國 立 清 華 大 學

碩 士 論 文

Hardy 空間的等價距之探討
Quasi-norm Equivalence of a class of Maximal Functions in Hardy space

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摘要

本文探討 Hardy 上的多個等價半距，有時我們無法直接得出最大函數 (maximal function) 之間的距等價關係，而且不容易看出這些函數彼此之間的關聯，這時我們可以先引進限制條件較強的輔助函數，再逐步減少輔助函數的限制條件，雖然因為引進的輔助函數的表示法會讓過程看起來很瑣碎，但最終藉由輔助函數之間大小關係的幫助，我們可以得到原來函數之間所想要的互相等價的結論。

Abstract

We study the quasi-norm equivalence of several maximal functions which serves the definition of Hardy space. We have difficulty that we can not obtain the result directly and need to introduce some auxiliary functions first. These auxiliary functions have nicer properties than original maximal functions and play the role to make connections between original maximal functions. By the help of auxiliary functions, we can obtain the equivalence of original maximal functions.

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Yi-Hsuan Huang
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Contents

1 Introduction

Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p < \infty$ are spaces of distribution have remarkable similarities to L^p. There exists an abundance of equivalent characterizations for Hardy spaces. This thesis provided the details of proof based on Grafakos Loukas[1] and Stein[3] on the equivalence of the Hardy space characterizations.

The organization of the thesis is as the following:

In Section 2, we introduce the definition of Hardy space and serveral kinds of maximal functions needed to construct the desired characterization.

In Section 3, we introduce the main theorem that all the maximal functions defined in the section 1 all have comparable L^p quasi-norms for all $0 < p < \infty$.

2 Settings and Definitions

In this section, we will introduce definitions used in later sections. First, we need some preparations for the definition of Hardy space.

We say that a tempered distribution v is *bounded* if $\varphi * v \in L^\infty(\mathbb{R}^n)$ whenever φ is in Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$.

We observe that if v is a bounded tempered distribution and $h \in L^1(\mathbb{R}^n)$, then the convolution $h*v$ can be defined as a distribution via the convergent integral

$$\langle h * v, \varphi \rangle := \langle \tilde{\varphi} * v, \tilde{h} \rangle = \int_{\mathbb{R}^n} (\tilde{\varphi} * v)(x)(\tilde{h})(x)dx,$$

where φ is a Schwartz function and $\tilde{\varphi}(x) = \varphi(-x)$, $\tilde{h}(x) = h(-x)$.

2.1 Definition of Hardy Spaces

Definition 2.1. *Let f be a bounded tempered distribution on $\mathbb{R}^n$ and let $0 < p < \infty$. We say that f lies in the Hardy space $H^p(\mathbb{R}^n)$ if $P(x) = \Gamma(\frac{n+1}{2})[\pi^{\frac{n+1}{2}}]^{-1}[(1 + |x|^2)^{\frac{n+1}{2}}]^{-1}$ is the Poisson kernel in $L^1(\mathbb{R}^n)$ and*

$$M(f;P)(x) = \sup_{t>0} |(P_t * f)(x)| \tag{2.1.1}$$

lies in $L^p(\mathbb{R}^n)$, where $P_t(x) = t^{-n}P(t^{-1}x)$. If this is the case, we set

$$\|f\|_{H^p} = \|M(f;P)(x)\|_{L^p}.$$

2.2 Definition of a class of Maximal Functions

Let $a, b > 0$. Let Φ be a Schwartz function and let f be a tempered distribution on $\mathbb{R}^n$, i.e. $\Phi \in \mathcal{S}(\mathbb{R}^n)$. We define the following class of maximal functions of f with respect to Φ.

Definition 2.2 (Smooth Maximal Function). *The smooth maximal function of f with respect to Φ is defined as*

$$M(f;\Phi)(x) := \sup_{t>0} |(\Phi_t * f)(x)|.$$

Definition 2.3 (Nontangential Maximal Function). *The nontangential maximal function (with aperture a) of f with respect to Φ is defined as*

$$M_a^*(f;\Phi)(x) := \sup_{t>0} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\le at}} |(\Phi_t * f)(y)|.$$

Definition 2.4 (Auxiliary Maximal Function). *The auxiliary maximal function is defined as*

$$M_b^{**}(f;\Phi)(x) := \sup_{t>0} \sup_{y\in\mathbb{R}^n} \frac{|(\Phi_t * f)(x-y)|}{(1+t^{-1}|y|)^b}.$$

Note that we have

$$M(f;\Phi)(x) \le M_a^*(f;\Phi)(x) \le (1+a)^b M_b^{**}(f;\Phi)(x), \tag{2.2.1}$$

where the first inequilty is quickly obtained from definitions and the second inequilty from viewing $M_b^{**}(f;\Phi)(x) = \sup_{t>0}\sup_{y\in\mathbb{R}^n} \frac{|(\Phi_t*f)(y)|}{(1+t^{-1}|x-y|)^b}$ and the restriction $|x-y| \le at$.

We need the bound of a Schwartz function to define our last maximal function.

Definition 2.5 (Schwartz function bound). *The bound of a Schwartz function (with respect to N) is defined as*

$$\mathfrak{N}_N(\varphi) = \int_{\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)| dx.$$

Definition 2.6 (Grand Maximal Function). *The grand maximal function of f (with respect to N) as*

$$\mathcal{M}_N(f)(x) = \sup_{\varphi\in\mathcal{F}_N} M_1^*(f;\varphi)(x),$$

where

$$\mathcal{F}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \mathfrak{N}_N(\varphi) \le 1\}.$$

3 Quasi-norm Equivalence of Sever Maximal Function

Before stating the main theorem, we first introduce the following Lemma.

3.1 Lemma

Lemma 3.1. *Let $m \in \mathbb{Z}^+$ and let Φ in $\mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Phi(x)dx = 1$, Then there exists a constant $C_0(\Phi, m)$ such that for any Ψ in $\mathcal{S}(\mathbb{R}^n)$, there are Schwartz functions $\Theta^{(s)}$, $0 \le s \le 1$, with the properties*

$$\Psi(x) = \int_0^1 (\Theta^{(s)} * \Phi_s)(x)ds \tag{3.1.1}$$

and

$$\int_{\mathbb{R}^n} (1+|x|)^m \left|\Theta^{(s)}(x)\right| dx \le C_0(\Phi, m) s^m \mathfrak{N}_m(\Psi). \tag{3.1.2}$$

Proof. We start with a smooth function ζ supported in $[0,1]$ that satisfies

$$\begin{aligned} 0 \le \zeta(s) \le \frac{2s^m}{m!} \qquad & \text{for all } 0 \le s \le 1 \\ \zeta(s) = \frac{s^m}{m!} \qquad & \text{for all } 0 \le s \le \frac{1}{2} \\ \frac{d^r\zeta}{dt^r}(1) = 0 \qquad & \text{for all } 0 \le r \le m+1. \end{aligned}$$

We define

$$\Theta^{(s)} = \Xi^{(s)} - \frac{d^{m+1}\zeta}{ds^{m+1}}(s) \left(\overbrace{\Phi_s * \cdots * \Phi_s}^{\text{m+1 terms}} \right) * \Psi, \tag{3.1.3}$$

where

$$\Xi^{(s)} = (-1)^{m+1} \zeta(s) \frac{d^{m+1}\zeta}{ds^{m+1}} \left(\overbrace{\Phi_s * \cdots * \Phi_s}^{\text{m+2 terms}} \right) * \Psi,$$

and we claim that (3.1.1) holds for this choice of $\Theta^{(s)}$. To verify this assertion, we apply integration by parts to write

$$\int_0^1 -\frac{d^{m+1}\zeta}{ds^{m+1}}(s)(\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi ds = -\overbrace{\frac{d^m\zeta}{ds^m}(1)}^{=0}(\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_1 * \Psi$$

$$+\overbrace{\frac{d^m\zeta}{ds^m}(0)}^{=1}\lim_{s\to 0^+}(\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi$$

$$+\int_0^1 \frac{d^m\zeta}{ds^m}(s)\frac{d}{ds}(\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi ds.$$

Therefore appling $m+1$ times integration by parts we rewrite (3.1.1) as

$$\int_0^1 \Theta^{(s)} * \Psi_s ds = \int_1^0 \Xi^{(s)} * \Psi_s ds + \frac{d^m\zeta}{ds^m}(0)\lim_{s\to 0^+}(\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi$$

$$-(-1)^{m+1}\int_0^1 \zeta(s)\frac{d^{m+1}}{s^{m+1}}\left(\overbrace{\Phi_s * \cdots * \Phi_s}^{\text{m+2 terms}}\right) * \Psi ds.$$

Noting that all the boundary terms vanish except for the term at $s = 0$ in the first integration by parts. The first and the third terms in the previous expression on the right add up to zero, while the second term is equal to Ψ, since Ψ has integral one. This implies that the family $\{(\Phi * \cdots * \Phi)_s\}_{s>0}$ is an appproximate identity as $s \to 0^+$. Specifically, from $\|\Phi * \Phi\|_1 \le \|\Phi\|_1\|\Phi\|_1 = 1$ and $\int_{\mathbb{R}^n} \Phi dx = \int_{\mathbb{R}^n} \Phi_s dx = 1$, we have

$$\int_{\mathbb{R}^n} (\Phi * \cdots * \Phi)_s dx = 1.$$

Since $\Psi \in L^\infty(\mathbb{R}^n)$, appling the statement from Wheeden and Zygmund[4].

Remark. *Let $f_\varepsilon = f * K_\varepsilon$, where $K \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} K = 1$. If $f \in L^\infty(\mathbb{R}^n)$, then $f_\varepsilon \to f$ as $\varepsilon \to 0$ at every point of continuity of f, and the convergence is uniform on any set where f is uniformly continuous.*

Therefore, the (3.1.1) follows. We now prove the estimate (3.1.2). Let Ω be the (m+1)-fold convolution of Φ. For the second term on the right in (3.1.3), we note that the (m+1)st derivative of $\zeta(s)$ vanish on $[0, \frac{1}{2}]$, so that we may write

$$\int_{\mathbb{R}^n} (1+|x|)^m \left|\frac{d^{m+1}\zeta(s)}{ds^{m+1}}\right| |\Omega_s * \Psi(x)| dx$$

$$\leq C_m \chi_{[\frac{1}{2},1]}(s) \int_{\mathbb{R}^n} (1+|x|)^m \left[\int_{\mathbb{R}^n} \frac{1}{s^n} \left|\Omega(\frac{x-y}{s})\right| |\Psi(y)|\, dy\right] dx$$

change of variables $\dfrac{x-y}{s} = x$

$$\leq C_m \chi_{[\frac{1}{2},1]}(s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|y+sx|)^m |\Omega(x)||\Psi(y)| dy dx$$

since $(1+|sx+y|) < (1+|sx|)(1+|y|)$

$$\leq C_m \chi_{[\frac{1}{2},1]}(s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|sx|)^m |\Omega(x)| (1+|y|)^m |\Psi(y)| dy dx$$

by $s \leq 1$ and Fubini's theorem

$$\leq C_m \chi_{[\frac{1}{2},1]}(s) \left(\int_{\mathbb{R}^n} (1+|x|)^m |\Omega(x)| dx\right) \left(\int_{\mathbb{R}^n} (1+|y|)^m |\Psi(y)| dy\right)$$

$$\leq C_0'(\Phi, m) s^m \mathfrak{N}_N(\Psi),$$

where the last inequality follows by $\chi_{[\frac{1}{2},1]}(s) \leq 1 \leq 2^m s^m$. To obtain a similar estimate for the first term on the right in (3.1.3), we argue that

$$\int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \left| \frac{d^{m+1}(\Omega_s * \Psi)}{ds^{m+1}}(x) \right| dx$$

$$= \int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \left| \frac{d^{m+1}}{ds^{m+1}} \int_{\mathbb{R}^n} \frac{1}{s^n} \Omega(\frac{x-y}{s}) \Psi(y) dy \right| dx$$

change of variables $\frac{x-y}{s} = y$ and move derivatives inside integral by DCT

$$= \int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \left| \int_{\mathbb{R}^n} \Omega(y) \frac{d^{m+1}\Psi(x-sy)}{ds^{m+1}} dy \right| dx$$

since derivatives on space of Schwartz functions can be viewed as tempered distribution

and bounded by its seminorms

$$\leq C'_m \int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \int_{\mathbb{R}^n} |\Omega(y)| \left[\sum_{|\alpha| \leq m+1} |\partial^\alpha \Psi(x-sy)| |y|^{|\alpha|} \right] dydx$$

change of variables $x - sy = x$ and use the fact $|y|^{|\alpha|} \leq (1+|y|)^{m+1}$ and Tonelli's theorem

$$\leq C'_m |\zeta(s)| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|x+sy|)^m |\Omega(y)| \left[\sum_{|\alpha| \leq m+1} |\partial^\alpha \Psi(x)| (1+|y|)^{m+1} \right] dydx$$

use $(1+|sx+y|) < (1+|sx|)(1+|y|)$ and $s \leq 1$ again

$$\leq C'_m |\zeta(s)| \int_{\mathbb{R}^n} (1+|y|)^m |\Omega(y)| (1+|y|)^{m+1} dy \int_{\mathbb{R}^n} (1+|x|)^m \sum_{|\alpha| \leq m+1} |\partial^\alpha \Psi(x)| dx$$

by the defnition of $0 \leq \zeta(s) \leq \frac{2s^m}{m!}$ for all $0 \leq s \leq 1$

$$\leq C''_0(\Phi, m) s^m \mathfrak{N}_N(\Psi).$$

We could let $C_0(\Phi, m) = C'_0(\Phi, m) + C''_0(\Phi, m)$ to obtain the desirable result (3.1.2).

□

3.2 The main theorem

We are now ready to state the main theorem.

Theorem 3.2. *Let $0 < p < \infty$. Then the following statements are valid:*

(a) There exists a Schwartz function Φ^o with $\int_{\mathbb{R}^n} \Phi^o(x)dx = 1$ such that

$$\|M(f; \Phi^o)\|_{L^p} \leq C_1 \|f\|_{H^p} \tag{3.2.1}$$

for all bounded distributions $f \in \mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of f.

(b) *For every $a > 0$, and every Φ in $\mathcal{S}(\mathbb{R}^n)$ there exists a constant $C_2(n,p,a,\Phi) < \infty$ such that*

$$\|M_a^*(f;\Phi)\|_{L^p} \leq C_2(n,p,a,\Phi)\|M(f;\Phi)\|_{L^p} \tag{3.2.2}$$

for all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$.

(c) *For every $a > 0$, $b > n/p$, and every Φ in $\mathcal{S}(\mathbb{R}^n)$ there exists a constant $C_3(n,p,a,b,\Phi) < \infty$ such that*

$$\|M_b^{**}(f;\Phi)\|_{L^p} \leq C_3(n,p,a,b,\Phi)\|M_a^*(f;\Phi)\|_{L^p} \tag{3.2.3}$$

for all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$.

(d) *For every $b > 0$, and every Φ in $\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x)dx = 1$ there exists a constant $C_4(b,\Phi) < \infty$ such that if $N = [b] + 1$ we have*

$$\|\mathcal{M}_N(f)\|_{L^p} \leq C_4(b,\Phi)\|M_b^{**}(f;\Phi)\|_{L^p} \tag{3.2.4}$$

for all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$.

(e) *For every positive integer N there exists a constant $C_5(n,N)$ such that every tempered distribution f with $\|\mathcal{M}_N(f)\|_{L^p} < \infty$ is a bounded distribution and satisfies*

$$\|f\|_{H^p} \leq C_5(n,N)\|\mathcal{M}_N(f)\|_{L^p} \tag{3.2.5}$$

that is, it lies in the Hardy space H^p.

Remark. *Choosing $\Phi = \Phi^o$ in parts (b), (c), and (d), $\frac{n}{p} < b < [\frac{n}{p}] + 1$, and $N = [\frac{n}{p}] + 1$, we conclude that for bounded distributions f we have*

$$\|f\|_{H^p} \approx \|\mathcal{M}_N(f)\|_{L^p}.$$

Moreover, for any Schwartz function Φ with $\int_{\mathbb{R}^n} \Phi(x)dx = 1$ and any bounded distribution f in $\mathcal{S}'(\mathbb{R}^n)$, the following quasi-norms are equivalent

$$\|f\|_{H^p} \approx \|M(f;\Phi)\|_{L^p}$$

with constants that depend only on Φ, n, p.

Proof. **(a)**We pick a continous and integrable function $\Psi(s)$ on the interval $[1,\infty)$ that decays faster than any negative power of s (i.e., $|\Psi(s)| \le C_N s^{-N}$ for all $N > 1$) and such that

$$\int_1^\infty s^k \Psi(s) ds = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = 1, 2, 3, ... \end{cases} \tag{3.2.6}$$

Such a function exists. In fact, we may take

$$\Psi(s) = \frac{e}{\pi}\frac{1}{s} e^{-\frac{\sqrt{2}}{2}(s-1)^{\frac{1}{4}}} \sin(\frac{\sqrt{2}}{2}(s-1)^{\frac{1}{4}}). \tag{3.2.7}$$

We now define the function

$$\Phi^o(x) = \int_1^\infty \Psi(s) P_s(x) ds, \tag{3.2.8}$$

where P_s is the Poisson kernel. Note that the double integral

$$\int_{\mathbb{R}^n} \int_1^\infty \frac{s}{(s^2 + |x|^2)^{\frac{n+1}{2}}} s^{-N} ds dx \tag{3.2.9}$$

converges. To check (3.2.9) converges, we may consider

$$A = \{x : |x| \le s\} \text{ and } B = \{x : |x| > s\}.$$

Then integral on these two set are

$$\begin{aligned} \int_1^\infty \int_A \frac{s}{(s^2 + |x|^2)^{\frac{n+1}{2}}} s^{-N} ds dx &\le \int_1^\infty \int_A \frac{s}{s^{n+1}} s^{-N} dx ds \\ &\le \omega_n \int_1^\infty \frac{s}{s^{n+1}} s^{-N} s^n ds \\ &= \omega_n \int_1^\infty s^{-N} ds = \omega_n \frac{1}{N-1} \end{aligned}$$

and

$$\begin{aligned} \int_1^\infty \int_B \frac{s}{(s^2 + |x|^2)^{\frac{n+1}{2}}} s^{-N} dx ds &\le \int_1^\infty \int_B \frac{s}{(|x|^2)^{\frac{n+1}{2}}} s^{-N} dx ds \\ &= \omega_n \int_1^\infty \int_s^\infty \frac{s^{-N+1}}{r^{n+1}} r^{n-1} dr ds \\ &= \omega_n \int_1^\infty \int_s^\infty \frac{s^{-N+1}}{r^2} dr ds \\ &= \omega_n \int_1^\infty s^{-N+1} s^{-1} ds \\ &= \omega_n \frac{1}{N-1}. \end{aligned}$$

And so it follows from (3.2.6) and (3.2.8) via Fubini's theorem that

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi^o(x)dx &= \int_{\mathbb{R}^n}\int_1^\infty \Psi(s)P_s(x)dsdx\\ &= \int_1^\infty \Psi(s)\int_{\mathbb{R}^n} P_s(x)dxds\\ &= \int_1^\infty s^0\Psi(s)ds\\ &= 1,\end{aligned}$$

since P_s is Poisson kernel and thus $\int_{\mathbb{R}^n} P_s(x)dx = 1 = s^0$.

Moreover, by the convergences of (3.2.9) again, another application of Fubini's theorem yields that

$$\begin{aligned}\widehat{\Phi^o}(\xi) &= \int_{\mathbb{R}^n} \Phi^o(x)e^{ix\cdot\xi}dx\\ &= \int_{\mathbb{R}^n}\int_1^\infty \Psi(s)P_s(x)e^{ix\cdot\xi}dsdx\\ &= \int_1^\infty \Psi(s)\int_{\mathbb{R}^n} P_s(x)e^{ix\cdot\xi}dxds\\ &= \int_1^\infty \Psi(s)\widehat{P_s}(\xi)ds\\ &= \int_1^\infty \Psi(s)e^{-2\pi s|\xi|}ds,\end{aligned}$$

by using that the fourier transform of Poisson kernel $\widehat{P_s}(\xi) = e^{-2\pi s|\xi|}$. This function is rapidly decreasing as $|\xi| \to \infty$ and the same is true for all the derivatives

$$\partial_\xi^\alpha \widehat{\Phi^o}(\xi) = \int_1^\infty \Psi(s)\partial_\xi^\alpha(e^{-2\pi s|\xi|})ds. \tag{3.2.10}$$

Moreover, the function $\widehat{\Phi^o}$ is smooth on $\mathbb{R}^n\backslash\{0\}$ and we will show that it is also smooth at the origin. Notice that for all multi-indices α we have

$$\partial_\xi^\alpha(e^{-2\pi s|\xi|}) = \left(\sum_{k=1}^{|\alpha|}(-2\pi s)^k p_k(\xi)|\xi|^{-m_k}\right)e^{-2\pi s|\xi|}, \tag{3.2.11}$$

where $p_k(\xi)$ is polynomial of ξ. On the other hand, by Taylor's theorem in one dimenson, for some function $v(s,|\xi|)$ with $0 \le v(s,|\xi|) \le 2\pi s|\xi|$, we have

$$e^{-2\pi s|\xi|} = \sum_{j=1}^{L}(-2\pi)^j\frac{|\xi|^j}{j!}s^j + \frac{(-2\pi s|\xi|)^{L+1}}{(L+1)!}e^{v(s,|\xi|)}.$$

Choosing $L > m_{|\alpha|}$ and substituting $e^{-2\pi s|\xi|}$ in (3.2.11) with Taylor's expension, we obtain

$$\begin{aligned}\partial_\xi^\alpha(e^{-2\pi s|\xi|}) = & \sum_{j=1}^{L}\left((-2\pi)^j\frac{|\xi|^j}{j!}s^j\left[\sum_{k=1}^{|\alpha|}(-2\pi s)^k p_k(\xi)|\xi|^{-m_k}\right]\right)\\ & + \left[\sum_{k=1}^{|\alpha|}(-2\pi s)^k p_k(\xi)|\xi|^{-m_k}\right]\frac{(-2\pi s|\xi|)^{L+1}}{(L+1)!}e^{v(s,|\xi|)}.\end{aligned}$$

For the case $|\alpha| > 0$, taking the first part above into $\int_1^\infty \Psi(s)\partial_\xi^\alpha(e^{-2\pi s|\xi|})ds$, we have zero integral by $\int_1^\infty s^k\Psi(s)ds = 0$ for $k = 1, 2, 3, ...$ and the second part above in $\int_1^\infty \Psi(s)\partial_\xi^\alpha(e^{-2\pi s|\xi|})ds$ tend to zero as $\xi \to 0$ by Lebesgue's dominated convergence theorem. For the case $|\alpha| = 0$, $\widehat{\Phi^o}(\xi) \to 1$ as $\xi \to 0$. This implies $\widehat{\Phi^o}$ is continuously differentiable and hence smooth at the origin; hence it lies in the Schwartz class, and thus so does Φ^o.

Since f is bounded distribtion, $P_t * f$ is a well-defined bounded function. Given a Schwartz funtion ϕ and using Fubini's theorem several times, we have the convulution of f starting with definition:

$$\begin{aligned}< \Phi_t^o * f,\ \phi > &=< \widetilde{\phi} * f,\ \widetilde{\Phi_t^o} >\\ &= \int_{\mathbb{R}^n}\widetilde{\Phi_t^o}(x)\left(\widetilde{\phi} * f\right)(x)dx\\ &= \int_{\mathbb{R}^n}\left[\int_1^\infty \Psi(s)\widetilde{P_s}(x)ds\right]\left(\widetilde{\phi} * f\right)(x)dx\\ &= \int_1^\infty \Psi(s)\left[\int_{\mathbb{R}^n}\widetilde{P_s}(x)\left(\widetilde{\phi} * f\right)(x)dx\right]ds\\ &= \int_1^\infty \Psi(s) < \widetilde{P_s},\ \widetilde{\phi} * f > ds\\ &= \int_1^\infty \Psi(s) < P_{ts} * f,\ \phi > ds\\ &= \int_1^\infty \Psi(s)\left[\int_{\mathbb{R}^n}(P_{ts} * f)(x)\phi(x)dx\right]ds\\ &= \int_{\mathbb{R}^n}\left[\int_1^\infty \Psi(s)(P_{ts} * f)(x)ds\right]\phi(x)dx.\end{aligned}$$

This implies

$$(\Phi_t^o * f)(x) = \int_1^\infty \Psi(s)(f * P_{ts})(x)ds.$$

Finally, we have the estimate

$$\begin{aligned}
M(f;\Phi^o)(x) &= \sup_{t>0} |(\Phi_t^o * f)(x)| \\
&= \sup_{t>0} \left| \int_1^\infty \Psi(s)(f * P_{ts})(x) ds \right| \\
&\leq \sup_{t>0} \int_1^\infty |\Psi(s)||(f * P_{ts})(x)| ds \\
&\leq \int_1^\infty |\Psi(s)| \left[\sup_{t>0} |(f * P_t)(x)| \right] ds \\
&\leq \int_1^\infty |\Psi(s)| ds \; M(f;P)(x) \\
&\leq C M(f;P)(x)
\end{aligned}$$

for some constant C. Therefore, we have the desired result

$$\|M(f;\Phi^0)\|_{L^p} \leq C_1 \|f\|_{H^p},$$

by taking L^P norm on both sides. $\square$

Proof. (b)The control of the nontagential maximal function $M_a^*(\cdot;\Phi)$ in terms of the vertical maximal function $M(\cdot;\Phi)$ is the hardest and most technical part of the proof. For matters of exposition, we present the proof in the case that $a = 1$ and we observe that the case of general $a > 0$ presents only notational differences. We derive

$$\|M_a^*(f;\Phi)\|_{L^p} \leq C_2(n,p,a,\Phi) \|M(f;\Phi)\|$$

as a consequence of the estimate

$$\|M_1^*(f;\Phi)\|_{L^p}^p \leq C_2''(n,p,\Phi)^p \|M(f;\Phi)\|_{L^p}^p + \frac{1}{2} \|M_1^*(f;\Phi)\|_{L^p}^p, \tag{3.2.12}$$

which is useful only if we know that

$$\|M_1^*(f;\Phi)\|_{L^p}^p < \infty.$$

This presents a significant hindrance that needs to be overcome by an approximation. For this reason we introduce a family of maximal functions

$$M_1^*(f;\Phi)^{\varepsilon,N}$$

for $0 \leq \varepsilon, N < \infty$ such that

$$\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p}^p < \infty$$

and such that

$$M_1^*(f;\Phi)^{\varepsilon,N} \nearrow M_1^*(f;\Phi)$$

as $\varepsilon \to 0$ and we prove (3.2.12) with $M_1^*(f;\Phi)^{\varepsilon,N}$ in place of $M_1^*(f;\Phi)$. In other words we prove

$$\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p}^p \leq C_2'(n,p,\Phi,N)^p \|M(f;\Phi)\|_{L^p}^p + \frac{1}{2}\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p}^p, \quad (3.2.13)$$

where there is an additional dependence on N in the constant $C_2'(n,p,\Phi,N)$ but there is no dependence on ε. The $M_1^*(f;\Phi)^{\varepsilon,N}$ are defined as follows: for a bounded distribution f in $\mathcal{S}'(\mathbb{R}^n)$ such that $M(f;\Phi) \in L^P$ we define

$$M_1^*(f;\Phi)^{\varepsilon,N}(x) = \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} |(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N}.$$

We first show that $M_1^*(f;\Phi)^{\varepsilon,N}$ lies in $L^P(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ if N is large enough depending on f. Indeed, using that $(\Phi_t * f)(x) =< f,\ \Phi_t(x-\cdot) >$[1] and the fact that f is in $\mathcal{S}'(\mathbb{R}^n)$, we obtain constants C_f and $m = m_f$ such that:

$$\begin{aligned}
|(\Phi_t * f)(y)| &\leq C_f \sum_{|\gamma|\leq m, |\beta|\leq m} \sup_{w\in\mathbb{R}^n} |w^\gamma (\partial^\beta \Phi_t)(y-w)| \\
&= C_f \sum_{|\gamma|\leq m, |\beta|\leq m} \sup_{z\in\mathbb{R}^n} |\,(y-z)^\gamma\,(\partial^\beta \Phi_t)(z)| \\
&\leq C_f \sum_{|\beta|\leq m} \sup_{z\in\mathbb{R}^n} 3^m\,(1+|y|^m+|z|^m)\,|(\partial^\beta \Phi_t)(z)| \\
&\leq C_f 3^m (1+|y|^m) \sum_{|\beta|\leq m} \sup_{z\in\mathbb{R}^n} (1+|z|^m)\,|(\partial^\beta \Phi_t)(z)| \\
&= C_f \frac{3^m(1+|y|^m)}{t^n} \sum_{|\beta|\leq m} \sup_{z\in\mathbb{R}^n} (1+|z|^m)\,|(\partial^\beta \Phi)(z/t)| \\
&\leq C_f \frac{3^m(1+|y|^m)}{t^n} \sum_{|\beta|\leq m} \sup_{z\in\mathbb{R}^n} [(1+|z/t|^m)\,(1+t^m)]\,|(\partial^\beta \Phi)(z/t)| \\
&\leq C_{f,\Phi}(1+|y|^m)\frac{(1+t^m)}{t^n} \\
&\leq C_{f,\Phi}(1+|y|)^m\frac{(1+t^m)}{t^n} \\
&\leq C_{f,\Phi}(1+\varepsilon|y|)^m \max\{\varepsilon^{-m},1\}\frac{(1+t^m)}{t^n}.
\end{aligned}$$

Multiplying by

$$\left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N}$$

[1] The property is shown in Loukas [2]

for some $0 < t < \frac{1}{\varepsilon}$ and $|y - x| < t$ yelds

$$\begin{aligned}
|(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} &\leq C_{f,\Phi} \left(\frac{t}{t+\varepsilon}\right)^N \frac{\max\{\varepsilon^{-m}, 1\}(t^{-n} + t^{m-n})}{(1+\varepsilon|y|)^{N-m}} \\
&\leq C_{f,\Phi} \frac{\max\{\varepsilon^{-m}, 1\}(t^{-n} + t^{m-n})}{(1+\varepsilon|y|)^{N-m}} \\
&\leq C_{f,\Phi} \frac{\max\{\varepsilon^{-m}, 1\}(\varepsilon^{n} + \varepsilon^{-(m-n)})}{(1+\varepsilon|y|)^{N-m}}.
\end{aligned}$$

And using $1 + \varepsilon|y| \geq \frac{1}{2}(1 + \varepsilon|x|)$ from

$$\begin{aligned}
&|x| - |y| \leq |y - x| < \frac{1}{\varepsilon} \\
&\Rightarrow 1 + \varepsilon|y| \geq \varepsilon|x| \\
&\Rightarrow 2 + \varepsilon|y| \geq 1 + \varepsilon|x| \\
&\Rightarrow 1 + \varepsilon|y| \geq \frac{1}{2}(2 + \varepsilon|y|) \geq \frac{1}{2}(1 + \varepsilon|x|),
\end{aligned}$$

we obtain for some $C''(f, \Phi, \varepsilon, m, n, N) < \infty$,

$$\begin{aligned}
M_1^*(f;\Phi)^{\varepsilon,N}(x) &= \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} |(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \\
&\leq \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} C_{f,\Phi} \frac{\max\{\varepsilon^{-m}, 1\}(\varepsilon^{n} + \varepsilon^{-(m-n)})}{(1+\varepsilon|y|)^{N-m}} \\
&\leq C_{f,\Phi} \frac{2\max\{\varepsilon^{-m}, 1\}(\varepsilon^{n} + \varepsilon^{-(m-n)})}{(1+\varepsilon|x|)^{N-m}} \\
&\leq \frac{C''(f, \Phi, \varepsilon, m, n, N)}{(1+\varepsilon|x|)^{N-m}}.
\end{aligned} \tag{3.2.14}$$

Taking $N > m + n/p$, we have that $M_1^*(f;\Phi)^{\varepsilon,N}$ lies in $L^P(\mathbb{R}^n)$. This choice of N depends on m and hence on the distribution f.

We now introduce functions

$$U(f;\Phi)^{\varepsilon,N}(x) = \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} t\,|\nabla(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \tag{3.2.15}$$

and

$$V(f;\Phi)^{\varepsilon,N}(x) = \sup_{0<t<\frac{1}{\varepsilon}} \sup_{y\in\mathbb{R}^n} |(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \left(\frac{t}{t+|x-y|}\right)^{[\frac{2n}{p}]+1}. \tag{3.2.16}$$

Let

$$C(n) = \|M\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)},$$

where M is the Hardy–Littlewood maximal operator. We need the norm estimate

$$\|V(f;\Phi)^{\varepsilon,N}\|_{L^p} \leq C(n)^{\frac{2}{p}} \|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p} \tag{3.2.17}$$

and the pointwise estimate

$$U(f;\Phi)^{\varepsilon,N}(x) \leq A(n,p,\Phi,N)V(f;\Phi)^{\varepsilon,N}(x), \tag{3.2.18}$$

where

$$A(n,p,\Phi,N) = 2^{[\frac{2n}{p}]+1} C_0(\partial_j\Phi, N+[\frac{2n}{p}]+1)\mathfrak{N}_{N+[\frac{2n}{p}]+1}(\partial_j\Phi).$$

To prove (3.2.17) we observe that when

$$z \in B(y,t) \subseteq B(x,|x-y|+t)$$

from definetion of

$$M_1^*(f;\Phi)^{\varepsilon,N}(x) = \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} |(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N},$$

we have

$$|(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \leq M_1^*(f;\Phi)^{\varepsilon,N}(z).$$

From above it follows that for any $y \in \mathbb{R}^n$

$$\begin{aligned}
|(\Phi_t * f)(y)| &\left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \\
&\leq \left(\frac{1}{|B(y,t)|}\int_{B(y,t)} \left[M_1^*(f;\Phi)^{\varepsilon,N}(z)\right]^{\frac{p}{2}} dz\right)^{\frac{2}{p}} \\
&\leq \left(\frac{1}{|B(y,t)|}\right)^{\frac{2}{p}} \left(\int_{B(x,|x-y|+t)} \left[M_1^*(f;\Phi)^{\varepsilon,N}(z)\right]^{\frac{p}{2}} dz\right)^{\frac{2}{p}} \\
&= \left(\frac{|B(x,|x-y|+t)|}{|B(y,t)|}\right)^{\frac{2}{p}} \left(\frac{1}{|B(x,|x-y|+t)|}\int_{B(x,|x-y|+t)} \left[M_1^*(f;\Phi)^{\varepsilon,N}(z)\right]^{\frac{p}{2}} dz\right)^{\frac{2}{p}} \\
&= \left(\frac{|x-y|+t}{t}\right)^{\frac{2n}{p}} \left(\frac{1}{|B(x,|x-y|+t)|}\int_{B(x,|x-y|+t)} \left[M_1^*(f;\Phi)^{\varepsilon,N}(z)\right]^{\frac{p}{2}} dz\right)^{\frac{2}{p}} \\
&\leq \left(\frac{|x-y|+t}{t}\right)^{\frac{2n}{p}} M\left(\left[M_1^*(f;\Phi)^{\varepsilon,N}\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}(x) \\
&\leq \left(\frac{|x-y|+t}{t}\right)^{[\frac{2n}{p}]+1} M\left(\left[M_1^*(f;\Phi)^{\varepsilon,N}\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}(x).
\end{aligned}$$

From above we have for any $y \in \mathbb{R}^n$

$$|(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \left(\frac{t}{|x-y|+t}\right)^{[\frac{2n}{p}]+1} \leq M\left(\left[M_1^*(f;\Phi)^{\varepsilon,N}\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}(x),$$

and thus

$$V(f;\Phi)^{\varepsilon,N}(x) \leq M\left(\left[M_1^*(f;\Phi)^{\varepsilon,N}\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}(x).$$

We now can use the boundedness of the Hardy–Littlewood maximal operator M on L^2 to obtain

$$\|V(f;\Phi)^{\varepsilon,N}\|_{L^p} \leq C(n)^{\frac{2}{p}}\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p},$$

which is the (3.2.17). In proving (3.2.18), we may assume that Φ has integral 1; otherwise we can multiply Φ by a suitable constant to arrange for this to happen. We note that

$$t|\nabla(\Phi_t * f)| = |(\nabla\Phi)_t * f| \leq \sqrt{n}\sum_{j=1}^{n}|(\partial_j\Phi)_t * f|$$

and it suffices to work with each partial derivative $\partial_j\Phi$ of Φ. Using Lemma (3.1.1), we can write

$$\partial_j\Phi(x) = \int_0^1 (\Theta^{(s)} * \Phi_s)(x)ds$$

for suitable Schwartz functions $\Theta^{(s)}$.

Fix $x \in \mathbb{R}^n$, $t > 0$ and y with $|y - x| < t < 1/\varepsilon$. Then we have

$$\begin{aligned}
&|((\partial_j\Phi)_t * f)(y)|\left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N}\\
&\qquad = \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N}\left|\int_0^1 ((\Theta^{(s)})_t * \Phi_{st} * f)(y)ds\right|\\
&\qquad \leq \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N}\int_0^1\int_{\mathbb{R}^n} t^{-n}|\Theta^{(s)}(t^{-1}z)||(\Phi_{st} * f)(y-z)|dzds.F
\end{aligned}$$

Inserting the factor 1 written as

$$\left(\frac{ts}{ts+|x-(y-z)|}\right)^{[\frac{2n}{p}]+1}\left(\frac{ts}{ts+\varepsilon}\right)^N\left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1}\left(\frac{ts+\varepsilon}{ts}\right)^N$$

in the preceding z-integral and using that

$$\begin{aligned}
&(1+\varepsilon|y-z|)^N \leq (1+\varepsilon|y|)^N(1+\varepsilon|z|)^N\\
\Rightarrow\ &\frac{1}{(1+\varepsilon|y|)^N} \leq \frac{(1+\varepsilon|z|)^N}{(1+\varepsilon|y-z|)^N}
\end{aligned}$$

and the fact that $|x-y| < t < 1/\varepsilon$, we obtain the estimate

$$\left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \int_0^1 \int_{\mathbb{R}^n} t^{-n} |\Theta^{(s)}(t^{-1}z)| |(\Phi_{st} * f)(y-z)| dz ds$$

$$= \left(\frac{t}{t+\varepsilon}\right)^N \int_0^1 \int_{\mathbb{R}^n} t^{-n} |\Theta^{(s)}(t^{-1}z)| \frac{|(\Phi_{st} * f)(y-z)|}{(1+\varepsilon|y|)^N} dz ds$$

$$\leq \left(\frac{t}{t+\varepsilon}\right)^N \int_0^1 \int_{\mathbb{R}^n} \left[\frac{|(\Phi_{st} * f)(y-z)|}{(1+\varepsilon|y|)^N} \left(\frac{ts}{ts+|x-(y-z)|}\right)^{[\frac{2n}{p}]+1} \left(\frac{ts}{ts+\varepsilon}\right)^N \right]$$
$$\left[\left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1} \left(\frac{ts+\varepsilon}{ts}\right)^N t^{-n} |\Theta^{(s)}(t^{-1}z)| \right] dz ds$$

$$\leq \left(\frac{t}{t+\varepsilon}\right)^N \int_0^1 \int_{\mathbb{R}^n} \left[\frac{|(\Phi_{st} * f)(y-z)|}{(1+\varepsilon|y-z|)^N} (1+\varepsilon|z|)^N \left(\frac{ts}{ts+|x-(y-z)|}\right)^{[\frac{2n}{p}]+1} \left(\frac{ts}{ts+\varepsilon}\right)^N \right]$$
$$\left[\left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1} \left(\frac{ts+\varepsilon}{ts}\right)^N t^{-n} |\Theta^{(s)}(t^{-1}z)| \right] dz ds$$

$$
\begin{aligned}
&= \left(\frac{t}{t+\varepsilon}\right)^N \int_0^1 \int_{\mathbb{R}^n} \left[\frac{|(\Phi_{st} * f)(y-z)|}{(1+\varepsilon|y-z|)^N} \left(\frac{ts}{ts+|x-(y-z)|}\right)^{[\frac{2n}{p}]+1} \left(\frac{ts}{ts+\varepsilon}\right)^N\right] \\
&\qquad \left[(1+\varepsilon|z|)^N \left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1} \left(\frac{ts+\varepsilon}{ts}\right)^N t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&\le \left(\frac{t}{t+\varepsilon}\right)^N \int_0^1 \int_{\mathbb{R}^n} V(f;\Phi)^{\varepsilon,N}(x) \\
&\qquad \left[(1+\varepsilon|z|)^N \left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1} \left(\frac{ts+\varepsilon}{ts}\right)^N t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&= V(f;\Phi)^{\varepsilon,N}(x) \int_0^1 \int_{\mathbb{R}^n} \left[\left(\frac{t}{t+\varepsilon}\right)^N \left(\frac{ts+\varepsilon}{ts}\right)^N\right] \\
&\qquad \left[(1+\varepsilon|z|)^N \left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1} t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&= V(f;\Phi)^{\varepsilon,N}(x) \int_0^1 \int_{\mathbb{R}^n} \left[\left(\frac{t}{t+\varepsilon}\right)^N \left(\frac{ts+\varepsilon}{t}\right)^N \frac{1}{s^N}\right] \\
&\qquad \left[(1+\varepsilon|z|)^N \left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1} t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&\le V(f;\Phi)^{\varepsilon,N}(x) \int_0^1 \int_{\mathbb{R}^n} \left[\frac{1}{s^N}\right] \\
&\qquad \left[(1+\varepsilon|z|)^N \left(\frac{ts+|x-(y-z)|}{ts}\right)^{[\frac{2n}{p}]+1} t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&= V(f;\Phi)^{\varepsilon,N}(x) \int_0^1 \int_{\mathbb{R}^n} \left[s^{-[\frac{2n}{p}]-1-N} \left(\frac{ts+|x-(y-z)|}{t}\right)^{[\frac{2n}{p}]+1}\right] \\
&\qquad \left[(1+\varepsilon|z|)^N t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&\le V(f;\Phi)^{\varepsilon,N}(x) \int_0^1 \int_{\mathbb{R}^n} \left[s^{-[\frac{2n}{p}]-1-N} \left(\frac{ts+|x-y|+|z|}{t}\right)^{[\frac{2n}{p}]+1}\right] \\
&\qquad \left[(1+\varepsilon|z|)^N t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&\le V(f;\Phi)^{\varepsilon,N}(x) \int_0^1 \int_{\mathbb{R}^n} \left[s^{-[\frac{2n}{p}]-1-N} \left(\frac{ts+t+|z|}{t}\right)^{[\frac{2n}{p}]+1}\right] \\
&\qquad \left[(1+\varepsilon|z|)^N t^{-n}|\Theta^{(s)}(t^{-1}z)|\right] dzds \\
&= V(f;\Phi)^{\varepsilon,N}(x) \int_0^1 \int_{\mathbb{R}^n} \left[s^{-[\frac{2n}{p}]-1-N} \left(\frac{ts+t+t|z|}{t}\right)^{[\frac{2n}{p}]+1}\right] (1+\varepsilon t|z|)^N|\Theta^{(s)}(z)|dzds
\end{aligned}
$$

$$
\begin{aligned}
&= V(f;\varPhi)^{\varepsilon,N}(x)\int_0^1\int_{\mathbb{R}^n}\left[s^{-[\frac{2n}{p}]-1-N}\,(s+1+|z|)^{[\frac{2n}{p}]+1}\right](1+\varepsilon t|z|)^N|\varTheta^{(s)}(z)|dzds\\
&\le V(f;\varPhi)^{\varepsilon,N}(x)\int_0^1\int_{\mathbb{R}^n}\left[s^{-[\frac{2n}{p}]-1-N}\,(1+s)^{[\frac{2n}{p}]+1}\,(1+|z|)^{[\frac{2n}{p}]+1}\right](1+\varepsilon t|z|)^N|\varTheta^{(s)}(z)|dzds\\
&\le V(f;\varPhi)^{\varepsilon,N}(x)\int_0^1\int_{\mathbb{R}^n}\left[s^{-[\frac{2n}{p}]-1-N}\,(1+s)^{[\frac{2n}{p}]+1}\,(1+|z|)^{[\frac{2n}{p}]+1}\right](1+|z|)^N|\varTheta^{(s)}(z)|dzds\\
&= V(f;\varPhi)^{\varepsilon,N}(x)\int_0^1\left[s^{-[\frac{2n}{p}]-1-N}\,(1+s)^{[\frac{2n}{p}]+1}\right]\left[\int_{\mathbb{R}^n}(1+|z|)^{[\frac{2n}{p}]+1}\,(1+|z|)^N|\varTheta^{(s)}(z)|dz\right]ds\\
&\le V(f;\varPhi)^{\varepsilon,N}(x)\int_0^1\left[s^{-[\frac{2n}{p}]-1-N}\,(1+s)^{[\frac{2n}{p}]+1}\right]\\
&\qquad\left[s^{[\frac{2n}{p}]+1+N}C_0(\partial_j\varPhi,[\frac{2n}{p}]+1+N)\mathfrak{N}_{[\frac{2n}{p}]+1+N}(\partial_j\varPhi)\right]ds\\
&= V(f;\varPhi)^{\varepsilon,N}(x)\left[C_0(\partial_j\varPhi,[\frac{2n}{p}]+1+N)\mathfrak{N}_{[\frac{2n}{p}]+1+N}(\partial_j\varPhi)\right]\int_0^1(1+s)^{[\frac{2n}{p}]+1}\,ds\\
&\le 2^{[\frac{2n}{p}]+1}V(f;\varPhi)^{\varepsilon,N}(x)\left[C_0(\partial_j\varPhi,[\frac{2n}{p}]+1+N)\mathfrak{N}_{[\frac{2n}{p}]+1+N}(\partial_j\varPhi)\right],
\end{aligned}
$$

where the last third inequlity is from the result of lemma (3.1.1).

In brief, we obtain the estimate

$$
\begin{aligned}
&|((\partial_j\varPhi)_t*f)(y)|\left(\frac{t}{t+\varepsilon}\right)^N\frac{1}{(1+\varepsilon|y|)^N}\\
&\qquad\le 2^{[\frac{2n}{p}]+1}V(f;\varPhi)^{\varepsilon,N}(x)\left[C_0(\partial_j\varPhi,[\frac{2n}{p}]+1+N)\mathfrak{N}_{[\frac{2n}{p}]+1+N}(\partial_j\varPhi)\right].
\end{aligned}
$$

Combining this estimate with

$$
t|\nabla(\varPhi_t*f)|=|(\nabla\varPhi)_t*f|\le\sqrt{n}\sum_{j=1}^n|(\partial_j\varPhi)_t*f|,
$$

we have

$$
U(f;\varPhi)^{\varepsilon,N}(x)\le A(n,p,\varPhi,N)V(f;\varPhi)^{\varepsilon,N}(x)
$$

for some constant $A(n,p,\varPhi,N)$, which is (3.2.18).

Putting estimates (3.2.17) and (3.2.18) together yield

$$
\|U(f;\varPhi)^{\varepsilon,N}\|_{L^p}\le A(n,p,\varPhi,N)\|V(f;\varPhi)^{\varepsilon,N}\|_{L^p}\le C(n)^{\frac{2}{p}}A(n,p,\varPhi,N)\|M_1^*(f;\varPhi)^{\varepsilon,N}\|_{L^p}. \tag{3.2.19}
$$

We now set

$$
E_\varepsilon=\{x\in\mathbb{R}^n:U(f;\varPhi)^{\varepsilon,N}(x)\le KM_1^*(f;\varPhi)^{\varepsilon,N}(x)\}
$$

for some constant K to be determined shortly. With $A = A(n, p, \Phi, N)$, we have

$$\begin{aligned} \int_{(E_\varepsilon)^c} \left[M_1^*(f;\Phi)^{\varepsilon,N}(x)\right]^p dx &\leq \frac{1}{K^p} \int_{(E_\varepsilon)^c} \left[U(f;\Phi)^{\varepsilon,N}(x)\right]^p dx \\ &\leq \frac{1}{K^p} \int_{\mathbb{R}^n} \left[U(f;\Phi)^{\varepsilon,N}(x)\right]^p dx \\ &\leq \frac{\left[C(n)^{\frac{2}{p}} A\right]^p}{K^p} \int_{\mathbb{R}^n} \left[M_1^*(f;\Phi)^{\varepsilon,N}\right]^p dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^n} \left[M_1^*(f;\Phi)^{\varepsilon,N}\right]^p dx, \end{aligned} \tag{3.2.20}$$

provided we choose K such that $K^p = 2\left[C(n)^{\frac{2}{p}} A(n, p, \Phi, N)\right]^p$. Obviously K is a function of n, p, Φ, N and in particular depends on N.

It remains to estimate the contribution of the integral of $\left[M_1^*(f;\Phi)^{\varepsilon,N}\right]^p$ over the set E_ε. We claim that the following pointwise estimate is valid:

$$M_1^*(f;\Phi)^{\varepsilon,N}(x) \leq 4C'(n, N, K)^{\frac{1}{q}} \left[M(M(f;\Phi)^q)(x)\right]^{\frac{1}{q}} \tag{3.2.21}$$

for any $x \in E_\varepsilon$ and $0 < q < \infty$ and some constant $C'(n, N, K)$, where M is the Hardy-Littlewood maximal operator. To prove (3.2.21) we fix $x \in E_\varepsilon$ and we also fix y such $|y - x| < t$.

By the definition of $M_1^*(f;\Phi)^{\varepsilon,N}$ there exists a point $(y_0, t) \in \mathbb{R}^{n+1}_+$ such that $|x - y_0| < t < \frac{1}{\varepsilon}$ and

$$|(\Phi_t * f)(y_0)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1 + \varepsilon|y_0|)^N} \geq \frac{1}{2} M_1^*(f;\Phi)^{\varepsilon,N}(x). \tag{3.2.22}$$

Also by the definitions of E_ε and $U(f;\Phi)^{\varepsilon,N}$, for any $x \in E_\varepsilon$ we have

$$t\,|\nabla(\Phi_t * f)(\xi)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1 + \varepsilon|\xi|)^N} \leq K M_1^*(f;\Phi)^{\varepsilon,N}(x) \tag{3.2.23}$$

for all ξ satisfying $|\xi - x| < t < \frac{1}{\varepsilon}$. It follows from (3.2.22) and (3.2.23) that

$$t\,|\nabla(\Phi_t * f)(\xi)| \leq 2K\,|(\Phi_t * f)(y_0)| \,\frac{1 + \varepsilon|\xi|)^N}{(1 + \varepsilon|y_0|)^N} \tag{3.2.24}$$

for all ξ satisfying $|\xi - x| < t < \frac{1}{\varepsilon}$. We let z be such that $|z - x| < t$. Applying the mean value theorem and using (3.2.24), we obtain for some ξ between y_0 and z,

$$
\begin{aligned}
|(\Phi_t * f)(z) - (\Phi_t * f)(y_0)| &= |\nabla(\Phi_t * f)(\xi)(z - y_0)| \\
&\leq |\nabla(\Phi_t * f)(\xi)||(z - y_0)| \\
&\leq \frac{2K}{t}\,|(\Phi_t * f)(y_0)|\,\frac{(1+\varepsilon|\xi|)^N}{(1+\varepsilon|y_0|)^N}|(z - y_0)| \\
&\leq \frac{2^{N+1}K}{t}|(\Phi_t * f)(y_0)||(z - y_0)| \\
&\leq \frac{1}{2}|(\Phi_t * f)(y_0)|,
\end{aligned}
$$

provided z also satisfies $|z - y_0| < \frac{1}{2}\frac{t}{2^{N+1}K}$ in addition to $|z - x| < t$. Therefore, for z satisfying $|z - y_0| < \frac{1}{2}\frac{t}{2^{N+1}K}$ and $|z - x| < t$ we have

$$
|(\Phi_t * f)(z)| \geq \frac{1}{2}|(\Phi_t * f)(y_0)| \geq \frac{1}{4}M_1^*(f;\Phi)^{\varepsilon,N}(x),
$$

where the last inequality uses (3.2.22). Thus we have

$$
\begin{aligned}
M(M(f;\Phi)^q)(x) &\geq \frac{1}{|B(x,t)|}\int_{B(x,t)} [M(f;\Phi)(w)]^q\,dw \\
&\geq \frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(y_0,\frac{1}{2}\frac{t}{2^{N+1}K})} [M(f;\Phi)(w)]^q\,dw \\
&\geq \frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(y_0,\frac{1}{2}\frac{t}{2^{N+1}K})} \left[\frac{1}{4}M_1^*(f;\Phi)(x)\right]^q dw \\
&\geq \frac{|B(x,t)\cap B(y_0,\frac{1}{2}\frac{t}{2^{N+1}K})|}{|B(x,t)|}\left[\frac{1}{4}M_1^*(f;\Phi)(x)\right]^q \\
&\geq C'(n,N,K)^{-1}4^{-q}\left[M_1^*(f;\Phi)(x)\right]^q,
\end{aligned}
$$

where we used the simple geometric fact that if $|x - y_0| \leq t$ and $\delta > 0$, then

$$
\frac{|B(x,t)\cap B(y_0,\frac{1}{2}\frac{t}{2^{N+1}K})|}{|B(x,t)|} \geq c_{n,\delta} > 0
$$

the minimum of this constant being obtained when $|x - y_0| = t$ and the value was still greater than zero.

This proves

$$
M_1^*(f;\Phi)^{\varepsilon,N}(x) \leq 4C'(n,N,K)^{\frac{1}{q}}\left[M(M(f;\Phi)^q)(x)\right]^{\frac{1}{q}},
$$

which is (3.2.21).

Taking $q = p/2$ and applying the boundedness of the Hardy–Littlewood maximal operator on L^2 yields

$$\begin{aligned}\int_{E_\varepsilon} \left[M_1^*(f;\Phi)^{\varepsilon,N}(x)\right]^p dx &\leq \int_{\mathbb{R}^n} \left[4C'(n,N,K)^{\frac{2}{p}} \left[M(M(f;\Phi)^{p/2})(x)\right]^{\frac{2}{p}}\right]^p dx \\ &\leq C_2'(n,p,\Phi,N) \int_{\mathbb{R}^n} [M(f;\Phi)(x)]^p \, dx.\end{aligned} \tag{3.2.25}$$

Combining this estimate with (3.2.20), we finally prove

$$\begin{aligned}\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p}^p &= \int_{\mathbb{R}^n} \left[M_1^*(f;\Phi)^{\varepsilon,N}(x)\right]^p dx \\ &= \int_{(E_\varepsilon)^c} \left[M_1^*(f;\Phi)^{\varepsilon,N}(x)\right]^p dx + \int_{E_\varepsilon} \left[M_1^*(f;\Phi)^{\varepsilon,N}(x)\right]^p dx \\ &\leq C_2'(n,p,\Phi,N)^p \|M(f;\Phi)\|_{L^p}^p + \frac{1}{2}\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p}^p,\end{aligned}$$

which is (3.2.13).

Recalling (3.2.14) the fact obtained earlier that $\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p} < \infty$, we deduce from (3.2.13) that

$$\|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p} \leq 2^{\frac{1}{p}} C_2'(n,p,\Phi,N)\|M(f;\Phi)\|_{L^p}. \tag{3.2.26}$$

The previous constant depends on f but is independent of ε. Notice that from $1+\varepsilon|y| \geq \frac{1}{2}(1+\varepsilon|x|)$ again, we have

$$\begin{aligned}M_1^*(f;\Phi)^{\varepsilon,N}(x) &= \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} |(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{1}{(1+\varepsilon|y|)^N} \\ &\geq \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} |(\Phi_t * f)(y)| \left(\frac{t}{t+\varepsilon}\right)^N \frac{2^{-N}}{(1+\varepsilon|x|)^N} \\ &\geq \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} |(\Phi_t * f)(y)| \frac{2^{-N}}{(1+\varepsilon|x|)^N} \\ &= \frac{2^{-N}}{(1+\varepsilon|x|)^N} \sup_{0<t<\frac{1}{\varepsilon}} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\leq t}} |(\Phi_t * f)(y)| \\ &= \frac{2^{-N}}{(1+\varepsilon|x|)^N} M_1^*(f;\Phi).\end{aligned}$$

And we take both sides $\varepsilon \to 0$ to get

$$\lim_{\varepsilon\to 0} M_1^*(f;\Phi)^{\varepsilon,N}(x) \geq 2^{-N} M_1^*(f;\Phi)$$

and observe that the preceding expression on the right increases to

$$2^{-N} M_1^*(f;\Phi).$$

Since the constant in (3.2.26) does not depend on ε, an application of the Lebesgue monotone convergence theorem yields

$$\lim_{\varepsilon\to 0} \|M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p} = \|\lim_{\varepsilon\to 0} M_1^*(f;\Phi)^{\varepsilon,N}\|_{L^p} \geq 2^{-N}\|M_1^*(f;\Phi)\|_{L^p}$$

and

$$\|M_1^*(f;\Phi)\|_{L^p} \leq 2^{N+\frac{1}{p}} C_2'(n,p,\Phi,N)\|M(f;\Phi)\|_{L^p}. \tag{3.2.27}$$

The problem with this estimate is that the finite constant $2^N C_2'(n,p,\Phi,N)$ depends on N and thus on f. However, we have managed to show that under the assumption

$$\|M(f;\Phi)\|_{L^p} < \infty,$$

one must necessarily have

$$\|M_1^*(f;\Phi)\|_{L^p} < \infty.$$

We repeat the preceding argument from the point where the functions $U(f;\Phi)^{\varepsilon,N}$ and $V(f;\Phi)^{\varepsilon,N}$ are introduced, setting $\varepsilon = N = 0$. Then we arrive at (3.2.12) with a constant

$$C_2''(n,p,\Phi) = C_2'(n,p,\Phi,0)$$

which is independent of N and thus of f. We conclude the validity of (3.2.2) with

$$C_2(n,p,1,\Phi) = 2^{\frac{1}{p}} C_2''(n,p,\Phi)$$

when $a = 1$. A similar constant (depending on a) is obtained for different values of $a > 0$. □

Proof. (c)Let $B(x,R)$ denote a ball centered at x with radius R. It follows from the definition of

$$M_a^*(f;\Phi)(z) = \sup_{t>0} \sup_{\substack{y\in\mathbb{R}^n \\ |w-z|\leq at}} |(\Phi_t * f)(w)|$$

that

$$|(\Phi_t * f)(x-y)| \leq M_a^*(f;\Phi)(z) \qquad \text{if } z \in B(x-y,at).$$

But the ball $B(x-y,at)$ is contained in the ball $B(x,|y|+at)$; hence it follows that

$$\begin{aligned}
|(\Phi_t * f)(x-y)|^{\frac{n}{b}} &\le \frac{1}{|B(y,at)|}\int_{B(y,at)} M_a^*(f;\Phi)(z)^{\frac{n}{b}}dz\\
&\le \frac{1}{|B(y,at)|}\int_{B(x,|y|+at)} M_a^*(f;\Phi)(z)^{\frac{n}{b}}dz\\
&= \frac{|B(x,|y|+at)|}{|B(y,at)|}\left(\frac{1}{|B(x,|y|+at)|}\int_{B(x,|y|+at)} M_a^*(f;\Phi)(z)^{\frac{n}{b}}dz\right)\\
&\le \frac{|B(x,|y|+at)|}{|B(y,at)|} M(M_a^*(f;\Phi)^{\frac{n}{b}}(x))\\
&= \left(\frac{|y|+at}{at}\right)^n M(M_a^*(f;\Phi)^{\frac{n}{b}}(x))\\
&\le \max(1,a^{-n})\left(\frac{|y|}{t}+1\right)^n M(M_a^*(f;\Phi)^{\frac{n}{b}}(x)),
\end{aligned}$$

from which we conclude that for all $x\in\mathbb{R}^n$ we have

$$\begin{aligned}
M_b^{**}(f;\Phi)(x) &\equiv \sup_{t>0}\sup_{y\in\mathbb{R}^n}\frac{|(\Phi_t * f)(x-y)|}{(\frac{|y|}{t}+1)^b}\\
&\le \frac{\left[\max(1,a^{-n})\left(\frac{|y|}{t}+1\right)^n M(M_a^*(f;\Phi)^{\frac{n}{b}}(x))\right]^{\frac{b}{n}}}{(\frac{|y|}{t}+1)^b}\\
&= \max(1,a^{-b})\left[M(M_a^*(f;\Phi)^{\frac{n}{b}}(x))\right]^{\frac{b}{n}}.
\end{aligned}$$

Raising to the power p and using the fact that $p>n/b$ and the boundedness of the Hardy–Littlewood maximal operator M on $L^{pb/n}$, we obtain

$$\begin{aligned}
\|M_b^{**}(f;\Phi)\|_{L^p} &\le C\|\left[M(M_a^*(f;\Phi)^{\frac{n}{b}}(x))\right]^{\frac{b}{n}}\|_{L^p}\\
&= C\|\left[M(M_a^*(f;\Phi)^{\frac{n}{b}}(x))\right]\|_{L^{pb/n}}\\
&\le C_3(n,p,a,b,\Phi)\|M_a^*(f;\Phi)^{\frac{n}{b}}\|_{L^{pb/n}}\\
&= C_3(n,p,a,b,\Phi)\|M_a^*(f;\Phi)\|_{L^p}
\end{aligned}$$

the required conclusion (3.2.3). □

Proof. (d)we may replace b by the integer $b_0=[b]+1$. Let Φ be a Schwartz function with integral equal to 1. Applying Lemma3.1.1 with $m=b_0$, we write any function $\varphi in \mathcal{F}_N$ as

$$\varphi(y)=\int_0^1(\Theta^{(s)} * \Phi_s)(y)ds$$

for some choice of Schwartz functions $\Theta^{(s)}$. Then we have

$$\varphi(y)_t = \int_0^1 (\Theta_t^{(s)} * \Phi_{ts})(y) ds$$

for all $t > 0$. Fix $x \in \mathbb{R}^n$. Then for y in $B(x,t)$ we have

$$\begin{aligned}
|(\varphi_t * f)(y)| &\leq \int_0^1 \int_{\mathbb{R}^n} |\Theta_t^{(s)}(z)||(\Phi_{ts} * f)(y-z)| dz ds \\
&= \int_0^1 \int_{\mathbb{R}^n} |\Theta_t^{(s)}(z)| \left[\frac{|(\Phi_{ts} * f)(y-z)|}{\left(\frac{|x-(y-z)|}{st} + 1\right)^{b_0}} \right] \left(\frac{|x-(y-z)|}{st} + 1 \right)^{b_0} dz ds \\
&\leq \int_0^1 \int_{\mathbb{R}^n} |\Theta_t^{(s)}(z)| M_{b_0}^{**}(f;\Phi)(x) \left(\frac{|x-(y-z)|}{st} + 1 \right)^{b_0} dz ds \\
&\leq \int_0^1 \int_{\mathbb{R}^n} |\Theta_t^{(s)}(z)| M_{b_0}^{**}(f;\Phi)(x) \left(\frac{|x-y|}{st} + \frac{|z|}{st} + 1 \right)^{b_0} dz ds \\
&\leq M_{b_0}^{**}(f;\Phi)(x) \int_0^1 s^{-b_0} \int_{\mathbb{R}^n} |\Theta_t^{(s)}(z)| \left(\frac{|x-y|}{t} + \frac{|z|}{t} + 1 \right)^{b_0} dz ds \\
&\leq M_{b_0}^{**}(f;\Phi)(x) \int_0^1 s^{-b_0} \int_{\mathbb{R}^n} |\Theta_t^{(s)}(z)| \left(\frac{|x-y|}{t} + 1 \right) \left(\frac{|z|}{t} + 1 \right)^{b_0} dz ds \\
&\leq 2^{b_0} M_{b_0}^{**}(f;\Phi)(x) \int_0^1 s^{-b_0} \int_{\mathbb{R}^n} |\Theta_t^{(s)}(z)| \left(\frac{|z|}{t} + 1 \right)^{b_0} dz ds \\
&= 2^{b_0} M_{b_0}^{**}(f;\Phi)(x) \int_0^1 s^{-b_0} \int_{\mathbb{R}^n} |\Theta^{(s)}(w)| \left(|w| + 1 \right)^{b_0} dz ds \\
&\leq 2^{b_0} M_{b_0}^{**}(f;\Phi)(x) \int_0^1 s^{-b_0} C_0(\Phi, b_0) s^{b_0} \mathfrak{N}_{b_0}(\varphi) ds,
\end{aligned}$$

where we applied conclusion (3.1.2) of Lemma3.1.1 to the last inequality. Setting $N = b_0 = [b] + 1$, we obtain for y in $B(x,t)$ and $\varphi \in \mathcal{F}_N$,

$$|(\varphi_t * f)(y)| \leq 2^{b_0} C_0(\Phi, b_0) M_{b_0}^{**}(f;\Phi)(x).$$

Taking the supremum over all y in $B(x,t)$, over all $t > 0$, and over all $\varphi \in \mathcal{F}_N$, we obtain the pointwise estimate

$$\mathcal{M}_N(f)(x) \leq 2^{b_0} C_0(\Phi, b_0) M_{b_0}^{**}(f;\Phi)(x), \quad x \in \mathbb{R}^n,$$

where $N = b_0 + 1$. This clearly yields

$$\|\mathcal{M}_N(f)\|_{L^p} \leq C_4(\Phi, b_0) \|M_b^{**}(f;\Phi)\|_{L^p},$$

which is (3.2.4) if we set $C_4(\Phi, b_0) = 2^{b_0} C_0(\Phi, b_0)$ □

Proof. (e)We fix an $f \in \mathcal{S}'(\mathbb{R}^n)$ that satisfies $\|\mathcal{M}_N(f)\|_{L^p} < \infty$ for some fixed positive integer N. To show that f is a bounded distribution, we fix a Schwartz function φ and we observe that for some positive constant C_φ, we have that $c\varphi$ is an element of $\mathcal{F}_N$ and thus $M_1^*(f; c\varphi) \leq \mathcal{M}_N(f)$. Then

$$\begin{aligned}
c^p|(\varphi * f)(x)|^p &\leq \inf_{|y-x|\leq 1} \sup_{|z-y|\leq 1} |(c\varphi * f)(z)|^p \\
&\leq \inf_{|y-x|\leq 1} \left[\sup_{t>0} \sup_{|z-y|\leq 1} |(c\varphi_t * f)(z)|^p\right] \\
&\leq \inf_{|y-x|\leq 1} M_1^*(f; c\varphi)(y)^p \\
&\leq \frac{1}{v_n} \int_{|y-x|\leq 1} M_1^*(f; c\varphi)(y)^p dy \\
&\leq \frac{1}{v_n} \int_{\mathbb{R}^n} M_1^*(f; c\varphi)(y)^p dy \\
&\leq \frac{1}{v_n} \int_{\mathbb{R}^n} \mathcal{M}_N(f)(y)^p dy < \infty,
\end{aligned}$$

which implies that $\varphi * f$ is a bounded function. We conclude that f is a bounded distribution. We now proceed to show that f is an element of H^p. We fix a smooth radial nonnegative compactly supported function θ such that

$$\theta(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 2. \end{cases}$$

We observe that the identity

$$\begin{aligned}
P(x) &= P(x)\theta(x) + \sum_{k=1}^{\infty} \left[\theta(2^{-k}x)P(x) - \theta(2^{-(k-1)}x)P(x)\right] \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} \left[\frac{\theta(2^{-k}x) - \theta(2^{-(k-1)}x)}{(1+|x|^2)^{\frac{n+1}{2}}}\right] \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} \left[\frac{\theta(2^{-k}x) - \theta(2^{-(k-1)}x)}{(1+|x|^2)^{\frac{n+1}{2}}}\right] \left(\frac{1}{2^{-2k}}\right)^{\frac{n+1}{2}} \left(\frac{1}{2^{2k}}\right)^{\frac{n+1}{2}} \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} \left[\frac{\theta(2^{-k}x) - \theta(2^{-(k-1)}x)}{(2^{-2k}+2^{-2k}|x|^2)^{\frac{n+1}{2}}}\right] \left(\frac{1}{2^{2k}}\right)^{\frac{n+1}{2}} \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} \left[\frac{\theta(2^{-k}x) - \theta(2^{-(k-1)}x)}{(2^{-2k}+2^{-2k}|x|^2)^{\frac{n+1}{2}}}\right] \left(\frac{1}{2^{kn}}\right) \left(\frac{1}{2^{k}}\right) \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} \left(\frac{\theta(\cdot) - \theta(2(\cdot))}{(2^{-2k}+|\cdot|^2)^{\frac{n+1}{2}}}\right)_{2^k} (x) \left(\frac{1}{2^{k}}\right)
\end{aligned}$$

is valid for all $x \in \mathbb{R}^n$. We set

$$\Phi(x) = (\theta(x) - \theta(2x)) \frac{1}{(2^{-2k} + |x|^2)^{\frac{n+1}{2}}}$$

and we claim that for all bounded tempered distributions f and for all $t > 0$ we have

$$P_t * f = (\theta P)_t * f + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t} * f, \tag{3.2.28}$$

where the series converges in $\mathcal{S}'(\mathbb{R}^n)$. To prove the claim, we observe that

$$\delta_0 = \phi + (\delta_0 - \phi),$$

where δ_0 is the Dirac delta function and $\phi \in \mathcal{S}'(\mathbb{R}^n)$ is a function whose Fourier transform is equal to 1 in a neighborhood of zero. We can write

$$\begin{aligned} P_t * f &= P_t * \delta_0 * f \\ &= P_t * \phi * f + P_t * (\delta_0 - \phi) * f \end{aligned}$$

and subtitude P_t with

$$P_t(x) = (P(x)\theta(x))_t + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t}$$

to obtain

$$\begin{aligned} P_t * f &= [(P\theta)_t * \phi * f + (P\theta)_t * (\delta_0 - \phi) * f] + \left(\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t} \right) * \phi * f \\ &\quad + \left(\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t} \right) * (\delta_0 - \phi) * f \\ &= (P\theta)_t * f \\ &\quad + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} \left((\Phi^{(k)})_{2^k t} * \phi * f \right) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} \left((\Phi^{(k)})_{2^k t} * (\delta_0 - \phi) * f \right). \end{aligned}$$

We obtain the second part on the right side by Lebesgue dominated convergence theorem and the definition of convulution. In fact, since $\phi * f \in L^\infty$ and the support of $\Phi^{(k)}$ is away from zero at least $\frac{1}{2}$ for all positive intergers k, we have

$$\sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t}$$

converges for all positive intergers k.

We can also obtain the third part by the use of Fourier transform from viewing

$$\sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t}(x) = (1-\theta_t(x))P_t(x).$$

Using the notation $\hat{\theta}$ as Fourier transform of θ, we have

$$\widehat{P_t * (\delta_0 - \phi)} = e^{-2\pi t|\xi|}(1-\hat{\phi}(\xi))$$

is a Schwartz function and thus $(1-\theta_t)P_t * (\delta_0 - \phi)$ is also a Schwartz function. So we have

$$\left(\sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)})_{2^k t}\right) * (\delta_0 - \phi) * f = (1-\theta_t)P_t * (\delta_0 - \phi) * f$$

is in L^∞ and converges.

we claim that for some fixed constant $c_0 = c_0(n, N)$, the function $c_0 \theta P$ and $c_0 \Phi^{(k)}$ lie in $\mathcal{F}_N$ uniformly in $k = 1, 2, 3, ...$ To verify this assertion for $|\alpha| \leq N+1$, we apply Leibniz's rule to write

$$\begin{aligned}\left|\partial^\alpha \left[\frac{\theta(x)-\theta(2x)}{(2^{-2k}+|x|^2)^{\frac{n+1}{2}}}\right]\right| &= \left|\sum_{\beta\leq\alpha} c_{\alpha,\beta} \partial_x^{\alpha-\beta}\left(\theta(x)-\theta(2x)\right)\partial_x^\beta\left(\frac{1}{(2^{-2k}+|x|^2)^{\frac{n+1}{2}}}\right)\right| \\ &\leq \sum_{\beta\leq\alpha}\left|c'_{\alpha,\beta}\right|\chi_{\frac{1}{2}\leq|x|\leq 2}\partial_x^\beta\left(\frac{1}{(2^{-2k}+|x|^2)^{\frac{n+1}{2}}}\right) \\ &\leq \sum_{\beta\leq\alpha}\left|c'_{\alpha,\beta}\right|\chi_{\frac{1}{2}\leq|x|\leq 2}\sup_x\left|\chi_{\frac{1}{2}\leq|x|\leq 2}\partial_x^\beta\left(\frac{1}{(|x|^2)^{\frac{n+1}{2}}}\right)\right|\end{aligned}$$

These estimates are uniform in $k = 0, 1, 2, ...$ and thus from the definition

$$\mathfrak{N}_N(\varphi) = \int_{\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\leq N+1} |\partial^\alpha \varphi(x)| dx,$$

we have

$$\mathfrak{N}_N(\theta P) + \mathfrak{N}_N(\Phi^{(k)}) \leq 1/c_0(n, N)$$

for all some constant $c_0 = c_0(n, N)$ for all $k = 0, 1, 2, ...$

Then we obtain

$$\begin{aligned}\sup_{t>0}|P_t * f| &\leq \sup_{t>0}|(P\theta)_t * f| + \frac{1}{c_0}\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^{\infty} 2^{-k}\sup_{t>0}\left|(c_0\Phi^{(k)})_{2^k t} * f\right| \\ &\leq C_5(n, N)\mathcal{M}_N(f),\end{aligned}$$

and thus have

$$\|f\|_{H^p} \leq C_5(n, N)\|\mathcal{M}_N(f)\|_{L^p},$$

which is the (3.2.5). $\square$

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