On Determinants

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The goal of this document is to understand determinant of the matrix in a better way.

First, view the matrix of $n \times n$ dimension as a linear transformation. It is helpful to think of determinant as a function that takes n many $\mathbb{R}^n$ dimensional vectors in the matrix and outputs a real number. The number it is supposed to tell us is the change in oreinted volume by the action of matrix (linear transformation), disregarding the shape of inputs and outputs.

This heuristics can be made more precise by characterization of determinant "function" in the following way.

1. Switching rows change the sign.
2. It is linear for each element (column) vector.
3. determinant of the identity matrix is 1.
4. Multiplying a row by a constant multiplies the determinant by that constnat.

This motivates the definition of wedge product in differential topology.

We define

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g)$$

where

$$Af = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \sigma f$$

and

$$f \otimes g(v_1, \ldots, v_{k+l}) = f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+l}).$$

It follows that

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) = det[\alpha^i(v_j)]$$

This is exactly an attempt to capture the characteristics of determinant; antilinearity corresponds to 1, and linearity corresponds to 2 and 4. Further, 3 corresponds to the normalizing constant $\frac{1}{k!l!}$.

Remark 1.

It is intuitively clear that under these characterizations $det(AB) = detA * detB$

In this perspective, we can alternatively view the determinant as the product of all singular values, which are simply the degree of stretch in each n direction.

Logarithm translates product to addition, and is a concave in itself. It follows that $\log det(A)$ where A is positive definite is concave. We may make this more rigorous by checking $g''(t) \leqslant 0$ where $g(t) = f(Z + tV)$ for $Z, V \in S^n$.