Real Analysis II: Homework 03

104021615 黃翊軒

March 14, 2016

Exercise 16. *p.86*

Proof. Given $f \geqslant 0$ and $|E|$ is not necessarily finite, we consider similar process to (5.46) to show if $\varphi(x) = |x|^p$, then $\int_{E_{ab}} \varphi(f) = -\int_a^b \varphi(\alpha)d\omega(\alpha)$. But we will use the **monotone convergence theorem** here instead of the **bounded convergence theorem** since $|E|$ could be $+\infty$.

Select a sequence of simple function $\{f_k\}$ like (4.13) on E_{ab} such that $f_k \nearrow f|_{E_{ab}}$. Since φ is continuous and nongative, it follows that $\varphi(f_k) \nearrow \varphi(f)$. So by the **monotone convergence theorem** we have

$$\int_{E_{ab}} \varphi(f_k) \nearrow \int_{E_{ab}} \varphi(f)$$

Moreover, since $\varphi(f_k)$ is a simple function on $[a, b]$, we have

$$\sum_j \varphi(\alpha_{j-1}^{(k)})[\omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)})] \leqslant \int_{E_{ab}} \varphi(f_k) \leqslant \sum_j \varphi(\alpha_j^{(k)})[\omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)})]$$

Since the norm of the partitions approach 0 as $k \to \infty$, we have $\int_{E_{ab}} \varphi(f) = -\int_a^b \varphi(\alpha)d\omega(\alpha)$ and the case that $\int_{E_{ab}} \varphi(f_k) \nearrow +\infty$ implies $-\int_a^b \varphi(\alpha)d\omega(\alpha) = +\infty$ is trival. Let $a \to 0^+$, $b \to +\infty$ the **monotone convergence theorem** show that

$$\int_E \varphi(f) = -\int_0^\infty \varphi(\alpha)d\omega(\alpha)$$

note that the equality hold without regard to the finiteness of either side.

Suppose that $-\int_0^\infty \varphi(\alpha)d\omega(\alpha)$ and hence $\int_E \varphi(f)$ is finite. Then $f \in L^p(E)$, so Exercise 14 and (5.50) state that $\lim_{a\to 0^+} a^p\omega(a)$ and $\lim_{b\to+\infty} b^p\omega(b) = 0$, so integrating by parts gives us

$$\int_a^b \alpha^p d\omega(\alpha) = \alpha^p\omega(\alpha)\Big|_a^b - p\int_a^b \alpha^{p-1}\omega(\alpha)d\alpha \to -p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$$

Conversely, suppose that $-p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite. Then Exercise 15 states that $\lim_{a\to 0^+} a^p\omega(a)$ and $\lim_{b\to+\infty} b^p\omega(b) = 0$. By integrating by parts, it follows

$$\alpha^p\omega(\alpha)\Big|_a^b - p\int_a^b \alpha^{p-1}\omega(\alpha)d\alpha = \int_a^b \alpha^p d\omega(\alpha)$$

Letting $a \to 0^+$, $b \to +\infty$,

$$-p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha = \int_0^\infty \alpha^p d\omega(\alpha) = \int_E \varphi(f)$$

It therefore follows that one integral is finite if and only if the other is finite, and if they are finite, then they are equal (so if they are not finite, they are also both equal to $+\infty$, as f is nonnegative. ☐

Exercise 1. *p.146*

Proof. If $\int_E f$ is finite, then both $\int_E f_1$ and $\int_E f_2$ are finite, so $\int_E |f_1|$ and $\int_E |f_2|$ are finite. Thus

$$\int_E |f| = \int_E |f_1 + if_2| \leqslant \int_E |f_1| + \int_E |f_2| < +\infty$$

Conversely, if $\int_E |f|$ is finite, then so are $\int_E |f_1|$ and $\int_E |f_2|$ since $|f_1|, |f_2| \leqslant |f|$. Thus, $\int_E f_1$ and $\int_E f_2$ are finite, so $\int_E f = \int_E f_1 + if_2$ is also finite.

Following the hint, chose α such that

$$\left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2} = \cos(\alpha)\int_E f_1 + \sin(\alpha)\int_E f_2$$

Then

$$\begin{aligned}\left|\int_E f\right| = \left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2} &= \cos(\alpha)\int_E f_1 + \sin(\alpha)\int_E f_2 \\ &= \int_E (f_1\cos(\alpha) + f_2\sin(\alpha)) \leqslant \int_E |f_1\cos(\alpha) + f_2\sin(\alpha)| \\ &\leqslant \int_E \sqrt{f_1^2 + f_2^2} = \int_E |f|\end{aligned}$$

□

Exercise 4. *p.146*

Proof. Observe that Hölder's inequality comes from Young's inequality. With the observation the **equality** of $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$(Young's inequality) hold if and only if $a^{p-1} = b$ and hence if and only if $a^p = b^q$, where p, q are conjugate exponents.

Inparticular, let $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$, then integrating both side of Young's inequality implies the Hölder's inequality. It follows that the equality of Hölder's inequality hold if and only if $a^p = b^q$ and $|f||g| = |fg|$ *a.e.* if and only if $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$ and fg has constant sign *a.e.*

Hence, the equality of Hölder's inequality hold if and only if fg has constant sign and $|f|^p = c \cdot |g|^q$ *a.e.*, where $c = \frac{\|f\|_p^p}{\|g\|_q^q}$.

Observe that the Minkowski's inequality comes from $|f+g| \leqslant |f| + |g|$ and Hölder's inequality for $|f|, |f+g|^{p-1}$ and $|g|, |f+g|^{p-1}$.

By the result of previous discussion, we quickly have fg has constant sign *a.e.* and $c_1|f|^p = c_2|g|^p = |f+g|^p$ *a.e.* Hence, the equality of Minkowski's inequality hold if and only if $fg \geqslant 0$ and $|f|^p = c \cdot |g|^p$ *a.e.*, where $c = c_2/c_1$. □

Exercise 6. *p.146*

Proof. We prove this by induction. The $k = 2$ case is a consequence of Hölder's inequality: if $1/p_1 + 1/p_2 = 1/r$, then $r/p_1 + r/p_2 = 1$, so

$$\|fg\|_r^r = \|f^r g^r\|_1 \leqslant \|f^r\|_{p_1/r}\|g^r\|_{p_2/r} = \|f\|_{p_1}^r \|g\|_{p_2}^r.$$

Now if $1/p_1 + \cdots + 1/p_k = 1/r$ for $k \geqslant 2$, we have

$$\|f_1 \cdots f_k\|_r \leqslant \|f_1 \cdots f_{k-1}\|_s \|f_k\|_{p_k} \leqslant \|f_1\|_{p_1} \cdots \|p_k\|_{p_k},$$

where $1/s = 1/r - 1/p_k = 1/p_1 + \cdots + 1/p_{k-1}$. □