# Numerical Optimization with applications: Homework 06

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December 7, 2016

**Exercise 1.** *The following example from [268] with a single variable $x \in \mathbb{R}$ and a single equality constraint shows that strict local solutions are not necessarily isolated. Consider*

$$\min_x x^2 \quad \text{subject to } c(x) = 0, \text{where } c(x) = \begin{cases} x^6 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad (12.96)$$

*(a) Show that the constraint function is twice continuously differentiable at all $x$ (including at $x = 0$) and that the feasible points are $x = 0$ and $x = 1/(k\pi)$ for all nonzero integers $k$.*

*(b) Verify that each feasible point except $x = 0$ is an isolated local solution by showing that there is a neighborhood $\mathcal{N}$ around each such point within which it is the only feasible point.*

*Proof.* (a) We first show directly that constraint function is twice continuously differentiable at all $x$.

If $x \neq 0$, then

$$\begin{aligned} c(x) &= x^6 \sin(1/x) \\ c'(x) &= 6x^5 \sin(1/x) - x^4 \cos(1/x) \\ c''(x) &= (30x^4 - x^2)\sin(1/x) - 10x^3 \cos(1/x) \end{aligned}$$

If $x = 0$, then by definition we obtain

$$\begin{aligned} c'(0) &= \lim_{h \to \infty} \frac{h^6 \sin(1/h) - 0}{h} = 0 \\ c''(0) &= \lim_{h \to \infty} \frac{\left[6h^5 \sin(1/h) - h^4 \cos(1/h)\right] - 0}{h} = 0 \end{aligned}$$

Hence, the constraint function is twice continuously differentiable at all $x$.

Second, we show that the feasible points are $x = 0$ and $x = 1/(k\pi)$ for all nonzero integers $k$.

If $x = 0$, then $c(x) = 0$ by definition. If $x = 1/(k\pi)$, then $\sin(1/x) = 0$ and thus we have $c(x) = 0$.

(b) If $x = 1/(k\pi)$ for some fixed nonzero integers $k$ and we choose $r = \left|\frac{1}{k\pi} - \frac{1}{(k+1\pi)}\right|$, then the open interval $\mathcal{N} = (x - r, x + r)$ contains only a feasible point, which is $x$ itself.

On the other hand, if $x = 0$, then for all $r > 0$ there exists nonzero positive integers $k$ such that $x_k = 1/(k\pi) < r$. However, $x_k$ are feasible points in the neighborhood $(-r, r)$.

Therefore, combining disscusion above, we have that each feasible point except $x = 0$ is an isolated local solution by showing that there is a neighborhood $\mathcal{N}$ around each such point within which it is the only feasible point.

□

**Exercise 15.** *Consider the following modification of (12.36), where t is a parameter to be fixed prior to solving the problem:*

$$\min_x (x_1 - \frac{3}{2})^2 + (x_2 - t)^4 \qquad s.t. \qquad \begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geq 0$$

*(a)For what value of t does the point $x^* = (1, 0)^T$ satisfy the KKT conditions?*
*(b)Show that when $t = 1$, only the first constraint is active at the solution, and find the solution.*

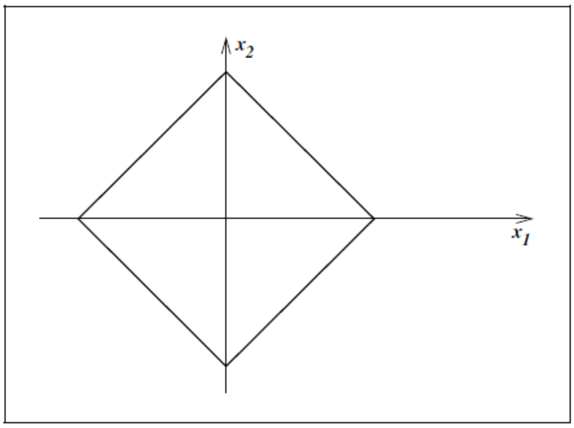
*Proof.* (a) First, we check the complementary condition of KKT, i.e. $\lambda_i c_i(x^*) = 0$ for $i = 1, 2, 3, 4$.

$$\begin{cases} \lambda_1(1 - 1 - 0) = 0 \\ \lambda_2(1 - 1 + 0) = 0 \\ \lambda_3(1 + 1 - 0) = 0 \\ \lambda_4(1 + 1 + 0) = 0 \end{cases} \Rightarrow \begin{cases} \lambda_3 = 0 \\ \lambda_4 = 0 \end{cases} \tag{1}$$

Obviously, $c(x^*) \geq 0$ holds. Consider

$$\begin{aligned} \triangledown_x L(x^*, \lambda) &= \triangledown_x L((0, 1)^T, \lambda) \\ &= \begin{bmatrix} -1 + \lambda_1 + \lambda_2 \\ -4t^3 + \lambda_1 - \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Since $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$, we have $\lambda_1 - \lambda_2 \in [-1, 1]$. Hence, $4t^3 \in [-1, 1]$, then $t \in [-\sqrt[3]{4}, \sqrt[3]{4}]$.




(b) We know the feasible set $E = \{x \in \mathbb{R}^2 \mid \|x\|_1 = 1\}$. Compute the gradient of $f(x)$

$$\triangledown f(x) = \begin{bmatrix} 2(x_1 - \frac{3}{2}) \\ 4(x_2 - 1)^3 \end{bmatrix} \leq 0 \qquad \forall x \in E$$

From above, $\forall x, y \in E$, if $x_1 \leq y_1$ and $x_2 \leq y_2$ then $f(y) \leq f(x)$. Therefore, we only have to consider the case that the first constraint is active: $1 - x_1 - x_2 = 0$. Substituting $x_2 = 1 - x_1$ into $f(x)$ and find the minimum of $f$:

$$\begin{aligned} f(x) &= (x_1 - \frac{3}{2})^2 + (-x_1)^4 \\ f'(x) &= 2(x_1 - \frac{3}{2}) + 4x_1^3 \\ &= 4x_1^3 + 2x_1 - 3 \end{aligned}$$

$f'(x^*) = 0$ if $x_1^* = \frac{\sqrt[3]{27+\sqrt{753}}}{2\times 3^{2/3}} - \frac{1}{\sqrt[3]{3(27+\sqrt{753})}}$. Consequently, the minimizer of $f$ is $(x_1^*, 1 - x_1^*)$.

□

**Exercise 19.** *Consider the problem*

$$\min_{x\in R^2} = -2x_1 + x_2 \quad \textit{subject to} \quad \begin{cases} (1-x_1)^3 - x_2, & \geq 0 \\ x_2 + 0.25x_1^2 - 1, & \geq 0 \end{cases}$$

*The optimal solution is $x^* = (0,1)^T$, where both constraints are active.*

*(a) Do the LICQ hold at this point?*

*(b) Are the KKT conditions satisfied?*

*(c) Write down the set $\mathcal{F}(x^*)$ and $\mathcal{C}(x^*, \lambda^*)$.*

*(d) Are the second-order necessary conditions satisfied? Are the second-order sufficient conditions satisfied?*

*Proof.* (a)

$$A(x^*) = [\bigtriangledown C_i(x^*)]_{i\in\mathcal{A}(x^*)} = \begin{bmatrix} -3(1-x_1)^2 & 0.5x_1 \\ -1 & 1 \end{bmatrix}_{x=x*} = \begin{bmatrix} -3 & 0 \\ -1 & 1 \end{bmatrix}$$

$A(x^*)$ is nonsingular. Therefore, the LICQ holds.

(b)

$$\bigtriangledown f(x^*) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \bigtriangledown c_1(x^*) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad \bigtriangledown c_2(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, the KKT conditions (12.34a)-(12.34e) are satisfied when we set

$$\lambda^* = \left(\frac{2}{3}, \frac{5}{3}\right)^T$$

(c)

$$\mathcal{F}(x^*) = \{d | \bigtriangledown c_i(x^*)^T d \geq 0\} = \{(d_1, d_2)^T | \begin{bmatrix} -3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \geq 0\} = \{(d_1, d_2)^T | d_2 \geq 0,\ 3d_1 + d_2 \leq 0\}$$

$$\begin{aligned} \mathcal{C}(x^*, \lambda^*) &= \{w \in \mathcal{F}(x^*) | \bigtriangledown c_i(x^*)^T w = 0 \ \forall i \in \mathcal{A}(x^*) \cap I \text{ with } \lambda_i^* > 0\} \\ &= \{w \in \mathcal{F}(x^*) | \bigtriangledown c_i(x^*)^T w = 0 \text{ for } i = 1, 2\} \\ &= \{(w_1, w_2)^T \in \mathcal{F}(x^*) | \begin{bmatrix} -3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0\} \\ &= \{(0,0)^T\} \end{aligned}$$

(d)

$$\forall w \in \mathcal{C}(x^*, \lambda^*) = \{(0,0)^T\} \qquad w^T \bigtriangledown_{xx} \mathcal{L}(x^*, \lambda^*) w = 0 \tag{2}$$

Since at $x^*$ LICQ holds and $(x^*, \lambda^*)$ satisfies KKT. By (2), the second-order necessary conditions is satisfied.
Since $x^*$ is a feasible solution and $(x^*, \lambda^*)$ satisfies KKT. By (2), the second-order sufficient conditions is satisfied.

□