Large Sparse Matrix Computations: Homework 01

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Exercise 1.

Proof. Suppose Y is another pseudoinverse of $A \in R^{m\times n}$, that is Y satisfies (a) $AYA = A$, (b) $YAY = Y$, (c)$(AY)^T = AY$, (d) $(YA)^T = YA$.

$$\begin{aligned}
X &= XAX = X\left(AYA\right)X = X\left(AYA\right)Y(AYA)X \\
&= \left(XA\right)\left(YA\right)Y\left(AY\right)\left(AX\right) = \left(XA\right)^T\left(YA\right)^T Y\left(AY\right)^T\left(AX\right)^T \\
&= \left(A^TX^TA^TY^T\right)Y\left(Y^TA^TX^TA^T\right) = \left(YAXA\right)^T Y\left(AXAY\right)^T \\
&= \left(YA\right)^T Y\left(AY\right)^T = \left(YA\right)Y\left(AY\right) = Y\left(AYA\right)Y = YAY \\
&= Y
\end{aligned}$$

Therefore, the pseudoinverse of A is unique. □

Exercise 2.

Proof. Claim 1:$\left\|A - A_k\right\|_2 = \sigma_{k+1}$.

$$\left\|A - A_k\right\|_2 = \left\|U\Sigma V^T - UDV^T\right\|_2 = \left\|U\left(\Sigma - D\right)V^T\right\|_2 = \left\|\Sigma - D\right\|_2,$$

where $\Sigma = diag\left(\sigma_1, \cdots, \sigma_r, 0, \cdots, 0\right)$ and $D = diag\left(\sigma_1, \cdots, \sigma_k, 0, \cdots, 0\right)$. Hence

$$\Sigma - D = diag\left(0, \cdots 0, \sigma_{k+1}, \cdots, \sigma_r, 0, \cdots, 0\right).$$

So $\left\|A - A_k\right\|_2$ equals the greatest eigenvalue of $\Sigma - D = \sigma_{k+1}$.

Claim 2: $\min\limits_{rank(B)=k} \left\|A - B\right\|_2 = \sigma_{k+1}$.

Since $A_k \in \{B \mid rank(B) = k\}$, $\sigma_{k+1} = \left\|A - A_k\right\|_2 \geq \min\limits_{rank(B)=k} \left\|A - B\right\|_2$ is obvious. So we need to show $\min\limits_{rank(B)=k} \left\|A - B\right\|_2 \geq \sigma_{k+1}$.

For any B with $rank(B) = k$, implies $nullity(B) = n - k$. Consider the set $S = null(B) \cap span\left\{v_1, v_2, \cdots, v_{k+1}\right\}$. Since $dim(S) = (n-k) + (k+1) = n + 1 > n$, S is a nonempty set. Choose $x \in S$ with $\left\|x\right\|_2 = 1$.

$$\begin{aligned}
\left\|A - B\right\|_2^2 &= \left\|A - B\right\|_2^2 \left\|x\right\|_2^2 \\
&\geq \left\|\left(A - B\right)x\right\|_2^2 \\
&= \left\|Ax - Bx\right\|_2 = \left\|Ax\right\|_2 \text{ (Since } x \in null(B)) \\
&= \left\|U\Sigma V^T x\right\|_2^2 = \left\|\Sigma V^T x\right\|_2^2 \text{ (Since } U \text{ is unitary)} \\
&= \left\|\sum_{i=1}^{n} \sigma_i v_i^T x\right\|_2^2 = \sum_{i=1}^{k+1} \sigma_i^2 \left(v_i^T x\right)^2 \text{ (Since } x \in span\left\{v_1, v_2, \cdots, v_{k+1}\right\}, v_i^T x = 0, \forall i = k+2, \cdots, n) \\
&\geq \sigma_{k+1}^2 \sum_{i=1}^{k+1} \left(v_i^T x\right)^2 = \sigma_{k+1}^2
\end{aligned}$$

Hence $\left\|A - B\right\|_2 \geq \sigma_{k+1}$, for any $rank(B) = k$, we get $\min\limits_{rank(B)=k} \left\|A - B\right\|_2 \geq \sigma_{k+1}$.

Therefore by claim 1 and claim 2, the proof is complete.

□