On Determinants

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The aim of this document is to delve deeper into the concept of the determinant of a matrix, enriching our understanding beyond the standard definitions.

Begin by visualizing an $n \times n$ matrix as a linear transformation. In mathematics, it's often enlightening to view objects not just for what they are, but for the roles they play—in this case, as functions. Through this lens, we can perceive the determinant as a function: it ingests n column vectors from $\mathbb{R}^n$ present in the matrix and produces a real number. But what does this number represent? At its essence, the determinant can be seen as an indicator of oriented volume.

1. **Sign Inversion**: Interchanging rows (or columns) of the matrix inverts the sign of the determinant.
2. **Linearity**: The determinant is linear in relation to each column and row.
3. **Identity Matrix**: The determinant of the identity matrix is 1.

With this understanding, we recognize the inherent logic in the definition of the determinant. Moreover, when extended to continuous domains, this understanding paves the way to the concept of the wedge product—an essential tool for generalizing integration, which is fundamentally about calculating the "volume" of more complex structures, often referred to as manifolds.

We define

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g)$$

where

$$Af = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f$$

and

$$f \otimes g(v_1, \ldots, v_{k+l}) = f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+l}).$$

It follows that

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) = det[\alpha^i(v_j)]$$

The formulation of $f \wedge g$ is meticulously designed to encapsulate the inherent attributes of the determinant. Specifically:

1. The anticommutative nature is reflected in the property 1.
2. The linearity is mirrored in property 2.

3. The normalization constant $\frac{1}{k!l!}$ embodies property 3.

Remark 1.

- The above characterizations intuitively and rigorously (by simply checking that $\det(A)\det(B)$ satisfies three characterizations) demonstrate why $\det(AB) = \det(A) \times \det(B)$.
- From this, we can see that the determinant of orthonormal matrix is -1 or 1, and in turn that the determinant is a multiplication of all singular values by SVD.
- These singular values represent the extent of stretching in each of the n directions.
- The logarithm function translates multiplication into addition and possesses inherent concavity. As a result, for a positive definite matrix A, $\log\det(A)$ is concave. A more rigorous justification can be derived by verifying $g''(t) \leqslant 0$ for the function $g(t) = f(Z + tV)$ where $Z, V \in S^n$.

Definition 1. *Moore-Penrose pseudo-inverse for a matrix $A \in \mathbb{C}^{m\times n}$ is defined as a matrix $X \in \mathbb{C}^{n\times m}$ satisfying*

- $(AX)^* = AX$
- $(XA)^* = XA$
- $XAX = X$
- $AXA = A$

Remark 2.

- Uniqueness can be seen by computing SVD and inverse each singular value, which is the most obvious thing to do.
- Geometrically, this is a least squares problem (L^2 norm): there exists a unique vector x such that Ax is closest to b.
- We can use different pseudo-inverse by using different norm, say L^∞ norm.
- Related concept is rank-r approximation in Young-Eckart theorem. The key of the proof is by exploring the space spanned by the right singular vectors $v_1, \ldots, v_{r+1}$ (and connecting it to kernel of B) and deriving contradiction from that (for example by dimensionality argument).