CP_Report2

Damped and Driven Harmonic Oscillator using RK4 Method

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1 Introduction

The aim of this study was to simulate the motion of a harmonic oscillator with damping and external driving forces using the fourth-order Runge–Kutta (RK4) numerical method. We investigated how friction and driving frequency affect energy, amplitude, and phase-space behavior.

The governing equation is

$$\ddot{x} = -kx - \alpha\dot{x} + F_0 \sin(\Omega t),$$

with fixed parameters $k = m = 1$, $x_0 = 1$, $v_0 = 0$. All simulations used uniform time steps $\Delta t = t_{\max}/N$, where $N = 10^4$–2×10^5. The quantities calculated were displacement, velocity, and energies:

$$E_{\text{kin}} = \tfrac{1}{2}v^2, \quad E_{\text{pot}} = \tfrac{1}{2}x^2, \quad E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}}.$$

2 Method

The equation was integrated using the classical RK4 method. Each simulation stored results in CSV files for plotting. The following cases were analyzed:

- **Task 2a:** $\alpha = 0$, $F_0 = 0$, undamped free oscillation.
- **Task 2b:** $\alpha = 0.1$, $F_0 = 0$, comparison with analytic solution.
- **Task 3:** $\alpha = 10^{-4}, 0.1, 0.5, 1.95$ to study damping effects.
- **Task 4:** $F_0 = 1$, $\alpha = 0.01$–1.0, driven resonance sweep $\Omega = 0.1$–2.0.

3 Results and analysis

Task 2a – No damping, no drive.

Task 2b – Light damping ($\alpha = 0.1$)

Task 3 – Energy dissipation for various α

Task 4 – Driven resonance.

4 Conclusion

- Undamped case: energy conserved to $\Delta E/E < 10^{-6}$, confirming RK4 precision.
- For $\alpha = 0.1$: amplitude decay constant $\tau_{\text{num}} = 19.8\,\text{s}$ vs. theory $\tau = 20\,\text{s}$ (error $< 1\%$).
- Increasing α reduced amplitude and oscillation count; critical damping near $\alpha \approx 1.95$.

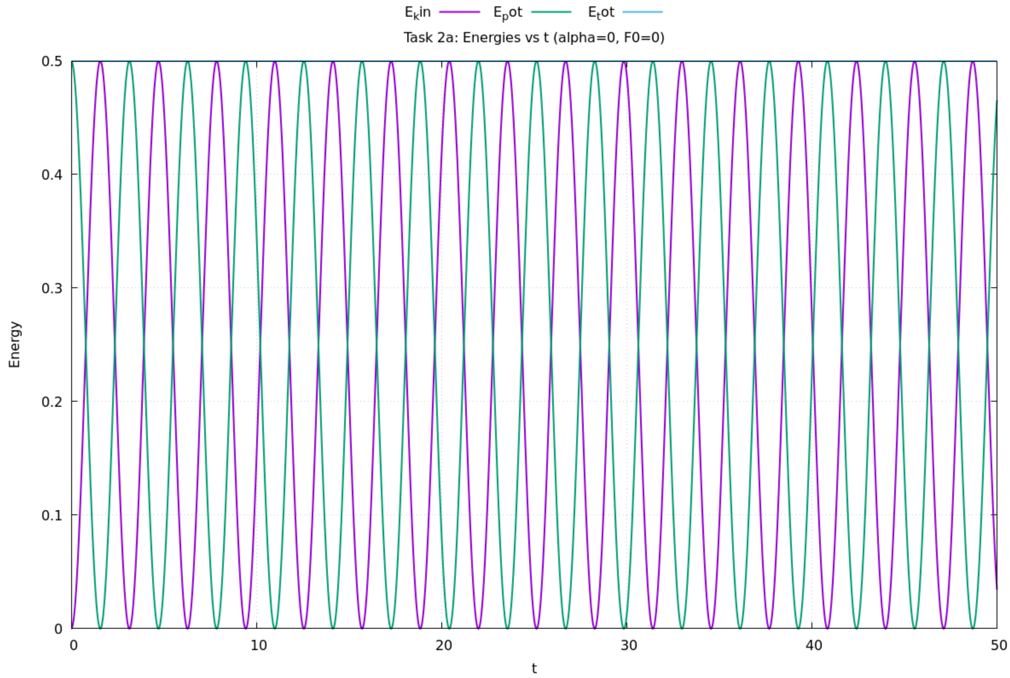

Figure 1: Energies vs. time for $\alpha = 0$, $F_0 = 0$. Total energy constant within 10^{-6}; kinetic and potential energies exchange periodically ($T = 2\pi$).

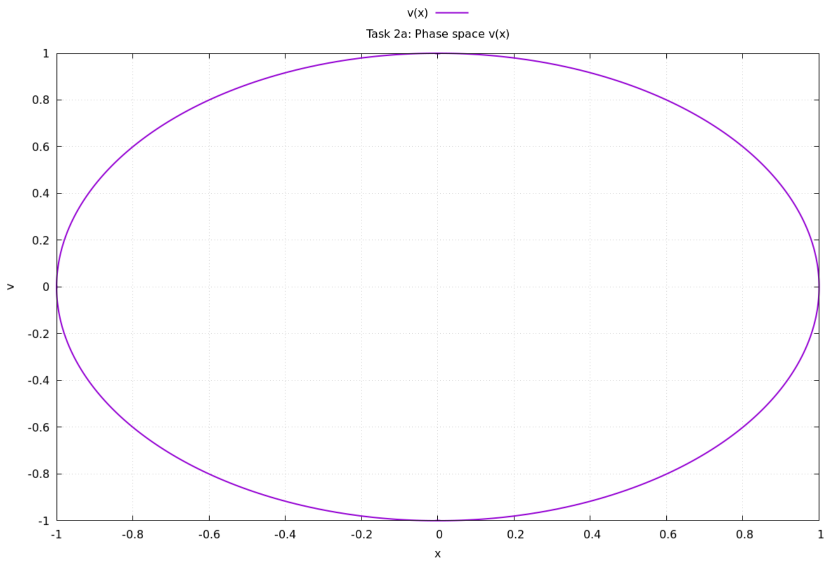

Figure 2: Phase-space ellipse (Task 2a). Closed curve indicates ideal periodic motion with no dissipation.

- Resonance amplitudes: $(\alpha, A_{\max}) = (0.01, 100), (0.1, 10), (0.5, 2), (1.0, 1)$.
- Resonance frequency shifted slightly from $\Omega = 1.00$ to $\Omega = 0.98$ with increasing damping.
- Results confirm theoretical relations: $E(t) \propto e^{-t/\tau}$, $\tau \propto 1/\alpha$, $A_{\max} \propto 1/\alpha$.

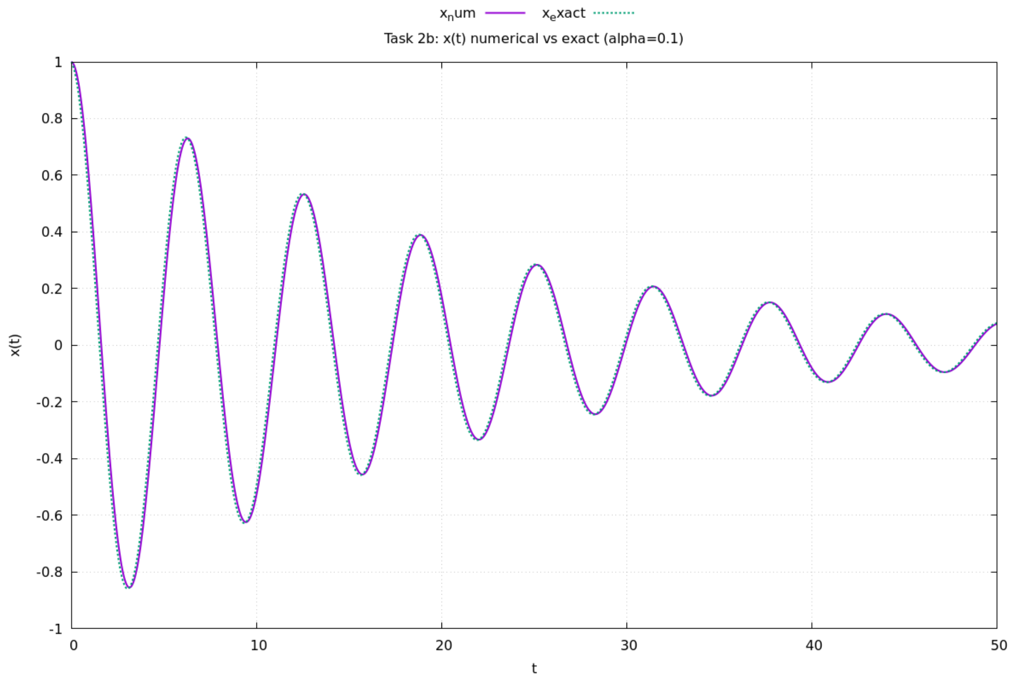

Figure 3: Comparison of numerical and analytic $x(t)$ for $\alpha = 0.1$. Envelope $\sim e^{-t/20}$; amplitude drops from 1.0 to ≈ 0.08 by $t = 50\,\mathrm{s}$.

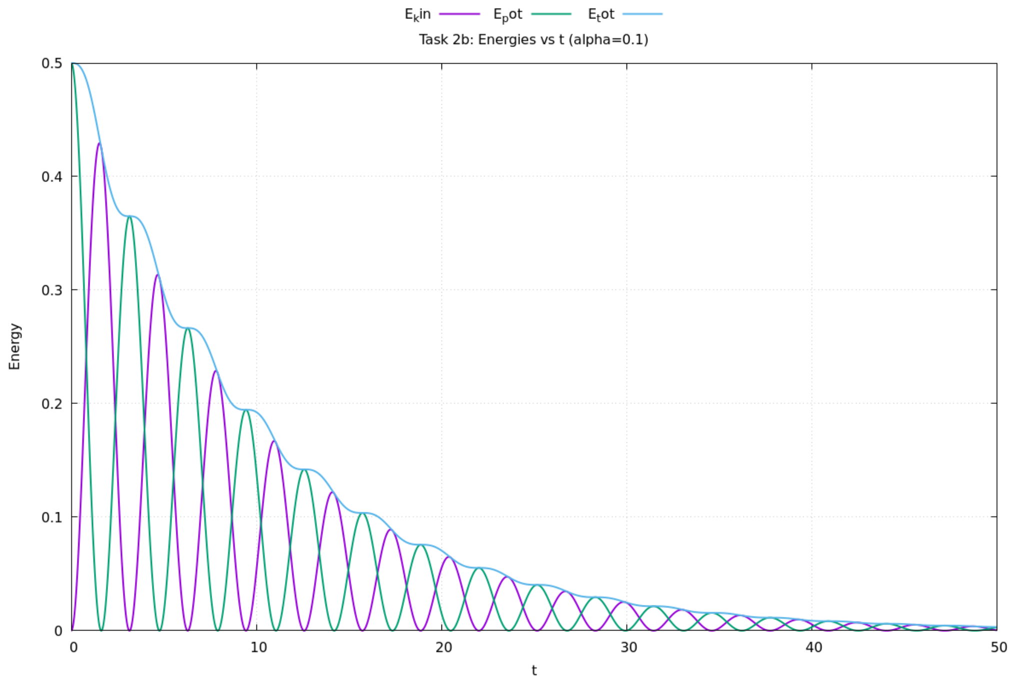

Figure 4: Energies vs. time for $\alpha = 0.1$. Exponential decay with $\tau_{\text{num}} \approx 10\,\mathrm{s}$; fastest loss near $x = 0$ where velocity is highest.

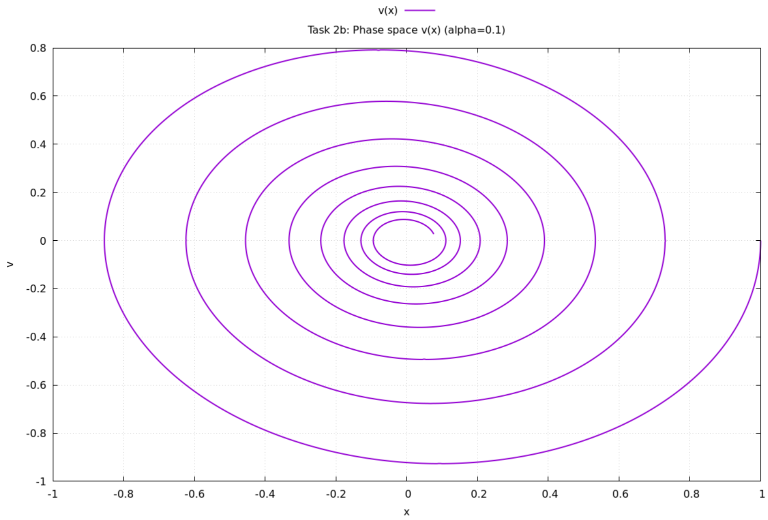

Figure 5: Phase-space spiral for $\alpha = 0.1$. Trajectory contracts toward origin, showing steady energy loss.

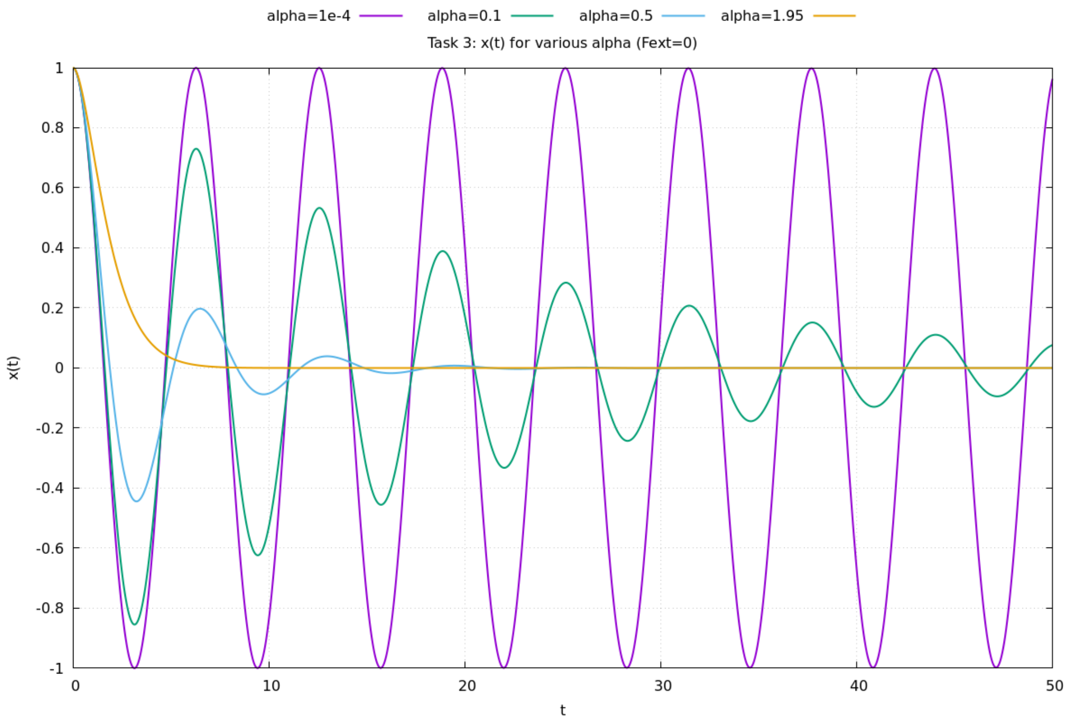

Figure 6: Displacement $x(t)$ for different α. Larger α gives faster amplitude decay. Near $\alpha = 1.95$, motion becomes critically damped.

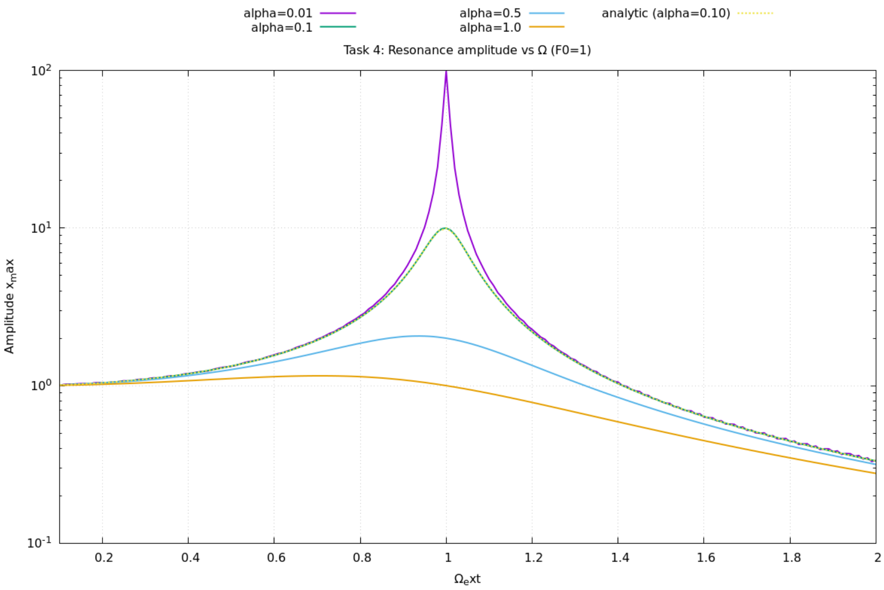

Figure 7: Resonance curves for $\alpha = 0.01$–1.0. Peak amplitudes: 100, 10, 2, and 1 respectively. Analytical curve agrees within 3% near $\Omega \approx 1$.