

Logit Choice and Perturbed Optimization

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This note is a brief summary of the well-known connection between the logit choice model and perturbed optimization.

1 Additive random utility models and the logit choice

Consider a decision maker (DM) facing a choice situation. There is a known set of finite alternatives $a \in A$ with known deterministic payoffs $v = (v_a)$. However, the actual payoff V_a of choosing a is subject to uncertainty, and may be expressed with random shocks such that

$$V_a = v_a + \epsilon_a, \quad (1)$$

where $\{\epsilon_a\}$ are random variables. It is assumed that ϵ_a are i.i.d. across alternatives. It is further assumed that the DM uses randomization, or mixed strategies, so that they choose each alternative a with the probability p_a that a is payoff-maximizing. That is,

$$p_a = \Pr[V_a \geq V_b \ \forall b \in A] = \Pr[v_a + \epsilon_a \geq v_b + \epsilon_b \ \forall b \in A]. \quad (2)$$

This framework is called *additive random utility models* (ARUM).

If ϵ_a is i.i.d. with a differentiable c.d.f. F , we have

$$p_a = \int F'(\epsilon_a) \prod_{b \neq a} F(v_a - v_b + \epsilon_a) d\epsilon_a. \quad (3)$$

Further suppose that every ϵ_a follows the Gumbel distribution with no location parameter and scale parameter $\eta > 0$, whose c.d.f. is given as

$$F(\epsilon) \equiv \exp(-\exp(-\eta^{-1}\epsilon)) \quad \epsilon \in (-\infty, \infty). \quad (4)$$

It is known that $\mathbb{E}[\epsilon] = \eta^{-1}\gamma$ with Euler’s constant $\gamma \approx 0.5772$, $\text{Var}[\epsilon] = \eta^2\pi^2/6$. The constant η thus represents the magnitude of randomness, and the deterministic payoff v is less (more) relevant for DM’s choice when η is large (small).

Under the Gumbel assumption, we obtain the *logit choice rule*:

$$p_a = \frac{\exp(\eta^{-1}v_a)}{\sum_{b \in A} \exp(\eta^{-1}v_b)}. \quad (5)$$

It is also known that the expected value of the maximized payoff, the *expected maximum utility* is

$$\lambda \equiv \mathbb{E} \left[\max_{a \in A} V_a \right] = \eta \log \sum_{a \in A} \exp(\eta^{-1}v_a) + \eta^{-1}\gamma. \quad (6)$$

It is well known that $\frac{\partial \lambda}{\partial v_a} = p_a$.

Note: Computation of p_a and λ

To compute p_a , we first note that

$$\begin{aligned} F'(\epsilon) &= \rho(\epsilon)F(\epsilon) \quad \text{with} \quad \rho(\epsilon) \equiv \eta^{-1} \exp(-\eta^{-1}\epsilon), \\ F(v + \epsilon) &= F(\epsilon)^{\exp(-\eta^{-1}v)}, \\ \{F(\epsilon)^t\}' &= tF(\epsilon)^{t-1}F'(\epsilon) = t\rho(\epsilon)F(\epsilon)^t. \end{aligned}$$

Then, noting that F is a c.d.f. ($\lim_{\epsilon \rightarrow 0} F(\epsilon) = 0$, $\lim_{\epsilon \rightarrow \infty} F(\epsilon) = 1$), we see

$$\begin{aligned} p_a &= \int F'(\epsilon_a) \prod_{b \neq a} F(v_b - v_a + \epsilon_a) d\epsilon_a \\ &= \int \rho(\epsilon_a) F(\epsilon_a) \prod_{b \neq a} F(\epsilon_a)^{\exp(\eta^{-1}(v_b - v_a))} d\epsilon_a \\ &= \int \rho(\epsilon_a) F(\epsilon_a)^{1 + \sum_{b \neq a} \exp(\eta^{-1}(v_b - v_a))} d\epsilon_a \\ &= \frac{1}{1 + \sum_{b \neq a} \exp(\eta^{-1}(v_b - v_a))} \left[F(\epsilon_a)^{1 + \sum_{b \neq a} \exp(\eta^{-1}(v_b - v_a))} \right]_{-\infty}^{\infty} \\ &= \frac{\exp(\eta^{-1}v_a)}{\sum_{b \in A} \exp(\eta^{-1}v_b)}. \end{aligned}$$

To compute λ , we observe that for $v^* \equiv \max_{a \in A} V_a$,

$$\begin{aligned} \Pr[v^* \leq x] &= \Pr[\epsilon_a \leq x - v_a \quad \forall a \in A] = \prod_{a \in A} F(x - v_a) \\ &= F(x)^{\sum_{a \in A} \exp(\eta^{-1}v_a)} = F(x)^{\exp(\eta^{-1}\lambda_0)} \quad \text{where} \quad \lambda_0 \equiv \eta \log \sum_{a \in A} \exp(\eta^{-1}v_a) \\ &= F(x - \lambda_0). \end{aligned}$$

Thus, v^* follows the Gumbel distribution with location parameter λ_0 and scale parameter η , implying $\lambda = \mathbb{E}[v^*] = \lambda_0 + \eta^{-1}\gamma$ as in (6).

2 Mixed-strategy best response and linear optimization problem

Next, consider a simple, deterministic approach. Given alternatives $a \in A$ and payoffs $v = (v_a)$, suppose that the DM's problem is to determine the payoff-maximizing mixed strategy by solving the following linear optimization problem:

$$\max_{y \in \Delta} \langle v, y \rangle \tag{7}$$

where $\Delta \equiv \{y \geq \mathbf{0} \mid \sum_{a \in A} y_a = 1\}$ is the probability simplex and $\langle x, y \rangle$ denotes the inner product of x and y . The solution y^* for this problem should satisfy

$$y_a^* > 0 \Rightarrow a \in \text{br}(v), \tag{8}$$

where $\text{br}(v) \equiv \arg \max_b \{v_b\}_{b \in A}$ is the set of payoff-maximizing alternatives given the payoff vector v . Such y^* form a convex set.

The dual problem is given as

$$\min_{\lambda} \quad \lambda \quad \text{s.t.} \quad \lambda \geq v_a \quad \forall a \in A. \tag{9}$$

The problem aims to obtain the best (smallest) upper bound for DM's attainable payoff. Evidently, the solution and the optimal value for the problem is $\lambda^* = \max_{a \in A} v_a$.

Note: Derivation of the dual problem

Let λ be the Lagrange multiplier for the constraint $\sum_{a \in A} y_a = 1$. The Lagrangian function is

$$L(y, \lambda) \equiv -\langle v, y \rangle + \lambda (\langle \mathbf{1}, y \rangle - 1) = \langle \lambda \mathbf{1} - v, y \rangle - \lambda \quad (10)$$

with $y \geq 0$. The Lagrangian dual problem is to maximize

$$\omega(v) = \inf_{y \geq 0} L(y, \lambda) = \inf_{y \geq 0} \langle \lambda \mathbf{1} - v, y \rangle - \lambda = \begin{cases} -\lambda & \text{if } \lambda \geq v_a \quad \forall a \in A, \\ -\infty & \text{otherwise.} \end{cases} \quad (11)$$

3 Perturbed optimization

As seen, the deterministic approach does not provide unique prediction regarding DM's choice. From the mathematical optimization perspective, this stems from the fact that (7) is a linear optimization problem. We can consider adding a regularization term to ensure the uniqueness of the predicted behavior.

Suppose that the DM's problem in (7) is modified as follows:

$$\max_{y \in \Delta} \langle v, y \rangle - H(y) \quad (12)$$

The function $H : \text{int}(\Delta) \rightarrow \mathbb{R}$ is assumed to be strictly convex and becomes infinitely steeper as y goes to the boundary of Δ . Since the objective function is strictly concave and the feasible region Δ is convex and compact, the modified problem has unique solution.

Below, as a representative case, suppose that H is the negative entropy

$$H(y) = \eta \sum_{a \in A} y_a \log y_a, \quad (13)$$

where we define $0 \log 0 \equiv 0$. As $\eta \rightarrow 0$, the problem (12) recovers the unperturbed problem (7).

The optimal solution y^* is the logit choice rule:

$$y_a^* = p_a = \frac{\exp(\eta^{-1} v_a)}{\sum_{b \in A} \exp(\eta^{-1} v_b)}. \quad (14)$$

The optimal value of the problem (12) is

$$\lambda(v) \equiv \langle v, y^* \rangle - \eta \langle y^*, \log y^* \rangle = \eta \log \sum_{a \in A} \exp(\eta^{-1} v_a). \quad (15)$$

We see that the optimal value can be seen as the expected maximum utility for the logit model. In fact,

$$\frac{\partial \lambda(v)}{\partial v_a} = \frac{\exp(\eta^{-1} v_a)}{\sum_{b \in A} \exp(\eta^{-1} v_b)} = p_a. \quad (16)$$

The optimal value function of (12) is nothing but the convex conjugate (Legendre transform) of H , which also implies the above formula.

The Lagrange dual problem for (12) is

$$\min_{\lambda} \quad \lambda \quad \text{s.t.} \quad \lambda = \eta \log \sum_{a \in A} \exp(\eta^{-1} v_a) \quad (17)$$

whose solution and optimal value is given by (15) and tends to $\lambda^* = \max_{a \in A} v_a$ as $\eta \rightarrow 0$. Observe the similarity between the dual problem for the unperturbed case.

Note: Derivations for y_a^* and the Lagrangian dual problem

The Lagrangian function is modified as

$$L(y, v) \equiv -\langle v, y \rangle + \lambda (\langle \mathbf{1}, y \rangle - 1) + H(y). \quad (18)$$

The optimality condition is

$$y_a \frac{\partial L(y, \lambda)}{\partial y_a} = 0, y_a \geq 0, \frac{\partial L(y, \lambda)}{\partial y_a} = -v_a + \lambda + \eta \log y_a + \eta \geq 0, \quad (19)$$

$$\frac{\partial L(y, \lambda)}{\partial \lambda} = \sum_a y_a - 1 = 0. \quad (20)$$

Since $\frac{\partial L(y, \lambda)}{\partial y_a} \rightarrow -\infty$ as $y_a \rightarrow 0$, $y_a = 0$ violates (19). Then, $y_a > 0$ and $\frac{\partial L(y, \lambda)}{\partial y_a} = 0$ for all a , implying $y_a = \exp(\eta^{-1}(v_a - \lambda) - 1)$. Thus, from $\sum_a y_a = 1$, we obtain

$$\lambda = \eta \log \sum_{a \in A} \exp(\eta^{-1} v_a) - \eta. \quad (21)$$

Since $\inf_{y \geq 0} L(y, \lambda) = \lambda + \eta$, the dual problem is equivalent to (17) where we redefine $\lambda := \lambda + \eta$.

Observe that when we take the limit $\eta \rightarrow 0$, $y_a > 0$ can occur only if $a \in \text{br}(v)$, and $y_a \rightarrow 0$ as $\eta \rightarrow 0$ if $a \notin \text{br}(v)$, which are consistent with the unperturbed case. To see this, observe

$$y_a = \frac{1}{\sum_{b \in A} \exp(\eta^{-1}(v_b - v_a))}. \quad (22)$$

If $a \notin \text{br}(v)$, $y_a \rightarrow 0$ because the denominator goes to infinity as $\eta \rightarrow 0$ when $v_b > v_a$ for some b . If $a \in \text{br}(v)$, y_a tends to $1/|\text{br}(v)|$ as $\eta \rightarrow 0$, which is slightly different from the unperturbed case where mixed-strategy best response can be nonunique.

Considering a different convex function for H induces a different choice rule. All practically used ARUM have such deterministically perturbed optimization representation but converse is not true.

4 Further readings

- Hofbauer and Sandholm (2002), Theorem 2.1; Hofbauer and Sandholm (2007), Appendix.
- Anderson et al. (1992)
- 土木学会 (1998), Ch.6
- Fudenberg et al. (2015)

References

Anderson, S. P., de Palma, A., and Thisse, J. F. (1992). *Discrete Choice Theory of Product Differentiation*. MIT Press.

- Fudenberg, D., Iijima, R., and Strzalecki, T. (2015). Stochastic choice and revealed perturbed utility. *Econometrica*, 83(6):2371–2409.
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