Prioritization

1. 括号：()
   - 括号内的表达式总是首先被计算，用来强制指定某些操作的优先级。
2. 否定（NOT）：¬
   - 例如：$\neg P$
   - 否定运算符对单个命题取反，优先于其他逻辑运算符。
3. 合取（AND）：∧
   - 例如：$P \wedge Q$
   - 两个命题都为真时结果才为真，优先级高于"或"运算。
4. 析取（OR）：∨
   - 例如：$P \vee Q$
   - 两个命题至少一个为真时结果为真，优先级低于"与"运算。
5. 异或（XOR）：⊕
   - 例如：$P \oplus Q$
   - 两个命题中仅有一个为真时结果为真，优先级一般与析取（OR）相同。
6. 蕴含（IMPLICATION）：→
   - 例如：$P \rightarrow Q$
   - 当 P 为真且 Q 为假时结果为假，其他情况为真，优先级低于"与"与"或"运算。
7. 双条件（BICONDITIONAL）：↔
   - 例如：$P \leftrightarrow Q$
   - 当 P 和 Q 具有相同真值时结果为真，先级最低。

1. Predicate（谓词）：

- 定义: 谓词是一个表达式，通常包含一个或多个变量，并且只有在这些变量被赋予具体值时才可以确定其真值。例如，"x 是偶数"就是一个谓词，只有当你给 x 一个具体值时（比如 x = 4），你才能判断这个谓词是真的或是假的。
- 作用: 谓词用来描述某些属性或关系，它们的真值依赖于变量的取值。例如，P(x) 可以表示"x 是奇数"，它的真值取决于具体的 x 值。

2. Quantifier（量词）：

- 定义: 量词是用于描述谓词在其定义域中的某种普遍性或存在性的逻辑运算符。量词有两种主要类型：
  - Universal Quantifier（全称量词，∀）：表示"对所有 x, P(x) 为真"（$\forall x P(x)$），即谓词 P(x) 在定义域的每个元素上都为真。
  - Existential Quantifier（存在量词，∃）：表示"存在一个 x，使得 P(x) 为真"（$\exists x P(x)$），即谓词 P(x) 在定义域中至少有一个元素上为真。
- 作用: 量词对谓词进行逻辑量化，决定了谓词如何在整个定义域上被评估。它们帮助我们表达普遍的或存在的陈述，并将谓词从依赖于具体值的表达式提升为更为抽象的逻辑命题。

- Predicate 是描述性质或关系的表达式，依赖于变量的值。
- Quantifier 是对谓词进行量化的工具，用来描述谓词在整个定义域上的适用范围。

例如，谓词 $P(x)$ 可以表示"x 是偶数"，而量词可以帮助我们形成命题：

- 全称量词: $\forall x P(x)$ 表示"所有的 x 都是偶数"。
- 存在量词: $\exists x P(x)$ 表示"存在一个 x 是偶数"。

2.1 Predicates and quantifiers

Many mathematical statements contain variables. The statement "x is an odd number" is not a proposition because the statement does not have a well-defined truth value until the value of x is specified. If $x = 5$, the statement is true. If $x = 4$, the statement is false. The truth value of the statement can be expressed as a function p of the variable x, as in $P(x)$. The expression $P(x)$ is read "P of x." A logical statement whose truth value is a function of one or more variables is called a ***predicate***. If $P(x)$ is defined to be the statement "x is an odd number," then $P(5)$ corresponds to the statement "5 is an odd number." $P(5)$ is a proposition because it has a well-defined truth value.

A predicate can depend on more than one variable. Define the predicates Q and R as:

Domain?

$$Q(x, y) : x^2 = y$$

$$R(x, y, z) : x + y = z$$

The proposition $Q(5, 25)$ is true because $5^2 = 25$. The proposition $R(2, 3, 6)$ is false because $2 + 3 \neq 6$.

The ***domain*** of a variable in a predicate is the set of all possible values for the variable. For example, a natural domain for the variable x in the predicate "x is an odd number" would be the set of all integers. If the domain of a variable in a predicate is not clear from context, the domain should be given as part of the definition of the predicate.

数学领域之外的陈述也可以是谓词。例如，考虑以下陈述："这座城市的人口超过100万。"这里，"城市"是变量，定义域是美国所有城市的集合。当城市是纽约时，该陈述变为："纽约市的人口超过100万"，并且该陈述是真的。当城市是托莱多时，陈述变为："托莱多的人口超过100万"，但该陈述是假的。

注意，即使陈述$P(x)$在定义域中的所有值下都为真，如果陈述包含一个变量，它仍然被认为是一个谓词，而不是一个命题。例如，如果$P(x)$是陈述"x + 1 > 1"，且定义域是所有正整数，那么该陈述在定义域中的每一个值下都为真。但是，由于$P(x)$包含一个变量，它仍然被认为是一个谓词，而不是一个命题。

x is even iff it can be partitioned into 2 equal subsets.

$x \bmod 2 = 0$

Quantifier $\forall x P(x)$ is a proposition

Universal quantifier

$\forall x P(x)$

If all the variables in a predicate are assigned specific values from their domains, then the predicate becomes a proposition with a well-defined truth value. Another way to turn a predicate into a proposition is to use a quantifier. The logical statement $\forall x P(x)$ is read "for all $x, P(x)$" or "for every $x, P(x)$." The statement $\forall x P(x)$ asserts that $P(x)$ is true for every possible value of x in its domain. The symbol $\forall$ is a ***universal quantifier***, and the statement $\forall x P(x)$ is called a ***universally quantified statement***. $\forall x P(x)$ is a proposition because it is either true or false. $\forall x P(x)$ is true if and only if $P(n)$ is true for every n in the domain of variable x.

If the domain is a finite set of elements $a_1, a_2, \ldots, a_k$, then the equality below holds for any predicate P defined on the domain:

$$\forall x P(x) = P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_k)$$

If the domain is the set of students in a class and the predicate $A(x)$ means that student x completed the assignment, then the proposition $\forall x A(x)$ means: "Every student completed the assignment." Establishing that $\forall x A(x)$ is true requires showing that each student in the class did complete the assignment.

Some universally quantified statements can be shown to be true by showing that the predicate holds for an arbitrary element from the domain. An "arbitrary element" means nothing is assumed about the element other than the fact that it is in the domain. In the following example, the domain is the set of all positive integers:

$$\forall x \left(\frac{1}{x+1} < 1\right)$$

The statement is true because when x is assigned any arbitrary value from the set of all positive integers, the inequality $\frac{1}{x+1} < 1$ holds.

证明 Universal Quantifier 的命题：

- 要证明**$\forall x P(x)$** 为真，我们需要证明谓词P(x)对定义域中的每一个元素都成立。
- 文中提到的例子：如果定义域是所有正整数的集合，命题**$\forall x \left(\frac{1}{x+1} < 1\right)$** 是真的，因为对于定义域中的每一个x值，$\frac{1}{x+1}$ 都确实小于1。

Counterexample 的定义：

- 一个全称量词语句（$\forall x P(x)$）中的反例是指在定义域中找到的一个元素，使得谓词P(x) 为假。找到一个反例就足以证明这个全称量词语句是假的。

例子：

- 例如，考虑陈述**$\forall x(x^2 > x)$，其中定义域是正整数的集合。如果我们取x=1，那么我们可以看到$1^2 = 1$，所以$x^2 > x$** 这个陈述对x=1时不成立。因此，x=1就是一个反例，说明这个全称量词语句是假的。

反例的作用：

- 反例在逻辑证明中起到了至关重要的作用，因为它们可以直接反驳一个全称量词语句的真实性。只需要找到一个反例就可以证明这个语句是假的，而不需要对定义域中的所有元素进行验证。

"Can't refute"

特殊情况：

if the domain for variable x is empty, $\forall x P(x)$ is True

- 文中还提到，如果定义域是空集（即没有任何元素），通常假定全称量词语句**$\forall x P(x)$** 是真的。这是因为在没有元素的情况下，没有可能找到使谓词P(x)为假的元素。

Proving ∀x (1/(x+1) < 1) is true for an arbitrary positive integer x.

$$\forall x \left(\frac{1}{(x+1)} < 1\right)$$

$0 < x$ True for all positive integers x

$+1 \quad +1$ Add 1 to both sides

$1 < x+1$

$\frac{1}{(x+1)} < \frac{x+1}{(x+1)}$ Divide both sides by positive (x+1)

$\forall x \left(\frac{1}{(x+1)} < 1\right)$ Is true for all positive integers x

Existential quantifier $\exists x\, P(x)$

The logical statement $\exists x P(x)$ is read "there exists an x, such that P(x)." The statement $\exists x P(x)$ asserts that $P(x)$ is true for at least one possible value for x in its domain. The symbol $\exists$ is an ***existential quantifier***, and the statement $\exists x P(x)$ is called an ***existentially quantified statement***. $\exists x P(x)$ is a proposition because it is either true or false. $\exists x P(x)$ is true if and only if $P(n)$ is true for at least one value n in the domain of variable x.

If the domain is a finite set of elements $a_1, a_2, \ldots, a_k$, then the following equality holds for any predicate P defined on the domain:

$$\exists x P(x) = P(a_1) \vee P(a_2) \vee \ldots \vee P(a_k)$$

If the domain is the set of students in a class and the predicate $A(x)$ means that student x completed the assignment, then $\exists x A(x)$ is the statement: "There is a student who completed the assignment." Establishing that $\exists x A(x)$ is true only requires finding one particular student who completed the assignment. An ***example*** for an existentially quantified statement is an element in the domain for which the predicate is true. A single example is sufficient to show that an existentially quantified statement is true. However, showing that $\exists x A(x)$ is false requires showing that every student in the class did not complete the assignment.

Some existentially quantified statements can be shown to be false by showing that the predicate is false for an arbitrary element from the domain. For example, consider the existentially quantified statement in which the domain of x is the set of all positive integers:

$$\exists x (x + 1 < x)$$

The statement is false because no positive integer satisfies the expression $x + 1 < x$.

Note that it is typically assumed that the domain contains at least one element. If the domain for variable x is empty, then the statement $\exists x P(x)$ is false because there is no element in the domain for which $P(x)$ is true.

证明与反驳：

- **证明：要证明$\exists x P(x)$为真，只需找到一个例子（example）使得P(x)为真。一个例子就足够证明存在量词语句为真。**
- **反驳：要证明$\exists x P(x)$为假，你必须展示对于定义域中的每一个x，P(x)都为假。这通常是更困难的，因为需要排除所有可能的值。**

特殊情况：

- **文中提到，如果定义域为空集（即没有元素），通常假定存在量词语句$\exists x P(x)$为假，因为在这种情况下，没有一个元素能使P(x)为真。**

"Can't proof"

To proof:

Show the following statement is false: $\exists x P(x)$.

We have to:

Show that for every element n in the domain, $P(n)$ is false.

The fact that $\exists x P(x)$ is false means that there is no element n in the domain that makes $P(n)$ true, which is in turn is equivalent to the fact that every element in the domain makes $P(n)$ false.

Correct

"proof by contradiction"

$\exists x \quad (x + 1 < x)$

$$\begin{array}{r l}
x + 1 < x & \\
-x \quad\; -x & \\
\hline
1 \nless 0 &
\end{array}$$

Subtract x

The inequality does not hold regardless of the value of x

when we reach a contradiction,

$\exists x \quad (x + 1 < x)$ Is false

The predicate P is defined as follows.

$$P(x, y) : y = 2x - 6$$

The domain for variables x and y is the set of all positive integers.

Is $P(1, 5)$ a proposition? If yes, then give its truth value.

Yes, False

1	2	3

Check Next

✔ Expected: Yes, False

The logical expression $P(1, 5)$ represents the statement $5 = 2(1) - 6$, which is a false statement. Since $P(1, 5)$ has a well-defined truth value, the expression is a proposition.

Quantified statements

- The **universal** and **existential** quantifiers are generically called **quantifiers**.
- A logical statement that includes a universal or existential quantifier is called a **quantified statement**.
- The quantifiers ∀ and ∃ are applied before the logical operations (∧, ∨, →, and ↔) used for propositions.
- This means that the statement ∀x P(x) ∧ Q(x) is equivalent to (∀x P(x)) ∧ Q(x) as opposed to ∀x (P(x) ∧ Q(x)).

- A variable x in the predicate P(x) is called a ***free variable*** because the variable is free to take on any value in the domain.

- The variable x in the statement ∀x P(x) is a ***bound variable*** because the variable is bound to a quantifier.

1. 自由变量 (Free Variable)：
   - 在命题 $P(x)$ 中，变量 x 被称为自由变量，因为它可以在定义的域（domain）内取任意值。也就是说，x 不是固定的，可以独立变化。
   - 例如，假设 $P(x)$ 表示"x 是偶数"，那么 x 可以取任何整数值，而不会受到其他约束，这时 x 是自由的。
2. 约束变量 (Bound Variable)：
   - 在陈述 $\forall x\ P(x)$ 中，变量 x 被称为约束变量，因为它受到了量词（这里是全称量词 $\forall$）的约束。量词决定了 x 的取值范围，即 x 不再是自由的，而是被绑定在某个范围之内。
   - 例如，$\forall x\ P(x)$ 表示"对于所有的 x，命题 $P(x)$ 为真。"这里的 x 是被全称量词 $\forall$ 限定的，所以 x 受到了量词的约束。

无free variable 则是 prop.

有free variable 的我们则可 plug in variable → 不同的可能性

We can make quantifiers by bounding free variables.

1) The expression $\exists x P(x)$ is a proposition.

x is bound

- [x] True
- [ ] False

- A statement with no free variables is a proposition because the statement's truth value can be determined.
- In the statement (∀x P(x)) ∧ Q(x), the variable x in P(x) is **bound** by the universal quantifier, but the variable x in Q(x) is **not bound** by the universal quantifier.
- Therefore the statement (∀x P(x)) ∧ Q(x) is not a proposition.
- In contrast, the universal quantifier in the statement ∀x (P(x) ∧ Q(x)) binds both occurrences of the variable x. Therefore ∀x (P(x) ∧ Q(x)) is a proposition.

- 命题是没有自由变量的陈述，可以确定真值。
- 如果一个表达式中有自由变量，它不是一个命题，因为自由变量的取值无法固定，无法确定其真值。
- 通过使用量词（例如全称量词 ∀ 或存在量词 ∃），可以将自由变量绑定，使其成为命题。

Translating

$P(x)$: x came to the party

$S(x)$: x was sick

∀x

The statement "Everyone was not sick" has the same meaning as "$\forall x \neg S(x)$" because the two statements have the same truth value regardless of who was invited to the party and whether they were sick.

3) Select the logical expression that is equivalent to the statement: "There is a new employee who met his deadline."

- [ ] $\exists x(N(x) \wedge D(x))$
- [x] $\exists x(N(x) \vee D(x))$

Incorrect

The statement is saying that there is someone who is both a new employee *and* who met his deadline, not *or* met the deadline.

Logical equivalence with quantified statements

Example 2.2.7

The domain is a group working on a project at a company. Define the following predicates.

prop. (bounded)

- $D(x)$: x missed the deadline.
- $N(x)$: x is a new employee.

Consider a situation in which there are five people in the group. The following table gives values for the predicates D and N for each member of the group. For example, Bert did not miss the deadline because the truth value in the row labeled Bert and the column labeled D(x) is F.

Name	D(x)	N(x)
Sam	T	F
Beth	T	T
Melanie	F	T
Al	T	T
Bert	F	T

3. 如何使用这些数据：

这个例子展示了如何使用定义域中的人以及他们的属性来验证带有全称量词或存在量词的逻辑语句。通过查看表中的每个成员，分别判断谓词 $D(x)$ 和 $N(x)$ 在定义域中的真假。

例如：

- 如果我们有一个全称语句 $\forall x(N(x) \to D(x))$，意思是“对于所有人，如果他们是新员工，则他们错过了截止日期”，我们可以通过检查表中的每个人来验证这个语句是否为真。
- 从表中可以看到，Melanie 是新员工（$N(\text{Melanie}) = T$），但她没有错过截止日期（$D(\text{Melanie}) = F$），所以这个全称命题是假的。

Example 2.2.7

- $D(x)$: x missed the deadlin
- $N(x)$: x is a new employee

- $\forall x\,(D(x) \lor N(x))$ True
- $\forall x\,((x \neq \text{Sam}) \to N(x))$ True
  if ur not Sam, you are new
- $\exists x\,(\neg D(x) \land N(x))$ Melanie / Bert True
- $\forall x\,(\neg D(x) \to \neg N(x))$ False M/Bert

False.

T → F

Name	D(x)	N(x)
Sam	T	F
Beth	T	T
Melanie	F	T
Al	T	T
Bert	F	T

$\boxed{\text{Easiest}}$

Show $\forall x\, P(x) \rightarrow$ false by counterexample.

Show $\exists x\, P(x) \rightarrow$ true by single example.

$\neg(p \vee q) = \neg p \wedge \neg q$

2.3 De Morgan's law

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

De Morgan's law for quantified statements

- The negation operation can be applied to a quantified statement, such as **¬∀x F(x)**. If the domain for the variable x is the set of all birds and the predicate F(x) is "**x can fly**", then the statement **¬∀x F(x)** is equivalent to:
  - "Not every bird can fly."
  - which is logically equivalent to the statement:
  - "There exists a bird that cannot fly.

$\neg \forall x F(x) \equiv \exists x \neg F(x)$

$\neg \exists x F(x) \equiv \forall x \neg F(x)$

Figure 2.4.1: De Morgan's law for universally quantified statements.

$\forall : \wedge$

$\exists : \vee$

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\equiv \qquad \equiv$$

$$\neg(P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_n)) \equiv \neg P(a_1) \vee \neg P(a_2) \vee \ldots \vee \neg P(a_n)$$

Figure 2.4.2: De Morgan's law for existentially quantified statements.

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

$$\equiv \qquad \equiv$$

$$\neg(P(a_1) \vee P(a_2) \vee \ldots \vee P(a_n)) \equiv \neg P(a_1) \wedge \neg P(a_2) \wedge \ldots \wedge \neg P(a_n)$$

$\neg\forall x(P(x) \wedge \neg Q(x))$

$\exists x(\neg P(x) \vee Q(x))$

De Morgan's law says that $\neg\forall x(P(x) \wedge \neg Q(x)) \equiv \exists x \neg(P(x) \wedge \neg Q(x))$.

De Morgan's law for propositions can be applied to show that

$\exists x \neg(P(x) \wedge \neg Q(x)) \equiv \exists x(\neg P(x) \vee \neg\neg Q(x))$.

Finally, the double negation law can be used to replace $\neg\neg Q(x)$ with $Q(x)$.

$\neg\exists x(P(x) \vee Q(x))$

$\forall x(\neg P(x) \wedge \neg Q(x))$

De Morgan's law says that

$\neg\exists x(P(x) \vee Q(x)) \equiv \forall x \neg(P(x) \vee Q(x))$.

De Morgan's law for propositions can be applied to show that
$\forall x \neg(P(x) \vee Q(x)) \equiv \forall x(\neg P(x) \wedge \neg Q(x))$.

$\neg\exists x(\neg P(x) \wedge Q(x))$

$\forall x[\neg(\neg P(x)) \wedge Q(x)]$

$\forall x[P(x) \vee \neg Q(x)]$

$\forall x(P(x) \vee \neg Q(x))$

De Morgan's law says that

$\neg\exists x(\neg P(x) \wedge Q(x)) \equiv \forall x \neg(\neg P(x) \wedge Q(x))$.

De Morgan's law for propositions can be applied to show that

$\forall x \neg(\neg P(x) \wedge Q(x)) \equiv \forall x(\neg\neg P(x) \vee \neg Q(x))$.

Finally, the double negation law can be used to replace $\neg\neg P(x)$ with $P(x)$.

Every patient was given the medication.

- $\forall x\ D(x)$
- Negation: $\neg\forall x\ D(x)$
- Applying De Morgan's law: $\exists x\ \neg D(x)$
- English: Some patient was not given the medication

Every patient was given the medication or the placebo or both.

this means the inclusive "or"

- $\forall x\ (D(x) \vee P(x))$
- Negation: $\neg\forall x\ (D(x) \vee P(x))$
- Applying De Morgan's law: $\exists x\ (\neg D(x) \wedge \neg P(x))$
- English: There is a patient who was not given the medication and not given the placebo.

$$\neg\forall x\,(D(x) \vee P(x))$$
$$\exists x\,\neg(D(x) \vee P(x))$$
$$\exists x\,(\neg D(x) \wedge \neg P(x))$$

- **Original statement**: "Every patient was given the medication or the placebo, or both."
- **Negation (after applying De Morgan's law)**: "There is a patient who was not given the medication and not given the placebo."

In summary, the original statement asserts that all patients received either the medication, the placebo, or both. The negation means that at least one patient did not receive either the medication or the placebo.

Select the proposition that is logically equivalent to $\forall x(P(x) \rightarrow Q(x))$.

$\exists x$ ($P(x)$ $\land$ $\neg Q(x)$)

1 | 2 | 3

Check   Next

✕ Each incorrect answer is highlighted.

Expected: $\neg\exists x(P(x) \land \neg Q(x))$

$\forall x(P(x) \rightarrow Q(x))$

$\equiv \forall x(\neg P(x) \lor Q(x))$ by conditional identity

$\equiv \neg\neg\forall x(\neg P(x) \lor Q(x))$ by double negation law

$\equiv \neg\exists x\neg(\neg P(x) \lor Q(x))$ by De Morgan's law

$\equiv \neg\exists x(\neg\neg P(x) \land \neg Q(x))$ by De Morgan's law

$\equiv \neg\exists x(P(x) \land \neg Q(x))$ by double negation law

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Select the proposition that is logically equivalent to $\exists x(\neg P(x) \rightarrow \neg Q(x))$.

$\exists x(\neg(\neg P(x) \lor \neg Q(x)))$

$\exists x\neg(\neg P(x) \land Q(x))$

$-\forall x(-P(x) \land Q(x))$

2.4 Nested Quantifier

Nested Quantifiers

- A logical expression with more than one quantifier that bind different variables in the same predicate is said to have **nested quantifiers**.

$\forall x \exists y P(x,y)$ x and y are both bound.

$\forall x P(x,y)$ x is bound and y is free.

$\exists y \exists z T(x,y,z)$ y and z are bound. x is free.

2. 例子解析：

例子 1： $\forall x \exists y P(x,y)$

- 解释：这里有两个量词，$\forall x$ 和 $\exists y$，它们分别绑定变量 x 和 y。
  - $\forall x$：表示对所有的 x，都存在一个 y，使得命题 $P(x,y)$ 成立。
  - $\exists y$：对于每一个 x，都存在一个特定的 y。
  - 结论：在这个表达式中，x 和 y 都是绑定变量。

例子 2： $\forall x P(x,y)$

- 解释：这个表达式只有一个量词 $\forall x$，它绑定了变量 x，但是 y 是自由变量，因为没有量词绑定它。
  - $\forall x$：表示对于所有的 x，命题 $P(x,y)$ 成立。
  - y 是自由的：因为 y 没有被量词绑定，它可以取任何值，但它的取值不影响 x 的范围。
  - 结论：在这个表达式中，x 是绑定变量，而 y 是自由变量。

例子 3： $\exists y \exists z T(x,y,z)$

- 解释：这里有两个存在量词 $\exists y$ 和 $\exists z$，它们分别绑定了 y 和 z 变量。
  - $\exists y$ 和 $\exists z$：表示存在某个 y 和某个 z，使得命题 $T(x,y,z)$ 为真。
  - x 是自由的：因为没有量词绑定 x，所以 x 是自由变量。
  - 结论：在这个表达式中，y 和 z 是绑定变量，而 x 是自由变量。

- Two quantifiers are nested if one is within the scope of the other.

- ∀x ∃y (x + y = 0)

$\forall x \exists y\, P(x, y)$

1. 嵌套量词的定义：

- 两个量词被称为嵌套量词，当一个量词位于另一个量词的作用范围（scope）内时。例如：

$$\forall x \exists y (x + y = 0)$$

在这个表达式中，$\exists y$ 的量词位于 $\forall x$ 的作用范围内，因此这是嵌套量词。

2. 表达式解释：

- $\forall x \exists y (x + y = 0)$ 的意思是："对于每个 x，都存在一个 y，使得 $x + y = 0$。" 这表示，对于每一个 x，我们都可以找到一个相应的 y，使得它们的和为 0。这个表达式中的 x 和 y 由两个不同的量词绑定，所以它们是嵌套的。

3. 使用谓词表示：

- 可以将这个逻辑表达式转换成使用谓词 $P(x, y)$ 的形式：

$$\forall x \exists y P(x, y)$$

其中 $P(x, y)$ 表示 $x + y = 0$。这表示 "对于每个 x，都存在一个 y，使得 $P(x, y)$ 为真。"

- 另一个表达式 $\forall x Q(x, y)$ 是在量词外部引入谓词 $Q(x, y)$，其中 $Q(x, y)$ 可能是另一个已经嵌套的逻辑表达式。比如这里的 $Q(x, y)$ 定义为 $\exists y P(x, y)$，代表了嵌套量词的存在。

4. 总结：

- 嵌套量词：当一个量词处于另一个量词的作用范围内时，它们是嵌套的。比如 $\forall x \exists y$，x 的范围在 y 之前，这就形成了嵌套量词。
- 形式表示：可以通过定义谓词 $P(x, y)$ 和 $Q(x, y)$ 来简化表达，便于理解逻辑表达式的嵌套结构。

If the predicate is true after all the variables are set, then the quantified statement is true. If the predicate is false after all the variables are set, then the quantified statement is false.

2. **例子：**
   - **考虑以下量词命题，其定义域是所有整数的集合：**

$$\underbrace{\forall x\exists y}_{\text{quantifier}}\underbrace{(x+y=0)}_{\text{predicate}}$$

     - **解释：在这个命题中，普遍选择者（universal player）首先选择 x 的值。无论普遍选择者选择哪个值 x，存在选择者（existential player）都可以选择 $y=-x$，这将导致 $x+y=0$ 成立。因为存在选择者总能成功让谓词为真，所以该命题 $\forall x\exists y(x+y=0)$ 是真的。**

3. **交换量词顺序：**
   - **如果我们交换量词的顺序，得到以下命题：**

$$\exists x\forall y(x+y=0)$$

     - **解释：现在，存在选择者首先选择一个 x 的值。无论选择哪个值 x，普遍选择者都可以选择某个 y，使得谓词为假。例如，如果 x 是整数，那么 $y=-x+1$ 也是一个整数，此时 $x+y=1\neq 0$。因此，普遍选择者总是可以找到一个值使得命题为假，所以命题 $\exists x\forall y(x+y=0)$ 是假的。**

English translation

1) $\forall x \exists y (xy = 1)$

Explanation:

- For every x, there exists a y such that $xy = 1$.

2) $\exists x \forall y (xy = 1)$

Explanation:

- There exists an x such that for every y, $xy = 1$.

---

1) $\forall x \forall y (xy = 1)$

Explanation:

- The statement says that for every x and every y, $xy = 1$ must hold true.

2) $\exists x \exists y (xy = 1)$

Explanation:

- The statement says that there exists some x and some y such that $xy = 1$ holds true.

---

1) $\forall x \exists y(xy = 1)$ F

- [x] True
- [ ] False

"for every x, there exists a y such that $xy=1$"

as $x=0$

Incorrect

If the universal player selects x to be 0, then there is no value that the existential player can select for y that can make $xy = 1$.

2) $\exists x \forall y(xy = 1)$ F

- [x] True
- [ ] False

"there exists a x such that for every y, $xy=1$"

Incorrect

If the existential player selects $x = 0$, then $xy \neq 1$ for any choice of y. If the existential player selects some $x \neq 0$, the universal player selects y to be $\frac{2}{x}$ and then $xy = 2$.

- **Why it is False**: This statement implies that there is a single value of x such that multiplying it by any y always results in 1. However, no such x exists.
  - If the existential player (who selects x) chooses $x = 0$, then for any y, $xy = 0$, which clearly does not satisfy $xy = 1$.
  - If the existential player chooses $x = 1$, then $1 \times y = y$, which only equals 1 when $y = 1$, but not for all values of y.
  - In fact, no matter what x is chosen, there will always be some value of y that makes $xy \neq 1$. For example, if $x = 2$, the universal player can choose $y = 2$, and $2 \times 2 = 4 \neq 1$.
- **Summary**: The statement is false because no matter what value of x is chosen, it cannot satisfy $xy = 1$ for all values of y.

$S(x,y)$ = "x and y are soul mates"

① $\forall x \exists y\, S(x,y)$, for every x, there is a y and x & y are sm

② $\exists x \forall y\, S(x,y)$, for a given x, for all y, x & y are sm

Order matters !!!

$O(x,y)$ = x is obsessed with y

$\forall_x \exists_y O(x,y)$ everyone has someone they are obsessed with

$\exists_y \forall_x O(x,y)$ There is someone that everyone is obsessed with.

给定逻辑表达式：

$$\forall x(A(x) \rightarrow \exists y M(x,y))$$

合适的移动

这个表达式的意思是："对于每一个 x，如果 x 是成年人 $A(x)$，那么存在一个 y 使得 x 与 y 结婚 $M(x,y)$。"

3. 移动量词的规则：

在某些情况下，量词可以在逻辑表达式中移动，而不改变表达式的含义。例如，上述表达式可以重写为：

$$\forall x \exists y(A(x) \rightarrow M(x,y))$$

- Let P(x,y) define "x has made friends with y"
- Domain is "The students in CS113"
- Translate ∀x∀yP(x,y):
  - Every Student in CS113 has made friends with every student in CS113, including themselves.
- Translate ∀y∀xP(x,y):
  - Every Student in CS113 has made friends with every student in CS113, including themselves.

- Let P(x,y) define "x has made friends with y"
- Domain is "The students in CS113"
- Translate $\forall x \exists y P(x,y)$:
  - Every Student in CS113 has made friends with some student in CS113.
- Translate $\exists y \forall x P(x,y)$:
  - There is a Student in CS113 that has made friends with every student in CS113, including themselves.

- Let P(x,y) denote "x + y = 0" with a domain of all real numbers.
  - $\forall x \forall y P(x,y)$
  - For all real numbers x and y, x + y = 0
  - True or False? False
    - Counterexample: let x = 10 and y = 1,000
    - 10 + 1,000 ≠ 0

$P(x,y)$ = x is a coworker of y

$\forall x \forall y P(x,y)$, for each x, all y's are coworkers of x

$$(x \neq y) \longrightarrow P(x,y)$$ someone else

$P(x,y)$ = x is jealous of y

$\forall x \exists y P(x,y)$, for each x, there is a y that x is jealous of (including themselves)

$$\left.\begin{array}{l} \forall x \exists y (x \neq y) \rightarrow P(x,y) \\ \forall x \exists y (x \neq y) \land P(x,y) \end{array}\right\}$$ excluding themselves.

De Morgan's laws for nested quantified statements.

" $\forall \to \exists$ "

" $\exists \to \forall$ "

$\neg\forall x\forall y P(x,y) \equiv \exists x\exists y\neg P(x,y)$
$\neg\forall x\exists y P(x,y) \equiv \exists x\forall y\neg P(x,y)$
$\neg\exists x\forall y P(x,y) \equiv \forall x\exists y\neg P(x,y)$
$\neg\exists x\exists y P(x,y) \equiv \forall x\forall y\neg P(x,y)$

The table below shows the values of predicates Q, R, and S for every possible input. The domain for the input to each function is $\{1, 2, 3, 4\}$.

Ex: $Q(3)$ is false because the entry in row 3, column Q is F.

	Q	R	S
1	F	F	F
2	T	T	F
3	F	T	F
4	F	F	F

Determine the truth value of each expression.

(1) $\exists x\forall y(\neg Q(x) \wedge ((x \neq y) \to Q(y)))$ False

(2) $\exists x\forall y(\neg R(x) \wedge ((x \neq y) \to R(y)))$ False

(3) $\exists x\forall y(\neg S(x) \wedge ((x \neq y) \to S(y)))$ False

1 2 3 4

Check Next

✔ Expected: False, False, False

- The domain for variables x and y is the set of students in a class.
- The predicate $R(x, y)$ means that student x knows the test results for student y.

Select the right items so that the logical expression is equivalent to the following sentence.

Everyone knows someone else's test results.

∀x ∃y $((x \neq y)$ → $R(x, y))$

1 2 3 4

Check Next

✗ Each incorrect answer is highlighted.
Expected: $\forall x \exists y((x \neq y) \wedge R(x, y))$

For each person x, there is a person y, such that x is different from y and x knows y's test results.
(For each person x), (there is a person y), ($x \neq y$ and x knows y's test results).
(For each person x), (there is a person y), ($x \neq y \wedge R(x, y)$).
$\forall x \exists y((x \neq y) \wedge R(x, y))$

- The domain for variables x and y is the set of students in a class.
- The predicate $R(x, y)$ means that student x knows the test results for student y.

Select the right items so that the logical expression is equivalent to the following sentence.

Someone knows someone else's test results.

∃x ∃y $((x \neq y)$ ∧ $R(x, y))$

1 2 3 4

Check Next

Done. Click any level to practice more. Completion is preserved.

✓ Expected: $\exists x \exists y((x \neq y) \wedge R(x, y))$

There is a person x, and there is a person y, such that x is different from y and x knows y's test results.
(There is a person x), (there is a person y), ($x \neq y$ and x knows y's test results).
(There is a person x), (there is a person y), ($x \neq y \wedge R(x, y)$).
$\exists x \exists y((x \neq y) \wedge R(x, y))$