ANALYSIS FOUNDATION

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Some transformation

(1) Fourier Series
- Some trigonometry; just remember that the cosine swaps the sign ($\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$)
- Fourier coefficient is given by $\hat{f}(n) = \frac{1}{L}\int_0^L f(x)e^{-2\pi inx/L}dx$ for $n \in \mathbb{Z}$.
- The Fourier series is given by

$$\begin{aligned} f &\sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx/L} \\ &= \hat{f}(0) + \sum_{n\geqslant 1} [\hat{f}(n) + \hat{f}(-n)]\cos(n\theta) + i[\hat{f}(n) - \hat{f}(-n)]\sin(n\theta). \end{aligned} \tag{1}$$

- As we see from the construction, it is only valid for a periodic function.
- $\hat{f}(0)$ is like the average value of the sound wave (like loudness of the chord), while cosine and sine may represent the different components of the notes, like rhythmic or melodic aspect.
  (a) Multiplicative formula: $\int_0^T f(x)g(x)dx = \frac{1}{T}\sum\limits_{n\in\mathbb{Z}} \hat{f}(n)\hat{g}(n)$
  (b) Convolution theorem: $\widehat{f * g}(n) = L\hat{f}(n)\hat{g}(n)$.
  (c) Plancherel: $\frac{1}{L}\int_0^L |f(x)|^2 dx = \sum\limits_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.
- We can now ask, 1) when does it converge? 2) in what sense? (mean square convergence, uniform convergence, etc) 3) is there any variation of foureir series that makes some convergence possible? (Abel, Cesaro)
- These are central theme in $Stein - Shakarchi$.

(2) Fourier transform
- $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$
- Fourier inversion formula states that $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi ix\xi}d\xi$.

  Example 1. Suppose f is a continuous function supported on an interval [-M, M], whose Fourier transform $\hat{f}$ is of moderate decrease. Let $L/2 > M$. Then we have

$$f(x) = \sum_n a_n(L)e^{2\pi inx/L}, \text{ where } a_n(L) = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-2\pi inx/L}dx$$

  and

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi ix\xi}d\xi$$

- Some fundamental results.
  (a) Poinsson summation formula: $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$; in praticular,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

  (b) Multiplicative formula: $\int_{-\infty}^{\infty} f(x)\hat{g}(x) = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy$.
  (c) Convolution theorem: $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.
  (d) Plancherel: $\|\hat{f}\| = \|f\|$.
- Of course, these theorems do not hold in free; we need some regularity condition on f. In general, Schwartz class and moderately decreasing function are discussed. Some generalization of L^2 is an optimal functional class but the discussion is a bit more involved (discussed for example in Evan's).

Hyperbolic function

- As in $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$ and $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$, we simply define $\sinh = \dfrac{e^{x} - e^{-x}}{2}$ and $\cos(x) = \dfrac{e^{x} + e^{-x}}{2}$.
- Similarly, we define $\tanh x = \dfrac{\sin h(x)}{\cosh(x)} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$.
- tangent hyperbolic function is the most useful since it has a range between -1 and 1, while having the desired nonlinearity. It also has a stronger gradient around 0.

Hilbert space

- Riesz representation theorem states that every linear functional can be expressed as an inner product with respect to some element in $\mathcal{H}$.

Orthogonality

Fact 1. *Orthogonal complement of a subset of $\mathcal{H}$ is a closed (linear) subset of $\mathcal{H}$.*

Proof. It suffices to show that if (y_n) is a convergent sequence in $A^{\perp}$, then the limit y also belongs to $A^{\perp}$. Let $x \in A$. By the continuity of inner product we have $\langle x, y \rangle = 0$. □

Theorem 1. *(projection) Let $\mathcal{M}$ be a closed linear subpsace of a Hilbert space $\mathcal{H}$. Then*

(1) For each $x \in \mathcal{H}$, there is a unique closest point $y \in \mathcal{M}$ such that

$$\|x - y\| = \min_{z \in \mathcal{M}} \|x - z\| \tag{2}$$

(2) The point $y \in \mathcal{M}$ closest to $x \in \mathcal{H}$ is the unique element of $\mathcal{M}$ with the property that $(x - y) \perp \mathcal{M}$.