Numerical PDE

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Lecture 1

- Three kinds of PDE we deal with in the class.
  - parabolic PDE (aka heat equation, $\delta = 0$)
  - hyperbolic ($\delta < 0$) (ex) wave equation
  - elliptic ($\delta > 0$)
    Note that in a way, a and c are disproportionately related.
- In numerical PDE, we care about i)accuracy, ii)stability, iii) quickness.
- Introduce the definition of vector space, linear functional, bilinear form, and norm. Note that for a finite Euclidean vector space, for every bilinear form we can find a matrix A so that $B(u, v) = u^T Av$. (consider $u^T Av = \sum_{i,j=1}^{n} u_i A_{ij} v_j$ where $A_{ij} = B(e_i, e_j)$ and u, v being expressed with respect to these basis).
- Symmetric, positive bilinear form is exactly vector space.

  Lemma 1. *There exists a unique $v_0 \in H_0$ (closed subspace of Hilbert) such that $\|v - v_0\| = d(v, H_0)$ (minimizes).*
- Using this, we can show that $v = v_0 + v_{0,\perp}$ where $(v_{0,\perp}, H_0) = 0$. (uniquely decomposed, parallel equation? check.)

Lecture 2

- In finite dimension, all norms are equivalent. $https://math.mit.edu/\ stevenj/18.335/norm-equivalence.pdf$, but the constant depends on dimension. The idea is to use $\|\|_1$. We crucially used here the fact that unit sphere is compact in finite dimension (Heine-Borel). We consider the norm as a continuous function where the domain is endowed by L_1 and the codomain by a random norm.
- If linear operator is bounded then it has a norm. The space of bounded linear operator from Banach to Banach is Banach space.
- We call B is linear functional if $B : V \to \mathbb{R}$ is bounded (which is same as continuity, using the fact that continuity iff continuity at the origin, by the linearity). The space of linear functional is called the dual space V^*.

Theorem 2. *(Riesz Representation Theorem) We have $H = H^*$ in the sense that $\|L\|_{H*} = \|u\|_H$. Here we used sup norm. For every $L \in H^*$, there exists $u \in H$ such that $L(v) = (v, u)$ for every $v \in H$.*

Theorem 3. *(Rudin p40, recast) X is not necessarily Hilbert, but locally compact Hausdorff space. We also consider positive linear functional on $C_c(X)$ (compactly supported complex functions). "we can find some measure" that represents linear functional. $L(f) = \int_X f d\mu$ for every f.*

Proposition 4. *(Stein and Shakarchi p13, recast) For every bounded functional l on L^p there exists $g \in L^q$ so that $l(f) = \int_X f(x)g(x)d\mu(x)$ for all $f \in L^p$. Moreover, $\|l\|_{B_*} = \|g\|_{L^q}$.*

Note that weak derivative is unique up to measure zero.($https://math.stackexchange.com/questions/538$ $of - a - weak - derivative$, check!)

Also, completion and closure can be equivalent if we have an appropriate ambient space/metric.

Definition 1. *Bilinear form a is bounded if there exists M such that $a(u, v) \leqslant M\|u\|\|v\|$. a is coercive if there exists $\alpha > 0$ such that $|a(v, v)| \geqslant \alpha\|v\|^2$*

Example 1. $V = \mathbb{R}^n$:$M = \sigma_1$, intuition is that $a(u, v) = v^T Au$ means we rotate v, scale at most by σ_1 then rotate u.

If A is symmetric, $\alpha = |\lambda_n| = \sigma_n$. Note that symmetric matrix is orthogonally diagonalizable;$A = V\Sigma V^T$, and eigenvalue can be negative.

Important note for symmetric positive definite matrix (`https://math.stackexchange.com/questions/1808799/positive-definite-if-and-only-if-determinants-are-positive`).

Theorem 5. *Let a be* ***positive definite symmetric*** *bounded bilinear form. Solution for $a(u, v) = L(v)$ exists iff $v = u$ minimizes $F(v) = \frac{1}{2}a(v, v) - L(v)$, $v \in H$.*

Theorem 6. *(Lax-Milgram) There is uniqueness. If a is bounded, $\alpha-$coercive, bilinear form and L is a bounded linear functional on H then solution exists with some boundedness of u by L with coefficient α.*

- Think of L as an input, u as a solution.

Lecture 3

- $\partial\bar{\partial}u(x_j) = u''(x_j) + O(h^2\|u\|_{C^4})$ (C^k norm here is a max of sup of each derivatives upto k) by Taylor approximations. How do we come up with this? I don't know. ($U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}$) $u''(x_j) = \bar{\partial}\partial u(x_j) + \bar{\partial}O(h) + O(h)$???

- Due to floating point error, the smallest h does not necessarily give the smallest error $((\frac{u(x_{j+1})-u(x_j)}{h}+\frac{\epsilon_1-\epsilon_2}{h}))$.
- Indeed, $(A_h)_j = -a_j\partial\bar{\partial}$, $j = 1, \ldots, M$.
- Essentially, we switched to matrix form, and reduce it to $-D_aLU = F$. D for diagonal, L is for operator, U is obvious.
- **Error analysis**
  - Maximum principle: $A_hU \leqslant 0 \to \max_j U_j = \max\{U_0, U_M\}$. Think of concave function. So now the important point is that $\frac{U_{j+1}-2U_j+U_{j-1}}{h^2}$ does not depend on h^2. That is, A_hR can be constant, not depending on h. Here in Lemma2, we have Poisson equation $A_hU = f$ but for the maximum principle, we have $A_hU \leqslant 0$.
  - $A_h(U_j + R_j)$
  - Question: how do we come up with $w(x)$? well, it is the ODE solution for $w(0) = w(1) = 0$ and $w'' = c$. I wonder where $O(h^2)$ goes.
  - In class, we only did 1d case.

Lecture 4

- Convergence = stability(Maximum principle, this helps taking off Laplacian) + convergence(Taylor expansion)
- To get the Eigenvalue/Eigenvectors, we have three separate equations. The first step is to guess $v_k = s^k$. Getting two solutions for s, we can let $v_k = c_+s_+^k + c_-s_-^k$. The second important point is to let $\lambda_0 = 2\cos\xi$.
- Essentially L^2 norm, so sub-multiplicativity holds. We just multiply by h.
- L^2 norm of the matrix, so we care about the maximum/minimum eigenvalues.
- The goal is to make discretized function approximate to the analytic in L^2. To this end, we change l^2 and inner product so both converge nicely.
- analogous result holds for 2d case.

Lecture 5

- Condition number can be seen as "how much A stretches and compresses vectors."

- Weak derivative does not always exist. If all weak derivative of n'th order exist, then all weak derivatives less than n exist. Sobolev norm is defined by the square root of the sum of squares of L^2 norms. H_0^1 is a completion of compactly supported smooth functions in H^1.

- LHS involves v' (and v) and RHS involves v; reduced to $a(u,v) = L(v)$. This is on H_0^1.

- not sure why $\|v'\| \leqslant \|v\|$ in H_0^1.

- Poincare inequality: original function bounded by its gradient (a bit sloppy).

- $f \in L^2$ since L^2 is closed under multiplication.

- Finite element method: discretize the function in a global sense! We have then the flexibility with the choice of v, that will ease the computation (this may be why we use it).

- Accuracy; Basically good algorithm. How close is our subspace to the original space in representing a true solution u. A should be well-conditioned. Both are inherent to the "algorithm." (easy to compute?)

- Easy to compute.

- Reduced to $AU = F$ (errata, g should be u, A is tridiagonal, so easy to solve)

- This is in fact $-\partial\bar{\partial}U_j = h^{-1}(f, \phi_j)$, that is, the "average" of f_j.

- Finitely many intervals, hence finite element method.

- By a in page 4, it means the function $a(x)$.

- Matrix A will be same as in finite difference method, but RHS will be different.

- On page 5, I don't understand the L^1 norm, and the idea of the proof at all.

- I think it is saying that A is tri-diagonal in general.

Lecture 8

- What is the point of bounding f_k? Then we can bound α_k.

Lecture 9

- We are solving ODE $u'(t) + au(t) = f(t)$ (can extend to when a is matrix and u,f are vectors).
- Duhamel's principle: think as if we are starting afresh at each time. This is like solving homogeneous equation each time. The homogeneous solution + Integrate from 0 to t a solution for different IVP where initial value is f, that is, $ve^{-at} + \int_0^t e^{-a(t-s)} f(s)ds$
- what is E? I don't know. I can't find it in the book...
- What do you mean that λ is a complex number?
- My understanding is that when nh is small, ODE does not make sense, even though numerically it is stable (converge?).
- The reason we sometimes prefer backward Euler method: does not need to solve nonlinear equation (versus for forward and Crank-Nicholson, we needed $f(t_{n+1}, u_{n+1})$).

Error analysis on the book

- Thm 5.1 says that u_h is an solution that is closest to u in "energy norm."; used Cauchy Schwrtz of inner product in inequality.
- when $v(0) = v(1) = 0$, we have $L2$ norm of a function $\leqslant L2$ norm of derivative, explaining $\|v\|_a \leqslant C\|v'\|$. It follows that the derivative of error is bounded by the original function. This is used in the convergence in Thm 5.2. Q: do we know that this is a tight bound?
- finite element: second derivative, finite difference: fourth derivative.
- Just be careful $\|\|_2$ is not L_2 norm.
- (p56) do not understand $\partial\bar{\partial}u(x_j)$

Chapter 2

- Strong maximum principle says that if interior has maximum then it is constant on the domain - gives uniqueness.
- Thm 2.1: why is linear term not so special?
  - essentially second order term dominates the first order term.
- Idea of the proof (i): first prove the case $Au < 0$ (use the fact that in maximum point, two derivative is nonpositive). In the case $Au \leqslant 0$, we perturb little by v with $Av < 0$ and $v \leqslant 0$ ($v = e^{\lambda x}$, with large enough λ).

- Idea of the proof (ii): we use the fact that $c \leqslant 0$. Pick the maximum interior point and construct largest subinterval containing x_0 in which $u > 0$. Apply the part (i) onto $Au - cu$, which does not have any linear term.
- The little subtlety is that, a, b, c depends on x.
- The stability condition (changing the data, i.e. changing A does not change the solution that much) in theorem 2.2 is derived by finding g such that $A(u(x) + g(x)) \leqslant 0$.
- Using strong maximum principle, monotonicity property can be derived (p18).
- In theory of ODE we said that there exists unique solution involving u^n with boundary conditions up to derivatives of $n-1$. (`https://en.wikipedia.org/wiki/Ordinary_differential_equation`)
- We get an explicit formula using Green's function. In order to show κ is constant, we differentiate it to find its value being zero, and use the theory in ODE to see that it's positive at 0.
- Using g, we get a precise constant for Thm 2.2.
- For general boundary, we can use $\bar{u}(x) = u_0(1-x) + u_1 x$.
- trivial fact: C_0^1 is dense in H_0^1. we can extend from C_0^1 to H_0^1 due to Cauchy Schwartz, and because the operators a and L involve integration.
- Poincare inequality involves H_0^1. It says that $\|v\| \leqslant C\|\nabla v\|$ for every $v \in H_0^1$. For the special case, we said that $\|v\| \leqslant \|v'\|$, proven by Cauchy-Schwarz (convert $v = \int v'$ to $1^2 \cdot v'^2$).
- Coercivity is proven by minimality of a.
- Boundedness is proven by clever Cauchy Schwarz($\mid v'w' \mid + \mid v'w \mid + \mid vw \mid, \|v\|_1 \|w\|_1$).
- not sure about the computation in p 21
- $\|u\|_1 \leqslant C\|f\|$ follows by coercivity (nice direction of inequality going from sobolev-1 norm and bilinear form a).
- Lax-Milgram concerns Hilbert space, which H_0^1 is.
- $f \in L_2$ is enough, which is used to prove that L is bounded.
- When b=0, a is symmetric positive definite (check).
- Discussion about H^2 regularity: we used the fact that a is smooth and $f \in L_2$. For example when $f \in H^{-1}$ (dual of H_0^1, in which case the seminorm for 1 becomes the norm, used in the definition!) u may not be a strong solution.

Chapter 3

- Equation is analogous to the one in chapter 2.

$$Au := -\nabla \cdot (a\nabla u) + b \cdot \nabla u + cu = f$$

  called Dirichlet problem. Here, we set g as a boundary value.

  (c is positive, a is bounded below, what do you mean by smooth boundary?, here we restricted to $(a_{ij}) = aI$)

- $a = 0, b = 0, c = 0$ is a poisson equation, $-\Delta u = f$
- Boundary condition can be Neumann + Dirichlet (called Robin).
- Maximum principle
- 1) Idea of maximum principle: same idea of compensating equal sign by arbitrary function $e^{\lambda x_1}$ for some large λ. The reason that $v = Au + \epsilon A\phi$'s maximum is achieved in the interior is because ϕ is bounded in Ω, so we can adjust ϵ to not exceed the maximum value.
- The second statement says that the maximum cannot be positive. We considered the largest possible open set containing some positive point x_0 and applied part (i).
- page 27, a little imprecise. We needed absolute value of v on the boundary. But the idea is to find ϕ so we can use the previous theorem. Intuitively, we need $\phi \geqslant 0$(needed for $u \leqslant v$) and $A\phi \leqslant -1$(needed to argue that Av is nonpositive).
- Poisson integral formula gives the solution for $-\Delta u = 0$ with g on a circle boundary, in 2d case.
- Change of variable formula for laplacian: $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2}$
- We consider the fourier series of g on the boundary.
- R is the radius of the boundary. r and ϕ are our variables "g appropriately smooth" concerns Fourier series of g. For example, coefficient power series is bounded by L_2 norm of the function. The idea of Thm 3.3 is to construct a nice function, use it as a building block for Fourier series, and get a coefficient using boundary condition.
- Plug in $r = 0$ (value at the "origin"); $P_R = 1$ so we are only left with g on the boundary, proving the mean value property. This is a special case of strong maximum principle (but is why connectedness assumption for Ω plays a role here).
- $(u, A\phi)$=(f, ϕ) because the linear term $cu\phi$ survives without any IBP.

- Fundamental solution is essentially any solution that has singular value at 0 but its integration does not blow up. In the sense of weak derivative, we have $AU = \delta$ (i.e. $(u, A\phi) = \phi(0))$). We also have some boundedness of all derivatives (why?).
- **Thm 3.4?**: not completely sure (justification of C^2, specifically differnetiation under integration), but we could use Fubini (since f and ϕ are nice) to get $(u, A\phi) = (f, \phi)$. Then we said that $u \in C^2$ and so by integration by parts, we know $(Au, \phi) = (u, A\phi)$. Combining these yields $Au - f = 0$
- The point is to realize that convolution makes it more smooth.
- We have fundamental solutions for $d = 2, 3$. We used here $\lim_{\epsilon\to 0} \int_{|x|>\epsilon}$ (check Green's formula, page 31)
- Convoluting with f yields the solution for Poisson PDE with 0 boundary.
- We can also derive a solution for Laplacian with a certain boundary, which is unique.
- Now, for the general A, weak formulation is always an extension of the original problem (depending on the data f and the domain Ω).
- Note that we only needed $f \in L_2$.
- Theorem 3.6: existence of a weak solution

  $$Au := \nabla \cdot (a\nabla u) + b \cdot \nabla u + cu = f$$

  , **vanishing on the boundary**, with $a(x) \geqslant a_0 > 0, c(x) - \frac{1}{2}\nabla \cdot b(x) \geqslant 0$.
- (Thm 3.6) Stability for weak formulation. Lax-Milgram says that u exists (for weak formulation $a(\cdot,\cdot) = L(\cdot)$). L is bounded by using Cauchy Schwarz multiple times. L is bounded by f due to Poincare inequality, which applies for H_{10}. Now $| u |_1 \leqslant C\|f\|$ follows by combining these two (stability).
- In the case $b = 0$, we have positive definite and symmetry of a so nice Dirichlet's principle holds (Thm 3.7, characterization, iff condition, of a weak solution, that is, when F is minimized).
- Inhomogeneous boundary condition, $u = g$ on Γ: used trace operator. page 35?

Chapter 4

- Just mindful about the nice cancellation for difference approximations of derivatives. We have better cancellation for central methods (coeffiacient of u_i is the largest) as the number of terms in the denominator minus the order of denominator plus one. we have $a_j \pm \frac{1}{2}hb_j$ and $A_hU_j \leqslant 0$ (since we deal with linear, we do not need any sophisticated method). Just use the intuition that the maximum cannot be achieved in the middle point by thinking average.

- (p48) Use linear interpolation; roughly speaking, $l_hu_j = 0$ in w_h and $U = 0$ on Γ_h mean that we have zero in the rims and linearly interpolate in the in-homogeneous parts.
- how to prove $\partial\bar{\partial}u - \partial^2u/\partial x^2 \leqslant Ch^2 \mid u \mid_{C^4}$? Taylor series with remainder! h^2 due to central method, C^4 due to nice cancellation.
- Lemma 4.1: think why
- Invertible due to diagonal dominance for h small

Chapter 5

- u_h is the one we get by solving $AU = b$ and I_h is a function in S_h that follows true u.
- This is called "finite element method," a special instance of Galerkin's method(replace H_0^1 with the finite dimensional subspace)
- $a(\cdot, \cdot)$ means that it is inner product!
- I was just confused about how $\|\|_{2,K_j}$ sums up to $\|\|_2$.
- $\partial\bar{\partial}$ equality is nicely cancelled when we use definition.
- In fact, I_h and u_h are identical in our specific case
- confused about ϕ_{ij} on page 56. Subinterval of interval (picture is illustrative). We include all polynomials whose degree is less than k.

triangulation

- $\phi_i(P_j) = 1$ if $i = j$, 0 if $i \neq j$.
- may use polynomial of degree k. In this case we need different number of nodes as before (page 59).
- We might get some trouble when Ω is a smooth curve. Triangulation method enables a flexible way to approximate Ω.
- $\tilde{S}_h$ is just a generalization of S_h, where it does not necessarily vanish on the boundary.

- If the angles are bounded below, then we have

$$\|I_h v - v\| \leqslant Ch^2 \|v\|_2$$

for every $v \in H^2$, for piecewise linear case.

- barycentric method: use only the value of barycentric. nodal method: average out three values of each nodes.
- Check the discussion on convex set and orthogonal projection.

Bramble Hilbert Lemma

- Approximation by polynomial.
- In Bramble Hilbert Lemma: Lipschitz domain means that it is locally the set of points located above the graph of some Lipschitz function.
- Our ultimate goal is to change from norm to seminorm.
- $\|v - q\|_{W^{k,p}} \leqslant C_0 \mid v \mid_{W^{k,p}}$. Remember that we had $\int D^\alpha (v - q) dx = 0$ and q is a polynomail of degree less than equal to $k - 1$.
- **Sobolev inequality**: when $k > d/2$, $H^k(\Omega) \subset C(\bar{\Omega})$ (have a bound accordingly). Big k means many restrictions, so better be regular.
- Bound of $\|I_h v - v\|$: Order of approximation of $I_h v$ depends on the regularity of the function v.

Chapter 7

- Notion of solution operator.
- derivative under infinite sum. Why were we able to do it? (p96)
- Integrand of $\int_0^t E(t-s) f(s) ds$ can be interpreted as a solution at $t - s$ when the initial value is $f(s)$.
- In general when $a = a(t)$, we have $\int_0^t E(t, s) f(s) ds$ where $E(t, s) = exp(-\int_s^t a(\tau) d\tau)$, we cannot take a outside the integral. In this case we introduced two variables. Think of $E(t, s)$ as an operator that takes the solution of $u' + A(t)u = 0$ from s to t. In other words, $u(t) = E(t, s)u(s)$.
- Usually we generalize scalar case to matrix case. Unstable if a is negative. For matrix case, unstable if the smallest eigenvalue(think $\exp -t\lambda_1$, our norm of exponential of $-tA$, as $t \to \infty$) is negative.
- $\cos B = \frac{1}{2}(e^{iB} + e^{-iB})$ is an operator. Similarly for $\sin B$. We can change $u'' + Au = 0$ into two equations by substituting $v = u'$.

- Euler's method: Iteratively approximate the value.
- "if K is larger, U_n grows with n, what do you mean? It means that when $1 - ak$ is less than -1 then we get in trouble. That is why stability condition is $ak \leqslant 2$.
- Error is given by $O(k)$, but only when $ak \leqslant 2$ (step size is small compared to a), noting that $1 - x - e^{-x} \leqslant \frac{1}{2}x^2$.
- $t_n = nk$, so $2t_n a^2 k \mid v \mid = C(t_n, a)k \mid v \mid$. Same for backward Euler error, noting that
$$\mid \frac{1}{1+x} - e^{-x} \mid \leqslant 2x^2, x \geqslant 0$$
- We get an error bound O(k^2) for Crank-Nicolson(because $\mid \frac{1-\frac{1}{2}x}{1+\frac{1}{2}x} - e^{-x} \mid \leqslant x^3$), better than O($k$) for forward/backward Euler method.
- For backward error, we can show an error $O(k)$ independent of the coefficient a. Trick is to bound the exponential function by fractional function, and vice versa (compare ak with 1).
- Unsure about the computation.
- When A is not well-conditioned backward Euler method can be useful.
- For second order ODE, implicit is more stable. We can also make stable irrespective of a by using
$$\frac{U^{n+1} - 2U^n + U^{n-1}}{k^2} + a(\frac{1}{4}U^{n+1} + \frac{1}{2}U^n + \frac{1}{4}U^{n-1}) = 0$$
(typo?)
- Symmetry increases accuracy in general. Implicit method is in general better.
- not completely sure why norm 1 and distinct solutions guarantee stability.
- Just a bit confused about the idea of stability. For analytic solution, stability means that solution is not affected by the perturbation of our input sources as $t \to \infty$. For stability of computational solution, it means that we actually have convergence to the analytic solution, that it does not blow up.
- a can be complex number or a matrix. k is a step size, which is always positive. We are concerned with $\lambda = ak$. The real part of λ determines the stability of analytic solution simply because step size is always given by positive value.

Projection

- Ritz projection is not the minimization because perhaps S_h is not a subspace - u_h is because we have this nice relation $a(u - u_h) = \chi$.

Big O, small O

- Big O $\rightarrow limsup \leqslant C$ as $h \rightarrow 0$
- Small O: lim $\rightarrow 0$ as $h \rightarrow 0$

Fourier transform

- $e_n = e^{-in\theta}$.
  $(e_n, e_m) = \frac{1}{2\pi}\int_0^{2\pi} e_n \bar{e}_m d\theta$ if $n = m$, 0 if $n \neq m$. We may put $\theta = 2\pi x$.
- Best approximation: If f is periodic, then
  $$\|f - S_N(f)\| \leqslant \|f - \sum_{|n|\leqslant N} c_n e_n\|$$
  . Indeed, if f is integrable, then $\frac{1}{2}\pi\int_0^{2\pi} \mid f(\theta) - S_N(f)(\theta) \mid^2 d\theta \rightarrow 0$ as $N \rightarrow \infty$.
- Parseval's identity
  $$\sum_{-\infty}^{\infty} \mid a_n \mid^2 = \frac{1}{2\pi}\int_0^{2\pi} \mid f(\theta) \mid^2 d\theta$$
  , in general
  $$\sum_{-\infty}^{\infty} \mid a_n \mid^2 \leqslant \|f\|^2$$
  .
- Fourier transform of Schwartz class $f \in S(\mathbb{R})$ is defined by $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$. $\hat{f}(\xi)$ is in Schwartz class as well.
- It is a bijective mapping on a Schwartz class.
- Multiplication formula ($\int \hat{f} \cdot g dx = \int \hat{g} \cdot f dx$), $f \hat{*} g = \hat{f} \cdot \hat{g}$.
- Fourier inversion, Plancheral ($\|\hat{f}\| = \|f\|$) hold when fourier transform of moderately decreasing function is moderately decreasing function.
- Intuition:
  - $a_n = \int_0^1 f(x)e^{-2\pi inx}dx \rightarrow \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$
  - $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx} \rightarrow f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi ix\xi}d\xi$
- Bridge between fourier series(periodic function) and transform(non-periodic function): Poisson summation formula $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$

Finite Difference Method

- Log-log scale plot works well since... $E(h) \approx Ch^p \to \log \mid E(h) \mid \approx \log \mid C \mid + p \log h$
-
$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + O(h^4)$$
$$u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + O(h^4)$$
- (p7) follows from odd formula in p4..
- first order: left side, right side, both sides, undetermined coefficients.
- second order: essentially adding these two or using undetermined coefficients.
- Method of undetermined coefficients: the objective is to approximate u' by using specific non-differential terms. We can let a_i as the coefficient for these. (page 9)

Stability, Consistency, Convergence

- Stability: the error does not get amplified (for example, maximum principle for the solution).
- Consistency: Scheme should tend to the differential equation.
- Convergence: Method approaches true solution as h goes to 0.

  `https://www.researchgate.net/post/What_is_the_difference_between_consistency_stability_and_convergence_for_the_numerical_treatment_of_any_PDE` I'm just very confused.. they all look similar..

Errors

- absolute error, relative error (x being in the denominator makes sense), point-wise error
- Machine epsilon (usage of infimum)
- Floating point representation (sign, mantissa, exponent)

  (1) roundoff errors: mantissa roundoff

  (2) overflows and underflows: exponent too big/small

  (3) cancellation errors: due to roundoff error we may get an inaccurate result.

- For $Ax = b$, if we know the upper bound of relative error for A and b, We may get an upper bound of relative error.
- Stability is about algorithm (input and output), condition is about modelling (in order to formulate the mathematical problem).
- Well-posed problem if uniqueness and existence hold. Otherwise, it is ill-posed.
- **The condition number of an instance of a problem is the reciprocal of the normalized distance to the nearest ill-posed instance.** $\min_{X\in M}\|A-X\| = \frac{1}{\|A^{-1}\|}$ follows directly by Eckart-Young (rank$X \leqslant r-1$ case, square sum after r). Note that this is a minimal SINGULAR value!

Boundary C^k

- Boundary is called C^k if for every point on the boundary, we can construct some appropriate function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ that sort of matches with the boundary. Precisely,

$$U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n > \gamma(x_1, \ldots, x_{n-1})\}$$

Green's formula

- First, normal derivative is defined by $\frac{\partial f}{\partial n} = \nabla f(x) \cdot n$, outward pointing.
- Start with

$$\int_U u_{x_i} dx = \int_{\partial U} uv^i dS (i = 1, \ldots, n)$$

then we can derive a divergence theorem

$$\int_U \mathrm{div} u dx = \int_{\partial U} u \cdot \nu dS$$

Applying it to uv, we get IBP,

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} uv\nu^i dS$$

it follows that, by taking $v = 1$ in IBP,

$$\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial v} dS$$

by taking $v \to v_i$ in IBP, we get

$$\int_U Dv \cdot Du dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS$$

Interchanging u and v and subtract, we get

$$\int_U u\Delta v - v\Delta u dx = \int_{\partial U} u\frac{\partial u}{\partial \nu} - v\frac{\partial v}{\partial \nu} dS$$

how is it really related to Green's theorem in Cal3? I really don't know...`https://math.stackexchange.com/questions/74843/when-integrating-how-do-i-choose-wisely-between-greens-stokes-and-divergence`

Trace Operator

- Question: Can we extend a function to the boundary and still make it be in the same functional space? Boils down to finding the existence of the norm $\|\gamma v\|_{(\Gamma)} \leqslant C\|v\|_1$ for every $v \in C^1(\bar{\Omega})$
- Lemma A.1: Function is bounded by Sobolev 1 norm (used Cauchy Schwarz).
- (p236) Trace theorem says that we can uniquely extend $W^{1,p}$ in the domain Ω to its boundary in $L_2 = p$ sense. I am wondering if we can say the trace is "well-defined" irrespective of the norm we use.
- Now we define $H_0^1(\Omega) = \{v \in H^1(\Omega) : \gamma v = 0\}$. Since γ is bounded from above inequality, this is a closed subspace (closed subset of complete space is complete, so this is a Hilbert space as well).
- Poincare: $\|v\| \leqslant C\|\nabla v\|$ for every $v \in H_0^1(\Omega)$, giving the equivalence with the semi-norm (which we use for the definition of dual).

Office hours

- For estimating $I_h - u$, we first used stability theorem for stability. We go from energy norm to $\|\dot{\|}_1$ norm.
- $| f_k |$ is really an error since its convergence measures the magnitude of coefficients.
- Rectangles - too many neighboring points, so too many degree of freedom. Particular structure of triangular due to our structure of subspace.
- Relationship between Taylor series and Brandle-Hilbert Lemma.
- Fourth order limit due to h^2.

remarks

- Coercivity due to $a > c$, boundedness of a due to Cauchy-Schwartz, boundedness of L due to Poincare inequality. We needed $f \in L^2$ (this is specifically for using H_0^1 norm, which is just L_2 norm of the derivative).
- Question: how do you use Green's function?
- Sobolev inequality only concerns continuity.
- parabolic looks like we have lower "degree" so it makes sense that delta is zero.
- Taylor expansion of f centered at the Barycenter and the exactness implies that the linear term vanishes if we intergrate over K!
- Question: for Bramble Hilbert, the reason for the change of variable is to get the extra factor h^2? For 1d, it makes sense that we use the lemma on an interval because we have a linear function. But on triangle, we do not have a polynomial. How do we resolve that issue?
- Is Bramble Hilbert lemma somehow related to Taylor's theorem? Since I don't have a good intuition of it, we seem to be able to prove 1d case by using Bramble-Hilbert or Taylor's series.
- Just not entirely sure about the second approach in the problem. He seems to have used change of variable for the first inequality, then I'm not entirely sure where he used Bramble Hilbert since we needed seminorm somewhere.
- To prove Friedrich's inequality, we had to use this function ϕ and use integration by parts. We needed a little trick by first using Cauchy-Schwartz and then used the fact $ab \leqslant \frac{a^2+b^2}{2}$.
- Poincare says that we do not change like crazy for H_0^1 since it vanishes at the boundary. Proof by Cauchy-Schwarz.

Parabolic Analytics

- Gauss Kernel, Fourier techniques, energy arguments, eigenfunction expansions, Duhamel's principle, variational formulation, maximum principle.
- Time is always positive, and the space is $\mathbb{R}^d$.
- We used Fourier transform, the fact that $\hat{e}^{-t|\xi|^2} = \pi^{d/2} e^{-|\xi|^2/4}$ and convolution formula to derive fundamental solution given by $u(x,t) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} v(y) e^{-|x-y|^2/4t} dy$ where v is our initial function depending on x. This is a convolution of this Gauss Kernel with fundamental solution.

- Maybe to verify that this satisfies PDE is straightforward calculation. To verify that this satisfies initial condition needs some analytic work involving ϵ and δ.
- We can view this as an act of solution operator called $E(t)$.
- Its operator norm is one.
- This gives a nice stability condition! Namely $\|E(t)v\|_C \leqslant \|v\|_C$.
- Uniqueness holds by maximum principle. Continuous dependence holds as well (which makes this problem well-posed).
- Also, $E(t)$ has a smoothing property essentially due to exponential term.
- Now for the weak variation formula, we can weaken the regularity of initial value function. That is, $v \in L_2(\Omega)$.
- Note that we have some kind of stability for weak formulation as well.

Parabolic Numerics

- When $\lambda < \frac{1}{2}$ (this is stability), we have a solution bounded by max of v, complete analogue for analytical solution. Parseval's theorem rescues us (p134). In the end, some inequality involving C holds for every n
- h in x (lower index), k in time (upper index). We have $\lambda = k/h^2$.
- When $\lambda > \frac{1}{2}$, the method is unstable (we saw in theorem 9.2 that for specific choice of v, the scheme will always be unstable).
- Initial data should be smooth enough in order for discrete solution to converge to exact solution as mesh tends to zero.
- Stability estimate and the truncation error estimate.
- Question on 132 computation. z is an error for the function itself, and τ is an error for the operator.
- E operator upgrades the time (need to specify step size of time). τ is a truncation error - we can bound by using Taylor's theorem. Further, this is just analytical solution acted by discrete operator.
- Error at time n is bounded linearly in n, precisely $Cnkh^2 \mid v \mid_{C^4}$. n since n many terms are bounded by the same $h^2 \mid v \mid^{C^4}$.
- This also says that total effect is second order accuracy with respect to the step size in x which is h.
- Where is this $U_j^1 = \epsilon \sum_p a_p e^{i(j-p)\xi_0}$ coming from? This is just using definition $v_j = e^{ij\xi_0}\epsilon$.

- We defined $\tilde{E}(\xi)$ in order to say something about stability. (necessary condition is when its maximum norm is less than equal to 1). This agrees with the previous discovery when we have $\tilde{E}(\xi) = 1 - 2\lambda + 2\lambda \cos\xi$ (we had $\tilde{E}(\xi) = \sum_p a_p e^{-ip\xi}$.

- We defined mesh-norm of v, discrete Fourier transform of v, and relate these two by Parseval's relation.

- h factor is necessary since if not we just add up things up by shrinking the mesh size.

- Let's get this straight about Parseval's theorem: sum of coefficients is L^2 norm of the function.

- So this is the definition, no extra h term.

- why $C = 1$? because it must hold for all n.

- Can treat U^n as a continuous function (do not discretize in x but we can still use discretized operator): then can use L_2 function.

- Why would the accuracy be defined that way?

- Since λ is constant, as $h \to 0$, we also have $k \to 0$.

- Check what it means to be consistent.

- Can be all generalized into θ method.

- $\|z^n\|_{\infty,h} \leqslant k\sum_{l=0}^{n-1}\|\tau^l\|_{\infty,h}$ is pretty important step since it enables us to go from $A(U-u)$ to $U-u$. This is how we have gotten convergence up till now. Where is this step coming from? In fact, this comes from $z_j^{n+1} = (E_k z^n)_j - k\tau_j^n$. Note that τ treats u.

- Forward Euler method has the problem in that we need to have $\lambda \leqslant \frac{1}{2}$. We can make this better by using backward Euler, which depends on $h^2 + k$. This is still first order accuracy in time so to make it second okrder accuracy in time, use Crank-Nicolson.

- For Crank-Nicolson, easy to show stability for $\lambda < 1$. In order to treat the other case, we need to go L^2 space and argue in $\tilde{E}$.

- Essentially to get a bound, using Taylor series is necessary.

- I'm not sure if I understand the heuristic in the lecture note. What does $e^{\delta u}$ tell us? It explains the rationale of why we used Fourier transform.

- Stability pertains the norm of the operator.

- Now, I'm unsure why we bound in terms of u or v. Well, this is because the book uses mixed initial boundary condition.

- (page 134) why if and only if? why do we care about $l_{2,h}$ stability?
- Maybe nice iff condition is due to Parseval and nice multiplicative property of hat.
- We can continuously define L^2 function(how?) and do analysis accordingly. Q: what is the maximum or minimum that we tried to bound? (in the homework...)
- When $\lambda > \frac{1}{2}$, unstable! (since $\|U^n\|_{\infty,h} = (4\lambda - 1)^n \epsilon \to \infty$)
- τ_j^n: truncation error
- $\tilde{E}$: Parseval
- Not sure how to prove (9.16).

Fourier series, Fourier discrete transform

- DTFT is a Fourier series. `https://en.wikipedia.org/wiki/Discrete-time_Fourier_transform`
- You can see that this is applied for l^2. Read up until page 136.
- We used Fourier series for stability and Fourier transform for accuracy. Convolution enables us to do great diagonalization. page 132
- $E_K u$ and $\partial_t u_j^n - \partial_x \bar{\partial}_x u_j^n$ is what made things complicated.
- Discrete max norm.

- ### Parabolic Equation

- Initial value problem: $\frac{\partial u}{\partial t} - \Delta u = 0$ and $u(\cdot, 0) = v$.
- Maybe regularity matters when we do Fourier transform.

Finite element method for Parabolic PDE

- discretize only in space.
- Then coefficient will depend on time.
- α: nodal value depending on t.
- τ_h is a number of interior nodes, not a number of triangles.
- Well d_j is just a constant, not a derivative!
- Duhamel's principle for discretized solution? (p151, check!)

- P_h has unit norm in L_2? Yes, since it is a projection.
- Unsure about a remark in stability.
- Basically used the fact that ρ is accessible.
- Theorem 10.1 already gives us the information that infinity norm is less than equal to L_2 norm, which gives us some kind of regularity. Usually L^∞ convergence is strong so this ($\|\|_\infty \leqslant \|\|_2$) is our new information.
- What is the rationale behind dividing into two parts in theorem 10.1? This usage of $a-$norm projection. Why is it a reasonable thing?
- The moral of the story of thm 10.1 is that it gives h^2 error!
- stationary problem - time independent problem.
- maximum principle gives stability, since we bound Au in terms of v. Now, for discrete case, maximum principle does not necessarily hold, but it does in the case of quasi-uniform triangulation ($h_K \geqslant ch$).
- Also, we could redefine B to be the row sum diagonal matrix to have maximum princ
- I'm unsure what it means by ρ_t and θ_t.
- It seems like there is a relationship between θ and ρ_t that I'm unaware of, see page 155.

Hyperbolic Equations

- Defined "Characteristic polynomial" of linear PDE.
- So this wedge thing is a matrix. If determinant is zero, we say that it is characteristic surface. characteristic direction at x is the one that is normal at each point of surface.

Hyperbolic Numerics

- Used stability and local truncation to have infinity norm convergence.
- truncation error comes from using PDE and mean value Taylor theorem.
- To finish up, another key ingredient is $k\tau^n = u^{n+1} - E_K u^n$.
- $\tau^n = \partial_t u^n - a\partial_x u^n = k^{-1}(u^{n+1} - E_k u^n) = O(h^r)$
- I am not sure why it is defined that way. Well, page 135 explains this; if we go back to E_k, this exactly means that $u^{n+1} - E_k u^n$ is of order h^r.

Finite Volume method

- On $\xi(t), t$, u is constant, and that's all it matters.
- Not understanding how we got slope.
- Where did we use the weak solution?
- not understanding the graph
- how do we restore uniqueness?
- So viscocity means smoothing process?

CFL condition

- domain of dependence of the finite difference scheme at (x, t) should contain the domain of dependence of the continuous problem .

Consistency, stability, convergence

- Note that in functional analysis textbook, it is about linear operator A.

Problem 4

- We do not have to normalize for Gram-Schmidt? I think this wikipedia gives me a hint how to tackle the other problem. Fascinating.
- If we do not normalize properly, we actually have different polynomials. But it's just that we do not have nomrmalized version in the wikipedia.
- Clearly we need to use orthogonality.
- (b) says that we have control over all polynomials at the point of the root of Legendre polynomial.
- I'm not understanding: for what vector space?
- this must be finite element method.
- How do I freaking use the function when I publish it?

problem 1

- The problem is that $v \in H_0^2$, so it is not three times differentiable.
- In fact, the Bramble-Hilbert treats both original function and its derivative so we are good.

Problem 2

- We are concerned about the original vector space. $\tilde{u}$ is a solution in the subspace.
- The problem is that this is not an inner product so we cannot use Cauchy-Schwarz.
- We can check if Cauchy Schwarz still holds.
- The variation of Cauchy Schwarz does not seem to work out; the problem is I don't know how to deal with $a(\tilde{u}-\chi, u-\tilde{u})$. This is not $a(u-\tilde{u}, u-\tilde{u})$. Also, is $\tilde{u} \in \tilde{V}$?
- I might need to look at the proof of Lax-Milgram; heavily used Riesz representation theorem. Given bound does not seem to help either.
- Projection theorem.
- Cauchy Schwarz does not work...
- Actually the inequality is straightforward...

Problem 3

- tools: Sobolev inequality, facts about angle and area, change of variable in multi-dimensions.
- How can I use Sobolev? Ask in office hours.
- We want to change $\tilde{v}$ to v and relate a'_{ij} to $\det A$: how?
- I think translation is obviously okay, and rotation should be okay since L^2 norm is rotationally invariant. We can actually get two exact triangles. The obvious problem is that we cannot really bound a_{11} and a_{22} by $\frac{C}{h}$ or in terms of the determinant.
- I was confused with gradient and diversion.
- Condition number comes up when we take the derivative. But I'm still not sure about this.

Problem 1

- Parseval's theorem, page 134.
- ξ should be $2d$. We take fourier transform to $E_k V$ and use Parseval's theorem - we get the condition for $\tilde{E}$.
- it is reasonable to use Parseval's theorem.

Problem 4

- What about matrix?

Lingering questions

- is E tilde being less than one a iff condition?
- p141 middle computation.