

## Logit Choice and Perturbed Optimization

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This note is a brief summary of the well-known connection between the logit choice model and perturbed optimization.

### 1. Additive random utility models and the logit choice

Consider a decision maker (DM) facing a choice situation. There is a finite set of alternatives,  $A$ . The payoff  $V_a$  of choosing an alternative  $a \in A$  is subject to uncertainty, and may be expressed as a random variable such that

$$V_a = v_a + \epsilon_a, \quad (1)$$

where  $v_a$  is known deterministic payoff, and  $\epsilon_a$  is a random payoff. It is assumed that  $\epsilon_a$  are i.i.d. across alternatives. It is further assumed that the DM uses randomization, or mixed strategies, so that they choose each alternative  $a$  with the probability  $p_a$  that  $a$  is payoff-maximizing. That is,

$$p_a = \Pr[V_a \geq V_b \ \forall b \in A] = \Pr[v_a + \epsilon_a \geq v_b + \epsilon_b \ \forall b \in A]. \quad (2)$$

This framework is called *additive random utility models* (ARUM).

If  $\epsilon_a$  is i.i.d. with a differentiable c.d.f.  $F$ , we have

$$p_a = \int F'(\epsilon_a) \prod_{b \neq a} F(v_a - v_b + \epsilon_a) d\epsilon_a. \quad (3)$$

Further suppose that every  $\epsilon_a$  follows the Gumbel distribution with scale parameter  $\eta > 0$  and no location parameter, whose c.d.f. is given as

$$F(\epsilon) \equiv \exp(-\exp(-\eta^{-1}\epsilon)) \quad \epsilon \in (-\infty, \infty). \quad (4)$$

It is known that  $\mathbb{E}[\epsilon] = \eta^{-1}\gamma$  with Euler’s constant  $\gamma \approx 0.5772$ ,  $\text{Var}[\epsilon] = \eta^2\pi^2/6$ . The constant  $\eta$  thus represents the magnitude of randomness, and the deterministic payoff  $v$  is less (more) relevant for DM’s choice when  $\eta$  is large (small).

Under the Gumbel assumption, we obtain the *logit choice rule*:

$$p_a = \frac{\exp(\eta^{-1}v_a)}{\sum_{b \in A} \exp(\eta^{-1}v_b)}. \quad (5)$$

The expected value of the maximized payoff, the *expected maximum utility* (EMU) is

$$\lambda \equiv \mathbb{E} \left[ \max_{a \in A} V_a \right] = \eta \log \sum_{a \in A} \exp(\eta^{-1}v_a) + \eta^{-1}\gamma. \quad (6)$$

It is well known that choice probabilities in the logit model satisfy  $\frac{\partial \lambda}{\partial v_a} = p_a$ , that is, choice probability vector is the gradient of EMU with respect to the deterministic payoffs. This result also extends to all ARUMs under mild conditions (Williams–Daly–Zachary Theorem) (see [Fosgerau et al., 2020](#)).

Note: Computation of  $p_a$  and  $\lambda$

To compute  $p_a$  under the Gumbel-distributed  $\epsilon_a$ , note that, for c.d.f. (4), we have

$$\begin{aligned} F'(\epsilon) &= \rho(\epsilon)F(\epsilon) \quad \text{with} \quad \rho(\epsilon) \equiv \eta^{-1} \exp(-\eta^{-1}\epsilon) \quad (= \text{p.d.f.}), \\ F(v + \epsilon) &= F(\epsilon)^{\exp(-\eta^{-1}v)}, \\ \{F(\epsilon)^t\}' &= tF(\epsilon)^{t-1}F'(\epsilon) = t\rho(\epsilon)F(\epsilon)^t. \end{aligned}$$

Then, noting that  $\lim_{\epsilon \rightarrow 0} F(\epsilon) = 0$  and  $\lim_{\epsilon \rightarrow \infty} F(\epsilon) = 1$ , we see

$$\begin{aligned} p_a &= \int_{-\infty}^{\infty} F'(\epsilon_a) \prod_{b \neq a} F(v_b - v_a + \epsilon_a) d\epsilon_a \\ &= \int_{-\infty}^{\infty} \rho(\epsilon_a) F(\epsilon_a) \prod_{b \neq a} F(\epsilon_a)^{\exp(\eta^{-1}(v_b - v_a))} d\epsilon_a \\ &= \int_{-\infty}^{\infty} \rho(\epsilon_a) F(\epsilon_a)^{1 + \sum_{b \neq a} \exp(\eta^{-1}(v_b - v_a))} d\epsilon_a \\ &= \frac{1}{1 + \sum_{b \neq a} \exp(\eta^{-1}(v_b - v_a))} \left[ F(\epsilon_a)^{1 + \sum_{b \neq a} \exp(\eta^{-1}(v_b - v_a))} \right]_{-\infty}^{\infty} \\ &= \frac{\exp(\eta^{-1}v_a)}{\sum_{b \in A} \exp(\eta^{-1}v_b)}. \end{aligned}$$

To compute  $\lambda$ , we observe that for  $\hat{V} \equiv \max_{a \in A} V_a$ ,

$$\begin{aligned} \Pr[\hat{V} \leq x] &= \Pr[\epsilon_a \leq x - v_a \quad \forall a \in A] = \prod_{a \in A} F(x - v_a) \\ &= F(x)^{\sum_{a \in A} \exp(\eta^{-1}v_a)} = F(x)^{\exp(\eta^{-1}\lambda_0)} \quad \text{where} \quad \lambda_0 \equiv \eta \log \sum_{a \in A} \exp(\eta^{-1}v_a) \\ &= F(x - \lambda_0). \end{aligned}$$

Thus,  $\hat{V}$  follows the Gumbel distribution with location parameter  $\lambda_0$  and scale parameter  $\eta$ , implying  $\lambda = \mathbb{E}[\hat{V}] = \lambda_0 + \eta^{-1}\gamma$  as in (6).

## 2. Mixed-strategy best response and linear optimization problem

Next, consider a simple, deterministic approach. Given alternatives  $a \in A$  and payoffs  $v = (v_a)$ , suppose that the DM's problem is to determine the payoff-maximizing mixed strategy by solving the following linear optimization problem:

$$\max_{y \in \Delta} \langle v, y \rangle \tag{7}$$

where  $\Delta \equiv \{y \geq \mathbf{0} \mid \sum_{a \in A} y_a = 1\}$  is the probability simplex and  $\langle x, y \rangle$  denotes the inner product of  $x$  and  $y$ . The solution  $y^*$  for this problem should satisfy

$$y_a^* > 0 \Rightarrow a \in \text{br}(v), \tag{8}$$

where  $\text{br}(v) \equiv \arg \max_b \{v_b\}_{b \in A}$  is the set of payoff-maximizing alternatives given the payoff vector  $v$ . Such  $y^*$  form a convex set but nonuniqueness is not guaranteed.

The dual problem is given as

$$\min_{\lambda} \quad \lambda \quad \text{s.t.} \quad \lambda \geq v_a \quad \forall a \in A. \quad (9)$$

The problem aims to obtain the best (smallest) upper bound for DM's attainable payoff. Evidently, the solution and the optimal value for the problem is  $\lambda^* = \max_{a \in A} v_a$ .

Note: Derivation of the dual problem

Let  $\lambda$  be the Lagrange multiplier for the constraint  $\sum_{a \in A} y_a = 1$ . The Lagrangian function is

$$L(y, \lambda) \equiv -\langle v, y \rangle + \lambda (\langle \mathbf{1}, y \rangle - 1) = \langle \lambda \mathbf{1} - v, y \rangle - \lambda \quad (10)$$

with  $y \geq 0$ . The Lagrangian dual problem is to maximize

$$\omega(\lambda) = \inf_{y \geq 0} L(y, \lambda) = \inf_{y \geq 0} \langle \lambda \mathbf{1} - v, y \rangle - \lambda = \begin{cases} -\lambda & \text{if } \lambda \geq v_a \quad \forall a \in A, \\ -\infty & \text{otherwise.} \end{cases} \quad (11)$$

### 3. Perturbed optimization

As seen, the deterministic approach does not provide unique prediction regarding DM's choice. From the mathematical optimization perspective, this stems from the fact that (7) is a linear optimization problem. We can consider adding a regularization term to ensure the uniqueness of the predicted behavior.

Suppose that the DM's problem in (7) is modified as follows:

$$\max_{y \in \Delta} \quad \langle v, y \rangle - H(y) \quad (12)$$

The function  $H : \text{int}(\Delta) \rightarrow \mathbb{R}$  is assumed to be strictly convex and becomes infinitely steeper as  $y$  goes to the boundary of  $\Delta$ . Since the objective function is strictly concave and the feasible region  $\Delta$  is convex and compact, the modified problem has unique solution.

Below, as a representative case, suppose that  $H$  is the negative entropy

$$H(y) = \eta \sum_{a \in A} y_a \log y_a, \quad (13)$$

where we define  $0 \log 0 \equiv 0$ . As  $\eta \rightarrow 0$ , the problem (12) recovers the unperturbed problem (7).

The optimal solution  $y^*$  is the logit choice rule:

$$y_a^* = p_a = \frac{\exp(\eta^{-1} v_a)}{\sum_{b \in A} \exp(\eta^{-1} v_b)}. \quad (14)$$

The optimal value of the problem (12) is

$$\lambda(v) \equiv \langle v, y^* \rangle - \eta \langle y^*, \log y^* \rangle = \eta \log \sum_{a \in A} \exp(\eta^{-1} v_a). \quad (15)$$

We see that the optimal value can be seen as the expected maximum utility for the logit model. In

fact,

$$\frac{\partial \lambda(v)}{\partial v_a} = \frac{\exp(\eta^{-1}v_a)}{\sum_{b \in A} \exp(\eta^{-1}v_b)} = p_a. \quad (16)$$

The optimal value function of (12) is nothing but the convex conjugate (Legendre transform) of  $H$ , which also implies the above formula.

The Lagrange dual problem for (12) is

$$\min_{\lambda} \quad \lambda \quad \text{s.t.} \quad \lambda = \eta \log \sum_{a \in A} \exp(\eta^{-1}v_a) \quad (17)$$

whose solution and optimal value is given by (15) and tends to  $\lambda^* = \max_{a \in A} v_a$  as  $\eta \rightarrow 0$ . Observe the similarity between the dual problem for the unperturbed case.

Note: Derivations for  $y_a^*$  and the Lagrangian dual problem

The Lagrangian function is modified as

$$L(y, v) \equiv -\langle v, y \rangle + \lambda (\langle \mathbf{1}, y \rangle - 1) + H(y). \quad (18)$$

The optimality condition is

$$y_a \frac{\partial L(y, \lambda)}{\partial y_a} = 0, y_a \geq 0, \frac{\partial L(y, \lambda)}{\partial y_a} = -v_a + \lambda + \eta \log y_a + \eta \geq 0, \quad (19)$$

$$\frac{\partial L(y, \lambda)}{\partial \lambda} = \sum_a y_a - 1 = 0. \quad (20)$$

Since  $\frac{\partial L(y, \lambda)}{\partial y_a} \rightarrow -\infty$  as  $y_a \rightarrow 0$ ,  $y_a = 0$  violates (19). Then,  $y_a > 0$  and  $\frac{\partial L(y, \lambda)}{\partial y_a} = 0$  for all  $a$ , implying  $y_a = \exp(\eta^{-1}(v_a - \lambda) - 1)$ . Thus, from  $\sum_a y_a = 1$ , we obtain

$$\lambda = \eta \log \sum_{a \in A} \exp(\eta^{-1}v_a) - \eta. \quad (21)$$

Since  $\inf_{y \geq 0} L(y, \lambda) = \lambda + \eta$ , the dual problem is equivalent to (17) where we redefine  $\lambda := \lambda + \eta$ .

Observe that when we take the limit  $\eta \rightarrow 0$ ,  $y_a > 0$  can occur only if  $a \in \text{br}(v)$ , and  $y_a \rightarrow 0$  as  $\eta \rightarrow 0$  if  $a \notin \text{br}(v)$ , which are consistent with the unperturbed case. To see this, observe

$$y_a = \frac{1}{\sum_{b \in A} \exp(\eta^{-1}(v_b - v_a))}. \quad (22)$$

If  $a \notin \text{br}(v)$ ,  $y_a \rightarrow 0$  because the denominator goes to infinity as  $\eta \rightarrow 0$  when  $v_b > v_a$  for some  $b$ . If  $a \in \text{br}(v)$ ,  $y_a$  tends to  $1/|\text{br}(v)|$  as  $\eta \rightarrow 0$ , which is slightly different from the unperturbed case where mixed-strategy best response can be nonunique.

Considering a different convex function for  $H$  induces a different choice rule. All practically used ARUM have such deterministically perturbed optimization representation but converse is not true.

## 4. Further readings

- Hofbauer and Sandholm (2002), Theorem 2.1; Hofbauer and Sandholm (2007), Appendix.
- Anderson et al. (1992)
- 土木学会 (1998), Ch.6

- [Fudenberg et al. \(2015\)](#)
- [Fosgerau et al. \(2020\)](#)

## References

- Anderson, S. P., de Palma, A., and Thisse, J. F. (1992). *Discrete Choice Theory of Product Differentiation*. MIT Press.
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