Large Sparse Matrix Computations: Homework 05

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Exercise 1.

Proof. 1. Use $\begin{cases} F(x) = \frac{1}{2}x^T Ax - bx \\ F(x_k) + \frac{1}{2}b^T A^{-1}b \le (\frac{\lambda_1-\lambda_n}{\lambda_1-\lambda_n})^2[F(x_{k-1}) + \frac{1}{2}b^T A^{-1}b] \end{cases}$,

$$\begin{aligned}
F(x_k) + \frac{1}{2}b^T A^{-1}b &= \frac{1}{2}x_k^T Ax_k - bx_k + \frac{1}{2}b^T A^{-1}b \\
&= \frac{1}{2}(x_{k-1}^T + tr_{k-1}^T)A(x_{k-1} + tr_{k-1}) - b(x_{k-1} + tr_{k-1}) + \frac{1}{2}b^T A^{-1}b \\
&= \frac{1}{2}(x_{k-1}^T Ax_{k-1} + tx_{k-1}^T Ar_{k-1} + tr_{k-1}^T Ax_{k-1} + t^2 r_{k-1}^T Ar_{k-1}) - (bx_{k-1} + btr_{k-1}) \\
&\quad + \frac{1}{2}b^T A^{-1}b \\
&= F(x_{k-1}) + \frac{1}{2}b^T A^{-1}b + \frac{1}{2}(tx_{k-1}^T Ar_{k-1} + tr_{k-1}^T Ax_{k-1} + t^2 r_{k-1}^T Ar_{k-1}) - btr_{k-1} \\
&= F(x_{k-1}) + \frac{1}{2}b^T A^{-1}b + \frac{1}{2}(tx_{k-1}^T A(b - Ax_{k-1}) + t(x_{k-1}^T A^T - b^T)Ax_{k-1} \\
&\quad + t^2(x_{k-1}^T A^T - b^T)A(b - Ax_{k-1})) - bt(b - Ax_{k-1}) \\
&= F(x_{k-1}) + \frac{1}{2}b^T A^{-1}b + \frac{1}{2}(t^2(x_{k-1}^T A^T - b^T)A(b - Ax_{k-1})) - bt(b - Ax_{k-1}) \\
&\le (\frac{\lambda_1 - \lambda_n}{\lambda_1 - \lambda_n})^2[F(x_{k-1}) + \frac{1}{2}b^T A^{-1}b]
\end{aligned}$$

$$\Longrightarrow \frac{1}{2}((t^2(x_{k-1}^T A^T - b^T)A - 2bt)(b - Ax_{k-1})) \le [(\frac{\lambda_1 - \lambda_n}{\lambda_1 - \lambda_n})^2 - 1][F(x_{k-1}) + \frac{1}{2}b^T A^{-1}b]$$

□

Exercise 2.

Proof. □

Exercise 3.

Proof. □

Exercise 4.

Proof. □

Exercise 5.

Proof. The Chebyshev differential equation is written as

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0,\ n = 0, 1, 2, \cdots$$

If we let $x = \cosh t$ we obtain

$$\frac{d^2y}{dt^2} - n^2 y = 0$$

whose general solution is

$$y = A\cosh nt + B\cosh nt$$

or as

$$y = A\cosh\left(n\cosh^{-1} x\right) + B\cosh\left(n\cosh^{-1} x\right), \quad |x| > 1$$

or equivalently

$$y = AT_n(x) + BU_n(x), \quad |x| > 1$$

the function $T_n(x)$ is a polynomial. For $|x| < 1$ we have

$$\begin{aligned} T_n(x) + iU(x) &= (\cos t + i\sin t)^n = \left(x + i\sqrt{1 - x^2}\right)^n \\ T_n(x) - iU(x) &= (\cos t - i\sin t)^n = \left(x + i\sqrt{1 - x^2}\right)^n \end{aligned}$$

from which we obtain

$$2T_n(x) = \left(x + i\sqrt{1 - x^2}\right)^n + \left(x + i\sqrt{1 - x^2}\right)^n$$

so

$$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1 - x^2}\right)^n + \left(x + i\sqrt{1 - x^2}\right)^n\right]$$

$\square$