國立清華大學數學系應用數學組

碩士論文

Hardy 空間的等價距之探討

Quasi-norm Equivalence of Several Maximal Functions in Hardy space

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摘要

本文探討 Hardy 上的多個等價半距，有時我們無法直接得出最大函數 (maximal function) 之間的距等價關係，可以先引進多個輔助函數，藉由輔助函數之間的大小關係來幫助我們得到原來函數之間所想要的結果

Abstract

This thesis consists of two parts. In the first part, bla bla.

In the second part, bla bla.

Acknowledgements

First of all, bla bla.

Last but not least, I am really indebted to my parents, bla bla.

Contents

1 Introduction

Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$ are spaces of distribution have remarkable similarities to L^p. There exists an abundance of equivalent characterizations for Hardy spaces. The major objective of this study was to investigate one of the characterizations from historical views.

The organization of the thesis is the following. In Section 2, we provide a introduction to definition of Hardy space and serveral kinds of maximal functions needed to construct the desired characterization.

In Section 3, we introduce main theorem that all the maximal functions of the section 1 have comparable L^p quasi-norms for all $0 < p < \infty$.

2 Settings and Definitions

In this section, we introduce definitions we would use in the later sections. First, we need some background to give the definition of Hardy space. We say that a tempered distribution v is *bounded* if $\varphi * v \in L^\infty(\mathbb{R}^n)$ whenever φ is in Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$.

We observe that if v is a bounded tempered distribution and $h \in L^1(\mathbb{R}^n)$, then the convolution $h*v$ can be defined as a distribution via the convergent integral

$$\langle h * v, \varphi \rangle = \langle \tilde{\varphi} * v, \tilde{h} \rangle = \int_{\mathbb{R}^n} (\tilde{\varphi} * v)(x)(\tilde{h})(x)dx,$$

where φ is a Schwartz function and $\tilde{\varphi}(x) = \varphi(-x)$, $\tilde{h}(x) = h(-x)$.

2.1 Definition of Hardy Spaces

Definition 2.1. *Let f be a bounded tempered distribution on $\mathbb{R}^n$ and let $0 < p < \infty$. We say that f lies in the Hardy space $H^p(\mathbb{R}^n)$ if the Poisson maximal function $P(x)$*

$$M(f;P)(x) = \sup_{t>0} |(P_t * f)(x)| \tag{2.1.1}$$

lies in $L^p(\mathbb{R}^n)$. If this is the case, we set

$$\|f\|_{H^p} = \|M(f;P)(x)\|_{L^p}.$$

2.2 Definition of Several Maximal Functions

Let $a, b > 0$. Let Φ be a Schwartz function and let f be a tempered distribution on $\mathbb{R}^n$. We define following maximal functions of f with respect to Φ.

Definition 2.2 (Smooth Maximal Function)**.** *The smooth maximal function of f with respect to Φ is defined as*

$$M(f;\Phi)(x) = \sup_{t>0} |(\Phi_t * f)(x)|$$

Definition 2.3 (Nontangential Maximal Function)**.** *The nontangential maximal function (with aperture a) of f with respect to Φ is defined as*

$$M_a^*(f;\Phi)(x) = \sup_{t>0} \sup_{\substack{y\in\mathbb{R}^n \\ |y-x|\le at}} |(\Phi_t * f)(x)|$$

Definition 2.4 (Auxiliary Maximal Function)**.** *The auxiliary maximal function is defined as*

$$M_b^{**}(f;\Phi)(x) = \sup_{t>0} \sup_{y\in\mathbb{R}^n} \frac{|(\Phi_t * f)(x-y)|}{(1+t^{-1}|y|)^b}$$

Note that we have

$$M(f;\Phi)(x) \le M_a^*(f;\Phi)(x) \le (1+a)^b M_b^{**}(f;\Phi)(x), \tag{2.2.1}$$

where the first inequilty is quickly from definitions and the second from viewing $M_b^{**}(f;\Phi)(x) = \sup_{t>0}\sup_{y\in\mathbb{R}^n} \frac{|(\Phi_t * f)(y)|}{(1+t^{-1}|x-y|)^b}$ and then restricting $|x-y| \le at$.

Finally, we also need to estimate the quantity of a Schwartz function.

Definition 2.5.

$$\mathfrak{N}_N(\varphi) = \int_{\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)| dx$$

Definition 2.6 (Grand Maximal Function)**.** *The grand maximal function of f (with respect to N) as*

$$\mathcal{M}_N(f)(x) = \sup_{\varphi\in\mathcal{F}_N} M_1^*(f;\varphi)(x)$$

3 Quasi-norm Equivalence of Sever Maximal Function

Before stating the main theorem, we first introduce the following Lemma.

3.1 Lemma

Lemma 3.1. *Let* $m \in \mathbb{Z}^+$ *and let* Φ *in* $\mathcal{S}(\mathbb{R}^n)$ *satisfy* $\int_{\mathbb{R}^n} \Phi(x)dx = 1$*, Then there exists a constant* $C_0(\Phi, m)$ *such that for any* Ψ *in* $\mathcal{S}(\mathbb{R}^n)$*, there are Schwartz functions* $\Theta^{(s)}$*,* $0 \leq s \leq 1$*, with the properties*

$$\Psi(x) = \int_0^1 (\Theta^{(s)} * \Phi_s)(x)ds \tag{3.1.1}$$

and

$$\int_{\mathbb{R}^n} (1+|x|)^m \left|\Theta^{(s)}(x)\right| dx \leq C_0(\Phi, m) s^m \mathfrak{N}_m(\Psi) \tag{3.1.2}$$

Proof. We start with a smooth function ζ supported in $[0,1]$ that satisfies

$$\begin{aligned} 0 \leq \zeta(s) \leq \frac{2s^m}{m!} \qquad & \text{for all } 0 \leq s \leq 1 \\ \zeta(s) = \frac{s^m}{m!} \qquad & \text{for all } 0 \leq s \leq \frac{1}{2} \\ \frac{d^r\zeta}{dt^r}(1) = 0 \qquad & \text{for all } 0 \leq r \leq m+1 \end{aligned}$$

We define

$$\Theta^{(s)} = \Xi^{(s)} - \frac{d^{m+1}\zeta}{ds^{m+1}}(s) \left(\overbrace{\Phi_s * \cdots * \Phi_s}^{\text{m+1 terms}} \right) * \Psi, \tag{3.1.3}$$

where

$$\Xi^{(s)} = (-1)^{m+1}\zeta(s) \frac{d^{m+1}\zeta}{ds^{m+1}} \left(\overbrace{\Phi_s * \cdots * \Phi_s}^{\text{m+2 terms}} \right) * \Psi,$$

and we claim that (3.1.1) holds for this choice of $\Theta^{(s)}$. To verify this assertion, we apply integration by parts to write

$$\begin{aligned} \int_0^1 -\frac{d^{m+1}\zeta}{ds^{m+1}}(s)(\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi ds = & - \overbrace{\frac{d^m\zeta}{ds^m}(1)}^{=0} \, (\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_1 * \Psi \\ & + \overbrace{\frac{d^m\zeta}{ds^m}(0)}^{=1} \lim_{s \to 0^+} (\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi \\ & + \int_0^1 \frac{d^m\zeta}{ds^m}(s) \frac{d}{ds} (\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi ds \end{aligned}$$

Therefore appling $m+1$ times integration by parts we rewrite (3.1.1) as

$$\int_0^1 \Theta^{(s)} * \Psi_s ds = \int_1^0 \Xi^{(s)} * \Psi_s ds + \frac{d^m \zeta}{ds^m}(0) \lim_{s\to 0^+} (\overbrace{\Phi * \cdots * \Phi}^{\text{m+2 terms}})_s * \Psi$$
$$-(-1)^{m+1} \int_0^1 \zeta(s) \frac{d^{m+1}}{s^{m+1}} \left(\overbrace{\Phi_s * \cdots * \Phi_s}^{\text{m+2 terms}} \right) * \Psi ds.$$

Noting that all the boundary terms vanish except for the term at $s = 0$ in the first integration by parts. The first and the third terms in the previous expression on the right add up to zero, while the second term is equal to Ψ, since Ψ has integral one. This implies that the family $\{(\Phi * \cdots * \Phi)_s\}_{s>0}$ is an appproximate identity as $s \to 0^+$. Specifically, from $\|\Phi * \Phi\|_1 \le \|\Phi\|_1 \|\Phi\|_1 = 1$ and $\int_{\mathbb{R}^n} \Phi dx = \int_{\mathbb{R}^n} \Phi_s dx = 1$, we have

$$\int_{\mathbb{R}^n} (\Phi * \cdots * \Phi)_s dx = 1$$

Since $\Psi \in L^\infty(\mathbb{R}^n)$, appling the statement from Zygmund's real analysis textbook

Theorem 3.2. *Let $f_\varepsilon = f * K_\varepsilon$, where $K \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} K = 1$. If $f \in L^\infty(\mathbb{R}^n)$, then $f_\varepsilon \to f$ as $\varepsilon \to 0$ at every point of continuity of f, and the convergence is uniform on any set where f is uniformly continuous.*

Therefore, the (3.1.1) follows. We now prove the estimate (3.1.2). Let Ω be the (m+1)-fold convolution of Φ. For the second term on the right in (3.1.3), we note that the (m+1)st derivative of $\zeta(s)$ vanish on $[0, \frac{1}{2}]$, so that we may write

$$\int_{\mathbb{R}^n} (1+|x|)^m \left| \frac{d^{m+1}\zeta(s)}{ds^{m+1}} \right| |\Omega_s * \Psi(x)| dx$$
$$\le C_m \chi_{[\frac{1}{2},1]}(s) \int_{\mathbb{R}^n} (1+|x|)^m \left[\int_{\mathbb{R}^n} \frac{1}{s^n} \left| \Omega(\frac{x-y}{s}) \right| |\Psi(y)| \, dy \right] dx$$

change of variables $\frac{x-y}{s} = x$

$$\le C_m \chi_{[\frac{1}{2},1]}(s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|y+sx|)^m |\Omega(x)||\Psi(y)| dy dx$$

since $(1+|sx+y|) < (1+|sx|)(1+|y|)$

$$\le C_m \chi_{[\frac{1}{2},1]}(s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|sx|)^m |\Omega(x)| (1+|y|)^m |\Psi(y)| dy dx$$

by $s \le 1$ and Fubini's theorem

$$\le C_m \chi_{[\frac{1}{2},1]}(s) \left(\int_{\mathbb{R}^n} (1+|x|)^m |\Omega(x)| dx \right) \left(\int_{\mathbb{R}^n} (1+|y|)^m |\Psi(y)| dy \right)$$
$$\le C_0'(\Phi, m) s^m \mathfrak{N}_N(\Psi),$$

where the last inequality follows by $\chi_{[\frac{1}{2},1]}(s) \leq 1 \leq 2^m s^m$. To obtain a similar estimate for the first term on the right in (3.1.3), we argue that

$$\int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \left| \frac{d^{m+1}(\Omega_s * \Psi)}{ds^{m+1}}(x) \right| dx$$

$$= \int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \left| \frac{d^{m+1}}{ds^{m+1}} \int_{\mathbb{R}^n} \frac{1}{s^n} \Omega(\frac{x-y}{s}) \Psi(y) dy \right| dx$$

change of variables $\frac{x-y}{s} = y$ and move derivatives inside integral by DCT

$$= \int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \left| \int_{\mathbb{R}^n} \Omega(y) \frac{d^{m+1}\Psi(x-sy)}{ds^{m+1}} dy \right| dx$$

since derivatives on space of Schwartz functions can be viewed as tempered distribution

and bounded by its seminorms

$$\leq C'_m \int_{\mathbb{R}^n} (1+|x|)^m |\zeta(s)| \int_{\mathbb{R}^n} |\Omega(y)| \left[\sum_{|\alpha| \leq m+1} |\partial^\alpha \Psi(x-sy)| |y|^{|\alpha|} \right] dydx$$

change of variables $x - sy = x$ and use the fact $|y|^{|\alpha|} \leq (1+|y|)^{m+1}$ and Tonelli's theorem

$$\leq C'_m |\zeta(s)| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1+|x+sy|)^m |\Omega(y)| \left[\sum_{|\alpha| \leq m+1} |\partial^\alpha \Psi(x)| (1+|y|)^{m+1} \right] dydx$$

use $(1+|sx+y|) < (1+|sx|)(1+|y|)$ and $s \leq 1$ again

$$\leq C'_m |\zeta(s)| \int_{\mathbb{R}^n} (1+|y|)^m |\Omega(y)| (1+|y|)^{m+1} dy \int_{\mathbb{R}^n} (1+|x|)^m \sum_{|\alpha| \leq m+1} |\partial^\alpha \Psi(x)| dx$$

by the defnition of $0 \leq \zeta(s) \leq \frac{2s^m}{m!}$ for all $0 \leq s \leq 1$

$$\leq C''_0(\Phi, m) s^m \mathfrak{N}_N(\Psi),$$

We now let $C_0(\Phi, m) = C'_0(\Phi, m) + C''_0(\Phi, m)$ to obtain the desirable result (3.1.2).

□

3.2 The main theorem

分隔線

3.3 Importance Sampling Method

Even with the aid of ever faster computers nowadays, the slow convergence speed is still the limit of basic Monte Carlo simulations. Bla bla.

$$\mathbb{P}\{X > c\} \tag{3.3.1}$$

$$\begin{aligned}\mathbb{P}\{X > c\} &= \tilde{\mathbb{E}}\left[\mathbb{I}_{\{X>c\}}\right] \\ &= \mathbb{E}\left[\mathbb{I}_{\{Y>c\}} e^{\frac{c^2}{2} - cY}\right] \end{aligned} \tag{3.3.2}$$

where $Y \sim N(c, 1)$. And Equation (3.3.2) is more easier to sample the rare events than the original one.

4 Pricing Contingent Claims

In this section, we will discuss bla bla.

4.1 European Options

European options are the fundamental options. As a result, they are also called "vanilla" or "plain" options.

4.1.1 Proof of Efficiency of Importance Sampling

In this section, we will prove that our importance sampling methods are bla bla.

Theorem 4.1. *If the following approximation holds, for large c*

$$\mathbb{E}\left[\left(\mathbb{I}_{\{Z>c\}} e^{\frac{c^2}{2} - cZ}\right)^2\right] \approx p^2,$$

then bla bla.

Proof.

$$
\begin{aligned}
q &= \mathbb{E}\left[e^{-2rT}S_0^2 e^{2\left(r-\frac{\sigma^2}{2}\right)T+2\sigma\sqrt{T}Z}\mathbb{I}_{\{Z>c\}}e^{\alpha^2-2\alpha Z}\right] \\
&\quad - 2K\mathbb{E}\left[e^{-2rT}S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z}\mathbb{I}_{\{Z>c\}}e^{\alpha^2-2\alpha Z}\right] \\
&\quad + K^2\mathbb{E}\left[e^{-2rT}\mathbb{I}_{\{Z>c\}}e^{\alpha^2-2\alpha Z}\right] \\
&= S_0^2\mathbb{E}\left[\mathbb{I}_{\{Z>c\}}e^{-\sigma^2T+\alpha^2-(2\alpha-2\sigma\sqrt{T})Z}\right] \\
&\quad - 2S_0Ke^{-rT}\mathbb{E}\left[\mathbb{I}_{\{Z>c\}}e^{-\frac{\sigma^2}{2}T+\alpha^2-(2\alpha-\sigma\sqrt{T})Z}\right] \\
&\quad + K^2e^{-2rT}\mathbb{E}\left[\mathbb{I}_{\{Z>c\}}e^{\alpha^2-(2\alpha)Z}\right] \\
&= S_0^2e^{-\alpha^2+4\alpha\sigma\sqrt{T}-3\sigma^2T}\mathbb{E}\left[\mathbb{I}_{\{Z>c\}}e^{\frac{(2\alpha-2\sigma\sqrt{T})^2}{2}-(2\alpha-2\sigma\sqrt{T})Z}\right] \\
&\quad - 2S_0Ke^{-rT}e^{-\alpha^2+2\alpha\sigma\sqrt{T}-\sigma^2T}\mathbb{E}\left[\mathbb{I}_{\{Z>c\}}e^{\frac{(2\alpha-\sigma\sqrt{T})^2}{2}-(2\alpha-\sigma\sqrt{T})Z}\right] \\
&\quad + K^2e^{-2rT}e^{-\alpha^2}\mathbb{E}\left[\mathbb{I}_{\{Z>c\}}e^{\frac{(2\alpha)^2}{2}-(2\alpha)Z}\right] \\
&= S_0^2e^{-\alpha^2+4\alpha\sigma\sqrt{T}-3\sigma^2T}e^{2\alpha^2-6\alpha\sigma\sqrt{T}}\mathbb{E}^{(1)}\left[\mathbb{I}_{\left\{Z^{(1)}>c\right\}}\right] \\
&\quad - 2S_0Ke^{-rT}e^{-\alpha^2+2\alpha\sigma\sqrt{T}-\sigma^2T}e^{2\alpha^2-3\alpha\sigma\sqrt{T}}\mathbb{E}^{(2)}\left[\mathbb{I}_{\left\{Z^{(2)}>c\right\}}\right] \\
&\quad + K^2e^{-2rT}e^{-\alpha^2}e^{2\alpha^2}\mathbb{E}^{(3)}\left[\mathbb{I}_{\left\{Z^{(3)}>c\right\}}\right] \\
&= S_0^2e^{\alpha^2-2\alpha\sigma\sqrt{T}-3\sigma^2T}\mathbb{E}^{(1)}\left[\mathbb{I}_{\left\{\tilde{Z}>c+\alpha-2\sigma\sqrt{T}\right\}}\right] \\
&\quad - 2S_0Ke^{-rT}e^{\alpha^2-\alpha\sigma\sqrt{T}-\sigma^2T}\mathbb{E}^{(2)}\left[\mathbb{I}_{\left\{\tilde{Z}>c+\alpha-\sigma\sqrt{T}\right\}}\right] \\
&\quad + K^2e^{-2rT}e^{\alpha^2}\mathbb{E}^{(3)}\left[\mathbb{I}_{\left\{\tilde{Z}>c+\alpha\right\}}\right] \\
&= S_0^2e^{\alpha^2-2\alpha\sigma\sqrt{T}-3\sigma^2T}N\left(-c-\alpha+2\sigma\sqrt{T}\right) \\
&\quad - 2S_0Ke^{-rT}e^{\alpha^2-\alpha\sigma\sqrt{T}-\sigma^2T}N\left(-c-\alpha+\sigma\sqrt{T}\right) \\
&\quad + K^2e^{-2rT}e^{\alpha^2}N\left(-c-\alpha\right) \qquad (4.1.1)
\end{aligned}
$$

□

4.1.2 Numerical Results

	BS Formula	Method 1 (Separation)		Method 2 (Final Stock Price)		Method 3 (Whole Stock Path)	
S_0	BS	MC(SE)	IS(SE)	MC(SE)	IS(SE)	MC(SE)	IS(SE)
20	0.0002	0.0000(0.0000)	0.0002(0.0000)	0.0000(0.0000)	0.0002(0.0000)	0.0008(0.0008)	0.0002(0.0000)
30	0.0120	0.0129(0.0057)	0.0122(0.0002)	0.0201(0.0071)	0.0117(0.0002)	0.0092(0.0035)	0.0120(0.0002)
40	0.1307	0.1394(0.0206)	0.1317(0.0015)	0.1390(0.0200)	0.1268(0.0017)	0.1258(0.0188)	0.1303(0.0017)
50	0.6201	0.5491(0.0422)	0.6247(0.0070)	0.5723(0.0426)	0.6257(0.0080)	0.6318(0.0493)	0.6119(0.0080)
60	1.8442	1.7626(0.0818)	1.8543(0.0206)	1.8550(0.0881)	1.8468(0.0243)	2.0688(0.1002)	1.8530(0.0244)
70	4.1112	4.1254(0.1347)	4.1390(0.0474)	4.2075(0.1330)	4.1501(0.0580)	4.1799(0.1376)	4.2504(0.0581)
80	7.5782	7.6457(0.1959)	7.5930(0.0940)	7.4929(0.1902)	7.5067(0.1162)	7.6309(0.1947)	7.5453(0.1164)
90	12.2497	12.4345(0.2561)	12.4287(0.1698)	12.6012(0.2548)	12.5674(0.2162)	12.4271(0.2575)	12.1825(0.2098)
100	18.0230	17.9212(0.3132)	17.9249(0.2855)	18.0691(0.3130)	18.0856(0.3645)	17.8826(0.3122)	18.3281(0.3754)
110	24.7413	24.5990(0.1470)	24.6509(0.1241)	25.0272(0.1516)	24.9238(0.1018)	24.8550(0.1507)	24.6892(0.1014)
120	32.2343	32.3142(0.1313)	32.2657(0.0882)	32.3424(0.1305)	32.3059(0.0728)	32.4487(0.1339)	32.1958(0.0729)
130	40.3420	40.2426(0.1123)	40.3097(0.0628)	40.4034(0.1154)	40.2937(0.0525)	40.2442(0.1126)	40.4022(0.0526)
140	48.9259	48.8750(0.0982)	48.8083(0.0453)	48.9334(0.0987)	48.9342(0.0388)	48.9562(0.0973)	48.9011(0.0388)
150	57.8728	57.8214(0.0827)	57.8653(0.0334)	57.8235(0.0843)	57.8834(0.0287)	57.8361(0.0842)	57.9287(0.0287)
160	67.0929	67.0749(0.0716)	67.0902(0.0247)	67.2048(0.0737)	67.0782(0.0213)	66.9499(0.0680)	67.0564(0.0215)
170	76.5166	76.4533(0.0610)	76.5135(0.0182)	76.4662(0.0604)	76.4883(0.0160)	76.4513(0.0598)	76.5503(0.0161)
180	86.0915	86.1164(0.0525)	86.0737(0.0136)	86.1268(0.0528)	86.0918(0.0122)	86.0650(0.0528)	86.0856(0.0121)

Table 1: Results of European Option Pricing: S_0 is underlying's initial price; BS is the price computed under BS model; Method 1, 2, 3 are we mentioned previously respectively, it consist of the results of basic Monte Carlo and importance sampling.

4.2 European Options with Stochastic Volatility

In these two model, bla bla.

4.2.1 Basic Monte Carlo Simulations

First, we focus on the Vasicek model, we define it as the following,

$$\begin{cases} dS_t = rS_t dt + e^{\frac{y_t}{2}} d\tilde{W}_{1t} \\ dy_t = \alpha \left(m - y_t \right) dt + \beta d\tilde{Z}_{2t} \end{cases}$$

4.2.2 Numerical Results

Heston Model			
S_0	MC(SE)	IS(SE)	SE Reduction
20	0.1525(0.0291)	0.1384(0.0023)	12.6387
30	0.6013(0.0771)	0.6554(0.0101)	7.6525
180	89.2501(0.1445)	89.1496(0.0672)	2.1513

Table 2: Results of European Option Pricing with Stochastic Volatility and Interest Rate: S_0 is underlying's initial price; Heston model is what we mentioned previously, it consist of the results of basic Monte Carlo and importance sampling.

5 Fourier Series Method

The second and the main topic of this thesis begins, bla bla.

5.1 Example

Figure 1 shows the results of our example. And meantime, the sum of absolute error of the simulated example is shown next.

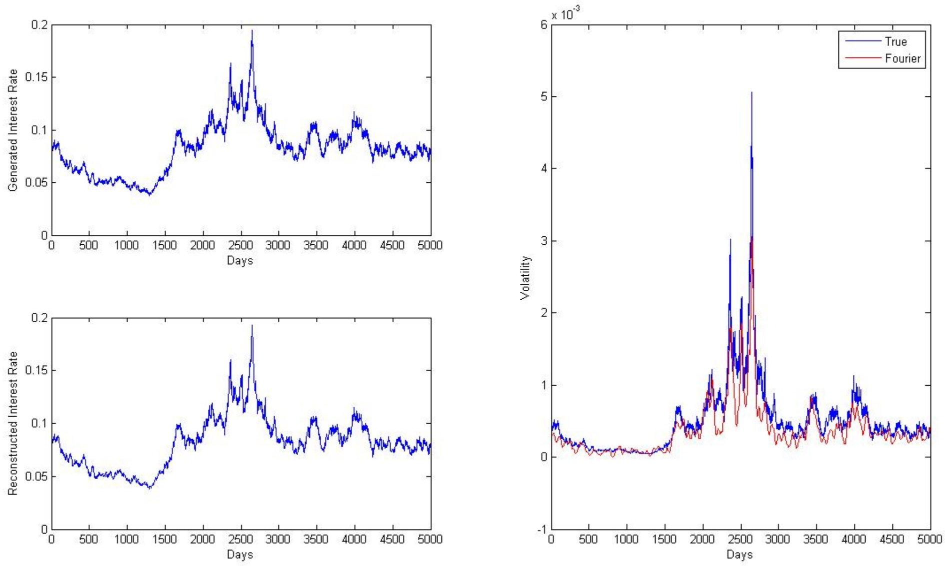

Figure 1: Estimate Volatility Process of Simulated Interest Rate by Fourier Series Method

6 Conclusions

We have done with option pricing and its inverse problem, model calibration, so far. Bla bla.

To sum up, bla bla.

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