Real Analysis II: Homework 02

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Exercise 14. *p.86*

Proof. Since $f = f^+ - f^-$, we may assume $f \geq 0$.
And by Theorem 3.26(ii), given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int_{\{0<\delta\}} f^p < \epsilon$$

Hence, the L^p version of Tchebyshev's inequality implies

$$a^p \left[\omega(a) - \omega(\delta)\right] \leq \int_{\{a<f\leq\delta\}} f^p < \epsilon \qquad \text{for} \quad 0 < a < \delta$$

Now let $a \to 0+$, we have

$$\lim_{a\to 0+} a^p \omega(a) - 0 < \epsilon \qquad \text{for all} \quad \epsilon > 0$$

which is equivalent to

$$\lim_{a\to 0+} a^p = 0$$

□

Exercise 15. *p.86*

Proof. Since th integral $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ converges, we know that for all $\epsilon > 0$, there exists a such that

$$\int_{\frac{a}{2}}^{a} \alpha^{p-1}\omega(\alpha)d\alpha \leq \int_0^a \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

Since α^{p-1} is a increasing function and $\omega(\alpha)$ is a decreasing function, we have

$$(\frac{a}{2})^p\omega(a) \leq \int_{\frac{a}{2}}^{a} \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

so

$$a^p\omega(a) < \epsilon$$

Hence,

$$\lim_{a\to 0+} a^p\omega(a) = 0$$

Similarly, for $b^p\omega(b)$:
Since th integral $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ converges, we know that for all $\epsilon > 0$, there exists b such that

$$\int_{\frac{b}{2}}^{b} \alpha^{p-1}\omega(\alpha)d\alpha \leq \int_{\frac{b}{2}}^{\infty} \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

Since α^{p-1} is a increasing function and $\omega(\alpha)$ is a decreasing function, we have

$$(\frac{b}{2})^p\omega(b) \leq \int_{\frac{b}{2}}^{b} \alpha^{p-1}\omega(\alpha)d\alpha \leq \frac{\epsilon}{2^p}$$

so

$$b^p\omega(b) < \epsilon$$

Hence,

$$\lim_{b\to\infty} b^p\omega(b) = 0$$

□

Exercise 3. *p.96*

Proof. Since $f(x) - f(y) \in L(I)$, where $I = [0,1] \times [0,1]$, by Fnbini's Theorem, we have:
For almost $y \in [0,1]$, $f(x) - f(y)$ is integrable on E_y with respect to x.

Inparticular, since $f(y)$ is finite a.e. on $[0,1]$, we may take an y such that $f(y) = a$ is finite. This implies $f(x) - a$ is integrable on $[0,1]$, which is equivalent to $f(x)$ is integrable on $[0,1]$. □

Exercise 4. *p.96*

Proof. Using the hint, set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t, \eta = -x + t$.
By assumption, we have

$$\int_0^1 \int_0^1 |f(x+t) - f(-x+t)| \, dtdx \le c$$

Change the variables to obtain

$$\frac{1}{2} \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| d\xi d\eta \le c$$

Since the periodicity of f, the inequality can be rewrited as

$$\int_0^1 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta \le \frac{c}{2}$$

Thus, $|f(\xi) - f(\eta)|$ and hence $f(\xi) - f(\eta)$ are integrable on $[0,1] \times [0,1]$. By the result of Exercise 6.3, f is integrable over $[0,1]$.

□