Real Analysis II: Homework 01

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Exercise 2. *P96.*

Proof. (a) Let $h_1(x,y) = f(x)$. As a function on $\mathbb{R}^{2n}$, h_1 is measurable since $f(x) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. And for any $a \in \mathbb{R}$, we have

$$\{(x,y) \in \mathbb{R}^{2n} : h_1(x,y) > a\} = \{x \in \mathbb{R}^n : f(x) > a\} \times \mathbb{R}^n$$

By viewing $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ and using Lemma 5.2, the RHS is a measurable set in $\mathbb{R}^{2n}$. Similarly, the function $h_2(x,y) = g(y)$ is also a measurable function on $\mathbb{R}^{2n}$. Then by Theorem 4.10, we know that

$$h_1(x,y) \cdot h_2(x,y) = f(x)g(y) : \mathbb{R}^{2n} \to \mathbb{R} \cup \{\pm\infty\}$$

is also a measurable function on $\mathbb{R}^{2n}$.

(b) Given E_1, E_2, both are measurable in $\mathbb{R}^n$. Since $\chi_{E_1}(x) \cdot \chi_{E_2}(y) = \chi_{E_1 \times E_2}(x,y)$, by above we know that $\chi_{E_1 \times E_2}(x,y)$ is a measurable function on $\mathbb{R}^{2n}$. Hence the set $E_1 \times E_2$ is measurable in $\mathbb{R}^{2n}$. By Tonelli's theorem,

$$\begin{aligned}
|E_1 \times E_2| &= \int_{E_1 \times E_2} \chi_{E_1 \times E_2}(x,y)dxdy \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_{E_1 \times E_2}(x,y)dxdy \\
&= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \chi_{E_1 \times E_2}(x,y)dx\right] dy \\
&= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \chi_{E_1}(x) \cdot \chi_{E_2}(y)dx\right] dy \\
&= \int_{\mathbb{R}^n} \chi_{E_1}(x)dx \int_{\mathbb{R}^n} \chi_{E_2}(y)dy \\
&= \int_{E_1} \chi_{E_1}(x)dx \int_{E_2} \chi_{E_2}(y)dy \\
&= |E_1||E_2|
\end{aligned}$$

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