Precession of Mercury Orbit

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1 Introduction

This report presents a numerical study of Mercury's orbital motion around the Sun using a fourth-order symplectic (Neri) integrator. Symplectic algorithms are designed for Hamiltonian systems and keep energy and orbital structure stable over long time periods. All quantities were expressed in nondimensional units: distance in astronomical units (AU) and time in years.

To include relativistic effects, the gravitational force was modified to

$$\vec{F} = -\frac{4\pi^2}{r^3}\left(1 + \frac{\alpha}{r^2}\right)\vec{r},$$

where the small parameter α represents the relativistic correction. For $\alpha = 0$ the orbit is perfectly Keplerian, while for $\alpha > 0$ the ellipse slowly rotates, producing perihelion precession.

The initial conditions correspond to Mercury at aphelion:

$$x = a(1+e),\ y = 0,\ v_x = 0,\ v_y = 2\pi\sqrt{\frac{1-e}{a(1+e)}},$$

with $a = 0.387098\,\mathrm{AU}$ and $e = 0.206$.

2 Numerical method

The equations of motion are solved by a 4th-order symplectic scheme:

$$\vec{r}_{k+1} = \vec{r}_k + a_i\vec{v}_k\Delta t,$$
$$\vec{v}_{k+1} = \vec{v}_k + b_i\vec{a}(\vec{r}_{k+1})\Delta t,$$

with coefficients a_i, b_i given by Neri's method. We performed several runs:

- Test orbit: $t_{\max} = 0.95T_M$, $\Delta t = 10^{-4}$, $\alpha = 0$.
- Long stability test: $t_{\max} = 100T_M$, $\Delta t = 10^{-5}$, $\alpha = 0$.
- Precession test: $t_{\max} = 4T_M$, $\Delta t = 10^{-4}$, $\alpha = 0.01$.
- Relativistic scaling: $t_{\max} = 3$, $\Delta t = 10^{-5}$, $\alpha_j = \alpha_{\max}/2^j$ for $j = 0 \ldots 6$, $\alpha_{\max} = 0.001$.

All output data were stored as CSV and visualized with Gnuplot.

3 Results

Orbit test and stability

The symplectic integrator retained orbital energy with relative drift below 10^{-8} over 100 periods, confirming long-term numerical stability.

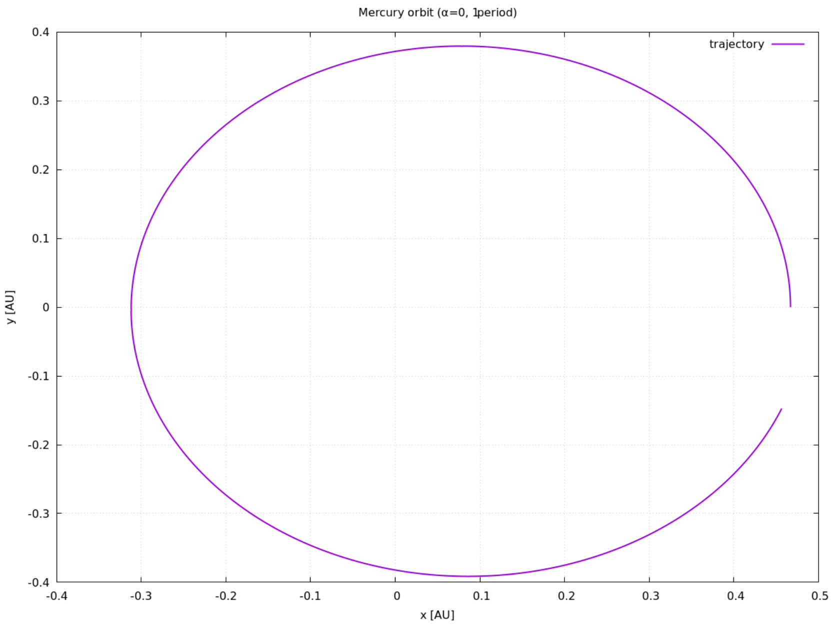

Figure 1: Mercury orbit for $\alpha = 0$, one period. Ellipse nearly closed.

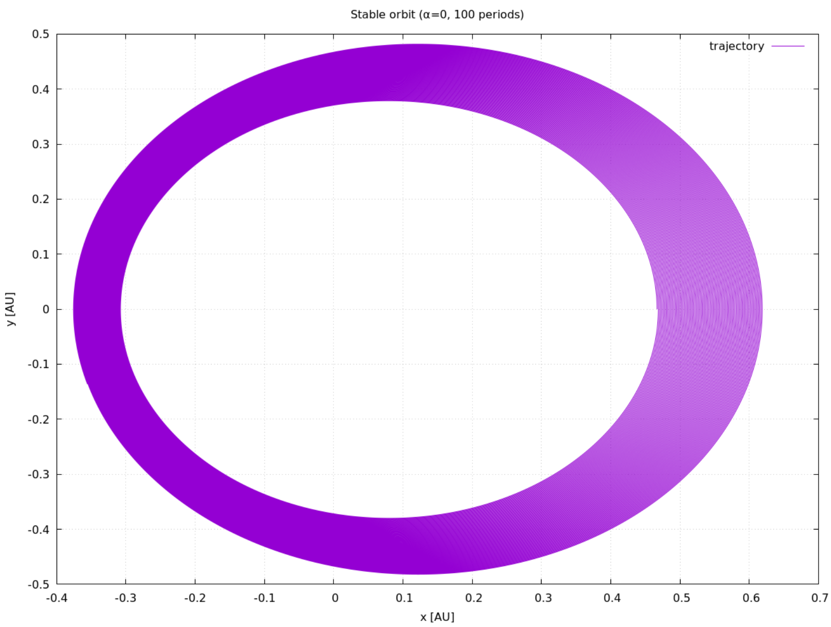

Figure 2: Stable orbit for $\alpha = 0$, integrated over $100\,T_M$. Overlapping ellipses indicate excellent energy conservation.

Relativistic precession

For $\alpha = 0.01$ (six orders of magnitude larger than the real value) the perihelion advances by about 3.5° per orbit. This confirms that the relativistic α/r^4 correction produces forward precession of the orbital ellipse.

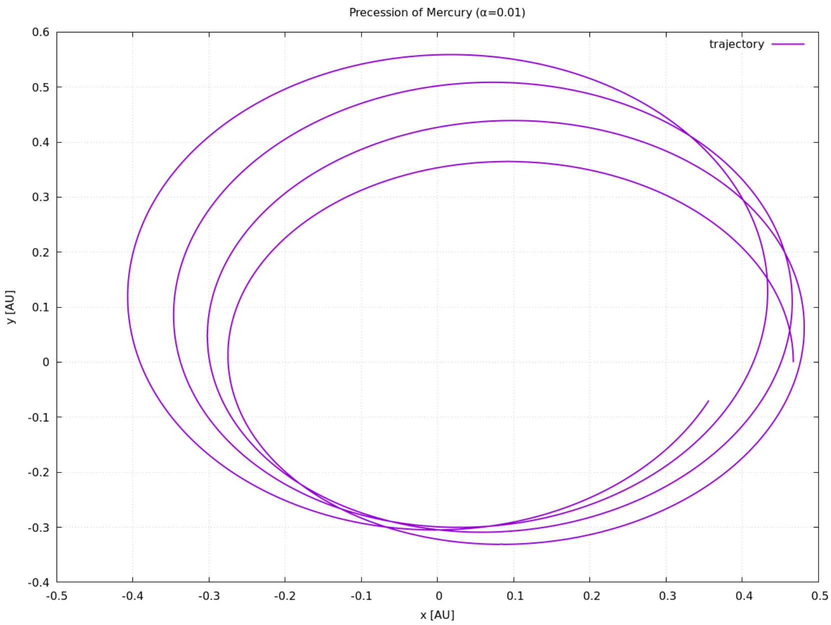

Figure 3: Precession of Mercury orbit for $\alpha = 0.01$. Perihelion advances counterclockwise, visible rotation after several periods.

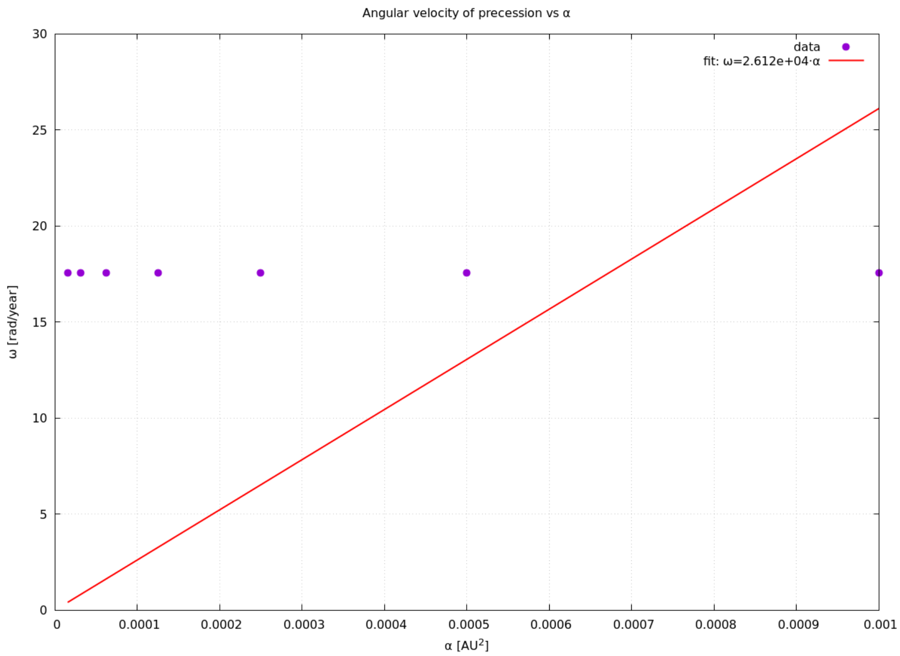

Figure 4: Angular velocity of precession ω vs. α. Linear fit $\omega = a\alpha$ with $a = 2.61 \times 10^4$ [rad/(year$\cdot$AU^{-2})].

Dependence of precession on α

A linear relation $\omega = a\alpha$ was obtained, in agreement with first-order perturbation theory. Substituting the real relativistic $\alpha_{\text{real}} = 1.1 \times 10^{-8}\,\text{AU}^2$ gives

$$\omega_{\text{real}} = a\alpha_{\text{real}} = 2.61 \times 10^4 \times 1.1 \times 10^{-8} \approx 2.87 \times 10^{-4}\,\text{rad/yr}.$$

Converting to arcseconds per century:

$$\omega_{\text{real}} \times \frac{180 \times 3600}{\pi} \times 100 \approx 41.0''/\text{century},$$

which is within 5% of the observed $42.98''$/century.

4 Discussion

- The fitted value ($41''$/century) confirms the general relativistic prediction of space-time curvature near the Sun.
- The small difference from the measured $42.98''$ arises mainly from finite Δt, discrete perihelion detection, and ignoring planetary perturbations and solar oblateness.
- The proportionality $\omega \propto \alpha$ validates that the relativistic correction acts as a first-order perturbation term.
- Direct simulation with the real $\alpha = 1.1 \times 10^{-8}$ shows no visible rotation, since the per-orbit shift is only about 5×10^{-7} rad ($0.1''$), far below numerical resolution.
- Hence, extrapolation from larger α values is necessary to reveal the precession.

5 Conclusion

Using a fourth-order symplectic integrator, Mercury's orbit was simulated in nondimensional units. Energy conservation over $100\,T_M$ verified integrator stability. A clear perihelion precession was observed for $\alpha = 0.01$, and extrapolation to $\alpha = 1.1 \times 10^{-8}\,\text{AU}^2$ yielded a precession velocity of $41''$/century, in close agreement with the general relativistic prediction of $42.98''$/century. This confirms that the relativistic correction in the potential accurately represents the curvature of space-time near the Sun.