applied functional analysis

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- compact operators are the ones well approximated by finite dimensional operators (there is a notion of dimension for operators).
- "Hamming distance": $d(x, y) = \#\{i \mid x_i \neq y_i\}$
- Remember that module is just a field replaced by a ring for a vector space.
- Intuition of complete is "small"
- (Question) what do you mean by $\tilde{X}$ as set of all classes of equivalence. Might be worth to check!
- In metric space, compact iff sequentially compact. There is a discussion of why this is so. $https : //math.stackexchange.com/questions/44907/whats-going-on-with-compact-implies-sequentially-compact$ Questionz; don't we need to consider "Any" subsequence in the set $\{f_i\}$? I don't understand the product topology on $\{0, 1\}^{[0,1]}$
- Continuity implies sequential continuity, but not the otherwise.
- Bolzano Weierstrass (bdd sequence has convergent subsequence) for $\mathbb{R}^2$ case can be proved by bisecting a square sequentially and using the completeness of $\mathbb{R}^2$.

Lecture 6

- L^1 and L^∞ not dual to each other (L^∞ is larger).

Lemma 1. *Q: how can we acquire q norm of g? When we know that it is in L^q or only know that it is integrable on finite measure...*

(i) $g \in L^q \to \|g\|_{L^q} = \sup_{\|f\|_{L^p} \leqslant 1} |\int fg|$.

(ii) If g is integrable on all sets of finite measure and $\sup_{\|f\|_{L^p} \leqslant 1,\ f \text{ is simple}} = |\int fg| = M < \infty$ *then* $\|g\|_{L^q} = M$.

Proof. The first conditions simply say about the existence a priori. Note that g integrable on all sets of finite measure does not necessarily imply $g \in L^q$. part (i): left direction is by Holder. Right direction is proven by considering specific functions, keeping in mind that the sign times the function is its absolute value (note about sigma finite: a space can be partitioned by finite sets: `https://math.stackexchange.com/questions/98965/significance-of-sigma-finite-measures`). Same spirit for part (ii). □

- Using this we can prove that the dual of L^p is L^q. We could directly prove in the case of finite measure. We defined a measure ν and used the absolute continuity with respect to μ (Radon-Nicodim). Then we use the above lemma. We can extend by limiting argument.
- Hahn-Banach theorem(page 20) has a little bit different flavor. l is bounded by some p satisfying certain conditions instead of asserting that $\|l\| = M$. I'll get back later (p18).
- Radon measure
- Hahn-Banach theorem
- Characterization of norm
- Bidual and weak convergence, how the topology generalizes.
- Homework, check uniqueness and the continuity on the boundary.

Digression

Baire category theorem

Definition 1. *E is nowhere dense: $(\bar{E})^\circ = \phi$, for example a point in $\mathbb{R}^d$ or Cantor set(it is closed set and the closure is itself). First category(idea: special) is the countable union of nowhere dense sets in X. Second category is something that is not first category. Generic(idea: typical) set is complement of first category.*

Theorem 2. *Complete metric (idea: continuum) space is second category (X cannot be written as the countable union of nowhere dense sets).*

Remark 1.
- From the theorem, we can prove that infinite dimensional Banach space is uncountable (`https://math.stackexchange.com/questions/217516/let-x-be-an-infinite-dimensional-banach-space-prove-that-every-hamel-basis-of`, `https://math.stackexchange.com/questions/854227/finite-dimensional-subspace-normed-vector-space-is-closed`)
- No relationship with measure
- Open dense set is generic (it is very large).

Heine Borel

- For $\mathbb{R}^n$, compact iff closed and bounded.
- For infinite dimensional Banach space, closed and boundedness does not imply compact: for example, consider the set of basis $S = (1, 0, \dots), (0, 1, \dots), (0, 0, \dots, 1, \dots)$ with the metric l^∞. This is clearly closed and bounded in this metric. But this is not compact (iff totally bounded(fail, this is a notion similar to compactness but whence not necessarily closed) and complete).
- Note for completeness: it is a notion about metric not of topology. Complete space can be homeomorphic to non-complete space.
- Another way is to see that closed and bounded subset of infinite dimensional Banach space is "too large" from the above remark (just realize that it has uncountably many elements and infinite basis so maybe it is reasonable to think that we cannot make it compact easily as in finite dimensional space).

Completing the space

Theorem 3. *Every metric space has a completion.* `https://en.wikipedia.org/wiki/Complete_metric_space`

- Cauchy sequence is each element, where the distance is defined by the distance as n goes to infinity. Isometry since the constant sequence is included. Choose $\leqslant \frac{1}{i}$ index for each x^i (kind of diagonal).`https://math.stackexchange.com/questions/2019077/the-set-of-all-equivalence-classes-from-cauchy-sequences-is-complete`

Ordinary Differential Equation

- We are considering ODE

$$u' = f(t,u), u(t_0) = u_0$$

  Intuitively, we think that the initial value gives us the nearby slope so nearby points, and we can iterate and so on.

- We can interpret $\frac{u'}{u}$ as per capita growth rate.

- (Thm 2.24) $f(t,u)$ is a continuous function. For (t_0, u_0), we can always find I and consequently $u : I \to \mathbb{R}$ ("local solution").

- $T_1 = Nh$ is our rectangle for u_ϵ. We need to choose good $T \leqslant T_1$.

- $M = sup\{| \ f(t,u) \ || \ (t,u) \in R_1\}$: essentially maximum of the slope, because f is given by a slope in ODE.

  $T = \min(T_1, L/M)$: if $L < T_1 M$, Then choose $\frac{L}{M}$ (just to be safe).

- Define u_ϵ. $u \leqslant \delta$ (corresponding to difference in x coordinate), $Mh \leqslant \delta$ (corresponding to difference in y coordinate).

- Check limiting argument regarding (2.26)! It is indeed easy to see when we use Lebesgue integration(via dominated convergence thm). Super useful fact: If f is bounded, then f is Riemann integrable iff f is almost surely continuous. If f is Riemann integrable (range $[a,b]$), then it is same as Lebesgue integral! (`https://math.stackexchange.com/questions/829927/general-condition-that-riemann-and-lebesgue-integrals-are-the-same`)

- Note also f does not have to be continuous on $\mathbb{R}^2$. It only has to be continuous in the domain including R_1.

- **When f is Lipschitz, we do have uniqueness, which leads to Gronwall's inequality**

- Gronwall's inequality: if certain condition holds, corresponding inequality holds as if it is equality.

- Thm 2.26: δ really corresponds to T before, and T corresponds to T_1 before.

- Goal: $| u(t) - u_0 | \leqslant L$ when $| t - t_0 | \leqslant \delta$. Define a function $D = \{0 \leqslant \eta \leqslant \delta \dots\}$ and use continuity to show that it is closed and open.
- Lipschitz concerns inequality of difference between two functions u and v. But in order to show uniqueness we set $w = | u - v |$. Peano theorem naturally follows since $w \geqslant 0$, and $u_0 = 0$.

Poisson equation with the rectangle boundary

- Green's formula - seem to work only for half space, sphere separation of variable - does not give me appropriate formula. The best way is to just guess!

  For finite difference approx method, we do not include boundary in the matrix, hence $m - 1 \times m - 1$ matrix.

L^p is complete

- First, take Cauchy Sequence. Take a sub-sequence. Construct two partial sums. Show the absolute value one by using convergence by monotonic convergence theorem. then the other one converges and define it as f. This is convergent almost everywhere, but we also prove that this subsequence converges in L^p (used dominated convergent theorem). Now use this index n_K to use triangle inequality in order to finish it.

The contraction Mapping Theorem

- Contraction map is a special case of Lipschitz function where it maps from metric space to itself and Lipschitz constant $K \leqslant 1$. It means we "contract" the perturbation as we impose T repeatedly.
- Intuition: $T(B_r(x)) \subset B_{cr}(T(x))$; $f = g + Kg + K^2 g + \dots$ (called Neumann series expansion), solution f exists only when imposing K gets smaller and smaller...so $\lim_{n\to\infty} K^n \to 0$

  Theorem 4. *Strict contraction on a complete metric space has a unique fixed point.*

  Proof. One way to construct Cauchy sequence is to show that for $n \geqslant m$, $d(x_n, x_m) \leqslant c_{n,m}$ and let $n \to \infty$, find that this sequence is indeed Cauchy. Uniqueness simply follows from the fact that the constant is less than 1. That is why it needs to be "strict." □

- Just be mindful when you apply, it should be $f_1 - f_2$, not $f(x_1) - f(x_2)$

- We can solve $f(x) = 0$ by recasting into $x = Tx$ (there are many ways). We know the uniqueness/existence, and we can solve this by using "iteration scheme" $x_{n+1} = Tx_n$.

  For example, $x^2 - a = 0 \to x = \frac{1}{2}(x + \frac{a}{x})$. So $Tx = \frac{1}{2}(x + \frac{a}{x})$. Express this in the form $|Tx_1 - Tx_2| = K \cdot |x_1 - x_2|$. (caveat: complete space is closed, but not the other way around ($\mathbb{Q}$), $https://math.stackexchange.com/questions/6750/difference-between-complete-and-closed-set$)

- $$f(x) = \int_a^b k(x,y)f(y)dy + g(x)$$

  , Fredholm Operator.

- $-v'' = f, v(0) = 0, v(1) = 0$ can be solved by integrating twice (integration by parts) directly. When $f \to -qv + f$, recast into the FPT.

- We can solve ODE

  $$u'(t) = f(t, u(t)), u(t_0) = u_0$$

  when f is globally Lipschitz continuous function (Obviously, recast into integration, use $\delta < 1/C$ and cover overlapping intervals, using uniqueness).

- Local existence theorem in 3.10, what's the difference with the previous one? In Theorem 3.10, the range of u is a ball around u_0 so when we tried to see that T maps from X to X, we have some restriction, namely η, the distance from t_0, has to be less than or equal to R/M. As we move t_0 to iterate, the value of R and M changes. For η being less than or equal to Lipschitz constant divided by two, and the value does not change as we move along. This is why we have $\delta = \min(T, R/M)$.

Exercise

- 3.7

  The problem here is $\sin u$ and $u'(c)$. We could just introduce the new function v. v can be simply obtained by integration by parts. We here used the crucial fact that twice derivative of sin is $-\sin$. Before solving, always think if we can simplify the given PDE/ODE. Now in order to see if $\sin x$ is Cauchy we simply just use the subtraction of sine functions to simply just multiplication of trigonometric functions. It's then easier to simplify. (`https://math.stackexchange.com/questions/2016731/how-to-prove-that-sin-x-is-a-lipschitz-continuous-function-on-the-real-line`)

- 3.5

  For matrix, in applying contraction mapping theorem, it may be useful to use L_2 norm instead of L_∞ norm. This is the biggest singular value. Remember the fact about the existence of norm bounded by slightly greater norm than spectral radius.

- In order to use diagonal dominance, we needed to extract $a_{p,p}$, so we let WLOG infinity norm of x is 1 and look at that specific element.

- 2.13

- for the case $0 \leqslant \alpha < 1$, we could just consider $c(\alpha)t^{g(\alpha)}$

- 2.3: In order to have continuous extension to closed set, we need to have some kind of regularity. In this case, uniform continuity is sufficient, since Cauchy sequence x_i will imply that $f(x_i)$ is Cauchy sequence. We can argue as follows the continuity on the boundary. If a and b are close, then a_n and b_n are close, then $f(a_n)$ and $f(b_n)$ are close, so $f(a)$ and $f(b)$ are close.

- Compact set in a metric space has convergent sub-sequence. Sometimes it is useful to construct non-convergent sub-sequence in order to show that the assumption is wrong (for example $x_n \nrightarrow x$)

- (Ex 5.10) I'm not sure how to use Ascoli-Arzela here. Equi-continuity seems too constrained.

Banach Space

- C^k norm is a Banach space with respect to the C^k norm.

- Sobolev space is Banach space.

Basis

- Schauder basis(exists sum with unique choice of scalars), Hamel basis(or algebraic basis, maximal linearly independent set of vectors): how do we know that these two are different?

- In page 118, we talked about Hamel basis of $C([0,1])$ which is an extension of the set of monomials. We can construct (sum of coefficient of monomials x^n multiplied by n) a discontinuous linear functional (since unbounded) on $C([0,1])$.

Holder

- Young's inequality intuition:

$$ab \leqslant \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$$

where f is strictly increasing function. Put $f(x) = x^{p-1}$

- For Minkowski, use the fact that $p'(p-1) = p$. In order to use this, we need to maybe try to bound exponent as p or $p-1$ so we can use Holder.

- Proof that L^p is Banach: identify the limit pointwise. Show that the identified series is in l^p. Now sho that $x^{(k)} \to y$ in l^p sense (just use the fact that $\sum \lim_{l\to\infty} \leqslant \limsup_{l\to\infty} \sum$ where $\sum$ is finite for the first one).

Open mapping theorem, closed mapping theorem

- Open mapping theorem: onto, bounded, Banach spaces, then the inverse is continuous (bounded).

  Proof. First reformulate in terms of ball. Use the fact that Y is complete (and Baire Category theory) and express in terms of countably many unions of balls. Specify one interior point. Play with it to show the claim, but using closeness (due to Baire Category).

  Now we need to prove $T(B_X(1)) \supset B_Y(1/2)$ using the fact that $\overline{T(B_X(2^{-k}))} \supset B_Y(2^{-k})$. We pick a point $y \in B_Y(1/2)$ and construct a Cauchy sequence in Y so it converges to zero. Also we use continuity of T and the completeness of X, when we consider $y - T(x_1) - \cdots - T(x_k) \in B_Y(1/2^{k+1})$. □

- Two corollaries: Two Banach spaces 1) if one norm dominates the other, then two norms are equivalent. 2)Two Banach spaces with the same Cauchy limits (can be different), then two norms are equivalent (use part 1) on $\|\cdot\|_1 + \|\cdot\|_2$).

- T^{-1} in page 13 does not work out.

- closed graph theorem: T is closed iff T is bounded, using the fact that closed set of complete space is complete (for nonlinear T, we demand Y be compact,`https://math.stackexchange.com/questions/45227/the-closed-graph-theorem-in-topology`).

- We have an example where if f is not continuous then the graph is not closed. `https://math.stackexchange.com/questions/137673/direct-approach-to-the-closed-graph-theorem`; range is closed but the graph is not closed.

- T bounded from Banach to Banach. Bounded from below iff range is closed and one-to-one.
- if Bounded below, then unique solution! (similar to coercivity)

Finite dimensional Banach Spaces

- Lemma 5.32: First, reduce when possible. Continuity because the element in the domain controls the outcome. Consider cube, where the sum of all elements is one, and construct a map $(x_1, \cdots, x_n) \mapsto \|f((x_1, \cdots, x_n))\|$.
- **Finite-dimensional normed linear space is a Banach space**: how did we use Lemma 5.32? well if we have $m \sum_{i=1}^{n} | x_i | \leqslant \|x\| \leqslant M \sum_{i=1}^{n} | x_i |$, obviously we can bound each element. We use inequality to see convergence of the Cauchy sequence. Since each element is Cauchy (using left inequality), we can find the actual y, and the right inequality shows the convergence of the norm, using the convergence of each element.
- We can also use Lemma 5.32 to show that linear operator on a finite dimensional linear space is bounded.
- We can also show the equivalence of two norms by leveraging two inequalities.

Bounded Operator

- Definition of uniform convergence.
- K_n acting on $f(x) \in C([0,1])$. We can normalize f so its maximum is 1. We find that $\|K_n\| = \max_{x \in [0,1]} \{\int_0^1 | k_n(x,y) | \, dy\}$.
- Proof that $B(X,Y)$ is complete (Thm 5.41), when Y is Banach: We need to prove that 1: construct value, 2: T is bounded regardless of x, 3: prove that the Cauchy sequence converges (need to use Cauchy of norm and convergence pointwise). We use here the fact that Cauchy sequence is bounded. Use the property of operator norm to remove the dependence on x of the index N_ϵ.
- Check again continuous function and l^p cases, similar!
- Proof that

compact operator

- Image of bounded set is pre-compact. In terms of sub-sequence, it does not need to converge to the image, but has got to converge in some point in Y.

- Note that in metric space, limit point compact(infinite subset has a limit point), compact, and sequentially compact are all the same.
- Check: it is two sided ideal of $B(X)$, uniform limit is compact, finite dimensional range is always compact (Heine Borel).
- Many compact operator can be approximated by finite rank operators. When we think of finite rank operator, it is important to think about the equivalence of norms, so we can use Bolzano-Weierstrass theorem (compactness iff closed and bounded, interesting read `https://math.stackexchange.com/questions/109733/are-two-norms-equivalent-if-they-induce-the-same-topology-on-a-vector-space`).
- Limit of compact operator is compact: subsequence of subsequence(identify the sequence) and use the triangle inequality(have to use the uniform convergence).
- Strong does not imply uniform convergence. Example: infinite dimensional projection, $P_n \to I$ strongly, but not uniformly.
- $\int_0^1 \sin(n\pi x) f(x) dx$ converges strongly to 0 ("averaging" effect).
- Norm convergent, absolutely convergent.
- some discussion about exponential of an operator, identifying three properties, abelian group structure, and uniformly continuous group.
- This is for linear equations!!

Approximation Scheme

- satisfy $Au = f$ and $Au_\epsilon = f_\epsilon$.
- f is given. f_ϵ may or may not converge to f. A_ϵ is our object of interest.
- Key inequality:

$$\|u - u_\epsilon\| \leqslant A_\epsilon^{-1}(\|A_\epsilon u - Au\| + \|f - f_\epsilon\|).$$

Dual

- Topological(continuous)/algebraic(including non-continuous, much larger) dual space.

Hw

- (5.15 b): it is not clear to me to use ODE method; I don't know the solution is unique
- Useful theorem:
$$y'(t) = f(t, y(t)), y(t_0) = y_0$$
If we have a nice regularity for f, then uniqueness/local existence holds.
`https://en.wikipedia.org/wiki/Picard%E2%80%93Lindel%C3%B6f_theorem`
- Use commutativity of $[A, B]$ and A, B?
- Whence sine function, converting to ODE is useful. Even when we try to get Kernel, we can somehow utilize the range (in this sense sine function is special).
- Exercise 5.6: what does it have to do with subspace? well we are concerned with one specific element! This says that weak derivative is unique.
- In order to show that it is compact operator, Ascoli-Arzela can be useful.
- Exercise 12.15: $a < b$ does not mean that $x^a < x^b$. How can we rectify it?

Hahn-Banach

- "It is possible to maintain boundedness of linear functional by suitable extension to the original X."
- bidual, $F_x(\phi) = \phi(x)$.
- this shows that weak convergence is unique.
- If $X = X^{**}$, we call that X is reflexive.
- weak convergence (for X), weak star convergence (for dual space, $\phi_n(x) \to \phi(x)$ as $n \to \infty$ for every $x \in X$).
- X^* closed unit ball in X^* is weak star compact (Analogue of Heine-Borel).
- Weak topology on X^*: the weakest(coarsest) topology on X^* making all maps $< x, \cdot >: X^* \to \mathbb{R}$ continuous. In other words, we choose specific linear functional corresponding to the original space.
- Radon measure: measure on a Hausdorff topological space that is finite on compact set, inner regular (compact sets), and outer regular (open sets)
- If we set continuous function as a test function, we get a nice result.
- Dual of continuous function is Radon measure. Well, measure can be viewed as a linear functional.

- Shouldn't it be ϕ_n? What does b stand for? When we say the dual is a function, we view it as a "test function." In the end, these are isomorphism as a vector space, due to one to one correspondence and the homogeneous norm.

- The dual of functional space does not have to be functional space (vector space). Can be identified as a test measure. `https://math.stackexchange.com/questions/1858615/what-is-actually-the-standard-definition-for-radon-measure`.

Measure theory

- Definition of measurable (inverse is in X), σ-algebra.
- Random variable is a measurable function whose codomain is real (for example we could use extended real number).
- Enough to simply verify generating set when we prove measurability.
- Countability of $\int A$ is by just using the same old trick (move from characteristic to simple to positive measurable function).
- Proving non-measurability is hard.
- f_n measurable converges pointwise to f then f is measurable.
- f_n measurable f_n converges pointwise a.e. to f AND (X, A, μ) is complete then f is measurable.
- We define $\{A_{N,k} = \frac{k-1}{2^N} \leqslant f(x) < \frac{k}{2^N}\}$
- $\phi_n(x) = \sum \frac{k-1}{2^N} \chi_{A_{n,k}}(x)$
- Positive measurable function can be approximated by increasing simple functions.
- Can approximate separately as well for negative and positive.
- δ_{x_0} (check)
- Counting measure $f : \mathbb{N} \to \mathbb{R}$.
- Counter-example: height n and interval $\frac{1}{n}$.
- Monotonic convergence theorem is also called Paul-Levy.
- Fatou's lemma think of it as losing mass (`https://math.stackexchange.com/questions/1890542/intuitively-understanding-fatous-lemma`).
- Question: Is $I(t) = \int f(x,t) d\mu(x)$ differentiable?

- f is differentiable in x, integrable in t, and dominated uniformly in x by $g(x)$.
- Use $\frac{\phi(t+\frac{1}{n})-\phi(t)}{\frac{1}{n}}$.
- Constructing product space is trivial, but making a measure on it needed work.
- Fubini's theorem: 1) integrable on product measure is same as (iff) integrate one and then the other.

  2) When integrable, integral value is exchanged.
- Wiggly L^p (too big) and $L^p(1 \leqslant p \leqslant \infty)$ are different. Latter is only up to measure zero.
- Now $\|\cdot\| = 0$ implying 0 is not trivial.
- Use approximation of the function $\phi_n \geqslant 0$, sandwich with $\int | f |^p \underset{\cdot}{\mu} = 0$, and use the fact the integrand is zero almost everywhere (need some work, but essentially sets are measure zero or the value c_i is zero).
- For the proof of dominated convergent theorem, we need some lemma about Cauchy: $\{f_n\}cauchy$ iff if the absolute sum of norm is finite then the finite sum converges (I'm not sure).
- Idea of the proof of MCT: One direction is direct. The other direction is by choosing $c < 1$, define E_n with it, and fix the simple function. Take the infimum on an appropriate inequality (p319 Rudin).
- Fatou can be proved by using MCT on $\inf\{f_n, f_{n+1}, \dots\}$
- DCT can be proved by using Fatou on $f + g$ and $-f + g$.
- L^p function can be approximated by simple functions in L^p sense.
- $1 \leqslant p < \infty$ $L^p(\mathbb{R}^n)$ is separable (countable dense set).
- C_C^∞ is dense in $L^p(\mathbb{R}^n)$.
- Polynomial with rational to real coefficients. Approximate compactly supported continuous function.
- L^∞ is not separable (too many variations, cannot approximate by some smooth functions).
- Of course, it does not hold that $\|f_n\| - \|f\| \to 0$ then $\|f_n - f\| \to 0$ (just plug in $-f$). But the other way around holds.
- Next class: Holder, Jensen, Chebyshev

Differentiation under integration

- The idea of the proof for Riemann integration: 1) mean value theorem and 2) uniform convergence; (d)(derivative continuous uniformly in x) says that ψ^t converges uniformly to $(D_2\phi)^s$.
- Riemann integrable in x says that it is "small enough." Uniformly continuity in t says that it is "regular enough."
- Same idea of using mean value theorem for Lebesgue case.
- Often times we get the integral by formulating it into differential equation.

L^p theory

- $L^p \subset L^q$ comes from the fact that p and q should both be greater than one.
- We did not say L^p is the same as L^q. The dual is!
- I'm not understanding the statement, "the dual is bigger than the original space."
- L^∞ does not have to be regular, so maybe that's why the dual of L^∞ is greater than L^1?
- The terminology of absolute continuity comes from the notion of "control." `https://en.wikipedia.org/wiki/Absolute_continuity`
- Simple functions are dense in L^p space.
- For the proof: We are given l. From finite measure proof, we can consider E_n and g_n. We know that there exists g such that $g_n \to g$, and this is integrable on all sets of finite measure. This is why we used (ii). We do not a priori know that it is in L^q.
- Definition of total variation of the signed measure: `https://en.wikipedia.org/wiki/Total_variation`; roughly speaking, it is the size of the measure.
- Dual of L^∞ is not L^1. It is a ba space(Banach space of finitely additive measure absolutely continuous with respect to μ whose norm is defined by total variation, `https://math.stackexchange.com/questions/47395/the-duals-of-l-infty-and-l-infty`), there is a natural embedding $d\nu = fd\mu$ when f is L^1 function.
- Bounded sequence that does not have a weakly convergent sub-sequence in $L^1([0,1])$. Well, clearly we are concerned about weak star compactness so we can clearly find this.
- Does reflexivity imply Banach-Alaoglu theorem for the original space?

Density

- simple is dense in $L^p (1 \leqslant p \leqslant \infty)$. We simply use simple function approximation.
- $L^p(\mathbb{R}^n)$ is separable for $p \in [1, \infty)$. Simply use cube with rational indexes. Same idea in the problem set that measure is a countable union of intervals.
- compactly supported continuous function is dense in $L^p(\mathbb{R}^n)$, not including ∞: suffices to approximate characteristic function of a bounded measurable set A by continuous compactly supported function. We can use theorem 12.10 ($K \subset A \subset G$ with $\lambda(G\ K) < \epsilon^p$). In order to generalized to compactly supported smooth functions, we need to use approximate identity to smooth it out.
- how do you show non-density (wrt $C[0,1]$) and non-separability? (in particular $L^\infty[0,1]$).

- ## Counting measure

- Interchanging of infinite sum is really a result of monotonic convergence theorem. `https://math.stackexchange.com/questions/1799766/relation-between-counting-measure-and-tonelli-theorem`

Zorn's Lemma, Axiom of Choice

- Discussion on axiom of choice: `https://math.stackexchange.com/questions/868787/dual-of-l-infty-is-not-l1`

Inequalities

- $< f >$ is like a mean value, $\frac{1}{\mu(x)} \int_x f \mu$
- Jensen's inequality in terms of expected value. We just expressed the lower bounded property and integrate over X (page 356).
- Chebyshev's inequality; take an inequality on the obvious direction. This gives the upper bound of measure and the lower bound of L^p space. We kind of lose a lot.
- Young's inequality: convolution is L^r where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.
- Convolution is like pseudo-product; when $f, g \in L^1$; give me an algebra structure.
- Another way to view convolution is a weighted average when the mass of g is one and is always positive. In general, weighted sum.

- This distinction would not matter when we investigate asymptotic behavior, smoothness, etc.
- So, be careful when we do the change of variable (if it's after or before Fubini): $\int [\int f(y)g(x-y)dy]dx = \int [\int f(y)g(x-y)dx]dy = [\int g(z)dz][\int f(y)dy]$
- $f, g \in L^2$ to L^∞.
- Always check if we can simplify(without loss of generality) when we prove the bound!
- The idea of the general case is to consider conjugate and use generalized Holder inequality. We want to kill by constructing $\|f\|_p$ and $\|f\|_q$ somehow.
- Delta function is like a kernel of identity operator. This says that it is "morally" L^1.

Hilbert Space

- Diagonalization of the operator is our final goal.
- Basis for Banach space is not very useful. Rule of finding coefficient should come from orthogonality.
- Antilinear with respect to the first variable. Change the position conjugates.
- There is a natural norm induced by an inner product. This is not necessarily complete.
- Completion of pre-Hilbert space is Hilbert space. This is Banach.
- Fact: Define $(f, g) = \int_a^b fg dx$. This can be viewed as a pre-Hilbert space defined for continuous function; Its completion is $L^2([a, b])$. not the sup-norm!
- Generalization: $C^k([a, b])$ norm defined by $\sum_{j=0}^k \int_a^b \bar{f}^{(j)}(x) g^{(j)}(x) dx$ where each superscript is the j'th derivatives; its completion is H^k.
- Cauchy-Schwarz. Proof idea: use the fact that $\|x - \lambda y\| \geqslant 0$ and put $| \lambda |= \|x\| \|y\|^{-1}$
- Parallelogram law; inner product space if and only if parallelogram law holds. Define an inner product by using a norm(sum of squares divided by four, this shows that inner product can be characterized by only diagonal elements). Prove that this is actually an inner product by using parallelogram law.

- Natural inner product of Cartesian product is to use the sum of each square of each norm, taking square root after that. Then it is Pre-Hilbert, check! (might not only be this way though)
- $X \times X \to \mathbb{C}$ defined by inner product is continuous. Proof is by using $x_1 + x_2 - x_2 = x_1$ and use Cauchy-Schwarz. This is locally Lipschiz continuous.
- Assume that we have Hilbert space. Definition of orthogonality.$A^\perp$ is a subspace. A does not need to be.
- Closed linear subspace by using the idea of continuity (topology generated by a norm generated by inner product). Similarly, kernel is a closed space.
- Projection theorem: minimum is attained for closed linear subspace of Hilbert space. This is a unique point that orthogonalizes. For Banach space, we may attain unique point but the complement of M may not be closed.
- "Separable Hilbert space": countably many orthonormal basis. Every Hilbert space has an orthonormal basis.
- Complement is closed.

Orthogonality

- Orthogonal direct sum of two Hilbert spaces. We can decompose nicely for closed subspaces.
- Cardinality of orthonormal basis is unique.
- Unconditional convergence (this is the notion for unordered sum) is needed for the uncountable sum. We say absolutely convergent if the sum of norms is unconditionally convergent.
- Of course absolute convergence is stronger notion. Crucial example is $\sum_{n=1}^{\infty} \frac{1}{n} x_n$.
- But in real number, absolute convergence iff unconditional convergence (why is real number special?).
- Inner product is continuous (like locally)!
- Unordered sum exists(that is, unconditional convergence) iff the order does not matter and it converges to the same point regardless ([1], interesting read, `https://en.wikipedia.org/wiki/Riemann_series_theorem`, `https://en.wikipedia.org/wiki/Riemann_series_theorem`).

[1] how do I prove? I think I can start by checking what's on the rudin

- More remarkable thing is that if the unordered set converges, then we have at most countably many nonzero elements (argue by using the existence of finite subset S_{J_n} having the gap less than $\frac{1}{n}$). We can now use this to utilize the continuity of inner product to say something about the Fubini type equality ($\langle \sum_{\alpha \in I} x_\alpha, \sum_{\beta \in J} y_\beta \rangle = \sum_{(\alpha,\beta) \in I \times J} \langle x_\alpha, y_\beta \rangle$). **Question: Does uncountably many orthonormal basis matter in some case? If unconditional sum has countably many non-zero element, shouldn't we just disregard it?**

- Cauchy can be defined on unordered set by using finite complement.

- This new notion of Cauchy works nicely with the original definition of Banach space ([2]).

- Comparison test is proven by Cauchy criterion. Check how completeness of real number and supremum property of real number are related.

- For Banach space, unordered sum converges iff it is Cauchy (in unordered sense).

- $\|\sum_{\alpha \in I} u_\alpha\|^2 = \sum_{\alpha \in I} \|u_\alpha\|^2$: first we need to consider finite case. Obviously from here we use the definition of Cauchy criterion, and the fact that unconditional convergence iff Cauchy. Take a limit to use the continuity of norm (maybe use the fact that countably many nonzero?).

- Bessel's inequality: how do I use orthonormality? always start with the finite case.

- if it's not conditionally convergent, then it is not interesting. Then we only care about countable sum. Ah, ordered sum is always countable. Also, we did not really define countably many sum of vectors.

- Closed linear span: we take into account countably many orthonormal basis.

- Parseval's identity says that the sum of square of inner product of the function with each basis is same as same as norm square of the original element.

- $\langle x, y \rangle = \sum_{\alpha \in I} \bar{a_\alpha} b_\alpha$.

- From Garm-Schmidt, we can at least construct orthonormal set whose linear span is dense.

- **Uncoutnably many basis example???**

- Important: Hermite polynomial ($w(x) = e^{-x^2/2}$), Tchebyschev polynomial ($w(x) = (1 - x^2)^{1/2}$), Legendre polynomial ($w(x) = 1$).

[2] check Proposition 6.22 in page 137

Sobolev spaces

- It can be generalized.
- Question: Embedding inequality; is it really necessary?
- Bounded by the derivative: need to be invariant under rescaling.
- Sobolev conjugate concerns the dimension, i.e., $\frac{1}{p}^* = \frac{1}{p} - \frac{1}{n}$.
- It follows that $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ when $p < n$. Well, p^* is always bigger than p and more derivative shrinks the space, so this makes sense.
- Perhaps p does not have to be smaller than q for theorem 12.70.
- Definition of $C^{k,r}$ norm (k'th order derivatives are Holder continuous)
- We are comparing p and n. When $p > n$, we are concerned about uniform norm.
- Sobolev spaces form an algebra (or Holder continuity with certain exponent) when they consist of continuous functions.
- Weak formulation: can be generalized where A is an operator from H to its dual.
- Dirichlet form associated with an operator A.
- Depending on the regularity of f, we have different regularity of u. For example, $\|u\|_{H^{k+2}} \leqslant C\|f\|_{H^k}$. This sometimes gives the classical solution when we have $H^{k+2} \subset C^2$.
- Holder continuity is sometimes special, like there is a difference between $f \in C$ and $f \in C^{k,r}$ where $r > 0$.
- Riesz representation theory gives unqiue solution.
- The point of Lax Milgram is that a does not need to be symmetric.
- Boundary has measure zero. How does that help for Sobolev space? Well, we are using new measure.

Bounded linear operator, projections

- If we have two linearly independent vectors, we have unique decomposition. No notion of orthogonality.
- Note that not only does M, but also N matters for the decomposition even in projection!

- Projection can be geometrically diagonal (the case when $\|A\| = \sqrt{2}$, in fact we can control as we want).
- We use $P^2 = P$ as projection it is more concise, without mentioning M and N. This is called projector is linear operator.
- Closeness of M and N yield, by closed graph theorem (well, this is reasonable, this says that the linear map between Banach space is cts iff the graph is closed), a bounded operator, and in particular in Hilbert space the value is one.
- Projector has the great orthogonal decomposition in terms of range and kernel, and the range is closed due to Hilbert space.
- $I - P$ is the orthogonal projection on the other one.
- There exists P such that its range and kernel is what is desired.
- For part one, we consider $x = Px + (x - Px)$. Uniqueness follows by considering $x = y + z$.
- Intuition about how the norm of P can be as big as we want, and its operator norm is not bounded in non-finite dimension. "$I = P$ onto N along M?
- Projection theorem now involves orthogonality. $H = M$ directsum $M^{\perp}$
- $\langle Px, y\rangle = \langle Px, Py\rangle = \langle x, Py\rangle$: now P is symmetric. (using the fact that $y - Py \in M^{\perp}$)
- This is the definition of orthogonal projector.
- Important property is that its operator norm is one unless $P = 0$ (trivial case); fact that its operator norm is greater than equal to one is obvious from the property of projection, for the other direction need orthogonality and Cauchy-Schwarz
- Hilbert space: the case when the range in the decomposition is closed for orthogonal projection.
- $M + M^{\perp}$ then unique projection exists such that its range is one of them.
- Part (b) was done before (and use the definition of orthogonal projection).
- Think how it goes wrong for infinite dimensions (angle shrinks).
- Examples of projecting operators: three examples; indicators, odd/even, rank one projector.
- Remember that Hahn Banach proves the uniqueness of weak convergence. Also $\phi(x) = \phi(y)$ for all x, y implies that $x = y$.

- Dual of Hilbert space, a priori we assume it's bounded.
- Inner product with each element defines linear functional in Hilbert space.
- By Cauchy Schwarz, we have a nice bound for this (by the norm of the element).
- Riesz representation says that this is all we have(isomorphism, identification). Think how dual of L^2 is L^2.
- We define P using ϕ and decompose H in terms of z and $Ker\phi$. Set $x = \alpha z + n$ then we can express α in terms of inner product and we can then calculate $\phi(x)$ in terms of good inner product; get $y = \frac{\phi(z)z}{\|z\|^2}$.
- Uniqueness how to prove? **check.**
- We can do all this without projectors but it is very helpful.
- Example 8.11: it is helpful to define fourier series so that its norm is one.
- Arbitrary direct sum can also turn into projection, but perhaps not an orthogonal one.
- For example 8.16, we used complex conjugate for L^2 norm.

step for Riesz representation theory

- We need to use something that is orthogonal to $ker\phi$. This is something tangible we can extract from ϕ. This gives us the definition of projection and the nice elimination.
- Riesz representation theory is a key component for adjoint.
- Used the fact that orthogonal projection has the property of uniqueness.
- why the hell $P^2 = P$? Well of course $\phi(\phi) = \phi$ because it's linear functional. Another key is to use $z \perp ker\phi$.

Adjoint operators

- Generalize symmetric matrices into infinite dimension (so we can use Riesz). Diagonalizability only works for normal matrix. If not, maybe use Jordan form but it does not work well for infinite dimension.
- $\langle x, Ay \rangle = \langle A^*x, y \rangle$
- Does this exist in Hilbert space? We could prove using Riesz representation theory.
- Linearity is followed by uniqueness.

- $(Kf)(x) = \int_0^1 k(x,y)f(y)dy$ for $L^2(0,1)$
- $(K^*f)(x) = \int_0^1 \bar{k}(y,x)f(y)dy$
- You may think taking closure is little awkward. But it is natural since it comes from the property of adjoint.
- We have $\langle y, z\rangle = \langle x, A^*z\rangle = 0$
- Constraint on y, In linear algebra, it's enough (Fredohlm alternative, which is always true in finite dimensional space)
- By rank nullity theorem, injectivity iif surjectivity.
- Adjoint preserves compactness, and also note the boundedness property of adjoint. `https://math.stackexchange.com/questions/1034278/if-a-linear-operator-has-an-adjoint-operator-it-is-bounded`

More on Bounded Linear Operator

- Note that adjoint can be useful in solving $Ax = y$ since we have $y \in (KerA^*)^{\perp}$.
- $ra\bar{n}A = (kerA^*)^{\perp}$
- The moral of the story is that range of A and kernel of A^* are similar.

Fredholm alternative

- Fredholm alternative concerns $Ax = 0$ and $A^*x = y$. I think it asserts solutions for both equations.
- Remember... orthogonal projection is not orthogonal matrix.
- 1) range need not be closed 2) we do not have a nice match between kernel of operator and the kernel of the adjoint operator.
- Invariant under compact perturbation.
- Bounded linear operator. Same Hilbert space?
- Orthogonal complement is already closed.
- Range is not always closed. We have rank nullity theorem kind of direct sum using adjoint and the original operator. $\langle x, Ay\rangle = \langle A^*X, Y\rangle$.
- When we take orthogonal to both sides of direct sum, we use projection theorem (uniqueness, check!).

- In the proof, we use the fact that the orthogonal is closed, so we can close the left hand side.
- Use the fact that taking orthogonal switches the direction.
- Then, if we take orthogonal, it will be closed, so we can close ($RanA^{\perp\perp} = R\bar{an}A$).
- Bounded operator has a closed range.
- Definition of Fredholm alternative: always works in finite dimension.
- Two issues in the infinite dimension: range of A is not closed ($\frac{1}{x}$ is not square integrable, L^2 to L^2), dimension of Kernel of A is not same as of A^*.
- Two examples
- Bounded operator is Fredholm if range is bounded and $KerA$ and $KerA^*$ are finite dimensional. In fact the latter implies the former.
- Can be characterized by index (dim ker A-dim Ker A^*) being zero (iff).
- dimension of Kernel is not stable upto perturbation.
- If you perturb by compact operator the property of Fredholm is preserved upto addition. (play around with Eigenvalue)
- Normal matrix is diagonalizable: self adjoint and unitary.
- Construct an inner product.
- Definition of sesquilinear form: construct another "inner product" related to A, $a(x, y) = \langle x, Ay \rangle$. This is an inner product so we have access to Riesz Representation theorem.
- We want eigenvalue of A be positive in order to definte this as a norm.
- Quadratic form is $q(x) = q(x, x) - \langle x, Ax \rangle$.
- A is nonnegative if $A = A^*$ and $\langle x, Ax \rangle \geqslant 0$.
- Why do we require $A = A^*$ for nonnegative and not for positive definite??
- Fact: if $A = A^*$ is positive definite, this is an inner product.
- Lemma: A bounded self-adjoint then $\|A\| = \sup_{\|x\|=1} \mid \langle x, Ax \rangle \mid$. (characterization of the norm)
- Inner product can be characterized by a norm, which is the idea of polarization formula in p198.
- Unitary operator is one to one and onto, and preserves the inner product (p199).

- normal: nice spectral theory.
- Theorem 8.35 mean Ergodic theorem (unitary operator, identity subspace M, projection on it, statement says that the average of $U^n x$ is Px) (p202); the key step of the proof is to separate into range and kernel.
- Characterization of unitary operator (rotation, preservation of the structure).
- Skew if $S = -S^*$. Then $S = iA$ for some A self adjoint.
- Normal if $T^*T = TT^*$, theory of diagonalization.
- Weak convergence. Since $H^* = H$, weak and weak star convergence are the same.
- Uniform boundedness (Banach Steinnhaus): boundedness implies uniform boundedness.
- Take a look at the book.
- For weak convergence, we do not need to test with all $y \in H$: operator is bounded and hold for dense set iff hold for all H.
- Idea: use uniform boundedness theorem. Use triangle inequality using the first condition. Morally the same proof for convergence=stability+consistency (I was thinking why the inverse should be bounded for the condition of stability but I'm unsure now).
- Next time a little more on chapter 8.
- Always think in terms of linear algebra.

Self-adjoint

- Usually in order to find adjoint operator, we can guess.
- We defined $(x, y) = \langle x, Ay \rangle$. Well this is coercive.
- $\|A\| = \sup_{\|x\|=1} \mid (x, x) \mid$

Unitary operator

- When you think of unitary operator, think of Parseval!
- This says that integrable functions on the circle is isomorphic to the Hilbert space of sequences on $\mathbb{Z}$.

- **Spectrum**
- definition of resolvent set and spectrum.
- continuous spectrum: $A - \lambda I$ is "almost" invertible. Well, think of this as "scaling." We could devise some approximating functions to consider density.
- intuition: there are pseudo-eigenvectors (for example non-normalized functions, distributions) `https://math.stackexchange.com/questions/3943120/intuition-behind-spectrum-of-an-operator`

Examples of spectrum

- Left/right shift
- $xf(x)$
- $\int_0^x$

Weak conlized functions, distributionvergence

- weak convergent in dense set and x_n bounded.
- Well, it is useful to note that finite combinations of an orthonormal set are dense.
- (page 112) for the norm we use L^∞ norm, whereas we use L^2 norm when we calculate Lf.
- We have a loss of mass in strong convergence (think of escaping to infinity).
- Weak topology is not metrizable, but sequential weak compactness implies weak compactness.
- Banach-Alaoglu: so we can use Heine Borel theorem in order to deal with weak topology. In order to go from dense set to the original space, we have unique extension due to continuity (worth checking). Riesz representation theory and the loss of mass is also useful in order to check if x belongs to unit ball.
- Weakly lower semicontinuous.
- how to deal with closure: think of dense set or the fact about perp.
- Banach-Alaoglu can be useful for minimization problem. Weakly lower semicontinuous function on a weakly closed bounded subset K of Hilbert space attains its minimum.
- smallest that makes it continuous.

Uniform boundedness

- if bounded for each x, then the norm is uniformly bounded, which is stronger.
- We used nested loop and contradiction, but cannot understand.

Discussion on Compact operator

- not sure about this; `https://math.stackexchange.com/questions/775330/compact-operator-whose-range-is-not-closed`

- ### Chapter 9

- We find that normal matrix is diagonalizable.
- Eigenvalue does not have to exist!
- Example of $Mf(x) = xf(x)$: resolvent set is the complement of $[0,1]$, which is spectrum. Every element in $[0,1]$ is continuous spectrum of M.
- Resolvent of $A : (\lambda I - A)^{-1}$
- Analyticity: $F(z) = \sum_{n=0}^{\infty} (z - z_0)^n F_n$.
- The resolvent set may or may not be smaller than or equal to $\|A\|$: this is false, because we neeed to have $\lambda I - A$ be invertible.
- Resolvent of A: a function $\mathbb{C} \to$ Operator, $R_\lambda = (\lambda I - A)^{-1}$.
- Neumann series gives the operator norm convergent series.
- Regarding Neumann series: Cauchy criterion is to be used in the comparison test, so completeness is necessary in the necessary condition $\|A\| < 1$. Note also that convergence and boundedness are two different things.
- If the below equality is true, then $r(A) \leqslant \|A\|$ and so spectrum is contained in the disc $\mid \lambda \mid \leqslant r(A)$.
- The way to get spectral radius: $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.
- (p222) Spectrum of bounded operator on a Hilbert space is nonempty: assuming it is nonempty, we can construct analytic function on $\mathbb{C}$ that is bounded and so it is constant by Liouville's Theorem; yet this is a contradiction since then our construction will say that the inverse is zero, which is impossible.
- this is why we were interested in analyticity.

Spectral theorem/self-adjoint operator

- Orthogonality of eigenvectors in self-adjoint operator come from the fact that we have nice transfer from the first component to second component so that we can cancel out in the final analysis.
- compact, self-adjoint operator; apply immediately. `https://math.stackexchange.com/questions/144204/compact-self-adjoint-operator-on-a-hilbert-space`

  Definition 2. *M is invariant subspace of H if $xM \subset H$ for every $x \in M$.*

- Invariant subspace.
- Prop 5.30 says that bounded from below iff closed range and one-to-one.
- The notion of Spectrum and spectral radius is different!
- Lemma 9.11 says that orthogonal space is also invariant under self-adjoint operator.
- Proposition 9.12 should have a bar in it too.
- Exposition of 9.14 is a bit misleading. I think the point is to realize that the conjugate of real is real, spectrum is real, but point and residual spectrum need to be disjoint.
- Spectral theorem pretty much boils down to $A = \lambda_i q_i q_i^T$.
- Generalized eigenvector is related to Jordan canonical form. `https://en.wikipedia.org/wiki/Generalized_eigenvector`
- Proof steps
  1. Find $\langle x_n, Ax_n \rangle \to \lambda = \|A\|$ or $-\|A\|$. Use compactness to extract subsequence of this so that Ax_n converges, denoting it by y.
  2. Show that $Ay = \lambda y$ (inequality using self-adjoint)
  3. The reason we use the notion of invariance is to define the operator on the smaller subspace.
  4. Not understanding why A_2 is compact operator. ($A : H \to \mathbb{C}$)
  5. Why is it given by a finite sum if we have finitely many nonzero eigenvalues?
  6. Used: zero accumulation point, each eigenvalue has finite multiplicity.
  7. Projection onto $\langle e_k \rangle$ is a compact operator.
  8. Unsure what this is trying to further prove in page 227.
- Compactness iff totally bounded and complete. Important example: discrete topology on the natural number.

- quadratic sum of inner product with orthonormal vector is small at the tail.
- If the set is bounded and satisfies the inequality for every ϵ, E is precompact.
- Defined the Hilbert-Schmidt norm on the operator (independent of the choice of orthonormal basis).

Compactness characterization

- Rellich's embedding theorem and Ascoli-Arzela give the characterization of compactness. We could generalize and indeed characterize precompact set for separable Hilbert space.
- Used some kind of contrapositive: assuming E is bounded, we proved that if the inequality does not hold then E is not precompact. This is possible since we can modify the original statement as needed. Now, negation of inequality involves "for each N,..." Using Parseval's inequality (bundedness), we can construct appropriate subsequence and use triangle inequality using these, ($[\|(I - P_{N_k} x_{N_k}\| - \|(I - PN_k)x_{N_l}\|]^2 \leqslant$) what we got.
- For part (b), use diagonalization argument to construct convergent subsequence.

Hilbert-Schmidt operator

- does not depend on the choice of orthonormal basis.
- this choosing $I - P$ is a bit enigmatic.

Spectral mapping theorem

Proof.

$\sigma(f(A)) = f(\sigma(A))$.

$f(A)$ is an operator, since f is a function from complex plane to complex plane. □

The adjoint of a differential operator

- Interested in $Au = f, Bu = 0$
- $Au = au'' + bu' + cu$ where a, b, and c are sufficiently smooth.

- boundary condition usually consists of two equations but not necessarily.
- Using Green's proposition of 10.8, we defined the "formal adjoint." I'm unsure how it is related to adjoint operator we defined beforehand.
- We are trying to find a function g which acts on f to give a solution u of the problem $Au = f, u(0) = u(1) = 0$.
- $Ag(x, y) = \delta(x - y), g(0, y) = g(1, y) = 0$
- We could formally characterize Green's function satisfying certain three properties. Not sure why they are defined that way. (a) asserts the existence of partial derivatives in each triangle. (b) says it actully satisfies PDE. (c) says something about the boundary line.
- Proposition 12: we should view $\partial u/\partial x$ as the addition of two integrations.
- Existence proved by Leibnitz rule.
- Check (10.22), using Wronskian.
- Green's function does not depend on f.
- How does this relate to Fredholm alternative, adjoint BVP?
- There is a "boundary condition" attached to it that defines the domain.

Unbounded operators

- Enough to have a range as dense set. We noted that we have unique extension of the operator defined on dense set.
- Four examples: C^2, vanishing on the boundary. C^2. H^2 space, vanishing on the boundary (by Sobolev inequality, since $1/2 < 2$ u is continuous). H^2 space (check why weak derivative is unique).
- Notion of closure, symmetric, closable, (essentially) self-adjoint
- closable: the limit in the graph should be unique!
- Closure vs. continuity. Continuity implies the closedness since the limit of Ax_n always exists and the desired equality always holds. Not the vice versa. unsure about the example.
- Closable gives the uniqueness of the limit when we take closure.
- WLOG, we can assume that the domain is dense subset.
- Adjoint operator is very naturally defined. Question: can't we always find z by Riesz representation theory? cannot use since unbounded.
- D is related to boundary condition for differential operators.

- Notion of extension and restriction: A and $\tilde{A}$ have same mapping, just the domain is different.
- Nice symmetry in the definition of self-adjoint operator ($D(A^*) = (A))$) and symmetric operator ($D(A) \subset D(A^*)$). The domain is defined by $D(A)$.
- Question: why would symmetric operator defined in such a way?
- difference between closedness and continuity.
- One to one and onto, and bounded $A - \lambda I$ iplies resolvent set.
- not quite sure about the remark in page 249.

Exercise 12.15

- Seems a lot like Young's inequality. But this is not convolution.

Digression on Riesz interpolation theorem

- Maybe this theorem implies that L^p is in $L^{p_0} + L^{p_1}$
- Simplify by considering ϕ
- In the end, we could just use Holder. The coefficient was more complicated than expected, but I made a little mistake when using Holder inequality.

Exercise 12.17

- I could use the argument using sequential compactness.
- Is it true that positive Lebesgue measurable set can be expressed by some kind of countable union?
- The reason that in the complete space we have $f_n \to f$ almost everywhere: first take liminf of f_n, which we know is measurable. Use null set N and N^c to complete the argument (we can see that $N^c \cap f^{-1}(I)$ is measurable due to completeness).

Exercise 6.2

- Mean value theorem?
- Maybe it's worthwhile to see how the textbook proves something is infimum.

- We did use parallelogram law. First we extracted the sequence y_n. Prove that it is Cauchy, use completeness to show that it converges, and use closeness to show that it is on that subspace. Continuity of norm to conclude.
- We are obtaining the actual infimum, so the argument in that proof is not enough.
- Use contradiction argument.
- positive part of n and negative part of n; well I could just directly prove by using the property of N.
- The crux is to realize that the first part of projection theorem is applied even for Banach space. We also suffice to have just convex closed subset.

Exercise 6.5

- I'm not sure how to prove something is closed linear subspace.
- The question is how I can make each term in infinite sum uniformly small; use orthogonality? Well, essentially we just prove that l^2 is Banach. This follows by a trick of fixing N for our sum, bound, and let $N \to \infty$.

Exercise 6.11

- converge unconditionally: $\|S_J - x\| < \epsilon$ for all finite subsets that contain J^ϵ.
- Closed linear span is defined by $[U] = \{\sum_{u \in U} c_u u\}$ where it converges unconditionally.
- The question is, can we extract orthonormal basis only on dense subset? The problem is that the dense subset is not Hilbert space, so I'm unsure if we can use the same argument in 6.29.
- I think if I can construct basis for dense set, then maximal argument from Theorem 6.26 to conclude.
- I can't really tell because intuitively, dense subset may not contain all the orthonormal set.
- I can try the most popular Hilbert space (just wondered how do you prove that something is NOT Hilbert space?).
- There are many ways of getting orthonormal set, so it might not matter if it's subspace of not.
- The key is to use Zorn's lemma, Gram-Schmidt, and π.

Exercise 6.14

- induction and integration by parts do not seem to work?
- I only know the factor $\frac{2}{n-1}$ and the recursion equation. Can I finish? I don't know...Check the computation again, this does not look promising.

Exercise 7.1

- This is not uniformly bounded.

Exercise 8.1

- how do I show that something is closed?
- Use inner product or the fact about finite space.
- Finite space enables us to look at each component.
- We can talk about closeness in Hilbert space.
- Motivation for Fredholm alternative:
- One thing you should think first when it comes to projection is the notion of kernel and range.

Exercise 7.2

- why "kernel"? `https://math.stackexchange.com/questions/2648516/intuition-about-dirichlet-kernel`

Exercise 8.14

- how to use polarization formula? I don't know what's special about it. It's going from inner product to the norm.
- Yes, complex vector space, hence conjugation is the key for this polarization thing.
- Adjoint is the extension of Hermitian.

Exercise 8.12

- remind me of closed graph theorem.
- Adjoint operator does not seem to be very useful.
- how is the inner product and the norm related?
- well, polarization law describes norm into inner products, but really we can manipulate this a bit further into inner product-inner product.
- Complex conjugation since $\langle ix, ix \rangle = i^2 \langle x, x \rangle$.

Exercise 8.20

- This says that f is bounded from below?

Exercise 9.3

- Resolvent equation: not sure how to proceed than just assuming some element in each set. Now, we only have the information that it is bounded.

Exercise 9.6

- Use the commutativity.

Exercise 9.7

`https://math.stackexchange.com/questions/320336/spectral-radius-of-the-volterra-operator`

- Idea: reverse engineering. The key step was using Cauchy-Schwarz and the change of variables. Change of variable is not special (it is 'symmetric') but it enables us to somehow cancel out the trigonometry term.
- Changing the order of integration is very handy.
- when we infer n'th order, we considered the fact that the unknown variable is in the integrand as well.

Exercise 9.8

- Well we can prove injectivity easily in (d).

Fourier Series

- we need to prove that e_n is complete.
- Approximate identity; in order to use (c) which says that $\lim_{n\to\infty}\int_{\delta\leqslant|x|\leqslant\pi}\phi_n(x)dx = 0$, we should guess that we need to divide the integration into two parts. Uniform continuity comes in naturally.

Exercise 9.22

- maybe use compact operator criterion?
- compact operator with respect to the norm in Hilbert space, not Hilbert-Schimidt norm.
- Hilbert Schmidt basis invariance (in separable): Use the fact about basis, noting that $\mid (Af_i, e_j) = (f_i, A^*e_j)$.
- I don't know. `https://math.stackexchange.com/questions/706525/positive-compact-operator-has-unique-square-root`