and assume there exists a positive constant T such that for any $t \in \mathbb{R}^+$,

$$\int_t^{\infty} E(s)ds \leq TE(t). \tag{II-1}$$

Then, for any $t \geq T$, the following holds

$$E(t) \leq E(0)e^{\frac{1-t}{T}}. \tag{II-2}$$

Proof: Observing the above equation, it is evident that for $0 \leq t < T$, $E(t) \leq E(0)$. Define a function $f(x) = e^{\frac{x}{T}} \int_x^{\infty} E(s)ds$ for $x \in \mathbb{R}^+$. Now, f is locally absolutely continuous and non-decreasing due to (II-1). Consequently,

$$f'(x) = T^{-1}e^{\frac{x}{T}}\left(\int_x^{\infty} E(s)ds - TE(x)\right) \leq 0 \quad \text{a.e. } x \in \mathbb{R}. \tag{II-3}$$

Reapplying (II-1), we obtain

$$f(x) \leq f(0) = \int_x^{\infty} E(s)ds \leq TE(0), \quad \forall x \in \mathbb{R}^+.$$

Rearranging, we get

$$TE(x+T) \leq \int_x^\infty E(s)ds \leq TE(0)e^{-\frac{x}{T}}, \quad \forall x \in \mathbb{R}^+.$$

Since E is non-negative and non-decreasing, we conclude

$$\int_x^\infty E(s)ds \geq \int_x^{x+T} E(s)ds \geq TE(x+T).$$

Substituting this into (II-3) and simplifying yields

$$E(x+T) \leq E(0)e^{-\frac{x}{T}}, \quad \forall x \in \mathbb{R}.$$

Let $t = x+T$, then we obtain (II-2).

4. Nonlinear Integral Inequality (Komornik 1994, 124-125)

Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function, and for any $t \in \mathbb{R}^+$, assume there exists $\alpha > 0$ and $T > 0$ such that

$$\int_t^\infty E^{\alpha+1}(s)ds \leq TE^\alpha(0)E(t). \tag{II-4}$$

Then, for any $t \geq 0$, the following holds

$$E(t) \leq E(0)\left(\frac{T+\alpha t}{T+\alpha T}\right)^{-\frac{1}{\alpha}}. \tag{II-5}$$

Proof: If $E(0) = 0$, then $E \equiv 0$, so there's no need to prove it. Otherwise, considering E as $\frac{E}{E(0)}$ and assuming $E(0) = 1$, we need to prove the inequality

$$E(t) \leq \left(\frac{T+\alpha t}{T+\alpha T}\right)^{-\frac{1}{\alpha}}. \tag{II-6}$$

Let's introduce a function $F : \mathbb{R}^+ \to \mathbb{R}^+$. Here, $F(t) = \int_t^\infty E^{\alpha+1}(s)ds$. This function is non-decreasing and locally absolutely continuous. Differentiating F and using (II-4), we get

$$F' \geq T^{-\alpha-1}F^{\alpha+1} \quad \text{a.e. in } (0,\infty).$$

Thus, $(F^{-\alpha})' \geq \alpha T^{-\alpha-1}$ a.e. in $(0,\infty)$, where $B := \sup\{t : E(t) > 0\}$. Integrating the above inequality over $[0,s]$, we obtain

$$F^{-\alpha}(s) - F^{-\alpha}(0) \geq \alpha T^{-\alpha-1}s, \quad \forall s \in [0,B).$$

Rearranging this gives

$$F(s) \leq \left(F^{-\alpha}(0) + \alpha T^{-\alpha-1} s\right)^{-\frac{1}{\alpha}}, \quad \forall s \in [0, B). \tag{II-7}$$

For $s \geq B$, $F(s) = 0$, so the inequality holds for any $s \in \mathbb{R}^+$. As $F(0) \leq TE(0)^{\alpha+1} = T$ by (II-4), the right-hand side of (II-7) is less than or equal to $T^{\frac{\alpha+1}{\alpha}}(T+\alpha s)^{-\frac{1}{\alpha}}$. On the other hand, as E is non-negative and non-decreasing, the left-hand side of (II-7) can be represented as

$$F(s) = \int_s^\infty E^{\alpha+1} dt \geq \int_s^{T+(\alpha+1)s} E^{\alpha+1} dt \geq (T + \alpha s) E \left(T + (\alpha+1)s\right)^{\alpha+1}.$$

Therefore, from (II-7),

$$(T + \alpha s) E \left(T + (\alpha+1)s\right)^{\alpha+1} \leq T^{\frac{\alpha+1}{\alpha}} \left(T + \alpha s\right)^{-\frac{1}{\alpha}}.$$

Simplifying this, we get

$$E\left(T + (\alpha+1)s\right) \leq \left(1 + \frac{\alpha s}{T}\right)^{-\frac{1}{\alpha}}, \quad \forall s \geq 0.$$

Letting $t = T + (\alpha+1)s$, we obtain (II-6).

III. Methodology and Results

1. Wave Equation with Damping Term

The form of the polynomial damping wave equation is as follows:

$$\rho u_{tt}(x,t) - T u_{xx}(x,t) + a|u_t(x,t)|^{p-2} u_t(x,t) = 0, \tag{III-1}$$

where $(x,t) \in (0,L) \times (0,\infty)$. Subject to the boundary and initial conditions

$$u(0,t) = u(L,t) = 0, \quad t \in (0,\infty), \tag{III-2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,L). \tag{III-3}$$

Here, a is the damping constant, $p \geq 2$, and u_0, $u_1 \in C[0,L]$. It is well-known that the problem (III-1), (III-2), (III-3) has a solution $u \in C^2$ $(0, \infty; C(0,L))$. We aim to investigate the energy decay rate to analyze the properties of the solution.

2. Symbols and Notations

For $v, w \in C(0, L)$ and $p \geq 2$,

$$\langle v, w \rangle = \int_0^L v(x)w(x)dx,$$

$$||v||_p = \left(\int_0^L (v(x))^p \, dx \right)^{\frac{1}{p}}.$$

Let $c_p > 0$ be a constant satisfying the Sobolev embedding $||v||_p \leq c_p||v_x||$ for $2 \leq p < \infty$. Here are the Holder inequality,

$$\langle v, w \rangle \leq ||v||_p ||w||_q \quad \text{for } p > 1, \; \frac{1}{p} + \frac{1}{q} = 1.$$

And the Young inequality,

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \quad \text{for } p > 1, \; \frac{1}{p} + \frac{1}{q} = 1.$$

The energy of the wave at time t, denoted as $E(t)$, is given by

$$E(t) = \frac{\rho}{2}||u_t(t)||_2^2 + \frac{T}{2}||u_x(t)||_2^2.$$

3. Verification of Energy Variation

Multiplying (III-1) by $u_t(x, t)$ and integrating the equation over $(0, L)$ and using (III-2), we obtain

$$\begin{aligned}
0 &= \rho \int_0^L u_{tt}(x,t)u_t(x,t)dx - T \int_0^L u_{xx}(x,t)u_t(x,t)dx + a \int_0^L |u_t(x,t)|^p dx \\
&= \frac{\rho}{2}\frac{d}{dt}||u_t(t)||_2^2 - T[u_x(x,t)u_t(x,t)]_0^L + T \int_0^L u_x(x,t)u_{xt}(x,t)dx + a||u_t||_p^p \\
&= \frac{\rho}{2}\frac{d}{dt}||u_t(t)||_2^2 + \frac{T}{2}\frac{d}{dt}||u_x(t)||_2^2 + a||u_t(t)||_p^p.
\end{aligned}$$

From the above equation, we can observe $E'(t) = -a||u_t(t)||_p^p \leq 0$. Thus, we confirm that the energy of the solution does not increase. However, obtaining information about the rate of energy decay from this equation is challenging.

4. Derivation of Energy Decay Rate

To investigate the energy decay rate, let $\theta > 0$, multiply (III-1) by $E^\theta(t)u(x, t)$ and integrate the equation over $(0, L)$ using (III-2),

$$
\begin{aligned}
2\int_S^R E^{\theta+1}(t)dt = &-\rho E^\theta(R)\langle u_t(R), u(R)\rangle + \rho E^\theta(S)\langle u_t(S), u(S)\rangle \\
&+ \rho\theta \int_S^R E^{\theta-1}(t)\frac{d}{dt}\langle u_t(t), u(t)\rangle dt \\
&- a\int_S^R E^\theta(t)\int_0^L |u_t(x,t)|^{p-2}u_t(x,t)u(x,t)dxdt \\
&+ 2\theta\int_S^R E^\theta(t)||u_t(t)||_2^2 dt \left(:= \sum_{k=1}^5 I_k \right).
\end{aligned}
$$

5. Derivation of Inequality between I_k and Energy Functions

Using Holder's inequality, Young's inequality, the Sobolev embedding theorem, and the fact that $E'(t) \le 0$ for any t, we derive the relations between I_k and the energy functions for $k \in \{1, 2, 3, 4, 5\}$.

5.1. $k = 1$: Inequality for I_1

For $k = 1$, the following relations can be observed:

$$
\begin{aligned}
I_1 &\le \rho E^\theta(R)||u_t(R)||_2^2||u(R)||_2^2 \\
&\le \rho c_2^2 E^\theta(R)||u_t(R)||_2^2||u_x(R)||_2^2 \\
&\le \rho c_2^2 E^\theta(R)\left(\frac{1}{2}||u_t(R)||_2^2 + \frac{1}{2}||u_x(R)||_2^2\right) \\
&\le \rho c_2^2 \max\left(\frac{1}{\rho}, \frac{1}{T}\right) E^\theta(R)\left(\frac{\rho}{2}||u_t(R)||_2^2 + \frac{T}{2}||u_x(R)||_2^2\right) \\
&= \rho c_2^2 \max\left(\frac{1}{\rho}, \frac{1}{T}\right) E^{\theta+1}(R) \le C_1 E^\theta(0)E(S).
\end{aligned}
$$

Here, C_i in the above and subsequent equations represents an appropriate positive constant.

5.2. $k = 2$: Inequality for I_2

For $k = 2$, a similar method as in I_1 yields

$$
I_2 \le C_1 E^\theta(0)E(S).
$$

5.3. $k = 3$: Inequality for I_3

For $k = 3$, the following relations hold

$$
\begin{aligned}
I_3 &\leq \rho\theta \int_S^R E^{\theta-1}(t)(-E'(t))||u_t(t)||_2||u(t)||_2 dt \\
&\leq \theta C_1 \int_S^R E^{\theta}(t)(-E'(t))dt \\
&= -\frac{\theta C_1}{\theta+1} \int_S^R \left(E^{\theta+1}(t)\right)' dt \\
&= -\frac{\theta C_1}{\theta+1} \left(E^{\theta+1}(R) - E^{\theta+1}(S)\right) \\
&\leq \frac{\theta C_1}{\theta+1} E^{\theta+1}(S) \\
&\leq C_2 E^{\theta}(0) E(S).
\end{aligned}
$$

5.4. $k = 4$: Inequality for I_4

For $k = 4$, a similar procedure as previously discussed can be demonstrated

$$
\begin{aligned}
I_4 &\leq a \int_S^R E^{\theta}(t) \int_0^L |u_t(x,t)|^{p-1}|u(x,t)|dxdt \\
&\leq \int_S^R E^{\theta}(t) \left(\int_0^L |u_t(x,t)|^p dx\right)^{\frac{p-1}{p}} \left(\int_0^L |u(x,t)|^p dx\right)^{\frac{1}{p}} dt \\
&= \int_S^R E^{\theta}(t)||u_t(t)||_p^{p-1}||u(t)||_p dt \\
&\leq \int_S^R c_p E^{\theta}(t)||u_t(t)||_p^{p-1}||u_x(t)||_2 dt \\
&\leq \int_S^R E^{\theta}(t) \left(C(\varepsilon)|u_t(t)_p^p + \varepsilon||u_x(t)||_2^p\right) dt \\
&= \int_S^R E^{\theta}(t) \left(-\frac{C(\varepsilon)}{a} E'(t) + \varepsilon||u_x(t)||_2^p\right) dt.
\end{aligned}
$$

Here, ε is a suitable positive constant, and $C(\varepsilon)$ is the Holder conjugate of ε.

5.5. $k = 5$: Inequality for I_5

For I_5, it can be divided into cases of $p = 2$ and $p > 2$.

5.5.1. Case $p = 2$

$$\begin{aligned} I_5 &= 2\theta \int_S^R E^\theta(t)||u_t(t)||_p^p dt \\ &= -\frac{2\theta}{a} \int_S^R E^\theta(t)E'(t)dt \\ &= -\frac{2\theta}{a(\theta+1)} \int_S^R \left(E^{\theta+1}(t)\right)' dt \\ &\leq C_4 E^\theta(0)E(S). \end{aligned}$$

5.5.2. Case $p > 2$

Let $\Omega_1 = \{x \in (0, L) : |u_t(x,t)| < 1\}$ and $\Omega_2 = \{x \in (0, L) : |u_t(x,t)| \geq 1\}$. Then the following relationship holds

$$\begin{aligned} ||u_t(t)||_2^2 &= \int_{\Omega_1} (u_t(x,t))^2 dx + \int_{\Omega_2} (u_t(x,t))^2 dx \\ &\leq \left(\int_{\Omega_1} 1dx\right)^{\frac{p-2}{p}} \left(\int_{\Omega_1} (u_t(x,t))^p dx\right)^{\frac{2}{p}} + \int_{\Omega_2} (u_t(x,t))^p dx \\ &\leq L^{\frac{p-2}{p}} \left(-\frac{1}{a}E'(t)\right)^{\frac{2}{p}} - \frac{1}{a}E'(t). \end{aligned}$$

From the above fact, the following relationship can be derived

$$I_5 \leq \varepsilon \int_S^R E^{\theta+1}(t)dt + C(\varepsilon) \int_S^R (-E'(t))^{\frac{2(\theta+1)}{p}} dt + C_5 E^\theta(0)E(S).$$

Here, similar to I_4, ε is a suitable positive constant, and $C(\varepsilon)$ is the Holder conjugate of ε.

6. $p = 2$: Relationship between $E(t)$ and $E(0)$

From the results of the preceding section, it can be confirmed that the following holds

$$2\int_S^R E^{\theta+1}(t)dt \leq \frac{2\varepsilon}{T} \int_S^R E^{\theta+1}(t)dt + (2C_1 + C_2 + C_3C(\varepsilon) + C_4)E^\theta(0)E(S).$$

Setting $\varepsilon = \frac{T}{2}$, the above equation is expressed as

$$\int_S^R E^{\theta+1}(t)dt \leq C(E(0))E(S).$$

Here, $C(E(0))$ is a positive constant determined by $E(0)$. Therefore, letting $q = 0$ and taking the limit $R \to \infty$, we obtain

$$\int_S^\infty E(t)dt \le C(E(0))E(S).$$

Defining $h(y) := e^{\frac{y}{C(E(0))}} \int_y^\infty E(s)ds$, the following result can be derived

$$E(t) \le E(0)e^{1-\frac{t}{C(E(0))}}.$$

7. $p > 2$: Relationship between $E(t)$ and $E(0)$

From the results of the preceding section, it can be confirmed that the following holds

$$\begin{aligned} 2\int_S^R E^{\theta+1}(t)dt \le{}& \varepsilon\left(\left(\frac{2}{T}\right)^{\frac{p}{2}} E^{\frac{p-2}{2}}(0)+1\right)\int_S^R E^{\theta+1}(t)dt \\ &+ C(\varepsilon)\int_S^R (-E'(t))^{\frac{2(\theta+1)}{p}}dt \\ &+ (2C_1+C_2+C_3C(\varepsilon)+C_5)E^\theta(0)E(S). \end{aligned}$$

Setting $\varepsilon\left(\left(\frac{2}{T}\right)^{\frac{p}{2}} E^{\frac{p-2}{2}}(0)+1\right) = 1$ for ε, and $\theta = \frac{p-2}{2}$, the above equation is expressed as

$$\begin{aligned} &\int_S^R E^{\theta+1}(t)dt \\ &\le C_6\int_S^R -E'(t)dt + cE^\theta(0)E(S) \\ &\le cE^\theta(0) + C_6E(S). \end{aligned}$$

Here, $C(E(0))$ is a positive constant determined by $E(0)$, and $c = 2C_1 + C_2 + C_3C(\varepsilon) + C_5$. Therefore, letting $q = 0$ and taking the limit $R \to \infty$, we obtain

$$\int_S^\infty E(t)dt \le C(E(0))E(S).$$

If $E(0) = 0$, obtaining $E(t) = 0$ results in a meaningless outcome. Hence, without loss of generality, assume $E(0) > 0$. Then, the following equation can be obtained

$$\int_S^R E^{\theta+1}(t)dt \le C(E(0))E^\theta(0)E(S).$$

Defining $F(S) := \int_S^\infty E^{\theta+1}(t)dt$, the following result is obtained

$$E(t) \le E(0)\left(\frac{C(E(0))+\theta t}{C(E(0))(1+\theta)}\right)^{-\frac{1}{\theta}},$$

where $\theta = \frac{p-2}{2}$ in the above equation.

IV. Research Results

As evident from the preceding sections, for the time variable t, the energy $E(t)$ in the linearly damped wave equation is bounded by the inequality

$$E(t) \leq E(0)e^{1-\frac{t}{C(E(0))}}$$

Likewise, in the case of the polynomially damped wave equation, the energy $E(t)$ is bounded by the inequality

$$E(t) \leq E(0)\left(\frac{C(E(0)) + \frac{p-2}{2}t}{C(E(0)) \cdot \frac{p}{2}}\right)^{-\frac{2}{2-p}}$$

For example, when $E(0) = 5$, $C(E(0)) = 0.3$, and $p = 14.8$, the graph of energy looks as depicted in Figure 2.

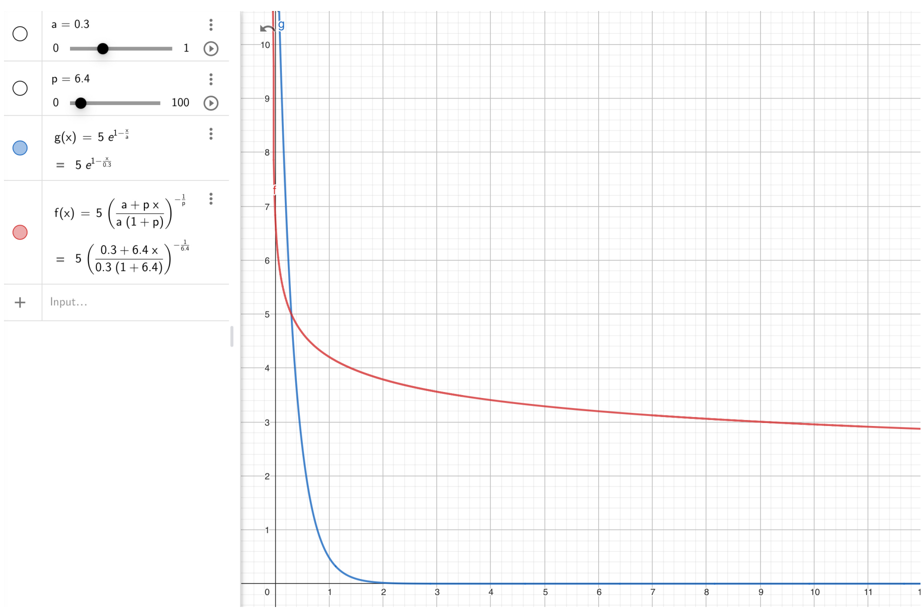

Figure 2: Graph for $(E(0), C(E(0)), p) = (5, 0.3, 14.8)$

In the above plot, the blue curve represents the graph for the wave equation with linear damping, while the red curve represents the graph for the wave equation with polynomial damping. Both curves illustrate the upper bounds of energy. From the graph and the inequalities, it's observed that compared to polynomial damping, linear damping shows a faster initial energy decay that gradually slows down over time.

V. Conclusion and Discussion

In this study, by utilizing integral inequality and the Lagrange multiplier method, we obtained inequalities regarding energy as a means to investigate changes in the energy of an elastic string from the perspective of energy in motion. To understand the more general phenomenon of the vibration dampening in the motion of the elastic string, we conducted research investigating the properties of solutions for wave equations with damping terms. Since directly obtaining solutions to wave equations for an elastic string with damping terms poses challenges, we explored the properties of solutions indirectly by computing the string's energy. It is expected that subsequent studies employing methods such as the multiplier method and other perturbation methods beyond those used in this study will yield more precise upper-bound inequalities. Moreover, by employing methods like the implicit method, narrowing down the range of the value of $E(T)$ for suitably large positive T is anticipated. Finally, we propose deriving an expression indicating the vertical displacement of the string through direct experiments for theoretical experimental validation.

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