Memo for the Graduate Course: "*Variational Inequality Approach for Economic Equilibrium Problems*"

Logit Choice and Perturbed Optimization

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1 Additive random utility models and the logit choice

Consider a decision maker (DM) facing a choice situation. There is a known set of finite alternatives $a \in A$ with known deterministic payoffs $v = (v_a)$. However, the actual payoff V_a of choosing a is subject to uncertainty, and may be expressed with random shocks such that

$$V_a = v_a + \epsilon_a, \tag{1}$$

where $\{\epsilon_a\}$ are random variables. It is assumed that ϵ_a are i.i.d. across alternatives. It is further assumed that the DM uses randomization, or mixed strategies, so that they choose each alternative a with the probability p_a that a is payoff-maximizing. That is,

$$p_a = \Pr[V_a \geq V_b \ \forall b \in A] = \Pr[v_a + \epsilon_a \geq v_b + \epsilon_b \ \forall b \in A]. \tag{2}$$

This framework is called *additive random utility models* (ARUM).

If ϵ_a is i.i.d. with a differentiable c.d.f. F, we have

$$p_a = \int F'(\epsilon_a) \prod_{b \neq a} F(v_a - v_b + \epsilon_a) \mathrm{d}\epsilon_a. \tag{3}$$

Further suppose that every ϵ_a follows the Gumbel distribution with no location parameter and scale parameter $\eta > 0$, whose c.d.f. is given as

$$F(\epsilon) \equiv \exp\left(-\exp\left(-\eta^{-1}\epsilon\right)\right) \qquad \epsilon \in (-\infty, \infty). \tag{4}$$

It is known that $\mathbb{E}[\epsilon] = \eta^{-1}\gamma$ with Euler's constant $\gamma \approx 0.5772$, $\mathrm{Var}[\epsilon] = \eta^2\pi^2/6$. The constant η thus represents the magnitude of randomness, and the deterministic payoff v is less (more) relevant for DM's choice when η is large (small).

Under the Gumbel assumption, we obtain the *logit choice rule*:

$$p_a = \frac{\exp(\eta^{-1}v_a)}{\sum_{b\in A}\exp(\eta^{-1}v_b)}. \tag{5}$$

It is also known that the expected value of the maximized payoff, the *expected maximum utility* is

$$\lambda \equiv \mathbb{E}\left[\max_{a\in A} V_a\right] = \eta \log \sum_{a\in A} \exp\left(\eta^{-1}v_a\right) + \eta^{-1}\gamma. \tag{6}$$

It is well known that $\frac{\partial \lambda}{\partial v_a} = p_a$.

Note: Computation of p_a and λ

To compute p_a, we first note that

$$F'(\epsilon) = \rho(\epsilon)F(\epsilon) \quad \text{with} \quad \rho(\epsilon) \equiv \eta^{-1}\exp\left(-\eta^{-1}\epsilon\right),$$

$$F(v+\epsilon) = F(\epsilon)^{\exp(-\eta^{-1}v)},$$
$$\{F(\epsilon)^t\}' = tF(\epsilon)^{t-1}F'(\epsilon) = t\rho(\epsilon)F(\epsilon)^t.$$

Then, noting that F is a c.d.f. ($\lim_{\epsilon\to 0} F(\epsilon) = 0$, $\lim_{\epsilon\to\infty} F(\epsilon) = 1$), we see

$$\begin{aligned}
p_a &= \int F'(\epsilon_a) \prod_{b\neq a} F(v_a - v_b + \epsilon_a)\mathrm{d}\epsilon_a \\
&= \int \rho(\epsilon_a)F(\epsilon_a) \prod_{b\neq a} F(\epsilon_a)^{\exp(\eta^{-1}(v_b - v_a))}\mathrm{d}\epsilon_a \\
&= \int \rho(\epsilon_a)F(\epsilon_a)^{1+\sum_{b\neq a}\exp(\eta^{-1}(v_b-v_a))}\mathrm{d}\epsilon_a \\
&= \frac{1}{1+\sum_{b\neq a}\exp(\eta^{-1}(v_b-v_a))}\left[F(\epsilon_a)^{1+\sum_{b\neq a}\exp(\eta^{-1}(v_b-v_a))}\right]_{-\infty}^{\infty} \\
&= \frac{\exp(\eta^{-1}v_a)}{\sum_{b\in A}\exp(\eta^{-1}v_b)}.
\end{aligned}$$

To compute λ, we observe that for $v^* \equiv \max_{a\in A} V_a$,

$$\begin{aligned}
\Pr[v^* \leq x] &= \Pr[\epsilon_a \leq x - v_a \;\; \forall a \in A] = \prod_{a\in A} F(x - v_a) \\
&= F(x)^{\sum_{a\in A}\exp(\eta^{-1}v_a)} = F(x)^{\exp(\eta^{-1}\lambda_0)} \qquad \text{where} \quad \lambda_0 \equiv \eta \log \sum_{a\in A} \exp(\eta^{-1}v_a) \\
&= F(x - \lambda_0).
\end{aligned}$$

Thus, v^* follows the Gumbel distribution with location parameter λ_0 and scale parameter η, implying $\lambda = \mathbb{E}[v^*] = \lambda_0 + \eta^{-1}\gamma$ as in (6).

2 Mixed-strategy best response and linear optimization problem

As a preparation, consider a simple deterministic setting. Given alternatives $a \in A$ and payoffs $v = (v_a)$, suppose that the DM's problem is to determine the payoff-maximizing mixed strategy by solving the following linear optimization problem:

$$\max_{y\in\Delta} \quad \langle v, y\rangle \tag{7}$$

where $\Delta \equiv \{y \geq \mathbf{0} \mid \sum_{a\in A} y_a = 1\}$ and $\langle x, y\rangle$ denotes the inner product of x and y. The solution y^* for this problem should satisfy

$$y_a^* > 0 \Rightarrow a \in \mathrm{br}(v), \tag{8}$$

where $\mathrm{br}(v) \equiv \arg\max_b \{v_b\}_{b\in A}$ is the set of payoff-maximizing alternatives given the payoff vector v. Such y^* form a convex set.

The dual problem is given as

$$\min_{\lambda} \quad \lambda \tag{9}$$
$$\text{s.t.} \quad \lambda \geq v_a \quad \forall a \in A. \tag{10}$$

The problem aims to obtain the best (smallest) upper bound for DM's attainable payoff. Evidently, the optimal solution and the optimal value for the problem is $\lambda^* = \max_{a\in A} v_a$.

Note: Derivation of the dual problem

Let λ be the Lagrange multiplier for the constraint $\sum_{a\in A} y_a = 1$. The Lagrangian function is

$$L(y,\lambda) \equiv -\langle v, y\rangle + \lambda\left(\langle \mathbf{1}, y\rangle - 1\right) = \langle \lambda\mathbf{1} - v, y\rangle - \lambda \tag{11}$$

with $y \geq 0$. The Lagrangian dual problem is to maximize

$$\omega(v) = \inf_{y\geq 0} L(y,\lambda) = \inf_{y\geq 0} \langle \lambda\mathbf{1} - v, y\rangle - \lambda = \begin{cases} -\lambda & \text{if } \lambda \geq v_a \quad \forall a \in A, \\ -\infty & \text{otherwise.} \end{cases} \tag{12}$$

3 Perturbed optimization

Suppose that the DM's problem in (7) is modified as follows:

$$\max_{y\in\Delta} \quad \langle v, y\rangle - H(y) \tag{13}$$

The function $H : \mathrm{int}(\Delta) \to \mathbb{R}$ is assumed to be convex and becomes infinitely steeper as y goes to the boundary of Δ. As a representative case, suppose that H is the negative entropy

$$H(y) = \eta \sum_{a\in A} y_a \log y_a, \tag{14}$$

where we define $0 \log 0 \equiv 0$.

The optimal solution y^* turns out to be the logit choice rule:

$$y_a^* = \frac{\exp(\eta^{-1} v_a)}{\sum_{b\in A} \exp(\eta^{-1} v_b)}. \tag{15}$$

The optimal value of the problem (13) is

$$\begin{aligned} \langle v, y^*\rangle - \eta\langle y^*, \log y^*\rangle &= \langle v, y^*\rangle - \eta\langle y^*, \eta^{-1} v\rangle + \eta \log \sum_{a\in A} \exp(\eta^{-1} v_a) \langle y^*, \mathbf{1}\rangle & (16) \\ &= \eta \log \sum_{a\in A} \exp(\eta^{-1} v_a) \quad (= \lambda + \eta). & (17) \end{aligned}$$

We observe that λ can be seen as the expected maximum utility for the logit model (up to a constant addition). We also confirm

$$\frac{\partial \lambda(v)}{\partial v_a} = \frac{\exp(\eta^{-1} v_a)}{\sum_{b\in A} \exp(\eta^{-1} v_b)} = y_a^*. \tag{18}$$

The optimal value function of (13) is the convex conjugate (Legendre transform) of H, which also implies the above formula.

The Lagrange dual problem for (13) is

$$\begin{aligned} &\min_{\lambda} \quad \lambda + \eta & (19) \\ &\text{s.t.} \quad \lambda = \eta \log \sum_{a\in A} \exp(\eta^{-1} v_a) - \eta, & (20) \end{aligned}$$

whose solution and optimal value tends to $\lambda^* = \max_{a\in A} v_a$ as $\eta \to 0$.

Note: Derivations for y_a^* and λ

The Lagrangian function is modified as

$$L(y,v) \equiv -\langle v, y\rangle + \lambda\left(\langle \mathbf{1}, y\rangle - 1\right) + H(y). \tag{21}$$

The optimality condition is

$$y_a \frac{\partial L(y,\lambda)}{\partial y_a} = 0, y_a \geq 0, \frac{\partial L(y,\lambda)}{\partial y_a} = -v_a + \lambda + \eta \log y_a + \eta \geq 0, \tag{22}$$

$$\frac{\partial L(y,\lambda)}{\partial v} = \sum_a y_a - 1 = 0. \tag{23}$$

Since $\frac{\partial L(y,\lambda)}{\partial y_a} \to -\infty$ as $y_a \to 0$, $y_a = 0$ violates (22). Then, $y_a > 0$ and $\frac{\partial L(y,\lambda)}{\partial y_a} = 0$ for all a, implying $y_a = \exp\left(\eta^{-1}\left(v_a - \lambda\right) - 1\right)$. Thus, from $\sum_a y_a = 1$,

$$\lambda = \eta \log \sum_{a \in A} \exp(\eta^{-1} v_a) - \eta \tag{24}$$

Remark 3.1. Observe that when we take the limit $\eta \to 0$, $y_a > 0$ can occur only if $a \in \mathrm{br}(v)$, and $y_a \to 0$ as $\eta \to 0$ if $a \notin \mathrm{br}(v)$, which are consistent with the unperturbed case. To see this, observe

$$y_a = \frac{1}{\sum_{b \in A} \exp(\eta^{-1}(v_b - v_a))}. \tag{25}$$

The denominator goes to infinity as $\eta \to 0$ when $v_b > v_a$ for some b, so that $y_a \to 0$ if $a \notin \mathrm{br}(v)$. If $a \in \mathrm{br}(v)$, y_a tends to $1/|\,\mathrm{br}(v)|$ as $\eta \to 0$, which is slightly different from the unperturbed case because mixed-strategy best response can be nonunique.

Remark 3.2. Considering a different convex function for H induces a different choice rule.

Further Readings:

- Hofbauer and Sandholm (2002), Theorem 2.1; Hofbauer and Sandholm (2007), Appendix.
- Anderson et al. (1992)
- 土木学会 (1998), Ch.6
- Fudenberg et al. (2015)

References

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