Numerical Optimization with applications: Homework 07

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Proof. Since all norms in $\mathbb{R}^n$ are equivalent.

$$\exists \alpha > 0 \quad \text{such that} \quad \|x\| \leq \alpha \|x\|_\infty \quad \forall x \in \mathbb{R}^n$$

We have,

$$\begin{aligned}
\|J(x_1) - J(x_2)\| &= \max_{\|y\|=1} \|(J(x_1) - J(x_2))y\| \\
&= \max_{\|y\|=1} \left\| \begin{bmatrix} (\bigtriangledown r_1(x_1) - \bigtriangledown r_1(x_2))^T y \\ \vdots \\ (\bigtriangledown r_m(x_1) - \bigtriangledown r_m(x_2))^T y \end{bmatrix} \right\| \\
&\leq \max_{\|y\|=1} \alpha \left\| \begin{bmatrix} (\bigtriangledown r_1(x_1) - \bigtriangledown r_1(x_2))^T y \\ \vdots \\ (\bigtriangledown r_m(x_1) - \bigtriangledown r_m(x_2))^T y \end{bmatrix} \right\|_\infty \\
&= \alpha \max_{\|y\|=1} \max_{1 \leq j \leq m} |(\bigtriangledown r_j(x_1) - \bigtriangledown r_j(x_2))^T y| \\
&\leq \alpha \max_{\|y\|=1} \max_{1 \leq j \leq m} |(\bigtriangledown r_j(x_1) - \bigtriangledown r_j(x_2))| \; |y| \\
&\leq \alpha \max_{\|y\|=1} \max_{1 \leq j \leq m} L \|x_1 - x_2\| \; |y| \\
&= \alpha L \|x_1 - x_2\|
\end{aligned}$$

We conclude that J is Lipschitz continuous with constant $\tilde{L} = \alpha L$. □

Proof. Given $x, \tilde{x}$ in $\mathcal{D}$, we estimate

$$\begin{aligned}
\|\nabla f(x) - \nabla f(\tilde{x})\| &= \|J(x)^T r(x) - J(\tilde{x})^T r(\tilde{x})\| \\
&= \| \left[J(x)^T r(x) - J(\tilde{x})^T r(x)\right] + \left[J(\tilde{x})^T r(x) - J(\tilde{x})^T r(\tilde{x})\right] \| \\
&= \| \left(J(x)^T - J(\tilde{x})^T\right) r(x) + J(\tilde{x})^T \left(r(x) - r(\tilde{x})\right) \| \\
&\leq \|J(x)^T - J(\tilde{x})^T\| |r(x)| + \|J(\tilde{x})^T\| \|r(x) - r(\tilde{x})\| \\
&\leq M\alpha L \|x - \tilde{x}\| + M' L \|x - \tilde{x}\| \\
&= \mathcal{L} \|x - \tilde{x}\|
\end{aligned}$$

where $\mathcal{L} = M\alpha L + M'L$ and $\|J(\tilde{x})^T\|$ is bounded since it is Lipschitz continuous on a compact set $\mathcal{D}$. □