

Real Analysis II: Homework 04

104021615 黃翊軒

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Exercise 5. p.143

Proof. We may assume $N_{p_1}[f] < +\infty$, otherwise the result is trivial by Theorem 8.2.

Let $q = p_2/p_1 > 1$, then the conjugate exponent of q is $p_2/(p_2 - p_1)$. By Hölder's Inequality, we get

$$\begin{aligned}\|f\|_{p_1}^{p_1} &= \int_E |f|^{p_1} = \int_E |f|^{p_1} \times 1 \\ &\leq \left(\int_E |f|^{p_1 q} \right)^{1/q} \left(\int_E 1^{p_2/(p_2-p_1)} \right)^{(p_2-p_1)/p_2} \\ &= \left(\int_E |f|^{p_2} \right)^{p_1/p_2} |E|^{(p_2-p_1)/p_2}\end{aligned}$$

Taking both sides to the power $1/p_1$, then

$$\begin{aligned}\|f\|_{p_1} &\leq \left(\int_E |f|^{p_2} \right)^{1/p_2} |E|^{(p_2-p_1)/p_1 p_2} \\ &= \|f\|_{p_2} |E|^{(p_2-p_1)/p_1 p_2}\end{aligned}$$

Multiplying both sides $1/|E|^{1/p_1}$, we have

$$N_{p_1}[f] = \frac{1}{|E|^{1/p_1}} \|f\|_{p_1} \leq \frac{1}{|E|^{1/p_2}} \|f\|_{p_2} = N_{p_2}[f]$$

Next we check other three properties.

From Minkowski's Inequality,

$$\begin{aligned}N_p[f+g] &= \frac{1}{|E|^{1/p}} \|f+g\|_p \\ &\leq \frac{1}{|E|^{1/p}} (\|f\|_p + \|g\|_p) \\ &= \frac{1}{|E|^{1/p}} \|f\|_p + \frac{1}{|E|^{1/p}} \|g\|_p \\ &= N_p[f] + N_p[g]\end{aligned}$$

From Hölder's Inequality,

$$\begin{aligned}N_1[fg] &= \frac{1}{|E|} \int |fg| \leq \frac{1}{|E|^{1/p+1/p'}} \int |f| \int |g| \\ &= N_p[f] N_{p'}[g]\end{aligned}$$

From Theorem 8.1,

$$\begin{aligned}\lim_{p \rightarrow \infty} N_p[f] &= \lim_{p \rightarrow \infty} \frac{1}{|E|^{1/p}} \|f\|_p \\ &= \lim_{p \rightarrow \infty} \frac{1}{|E|^{1/p}} \cdot \lim_{p \rightarrow \infty} \|f\|_p \\ &= 1 \cdot \lim_{p \rightarrow \infty} \|f\|_p \\ &= \|f\|_\infty\end{aligned}$$

□

Exercise 7. *p.143*

Proof. We quickly have

$$\|f\|_p = \left(\int_{(0, \varepsilon^p)} |1|^p \right)^{1/p} = \varepsilon$$

and similarly $\|g\|_p = \varepsilon$. However,

$$\|f/2 + g/2\|_p = \left(\int_{(0, 2\varepsilon^p)} |1/2|^p \right)^{1/p} = (2\varepsilon^p 2^{-p})^{1/p} = 2^{\frac{1}{p}-1} \varepsilon > \varepsilon$$

since $1/p > 1$. So the neighborhood $B_{\varepsilon+\eta}(0)$ is not convex for sufficiently small η . \square

Exercise 9. *p.143*

Proof. Suppose $\text{ess inf}_E f = 0$. Then for every $\alpha > 0$ we have $|\{x \in E : f(x) < \alpha\}| > 0$. Thus, for every $0 < \beta < +\infty$ we have $|\{x \in E : 1/f(x) > \beta\}| > 0$, so $\text{ess sup}_E 1/f = +\infty$. We may interpret $+\infty^{-1} = 0$, so the proposition still holds.

Now suppose $\text{ess inf}_E f > 0$, so there exists $\alpha > 0$ such that $|\{x \in E : f(x) < \alpha\}| > 0$. Then

$$\begin{aligned} \text{ess inf}_E f &= \sup\{\alpha > 0 : |\{x \in E : f(x) < \alpha\}| > 0\} \\ &= \sup\{\alpha > 0 : |\{x \in E : 1/f(x) > 1/\alpha\}| > 0\} \\ &= \sup\{1/\beta > 0 : |\{x \in E : 1/f(x) > \beta\}| > 0\} \\ &= (\inf\{\beta > 0 : |\{x \in E : 1/f(x) > \beta\}| > 0\})^{-1} \\ &= \left(\text{ess sup}_E (1/f) \right)^{-1} \end{aligned}$$

\square

Exercise 10. *p.143*

Proof. Let $\varepsilon_k = 1/k$, then by Lemma 3.22. We have for every ε_k , there exists a closed set F_k such that

$$\varepsilon_{k+1} < |E - F_k| < \varepsilon_k.$$

Note that such F_k is possible by reversely using Lemma 3.22 again if needed. Then we have a sequence of sets of strictly decreasing measure $|E - F_{k+1}| < |E - F_k|$.

The difference of any two of these sets must then have positive measure. Let A be the set of all possible differences of sets $|E - F_k|$. The set of all possible unions of sets in A is the power set of A , $P(A)$. Note that the power set is uncountable. Taking the characteristic function of any two distinct sets in $P(A)$ gives

$$\|\chi_{B_1} - \chi_{B_2}\|_\infty = 1,$$

because B_1 and B_2 must differ by a set of positive measure.

Since there are an uncountable number of such functions and their norms are all different by 1, they can be contained in disjoint open balls of radius $1/3$ in the space $L^\infty(E)$. Then the space $L^\infty(E)$ cannot be separable, because no countable set can intersect this uncountable family of disjoint balls. \square