Large Sparse Matrix Computations: Homework 01

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Exercise 1.

Proof. Since $A \in \mathbb{R}^{n\times n}$ is diagonal dominant, we have $|a_{jj}| > \sum_{i\neq j}^{n} |a_{ij}|, \forall 1 \le j \le n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} \xrightarrow{\text{1st step Gaussian Elimination}} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix} =$$

$A^{(2)}$, where $a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}, \forall 2 \le i, j \le n$. We are now prepared to show $A^{(2)}(2:n, 2:n)$ is also diagonal dominant :

$$\begin{aligned} \sum_{i=2, i\neq j}^{n} \left|a_{ij}^{(2)}\right| = \sum_{i=2, i\neq j}^{n} \left|a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}\right| &\le \sum_{i=2, i\neq j}^{n} |a_{ij}| + \sum_{i=2, i\neq j}^{n} \left|\frac{a_{1j}}{a_{11}} a_{i1}\right| \\ &= \sum_{i=2, i\neq j}^{n} |a_{ij}| + \left|\frac{a_{1j}}{a_{11}}\right| \sum_{i=2, i\neq j}^{n} |a_{i1}| \\ &< (|a_{jj}| - |a_{1j}|) + \left|\frac{a_{1j}}{a_{11}}\right| (|a_{11}| - |a_{j1}|) \\ &= |a_{jj}| - |a_{1j}| + |a_{1j}| - \left|\frac{a_{1j}}{a_{11}}\right| |a_{j1}| \\ &= |a_{jj}| - \left|\frac{a_{1j}}{a_{11}} a_{j1}\right| \\ &< \left|a_{jj} - \frac{a_{1j}}{a_{11}} a_{j1}\right| = \left|a_{jj}^{(2)}\right|, \forall 2 \le j \le n. \end{aligned}$$

where $\sum_{i=2, i\neq j}^{n} |a_{ij}| + |a_{1j}| = \sum_{i=1, i\neq j}^{n} |a_{ij}| < |a_{jj}|$ imply

$$\sum_{i=2, i\neq j}^{n} |a_{ij}| < |a_{jj}| - |a_{1j}|$$

and

$$\sum_{i=2, i\neq j}^{n} |a_{i1}| + |a_{j1}| = \sum_{i=2}^{n} |a_{i1}| < |a_{11}|$$

imply

$$\sum_{i=2, i\neq j}^{n} |a_{i1}| < |a_{11}| - |a_{j1}|.$$

Therefore the proof is completed. □

Exercise 2.

Proof. First, we consider $A^{(2)}$. Fix j,

$$\begin{aligned}
\sum_{i=2}^{n}\left|a_{i,j}^{(2)}\right| &= \sum_{i=2}^{n}\left|a_{i,j}-\frac{a_{i,1}}{a_{1,1}}a_{1,j}\right| \\
&\leq \sum_{i=2}^{n}|a_{i,j}| + \left|\frac{a_{1,j}}{a_{1,1}}\right|\sum_{i=2}^{n}|a_{i,1}| \\
&< \sum_{i=2}^{n}|a_{i,j}| + \left|\frac{a_{1,j}}{a_{1,1}}\right||a_{1,1}| \,\text{(Since A is diagonal dominant.)} \\
&= \sum_{i=1}^{n}|a_{i,j}|
\end{aligned}$$

By induction, we have

$$\sum_{i=k}^{n}\left|a_{i,j}^{(k)}\right| < \sum_{i=k-1}^{n}\left|a_{i,j}^{(k-1)}\right| < .... < \sum_{i=1}^{n}|a_{i,j}|.$$

Then

$$\begin{aligned}
\max_{k\leq i,j\leq n}\left|a_{i,j}^{(k)}\right| \leq \max_{k\leq j\leq n}\sum_{i=k}^{n}\left|a_{i,j}^{(k)}\right| \leq \max_{k\leq j\leq n}\sum_{i=1}^{n}|a_{i,j}| = \max_{k\leq j\leq n}(|a_{j,j}| + \sum_{i=1,i\neq j}^{n}\left|a_{i,j}^{(k)}\right|) \\
\leq \max_{k\leq j\leq n}(2\,|a_{j,j}|) \leq 2\max_{1\leq i,j\leq n}(|a_{i,j}|).
\end{aligned}$$

□

Exercise 3.

Proof. (a) Since

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}.$$

We have

$$\begin{aligned}
A_{11} &= L_{11}L_{11}^T \\
A_{12} &= L_{11}L_{21}^T \\
A_{21} &= L_{21}L_{11}^T \\
A_{22} &= L_{21}L_{21}^T + L_{22}L_{22}^T.
\end{aligned}$$

Then

$$\begin{aligned}
S &= A_{22} - A_{21}A_{11}^{-1}A_{21}^T \\
&= (L_{21}L_{21}^T + L_{22}L_{22}^T) - (L_{21}L_{11}^T)(L_{11}^{-T}L_{11}^{-1})(L_{11}L_{21}^T) \\
&= L_{22}L_{22}^T
\end{aligned}$$

Then we have

$$\kappa_2(S) = \|L_{22}L_{22}^T\|_2\|(L_{22}L_{22}^T)^{-1}\|_2 \leq \|A\|_2\|A^{-1}\|_2 \leq \kappa_2(A).$$

(b)Let $A = LL^T$ be the Cholesky decomposition of A. By writing

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}.$$

we have

$$A_{21} = L_{21}L_{11}^T.$$

By the same argument, we obtain
$$A_{11}^{-1} = L_{11}^{-T} L_{11}^{-1}.$$
Combining above, we can write
$$A_{21} A_{11}^{-1} = L_{21} L_{11}^{T} L_{11}^{-T} L_{11}^{-1} = L_{21} L_{11}^{-1}.$$
Taking 2-norm on both side, we have
$$\begin{aligned} \|A_{21} A_{11}^{-1}\|_2 = \|L_{12} L_{11}^{-1}\|_2 &\leq \|L_{12}\|_2 \|L_{11}^{-1}\|_2 \\ &\leq \|L\|_2 \|L^{-1}\|_2 \\ &= \kappa_2(L) = \kappa_2(A)^{(1/2)} \end{aligned}$$
$\square$