Large Sparse Matrix Computations: Homework 03

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May 11, 2017

Exercise 1.

Proof. 1.

(a-1)

$$M = \begin{bmatrix} 2 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 2 \end{bmatrix} \Longrightarrow M^{-1} = \begin{bmatrix} \frac{1}{2} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \frac{1}{2} \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & 1 & & & \\ 1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & 1 \\ & & & 1 & 0 \end{bmatrix} \Longrightarrow M^{-1}N = \begin{bmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \frac{1}{2} \\ & & & \frac{1}{2} & 0 \end{bmatrix} \equiv A$$

Since $A^k \longrightarrow 0$ when $k \longrightarrow \infty$, then $\rho(A) < 1 \implies$ Jacobi iteration of (a) converges.

(a-2)

$$M = \begin{bmatrix} 2 & & & & \\ -1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & -1 & 2 \end{bmatrix} \Longrightarrow M^{-1} = \begin{bmatrix} \frac{1}{2} & & & & \\ \frac{1}{4} & \cdot & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \frac{1}{2^n} & \cdot & \cdot & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{bmatrix} \Longrightarrow M^{-1}N = \begin{bmatrix} 0 & \frac{1}{2} & & & \\ 0 & \frac{1}{4} & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{2} \\ 0 & \frac{1}{2^n} & \cdot & \frac{1}{8} & \frac{1}{4} \end{bmatrix} \equiv A.$$

Since $A^k \longrightarrow 0$ when $k \longrightarrow \infty$, then $\rho(A) < 1 \implies$ GS iteration of (a) converges.

(b-1)

$$M = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \Longrightarrow M^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \Longrightarrow M^{-1}N = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \equiv A.$$

Since $\rho(A) = 0 < 1$, then Jacobi iteration of (b) converges.

(b-2)

$$M = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 2 & 1 \end{bmatrix} \Longrightarrow M^{-1} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -2 & 1 \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow M^{-1}N = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix} \equiv A.$$

Since $\rho(A) = 2 > 1$, then GS iteration of (b) does not converge.

(c-1)

$$M = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} \Longrightarrow M^{-1} = \begin{bmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix} \implies M^{-1}N = \begin{bmatrix} 0 & \frac{1}{2} & \frac{-1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \equiv A.$$

Since $\rho(A) = \frac{\sqrt{5}}{2} > 1$, then Jacobi iteration of (c) does not converge.

(c-2)

$$M = \begin{bmatrix} 2 & & \\ 2 & 2 & \\ -1 & -1 & 2 \end{bmatrix} \implies M^{-1} = \begin{bmatrix} \frac{1}{2} & & \\ \frac{-1}{2} & \frac{1}{2} & \\ & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \implies M^{-1}N = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix} \equiv A.$$

Since $\rho(A) = 5 > 1$, then GS iteration of (c) does not converge. □

Exercise 2.

Proof. (a) $\implies$ (b)

By theorem: A is M-matrix $\iff$ There exists $v > 0$ such that $Av > 0$.

$(A + D)v = Av + Dv > 0$. (Since $Av > 0$ and $Dv > 0$).

And $A + D \in Z^{n\times n} \implies A + D$ is M-matrix $\implies A + D$ is invertible.

□

Exercise 3.

Proof. (a)

Since $A \geq 0$ irreducible, $(I + A)^{n-1}$ is positive.

$$(I + A^T)^{n-1} = ((I + A)^{n-1})^T$$

is also positive.

By Perron Lemma there is an $y > 0$ such that

$$y^T(I + A)^{n-1} = \rho((I + A)^{n-1})y^T$$

Let λ be the eigenvalue satisfying $|\lambda| = \rho(A)$ and $Ax = \lambda x$, $x \neq 0$. Further,

$$\rho^2(A)\,|x| \leq \rho(A)A\,|x| = A\rho(A)\,|x| \leq A^2\,|x|\,.$$

and in general,

$$\rho^k(A)\,|x| \leq A^k\,|x|\,,\ \text{for } k = 1, 2, ...$$

Hence

$$(1 + \rho(A))^{n-1}\,|x| \leq (I + A)^{n-1}\,|x|\,.$$

Multiplying y^T from left it implies

$$(1 + \rho(A))^{n-1}(y^T\,|x|) \leq y^T(I + A)^{n-1}\,|x| = \rho((I + A)^{n-1})y^T\,|x|\,.$$

Since $y^T\,|x| > 0$, it implies

$$(1 + \rho(A))^{n-1} \leq \rho((I + A)^{n-1}).$$

The eigenvalue of $(I + A)^{n-1}$ are of the form $(1 + \alpha)^{n-1}$, where α is an eigenvalue of A. Hence there is an eighenvalue μ of A such that

$$\left|(1 + \mu)^{n-1}\right| = \rho((I + A)^{n-1}).$$

On the other hand, we have $|\mu| \leq \rho(A)$. We have

$$(1 + \rho(A))^{n-1} \leq \left|(1 + \mu)^{n-1}\right|$$

and further

$$1 + \rho(A) \leq |1 + \mu| \leq 1 + |\mu| \leq 1 + \rho(A).$$

Thus $\mu \geq 0$ and hence $\mu = \rho(A)$.

For $k = 1$, it follows

$$A\,|x| = \rho(A)\,|x| \text{ or } A\,|x| = \mu\,|x|\,.$$

and further

$$(I + A)^{n-1}\,|x| = \left|(1 + \mu)^{n-1}\right|\,|x| = \rho((I + A)^{n-1})\,|x|\,.$$

Using Perron's Lemma, we get $|x| > 0$.

$\implies$ There is only one linearly independent eigenvector belonging to eigenvalue μ.

Moreover $\rho(A) > 0$ as A is distinct from the null matrix.

We want to claim: $\rho(A)$ is a simple eigenvalue of A if and only if:

(i)there is a unique linearly independent eigenvector of A to λ, say μ and also only one linearly independent eigenvector of A^T belonging to λ, say v.

(ii)$v^T u \neq 0$.

Only one linearly independent eigenvector of A, say u, belongs to $\rho(A)$. Moreover $u > 0$.

Similarly, $A^T \geq 0$ irreducible. The respective eigenvector v of A^T (to $\rho(A)$) can be chosen positive as well $v > 0$.

Therefore $v^T u > 0$ and $\rho(A)$ is simple.

Suppose $Az = \xi z$, $z \geq 0$ and $\xi \neq \rho(A)$. We have shown that A^T has a positive eigenvector, say $w > 0$. Then

$$A^T w = \rho(A) w.$$

But

$$w^T A z = w^T \xi z = \xi(w^T z),$$

i.e.,

$$w^T A z = \rho(A)(w^T z),$$

which is a contradiction in view of $\rho(A) - \xi \neq 0$ and $w^T z > 0$.

(b)

Suppose $x > 0$, such that $Ax = 0 = 0 \cdot x$.

So x is a positive eigenvector.

By (a) $\implies$ $Ax = \rho(A) \cdot x$.

Since $\rho(A) > 0$, then $\rho(A) \cdot x > 0 = Ax$. It is a contradiction.

$\implies$ $x = 0$. □

Exercise 4.

Proof. □

Exercise 5.

Proof. □

Exercise 6.

Proof. □