Real Analysis II: Homework 01

104021615 黃翊軒

March 01, 2016

Exercise 2. *P96.*

Proof. (a) Let $h_1(x, y) = f(x)$. As a function on $\mathbb{R}^{2n}$, h_1 is measurable since $f(x) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. And for any $a \in \mathbb{R}$, we have

$$\{(x, y) \in \mathbb{R}^{2n} : h_1(x, y) > a\} = \{x \in \mathbb{R}^n : f(x) > a\} \times \mathbb{R}^n$$

By viewing $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ and using Lemma 5.2, the RHS is a measurable set in $\mathbb{R}^{2n}$. Similarly, the function $h_2(x, y) = g(y)$ is also a measurable function on $\mathbb{R}^{2n}$. Then by Theorem 4.10, we know that

$$h_1(x, y) \cdot h_2(x, y) = f(x)g(y) : \mathbb{R}^{2n} \to \mathbb{R} \cup \{\pm\infty\}$$

is also a measurable function on $\mathbb{R}^{2n}$.

(b) Given E_1, E_2, both are measurable in $\mathbb{R}^n$. Since $\chi_{E_1}(x) \cdot \chi_{E_2}(y) = \chi_{E_1 \times E_2}(x, y)$, by above we know that $\chi_{E_1 \times E_2}(x, y)$ is a measurable function on $\mathbb{R}^{2n}$. Hence the set $E_1 \times E_2$ is measurable in $\mathbb{R}^{2n}$. By Tonelli's theorem,

$$\begin{aligned}
|E_1 \times E_2| &= \int_{E_1 \times E_2} \chi_{E_1 \times E_2}(x, y) dx dy \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_{E_1 \times E_2}(x, y) dx dy \\
&= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \chi_{E_1 \times E_2}(x, y) dx \right] dy \\
&= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \chi_{E_1}(x) \cdot \chi_{E_2}(y) dx \right] dy \\
&= \int_{\mathbb{R}^n} \chi_{E_1}(x) dx \int_{\mathbb{R}^n} \chi_{E_2}(y) dy \\
&= \int_{E_1} \chi_{E_1}(x) dx \int_{E_2} \chi_{E_2}(y) dy \\
&= |E_1||E_2|
\end{aligned}$$

□

Real Analysis II: Homework 02

104021615 黃翊軒

March 07, 2016

Exercise 14. *p.86*

Proof. Since $f = f^+ - f^-$, we may assume $f \geq 0$.
And by Theorem 3.26(ii), given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int_{\{0<\delta\}} f^p < \epsilon$$

Hence, the L^p version of Tchebyshev's inequality implies

$$a^p \left[\omega(a) - \omega(\delta)\right] \leq \int_{\{a<f\leq\delta\}} f^p < \epsilon \qquad \text{for} \quad 0 < a < \delta$$

Now let $a \to 0+$, we have

$$\lim_{a\to 0+} a^p\omega(a) - 0 < \epsilon \qquad \text{for all} \quad \epsilon > 0$$

which is equivalent to

$$\lim_{a\to 0+} a^p = 0$$

□

Exercise 15. *p.86*

Proof. Since th integral $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ converges, we know that for all $\epsilon > 0$, there exists a such that

$$\int_{\frac{a}{2}}^{a} \alpha^{p-1}\omega(\alpha)d\alpha \leq \int_0^a \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

Since α^{p-1} is a increasing function and $\omega(\alpha)$ is a decreasing function, we have

$$(\frac{a}{2})^p\omega(a) \leq \int_{\frac{a}{2}}^{a} \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

so

$$a^p\omega(a) < \epsilon$$

Hence,

$$\lim_{a\to 0+} a^p\omega(a) = 0$$

Similarly, for $b^p\omega(b)$:
Since th integral $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ converges, we know that for all $\epsilon > 0$, there exists b such that

$$\int_{\frac{b}{2}}^{b} \alpha^{p-1}\omega(\alpha)d\alpha \leq \int_{\frac{b}{2}}^{\infty} \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

Since α^{p-1} is a increasing function and $\omega(\alpha)$ is a decreasing function, we have

$$(\frac{b}{2})^p\omega(b) \leq \int_{\frac{b}{2}}^{b} \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

so

$$b^p\omega(b) < \epsilon$$

Hence,

$$\lim_{b\to\infty} b^p\omega(b) = 0$$

□

Exercise 3. *p.96*

Proof. Since $f(x) - f(y) \in L(I)$, where $I = [0,1] \times [0,1]$, by Fnbini's Theorem, we have:
For almost $y \in [0,1]$, $f(x) - f(y)$ is integrable on E_y with respect to x.

Inparticular, since $f(y)$ is finite a.e. on $[0,1]$, we may take an y such that $f(y) = a$ is finite. This implies $f(x) - a$ is integrable on $[0,1]$, which is equivalent to $f(x)$ is integrable on $[0,1]$. $\square$

Exercise 4. *p.96*

Proof. Using the hint, set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t, \eta = -x + t$.
By assumption, we have

$$\int_0^1 \int_0^1 |f(x+t) - f(-x+t)|\, dt dx \leq c$$

Change the variables to obtain

$$\frac{1}{2} \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| d\xi d\eta \leq c$$

Since the periodicity of f, the inequality can be rewrited as

$$\int_0^1 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta \leq \frac{c}{2}$$

Thus, $|f(\xi) - f(\eta)|$ and hence $f(\xi) - f(\eta)$ are integrable on $[0,1] \times [0,1]$. By the result of Exercise 6.3, f is integrable over $[0,1]$.

$\square$

Real Analysis II: Homework 03

104021615 黃翊軒

March 14, 2016

Exercise 16. *p.86*

Proof. Given $f \geqslant 0$ and $|E|$ is not necessarily finite, we consider similar process to (5.46) to show if $\varphi(x) = |x|^p$, then $\int_{E_{ab}} \varphi(f) = -\int_a^b \varphi(\alpha) d\omega(\alpha)$. But we will use the **monotone convergence theorem** here instead of the **bounded convergence theorem** since $|E|$ could be $+\infty$.

Select a sequence of simple function $\{f_k\}$ like (4.13) on E_{ab} such that $f_k \nearrow f|_{E_{ab}}$. Since φ is continuous and nongative, it follows that $\varphi(f_k) \nearrow \varphi(f)$. So by the **monotone convergence theorem** we have

$$\int_{E_{ab}} \varphi(f_k) \nearrow \int_{E_{ab}} \varphi(f)$$

Moreover, since $\varphi(f_k)$ is a simple function on $[a, b]$, we have

$$\sum_j \varphi(\alpha_{j-1}^{(k)})[\omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)})] \leqslant \int_{E_{ab}} \varphi(f_k) \leqslant \sum_j \varphi(\alpha_j^{(k)})[\omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)})]$$

Since the norm of the partitions approach 0 as $k \to \infty$, we have $\int_{E_{ab}} \varphi(f) = -\int_a^b \varphi(\alpha) d\omega(\alpha)$ and the case that $\int_{E_{ab}} \varphi(f_k) \nearrow +\infty$ implies $-\int_a^b \varphi(\alpha) d\omega(\alpha) = +\infty$ is trival. Let $a \to 0^+$, $b \to +\infty$ the **monotone convergence theorem** show that

$$\int_E \varphi(f) = -\int_0^\infty \varphi(\alpha) d\omega(\alpha)$$

note that the equality hold without regard to the finiteness of either side.

Suppose that $-\int_0^\infty \varphi(\alpha) d\omega(\alpha)$ and hence $\int_E \varphi(f)$ is finite. Then $f \in L^p(E)$, so Exercise 14 and (5.50) state that $\lim_{a\to 0^+} a^p \omega(a)$ and $\lim_{b\to+\infty} b^p \omega(b) = 0$, so integrating by parts gives us

$$\int_a^b \alpha^p d\omega(\alpha) = \alpha^p \omega(\alpha)\Big|_a^b - p\int_a^b \alpha^{p-1}\omega(\alpha) d\alpha \to -p\int_0^\infty \alpha^{p-1}\omega(\alpha) d\alpha$$

Conversely, suppose that $-p\int_0^\infty \alpha^{p-1}\omega(\alpha) d\alpha$ is finite. Then Exercise 15 states that $\lim_{a\to 0^+} a^p \omega(a)$ and $\lim_{b\to+\infty} b^p \omega(b) = 0$. By integrating by parts, it follows

$$\alpha^p \omega(\alpha)\Big|_a^b - p\int_a^b \alpha^{p-1}\omega(\alpha) d\alpha = \int_a^b \alpha^p d\omega(\alpha)$$

Letting $a \to 0^+$, $b \to +\infty$,

$$-p\int_0^\infty \alpha^{p-1}\omega(\alpha) d\alpha = \int_0^\infty \alpha^p d\omega(\alpha) = \int_E \varphi(f)$$

It therefore follows that one integral is finite if and only if the other is finite, and if they are finite, then they are equal (so if they are not finite, they are also both equal to $+\infty$, as f is nonnegative. ☐

Exercise 1. *p.146*

Proof. If $\int_E f$ is finite, then both $\int_E f_1$ and $\int_E f_2$ are finite, so $\int_E |f_1|$ and $\int_E |f_2|$ are finite. Thus

$$\int_E |f| = \int_E |f_1 + if_2| \leqslant \int_E |f_1| + \int_E |f_2| < +\infty$$

Conversely, if $\int_E |f|$ is finite, then so are $\int_E |f_1|$ and $\int_E |f_2|$ since $|f_1|, |f_2| \leqslant |f|$. Thus, $\int_E f_1$ and $\int_E f_2$ are finite, so $\int_E f = \int_E f_1 + if_2$ is also finite.

Following the hint, chose α such that

$$\left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2} = \cos(\alpha)\int_E f_1 + \sin(\alpha)\int_E f_2$$

Then

$$\begin{aligned}\left|\int_E f\right| = \left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2} &= \cos(\alpha)\int_E f_1 + \sin(\alpha)\int_E f_2 \\ &= \int_E (f_1\cos(\alpha) + f_2\sin(\alpha)) \leqslant \int_E |f_1\cos(\alpha) + f_2\sin(\alpha)| \\ &\leqslant \int_E \sqrt{f_1^2 + f_2^2} = \int_E |f|\end{aligned}$$

□

Exercise 4. *p.146*

Proof. Observe that Hölder's inequality comes from Young's inequality. With the observation the **equality** of $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$(Young's inequality) hold if and only if $a^{p-1} = b$ and hence if and only if $a^p = b^q$, where p, q are conjugate exponents.

Inparticular, let $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$, then integrating both side of Young's inequality implies the Hölder's inequality. It follows that the equality of Hölder's inequality hold if and only if $a^p = b^q$ and $|f||g| = |fg|$ *a.e.* if and only if $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$ and fg has constant sign *a.e.*

Hence, the equality of Hölder's inequality hold if and only if fg has constant sign and $|f|^p = c \cdot |g|^q$ *a.e.*, where $c = \frac{\|f\|_p^p}{\|g\|_q^q}$.

Observe that the Minkowski's inequality comes from $|f + g| \leqslant |f| + |g|$ and Hölder's inequality for $|f|, |f+g|^{p-1}$ and $|g|, |f+g|^{p-1}$.

By the result of previous discussion, we quickly have fg has constant sign *a.e.* and $c_1|f|^p = c_2|g|^p = |f+g|^p$ *a.e.* Hence, the equality of Minkowski's inequality hold if and only if $fg \geqslant 0$ and $|f|^p = c \cdot |g|^p$ *a.e.*, where $c = c_2/c_1$. □

Exercise 6. *p.146*

Proof. We prove this by induction. The $k = 2$ case is a consequence of Hölder's inequality: if $1/p_1 + 1/p_2 = 1/r$, then $r/p_1 + r/p_2 = 1$, so

$$\|fg\|_r^r = \|f^r g^r\|_1 \leqslant \|f^r\|_{p_1/r}\|g^r\|_{p_2/r} = \|f\|_{p_1}^r \|g\|_{p_2}^r.$$

Now if $1/p_1 + \cdots + 1/p_k = 1/r$ for $k \geqslant 2$, we have

$$\|f_1 \cdots f_k\|_r \leqslant \|f_1 \cdots f_{k-1}\|_s \|f_k\|_{p_k} \leqslant \|f_1\|_{p_1} \cdots \|p_k\|_{p_k},$$

where $1/s = 1/r - 1/p_k = 1/p_1 + \cdots + 1/p_{k-1}$. □

Real Analysis II: Homework 04

104021615 黃翊軒

March 28, 2016

Exercise 5. *p.143*

Proof. We may assume $N_{p_1}[f] < +\infty$, otherwise the result is trivial by Theorem 8.2.

Let $q = p_2/p_1 > 1$, then the conjugate exponent of q is $p_2/(p_2 - p_1)$. By Hölder's Inequality, we get

$$\begin{aligned}\|f\|_{p_1}^{p_1} &= \int_E |f|^{p_1} = \int_E |f|^{p_1} \times 1 \\ &\leqslant \left(\int_E |f|^{p_1 q}\right)^{1/q} \left(\int_E 1^{p_2/(p_2-p_1)}\right)^{(p_2-p_1)/p_2} \\ &= \left(\int_E |f|^{p_2}\right)^{p_1/p_2} |E|^{(p_2-p_1)/p_2}\end{aligned}$$

Taking both sides to the power $1/p_1$, then

$$\begin{aligned}\|f\|_{p_1} &\leqslant \left(\int_E |f|^{p_2}\right)^{1/p_2} |E|^{(p_2-p_1)/p_1p_2} \\ &= \|f\|_{p_2} |E|^{(p_2-p_1)/p_1p_2}\end{aligned}$$

Multipling both sides $1/|E|^{1/p_1}$, we have

$$N_{p_1}[f] = \frac{1}{|E|^{1/p_1}} \|f\|_{p_1} \leqslant \frac{1}{|E|^{1/p_2}} \|f\|_{p_2} = N_{p_2}[f]$$

Next we check other three properties.

From Minkowski's Inequilty,

$$\begin{aligned}N_p[f+g] &= \frac{1}{|E|^{1/p}} \|f+g\|_p \\ &\leqslant \frac{1}{|E|^{1/p}} (\|f|_p + \|g\|_p) \\ &= \frac{1}{|E|^{1/p}} \|f\|_p + \frac{1}{|E|^{1/p}} \|g\|_p \\ &= N_p[f] + N_p[g]\end{aligned}$$

From Hölder's Inequality,

$$\begin{aligned}N_1[fg] = \frac{1}{|E|} \int |fg| &\leqslant \frac{1}{|E|^{1/p+1/p'}} \int |f| \int |g| \\ &= N_p[f] N_{p'}[g]\end{aligned}$$

From Theorem 8.1,

$$\begin{aligned}\lim_{p\to\infty} N_p[f] &= \lim_{p\to\infty} \frac{1}{|E|^{1/p}} \|f\|_p \\ &= \lim_{p\to\infty} \frac{1}{|E|^{1/p}} \cdot \lim_{p\to\infty} \|f\|_p \\ &= 1 \cdot \lim_{p\to\infty} \|f\|_p \\ &= \|f\|_\infty\end{aligned}$$

$\square$

Exercise 7. *p.143*

Proof. We quickly have

$$\|f\|_p = \left(\int_{(0,\varepsilon^p)} |1|^p \right)^{1/p} = \varepsilon$$

and similarly $\|g\|_p = \varepsilon$. However,

$$\|f/2 + g/2\|_p = \left(\int_{(0,2\varepsilon^p)} |1/2|^p \right)^{1/p} = \left(2\varepsilon^p 2^{-p}\right)^{1/p} = 2^{\frac{1}{p}-1}\varepsilon > \varepsilon$$

since $1/p > 1$. So the neighborhood $B_{\varepsilon+\eta}(0)$ is not convex for sufficiently small η. □

Exercise 9. *p.143*

Proof. Suppose $\operatorname{ess\,inf}_E f = 0$. Then for every $\alpha > 0$ we have $|\{x \in E : f(x) < \alpha\}| > 0$. Thus, for every $0 < \beta < +\infty$ we have $|\{x \in E : 1/f(x) > \beta\}| > 0$, so $\operatorname{ess\,sup}_E 1/f = +\infty$. We may interpret $+\infty^{-1} = 0$, so the proposition still holds.

Now suppose $\operatorname{ess\,inf}_E f > 0$, so there exists $\alpha > 0$ such that $|\{x \in E : f(x) < \alpha\}| > 0$. Then

$$\begin{aligned}
\operatorname*{ess\,inf}_E f &= \sup\{\alpha > 0 : |\{x \in E : f(x) < \alpha\}| > 0\} \\
&= \sup\{\alpha > 0 : |\{x \in E : 1/f(x) > 1/\alpha\}| > 0\} \\
&= \sup\{1/\beta > 0 : |\{x \in E : 1/f(x) > \beta\}| > 0\} \\
&= (\inf\{\beta > 0 : |\{x \in E : 1/f(x) > \beta\}| > 0\})^{-1} \\
&= \left(\operatorname*{ess\,sup}_E (1/f) \right)^{-1}
\end{aligned}$$

□

Exercise 10. *p.143*

Proof. Let $\varepsilon_k = 1/k$, then by Lemma 3.22. We have for every ε_k, there exists a closed set F_k such that

$$\varepsilon_{k+1} < |E - F_k| < \varepsilon_k.$$

Note that such F_k is possible by reversely using Lemma 3.22 again if needed. Then we have a sequence of sets of strictly decreasing measure $|E - F_{k+1}| < |E - F_k|$.

The difference of any two of these sets must then have positive measure. Let A be the set of all possible differences of sets $|E - F_k|$. The set of all possible unions of sets in A is the power set of A, $P(A)$. Note that the power set is uncountable. Taking the characteristic function of any two distinct sets in $P(A)$ gives

$$\|\chi_{B_1} - \chi_{B_2}\|_\infty = 1,$$

because B_1 and B_2 must differ by a set of positive measure.

Since there are an uncountable number of such functions and their norms are all different by 1, they can be contained in disjoint open balls of radius 1/3 in the space $L^\infty(E)$. Then the space $L^\infty(E)$ cannot be separable, because no countable set can intersect this uncountable family of disjoint balls. □