國立清華大學數學系應用數學組

碩士論文

一般化Radon變換最佳化函數的擬最佳方法

A Quasi-Extreme Method for An Extremizer of a Radon-like Transform

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摘要

本文考慮一種特殊的算子$\mathbf{T}: L_p \to L_q$。我們透過理解它的擬最佳函數的性質來尋找最佳化函數。這類算子是一般化Radon變換的基本的例子。我們首先引進了重要的幾何物件---- 拋物球，使得稍後的估計更為簡易。接著，這些性質幫助我們處理不同型的擬最佳化函數。最後，一個好的擬最佳化序列將會趨近最佳化函數。

Abstract

We study the quasi-extremizers of a specified linear operator $\mathbf{T} : L_p \to L_q$ in order to find an extremizer. This operator is a basic example of the generalized Radon transforms. We first includes some properties of paraballs for a better situation of approximation. Then these properties help us to deal with different type of quasi-extremizers. Finally, a good quasi-extremizing sequence appraoches to an extremizer, by its inner product form.

Acknowledgements

I would express the deepest appreciation to my advisor, Professor. Jin-Cheng Jiang. This is a best chance for self-training to study a series of the newest topics in analysis. Although there're many difficuties in this year, due to Prof. Jiang's inspiration, I would try and conquer many of the problems. At the same time, I got acquainted with these concepts, objects, definitions, and main idea in the paper I read. Besides, I realized the power of discussion, which is proven again to be able to gain new idea, and even to reach a juncture to break through a bottleneck, although it is not immediate.

There is a period of me in which I felt upset when doing mathematics. It's like I'm walking in a labyrinth, but never feel hope to find an exit. I lose I myself in all textbooks of mathematics—— modern algebra, measure and integrations, functional analysis, complex analysis, Calculus, Senior high mathematics, and Junior high mathematics. I jumped into the history of any topic I doubted that time, and find out any connection about them because I wondered how the topics I studied these years become what I looked that time. Many books, articles, courses helped me, inclusive of Prof. Jiang's Harmonic Analysis.

It is unreal for me to have come back to pure mathematics, and to have roughly touched Prof. Christ's almost newest works. I thank to Prof. Jiang's patient on every-week meeting, for details of a proof or repair of any deep gap of [1]. I hope the method in "On Extremals of Radon-like Transforms" would help in further studies.

Contents

0 Introduction

The report is based on the work of Professor M. Christ. Our goal of obtaining an extemizer divids into several parts. Section 1 describes its nature, i.e. $\mathbf{T}(f) := f * \sigma$, the convolution with a measure on a $d-1$ dimensional paraboloid. Also, we introduce the known property of boundedness.

Section 2 deals with elementary geometric objects that arise from the measure σ. The group $\mathcal{G}_d$ of transitive actions and paraballs are the most important. They form the geometric part here.

Section 3 lists several known properties, inclusive of Lorentz space and related properties, and some critical estimation properties. Many constants (quantities only depend on d) emerge and change here, and are frequently used in section 6,7,8.

Section 4 supports Lemma 7.1. It is an estimate of the form $\langle \chi_F, \mathbf{T}_B \rangle$ where F is measurable, and B is a paraball and is supported by Section 2.

Section 5 gives an approximation of $\|\mathbf{T}f\|_q$. We can also control $\|\tilde{f} - f\|_p$, where $\tilde{f}$ is a part of f and is to be determined here.

Section 6 gives a distribution about the level decomposition. In fact, it gives a rough appearance about an extremizer. Proposition 6.1 can be regarded as the main rule of this work.

Section 7 combines results of Section 4 and Section 6. It forms a clearer picture about a quasi-extremizer by a connection to baraballs.

Section 8 uses the connection above so that the form of the optimal problem is changed from norms into inner products. Finally, functional analysis and harmonic analysis is applied to obtain its extremizer.

There is a feature in the series of papers from Prof. Christ. The theorems assert existence of some constants C, c, A which depend only on dimension d, so that some properties of a function is estimated with respect to a small quantity operated with these constants. Constants may differ in different theorem, so I hope to mark C_{52} to mean the constant for Lemma 5.2. for example.

We think this paper ([1]) has some ambiguity in its own conditions. So we strengthen them—— related objects has to be **good** enough, then an extremizer exists with conditions.

1 Settings and Definitions

In this thesis, we have basic settings:

- Let $d \in \mathbb{Z}_{\geq 2}$, $p = \frac{d+1}{d}$, $q = d+1$. Then $1 < p \leq 2 \leq q < \infty$.
- Let $\mathbf{T}: L_p(\mathbb{R}^d) \to L_q(\mathbb{R}^d)$, with explicit expression:

$$\mathbf{T}f(x) = \int_{\mathbb{R}^{d-1}} f(x' - t', x_d - |t'|^2)\mathrm{d}t',$$

  where $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

Then we know that $\mathbf{T}$, p, q depends only on d. This is the convolution of a function with the measure on a paraboloid, and in fact, there're many things to do with such a type of transforms, as in [1],[2].

Remark. $\mathbf{T}f \geq 0$ in $\mathbb{R}^d$ if $f \geq 0$.

Theorem 1.1 (The Operator inequality, sharpened)**.** $\|\boldsymbol{T}(f)\|_q \leq \boldsymbol{A}\|f\|_p$.

A proof requires methods for FIO to show boundedness, and then $\mathbf{A}$ is defined as $\sup_{\|f\|_p=1} \|\mathbf{T}f\|_q$.

Remark. The operator inequality has corresponding inner product form: $\langle \mathbf{T}f, g\rangle \leq \mathbf{A}\|f\|_p\|g\|_p$. It's easily proved by Hölder's inequality and operator inequality.

Definition 1.1. *(1) By an extremizer we mean an $f \in L_p(\mathbb{R}^d)$ so that $\|\boldsymbol{T}f\|_q = \boldsymbol{A}\|f\|_p$. (2)An $(1-\delta)$-quasiextremizer is a function $f \in L_p(\mathbb{R}^d)$ so that $\|\boldsymbol{T}f\|_q \geq (1-\delta)\boldsymbol{A}\|f\|_p$. (3) An extremizing sequence: $\|f_\nu\|_p = 1$ so that $\|\boldsymbol{T}f_\nu\|_q \to \boldsymbol{A}$. Hence an extremizing sequence exists.*

Remark. Technically, $f \in L_p(\mathbb{R}^d)$ is often assumed $\|f\|_p = 1$.

2 Geometric Part

The integrand above leads to a consideration of geometric objects

- A group $\mathcal{G}_d$ of diffeomorphisms ;
- Paraballs $B = \mathbf{B}(z, \mathbf{e}, \mathbf{r}, \rho)$.

2.1 The Symmetric Group $\mathcal{G}_d$

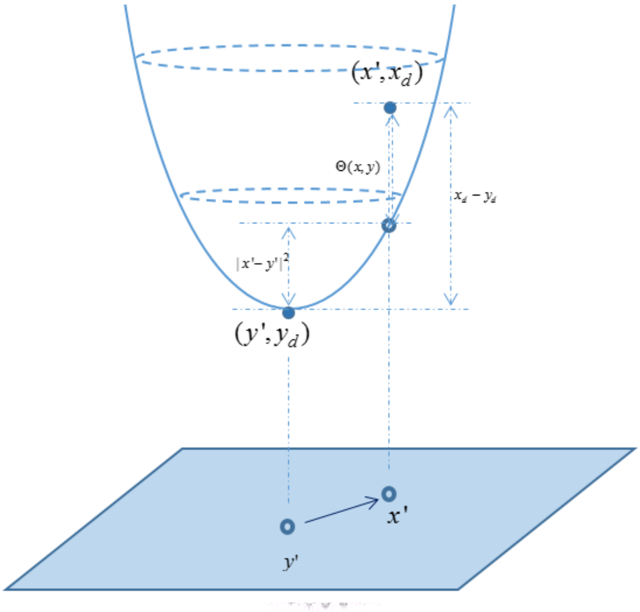
From now on, points in $\mathbb{R}^d$ are expressed by $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

Definition 2.1. *Let $x = (x', x_d), y = (y', y_d) \in \mathbb{R}^d$. We defind the* vertical difference *from y to x by*

$$\Theta(x, y) = x_d - y_d - |x' - y'|^2.$$

We also denote $\mathcal{I} = \{(x, y) \in \mathbb{R}^{d+d} \,|\, \Theta(x, y) = 0\}$.

The following is an illustrating graph in case $d = 3$.



Let Diff $(\mathbb{R}^d)$ denotes the set of all diffeomorphisms on $\mathbb{R}^d$. Let $\tau(x) = -x$.

Definition 2.2 ($\mathcal{G}_{d,d}$ and $\mathcal{G}_d$)**.** *Define $\mathcal{G}_{d,d} = \{(\phi, \psi) \in \text{Diff } \mathbb{R}^d \times \text{Diff } \mathbb{R}^d \;:\; \exists \lambda \ni \forall x, y \in \mathbb{R}^{2d},\ \Theta(\phi(x), \psi(y)) = \lambda \Theta(x, y)\}$, and $\mathcal{G}_d$ collects their first components, namely, $\mathcal{G}_d = \{\phi \in \text{Diff } \mathbb{R}^d \;:\; \exists \phi \in \text{Diff } \mathbb{R}^d \ni (\phi, \psi) \in \mathcal{G}_{d,d}\}$.*

Proposition 2.1. *If $(\phi, \psi) \in \mathcal{G}_{d,d}$, then $(\tau\psi\tau, \tau\phi\tau) \in \mathcal{G}_{d,d}$.*

Proof. By computation:

$$\begin{aligned}
\Theta(\tau\psi\tau(x), \tau\phi\tau(y)) &= \Theta(\tau\psi(-x), \tau\phi(-y)) = \Theta\left(\tau[G(-x), g(-x)], \tau[F(-y), f(-y)]\right) \\
&= \Theta(-G(-x), -g(-x);\ -F(-y), -f(-y)) \\
&= f(-y) - g(-x) - |F(-y) - G(-x)|^2 \\
&= \Theta(F(-y), f(-y);\ G(-x), g(-x)) \\
&= \Theta(\phi\tau(y), \psi\tau(x)) = \lambda\Theta(-y, -x) = \lambda[x_d - y_d - |x' - y'|^2] \\
&= \lambda\Theta(x, y).
\end{aligned}$$

Thus, $(\tau\psi\tau, \tau\phi\tau) \in \mathcal{G}_{d,d}$. $\square$

Proposition 2.2. *$\mathcal{G}_d = \{\phi : (x', x_d) \mapsto (Ax' + u, ax_d + Q(x')) \mid A \in GL_{d-1}, a \neq 0, Q$ a quadratic form on $x'\}$.*

Proof. Let $(\phi, \psi) \in \mathcal{G}_{d,d}$, expressed by $\begin{cases} \phi(x) = (F(x), f(x)) \\ \psi(x) = (G(x), g(x)) \end{cases}$, with $F, G : \mathbb{R}^d \to \mathbb{R}^{d-1}$ and $f, g : \mathbb{R}^d \to \mathbb{R}$. Then for all $x = (x', x_d), y = (y', y_d)$,

$$f(x) - g(x) - \langle F(x) - G(y), F(x) - G(y) \rangle = \lambda(x_d - y_d - \langle x' - y', x' - y' \rangle).$$

We obtain $\frac{\partial}{\partial y_j} g(y) - 2\langle G(y), \frac{\partial}{\partial y_j} G(y) \rangle + 2\langle F(x), \frac{\partial}{\partial y_j} G(y) = \lambda \begin{cases} 2\langle x' - y', e_j \rangle & \text{if } j \neq d \\ -1 & \text{if } j = d \end{cases}$ by taking the j-th partial derivative on y. Then take i-th partial derivative on x to obtain $2\langle \frac{\partial}{\partial x_i} F(x), \frac{\partial}{\partial y_j} G(y) \rangle = \begin{cases} 2\langle e_i, e_j \rangle = 2\delta_{ij} & \text{if } j \neq d \\ 0 & \text{if } j = d \end{cases}$, independent of x, y.

We know $\frac{\partial}{\partial x_i} F(x) \neq 0$ for all $x \in \mathbb{R}^d$ and all $i = 1, 2, ..., d-1$, for otherwise it would contradict to the last sentence of the last paragraph. Thus

$$\begin{pmatrix} \frac{\partial}{\partial x_1} F(x) \\ \vdots \\ \frac{\partial}{\partial x_{d-1}} F(x) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1} G(y) & \cdots & \frac{\partial}{\partial y_{d-1}} G(y) \end{pmatrix} = I.$$

Write above identity as $AB = I$. Take $\frac{\partial}{\partial y_\ell}$, $A\partial_{y_\ell} B = 0$. Multiply A^{-1}, B is a constant matrix. Also, $A\partial_{y_d} G(y) = 0$, so $\partial_{y_d} G(y) = 0$ and G is independent of y_d. Thus $\nabla_{y'} G(y) =$ constant and then $F(x) = Ax' + u$ for some A, u. Similarly for some M, v, $G(y) = My' + v$. Moreover, still from above, Fix i, we know $\langle \partial_{x_i} F(x), \partial_{y_j} G(y) \rangle = \delta_{ij}$. $\langle \partial_{x_i} F(x), G(y) \rangle$ is a linear function of y_i, say $ay_i + b$. So $\langle F(x), G(y) \rangle = \lambda \langle x', y' \rangle$ plus an affine function of x', y'. Hence $M = \lambda B = \lambda A^{-1}$ for some B.

Now $f(x) - g(x)$ equals $\lambda x_d - \lambda y_d$ plus a quadratic polynomial in x', y'. So f, g are quadratic polynomials with no $x_d x_j$ term with $1 \leq j \leq d$. Thus $f = ax_d + Q(x')$ with $a \neq 0$. $\square$

2.2 The conjugation ϕ^*

The following property of "changing variable" holds by proposition 2.2, and advanced calculus.

Proposition 2.3. *For each $\phi \in \mathcal{G}_d$, the Jacobian determinant $J_\phi : \mathbb{R}^d \to (0, \infty)$ is constant. For this ϕ, the corresponding function $\phi^* : L^p \to L^p$, $f(x) \mapsto f(\phi(x))J_\phi^{1/p}$ satisfies for all $f, g \in L^p$,*

$$\|\phi^* f\|_p = \|f\|_p,$$
$$\langle \psi^* g, \boldsymbol{T}(\phi^* f)\rangle = \langle g, \boldsymbol{T} f\rangle$$

2.3 Paraballs

Definition 2.3 (A Paraball). *Let $\boldsymbol{r} = (r_1, r_2, ..., r_d) \in (\mathbb{R}^+)^{d-1}$ with $r_1 > r_2 > ... > r_{d-1} > 0$, and $\rho > 0$. Let $\boldsymbol{e} = \{e_1, ..., e_{d-1}\}$ be an orthonormal basis for $\mathbb{R}^{d-1}$ and $z = (\bar{x}, \bar{x}_\star) \in \mathcal{I}$. A paraball $B = \boldsymbol{B}(z, \boldsymbol{e}, \boldsymbol{r}, \rho)$ is the set of all points $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ so that*

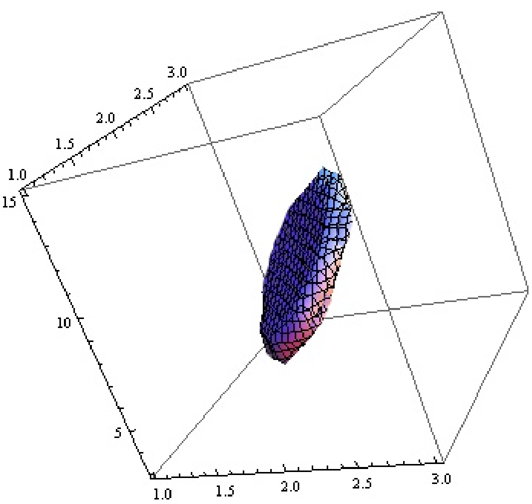
$$\sum_{j=1}^{d-1} \frac{|\langle x' - \bar{x}', e_j\rangle|}{r_j^2} < 1 \tag{1}$$

$$|x_d - \bar{x}_{\star d} - |x' - \bar{x}'_\star|^2| < \rho. \tag{2}$$

Here $\bar{x} = (\bar{x}', \bar{x}_d)$, $\bar{x}_\star = (\bar{x}'_\star, \bar{x}_{\star d}) \in \mathbb{R}^d$. At this time, we say the paraball is centered at $(\bar{x}, \bar{x}_*)$.

We can rewrite the $(d-1)$-ellipse (1) into the form $\|D_0(O_0(\bar{x} - \bar{x}'))\|^2$. Here D_0 is diagonal and O_0 is orthogonal. So there's an alternative expression for a paraball $\boldsymbol{B}(z, \mathbf{e}, \mathbf{r}, \rho) \leftrightarrow \mathcal{B}(z, D_0, O_0, \rho)$, namely, $\mathfrak{B}(z, D_0, O_0, \rho) := \boldsymbol{B}\Big(z, (O_0^{-1}(1,0,...,0)^T, O_0^{-1}(0,1,...,0)^T, ..., O_0^{-1}(0,0,...,1)^T), (\frac{1}{(1,0,...,0)D_0(1,0,...,0)^T}, \frac{1}{(0,1,...,0)D_0(0,1,...,0)^T}, ..., \frac{1}{(0,0,...,1)D_0(0,0,...,1)^T}), \rho\Big)$. Here $\cdot^T$ denotes matrix transpose.

This gives a graph of a paraball for $d = 3$:



Proposition 2.4. *Let B be a paraball centered at $z = (\bar{x}, \bar{x}_*)$ and $\phi \in \mathcal{G}_d$ with conjugate ψ. Then ϕB is also a paraball, and is centered at $(\phi\bar{x}, \psi\bar{x}_*)$.*

Proof. By above discussion, $\phi(x) = \phi(x', x_d) = v + (Lx', \lambda x_d + Qx') = (v', v_d) + (Lx', \lambda x_d + Qx')$, where $L \in G\ell(d-1)$, $\lambda \neq 0$, Q a quadratic form for x'.

Let $x = (x', x_d) \in \boldsymbol{B}(z, \mathbf{e}, \mathbf{r}, \rho)$.

Of course $\|D_0 O_0(\bar{x} - \bar{x}')\|^2 < 1 \Leftrightarrow \|D_0 O_0 L^{-1} L(\bar{x} - \bar{x}')\|^2 < 1$. Now $L_1 = D_0 O_0 L^{-1} \in \mathrm{G}\ell(d-1)$, so $L_1^* L_1 = O^{-1} D O$ for some orthogonal O and diagonal D with $\det D > 0$. By linear algebra this implies $\|\sqrt{D} O(L\bar{x} - L\bar{x}')\|^2 < 1$.

The second part relies on (2). This is trivial by definition of $\mathcal{G}_{d,d}$. $\square$

For each paraball $\boldsymbol{B}((\bar{x}, \bar{x}_*), \mathbf{e}, \mathbf{r}, \rho)$, we define its base ellipse $\mathcal{E} = \bar{x}' + \mathcal{C} = \{x' \in \mathbb{R}^{d-1} \mid \sum_{j=1}^{d-1} \frac{|\langle x' - \bar{x}', e_j \rangle|}{r_j^2} < 1\}$ and its translated base $\mathcal{C} = \{x' \in \mathbb{R}^{d-1} \sum_{j=1}^{d-1} \frac{|\langle x', e_j \rangle|}{r_j^2} < 1\}$.

Remark. We give the following relationship:

$$\begin{array}{ccc} \mathbf{B}(z, e, r, \rho) & \leftrightarrow & \mathfrak{B}(z, D_0, O_0, \rho). \\ \downarrow \text{Take } \phi & & \downarrow \text{Take } \phi \\ \mathbf{B}(w, e_1, r_1, \lambda\rho) & \leftrightarrow & \mathfrak{B}(w', \sqrt{D}, O, \lambda\rho) \end{array}$$

Here these O's are orthogonal matrices, and D's are diagonal.

A paraball is **good** if $r_1/r_{d-1} < c_0$. Here c_0 is fixed, depending only on d.

2.4 Dual Paraballs

For a paraball $B = \mathbf{B}(z, \mathbf{e}, \mathbf{r}, \rho)$, its dual paraball $B_\star = \mathbf{B}_\star(z, \mathbf{e}, \mathbf{r}, \rho)$ is given by the set of all $(x', x_d) \in \mathbb{R}^d$ so that

$$\sum_{j=1}^{d-1} \frac{|\langle x' - \bar{x}_\star{}', e_j \rangle|}{(\rho/r_j)^2} < 1 \tag{3}$$

$$|x_d - \bar{x}_d - |x' - \bar{x}'|^2| < \rho. \tag{4}$$

Moreover, $\mathcal{B}(z, e, r, \rho) := (B(z, e, r, \rho), B_\star(z, e, r, \rho))$.

2.5 Quasi-Distance

This sub-section gives a description about the separation/distance of given two paraballs, which is composed of several information:

- The bigger thickness ;
- The sum of the proportions of stretch/shrink from $\mathcal{C}^\sharp$ to cover $\mathcal{C}^\flat$ and reverse;
- The sum of the "distance"-squares of of center-differences with respect to $\mathcal{C}^\sharp$ and $\mathcal{C}^\flat$;
- The last term for dual paraballs (to measure the separation for verteces of paraboloids.);
- The sum of the proportions of the Θ-value from paraball center of $B^\sharp$ to the paraboloid of $B^\flat$ and $\rho^\flat$ and reverse.

That is,

Definition 2.4. *Let $B^\sharp = \boldsymbol{B}((\bar{x}^\sharp, \bar{x}^\sharp_*), e^\sharp, r^\sharp, \rho^\sharp)$ and $B^\flat = \boldsymbol{B}((\bar{x}^\flat, \bar{x}^\flat_*), e^\flat, r^\flat, \rho^\flat)$. We define*

$$\begin{aligned}
\varrho(B^\sharp, B^\flat) = & \frac{\max\{\rho^\sharp, \rho^\flat\}}{\min\{\rho^\sharp, \rho^\flat\}} + \sup_{v \in C^\sharp} \sum_{j=1}^{d-1} {r_j^\flat}^{-2} |\langle v, e_j^\flat \rangle|^2 + \sup_{v \in C^\flat} \sum_{j=1}^{d-1} {r_j^\sharp}^{-2} |\langle v, e_j^\sharp \rangle|^2 \\
& + \sum_{j=1}^{d-1} {r_j^\flat}^{-2} |\langle \bar{x}^{\sharp\prime} - \bar{x}^{\flat\prime}, e_j^\flat \rangle|^2 + \sum_{j=1}^{d-1} {r_j^\sharp}^{-2} |\langle \bar{x}^{\flat\prime} - \bar{x}^{\sharp\prime}, e_j^\sharp \rangle|^2 \\
& + \sum_{j=1}^{d-1} (\rho^\flat / r_j^\flat)^{-2} |\langle \bar{x}_*^{\sharp\prime} - \bar{x}_*^{\flat\prime}, e_j^\flat \rangle|^2 + \sum_{j=1}^{d-1} (\rho^\sharp / r_j^\sharp)^{-2} |\langle \bar{x}_*^{\flat\prime} - \bar{x}_*^{\sharp\prime}, e_j^\sharp \rangle|^2 \\
& + \frac{\left| \bar{x}^\sharp_d - \bar{x}^\flat_{*\,d} - |\bar{x}^{\sharp\,\prime} - \bar{x}^{\flat}_*{}'| \right|}{\rho^\flat} + \frac{\left| \bar{x}^\flat_d - \bar{x}^\sharp_{*\,d} - |\bar{x}^{\flat\,\prime} - \bar{x}^{\sharp}_*{}'| \right|}{\rho^\sharp}
\end{aligned}$$

It is understood as

$$\begin{aligned}
\varrho(B^\sharp, B^\flat) = & \frac{\max\{\rho^\sharp, \rho^\flat\}}{\min\{\rho^\sharp, \rho^\flat\}} + \sup_{v \in C^\sharp} |D_0^\flat O_0^\flat(v)|^2 + \sup_{v \in C^\flat} |D_0^\sharp O_0^\sharp(v)|^2 \\
& + |D_0^\sharp O_0^\sharp(\bar{x}^{\flat\,\prime} - \bar{x}^{\sharp\,\prime})|^2 + |D_0^\flat O_0^\flat(\bar{x}^{\sharp\,\prime} - \bar{x}^{\flat\,\prime})|^2 \\
& + |D_0^{\star\,\sharp} O_0^\sharp(\bar{x}_*^{\flat\,\prime} - \bar{x}_*^{\sharp\,\prime})|^2 + |D_0^{\star\,\flat} O_0^\flat(\bar{x}_*^{\sharp\,\prime} - \bar{x}_*^{\flat\,\prime})|^2 \\
& + \frac{\Theta(\bar{x}^\sharp, \bar{x}^\flat_*)}{\rho^\flat} + \frac{\Theta(\bar{x}^\flat, \bar{x}^\sharp_*)}{\rho^\sharp}
\end{aligned}$$

if we write $(B^\sharp, B^\flat) = (\mathcal{B}((\bar{x}^\sharp, \bar{x}^\sharp_*), D_0^\sharp, O_0^\sharp, \rho^\sharp), \mathcal{B}((\bar{x}^\flat, \bar{x}^\flat_*), D_0^\flat, O_0^\flat, \rho^\flat))$ and is convenient to give a proof of some properties:

Proposition 2.5. *Let $\phi \in \mathcal{G}_d$. Let $B^\sharp, B^\flat$ be paraballs with dual paraballs $B^\sharp_\star, B^\flat_\star$ respectively. Then*

1. $\varrho(\phi B^\sharp, \phi B^\flat) = \varrho(B^\sharp, B^\flat)$.

2. $\varrho(B^\sharp, B^\flat) = \varrho(B^\sharp_\star, \phi B^\flat_\star)$.

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Given a pair of good paraballs $\boldsymbol{B}((\bar{x}, \bar{x}_*), \mathbf{e}, \mathbf{r}, \rho)$, with translated bases $\mathcal{C}^\sharp, \mathcal{C}^\flat$.

Definition 2.5. *We call this pair* ***good*** *if* $\langle e^\sharp_1, e^\flat_{d-1}\rangle > c_0$ *and* $\langle e^\flat_1, e^\sharp_{d-1}\rangle > c_0$ *and* $\sum_{j=1}^{d-1} \frac{|\langle \bar{x}^{\sharp\,\prime} - \bar{x}^{\flat\,\prime}, e^\sharp_j\rangle|}{r^{\sharp\,2}_j} + \sum_{j=1}^{d-1} \frac{|\langle \bar{x}^{\sharp\,\prime} - \bar{x}^{\flat\,\prime}, e^\flat_j\rangle|}{r^{\flat\,2}_j} \leq C \frac{\max(|\mathcal{C}^\sharp|, |\mathcal{C}^\flat|)}{|(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)|}$. *Here* c_0, C *depends only on* d.

Lemma 2.1 ($C\varepsilon^C$ Estimation for Convex $\mathcal{C}$)**.** *$\exists 0 < C, c < \infty$, depending only on d, with the property: Let $\mathcal{C}$ be a $(d-1)$ dimensional ellipse, and Q a quadratic polynomail so that $Q(x') = 0$ represents an ellipse or a hyper-plane. Let $\varepsilon > 0$. Then*

$$|\{y \in \mathcal{C} \ : \ |Q(y)| < \varepsilon \sup_{\mathcal{C}} |Q|\}| \leq C\varepsilon^c |\mathcal{C}|.$$

Proposition 2.6. *Let $B^\sharp, B^\flat$ be a good pair of good paraballs. Then $\varrho = \varrho(B^\sharp, B^\flat) \leq C\left(\frac{\max\{|B^\sharp|, |B^\flat|\}}{|B^\sharp \cap B^\flat|}\right)^C$.*

Proof. Case I. If $\frac{\max\{\rho^\sharp, \rho^\flat\}}{\min\{\rho^\sharp, \rho^\flat\}} \geq \frac{1}{7}\varrho(B^\sharp, B^\flat)$ or $\frac{\max\{\rho^\sharp, \rho^\flat\}}{\min\{\rho^\sharp, \rho^\flat\}} \geq \sqrt{\varrho(B^\sharp, B^\flat)}$, then by definition of paraballs, heights, and tranlated bases, $|B^\sharp \cap B^\flat| \leq \rho^\sharp |(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)|$ and $|B^\sharp \cap B^\flat| \leq \rho^\flat |(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)|$. Moreover, $|(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)| \leq \{|\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp|, |\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat|\} = \{\frac{|B^\sharp|}{\rho^\sharp}, \frac{|B^\flat|}{\rho^\flat}\} \leq \{\frac{1}{\rho^\sharp}, \frac{1}{\rho^\flat}\} \cdot \max\{|B^\sharp|, |B^\flat|\}$. Thus

$$\begin{aligned} |B^\sharp \cap B^\flat| &\leq \min\{\rho^\sharp, \rho^\flat\} |(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)| \\ &\leq \frac{\min\{\rho^\sharp, \rho^\flat\}}{\max\{\rho^\sharp, \rho^\flat\}} \max\{|B^\sharp|, |B^\flat|\}. \end{aligned}$$

We obtain $\varrho(B^\sharp, B^\flat) \leq 7\left(\frac{\max\{|B^\sharp|, |B^\flat|\}}{|B^\sharp \cap B^\flat|}\right)^1$ or $\leq 1 \cdot \left(\frac{\max\{|B^\sharp|, |B^\flat|\}}{|B^\sharp \cap B^\flat|}\right)^2$.

Case II. If $\sup_{v \in \mathcal{C}^\sharp} \sum_{j=1}^{d-1} \frac{|\langle v, e^\flat_j\rangle|^2}{r^{\flat\,2}_j} \geq \frac{1}{7}\varrho(B^\sharp, B^\flat)$ and it's similar in case $\sup_{v \in \mathcal{C}^\flat} \sum_{j=1}^{d-1} \frac{|\langle v, e^\sharp_j\rangle|^2}{r^{\sharp\,2}_j} \geq \frac{1}{7}\varrho(B^\sharp, B^\flat)$, and w.l.o.g assume $|\mathcal{C}^\sharp| > |\mathcal{C}^\flat|$, then

$$\frac{|\mathcal{C}^\sharp|}{|\mathcal{C}^\flat|} = \frac{r^\sharp_1 \cdots r^\sharp_{d-1}}{r^\flat_1 \cdots r^\flat_{d-1}} \geq \frac{r^{\sharp\;d-1}_{d-1}}{r^{\flat\;d-1}_1} \geq c\left(\frac{r^\sharp_1}{r^{\flat\;d-1}_{d-1}}\right)^{d-1} = c'\left(\sup_{v \in \mathcal{C}^\flat} \sum_{j=1}^{d-1} \frac{|\langle v, e^\sharp_j\rangle|^2}{r^{\sharp\,2}_j}\right)^{d-1}.$$

On the other hand,

$$|B^\sharp \cap B^\flat| \leq |(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)| \cdot \{\rho^\sharp, \rho^\flat\} = |(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)| \{\frac{|B^\sharp|}{|\mathcal{C}^\sharp|}, \frac{|B^\flat|}{|\mathcal{C}^\flat|}\}$$

$$\leq \frac{|(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)|}{\max\{|\mathcal{C}^\sharp|, |\mathcal{C}^\flat|\}} \{|B^\sharp|, |B^\flat|\} \leq \frac{\min\{|\mathcal{C}^\sharp|, |\mathcal{C}^\flat|\}}{\max\{|\mathcal{C}^\sharp|, |\mathcal{C}^\flat|\}} \max\{|B^\sharp|, |B^\flat|\}.$$

Thus $\varrho(B^\sharp, B^\flat) \leq c'' \left(\frac{\max\{|B^\sharp|, |B^\flat|\}}{|B^\sharp \cap B^\flat|} \right)^{1/(d-1)}$.

Case III. In case the 5^{th} term is the largest and similar for the 6^{th} term, our additional condition for a good pair and the discussion in Case II imply $\varrho(B^\sharp, B^\flat) \leq 9c \left(\frac{\max\{|B^\sharp|, |B^\flat|\}}{|B^\sharp \cap B^\flat|} \right)^C$

Case IV. Suppose that $|\langle \bar{x}_*^{\sharp\,\prime} - \bar{x}_*^{\flat\,\prime}, e_i^\sharp \rangle| \geq c\varrho \cdot \rho^\sharp / r_i$ for some i(We may assume the biggest term). Denote for convenience that $u = \bar{x}_*^{\sharp\,\prime}$ and $v = \bar{x}_*^{\flat\,\prime}$. If $y = (w, t) \in \mathbb{R}^{d-1} \times \mathbb{R}$ belongs to $B^\sharp \cap B^\flat$ then by inequality (2) twice we obtain $\begin{cases} |t - \bar{x}_{*\,d}^\sharp - |w-u|^2| < \rho^\sharp \\ |t - \bar{x}_{*\,d}^\flat - |w-v|^2| < \rho^\flat \end{cases}$. Subtracting the second one by first one one can write down the following

$$|2\langle w, u - v \rangle - s| < 2 \max\{\rho^\sharp, \rho^\flat\}$$

where $s = 2|u|^2 - 2u \cdot v$. Note that

- The i-th component of $u - v$ has absolute value $\geq c\varrho \cdot \rho^\sharp / r_i$.
- We set $w \in \bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp$.
- we then have the sole constraint $\langle w, e_i^\sharp \rangle < r_i$, $\forall w \in \bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp$.

We want to apply Lemma 2.1. Thus

$$|\{w \in \bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp \ : \ |2\langle w, u - v \rangle - s| < 2 \max\{\rho^\sharp, \rho^\flat\}\}| \leq C\varrho^{-1} |C^\sharp|.$$

This holds for all $s \in \mathbb{R}$. Copy (5) in the next case, we reach a bound.

Case V. Suppose that above cases all fail and the eighth term $> \frac{1}{2}\varrho$. Let $Q^\sharp(y) = \Theta^\sharp(y) = y_d - \bar{x}_{\star\,d}^\sharp - |y'^2 - \bar{x}_\star^{\sharp\,\prime}|^2$ and $Q^\flat(y) = \Theta^\flat(y) = y_d - \bar{x}_{\star\,d}^\flat - |y'^2 - \bar{x}_\star^{\flat\,\prime}|^2$. So $\max / \min \leq \sqrt{\varrho(B^\sharp, B^\flat)}$ and if ϱ is large then $\bar{x}^\flat \notin B^\sharp$ by definition.

Let $P(z) = Q^\sharp(z, t(z))$, where $t(z)$ is chosen so that $Q^\flat(z, t(z)) \equiv 0$. In fact, $t(z) = \bar{x}_{\star\,d}^\flat + |z - \bar{x}_\star^{\flat\,\prime}|^2$. Then $P : \mathbb{R}^d \to \mathbb{R}$ is a polynomial. Note that $|P(z)| > \rho^\sharp + \rho^\flat$. This implies $B^\sharp \cap B^\flat \cap (\{z\} \times \mathbb{R}) = \varnothing$.

We want to apply Lemma 2.1. Set $\varepsilon = \frac{6}{\varrho}\frac{\max}{\rho^\sharp} \le \frac{6}{\varrho}\sqrt{\varrho} = \frac{6}{\sqrt{\varrho}}$. Let $A = \{z \in \mathcal{C}^\flat : |P(z)| \ge \varepsilon \sup_{\mathcal{C}^\flat}|P|\}$ and $A' = \{z \in \mathcal{C}^\flat : |P(z)| > \rho^\flat + \rho^\sharp\}$. Thus $|A^{\mathrm{c}}| \le C\varepsilon^C|\mathcal{C}^\flat|$ and $A \subset \{z \in \mathcal{C}^\flat : |P(z)| \ge \varepsilon \cdot \frac{1}{2}\varrho\rho^\sharp\} \subset \{z \in \mathcal{C}^\flat : 3\max\} \subset A' \subset \big((\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)\big)^{\mathrm{c}}$.

$$
\begin{aligned}
|B^\sharp \cap B^\flat| &\le \underbrace{|(\bar{x}^{\sharp\,\prime} + \mathcal{C}^\sharp) \cap (\bar{x}^{\flat\,\prime} + \mathcal{C}^\flat)|}_{base} \cdot \underbrace{\min(\rho^\sharp, \rho^\flat)}_{height} \\
&\le |A^{\mathrm{c}}|\rho^\flat \le c(\varepsilon/\sqrt{\varrho})^c|\mathcal{C}^\flat|\rho^\flat \le \frac{C}{\varrho^c}|B^\flat| \le \frac{C}{\varrho^c}\max\{|B^\sharp|, |B^\flat|\}. \qquad (5)
\end{aligned}
$$

Summary. Each of case 1 to case 4 is related to a term and ϱ, while the last case assumes either case fails, and reach a bound. Thus it is end. $\square$

Proposition 2.7 (Quasi-Triangle Inequality)**.** *For any three good paraballs, each pair is also good, we have*

$$\varrho(B^\sharp, B^\flat) \le C\varrho(B^\sharp, B^\natural)^c + C\varrho(B^\natural, B^\flat)^c.$$

Sketch. Applying transitive action of $\mathcal{G}_d$, we may assume w.l.o.g. that $B^\natural = \mathbf{B}((0,0)$,standard basis of $\mathbb{R}^{d-1}, (1, ..., 1), 1)$. Mimicing the proof of Lemma 2.6 then control the parameters specifying $B^\sharp$ in terms of $\eta^{-1} := \varrho(B^\sharp, B^\natural)$. Then we can show $\varrho(B^\sharp, B^\flat) \le C\eta^{-c}$, and the case is similar for $B^\flat$. $\square$

3 Analytic Part

In this section, we introduce some usual methods and known properties.

Definition 3.1 (Quasi-norm for the Lorentz space $L^{p,r}(\mathbb{R}^d)$)**.** *A function $f \in L_p(\mathbb{R}^d)$ here is always non-negative and represented by levels: $f = \sum_{j\in\mathbb{Z}} 2^j f_j$, where $1 \le f_j < 2$, $E_j = \{2^j \le f \le 2^{j+1}\}$, and $f_j = \chi_{E_j} \cdot f/2^j$. Define*

$$\|f\|_{p,r} = (\sum_j (2^j|E_j|^{1/p})^r)^{1/r}.$$

Remark When $r = p$, the space $L^{p,r} = L^p$.

We have known from [1]:

Proposition 3.1. *There are $C, c, \gamma \in (0,\infty)$ depending only on d, with the property: Let $|E|, |E^*| \in (0,\infty)$, both measurable in $\mathbb{R}^d$. Let $\varepsilon = \frac{\langle \chi_{E^*}, \boldsymbol{T}\chi_E\rangle}{|E|^{1/p}|E^*|^{1/p}}$. Then* *there are a pair* $\mathcal{B} = (B, B^*)$ *of dual paraballs so that*

$$|B| \le |E| \qquad\qquad |E \cap B| \ge c\varepsilon^{\gamma}|E|$$
$$|B^*| \le |E^*| \qquad\qquad |E^* \cap B^*| \ge c\varepsilon^{\gamma}|E^*|.$$

Proposition 3.2. *There exist* C, C'' *such that:* *Let* $E_1, E_2, F \subset \mathbb{R}^d$, $|E_1|, |E_2|, |F| \in (0, \infty)$. *Let* $\eta \in (0, \boldsymbol{A}]$. *If* $\boldsymbol{T}\chi_{E_\nu}(x) \ge \eta |E_\nu|^{1/p}|F|^{-1+1/p}$, $\nu = 1, 2$, *for every* $x \in F$. *Then* $|E_2| \le C''\eta^{-C}|E_1|$.

Proposition 3.3.

Proposition 3.1 is applied to section 4, proposition 3.2 to Lemma 6.1, while proposition 3.3 is to Proposition 7.1.

4 Levels Estimate by Many Paraballs

Proposition 4.1. *Let* $\eta \in (0, 1]$ *and* $B_1, ..., B_N$ *be a good collection of paraballs not too close to each other, i.e.* $\varrho(B_\alpha, B_\beta) \ge C\eta^{-c}$ *for different* α, β. *Let* $|F| < \infty$, $\subset \mathbb{R}^d$. *Then there's a decomposition* $F = F_1 \cup ... \cup F_N$ *of* F *into measurable sets so that whenever* $\alpha \ne \beta$, $\langle \chi_{F_\beta}, \boldsymbol{T}\chi_{B_\alpha}\rangle$ *is relatively small, i.e.*

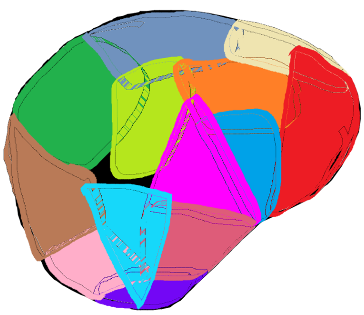
$$\langle \chi_{F_\beta}, \boldsymbol{T}\chi_{B_\alpha}\rangle \le \eta |F|^{1/p}|B_\alpha|^{1/p}. \tag{6}$$

Here C, c *depend only on* d.

Proof. Step 1. Define a small quantity $\gamma_\beta = \frac{1}{3}\eta|F|^{-1+1/p}|B_\beta|^{1/p}$ for $\mathbf{T}\chi_{B_\beta}$. Let $\tilde{F}_\beta = \{x \in F \mid \mathbf{T}\chi_{B_\beta} > \gamma_\beta\}$. Then

$$\langle \chi_{F\setminus\tilde{F}_\beta}, \mathbf{T}\chi_{B_\beta}\rangle \le \int_{F\setminus\tilde{F}_\beta} \gamma_\beta \le \gamma_\beta|F| = \frac{1}{3}\eta|F|^{1/p}|B_\beta|^{1/p}.$$

Choose pairwise disjoint measurable sets $F_\beta \subset \tilde{F}_\beta$ so that $\cup_\beta \tilde{F}_\beta = \cup_\beta F_\beta$.



The black part represents F, while the colored part represent F_β's, in which we know how did F_β's intersect. There may be some black part remained, meaning $F^\dagger := F \setminus \cup_\beta F_\beta$ might not be empty. However we have $\langle \chi_{F^\dagger}, \mathbf{T}\chi_{B_\alpha}\rangle \leq \frac{1}{3}\eta|F|^{1/p}|B_\beta|^{1/p}$ for all β. Thus it suffices to show

$$\langle \chi_{F_\beta}, \mathbf{T}\chi_{B_\alpha}\rangle \leq \frac{2}{3}\eta|F|^{1/p}|B_\alpha|^{1/p}. \tag{7}$$

Then the decomposition of F will be $F = (F^\dagger \cup F_1) \cup (\cup_{\beta\geq 2}F_\beta)$ and (6) holds.

Step 2. We prove (7) by contradiction. Suppose that $\alpha_0 \neq \beta_0$ are such that (7) fails to hold. Then we want to show $\varrho(B_{\alpha_0}, B_{\beta_0})$ is too small, contradicting the hypothesis.

The key point here is to apply 3.1. Let $\mathcal{F} = F_{\beta_0} \cap \tilde{F}_{\alpha_0}$. Then $\langle \chi_{F_{\beta_0}\setminus\tilde{F}_{\alpha_0}}, \mathbf{T}\chi_{B_{\alpha_0}}\rangle \leq \int_{F_{\beta_0}\setminus\tilde{F}_{\alpha_0}} \gamma_{\alpha_0} \leq \gamma_{\alpha_0}|F_{\beta_0}| = \frac{1}{3}\eta|F|^{1/p}|B_{\alpha_0}|^{1/p}$. So

$$\langle \chi_{\mathcal{F}}, \mathbf{T}\chi_{B_{\alpha_0}}\rangle = \langle(\chi_{F_{\beta_0}} - \chi_{F_{\beta_0}\setminus\tilde{F}_{\alpha_0}}), \mathbf{T}\chi_{B_{\alpha_0}} \geq \frac{1}{3}\eta|F|^{1/p}|B_{\alpha_0}|^{1/p}.$$

Moreover, we combine the operator inequality: $\langle \chi_{\mathcal{F}}, \mathbf{T}\chi_{B_{\alpha_0}}\rangle \leq \mathbf{A}|\mathcal{F}|^{1/p}|B_{\alpha_0}|^{1/p}$, then

$$|\mathcal{F}| \geq \left(\frac{\eta}{3\mathbf{A}}\right)^p |F|. \tag{8}$$

Let $E = B_{\alpha_0}, E^\star = \mathcal{F}$ in 3.1. Then $\varepsilon := \frac{\langle \chi_{\mathcal{F}}, \mathbf{T}\chi_{B_{\alpha_0}}\rangle}{|\mathcal{F}|^{1/p}|B_{\alpha_0}|^{1/p}} \geq \frac{1}{3}\eta$ and we obtain a dual pair $\mathcal{B}^{\alpha_0}(z_{\alpha_0}, e_{\alpha_0}, r_{\alpha_0}, \rho_{\alpha_0}) = (B^{\alpha_0}, B^{\alpha_0}_\star)$ so that

$$\begin{aligned}
&(1)\ |B^{\alpha_0}| \leq |B_{\alpha_0}| && (2)\ |B^{\alpha_0} \cap B_{\alpha_0}| \geq c\eta^\gamma|B_{\alpha_0}|,\\
&(3)\ |B^{\alpha_0}_\star| \leq |\mathcal{F}| \leq |F| && (4)\ |B^{\alpha_0}_\star \cap \mathcal{F}| \geq c\eta^\gamma|\mathcal{F}| \geq c\eta^\gamma|F|. \text{ (by (8))}
\end{aligned} \tag{9}$$

Here the independent constant c, γ is a little modified from 3.1.

Step 3. We're going to apply 3.1 again. Let $\tilde{\mathcal{F}} = \mathcal{F} \cap B^{\alpha_0}_\star$. Since $\forall x \in F_{\beta_0} \supset \mathcal{F} \supset \tilde{\mathcal{F}}$, we have $\mathbf{T}\chi_{B_{\beta_0}}(x) > \gamma_{\beta_0}$, so we mimic the above step and obtain $|\tilde{\mathcal{F}}| \geq c\eta^\gamma|F|$, and get another dual pair of paraballs $\mathcal{B}^{\beta_0}(z_{\beta_0}, e_{\beta_0}, r_{\beta_0}, \rho_{\beta_0}) = (B^{\beta_0}, B^{\beta_0}_\star)$ so that

$$\begin{aligned}
&(1)\ |B^{\beta_0}| \leq |B_{\beta_0}| && (2)\ |B^{\beta_0} \cap B_{\beta_0}| \geq c\eta^\gamma|B_{\beta_0}|,\\
&(3)\ |B^{\beta_0}_\star| \leq |\tilde{\mathcal{F}}| \leq |F| && (4)\ |B^{\beta_0}_\star \cap \tilde{\mathcal{F}}| \geq c\eta^\gamma|\tilde{\mathcal{F}}| \geq c\eta^\gamma|F|.
\end{aligned} \tag{10}$$

Step 4. Consider $B^{\alpha_0}_\star \cap B^{\beta_0}_\star$. By (10) we have

$$|B^{\alpha_0}_\star \cap B^{\beta_0}_\star| \geq |B^{\alpha_0}_\star \cap B^{\beta_0}_\star \cap \mathcal{F}| = |B^{\beta_0}_\star \cap \tilde{\mathcal{F}}| \geq c\eta^\gamma|F| \geq c\eta^\gamma \max\{B^{\alpha_0}_\star, B^{\alpha_0}_\star\}.$$

Take dual, use proposition 2.6, $\varrho(B^{\alpha_0}, B^{\beta_0}) \leq C\eta^{-C}$. Use (1),(2) of (9), (10), we know both $\varrho(B^{\alpha_0}, B_{\alpha_0})$, $\varrho(B^{\beta_0}, B_{\beta_0}) \leq C\eta^{-C}$. By quasi-triangle inequality,

$$\varrho(B_{\alpha_0}, B_{\beta_0}) \leq C\eta^{-C},$$

too small, a contradiction. $\square$

Remark. The idea of this proposition follows from [1], in which a quasi-extremizer is always investigated through level decomposition. In other words, it's natural to estimate $\langle \chi_E, \mathbf{T}\chi_{E'} \rangle$. The proposition is applied in 7.1.

5 Estimate

The following lemmas are modified version from origin paper because we want to confirm uniformity.

Lemma 5.1. *Let $f \geq 0, \in \boldsymbol{L}^p(\mathbb{R}^d)$, $\|f\| = 1$, expressed as $f = \sum_{\mathbb{Z}} 2^j \cdot f_j$. For any $\varepsilon > 0$, there is an $|S| \leq \tilde{C}_{5.1}\varepsilon^{-C_{5.1}}$ so that $\|\mathbf{T}\tilde{f}\| \geq \|\mathbf{T}f\| - \varepsilon$, where $\tilde{f} = \sum_S 2^j f_j$. Here $\tilde{C}_{51}, C_{51}$ are constants depending only on d.*

Note that C_{35} is an "upper bound" for $\mathbf{T}$ from $\mathrm{L}^r_{p,r}$ to L^q when $r = \frac{p+q}{2}$.

Proof. Let $r = \frac{p+q}{2} \in (p, q)$, $\eta = \left(\frac{\varepsilon}{C_{35}}\right)^{\frac{r}{r-p}} > 0$. Define $S = \{j : 2^j|E_j|^{1/p} > \eta\}$ and $\tilde{f} = \sum_S 2^j f_j$. Then

$$\|f - \tilde{f}\|^r_{p,r} = \sum_{j\notin S}(2^j|E_j|^{1/p})^r = \sum_{j\notin S}(2^j|E_j|^{1/p})^{r-p}(2^j|E_j|^{1/p})^p$$
$$\leq \sum_{j\notin S}\eta^{r-p}(2^j|E_j|^{1/p})^p \leq \eta^{r-p}\sum_{\mathbb{Z}}(2^j|E_j|^{1/p})^p = \eta^{r-p}\|f\|^p = \eta^{r-p}.$$

Thus,

$$\|\mathbf{T}(f - \tilde{f})\| \leq C_{35}\|f - \tilde{f}\|_{p,r} \leq C_{35}\eta^{1-\frac{p}{r}} = \varepsilon.$$

Moreover,

$$\eta^p|S| = \sum_{j\in S}\eta^p \leq \sum_{j\in\mathbb{Z}} 2^{jp}|E_j| \leq 1.$$

Then

$$|S| \leq \tfrac{1}{\eta^p} = C_{35}{}^{\frac{rp}{r-p}}\varepsilon^{\frac{-rp}{r-p}} = \tilde{C}_{51}\varepsilon^{C_{51}}.$$

Here $\tilde{C}_{51} = C_{35}{}^{(p+q)/(q-p)}, C_{51} = \frac{p+q}{q-p}p$.

$\square$

Lemma 5.2. *Let $\|f\|_p = 1$ be $(1-\delta)$-quasiextremized with $\delta \in (0, 0.5)$. Then the function $\tilde{f}$ in preceding lemma can be chosen so that*

$$\|f - \tilde{f}\| \leq C_{52}(\varepsilon + \delta)^{1/p}.$$

Here C_{52} depends only on d.

Proof. Note that we have

$$\begin{cases} \text{optimal inequality:} & \|\mathbf{T}\tilde{f}\| \leq \mathbf{A}\|\tilde{f}\| \\ \text{Approximation of } \|\mathbf{T}f\|_q \text{ from 5.1:} & \|\mathbf{T}\tilde{f}\| \geq \|\mathbf{T}f\| - \varepsilon \\ \text{quasi-extremizer:} & \|\mathbf{T}f\| \geq (1-\delta)\mathbf{A}\|f\| \end{cases}$$

Since $\tilde{f}$, $f - \tilde{f}$ have disjoint supports, by Bernoulli's inequality,

$$\begin{aligned} \|f - \tilde{f}\| &= \|f\|^p - \|\tilde{f}\|^p \leq \|f\|^p - \mathbf{A}^{-p}(\|\mathbf{T}f\| - \varepsilon)^p = 1 - \mathbf{A}^{-p}\|\mathbf{T}f\|^p(1 - \tfrac{\varepsilon}{\|\mathbf{T}f\|})^p \\ &\leq 1 - \mathbf{A}^{-p}\|\mathbf{T}f\|^p(1 - \tfrac{p\varepsilon}{\|\mathbf{T}f\|}) = 1 - \mathbf{A}^{-p}(\|\mathbf{T}f\|^p - p\varepsilon\|\mathbf{T}f\|^{p-1}) \\ &\leq 1 - \mathbf{A}^{-p}((1-\delta)^p\mathbf{A}^p - p\varepsilon\mathbf{A}^{p-1}) = 1 - (1-\delta)^p(1 - \tfrac{p\varepsilon}{\mathbf{A}(1-\delta)^p}) \leq 1 - (1-\delta)^p(1 - \tfrac{2^p p\varepsilon}{\mathbf{A}}) \\ &\leq 1 - (1 - p\delta)(1 - \tfrac{2^p p\varepsilon}{\mathbf{A}}) = p\delta + \tfrac{2^p p}{\mathbf{A}}\varepsilon \leq C_{52}(\varepsilon + \delta). \end{aligned}$$

Here C_{52} is either p or $\frac{2^p p}{\mathbf{A}}$. □

So original Lemma 5.3 changes a little.

Lemma 5.3. *Let $0 \leq f \in L^p$, $\|f\|_p = 1$, $(1-\delta)$-quasiextremized with $\delta \in (0, 0.5)$, and $f = \sum_j 2^j f_j$ with $f_j \leftrightarrow E_j$. Then for $\eta \in (0, 1]$,*

$$\|\sum_{j:2^j|E_j|^{1/p} < \eta} 2^j f_j\| \leq C_{53}(\delta^{1/p} + \eta^{c_{53}}).$$

Proof. Let $\varepsilon = C_{35}\eta^{1-2p/(p+q)}$ as Lemma 5.1. Then preceding lemmas say

$$\|\sum_{j:2^j|E_j|^{1/p} < \eta} 2^j f_j\| = \|f - \tilde{f}\| \leq C_{52}^{1/p}(C_{35}\eta^{1-2p/(p+q)} + \delta)^{1/p}$$

$$\leq C_{52}^{1/p}(C_{35}^{1/p}\eta^{1/p-2/(p+q)} + \delta^{1/p}) \leq C_{53}(\delta^{1/p} + \eta^{c_{53}}).$$

Here $C_{53} = C_{52}^{1/p}\mathrm{Max}(C_{35}^{1/p}, 1)$, $c_{53} = \frac{1}{p} - \frac{2}{p+q} = \frac{3d^2-1}{d^2-1}$. □

Remark for Constants. Here we denote c_{mn} the constant c from **Lemma m.n**.

6 Quasi-Extremals

In this section we want to roughly decompose a function f into two parts. The central parts consist of the indeces near a fixed integer, and the farther parts consist of those indeces far enought from the integer.

6.1 Represention of a $(1-\delta)$- quasiextremized function

Goal: If f is $(1-\delta)$-quasiextremized, then we want to find h to establish

1. Representation by inner product of the norms.

2. Six Term Decomposition.

Progress. Step 1. From $\tilde{f}$ to a supporting h By Lemma 5.1 and 5.3, let $\eta = \rho/2$, then all i so that $2^j 2|E_j|^{1/p} \geq \|2^j f_j\|_p \geq \rho$ satisfy $i \in S$ (Note that $\rho/2 \leftrightarrow \eta \leftrightarrow \varepsilon = C\rho^C$).

We've defined $\tilde{f} = \sum_S 2^j f_j$ and $\|\mathbf{T}\tilde{f}\| \geq \|\mathbf{T}f\| - C\rho^C$. By a representation theorem,

$$\|\mathbf{T}\tilde{f}\|_q = \sup_{\|h\|_p=1} \langle \mathbf{T}\tilde{f}, h\rangle.$$

So we choose an $\|h_0\|_p = 1$ with $h_0 = \sum_{\mathbb{Z}} 2^k h_k$, $h_k \leftrightarrow F_k$ and by 5.3 an $\|h\|_p < 1$, $h = \sum_{S'} 2^k h_k$ so that

$$\langle \mathbf{T}\tilde{f}, h_0\rangle \geq (1-\delta)\|\mathbf{T}\tilde{f}\|_q,$$

$$\|\mathbf{T}h\| > \|\mathbf{T}h_0\| - \varepsilon = \|\mathbf{T}h_0\| - C\rho^C,$$

$$\|h_0 - h\|_p \leq C\rho^c.$$

So

$$\begin{aligned}\langle h_0, \mathbf{T}\tilde{f}\rangle &> (1-\delta)\|\mathbf{T}\tilde{f}\| \geq (1-\delta)(\|Tf\| - C\rho^C)\\ &\geq (1-\delta)((1-\delta)A - C\rho^C) > (1-\delta)(1-C\rho^C)A.\end{aligned}$$

But we estimate (by Hölder's inequality)

$$\langle h_0, \mathbf{T}\tilde{f}\rangle = \langle h, \mathbf{T}\tilde{f}\rangle + \langle h_0 - h, \mathbf{T}\tilde{f}\rangle \leq \langle h, \mathbf{T}\tilde{f}\rangle + C\rho^c\|\tilde{f}\| \leq \langle h, \mathbf{T}\tilde{f}\rangle + C\rho^c.$$

So

$$\langle h, \mathbf{T}\tilde{f}\rangle \geq [(1-\delta)(1-C\rho^C) - C\rho^c]A \approx (1-\delta-2C\rho^C)\mathbf{A}.$$

<u>Step 2. To divide S, S' and $\tilde{f}, h$</u> Let $L := \ell\text{ength}(S)$, $N := \#S + \#\tilde{S}$. Fix a small η, Divide $S = S^\sharp \sqcup S^\flat$ by $|i-j| \geq L/N$ $\forall (i,j) \in S^\sharp \times S^\flat$ and each $F_k = F_k^\sharp \sqcup F_k^\flat \sqcup F_k^\natural$ by

$$\begin{cases} \text{For any } x \in F_k^\sharp, \text{ there is a } j \in S^\sharp, \text{ so that } \mathbf{T}\chi_{E_j}(x) > \eta |F_k|^{-1+1/p}|E_j|^{1/p}, \\ \text{For any } x \in F_k^\flat, \text{ there is a } j \in S^\flat, \text{ so that } \mathbf{T}\chi_{E_j}(x) > \eta |F_k|^{-1+1/p}|E_j|^{1/p}, \\ \text{For any } x \in F_k^\natural, \text{ and } j \in S, \text{ it holds that } \mathbf{T}\chi_{E_j}(x) \leq \eta |F_k|^{-1+1/p}|E_j|^{1/p}. \end{cases}$$

$$\text{Let } \begin{cases} f^\sharp = \sum_{S^\sharp} 2^j f_j \\ f^\flat = \sum_{S^\flat} 2^j f_j \end{cases}, \begin{cases} h^\sharp = \sum_{S'} 2^k h_k \chi_{F_k^\sharp} \\ h^\flat = \sum_{S'} 2^k h_k \chi_{F_k^\flat} \\ h^\natural = \sum_{S'} 2^k h_k \chi_{F_k^\natural} \end{cases}. \text{ So } \begin{cases} \tilde{f} = f^\sharp + f^\flat \\ h = h^\sharp + h^\natural + h^\flat \end{cases}.$$

In fact, we choose for $j \in S$, $F_{k,j} = \{\mathbf{T}\chi_{E_j}(x) > \eta |F_k|^{-1+1/p}|E_j|^{1/p}\}$ and then define $F_k^\sharp = \cup_{j\in S^\sharp} F_j$ and $F_k^\flat = \cup_{j \in S^\flat} F_j$ and adjust them so that they are disjoint. $F_k^\natural$ will be the rest part of F_k. Thus

$$\langle h, \mathbf{T}\tilde{f}\rangle = \langle h^\sharp, \mathbf{T}f^\sharp\rangle + \langle h^\natural, \mathbf{T}f^\sharp\rangle + \langle h^\flat, \mathbf{T}f^\sharp\rangle + \langle h^\sharp, \mathbf{T}f^\flat\rangle + \langle h^\natural, \mathbf{T}f^\flat\rangle + \langle h^\flat, \mathbf{T}f^\flat\rangle \quad \square$$

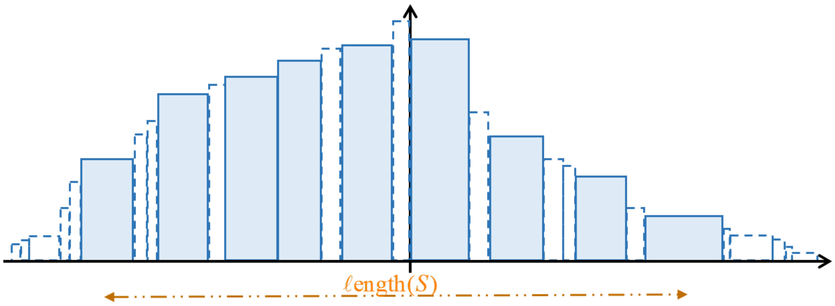
Remark. This is a part of the proof of Lemma 6.1 in Christ's paper.

6.2 The ρ-M statement and The Bound Ψ

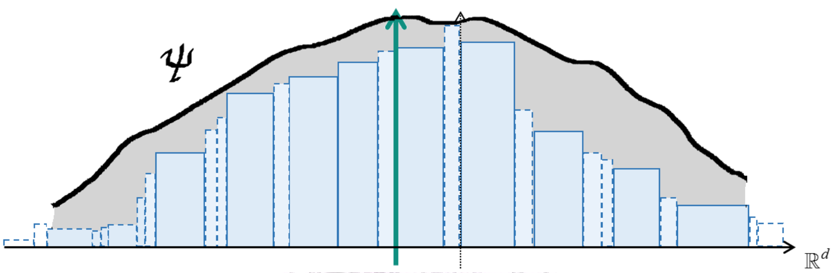
Lemma 6.1. *C', C, c exist for the approximation: Let $\rho \in (0,1)$. Then the $M = C'\rho^{-C}$ is such that when $\|f\|_p = 1$ is a good $(1-\delta)$-quasiextremizer with $\delta < c\rho^C$,*

o if $i, j \in \mathbb{Z}$, $\|2^i f_i\|_p, \|2^j f_j\|_p \geq \rho$, it has $|i-j| \leq M$.

o' whenever $|i-j| > M$, either $\|2^i f_i\|_p$ or $\|2^j f_j\|_p < \rho$.



Lemma 6.2 (The function Ψ). *There are $C_0 > 0$ and Ψ so that $\lim_{t\to\infty} \Psi(t)/t^p = \lim_{t\to 0} \Psi(t)/t^p = \infty$ with the property: For any $\rho > 0$ there exists $\delta > 0$ so that for any $f \geq 0$, $\|f\|_p = 1$, $(1-\delta)$-quasiextremal , there exists $\phi \in \mathcal{G}_d$ and a decomposition $\phi^* f = g + h$ so that $g, h \geq 0$ and $\|h\|_p < \rho$ and $\int \Psi \circ g \leq C_0$.*



6.3 A proof of lemma 6.1

We show $\ell\text{ength}(S) := \max_{i,j\in S} |i-j| \leq \tilde{C}\rho^{-\tilde{C}}$.

Step 3. Contruction of a gap First, we note

$$\begin{aligned}
\langle h^\natural, \mathbf{T} f^\sharp\rangle + \langle h^\natural, \mathbf{T} f^\flat\rangle &\leq \sum_{j\in S}\sum_{k\in S'} 2^{j+k}\langle h_k \chi_{F_k^\natural}, \mathbf{T} f_j \chi_{E_j}\rangle < \sum_{j\in S}\sum_{k\in S'} 2^{j+k+2}\langle \chi_{F_k^\natural}, \mathbf{T}\chi_{E_j}\rangle \\
&\leq \sum_{j\in S}\sum_{k\in S'} 2^{j+k+2}\langle \chi_{F_k^\natural}, \eta |F_k|^{-1+1/p}|E_j|^{1/p}\rangle \\
&\leq \sum_{j\in S}\sum_{k\in S'} 2^{j+k+2}\eta |F_k|^{1/p}|E_j|^{1/p}\rangle \\
&\leq \sum_{j\in S}\sum_{k\in S'} 4\eta \|f_j\|_p \|h_k\|_p \leq 4\eta N^2. \qquad (11)
\end{aligned}$$

Second, we recall the ℓog-convexity of the norms: For given $p \in [1,2]$ there is a corresponding $\theta \in (0,1)$ so that the different norms of a function/sequence F satisfy

$$\|F\|_2 \leq \|F\|_p^{1-\theta}\|F\|_\infty^\theta.$$

Thus,

$$\begin{aligned}
\langle \mathbf{T}f^\sharp, h^\sharp\rangle + \langle \mathbf{T}f^\flat, h^\flat\rangle &\le \mathbf{A}(\|f^\sharp\|_p\|\|h^\sharp\|_p + \|f^\flat\|_p\|h^\flat\|_p)\\
&\le \mathbf{A}\|(\|f^\sharp\|_p, \|f^\flat\|_p)\|_{\ell^2}\|(\|h^\sharp\|_p, \|h^\flat\|_p)\|_{\ell^2}\\
&\le \mathbf{A}\|(\|f^\sharp\|_p, \|f^\flat\|_p)\|_{\ell^p}^{1-\theta}\|(\|f^\sharp\|_p, \|f^\flat\|_p)\|_{\ell^\infty}^{\theta}\\
&\qquad \|(\|h^\sharp\|_p, \|h^\flat\|_p)\|_{\ell^p}^{1-\theta}\|(\|h^\sharp\|_p, \|h^\flat\|_p)\|_{\ell^\infty}^{\theta}\\
&\le \mathbf{A}(\max\{\|f^\sharp\|_p, \|f^\flat\|_p\})^\theta\\
&\le \mathbf{A}\|\tilde{f}\|_p^\theta < \mathbf{A}(1-c\rho^C)^\theta < \mathbf{A}(1-c'\rho^{C_1}).
\end{aligned}$$

Then we estimate

$$\begin{aligned}
(1-\delta-2C\rho^C)\mathbf{A} \le \langle h, \mathbf{T}\tilde{f}\rangle &= \langle h^\sharp, \mathbf{T}f^\sharp\rangle + \langle h^\natural, \mathbf{T}f^\sharp\rangle + \langle h^\flat, \mathbf{T}f^\sharp\rangle\\
&\quad + \langle h^\sharp, \mathbf{T}f^\flat\rangle + \langle h^\natural, \mathbf{T}f^\flat\rangle + \langle h^\flat, \mathbf{T}f^\flat\rangle\\
&\le CN^2\eta + \mathbf{A}(1-C\rho^c) + \langle h^\flat, \mathbf{T}f^\sharp\rangle + \langle h^\sharp, \mathbf{T}f^\flat\rangle\\
&\le \quad (1-\delta-3C\rho^C)\mathbf{A} \quad + \langle h^\flat, \mathbf{T}f^\sharp\rangle + \langle h^\sharp, \mathbf{T}f^\flat\rangle
\end{aligned}$$

in case $\eta = c_0\rho^{C_0}$ above is chosen small enough. In fact, the last inequality holds only if $1 - C\rho^C \le 1 - \delta - 3C\rho^C$. So it is expected that

$$\delta \le C\rho^C - 3C\rho^C \approx C\rho^C.$$

Thus above gap forces one of them, say $\langle h^\flat, \mathbf{T}f^\sharp\rangle$, $> \mathrm{A}C\rho^c$ so one of them conquers half, say, another $C\rho^C$.

<u>Step 4. To reach 3.2 by choosing $(k, j) \in \tilde{S} \times S^\sharp$</u>
<u>$\to F_k^\flat \supset F_k^{\flat,\star} \to i \in S^\flat \to F_k^{\flat,\star} \supset \mathcal{F} \notin$ null</u>

CLAIM: There is a $(k, j) \in \tilde{S} \times S^\sharp$ with $\langle \chi_{F_k^\flat}, \mathbf{T}\chi_{E_j}\rangle \ge \eta|F_k|^{1/p}|E_j|^{1/p}$.

Proof. For otherwise we estimate

$$c\rho^C < \langle h^\flat, \mathbf{T}f^\sharp\rangle \le C\eta N^2 = Cc_0\rho^{C_0}N^2 = c_1\rho^{C_0},$$

we can make c_0, C_0 smaller to get contradiction. □

CLAIM: $\exists F_k^{\flat,\star}$ so that $\mathbf{T}\chi_{E_j}(x) \ge c\rho^C|F_k^{\flat,\star}|^{-1+1/p}|E_j|^{1/p} \qquad \forall x \in F_k^{\flat,\star}$.

Proof. Let $F_k^{\flat,\dagger} = \{x \in F_k^\flat \,:\, \mathbf{T}\chi_{E_j}(x) < \frac{1}{2}\eta |F_k^\flat|^{-1}|F_k|^{1/p}|E_j|^{1/p}\}$, $F_k^{\flat,\star} = F_k^\flat \setminus F_k^{\flat,\dagger}$.

$$\langle \chi_{F_k^{\flat,\dagger}}, \mathbf{T}\chi_{E_j}\rangle \leq \tfrac{1}{2}\eta |F_k|^{1/p}|E_j|^{1/p}.$$

$$\mathbf{A}|F^{\flat,\star}|^{1/p}|E_j|^{1/p} \geq \langle \chi_{F_k^{\flat,\star}}, \mathbf{T}\chi_{E_j}\rangle \geq \tfrac{1}{2}\eta |F_k|^{1/p}|E_j|^{1/p}.$$

$$|F_k^{\flat,\star}| \geq c\rho^C |F_k|.$$

Thus in $F_k^{\flat,\star}$,

$$\mathbf{T}\chi_{E_j} \geq c\rho^C |F_k|^{-1+1/p}|E_j|^{1/p} \qquad (\geq c\rho^C|\mathcal{F}|^{-1+1/p}|E_j|^{1/p} \text{ later}),$$

as desired. □

This is half for 3.2. The other is similar:

CLAIM: $\exists i \in S^\flat\ \exists |\mathcal{F}| > 0$ so that $\mathbf{T}\chi_{E_j}(x) \geq c\rho^C|\mathcal{F}|^{-1+1/p}|E_j|^{1/p} \qquad \forall x \in \mathcal{F}$.

Proof. Choose $x_0 \in F_k^{\flat,*}$. Then $\exists i \in S^\flat$ so that $\mathbf{T}\chi_{E_i}(x_0) > c_0\rho^{C_0}|F_k|^{-1+1/p}|E_i|^{1/p} \geq c_0c\rho^{C_0+C}|F_k^{\flat,*}|^{-1+1/p}|E_i|^{1/p}$ and $\langle \chi_{F_k^{\flat,*}}, \mathbf{T}\chi_{E_i}\rangle \geq c_0c\rho^{C_0+C}|F_k^{\flat,*}|^{1/p}|E_i|^{1/p}$. $\mathcal{F}_0 := \{x \in F_k^{\flat,*} \,:\, \mathbf{T}\chi_{E_i}(x) < \alpha |F_k^{\flat,*}|^{-1+1/p}|E_j|^{1/p}\}$. $\mathcal{F} := F_k^{\flat,*} \setminus \mathcal{F}_0$. Then

$$\langle \chi_{\mathcal{F}}, \mathbf{T}\chi_{E_i}\rangle \geq (c_0c\rho^{C_0+C} - \alpha)|F_k^{\flat,*}|^{1/p}|E_i|^{1/p}.$$

By operator inequality, $\langle \chi_{\mathcal{F}}, \mathbf{T}\chi_{E_i}\rangle \leq \mathbf{A}|\mathcal{F}|^{1/p}|E_i|^{1/p}$. Thus, take $\alpha = \frac{c_0\rho^{C_0}}{2}c\rho^{-C}$, then

$$\mathbf{A}|\mathcal{F}|^{1/p} \geq \frac{c_0\rho^{C_0}}{2}c\rho^{-C}|F_k^{\flat,*}|^{1/p}$$

and in $\mathcal{F}$ we have

$$\mathbf{T}\chi_{E_i}(x) \geq c\rho^C|F_k^{\flat,*}|^{-1+1/p}|E_j|^{1/p} \geq c\rho^C|\mathcal{F}|^{-1+1/p}|E_j|^{1/p},$$

as desired. □

Recall from 3.2 that we have collected $|\mathcal{F}|, |E_j|, |E_i| > 0$ with required inequalities. Thus $|E_i| \leq c\rho^{-C}|E_j|$. ($|E_j| \neq 0$ because $2^{j+1}|E_j|^{1/p} > \|2^j f_j\|_p > \rho$)

Step 5. Bound for this $|i-j|$ and for arbitrary $|i-j|$ Since $2^{i+1}|E_j|^{1/p} \geq \|2^i f_i\| \geq \rho$, we know $\rho^p \leq 2^p 2^{pi}|E_i|$. On the other hand, we have $2^{pj}|E_j| \leq \|\tilde{f}\|_p^p = 1$. Thus

$$c\rho^C|E_i| \leq |E_j| \leq \frac{1}{2^{pj}} \leq \frac{2^{p(i+1)}|E_i|}{\rho^p 2^{pj}} = \frac{2^{(i-j)p}}{\rho^p}|E_i|.$$

Elemantary computation yields $2^{(j-i)p} \leq c\left(\frac{1}{\rho}\right)^{C+p}$. $j - i \leq \log c + C\log\left(\frac{1}{\rho}\right) \leq C\rho^{-1}$ if ρ is so small that $\log\left(\frac{1}{\rho}\right) \geq 1$.

But $\ell\text{ength}(S) = M \leq N|i-j| \leq C\rho^{-C} \cdot C\rho^{-1} \leq C\rho^{-C}$. $\square$

6.4 A Sketch of Lemma 6.2

0.1^o By Lemma 5.3, the function $C'\eta^c < 1/2$ for η small. The range for η to be small is decided by C' and c and hence by d. Choose such a $\eta = \eta_0$. Then produce S and $\tilde{f}$, and then for small δ $(< C'\eta^c)$, it's sure that $\|f - \tilde{f}\| < C'\eta^c < (1/2)^{1/p}$. Now $\|\tilde{f}\|^p \geq 1/2$. But $\|\tilde{f}\|^p = \sum_S \|2^j f_j\|^p \leq |S| \max_{j\in S} \|2^j f_j\|^p$. So

$$\max_{j\in S} \|2^j f_j\|^p \geq \frac{1}{2|S|} \geq \eta_0/2 =: c_0.$$

1^o After adjusted, Lemma 6.1 implies $\|2^j f_j\| < \eta$ for all $|j| \geq M$. Define $\begin{cases} h = \sum_{|j|>M} 2^j f_j \\ g = \sum_{|j|\leq M} 2^j f_j \end{cases}$. Then $\|h\|_p < \rho$.

2^o Next, we estimate $\int \Psi \circ g$ in case $\Psi \uparrow$. Since f_j's are disjoint supported,

$$\int \Psi \circ g = \sum_{j\leq M} \int \Psi(2^j f_j) \leq \sum_{|j|\leq M} \Psi(2^{j+1})|E_j|.$$

4^o Let $S_k = \{j : |j| < M \text{ and } \|2^j f_j\|_p \in (2^{-k-1}, 2^{-k}]\}$. By Lemma 6.1 and Lemma 5 we estimate as follow:

$$\sum_{|j|\leq M} \Psi(2^{j+1})|E_j| = \sum_k \sum_{j\in S_k} \Psi(2^{j+1})|E_j| \leq \sum_k \max_{|j|\leq C2^{kC}} \tfrac{\Psi(2^{j+1})}{2^{p(j+1)}} \cdot C(\delta + 2^{-ck})$$
$$\leq \underbrace{\sum_{k=1}^{\infty} 2^{-ck} C \max_{|j|\leq C2^{kC}} \tfrac{\Psi(2^{j+1})}{2^{p(j+1)}}}_{(I)} + \delta \underbrace{\sum_k C \max_{|j|\leq C2^{kC}} \tfrac{\Psi(2^{j+1})}{2^{p(j+1)}}}_{(II)}$$

We want $(I) + (II) < \infty$. Since that $|\{S_k\}_k| < \infty$, it suffices to let $(I) < \infty$, obtaining Ψ, and $(II) \leq 1$, obtaining a small enough δ.

7 Estimation using paraballs

Lemma 7.1. *For any $\varepsilon > 0$, there is a $0 < \delta \,\&\, N, K < \infty$ so that for any $\|f\|_p = 1$, $(1-\delta)$ - quasiextremized, $\exists F = \sum_{j\in S} 2^j F_j, F_j \leftrightarrow E_j$ satisfying*

- $0 \leq F \leq f$
- $\|\boldsymbol{T}F\|_q \geq (1-\varepsilon)\boldsymbol{A}$.
- $|i-j| \leq K$ *for all* $i, j \in S$.

and for $j \in S$, $\exists$ *paraballs* $B_{j1}, \ldots, B_{jN}$ *so that*

- $E_j \subset \cup_{i=1}^{N} B_{j,i}$,
- $\sum |B_{j,i}| \leq C(\varepsilon)|E_j|$.

Lemma 7.2. *For any* $\varepsilon > 0$, *there is a* $0 < \delta, K, \lambda < \infty$ *so that for any* $\|f\|_p = 1$, $(1-\delta)$ *- quasiextremized. Then* $\exists 0 \leq \tilde{f} \leq f$, $\exists \mathcal{B}$ *such that* $\|\tilde{f}\|_p \geq 1-\varepsilon$, $\|\boldsymbol{T}\tilde{f}\|_q \geq (1-\varepsilon)\boldsymbol{A}$, *and if* $\tilde{f} = \sum_S 2^j f_j$, *then for some* $J \in \mathbb{Z}$,

- $|j - J| \leq K$ $\quad\quad$ $\forall j \in S$.
- f_j *is supported in* $\lambda\mathcal{B}$ $\quad\quad$ $\forall j \in S$,
- $2^J |\mathcal{B}|^{1/p} \leq C\|f\|_p$.

8 Existence of an Extremizer

Proposition 8.1. *The same* $C_0 > 0$ *and* Ψ *together with* ρ *such that* $\lim_{R\to\infty} \rho(R) = 0$, *enjoy the property: For any* $\varepsilon > 0$, $\exists \delta > 0$ *such that if* $f \geq 0$, $\|f\|_p = 1$, $\|\boldsymbol{T}f\|_q \geq (1-\delta)\boldsymbol{A}$, *then* $\exists \phi \in \mathcal{G}_d$ *and a decomposition* $\phi^* f = g + h$, $g, h \geq 0$, *and* $\exists \|F\|_p = 1$, *so that*

$$\begin{array}{lll} \|h\|_p < \varepsilon, & \int_{\mathbb{R}^d} \Psi \circ g \leq C_0, & \int_{|x|\geq R} g^p \leq \rho(R). \\ \langle F, \boldsymbol{T}g \rangle \geq (1-\varepsilon)\boldsymbol{A}, & \int_{\mathbb{R}^d} \Psi \circ F \leq C_0, & \int_{|x|\geq R} F^p \leq \rho(R). \end{array}$$

We say that an extremizing sequence is **good** if each f_ν is good. We wonder:

Theorem 8.1. *If* T *admits a good extremizing sequence, then it admids an extremizer:* $\|\boldsymbol{T}f\|_q \leq \boldsymbol{A}\|f\|_p$.

Sketch. $\|\mathbf{T}f_\nu\|_q \to \mathbf{A}$ and 8.1 implies $\langle F_\nu, \mathbf{T}g_\nu\rangle \to \mathbf{A}$. By Banach-Alaoglu theorem $\exists(F,g)$ with $F_\nu^p \rightharpoonup F^p, g_\nu^p \rightharpoonup g^p$. The harder step is $\langle F_\nu, \mathbf{T}g_\nu\rangle \to \langle F, \mathbf{T}g\rangle$. Then $\|F\|_p = \|g\|_p = 1$ and g is extremal.

For this step, define $\begin{cases} g_{\nu,\lambda}(x) = g_\nu(x)\chi_{|x|\le\lambda}(x)\chi_{g_\nu(x)\le\lambda}(x) \\ g^{(\lambda)}(x) = g(x)\chi_{|x|\le\lambda}(x)\chi_{g(x)\le\lambda}(x) \end{cases}$ and $\begin{cases} F_{\nu,\lambda}(x) = F_\nu(x)\chi_{|x|\le\lambda}(x)\chi_{F_\nu(x)\le\lambda}(x) \\ F^{(\lambda)}(x) = F(x)\chi_{|x|\le\lambda}(x)\chi_{F(x)\le\lambda}(x) \end{cases}$. Let $\eta \in C_o^1(\mathbb{R}^d)$, the operator $L^2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$, $f \mapsto \eta\mathbf{T}\eta f$is smooth and bounded for $s = (d-1)/2$. H^s empbeds compactly into L^2 in any bounded region. Thus the weak convergence of $g_{\nu,\lambda}$ to $g^{(\lambda)}$ implies $\mathbf{T}(g_{\nu,\lambda}) \to_{L^2} \mathbf{T}(g^{(\lambda)})$ for any fixed λ. Thus $\langle F_{\nu,\lambda}, \mathbf{T}g_{\nu,\lambda}\rangle \to \langle F^{(\lambda)}, \mathbf{T}g^{(\lambda)}\rangle$.

by Proposition 8.1 $g_{\nu,\lambda} \to_{L^p} g_\nu$ and $F_{\nu,\lambda} \to_{L^p} F_\nu$ uniformly. Last sentences of both last two paragraphs, obtain the harder step. $\square$

9 Further Discussion

We hope to remove any additional conditions posed in theorems and then the final theorem is free: $\mathbf{T}$ admits an extremizer. Moreover, I also hope to absorb the method and then apply it to another topic—— near-extremizers of Young convolution inequality.

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