Lec1

Begin: Special Relativity in a form that emphasizes its geometric nature

Spacetime: A manifold of events endowed w. a metric

① Manifold: a set of points with well understood connectedness properties

More rigorous discussion: Carroll pp 54-62

② Event: when & where something happens

Label with coordinates but event itself exists indpt of these labels

③ Metric: encode gravity

A notion of distance between events in manifold

Without this, a manifold has no notion of distance encoded in it.

Special Relativity: Simplest theory of spacetime, correspond to general relativity in no-gravity limit

key notion: inertial reference frame

Lattice of clocks & measuring rods that allows us to assign coordinates to spacetime events

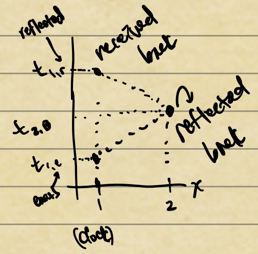
properties:

(i) Lattice moves freely through spacetime
no force acts on it at all

(ii) Measuring rods are orthogonal to each other (ex) x, y, z - axis
tick marks are uniformly spaced (irrespective of time)

(iii) Clocks tick uniformly (no evolution of time standard)

(iv) Clocks synchronized using Einstein synchronization procedure
Takes advantage of the fact that the speed of light is the same to all observers



$t_{1,e}$: Clock 1 emits a pulse of light

$t_{1,r}$: Clock 1 receives a reflected pulse

$$t_{2,0} = \frac{1}{2}\left(t_{1,e} + t_{1,r}\right)$$

reflect

Units Choose basic unit of length to be the distance light travels in your basic unit of time

If time unit is 1 second, length unit is 1 light second

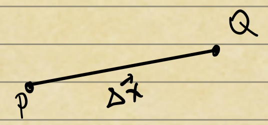
If time unit is 1 nanosecond, length unit is 1 light nano-second = 1 foot

$$c = \frac{1 \text{ light-time unit}}{\text{time unit}}$$

conversion factor

observes two events P Q

O is an observer in the inertial reference frame defined earlier (IRF)



displacement from P to Q

$$\Delta\vec{x} \underset{O}{\doteq} (t_Q - t_P,\ x_Q - x_P,\ y_Q - y_P,\ z_Q - z_P)$$

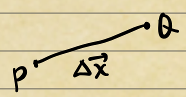
$\Delta\vec{x} \to \Delta x^{\mu}$ $\mu \in [t, x, y, z]$

$\in [0, 1, 2, 3]$

Greek indices tend to be used to label spacetime indices x^{μ}

Latin indices are often used to pick out spatial component at a moment x^1, x^2, x^3

Different Inertial observer



Measured by $\bar{O}$

P, Q, Δx are geometric objects exist indpt of representation

$$\Delta\vec{x} \xrightarrow[\bar{O}]{} \Delta x^{\bar{\mu}}$$

$\Delta\vec{x} \xrightarrow[\mathcal{O}]{} \Delta x^{\mu}$

Lorentz transformation relates the components in the two representations

$$\Delta x^{\bar{0}} = \gamma \Delta x^{0} - \gamma v \Delta x^{1}$$

$$\Delta x^{\bar{1}} = -\gamma v \Delta x^{0} + \gamma \Delta x^{1}$$

$$\Delta x^{\bar{2}} = \Delta x^{2} \quad , \Delta x^{\bar{3}} = \Delta x^{3}$$

$\bar{\mathcal{O}}$ moves with v along axis 1

with speed v as seen by $\mathcal{O}$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

Better notation : $\Delta x^{\bar{\mu}} = \sum_{\nu=0}^{3} \Lambda^{\bar{\mu}}{}_{\nu} \Delta x^{\nu}$

-or- $= \Lambda^{\bar{\mu}}{}_{\nu} \Delta x^{\nu}$ Einstein summation convention

Notice $\Lambda^{\bar{\mu}}{}_{\nu} = \dfrac{dx^{\bar{\mu}}}{dx^{\nu}}$ More general form

Notice $\Delta x^{\bar{\mu}} = \Lambda^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\nu} \Delta x^{\nu}$

calling this ν is not critical

$= \Lambda^{\bar{\mu}}{}_{\alpha} \Delta x^{\alpha}$

In this equation, ν (or α) is called a "dummy index"

$\bar{\mu}$ is not dummy! Call this "free index"

Spacetime vector: Any quartet of numbers ("components")

which transforms between inertial reference frames like displacement vector

$\vec{A} \underset{\mathcal{O}}{=} (A^0, A^1, A^2, A^3)$ $\left(\xrightarrow[\mathcal{O}]{} A^{\alpha}\right)$

if $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\alpha} A^{\alpha}$ describes components for $\bar{\mathcal{O}}$, then vector

Also require linearity rules

$\vec{A}$ is a vector $\vec{B}$ is a vector then $\vec{C} = \vec{A} + \vec{B}$ a vector

$\vec{A}$ is a vector, a is a scalar (same to all observers)

then $\vec{D} = a\vec{A}$ vector

(four vector $\leftrightarrow$ tangent space)

2 Basis vectors

In frame $\mathcal{O}$, write down 4 special vectors

$\vec{e}_0 \underset{\mathcal{O}}{=} (1,0,0,0)$

$\vec{e}_1 \underset{\mathcal{O}}{=} (0,1,0,0)$

$\vec{e}_2 \underset{\mathcal{O}}{=} (0,0,1,0)$

$\vec{e}_3 \underset{\mathcal{O}}{=} (0,0,0,1)$

Compactly, $(\vec{e}_\alpha)^\beta \underset{\mathcal{O}}{=} \delta_\alpha{}^\beta$

Utility of this

$\vec{A} = A^\alpha \vec{e}_\alpha$

$\hookrightarrow$ an actual equal sign, not representation symbol

* How do basis vectors transform?

$\vec{A} = A^\alpha \vec{e}_\alpha = A^{\bar{\mu}} \vec{e}_{\bar{\mu}}$

$= (\Lambda^{\bar{\mu}}{}_\beta A^\beta) \vec{e}_{\bar{\mu}}$

$= (A^\beta \Lambda^{\bar{\mu}}{}_\beta) \vec{e}_{\bar{\mu}}$

β is dummy index, so can rewrite

$= A^\alpha \Lambda^{\bar{\mu}}{}_\alpha \vec{e}_{\bar{\mu}}$

$A^\alpha (\vec{e}_\alpha - \Lambda^{\bar{\mu}}{}_\alpha \vec{e}_{\bar{\mu}}) = 0$

Means $\boxed{\vec{e}_\alpha = \Lambda^{\bar{\mu}}{}_\alpha \vec{e}_{\bar{\mu}}}$

Recall $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_\alpha A^\alpha$

added

Simple algorithm "Line up the indices"

Inverse Lorentz transformation just reverse the velocity

$$\vec{e}_\alpha = \Lambda^{\bar{\mu}}{}_{\alpha}(\underset{\sim}{v})\,\vec{e}_{\bar{\mu}}$$

3-vector

$$\vec{e}_{\bar{\mu}} = \Lambda^{\nu}{}_{\bar{\mu}}(-\underset{\sim}{v})\,\vec{e}_{\nu}$$

$$\vec{e}_\alpha = \Lambda^{\bar{\beta}}{}_{\alpha}(\underset{\sim}{v})\,\vec{e}_{\bar{\beta}}$$

$$= \Lambda^{\bar{\beta}}{}_{\alpha}(\underset{\sim}{v})\left(\Lambda^{\nu}{}_{\bar{\beta}}(-\underset{\sim}{v})\,\vec{e}_{\nu}\right)$$

$$= \left[\Lambda^{\bar{\beta}}{}_{\alpha}(\underset{\sim}{v})\,\Lambda^{\nu}{}_{\bar{\beta}}(-\underset{\sim}{v})\right]\vec{e}_{\nu}$$

Requires $\Lambda^{\bar{\beta}}{}_{\alpha}(\underset{\sim}{v})\,\Lambda^{\nu}{}_{\bar{\beta}}(-\underset{\sim}{v}) = \delta^{\nu}{}_{\alpha}$

Likewise, $\delta^{\bar{\nu}}{}_{\bar{\alpha}} = \Lambda^{\beta}{}_{\bar{\alpha}}\,\Lambda^{\bar{\nu}}{}_{\beta}$

invar. btwn reference frames

Scalar product

Recall that $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$

invariant to inertial reference

$$\Delta s^2 \equiv \Delta\vec{x}\cdot\Delta\vec{x}$$

$$\underset{\theta}{=} -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

Since 4-vectors have the same transformation properties as Δx, we similarly define

$$\vec{A}\cdot\vec{A} \underset{\theta}{=} -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$$

This must be Lorentz invariant

Terminology : $\vec{A}\cdot\vec{A} < 0$: $\vec{A}$ is "time-like" (can find an observer, same location, diff in time)

$\vec{A}\cdot\vec{A} > 0$: $\vec{A}$ is "spacelike" (can find an observer, same time, diff location)

$\vec{A}\cdot\vec{A} = 0$: $\vec{A}$ is "light like" or "null"

tangent to the trajectory of light beam at spacetime

Eyeballer → can connect two events

More general notion

$\vec{A}\cdot\vec{B} \underset{\theta}{\doteq} -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$

This is invariant too

$\underset{\vec{A}+\vec{B}}{\vec{C}}\cdot\vec{C} = \vec{A}\cdot\vec{A} + \vec{B}\cdot\vec{B} + 2\,\vec{A}\cdot\vec{B}$

invar. since other terms are!

$\vec{A}\cdot\vec{B} = (A^\alpha \vec{e}_\alpha)\cdot(B^\beta \vec{e}_\beta)$

$= A^\alpha B^\beta \underbrace{\vec{e}_\alpha\cdot\vec{e}_\beta}$

$= A^\alpha B^\beta \eta_{\alpha\beta}$

$\eta_{\alpha\beta} = I_{4\times4} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \rightarrow$ the "metric" tensor

$\Delta s^2 = \Delta\vec{x}\cdot\Delta\vec{x}$ fundamentally, "distance"

$ds^2 = d\vec{x}\cdot d\vec{x}$

$= \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta$

$d\vec{x} = dx^\alpha\, \vec{e}_\alpha$

When this is true, we say that $\vec{e}_\alpha$ is a "coordinate basis vector".

Not an interesting fact in cartesian coordinates

How about curvilinear coordinates?

$d\underset{\sim}{x} = dx^i\, \hat{e}_i$ $\quad$ $(\vec{r} = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta))$

$= \underset{\text{length}}{\underset{\sim}{dr}}\,\vec{e}_r + \underset{\text{angle}}{\underset{\sim}{d\theta}}\,\vec{e}_\theta + \underset{\text{angle}}{\underset{\sim}{d\phi}}\,\vec{e}_\phi$

$= dr\,\vec{e}_{\hat{r}} + r\,d\theta\,\vec{e}_{\hat{\theta}} + r\sin\theta\, d\phi\,\vec{e}_{\hat{\phi}}$

Basis $\vec{e}_{\hat{i}}$ is an orthonormal basis

$: \vec{e}_{\hat{i}}\cdot\vec{e}_{\hat{j}} = \delta_{\hat{i}\hat{j}}$

Our basis has $\vec{e}_r\cdot\vec{e}_r = 1$

$\vec{e}_\theta\cdot\vec{e}_\theta = r^2$

$\vec{e}_\phi \cdot \vec{e}_\phi = r^2 \sin^2\theta$

Important 4-vectors

$\vec{u} = \frac{d\vec{x}}{d\tau}$ → 4-velocity

$\doteq (\gamma, \gamma \underline{v})$ $d\tau$ = time interval as measured along the trajectory of observer w/ $\vec{u}$.

$\gamma = \frac{1}{\sqrt{1-v^2}}$ = interval of proper time

In the rest frame of this observer, $\vec{u} = (1, \underline{0})$

4-momentum $\vec{p} = m \vec{v}$

"rest mass" of object : Lorentz invariant

$= (\gamma m, \gamma m \underline{v})$

γm ← relativistic mass (not used anymore)

$\doteq (E, \underline{p})$

Couple these to scalar product

$\vec{u} \cdot \vec{u} = -\gamma^2 + \gamma^2 v^2 = -1$ (easier: $-(1)^2 + 0 = -1$)

$\vec{p} \cdot \vec{p} = m^2 \, \vec{u} \cdot \vec{u}$

$= -m^2$

$= -E^2 + |\underline{p}|^2$

$\rightarrow \boxed{E^2 - |p|^2 = m^2}$ — or —

$E^2 - p^2 c^2 = m^2 c^4$

Conservation of four-momentum

N particles that interact, then

$\vec{P}_{TOT} = \sum_{i=1}^{N} \vec{P}_i$ is conserved in the interaction

Algebra often simplifies by choosing "center of momentum frame"

$\vec{P}_{Tot} \underset{COM}{\doteq} (E, \underline{0})$

Very useful for studying particle collisions

$\vec{p}_A \quad \vec{p}_B \Rightarrow \dot{m}_C \quad \dot{m}_D$ (pset)

Very useful result follows from invariance of scalar product

$\begin{cases} \text{Let } \vec{p} \text{ be 4-momentum of particle A} \\ \quad'' \ \vec{u} \ '' \text{ 4-velocity of observer } \theta \end{cases}$

What does θ measure as energy of particle A?

so $\vec{p} \underset{\theta}{\doteq} (E_\theta, \vec{p}_\theta)$

But $\vec{u} \underset{\theta}{=} (1, \vec{0})$

$$\boxed{-\vec{p} \cdot \vec{u} = E_\theta}$$

Invariance guarantees that this holds no matter what frame is used for $(\vec{p}, \vec{u})$'s representation

* Another imp. vector

$\vec{a} = \dfrac{d\vec{u}}{d\tau}$

Always $\vec{a} \cdot \vec{u} = 0$

pf) $0 = \dfrac{d\overbrace{\vec{u} \cdot \vec{u}}^{(=-1)}}{d\tau} = 2\vec{u} \cdot \vec{a}$

* Tensors more generally

A tensor of type $\binom{0}{N}$ as a function (or mapping) of N 4-vectors into Lorentz-invariant scalars,

which is linear in each of its N arguments

Lec 3

Tensor $\vec{A} \cdot \vec{B} = \underset{\sim}{\eta_{\alpha\beta}} A^\alpha B^\beta = a$

$\gamma \vec{A} \cdot \vec{B} = \eta_{\alpha\beta} (\gamma A^\alpha)(B^\beta) = \gamma a \quad \rightsquigarrow \quad \eta(\gamma A, B) = \gamma \eta(A, B)$

$\vec{A} \cdot (\vec{B} + \vec{C}) = \eta_{\alpha\beta} A^\alpha B^\beta + \eta_{\alpha\beta} A^\alpha C^\beta \rightsquigarrow \eta(A, B+C) = \eta(A, B) + \eta(A, C)$

Abstractly define tensor as a two-slot mathematical machine

$\bar{\bar{\eta}}(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} = \eta_{\alpha\beta} A^\alpha B^\beta = a$

frame - independent geometric object

Put geometric objects into slots, get geometric object out. tensor must be a frame-independent geometric object as well!

Different representations of the tensor are used by different observers

To get the components used by a particular observer, plug basis vectors into its slots

$$\eta(\vec{e}_\alpha, \vec{e}_\beta) = \eta_{\alpha\beta}$$

$$\bar{\bar{\eta}}(\vec{e}_{\bar{\alpha}}, \vec{e}_{\bar{\beta}}) = \eta_{\bar{\alpha}\bar{\beta}}$$

$$\eta_{\bar{\alpha}\bar{\beta}} = \bar{\bar{\eta}}(\Lambda^{\mu}{}_{\bar{\alpha}}\, \vec{e}_\mu, \Lambda^{\nu}{}_{\bar{\beta}}\, \vec{e}_\nu)$$

free

$\Lambda^{\circ}{}_{\circ}$

index

$$= \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}}\, \bar{\bar{\eta}}(\vec{e}_\mu, \vec{e}_\nu)$$

$$\boxed{\eta_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}}\, \eta_{\mu\nu}}$$

Note: $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ in all frames!

$\binom{0}{1}$ tensors: special subset "1-forms"

(also "dual vectors")

1-form is mapping from a single vector to f.I. scalars

$\tilde{p}(\vec{A})$ = some scalar

Components come from putting in basis vectors

$$\tilde{p}(\vec{e}_\alpha) = p_\alpha$$

$$\tilde{p}(\vec{A}) = \tilde{p}(A^\alpha \vec{e}_\alpha)$$

$$= A^\alpha p_\alpha$$

$$\boxed{p_{\bar{\alpha}} = \Lambda^{\mu}{}_{\bar{\alpha}}\, p_\mu}$$

Basis 1-forms: want a set of geometric objects $\{\tilde{w}^\alpha\}$ such that

$$\tilde{p} = p_\alpha \tilde{w}^\alpha = \tilde{p}(\vec{e}_\alpha)\tilde{w}^\alpha$$

Combine what I want to get with contraction

$$\tilde{p}(\vec{A}) = p_\beta \tilde{w}^\beta(A^\alpha \vec{e}_\alpha)$$

$$= p_\beta A^\alpha \tilde{w}^\beta(\vec{e}_\alpha)$$

Require that $\tilde{w}^\beta(\vec{e}_\alpha) = \delta^\beta_\alpha$

Leads to (for example) $\tilde{w}^0 \underset{\theta}{\doteq} (1,0,0,0)$ row vectors

$\tilde{w}^1 \underset{\theta}{\doteq} (0,1,0,0)$

$\vdots$

look a lot like basis vectors, entered in a dual way akin to row vectors verses column vectors

(ex) $\sum_{\mu=0}^{3} A^\mu B^\mu \longrightarrow$ plays no role in our physics

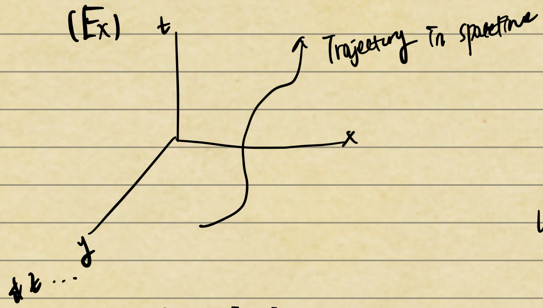
$\sum_\mu p_\mu A^\mu$ is important

Quantum Wavefunction $\psi(\vec{x}), \phi(\vec{x})$

$$\int \psi(\vec{x})\,\phi(\vec{x})\,d^3x$$

$$\int \psi^*(\vec{x})\,\phi(x)\,d^3x = \langle \psi, \phi \rangle$$

(Ex)



$\vec{u} = \frac{d\vec{x}}{d\tau}$ τ: proper time

Suppose spacetime is filled with some field $\phi(t, x, y, z)$

What is the rate of change of ϕ along the trajectory?

3-space intuition

$$\frac{d\phi}{dt} = \frac{dx}{dt}\frac{d\phi}{dx} + \frac{dy}{dt}\frac{d\phi}{dy} + \frac{dz}{dt}\frac{d\phi}{dz}$$

$$= \underset{\sim}{v} \cdot \nabla\phi$$

Spacetime generalization

$$\frac{d\phi}{d\tau} = \frac{dt}{d\tau}\frac{d\phi}{dt} + \frac{dx}{d\tau}\frac{d\phi}{dx} + \frac{dy}{d\tau}\frac{d\phi}{dy} + \frac{dz}{d\tau}\frac{d\phi}{dz}$$

$$= u^t \frac{d\phi}{dt} + u^x \frac{d\phi}{dx} + \cdots$$

$$\boxed{\frac{d\phi}{d\tau} = u^\alpha \frac{d\phi}{dx^\alpha} \equiv u^\alpha \partial_\alpha \phi}$$

u^α: component of four vector

$\partial_\alpha \phi$: component of 1-form ↙ dx^α in would, perhaps...

Gradient of a scalar field is a 1-form

$$\tilde{d}\phi \doteq \{\partial_\alpha \phi\} \qquad [\tilde{\nabla}\phi = \{\partial_\alpha \phi\}]$$

will change its meaning later

$\frac{dt}{d\tau}, \frac{dx}{d\tau} \ldots$

Notation $\frac{d\phi}{d\tau} = u^\alpha \partial_\alpha \phi \equiv \nabla_{\vec{u}} \phi$ ← directional derivative

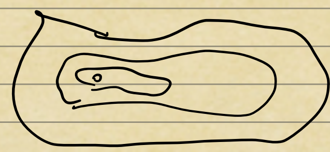
Notion of gradient as a 1-form gives us a nice way to think about basis 1-forms.

dual basis → $\tilde{w}^\alpha(\vec{e}_\beta) = \delta^\alpha{}_\beta$

We also know $\partial_\beta x^\alpha = \delta^\alpha{}_\beta$

$$\tilde{d} x^\alpha \equiv \tilde{w}^\alpha$$

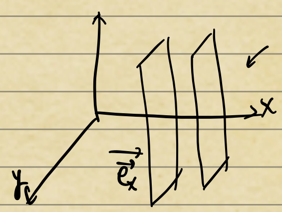
Draw level of a function in 2-D $h(x,y)$



$\Delta\vec{x}$ = displacement vector

$\tilde{d}h$ = 1-form of height function

$\tilde{d}h(\Delta\vec{x}) = \Delta x^\alpha \partial_\alpha h \equiv \Delta h$



$\tilde{w}^x$ thick sheets filling all of space

Metric with both "slots" filled with vectors Lorentz invariant number $\vec{A}\cdot\vec{B} = \bar{\bar{\eta}}(\vec{A},\vec{B})$

Only one slot?

$\bar{\bar{\eta}}(A, \cdot)$ object that takes a vector, yield L.I. number

⇒ 1-form

Define: $\tilde{A}(\cdot) = \bar{\bar{\eta}}(\vec{A}, \cdot)$

$$A_\alpha = \bar{\bar{\eta}}(\vec{A}, \vec{e}_\alpha) = \bar{\bar{\eta}}(A^\beta \vec{e}_\beta, \vec{e}_\alpha)$$

$$= \eta_{\alpha\beta} A^\beta$$

Metric converts vectors into 1-forms by "lowering" the indices

Define $\eta^{\alpha\beta}$ by $\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$ "inverse metric"

Then. $\eta^{\alpha\beta}p_\beta = p^\alpha$ $\quad p_\beta = \eta_{\beta\gamma}p^\gamma$ $\quad (\eta^{\alpha\beta}\eta_{\beta\gamma}p^\gamma = \delta^\alpha{}_\gamma p^\gamma = p^\alpha)$

$(= \langle \cdot, e_\gamma p^\gamma \rangle)$

$\vec{A}\cdot\vec{B} = \bar{\bar{\eta}}(\vec{A},\vec{B})$

$= \tilde{A}(\vec{B}) = \tilde{B}(\vec{A}) = \vec{A}(\tilde{B}) = \vec{B}(\tilde{A}) = \underline{\eta_{\alpha\beta}}A^\alpha B^\beta = \eta^{\alpha\beta}A_\alpha B_\beta$

Distinction is lost once we know how to do this! $(= \eta_{\alpha\beta}\eta^{\alpha\gamma}A_\gamma\eta^{\beta\tau}B_\tau)$
geometric

New definition

A tensor of type $\binom{M}{N}$ is a linear mapping of M 1-forms and N vectors

to the Lorentz scalars

$\binom{0}{1} \to$ 1-forms $\qquad \binom{1}{0} \to$ vectors

Metric lets us convert nature of slots on a tensor

$\binom{M}{N} \xrightarrow{\text{lower}} \binom{M-1}{N+1}$

$\eta_{\alpha\mu}R^{\mu}{}_{\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$

Inverse metric $\rightsquigarrow$ can raise

$\binom{M}{N} \longrightarrow \binom{M+1}{N-1}$

$\eta^{\alpha\mu}S_{\mu\beta\gamma} = S^{\alpha}{}_{\beta\gamma}$

*Do we need $\bar{\bar{\eta}} = \eta_{\alpha\beta}\,\tilde{\tilde{w}}^{\alpha\beta}$ —or—

$= \eta^{\alpha\beta}\,\bar{\bar{e}}_{\alpha\beta}$

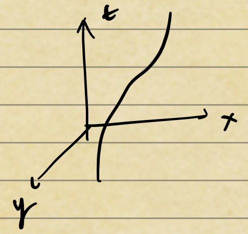
No: Basis 1-forms and vectors are sufficient

$\tilde{\tilde{w}}^{\alpha\beta} = \tilde{w}^\alpha \otimes \tilde{w}^\beta \qquad \bar{\bar{e}}_{\alpha\beta} = \vec{e}_\alpha \otimes \vec{e}_\beta$

Example $\bar{\bar{T}} = T^{\alpha\beta}\,\vec{e}_\alpha \otimes \vec{e}_\beta$

$\bar{\bar{R}} = R^{\alpha}{}_{\beta\gamma\delta}\,\vec{e}_\alpha \otimes \tilde{w}^\beta \otimes \tilde{w}^\gamma \otimes \tilde{w}^\delta$

Role of basis objects important when we calculate derivatives



$$\vec{u} = \frac{d\vec{x}}{d\tau}$$

$$\overset{=}{T} = T^{\alpha}{}_{\beta}\, \vec{e}_\alpha \otimes \tilde{w}^\beta$$

Derivative in principle (and often in practice!) depend on how $\vec{e}_\alpha$, $\tilde{w}$ vary in space and time

Lec 4

So far, quantities with physics content are good for particles

$\vec{u} \doteq (\gamma, \gamma \underset{\sim}{v})$

$\vec{u} \cdot \vec{u} = -1$ ← timelike, normalized
(-1) * (1)

4-momentum $\vec{p} = m\vec{u}$ $\quad \vec{p} \cdot \vec{m} = -m^2$

$\doteq (E, \underset{\sim}{p})$

$\Rightarrow E^2 - |\underset{\sim}{p}|^2 = m^2$

good for m=0 (photon)

$\vec{p} = \hbar\omega\,(1, \hat{n})$ $\quad \hat{n}$: unit vector in direction of propagation

More interesting matter. Simplest continuum Dust

particles with mass + energy, no interaction

Each element of dust has its own rest frame. In a given cloud different elements may have different rest frames

Characterization : 1st thing is counting how many bits of dust are in element per volume : number density

n_0 ≡ number density in rest frame of that element

$\left(= \frac{\#\text{ of particles}}{\Delta V_{rest}} \right)$

Move out of rest frame 2 things happen

1. Number in volume stays same while volume Lorentz contracts

n = # density in new frame

$= \gamma n_0$ $\quad$ contracts only one direction (not $\sim v^3$)

$\frac{1}{\sqrt{1-v^2}}$

2. Dust is flowing through space

$\underset{\sim}{n}$ = # of particles crossing unit area in unit time

$= n \underset{\sim}{v} = \gamma n_0 \underset{\sim}{v}$

$\vec{N} = (n, n\underline{v}) = (n_0\gamma,\ n_0\gamma\underline{v})$

number current four vector ⇝ $= n_0 \vec{U}$

$\vec{N}\cdot\vec{N} = -n_0^2$

— or —

$n_0 = \sqrt{-\vec{N}\cdot\vec{N}}$

Flux in a more general case?

Systematic way to pick out flux across a surface

Recall $\tilde{d}x^\alpha$ = basis 1-form

(p41)

= level surfaces at unit ticks of x^α

Flux of $\vec{N}$ in the x^α direction

$(\tilde{d}x^\alpha)_\beta N^\beta$ = this flux

$(\tilde{d}x^\alpha)_\beta = \delta^\alpha_\beta$

Note: $(\tilde{d}t)_\beta N^\beta = N^0 = n$ flux in time

More generally define surface as solution of some scalar function

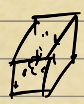
$\psi(t, x, y, z) = 0$

(ex) $\sqrt{x^2+y^2+z^2} = 5$

$\tilde{d}\psi$ = 1-form (& normalize)

→ $(\tilde{d}\psi)_\alpha N^\alpha$: Flux through this surface

Conservation laws



The flux out of the sides must come at the expense of the density of dust there

$$\frac{dn}{dt} = -\nabla\cdot\underset{\sim}{n}$$

$$\Rightarrow \boxed{\partial_\alpha N^\alpha = 0}$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}$$

Integral form of conservation law

Diff form equivalent to

Time & value of that observer

$$\frac{d}{dt}\int_{V^3} n \, dV = -\int_{\partial V^3} \underset{\sim}{n} \cdot d\underset{\sim}{a} \qquad \text{(divergence thm)}$$

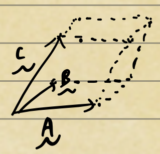
V^3 : Some 3-volume

∂V^3 : bdry of that 3-volume

How to make it frame-indpt as much as possible so we can generalize?

* Volumes & Volume integrals

Begin in 3-D. Consider a parallelepiped with sides $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}$.



3-volume

$$\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = \underset{\sim}{B} \cdot (\underset{\sim}{C} \times \underset{\sim}{A}) = \underset{\sim}{C} \cdot (\underset{\sim}{A} \times \underset{\sim}{B})$$

Equivalent way to write this:

$$3\text{-vol} = \varepsilon_{ijk} A^i B^j C^k \qquad \text{(determinant)}$$

Levi Civita Symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & i=1 \;\; j=2 \;\; k=3 \\ & \text{even permutations} \\ -1 & \text{odd permutations} \\ 0 & \text{repeated index} \end{cases}$$

Levi Civita components of a $\binom{0}{3}$ tensor takes in vectors, produces volume of figure bdd by those vectors

$$V^3 = \overset{=}{\varepsilon}(\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C})$$

Only put in 2-vectors $\overset{=}{\varepsilon}(-, \underset{\sim}{B}, \underset{\sim}{C}) = \varepsilon_{ijk} B^j C^k$

1-form whose magnitude is the area spanned by $\underset{\sim}{B}$ and C

" $= \Sigma_i$ "

Can use this to write out Gauss's theorem in geometric language

$$\int_{V^3} (\nabla \cdot A)\, dV = \int_{dV^3} \underset{\sim}{A} \cdot d\underset{\sim}{\Sigma}$$

Define a differential triple: $d\vec{x}_1,\ d\vec{x}_2,\ d\vec{x}_3$

$$dV = \varepsilon_{ijk}\, dx_1^i\, dx_2^j\, dx_3^k$$ literature, 1-form

Generalize to space-time

Imagine a parallelepiped w. sides $\vec{A}, \vec{B}, \vec{C}, \vec{D}$

$$\text{4-vol} \equiv \varepsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta$$

defined same way

The area of each "face" of this figure is a 3-volume

$$\Sigma_\alpha = \varepsilon_{\alpha\beta\gamma\delta} B^\beta C^\gamma D^\delta$$

Generalization of Gauss Theorem

$$\underbrace{\int_{V^4} (\partial_\alpha V^\alpha)\, d^4x = \int_{dV^4} V^\alpha d\Sigma_\alpha}$$

V defined by 4

4(x), four vector

Recall...

$$\frac{d}{dt} \int_{V^3} n\, dV = - \int_{dV^3} \underset{\sim}{n}\, d\underset{\sim}{a}$$ (divergence thm)

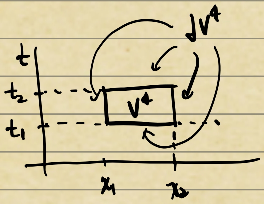
depends on frame

number density is conserved

$$\int_{V^4} (\partial_\alpha N^\alpha)\, d^4x = 0$$

$$\rightarrow \int_{dV^4} N^\alpha d\Sigma_\alpha = 0$$

Examine this integral:



$$\int_{dV^4} N^\alpha d\Sigma_\alpha = \underset{\text{(future)}}{\int_{t=t_2} N^0\, dx\, dy\, dz} - \underset{\text{(past)}}{\int_{t=t_1} N^0\, dx\, dy\, dz} + \int_{x=x_2} N^1\, dt\, dy\, dz - \int_{x=x_1} N^1\, dt\, dy\, dz + \dots$$

$$= 0$$

Let $t_2 \to t_1 + dt$ rearrange

$$\int_{t_1+dt} N^0\, dx\, dy\, dz - \int_{t_1} N^0\, dx\, dy\, dz$$

$= -dt \left[\int_{x_2} N^1 \, dy\, dz - \int_{x_1} N^1 \, dy\, dz + \dots \right]$

Divide by dt, take limit

$$\lim_{\Delta t \to 0} \frac{\int_{t_1+\Delta t} N^0 \, dx\, dy\, dz - \int_{t_1} N^0 \, dx\, dy\, dz}{\Delta t}$$

$$= \frac{d}{dt} \int_{t_1} N^0 \, dx\, dy\, dz$$

$$= -\left[\int_{x_2} N^1 \, dy\, dz - \int_{x_1} N^1 \, dy\, dz + \dots \right]$$

(get flux of $\underset{\sim}{N}$ through the six sides)

$\rightarrow \int_{V^4} (\partial_\alpha N^\alpha)\, d^4 = 0$ becomes

$$\frac{d}{dt} \int_{V^3} N^0 \, dV = -\int_{\partial V^3} \underset{\sim}{N} \cdot d\underset{\sim}{a}$$

once we've chosen (t, x, y, z) & used Gauss

* Dust is important in cosmology!

Another important example of matter: Electric current

$\vec{J} = (\rho, \underset{\sim}{J})$

charge density $\cdot c$ ($=1$), current density

Conservation of charge $\frac{d\rho}{dt} = -\underset{\sim}{\nabla} \underset{\sim}{J}$

$$\rightarrow \partial_\alpha J^\alpha = 0$$

Symmetric 4×4 tensor 4 on diagonal, 6 off $\rightarrow$ too many

Anti Symmetric $F^{\alpha\beta} = -F^{\beta\alpha}$ diagonal becomes 0 & only 6 off diagonal survives

Representation $F^{\mu\nu} \doteq \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$

according to one particular Lorentz observer

Maxwell's equations

$$\frac{d}{dt} = c\, \partial_0$$

$$\begin{cases} \partial_\nu F^{\mu\nu} = 4\pi J^\mu \\ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \end{cases}$$

$\mu = 0$ Gauss law

$\mu = 1 \sim 3$ Ampere's law

("Bianchi identity")

Space : $\nabla \cdot B = 0$ (Gauss for B)

time : Faraday

Divergence of current

$$4\pi\, \partial_\mu J^\mu = \partial_\mu \partial_\nu F^{\mu\nu}$$

$$= \partial_\nu \partial_\mu F^{\nu\mu}$$

$$= -\partial_\nu \partial_\mu F^{\mu\nu}$$

$$= -\partial_\mu \partial_\nu F^{\mu\nu}$$

$$= 0$$

Note... $A^{\alpha\beta} S_{\alpha\beta} = 0$

antisymmetric   symmetric

Lec 5

Consider energy & momentum of dust

Suppose each dust particle has the same rest mass m.

In the rest frame of the dust element, its rest energy density is

$$\rho_0 = m n_0 \qquad (c=1)$$

energy

Go into frame moving with $\underset{\sim}{v}$ relative to this frame

$$\text{Energy density} = \rho = (\gamma m)(\gamma n_0)$$

energy

$$= \gamma^2 \rho_0$$

This is not the transformation law of a 4-vector component nor scalar.

We assembled ρ by combining energy (timelike component of the four momentum) w. number density (timelike component of the number vector)

$$\rho = p^t N^t = T^{tt}$$

This is one component of tensor made by combining 2 4-vectors

$$\overset{=}{T} = \vec{N} \otimes \vec{p}$$

all same mass

$= n_0 \vec{u} \otimes m\vec{u}$

$= \underset{\sim}{\rho_0}\, \vec{u} \otimes \vec{u}$

rest energy density

$T^{\alpha\beta} = \rho_0 u^\alpha u^\beta$

$T^{\alpha\beta} = \overset{=}{T}(\underbrace{\tilde{d}x^\alpha, \tilde{d}x^\beta}_{\text{basis 1 form}})$

$=$ flux of 4-momentum p^α in the β direction

$T^{00} = \rho_0 u^t u^t \equiv \rho =$ flux of p^t in $\underline{t \text{ direction}}$ — # per unit volume

$=$ energy density — flux across a surface of constant time

$T^{0i} =$ flux of p^t in x^i direction

$=$ energy flux

$T^{i0} =$ flux of p^i in t direction

$=$ momentum density

$T^{ij} =$ flux of p^i in x^j direction

$=$ momentum flux

ith cpnt of $\underline{v}$: $u = (\gamma, \gamma\underline{v})$

$$\begin{cases} T^{0i} = \gamma^2 \rho_0 v^i \\ \quad \shortparallel \\ T^{0i} = \gamma^2 \rho_0 v^i \\ T^{ij} = \gamma^2 \rho_0 v^i v^j \end{cases}$$

energy flux = momentum density for most physical matter fields (where T symmetric)

$\underline{\underline{T}}$ symmetric

if this is not true, physical absurdity can be set up.

For now, deduce stress-energy tensor by applying definition of components filling in physical meaning

Example: perfect fluid — No energy flow in "rest" frame

No lateral stresses — T^{ij} $i \neq j$

→ No viscosity

Characterized by ρ, P

ρ: energy density

P: ↳ isotropic spatial stress

perfect fluid in rest frame.

$$\rightsquigarrow \quad T^{\alpha\beta} \doteq \begin{pmatrix} \overset{\text{energy density}}{\rho} & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

energy flux →

momentum density ↓

isotropic pressure

$= \text{diag}(\rho, p, p, p)$

covariant formula

$$\overset{=}{T} = \rho\, u \otimes u + \underbrace{(\overset{\text{metric}}{\eta} + \vec{u} \times \vec{u})}_{\text{projection tensor}} p$$

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p(\eta^{\alpha\beta} + u^\alpha u^\beta)$$

$$= (\rho + p) u^\alpha u^\beta + p\eta^{\alpha\beta}$$

(ex) $u = (1\ 0\ 0\ 0)$ in rest frame

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{=}{T} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

In general relativity, just modify what metric means

(ex) Dust $p = 0$.

Newtonian field equation

$$\nabla^2 \Phi_g = 4\pi G \rho$$

Φ_g: gravitational potential; G: gravitational constant; ρ: mass density

$\rightsquigarrow$ Newtonian way

$$\begin{cases} F = G\dfrac{Mm}{r^2}\hat{r} = m g(r) \\ g(r) = G\dfrac{M}{r^2}\hat{r} = -\nabla\phi_g(r) \\ \phi_g(r) = -\dfrac{GM}{r} \\ \oint_S g \cdot dA = -4\pi G M_{enc} \\ \nabla \cdot g = -4\pi G \rho(r) \\ \nabla \cdot g = \nabla(-\nabla\phi_g) \end{cases}$$

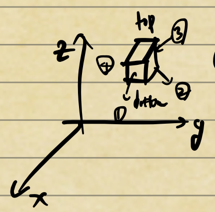
(ρ is component of a tensor, not a scalar! unnatural!)

"$=$" $T^{\alpha\beta}$

Goal: Two derivative turns out to be metric of spacetime.

* Physics of stress energy

Motivate the symmetry of tensor



Consider cube immersed in $T^{\alpha\beta}$

look at momentum flux into and out of box

Force on face 1

$$\underset{\sim}{F_1} = \{ T^{ix} \ell^2 \}$$

Force on face 2

$\underset{\sim}{F_2} = \{ T^{iy} \ell^2 \}$

$\underset{\sim}{F_3} \approx -\underset{\sim}{F_1}$

$\underset{\sim}{F_4} \approx -\underset{\sim}{F_2}$

Consider torques about an axis that goes through the center of this cube:

Sum up $\underset{\sim}{r} \times F$ over each face

$$\tau^z = \ell^3 (T^{xy} - T^{yx})$$

Moment of inertia: $I = \alpha (\rho \ell^3) \ell^2 = \alpha \rho \underline{\underline{\ell^5}}$

Angular acceleration $\ddot{\theta} = \tau^z / I$

$$\propto (T^{xy} - T^{yx}) / \ell^2$$

physics indicates $T^{xy} = T^{yx}$ to prevent absurdity!

Repeat for other axes $T^{\bar{i}\bar{j}} = T^{\bar{j}\bar{i}}$

*

Conservation of energy and momentum

$$\boxed{\partial_\alpha T^{\alpha\beta} = 0}$$

pick a frame

$\partial_\alpha T^{\alpha 0} = 0$ —or—

$\dfrac{\partial T^{00}}{\partial t} = -\dfrac{\partial T^{0i}}{\partial x^i}$ energy

$\dfrac{\partial T^{i0}}{\partial t} = -\dfrac{\partial T^{ij}}{\partial x^j}$ momentum

Can recast these as integral equations:

$$\frac{d}{dt} \int_{V^3} T^{00} \, d^3x = -\int_{\partial V^3} T^{0i} \, d\Sigma_i$$

Another example point particle of rest mass m_0

moving on world line $\vec{z}(\tau)$

$$T^{\mu\nu} = m_0 \int u^\mu u^\nu \underbrace{\delta^4(\vec{x} - \overset{\text{particle's parametric 4-position}}{\overbrace{\vec{z}(\tau)}})}_{\text{4D Dirac Delta}} \, d\tau$$

$$\delta^4 [\vec{x} - \vec{z}(\underset{\text{proper time}}{\underline{\underline{\tau}}})] = \delta [t - z^0(\tau)] \, \delta [x - z^1(\tau)] \cdots$$

Plus rule $\int f(x)\,\delta[g(x)]\,dx$

$$= f(x_0) / |g'|_{x=x_0}$$

$\left(\delta(ax) = \frac{\delta(x)}{|a|} \right)$

$$T^{\mu\nu} = \frac{m\, u^\mu u^\nu}{u^0} \delta^{(3)}[\underset{\sim}{x} - \underset{\sim}{z}(\tau)]$$ (Integrate out time)

Another example $u^0(\tau) = \frac{dz^0(\tau)}{d\tau}$

$$T^{\mu\nu}_{EM} = \frac{1}{4\pi}\left[F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}\right]$$

Look at component $T^{00} = \frac{1}{8\pi}(\underset{\sim}{E}\,\underset{\sim}{E} + \underset{\sim}{B}\,\underset{\sim}{B})$

$$T^{0i} = (\underset{\sim}{E} \times \underset{\sim}{B})^i / 4\pi$$

$$T^{ij} = \frac{1}{8\pi} \left[(\underset{\sim}{E}\cdot\underset{\sim}{E} + \underset{\sim}{B}\cdot\underset{\sim}{B})\,\delta^{ij} - 2\,(E^i \cdot E^j + B^i B^j)\right]$$

Example $\underset{\sim}{E} = E^x \underset{\sim}{e}_x$ $B = 0$

$$T^{\mu\nu}_{EM} = \frac{(E^x)^2}{8\pi} \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

prelude to curvature : flat spacetime in curvilinear coordinate

$$(t, x, y, z) \longrightarrow (t, r, \phi, z)$$

$x = r\cos\phi$

$y = r\sin\phi$

Continue to use a coordinate basis

$d\vec{x} = dx^\alpha\, \vec{e}_\alpha$ ← basis vectors

dim length

$$= dt\, e_t + dr\, \vec{e}_r + d\phi\, \vec{e}_\phi + dz\, \vec{e}_z$$

angle length!

not a "normal" basis $\vec{e}_\phi \cdot \vec{e}_\phi \neq 1$

Transformation between representation $L^\alpha{}_{\bar\mu} = \frac{dx^\alpha}{dx^{\bar\mu}}$ Reserve Λ for Lorentz transformation

Barred = polar , unbarred = cartesian

$$\frac{dx^\alpha}{dx^{\bar\mu}} = \begin{array}{c|cccc} & t & x & y & z \\ \hline t & 1 & 0 & 0 & 0 \\ r & 0 & \cos\phi & \sin\phi & 0 \\ \phi & 0 & -r\sin\phi & r\cos\phi & 0 \\ z & 0 & 0 & 0 & 1 \end{array} \to L^\alpha{}_{\bar\mu}$$

$$\frac{dx^{\bar{\mu}}}{dx^{\alpha}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi/r & 0 \\ 0 & \sin\phi & \cos\phi/r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = L^{\bar{\mu}}{}_{\alpha}$$

Lec 6

Basis vectors

$$\vec{e}_r = \cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y = L^{\alpha}{}_{r}\,\vec{e}_\alpha$$

$$\vec{e}_\phi = -r\sin\phi\,\vec{e}_x + r\cos\phi\,\vec{e}_y = L^{\alpha}{}_{\phi}\,\vec{e}_\alpha$$ not a unit vector

Before (Cartesian) $\eta_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$

Now: $\vec{e}_\alpha \cdot \vec{e}_\beta = g_{\alpha\beta} \underset{\text{p.p.c}}{\doteq} \mathrm{diag}(-1, 1, r^2, 1)$

(plane polar coordinates)

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2$$

Basis 1-forms $\tilde{d}r = L^{r}{}_{\alpha}\,\tilde{d}x^\alpha$

$$= \cos\phi\,\tilde{d}x + \sin\phi\,\tilde{d}y$$

$$\tilde{d}\phi = -\frac{\sin\phi}{r}\tilde{d}x + \frac{\cos\phi}{r}\tilde{d}y$$

Where this matters: Derivatives

Need to note that bases vary with coordinates

$$\frac{d\vec{e}_r}{dr} = 0 \qquad \frac{d\vec{e}_r}{d\phi} = \frac{\vec{e}_\phi}{r} \qquad \frac{d\vec{e}_\phi}{dr} = \frac{\vec{e}_\phi}{r} \qquad \frac{d\vec{e}_\phi}{d\phi} = -r\vec{e}_r$$

$\vec{V} = V^\alpha \vec{e}_\alpha$ PPC basis vectors

$$\vec{\nabla}\vec{V} = \underbrace{\partial_\beta (V^\alpha \vec{e}_\alpha)}\,\tilde{\omega}^\beta$$

Components of 1-form

Tensorial object!

$$\frac{d\vec{V}}{dx^\beta} = \left(\frac{dV^\alpha}{dx^\beta}\right)\vec{e} + V^\alpha \frac{d\vec{e}_\alpha}{dx^\beta}$$

Sum is tensorial

Individually, not.

$\partial_\beta \vec{e}_\alpha$ can be written as a linear combination of basis vectors

$$\partial_\beta \vec{e}_\alpha = \underset{\sim}{\Gamma^\mu_{\beta\alpha}}\, \vec{e}_\mu$$

"Christoffel Symbol : not a component of a tensor

For PP coordinates, $\Gamma^\phi_{r\phi} = \frac{1}{r} = \Gamma^\phi_{\phi r}$

$\Gamma^r_{\phi\phi} = -r$ else 0

Derivative of vector

$$\partial_\beta \vec{V} = (\partial_\beta V^\alpha \vec{e}_\alpha + V^\alpha \Gamma^\mu_{\beta\alpha} \vec{e}_\mu)$$

relabel α and μ

$$= \underbrace{(\partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\beta\mu})}_{\text{covariant derivative}} \vec{e}_\alpha$$

$$\equiv (\nabla_\beta V^\alpha) \vec{e}_\alpha$$

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha_{\beta\mu} V^\mu$$

$$\vec{\nabla} \vec{V} = (\nabla_\beta V^\alpha)\, \tilde{w}^\beta \otimes \vec{e}_\alpha$$

Application : Divergence

$$\nabla_\alpha V^\alpha = \partial_\alpha V^\alpha + \Gamma^\alpha_{\alpha\mu} V^\mu$$

$$= \partial_t V^t + \partial_r V^r + \partial_z V^z + \underbrace{\partial_\phi V^\phi}_{\text{weird unit}} + \underbrace{\frac{V^r}{r}}_{\text{divide by length}}$$

Recall e_ϕ dimension of length

$\Rightarrow V_\phi$ dimension V/length

So no contradiction of unit for $\partial_\phi V^\phi$

Scalars : No basis objects!

$$\nabla_\alpha \Phi = \partial_\alpha \Phi$$

1-form: Use the fact that 1-form contracted w/ vector is a scalar

$$\nabla_\beta (P_\alpha A^\alpha) = \partial_\beta (P_\alpha A^\alpha)$$

Leib law $= (\partial_\beta P_\alpha) A^\alpha + P_\alpha (\partial_\beta A^\alpha)$

upper : vector

lower : form

Use $\partial_\beta A^\alpha = \nabla_\beta A^\alpha - \Gamma^\alpha_{\beta\mu} A^\mu$

$$\nabla_\beta (P_\alpha A^\alpha) = A^\alpha \partial_\beta P_\alpha + P_\alpha (\nabla_\beta A^\alpha) - P_\alpha (\Gamma^\alpha_{\beta\mu} A^\mu)$$

$$\nabla_\beta (P_\alpha A^\alpha) = P_\alpha (\nabla_\beta A^\alpha) + A^\alpha \left(\partial_\beta P_\alpha - P_\mu \Gamma^\mu_{\beta\alpha} \right)$$

Require $\nabla_\beta (P_\alpha A^\alpha) = P_\alpha \nabla_\beta A^\alpha + A^\alpha \nabla_\beta P_\alpha$

form

$$\nabla_\beta P_\alpha = \partial_\beta P_\alpha - P_\mu \Gamma^\mu_{\beta\alpha}$$

Simple to find an an analysis of this form

$$\partial_\beta \tilde{\omega}^\alpha = -\Gamma^\alpha_{\beta\mu} \tilde{\omega}^\mu$$

Minus sign enforces $\underset{\text{pairing}}{\langle \tilde{\omega}^\alpha, \vec{e}_\beta \rangle} = \delta^\alpha_\beta$

Further generalization

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu_{\beta\alpha} T^{\alpha\nu} + \Gamma^\nu_{\beta\alpha} T^{\mu\alpha}$$

$$\nabla_\beta T_{\mu\nu} = \partial_\beta T_{\mu\nu} - \Gamma^\alpha_{\beta\mu} T_{\alpha\nu} - \Gamma^\alpha_{\beta\nu} T_{\mu\alpha}$$

$$\nabla_\beta T^{\mu\nu}{}_{\rho\tau} = \partial_\beta T^{\mu\nu\ldots}{}_{\rho\tau\ldots} + \Gamma^{\mu\ldots}_{\beta\alpha\ldots} T^{\alpha\nu\ldots}{}_{\rho\tau\ldots} + \Gamma^{\nu\ldots}_{\beta\alpha\ldots} T^{\mu\alpha\ldots}{}_{\rho\tau\ldots} - \Gamma^{\alpha\ldots}_{\beta\rho\ldots} T^{\mu\ldots}{}_{\alpha\tau\ldots} - \Gamma^{\alpha\ldots}_{\beta\tau\ldots} T^{\mu\nu\ldots}{}_{\rho\alpha\ldots}$$

* Better way to get Christoffels via metric

Derivation refers a a key property of tensor relationships

A tensorial equation that holds in one representation must hold in <u>all</u> representations

Changing representation cannot change equation

Exercise. Double gradient of a scalar

$$\vec{\nabla}\vec{\nabla}\phi = \partial_\alpha \partial_\beta \phi \; \tilde{w}^\alpha \otimes \tilde{w}^\beta$$

components of tensor — cartesian rep.

symmetric on exchange of α, β

General representation

$$\vec{\nabla}\vec{\nabla}\phi = \nabla_\alpha \nabla_\beta \phi \; \tilde{w}^\alpha \otimes \tilde{w}^\beta$$

Must also be symmetric in general!

$$\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi$$

$$\nabla_\alpha(\overbrace{\partial_\beta \underbrace{\phi}_{\text{scalar}}}^{\text{1-form}}) = \nabla_\beta(\partial_\alpha \phi)$$

$$\partial_\alpha \partial_\beta \phi - \Gamma^\mu_{\alpha\beta}\partial_\mu \phi = \partial_\beta \partial_\alpha \phi - \Gamma^\mu_{\beta\alpha}\partial_\mu \phi$$

$$\boxed{(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha})\partial_\mu \phi = 0}$$

Notation $\quad A_{(\alpha\beta)} = \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})$ symmetrization

$A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})$ antisymmetrization

$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{(\alpha\beta)} \qquad \Gamma^\mu_{\alpha\beta} A^{\alpha\beta} = 0$

antisymmetric

Gradient of the metric

$$\vec{\nabla}\bar{\bar{g}} = \nabla_\gamma g_{\alpha\beta} \; \tilde{w}^\alpha \otimes \tilde{w}^\beta \otimes \tilde{w}^\gamma$$

metric tensor

$$\underset{\text{cartesian}}{=} \nabla_\gamma \eta_{\alpha\beta} \; \tilde{w}^\alpha \otimes \tilde{w}^\beta \otimes \tilde{w}^\gamma$$

$\longrightarrow$ Require that $\nabla_\gamma g_{\alpha\beta} = 0$

Ⓘ $\nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} - \Gamma^\mu_{\gamma\alpha} g_{\mu\beta} - \Gamma^\mu_{\gamma\beta} g_{\alpha\mu} = 0$

Ⓘⓘ $\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\mu_{\alpha\beta} g_{\mu\gamma} - \Gamma^\mu_{\alpha\gamma} g_{\beta\mu} = 0$

Ⓘⓘⓘ $\nabla_\beta g_{\gamma\alpha} = \partial_\beta g_{\gamma\alpha} - \Gamma^\mu_{\beta\gamma} g_{\mu\alpha} - \Gamma^\mu_{\beta\alpha} g_{\gamma\mu} = 0$

Construct 1 - 2 - 3

$$\partial_\gamma g_{\alpha\beta} - \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\gamma\alpha} - g_{\mu\beta}\left(\cancel{\Gamma^{\mu}_{\gamma\alpha} - \Gamma^{\mu}_{\alpha\gamma}}\right) + g_{\mu\gamma}\left(\Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\beta\alpha}\right) + g_{\mu\alpha}\left(\cancel{\Gamma^{\mu}_{\beta\gamma} - \Gamma^{\mu}_{\gamma\beta}}\right) = 0$$

$$g_{\mu\gamma}\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right)$$

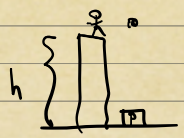
$$\equiv \Gamma_{\gamma\alpha\beta}$$

$$\boxed{\Gamma^{\mu}_{\alpha\beta} = g^{\mu\gamma}\Gamma_{\gamma\alpha\beta}}$$

Special relativity ; The theory which allows us to cover the entire spacetime manifold using inertial reference frames "Global inertial frame"

Gravity breaks this. No longer have global inertial frames. Local inertial frames are ok

part 1. Gravitational redshift exists



1. Drop rock of rest mass m off top of tower

2. At bottom photonulator converts rock into a single photon, conserving energy

$E_{BOT} = m + mgh$

$E_{BOT} = \hbar \omega_B = m(1+gh)$

3. At top, re-rockulator converts photon back into rock

What is energy at top? $E_T = \hbar \omega_T$

prevent perpetual machines $E_T = m$

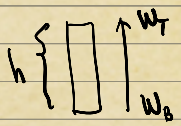
$$\frac{E_T}{E_B} = \frac{m}{m(1+gh)} = \frac{\omega_T}{\omega_B} \longrightarrow \omega_T = \omega_B \quad \times$$

lost energy

Lecture 7

When we have gravity, we cannot cover all of spacetime with an inertial frame

1. Gravitational redshift exists. Highly tested experimental fact

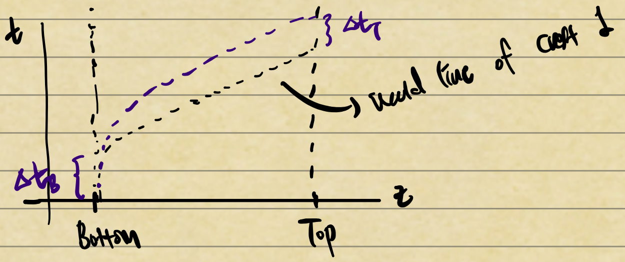


$$\omega_T = \omega_B\left(1 - \frac{gh}{c^2}\right)$$

quantum ⟷ particle
quantum field theory

2. Suppose we could cover a large region with a single Lorentz frame

Consider world line of successive crests of wave:



Crest 2: If we can use a global Lorentz frame, spacetime is translation invariant (w. respect to offsets time and space)

Crest 2 must be congruent with crest 1

$\rightarrow$ Must have $\Delta t_B = \Delta t_T$ if globally Lorentz

But $\Delta t = \frac{2\pi}{\omega} \rightarrow \Delta t_B \neq \Delta t_T$

Cannot have global Lorentz frame with gravity (argument due to Alfred Schild)

No global Lorentz frames....

Can have local $\underbrace{\text{Lorentz}}_{\text{Inertial}}$ frame

Inertial frame means no accelerations on observers at rest in that frame...

No forces are acting

Next best thing: A freely falling frame. Fact that $F = ma$ & $F_g \propto m$ means all objects in that freely falling frame (FFF) experience zero relative acceleration

relative acceleration (in absence of other forces)

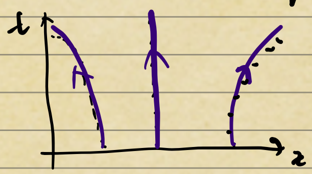
Within this frame, objects maintain their relative velocities

Tides break down the notion of uniform freely falling frames

middle person sees: top goes up, bottom goes down

tall elevator will see separation of freefall since gravity is not uniform

In a spacetime diagram



Not congruent trajectory

Tangent do not remain parallel

Euclid's parallelism axiom: 2 lines that start parallel remain parallel

Only true in a flat manifold

principle of equivalence

Over sufficiently small regions, the motion of freely falling particles due to gravity cannot be distinguished from uniform acceleration

$\Rightarrow$ "weak equivalence principle"

Free fall experiment WEP valid to $\sim 10^{-13}$

Can we now do physics by just applying S.R in our new notion of inertial frame?

Tides prevent this from working!

- Different FFFs exist at different locations. Have to knit them up

Reformulation of equivalent principle

In sufficiently small regions of spacetime, we can find a representation such that the laws of physics reduce to those of special relativity - Einstein Equivalence principle

(Strong equivalence principle: Gravity falls in gravitational field in a way indistinguishable from mass)

Existence of a local Lorentz frame

Want to show that we can put metric in S.R. form over some finite region

Let $\{x^\alpha\}$ be our starting coords, metric is $g_{\alpha\beta}$

Let $\{x^{\bar{\alpha}}\}$ be coordinates in which spacetime is Lorentz in vicinity of event P

Assume mapping between coordinates $x^\alpha = x^\alpha(x^{\bar{\alpha}})$

$$L^{\alpha}_{\bar{\mu}} = \partial x^{\alpha} / \partial x^{\bar{\mu}}$$

GOAL: Show we can find coordinate system such that $g_{\bar{\mu}\bar{\nu}} = L^{\alpha}_{\bar{\mu}} L^{\beta}_{\bar{\nu}} g_{\alpha\beta}$

$= \eta_{\bar{\mu}\bar{\nu}}$ over as large region as possible

Logic of calculation

Expand $L^{\alpha}_{\bar{\mu}} L^{\beta}_{\bar{\nu}} g_{\alpha\beta}$ in a Taylor expansion about P

Compare the degrees of freedom offered by coordinate transformation (which we select) to the constraints imposed by metric & its derivatives which we are given

$$g_{\alpha\beta} = g_{\alpha\beta}|_P + (x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)\, \partial_{\bar{\gamma}} g_{\alpha\beta}|_P + \frac{1}{2}(x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(x^{\bar{\varepsilon}} - x^{\bar{\varepsilon}}_P)\, \partial_{\bar{\gamma}} \partial_{\bar{\varepsilon}} g_{\alpha\beta}|_P + \cdots$$

$$L^{\alpha}_{\bar{\mu}} = L^{\alpha}_{\bar{\mu}}|_P + (x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)\, \partial_{\bar{\gamma}} L^{\alpha}_{\bar{\mu}}|_P + \frac{1}{2}(x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(x^{\bar{\varepsilon}} - x^{\bar{\varepsilon}}_P)\, \partial_{\bar{\gamma}} \partial_{\bar{\varepsilon}} L^{\alpha}_{\bar{\mu}}|_P$$

$g_{\alpha\beta}|_P \quad \partial g_{\alpha\beta}|_P \quad \partial^2 g_{\alpha\beta}|_P$ $\qquad (g = g_{\alpha\beta}\, e^{\alpha} \otimes e^{\beta})$

→ Handed to us, constraints

$L^{\alpha}_{\bar{\mu}} \quad \partial L^{\alpha}_{\bar{\mu}} \quad \partial^2 L^{\alpha}_{\bar{\mu}}$ freely specifiable, degrees of freedom

Want

$$L^{\alpha}_{\bar{\mu}} L^{\beta}_{\bar{\nu}} g_{\alpha\beta} = (L^{\alpha}_{\bar{\mu},P})(L^{\beta}_{\bar{\nu},P})(g_{\alpha\beta})_P + (x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(\text{terms involving } \partial g_{\alpha\beta}|_P, \partial L^{\alpha}_{\bar{\mu}})$$
$$+ \frac{1}{2}(x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(x^{\bar{\delta}} + x^{\bar{\delta}}_P)(\text{terms with } \partial^2 g, \partial^2 L)$$

0^{th} order: $g_{\alpha\beta}|_P$ symmetric 4×4 10 constraints to satisfy

$$L^{\alpha}_{\bar{\mu}}|_P = \left.\frac{\partial x^{\alpha}}{\partial x^{\bar{\mu}}}\right|_P$$ not symmetric 16 constraints

6 leftover: 3 rotations, 3 boosts

1st order $\partial_{\bar{\gamma}} g_{\alpha\beta}|_p = (4\times4 \text{ Symmetric}) \times 4$ components

$= 40$ constraints

$$\partial_{\bar{\gamma}} \Lambda^{\alpha}_{\bar{\mu}} = \frac{\partial^2 x^{\alpha}}{\partial x^{\bar{\gamma}} \partial x^{\bar{\mu}}} = (4\ \alpha \text{ component}) \times (4\times4 \text{ symmetric})$$

$= 40$ degrees of freedom

2nd order $\partial_{\bar{\gamma}} \partial_{\bar{\sigma}} g_{\alpha\beta}|_p = (\text{symmetric } 4\times4 \text{ on } \bar{\gamma}, \bar{\sigma}) \times (\text{symmetric } 4\times4 \text{ on } \alpha, \beta)$

$= 100$ constraints

$$\partial_{\bar{\gamma}} \partial_{\bar{\delta}} \Lambda^{\alpha}_{\bar{\mu}}\Big|_p = \frac{\partial^3 \alpha}{\partial x^{\bar{\mu}} \partial x^{\bar{\delta}} \partial x^{\bar{\gamma}}}$$

$$= 4\alpha \times \frac{n(n+1)(n+2)}{3!}\Big|_{n=4}$$

$= 80$ Degrees of Freedom

We can put spacetime representation into the form

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} + \mathcal{O}\left((\partial^2 g)(\delta x)^2\right)$$

2nd der's of metric

coordinate distance from P

Size of region over which spacetime is inertial in this representation is $\ell \sim \frac{1}{\sqrt{\partial^2 g}}$

Curvature & Curved manifold

- Curved manifold one on which initially parallel trajectories do not remain parallel

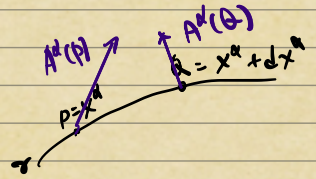
(ex) surface of a sphere

(Non-ex) Surface of a cylinder

Vectors reside in a "tangent space" Flat manifolds all points have the same tangent space

curved manifolds: Not!

Consider a curve γ in a curved manifold



How to differentiate this?

1^{st} guess : $\dfrac{\partial A^\alpha}{\partial x^\beta} = \lim_{dx^\alpha \to 0} \dfrac{A^\alpha(P) - A^\alpha(Q)}{dx^\alpha}$

P Q don't have the same tangent space, basis vectors vary as we move along

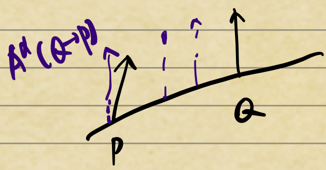
$$\partial_{\beta'} A^{\alpha'} \overset{?}{=} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} \partial_\beta A^\alpha$$

But ... $A^{\alpha'} = \dfrac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha$

$$\partial_{\beta'} = \frac{\partial x^\beta}{\partial x^{\beta'}} \partial_\beta$$

$$\partial_{\beta'} A^{\alpha'} = \frac{\partial x^\beta}{\partial x^{\beta'}} \left(\frac{\partial x^{\alpha'}}{\partial x^\alpha} \partial_\beta A^\alpha + \underbrace{\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\alpha} A^\alpha}_{\text{spoil tensorialness}} \right)$$

Need a notion of transporting objects to a location where they can be compared



Notion 1 : parallel transport

Lec 8

"Weak equivalence principle"

Gravitational charge = inertial mass

"Einstein equiv principle"

Small region → Special relativity

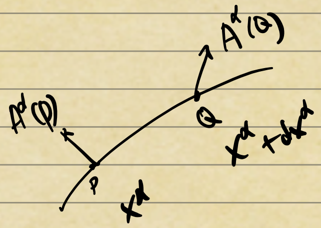
Calculation $g_{\mu\nu} \longrightarrow \eta_{\mu\nu} + O\left((dx)^2 \cdot (\partial^2 g) \right)$

curvature scale $\sim \dfrac{1}{\sqrt{\partial^2 g}}$ (radius)

$R_c \sim \dfrac{1}{\sqrt{\text{curvature}}}$ "curvature scale"

Where we stopped: partial derivatives of tensors do not yield tensors

Transformation of vector derivatives $\partial_{\beta'} A^{\alpha'} = \frac{\partial x^{\beta}}{\partial x^{\beta'}} \left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \partial_\beta A^\alpha + \underbrace{\frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\alpha}}}_{\text{spoiler}} A^\alpha \right)$



Transport notion (. parallel transport

Want to define a notion of how to transport vector from P to Q

Assume we can define an object

$\Pi^\alpha_{\beta\mu}$ which does the following

$$A^\alpha_T (P \to Q) = A^\alpha(P) - \Pi^\alpha_{\beta\mu}\, dx^\beta A^\mu$$

Define a derivative by comparing the transported vector field to the field at Q

$$D_\beta A^\alpha = \frac{A^\alpha(Q) - A^\alpha_T(P \to Q)}{dx^\beta}$$

$$= \partial_\beta A^\alpha + \Pi^\alpha_{\beta\mu} A^\mu$$

connection (connect P & Q)

for Π to be tensor, demand that

$$\Pi^{\alpha'}_{\beta'\mu'} A^{\mu'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \Pi^\alpha_{\beta\mu} A^\mu - \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\mu}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} A^\mu$$

cancels out the "bad" term in the partial deriv. transformation

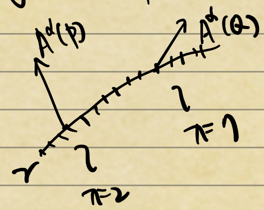
$$D_\beta g_{\mu\nu} = 0$$

$$\Rightarrow \Pi^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu}$$

This leads to connection being the christoffel. this derivative is the covariant derivative worked out earlier

$D_\beta \to \nabla_\beta$

physical interpretation of this transport rule



τ uniform tickmarks along γ

$u^\alpha = \dfrac{dx^\alpha}{d\tau}$ tangent vector to this curve

$u^\beta (\nabla_\beta A^\alpha) \overset{\text{def}}{\equiv} \dfrac{DA^\alpha}{d\tau}$ $\rightsquigarrow$ tells us how A^α changes as it is transported along the curve

parallel transport: Require $\dfrac{DA^\alpha}{d\tau} = 0$

Why? $u^\beta \nabla_\beta A^\alpha = u^\beta (\partial_\beta A^\alpha + \Gamma^\alpha_{\beta\mu} A^\mu)$

Imagine P & Q are close enough that they fit in the same LLF.

No Γs in that frame!

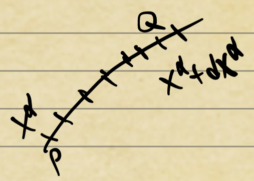
In this frame $\dfrac{DA^\alpha}{dx} = 0 \longrightarrow \dfrac{dA^\alpha}{d\tau} = 0$

Hold components constant as we slide vector along γ!

parallel transport!! As parallel as possible given curvature

analogous to moving in inertial frame $\rightsquigarrow$ can describe kinetics of bodies in spacetime

Another notion of transport



$u^\alpha = \dfrac{dx^\alpha}{d\tau}$

$$x^\alpha + dx^\alpha = x^\alpha + u^\alpha d\tau$$
$$\equiv (x')^\alpha$$

Regard the shift from P to Q as a coordinate transformation

$$A^\alpha_{LT}(P \to Q) = \frac{\partial (x')^\alpha}{\partial x^\beta} A^\beta(P)$$

$$= (\delta^\alpha_\beta + (\partial_\beta u^\alpha)\, d\tau)\, A^\beta(P)$$

$$A^\alpha_{LT}(P \to Q) = A^\alpha(P) + (\partial_\beta u^\alpha) A^\beta(P)\, d\tau$$

Can express field at Q in terms of field at P using Taylor expansion

$$A^\alpha(Q) = A^\alpha(x^\beta + dx^\beta)$$

$$= A^\alpha(x^\beta) + dx^\beta (\partial_\beta A^\alpha)|_P$$

$$= A^\alpha(x^\beta) + (u^\beta d\lambda)(\partial_\beta A^\alpha)|_P$$

So suppose I look at $\dfrac{A^\alpha(Q) - A^\alpha_{LT}(P\to Q)}{d\lambda} = \mathcal{L}_{\vec{u}} A$

Lie Derivative of $\vec{A}$ along $\vec{u}$

$\to \mathcal{L}_{\vec{u}} A^\alpha = u^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta u^\alpha$

$= u^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta u^\alpha$ $\rightsquigarrow$ tensorial

$\mathcal{L}_{\vec{u}} \vec{A} = [\vec{u}, \vec{A}]$ Common notation

Repeat for a scalar : $\mathcal{L}_{\vec{u}} \Phi = u^\alpha \partial_\alpha \Phi = u^\alpha \nabla_\alpha \Phi$

——— " ——— 1-form : $\mathcal{L}_{\vec{u}} P_\alpha = u^\beta \nabla_\beta P_\alpha + P_\beta \nabla_\alpha u^\beta$

(∂_β) (∂_α)

——— " ——— Tensor : $\mathcal{L}_{\vec{u}} T^\alpha{}_\beta = u^\mu \nabla_\mu T^\alpha{}_\beta - T^\mu{}_\beta \nabla_\mu u^\alpha + T^\alpha{}_\mu \nabla_\beta u^\mu$

key application : the properties of tensors for which $\mathcal{L}_{\vec{u}}(\text{tensor}) = 0$

If this holds, the tensor is "Lie transported" $\leftarrow$ fluid dynamics eg

Suppose a tensor is Lie transported

If that's the case, define coordinates centered on curve for which $\vec{u}$ is tangent

$x^0 = \lambda$ on the curve

x^1, x^2, x^3 constant on the curve

Then $u^\alpha = \delta^\alpha{}_0 \to \partial_\mu u^\alpha = 0$ (recall $u^\alpha = \dfrac{dx^\alpha}{d\lambda}$)

$\to$ Then $\mathcal{L}_{\vec{u}}(\text{tensor}) = \dfrac{\partial(\text{tensor})}{\partial x^0} = 0$ $\left(\mathcal{L}_{\vec{u}} T^\alpha{}_\beta = u^\mu \nabla_\mu T^\alpha{}_\beta - T^\mu{}_\beta \nabla_\mu u^\alpha + T^\alpha{}_\mu \nabla_\beta u^\mu\right)$

& can make christoffel vanish

$\to$ The tensor does not vary with this parameter along the curve

Suppose "tensor" is the metric

Let us say a vector $\underbrace{\vec{\xi}}_{\text{killing vector}}$ exists such that $\mathcal{L}_{\vec{\xi}}\, g_{\mu\nu} = 0$

There exists some coordinate such that $\partial g_{\alpha\beta} / \partial x^0 = 0$

(Convert is also true: If there is a coordinate such that $\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0$, then $\exists$ $\vec{\xi}$ exists)

2. Expand the Lie derivative

$$\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0$$

$$= \xi^\gamma \underbrace{\nabla_\gamma g_{\alpha\beta}}_{=0} + g_{\alpha\gamma} \nabla_\beta \xi^\gamma + g_{\beta\gamma} \nabla_\alpha \xi^\gamma$$

$$= \nabla_\beta (g_{\alpha\gamma} \xi^\gamma) + \nabla_\alpha (g_{\beta\gamma} \xi^\gamma)$$

$$= \nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta$$

$$" = " \quad \nabla_{(\alpha,} \xi_{\beta)} = 0 \quad \left(\text{Recall } A_{(\alpha\beta)} = \tfrac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})\right)$$

known as Killing's equation

$\vec{\xi}$ is a Killing vector

"$\nabla_{\vec{u}} \vec{u} = 0$"

(Thursday)

Consider a body that is freely falling. Trajectory for which $u^\alpha \nabla_\alpha u^\beta = 0$

trajectory that parallel transports its own tangent vector

Suppose space time has a Killing vector

$C = u^\alpha \xi_\alpha$ constant of a motion (can show from Killing equation)

pset 4

Suppose metric that is time independent

$\Downarrow$

A Killing vector exists corresponding to this property!

C for this Killing vector is energy

angular momentum

$\Rightarrow$ conservation law is put into GR

* Tensor densities   Quantities that transform almost like tensors off by a factor that is the determinant of coordinate transformation matrix (compensate for the volume change, hence density)

Most important tensor densities

Levi-Civita symbol

Determinant of metric

Levi-Civita $\tilde{\varepsilon}_{\alpha\beta\gamma\delta} = \begin{cases} +1 & 0123 \text{ \& even permutations} \\ -1 & \text{odd permutations} \\ 0 & \text{index repeated} \end{cases}$

Theorem for any 4x4 matrix $M^{\alpha}{}_{\mu}$

$$\tilde{\varepsilon}_{\alpha\beta\gamma\delta} M^{\alpha}{}_{\mu} M^{\beta}{}_{\nu} M^{\gamma}{}_{\delta} M^{\delta}{}_{\sigma}$$

$$= \tilde{\varepsilon}_{\mu\nu\delta\sigma} \, |M|$$

↳ determinant of M

Change $M \to \frac{dx^{\mu}}{dx^{\mu'}}$ coordinate transformation matrix

Then, $\tilde{\varepsilon}_{\alpha'\beta'\gamma'\delta'} = \tilde{\varepsilon}_{\alpha\beta\gamma\delta} \frac{dx^{\alpha}}{dx^{\alpha'}} \frac{dx^{\beta}}{dx^{\beta'}} \frac{dx^{\gamma}}{dx^{\gamma'}} \frac{dx^{\delta}}{dx^{\delta'}} \left| \frac{dx^{\mu'}}{dx^{\mu}} \right|^{w=1}$ ← det.

extra factor pushes away from a tensor relationship

One power of Jacobian "Tensor density of weight 1"

(ex) $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$g_{\mu\nu} = \text{diag}(1, r^2, r^2 \sin^2\theta)$

det → $g = r^4 \sin^2\theta$

$\sqrt{g} = r^2 \sin\theta$ weight 1

Lec 9

Levi-Civita $\tilde{\varepsilon}_{\alpha'\beta'\gamma'\delta'} = \left| \frac{dx^{\mu'}}{dx^{\mu}} \right| \varepsilon_{\alpha\beta\gamma\delta} \frac{dx^{\alpha}}{dx^{\alpha'}} \frac{dx^{\beta}}{dx^{\beta'}} \frac{dx^{\gamma}}{dx^{\gamma'}} \frac{dx^{\delta}}{dx^{\delta'}}$

Tensor density of weight 1

$g_{\alpha'\beta'} = \frac{dx^{\alpha}}{dx^{\alpha'}} \frac{dx^{\beta}}{dx^{\beta'}} g_{\alpha\beta}$ ← tensor

Take det on both sides

$$g' = \left|\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\right|^{2} g$$

$$= \left|\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|^{-2} g$$

Det of $g_{\mu\nu}$ is tensor density of weight -2

Convert any tensor density w. weight w into a proper tensor by multiplying w. $|g|^{\frac{w}{2}}$

Abs value so square root is ok

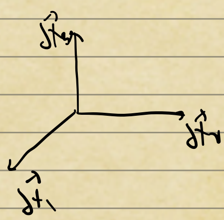
Example

Levi Civita tensor

$$\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{|g|}\;\tilde{\varepsilon}_{\alpha\beta\gamma\delta}$$

$$\varepsilon^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{|g|}}\;\tilde{\varepsilon}^{\alpha\beta\gamma\delta}$$

We use it to form covariant volume operator



$$d\tilde{V} = \sqrt{|g|}\;\tilde{\varepsilon}_{\alpha\beta\gamma\delta}\; dx_0^{\alpha}\; dx_1^{\beta}\; dx_2^{\gamma}\; dx_3^{\delta}$$

up to permutations $= \sqrt{|g|}\; dx^0\; dx^1\; dx^2\; dx^3$

4 orthogonal

3-D spherical $\quad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

$$g_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$$

$$dV^3 = \sqrt{|g|}\; dr\,d\theta\,d\phi$$

$$= r^2 \sin\theta\; dr\,d\theta\,d\phi$$

* Party trick Using determinant of $g_{\mu\nu}$ to compute some Christoffel symbol

$$\Gamma^{\mu}_{\mu\alpha} = g^{\mu\beta}\,\Gamma_{\beta\mu\alpha}$$

α, μ anti symmetric & g symmetric $\rightarrow$ cancelled out

$$= \frac{1}{2} g^{\mu\beta}\left(\partial_\mu \underbrace{g_{\alpha\beta}}_{\beta\alpha} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\alpha}\right)$$

Recall $\left[\begin{aligned} g_{\mu\gamma} \Gamma^{\mu}{}_{\alpha\beta} &= \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}) \\ &\equiv \Gamma_{\gamma\alpha\beta} \end{aligned} \right]$

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta}$$

To show $= \frac{1}{\sqrt{|g|}} \partial_\alpha \sqrt{|g|}$

$= \partial_\alpha \left(\ln \sqrt{|g|} \right)$

pf) Consider matrix M

Consider the variation

$\delta \ln(\det M)$

$= \ln[\det(M + \delta M)] - \ln[\det M]$

$= \ln \left(\frac{\det(M+\delta M)}{\det M} \right)$

$\delta \ln [\det M]$

$= \ln \det [[I + M^{-1} \cdot \delta M]]$

If ε is a "small" matrix then

$\det (I + \varepsilon) \simeq 1 + Tr(\varepsilon)$

$M^{-1} \delta M$ is our ε

$\rightsquigarrow \delta \ln [\det M] = \ln (1 + Tr(M^{-1} \delta M)) \approx Tr(M^{-1} \delta M)$

Take $M \to g_{\alpha\beta}$ $M^{-1} \to g^{\alpha\beta}$

Look at how functional variations go through

$\delta \ln |g| = Tr [g^{\mu\beta} \delta g_{\beta\nu}]$

Aside what is trace of matrix ε?

$Tr(\varepsilon) = g^{\alpha\beta} \varepsilon_{\alpha\beta} = \varepsilon^{\beta}{}_{\beta}$

So $\delta \ln |g| = g^{\mu\beta} \delta g_{\beta\mu}$

Divide δx^ν, take limit

$\partial_\alpha(\ln|g|) = g^{\mu\beta}\,\partial_\alpha(g_{\beta\mu})$

Recall $\Gamma^\mu_{\mu\beta} = \frac{1}{2} g^{\mu\beta}\partial_\alpha g_{\beta\mu}$

$= \frac{1}{2}\partial_\alpha(\ln|g|)$

$= \partial_\alpha(\ln\sqrt{|g|})$ ✓

Utility: Spacetime divergence of a vector field

$\nabla_\alpha A^\alpha = \partial_\alpha A^\alpha + \Gamma^\alpha_{\alpha\beta} A^\beta$

$= \partial_\alpha A^\alpha + \Gamma^\beta_{\beta\alpha} A^\alpha$

from above

$= \partial_\alpha A^\alpha + \frac{A^\alpha}{\sqrt{|g|}}\partial_\alpha\sqrt{|g|}$

$= \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}\,A^\alpha)$ ← no christoffel symbol

only involves partial derivatives!

Gaussian integral

$$\int_{V^4}(\nabla_\alpha A^\alpha)\sqrt{|g|}\,d^4x = \int_{V^4}\partial_\alpha(\sqrt{|g|}\,A^\alpha)\,d^4x$$

Can we do this for tensors?

1. No $\nabla_\alpha A^{\alpha\beta} = \partial_\alpha A^{\alpha\beta} + \Gamma^\alpha_{\alpha\nu} A^{\nu\beta} + \underline{\Gamma^\beta_{\alpha\nu} A^{\alpha\nu}}$

when antisymmetric ⇝ $=0$

2. Not as useful:

equivalence principle $\nabla_\mu T^{\mu\beta} = 0$

(Recall, Conservation of energy and momentum)

in special relativity $\boxed{\partial_\alpha T^{\alpha\beta} = 0}$: Conservation law

How do we formulate kinematics of bodies in curved spacetime?

Go into a freely falling frame, use locally Lorentz representation (consider a "test body")

no charge, no spatial extent, no spin, ⋯ nothing but mass

In this frame, body moves on a purely inertial trajectory

$$x^\alpha = \underset{\text{at } \tau=0}{\underline{\underline{x_0^\alpha}}} + \underset{\text{4-velocity}}{\underline{\underline{u^\alpha}}}\,\underset{=\text{ proper time}}{\tau}$$

"straight line" in this representation

More geometric way of thinking about its motion. Body is parallel transporting its tangent vector

Trajectory, parameterized by π $x^\alpha(\pi)$

Tangent is $u^\alpha = \dfrac{dx^\alpha}{d\pi}$

parallel transporting tangent vector $u^\alpha \nabla_\alpha u^\beta = 0$

$$\left(\nabla_{\vec{u}} \vec{u} = 0 \quad \text{—or—} \quad \frac{Du^\alpha}{d\pi} = 0\right)$$

Expand $u^\alpha \partial_\alpha u^\beta + \Gamma^\beta_{\alpha\mu} u^\alpha u^\mu = 0$

$$\rightarrow \boxed{\frac{du^\beta}{d\pi} + \Gamma^\beta_{\alpha\mu} u^\alpha u^\mu = 0 \quad \text{—or—} \quad \frac{d^2x^\beta}{d\pi^2} + \Gamma^\beta_{\alpha\mu} \frac{dx^\alpha}{d\pi}\frac{dx^\mu}{d\pi} = 0}$$

"geodesics equation" "second order approximation"

The geodesic equation solutions for trajectories are geodesics

: Suppose we allow the vector's normalization to change as it transports

$$\frac{Du^\alpha}{d\pi^*} = K(\pi^*)\, u^\alpha$$

Homework exercise

We can always reparameterize this such that RHS is zero.

Imagine $V^\alpha = \dfrac{dx^\alpha}{d\pi}$ such that $V^\alpha \nabla_\alpha V^\beta = 0$ $\left(\dfrac{dV^\beta}{d\pi} = 0\right)$

$$\frac{d\pi}{d\pi^*} = \exp\left(\int K(\pi^*)\, d\pi^*\right)$$

$\rightsquigarrow$ you can adjust parameterization and make it 0

parameterization with RHS=0

$\Rightarrow$ "Affine parameterization"

Intuition: Affine parameters correspond to the "tickmarks" on worldline being uniformly spaced in LLF.

For timelike trajectories, proper τ is a good affine parameter

If we adjust parameterization linearly.

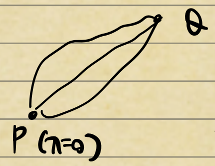
$$\lambda \to \lambda' = \underset{=}{\overset{\text{unit}}{a}}\lambda + \underset{=}{b} \leftarrow \text{~~strictly true~~} \quad \text{for proper time}$$

constant

we get a new affine parameterization (only)

* 2^{nd} path to geodesics Based on intuition that shortest path between 2 points is a straight line

The freefall path is the one on which the observer is maximally aged



$$\Delta \tau \equiv \int_0^{\lambda_Q} d\lambda \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}$$

In notes to be posted, show that

$\Delta\tau$ can be used to define an action $I = \frac{1}{2}\int \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}\right) d\tau$

Require $\delta I = 0$ as $x^\alpha \to x^\alpha + \delta x^\alpha$

$$g_{\alpha\beta} \to g_{\alpha\beta} + (\partial_\gamma g_{\alpha\beta})\delta x^\gamma \qquad g_{\alpha\beta}(x^\gamma + \delta x^\gamma) \cong g_{\alpha\beta}(x^\gamma) + \overset{\partial_\gamma g_{\alpha\beta}}{\boxed{\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}}}\delta x^\gamma$$

$$\delta I = -\int d\tau \left[g_{\gamma\alpha}\frac{du^\alpha}{d\tau} + \underbrace{\frac{1}{2}u^\alpha u^\beta (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})}_{\Gamma_{\gamma\alpha\beta}} \right]\delta x^\gamma$$

$$\delta I = -\int d\tau \left[g_{\gamma\alpha}\frac{du^\alpha}{d\tau} + \Gamma_{\gamma\alpha\beta} u^\alpha u^\beta \right]\delta x^\gamma$$

Require $\delta I = 0$ for any δx^γ, then bracketed term $= 0$

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\ \alpha\beta} u^\alpha u^\beta = 0$$

Geodesics generalize straight line to curved spacetime and give trajectory of "extremal aging"

Lec 10

Can rewrite this in terms of momentum

$p^\alpha = m u^\alpha$ (timelike trajectory, rest mass m)

$$p^\alpha \nabla_\alpha p^\beta = 0$$

$$m\frac{dp^\beta}{d\tau} + \Gamma^\beta_{\ \mu\nu} p^\mu p^\nu = 0$$

Change parameter $\Delta\lambda = \Delta\tau/m$

$$p^\alpha = m\frac{dx^\alpha}{d\tau} \rightarrow \frac{dx^\alpha}{d\lambda}$$

$$\frac{dp^\beta}{d\lambda} + \Gamma^\beta_{\mu\gamma}p^\mu p^\gamma = 0$$

Can limit in which $m\to 0$ as long as $\Delta\tau\to 0$ at rate such that $\frac{\Delta\tau}{m}$ is constant

perfectly good for null or light like trajectories! $(p^\alpha p_\alpha = -m^2 = 0)$

One further trick. Write momentum equation as

$$\underline{p^\alpha(\nabla_\alpha p_\gamma) = 0}$$

$$p^\alpha(\nabla_\alpha p^\beta)g_{\beta\gamma} = 0$$

$$m\frac{dp_\beta}{d\tau} - \Gamma^\gamma_{\beta\alpha}p^\alpha p_\gamma = 0$$

$$\rightarrow m\frac{dp_\alpha}{d\tau} = \Gamma_{\gamma\beta\alpha}p^\alpha p^\gamma$$

$$= \frac{1}{2}(\partial_\beta g_{\alpha\gamma} + \cancel{\partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}})p^\alpha p^\gamma$$

$$\boxed{m\frac{dp_\alpha}{d\tau} = \frac{1}{2}\partial_\beta g_{\alpha\gamma}p^\alpha p^\gamma}$$

Suppose $\partial_\beta g_{\alpha\gamma} = 0$ for some particular coordinate

with the choice of coordinate

P → Then $m\frac{dp_\beta}{d\tau} = 0 \rightarrow p_\beta =$ constant on this geodesic

We also know if $\partial_\beta g_{\alpha\gamma} = 0$ there exists a killing vector ξ^β

Look at how $(p^\beta\xi_\beta)$ evolves on geodesic worldline

Examine $m\frac{D}{d\tau}(p^\beta\xi_\beta) = p^\alpha\nabla_\alpha(p^\beta\xi_\beta) = 0$

$$= \xi_\beta\underbrace{(p^\alpha\nabla_\alpha p^\beta)}_{=0 \text{ (geodesic)}} + p^\alpha p^\beta\nabla_\alpha\xi_\beta$$

$$M_{\alpha\beta} = M_{(\alpha\beta)} + M_{[\alpha\beta]}$$

$$\tfrac{1}{2}(M_{\alpha\beta}+M_{\beta\alpha}) \qquad \tfrac{1}{2}(M_{\alpha\beta}-M_{\beta\alpha})$$

$$= p^\alpha p^\beta[\nabla_{(\alpha}\xi_{\beta)} + \nabla_{[\alpha}\xi_{\beta]}]$$

$$= p^\alpha p^\beta \nabla_{(\alpha}\xi_{\beta)}$$

indept of coordinates

② $\rightarrow$ $$= \frac{1}{2}p^\alpha p^\beta (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) = 0$$ by killing's equation

Example. $\partial_t g_{\alpha\beta} = 0$

$\vec{\xi}^t$, 'timelike' Killing vector exists

0'th cpnt of four momentum $= $ energy

$\partial_{tt} p^t$

$$P_t = \text{Constant}$$

$$= -E \quad (\text{If spacetime is asymptotically flat})$$

Example. geodesic If spacetime is

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)$$

$\Phi \ll 1 \quad \Phi = \Phi(x,y,z)$ no time dependence

Slow motion: $p^\alpha = (E, \vec{p}) \quad E \gg |\vec{p}|$

$E \simeq m \quad (E^2 = m^2 + |\vec{p}|^2)$

Geodesics for free-fall

$$m\frac{dp^\beta}{d\tau} + \underbrace{\Gamma^\beta_{\mu\nu}}_{} p^\mu p^\nu = 0$$

dominated by $\mu = \nu = 0$

$$m\frac{dp^\beta}{d\tau} \simeq -\Gamma^\beta_{00} p^0 p^0 \simeq -m^2 \Gamma^\beta_{00}$$

Focus on $\beta = i$

$$\Gamma^i_{00} = \frac{1}{2} g^{i\alpha}(\cancel{\partial_t} g_{0\alpha} + \cancel{\partial_t} g_{\alpha 0} - \partial_\alpha g_{00})$$

$$g^{i\alpha} = (1-2\Phi)^{-1}\delta^{i\alpha}$$

$$\Gamma^i_{00} = -\frac{1}{2}(1-2\Phi)^{-1}\delta^{ij}\partial_j(-2\Phi)$$

$$\rightarrow \Gamma^i_{00} = \delta^{ij}\partial_j\Phi + \mathcal{O}(\Phi^2)$$

$$\rightarrow \cancel{m}\frac{dp^i}{d\tau} = -m\delta^{ij}\partial_j\Phi$$

gradient of potential

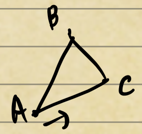
rate of change of momentum

later plug in Φ for whatever it is.

Quantifying Curvature: Breakdown of parallelism between initially parallel trajectories

Consider a vector parallel transported around a closed figure on a curved manifold

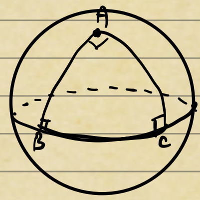
Example



Go around triangle, points along initial path

Triangle sum of angles = 180°

Embed triangle on surface of a sphere

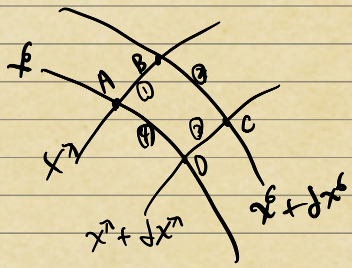


Internal angles = 270°

parallel transport rotates the vector

rotates by an angle of (int angle) − 180°

Holonomy: see wikipedia or mathworld.wolfram.com



parallel transport around loop

$A \to B \to C \to D \to A$

$A \to B$ $\quad x^\lambda$ constant, moving along $\vec{e}_\sigma$

$$\nabla_{\vec{e}_\sigma} V^\alpha = 0$$

$$\to \partial_\sigma V^\alpha + \Gamma^\alpha_{\sigma\mu} V^\mu = 0$$

or $$\frac{dV^\alpha}{dx^\sigma} = -\Gamma^\alpha_{\sigma\mu} V^\mu$$

Integrate this up

$$V^\alpha(B) = V^\alpha_{\text{init}} - \int_{①} \Gamma^\alpha_{\sigma\mu} \, dx^\sigma$$

continue:

$$V^\alpha(C) = V^\alpha(B) - \int_{②} \Gamma^\alpha_{\lambda\mu} V^\mu \, dx^\lambda$$

$$V^\alpha(D) = V^\alpha(C) + \int_{③} \Gamma^\alpha_{\sigma\mu} V^\mu \, dx^\sigma$$ ← sign switch due to decreasing direction

$$V^\alpha_{final} = V^\alpha(0) + \int_{\oplus} \Gamma^\alpha_{\lambda\mu} V^\mu dx^\lambda$$

$$V^\alpha_{final} - V^\alpha_{init} = \delta V^\alpha$$

$$= \int_{\oplus} \Gamma^\alpha_{\lambda\mu} V^\mu dx^\lambda - \int_{\ominus} \Gamma^\alpha_{\lambda\mu} V^\mu dx^\lambda$$

parallel but offset paths

$$+ \int_{\ominus} \Gamma^\alpha_{\sigma\mu} V^\mu dx^\sigma - \int_{\oplus} \Gamma^\alpha_{\sigma\mu} dx^\sigma$$

$$\int_{\oplus} \underbrace{\Gamma^\alpha_{\lambda\mu} V^\mu}_{\text{eval at } x^\sigma} dx^\lambda - \int_{\ominus} \underbrace{\Gamma^\alpha_{\lambda\mu} V^\mu}_{\text{eval at } x^\sigma + dx^\sigma} dx^\lambda$$

$$\simeq \int_{\ominus} (-\delta x^\sigma) \frac{d}{dx^\sigma} (\quad) \, dx^\lambda$$

$$\int_{\ominus} \underbrace{\Gamma^\alpha_{\sigma\mu} V^\mu}_{\substack{\text{eval at} \\ x^\lambda + dx^\lambda}} dx^\sigma - \int_{\oplus} \underbrace{\Gamma^\alpha_{\sigma\mu}}_{\text{eval at } x^\lambda} dx^\sigma$$

$$\simeq \int_{\ominus} \delta x^\lambda \frac{d}{dx^\lambda} (\quad) \, dx^\sigma$$

$$\delta V^\alpha \approx \int_{x^\sigma}^{x^\sigma + dx^\sigma} \delta x^\lambda \frac{d}{dx^\lambda} \left(\Gamma^\alpha_{\sigma\mu} V^\mu \right) dx^\sigma - \int_{x^\lambda}^{x^\lambda + dx^\lambda} \delta x^\sigma \frac{d}{dx^\sigma} \left(\Gamma^\alpha_{\lambda\mu} V^\mu \right) dx^\lambda$$

Evaluate infinitesimal integrals, expand derivs

$$\delta V^\alpha = \delta x^\lambda \delta x^\sigma \left[\partial_\lambda \Gamma^\alpha_{\sigma\mu} V^\mu - \partial_\sigma \Gamma^\alpha_{\lambda\mu} V^\mu + \Gamma^\alpha_{\sigma\mu} \partial_\lambda V^\mu - \Gamma^\alpha_{\lambda\mu} \partial_\sigma V^\mu \right]$$

parallel transport! $\partial_\lambda V^\mu = -\Gamma^\mu_{\lambda\nu} V^\nu$

$\partial_\sigma V^\mu = -\Gamma^\mu_{\sigma\nu} V^\nu$

$$\delta V^\alpha = \delta x^\lambda \delta x^\sigma \left[\left(\partial_\lambda \Gamma^\alpha_{\sigma\mu} - \partial_\sigma \Gamma^\alpha_{\lambda\mu} \right) V^\mu + \left(\Gamma^\alpha_{\lambda\mu} \Gamma^\mu_{\sigma\nu} - \Gamma^\alpha_{\sigma\mu} \Gamma^\mu_{\lambda\nu} \right) V^\nu \right]$$

$\mu \leftrightarrow \nu$

$$= \delta x^\lambda \delta x^\sigma V^\mu R^\alpha_{\ \mu\lambda\sigma}$$

$$R^\alpha_{\ \mu\lambda\sigma} = \partial_\lambda \Gamma^\alpha_{\sigma\mu} - \partial_\sigma \Gamma^\alpha_{\lambda\mu} + \Gamma^\alpha_{\lambda\nu} \Gamma^\nu_{\sigma\mu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\nu_{\lambda\mu}$$

"Riemann curvature tensor"

Truly tensor

Equivalent definition

$$[\nabla_\pi, \nabla_\sigma] V^\alpha = R^\alpha{}_{\mu\pi\sigma} V^\mu$$

$$[\nabla_\pi, \nabla_\sigma] P_\alpha = -R^\mu{}_{\alpha\pi\sigma} P_\mu$$ ← wrong in Schutz 1st edition

4 index tensor, 4 values of each

256 components

Symmetries reduced Riemann in n dim from $n^4 \to \frac{n^2(n^2-1)}{12}$

$\frac{n^2(n^2-1)}{12}$

$n=1$: 0

$n=2$: 1 radius of curvature

$n=4$: 20 Exactly the number leftover constraints at second order in a freely falling frame

Lec 11

Symmetries

$$R^\alpha{}_{\mu\pi\sigma} = -R^\alpha{}_{\mu\sigma\pi}$$

Corresponds to reversing direction of transport

To get others, lower index

$$R_{\alpha\mu\pi\sigma} = g_{\alpha\nu} R^\nu{}_{\mu\pi\sigma}$$

Go into LLF metric at point is flat $\Gamma s \to 0$ derivatives do not vanish

$$(R_{\alpha\mu\pi\sigma})^{LLF} = \partial_\pi \Gamma_{\alpha\sigma\mu} - \partial_\sigma \Gamma_{\alpha\pi\mu}$$

Insert definition of Christoffel

$$(R_{\alpha\mu\pi\sigma})^{LLF} = \frac{1}{2}(\partial_\pi \partial_\mu g_{\alpha\sigma} - \partial_\pi \partial_\alpha g_{\sigma\mu} - \partial_\sigma \partial_\mu g_{\alpha\pi} + \partial_\sigma \partial_\alpha g_{\pi\mu})$$

Stare at this: ① $R_{\alpha\mu\pi\sigma} = -R_{\alpha\mu\sigma\pi}$

② $R_{\alpha\mu\pi\sigma} = -R_{\mu\alpha\pi\sigma}$

③ $R_{\alpha\mu\pi\sigma} = R_{\pi\sigma\alpha\mu}$

④ $R_{\alpha\underline{\mu\pi\sigma}} + R_{\alpha\underline{\pi\sigma\mu}} + R_{\alpha\underline{\sigma\mu\pi}} = 0$

④ $R_{\alpha[\mu\pi\sigma]} = 0$

$\rightsquigarrow \; n^4 \to \frac{n^2(n^2-1)}{12}$

$n=4 : 20$

$g_{tt} = -1 - R_{tjtl}\, x^j x^l$

$g_{tj} = -\frac{2}{3} R_{tijl}\, x^i x^l$

$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}\, x^k x^l$

Riemann normal coordinates (poisson: A relativist's toolkit)

Variants of curvature tensor

Ricci 1. take the trace of Riemann on indices 1 & 3

$R^{\alpha}{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu}$

$1 \& 2,\; 3 \& 4 \rightsquigarrow 0$

$\equiv R_{\mu\nu}$

"Ricci curvature tensor"

Easy to see symmetric on μ & ν

$R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu}$

$\searrow \partial_\nu \partial_\mu (\ln |g|^{\frac{1}{2}})$

$+ \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\nu\mu} - \Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\alpha\mu}$

Symmetric 4x4 has 10 independent components

1' Trace of Ricci $R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \equiv R$

Ricci scalar, curvature scalar

Weyl 2. $\underbrace{C_{\alpha\mu\pi\sigma}}_{\text{four index tensor}} = R_{\alpha\mu\pi\sigma} - \frac{2}{n-2}\big(\overbrace{g_{\alpha[\pi}R_{\sigma]\mu}}^{g_{\alpha\pi}R_{\sigma\mu} - g_{\alpha\sigma}R_{\pi\mu}} - g_{\mu[\pi}R_{\sigma]\alpha}\big)$

$+ \frac{2}{(n-2)(n-1)} g_{\alpha[\pi} g_{\sigma]\mu} R$

Has symmetries of Riemann but has no trace

10 Indpt components

"Riemann" $\overset{\text{same info}}{\longleftrightarrow}$ "Ricci" + "Weyl"

Ricci is very closely related to sources of gravity

Roughly speaking...

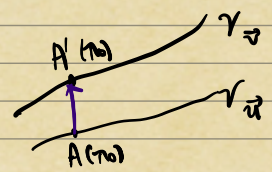
Weyl: free / radiative part of gravitational field

Ricci: gravitational field directly sourced by matter / energy

* Breakdown of parallelism

Initially parallel geodesics, characterize how they deviate as one moves along

Consider two nearby geodesics each parametrized by λ



$\vec{u}$ = tangent vector to $\gamma_{\vec{u}}$

$\vec{v}$ = —— " —— $\gamma_{\vec{v}}$

A is at λ_0 on $\gamma_{\vec{u}}$

A' is at λ_0 on $\gamma_{\vec{v}}$

$\vec{\xi}$ points from event at λ on $\gamma_{\vec{u}}$ to the event at λ on $\gamma_{\vec{v}}$

$$\vec{\xi} = \vec{x}(\gamma_{\vec{v}}, \lambda) - \vec{x}(\gamma_{\vec{u}}, \lambda)$$

Assume begin parallel

$$\vec{u}(\lambda_0) = \vec{v}(\lambda_0)$$

Can use $(u^\alpha \nabla_\alpha \xi^\beta)\big|_{\lambda=\lambda_0} = 0$ as boundary condition

Develop some intuition for what is happening by working in LLF, centered on A

$$g_{\mu\nu}\big|_A = \eta_{\mu\nu} \qquad \Gamma^\mu_{\alpha\beta}\big|_A = 0$$

same inner product rule

$$g_{\mu\nu}\big|_{A'} = \eta_{\mu\nu} \qquad \Gamma^\mu_{\alpha\beta}\big|_{A'} = (\partial_\nu \Gamma^\mu_{\alpha\beta})_A \xi^\nu + O(\xi^2)$$

- Geodesic equation along curve $\gamma_{\vec{v}}$ at A

$$\left.\frac{d^2x^\alpha}{d\tau^2}\right|_A = 0$$

- Geodesic eqn along $\gamma_{\vec{v}}$ at A'

$$\left.\frac{d^2x^\alpha}{d\tau^2}\right|_{A'} + \Gamma^\alpha_{\mu\nu}\left[\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]_{A'} = 0$$

$$\left.\frac{dx^\mu}{d\tau}\right|_{A'} = v^\mu\Big|_{A'} = u^\mu$$

$$\left.\frac{d^2x^\alpha}{d\tau^2}\right|_{A'} = -\left(\partial_\beta \Gamma^\alpha_{\mu\nu}\right)_A \underbrace{u^\mu u^\nu}_{\substack{\text{motion}\\ \text{along worldline}}} \underbrace{\xi^\beta}_{\text{displacement}}$$

$$\left.\frac{d^2x^\alpha}{d\tau^2}\right|_{A'} - \left.\frac{d^2x^\alpha}{d\tau^2}\right|_A = \frac{d^2\xi^\alpha}{d\tau^2} = -\underbrace{\left(\partial_\beta \Gamma^\alpha_{\mu\nu}\right)_A}_{\text{"second derivative of metric"}} u^\mu u^\nu \xi^\beta$$

Want to make tensorial

$$\frac{d}{d\tau} = \underline{\underline{u^\alpha \partial_\alpha}} \rightarrow \text{rewrite using } \frac{D}{d\tau} = u^\alpha \nabla_\alpha$$

$\left(\frac{dx^\alpha}{d\tau}\frac{\partial}{\partial x^\alpha}\text{ product rule}\right)$

$$\frac{D\xi^\alpha}{d\tau} = u^\beta \nabla_\beta \xi^\alpha + u^\beta \Gamma^\alpha_{\beta\mu}\xi^\mu$$

$$= \frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\beta\mu} u^\beta \xi^\mu$$

$$\frac{D^2\xi^\alpha}{d\tau^2} = u^\nu \nabla_\nu\left(\frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\beta\mu} u^\beta \xi^\mu\right)$$

$$= \frac{d^2\xi^\alpha}{d\tau^2} + \cancel{u^\nu \Gamma^\alpha_{\nu\mu}}\frac{d\xi^\mu}{d\tau} + u^\nu \nabla_\nu\left(\Gamma^\alpha_{\beta\mu} u^\beta \xi^\mu\right)$$

- In LLF
- Near point A

$$\left\{\begin{aligned} &+ u^\nu \nabla_\nu \Gamma^\alpha_{\beta\mu}\, u^\beta \xi^\mu \\ &+ \Gamma^\alpha_{\beta\mu}\left(\cancel{u^\nu \nabla_\nu u^\beta}\right)\xi^\mu \\ &+ \Gamma^\alpha_{\beta\mu}\, u^\beta\left(\cancel{u^\nu \nabla_\nu \xi^\mu}\right)\end{aligned}\right.$$

(parallel)

$$= \frac{d^2\xi^\alpha}{d\tau^2} + \partial_\gamma \Gamma^\alpha_{\beta\mu} u^\beta u^\gamma \xi^\mu$$

$$= \partial_\gamma \Gamma^\alpha_{\beta\mu} u^\beta u^\gamma \xi^\mu - \partial_\beta \Gamma^\alpha_{\mu\nu} u^\mu u^\nu \xi^\beta$$

Relabel dummy indices

On 2nd term $\beta \to \mu \quad \mu \to \gamma \quad \nu \to \beta$

$$\frac{D^2\xi^\alpha}{d\tau^2} = \partial_\gamma \Gamma^\alpha_{\beta\mu} u^\beta u^\gamma \xi^\mu - \partial_\mu \Gamma^\alpha_{\gamma\beta} u^\gamma u^\beta \xi^\mu$$

$$= (\partial_\gamma \Gamma^\alpha_{\beta\mu} - \partial_\mu \Gamma^\alpha_{\gamma\beta}) u^\beta u^\gamma \xi^\mu$$

$$= R^\alpha_{\beta\gamma\mu} u^\beta u^\gamma \xi^\mu$$

"Equation of geodesic deviation"

Recall Riemann is commutator of covariant derivs

$$[\nabla_\eta, \nabla_\sigma] V^\alpha = R^\alpha_{\mu\eta\sigma} V^\mu$$

Generally, $\quad [\nabla_\eta, \nabla_\sigma] F^\alpha{}_\beta = R^\alpha_{\mu\eta\sigma} F^\mu{}_\beta - R^\mu_{\beta\eta\sigma} F^\alpha{}_\mu$

Using these...

(A) $\quad [\nabla_\alpha, \nabla_\beta] \nabla_\gamma P_\delta$

$$= -R^\mu{}_{\gamma\alpha\beta} \nabla_\mu P_\delta - R^\mu{}_{\delta\alpha\beta} \nabla_\gamma P_\mu$$

(B) $\quad \nabla_\alpha [\nabla_\beta, \nabla_\gamma] P_\delta = -P^\mu \nabla_\alpha R_{\beta\gamma\mu\delta} - R^\mu_{\delta\beta\gamma} \nabla_\alpha P^\mu$

Anti Symmetrize on α, β, γ

$$[\nabla_{[\alpha}, \nabla_\beta] \nabla_{\gamma]} P_\delta = \frac{1}{3!}\Big([\nabla_\alpha, \nabla_\beta]\nabla_\gamma + [\nabla_\beta, \nabla_\gamma]\nabla_\alpha + [\nabla_\gamma, \nabla_\alpha]\nabla_\beta$$

$$- ([\nabla_\alpha, \nabla_\gamma]\nabla_\beta - [\nabla_\gamma, \nabla_\beta]\nabla_\alpha - [\nabla_\beta, \nabla_\alpha]\nabla_\gamma\Big) P_\delta$$

Not hard to show

$$= \frac{1}{3!}\Big(\nabla_\alpha [\nabla_\beta, \nabla_\gamma] + \nabla_\beta [\nabla_\gamma, \nabla_\alpha] + \nabla_\gamma [\nabla_\alpha, \nabla_\beta]$$

$$- \nabla_\alpha [\nabla_\gamma, \nabla_\beta] - \nabla_\beta [\nabla_\alpha, \nabla_\gamma] - \nabla_\gamma [\nabla_\beta, \nabla_\alpha]\Big) P_\delta$$

$$= \nabla_{[\alpha} [\nabla_\beta, \nabla_{\gamma]}] P_\delta$$

Antisymmetrize LHS of A → LHS of B

Apply to RHS

$$\cancel{R^{\mu}{}_{[\gamma\alpha\beta]}}\nabla_\mu P_\delta + R^{\mu}{}_{\delta[\alpha\beta}\nabla_{\gamma]}P_\mu = P^\mu \nabla_{[\alpha} R_{\beta\gamma]\mu\delta} + R^{\mu}{}_{\delta[\beta\gamma}\nabla_{\alpha]}P^\mu$$

0 Riemann Symmetry

$$\Rightarrow P^\mu \nabla_{[\alpha} R_{\beta\gamma]\mu\delta} = 0 \qquad \forall P$$

$$\boxed{\nabla_{[\alpha} R_{\beta\gamma]\mu\delta} = 0}$$ Bianchi identity

Another form (expand & use symmetry)

$$\nabla_\alpha R_{\beta\gamma\mu\delta} + \nabla_\beta R_{\gamma\alpha\mu\delta} + \nabla_\gamma R_{\alpha\beta\mu\delta} = 0$$

Contract Bianchi identity

$$\nabla^\mu (R_{\alpha\mu} - \tfrac{1}{2} g_{\alpha\mu} R) = 0$$

$$\nabla^\mu G_{\alpha\mu} = 0$$

Einstein curvature tensor

Lec 12

Contract Bianchi: Contract using $\underline{g^{\beta\mu}}$

metric commutes w. covariate derivative

$$\nabla_\alpha R_{\gamma\nu} + \nabla^\mu R_{\gamma\alpha\mu\nu} - \nabla_\gamma R_{\alpha\nu} = 0$$

Contract once more using $g^{\gamma\nu}$

$$\nabla_\alpha R - \nabla^\mu R_{\alpha\mu} - \nabla^\nu R_{\alpha\nu} = 0$$

$$\nabla_\alpha R - 2\nabla^\mu R_{\alpha\mu} = 0$$

—or— $$\nabla^\mu \underbrace{(R_{\alpha\mu} - \tfrac{1}{2} g_{\alpha\mu} R)}_{= G_{\alpha\mu}} = 0$$

$= G_{\alpha\mu}$ Einstein tensor

$$G^\mu{}_\mu = g^{\mu\nu} G_{\mu\nu} \equiv G$$

$\Downarrow$

$$= R - \frac{1}{2}\delta^{\mu}_{\ \mu} R$$

$$= -R$$ Einstein tensor is a trace-reversed Ricci Tensor

* 2 ingredients for making a theory of gravity

1. Principle of equivalence

"minimal coupling principle"

Take a law of physics that is valid in inertial coordinates, flat spacetime (or in LLF ⟺ FFF)

- Write that law in coordinate invariant tensorial form.

- Assert that the resulting law holds in curved spacetime

Example. force-free motion

In LLF $\frac{d^2 x^\mu}{d\tau^2} = 0$

Tensorial version $u^\alpha \nabla_\alpha u^\beta = 0$

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

Another: local conservation of energy and momentum

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{in LLF} \longrightarrow \nabla_\mu T^{\mu\nu} = 0$$

2. A field equation which connects spacetime to sources of matter and energy

We require that whatever emerges must recover Newtonian gravity in limit!

$$\nabla^2 \Phi = \delta^{ij}\partial_i \partial_j \Phi = 4\pi G\rho$$

Equation of motion $\frac{d^2 x^i}{dt^2} = -\delta^{ij}\partial_j \Phi$

For equation of motion: Start with $\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$

Slow motion limit: $\frac{dx^0}{d\tau} = \frac{dt}{d\tau} \gg \frac{dx^i}{d\tau} = \underbrace{\frac{dx^i}{dt}}_{\ll 1}\frac{dt}{d\tau}$

① $$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\mu\nu}\left(\partial_0 g_{\nu 0} + \partial_0 g_{0\nu} - \partial_\nu g_{00}\right)$$

Neglect time derivatives (since static)

$$\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \|h_{\mu\nu}\| \ll 1$$

metric on flat spacetime

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2) \qquad h^{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\alpha\beta}$$

② $$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} + O(h^2)$$

Motion in this limit $\quad \Gamma^{0}_{00} = 0$

$$\rightarrow \frac{d^2 t}{d\tau^2} = 0$$

①+②: $$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\eta^{ij}\partial_j h_{00}\left(\frac{dt}{d\tau}\right)^2$$

$$\rightarrow \frac{d^2 x^i}{dt^2} = \frac{1}{2}\delta^{ij}\partial_j h_{00} \qquad \left(\text{vs. } \frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_j \Phi\right)$$

Identical to Newton provided

$$h_{00} = -2\Phi$$

— or —

Newtonian gravitational potential

$$g_{00} = \eta_{00} + h_{00} = -(1+2\Phi)$$

Field equation

$$\eta^{ij}\partial_i\partial_j\Phi = 4\pi G\rho$$

mass density

ρ mass density

→ energy density

→ T_{00} A tensor component, not a tensor

Want to promote this to something tensorial

as $g_{00} = -(1+2\Phi)$

(metric stands for potential ... expect 2 derivatives of metric entering here) $= T_{\mu\nu}$

"curvature?"

2 derivatives of metric corresponds to curvature

[curvature tensor] = $T_{\mu\nu}$

To guide us, note that $\nabla_\mu T^{\mu\nu} = 0$

divergence free 2 index tensor

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

$\hookrightarrow$ constant to match dimensions

—or— $R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$ $\quad T = g^{\mu\nu} T_{\mu\nu}$

Well...

$$\begin{cases} -R = G = \kappa T \\ R = -\kappa T \\ G_{\mu\nu} =: R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \end{cases}$$

$$= R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \kappa T$$

$g_{0i} = 0 \quad \partial_0 g_{\mu\nu} = 0$

pick a static perfect fluid as our source of gravity

$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}$ $\qquad \rho \gg P$ in limit we're studying

static fluid: $u^\mu = (u^0, 0, 0, 0)$

$\leftarrow$ slow velocity $\frac{dx}{dt} = 0,\ \frac{dx}{d\tau} \neq 0$

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$\to g_{00} (u^0)^2 = -1$$

$\uparrow$ only matter time

$g_{00} = -1 + h_{00}$ (recall $g_{00} = \eta_{00} + h_{00}$)

$$\to u^0 \approx 1 + \frac{1}{2} h_{00} \left(+ O(h^2)\right)$$

$g_{\mu\nu} u^\mu = u_\nu, \quad u_0 = -u^0$

$$T_{00} = \rho u_0 u_0$$
$$= \rho (1 + h_{00})$$

$T = g^{\mu\nu} T_{\mu\nu} = (\rho u_0 u_0 g^{00} =) \rho u^\mu u_\mu = -\rho$

$R_{00} = k(T_{00} - \frac{1}{2} g_{00} T)$

$T_{00} - \frac{1}{2} g_{00} T = \rho(1+h_{00}) - \frac{1}{2}(-1+h_{00})(-\rho)$

$= \frac{1}{2}\rho + \frac{3}{2} h_{00}\rho$

$\overset{h \text{ small}}{\approx} \frac{1}{2}\rho$

$\Rightarrow R_{00} = \frac{1}{2} k\rho$

$R_{00} = R^{\mu}{}_{0\mu 0} = R^{i}{}_{0i0}$

$= \partial_i \Gamma^i{}_{00} - \cancel{\partial_0 \Gamma^i{}_{00}} + \mathcal{O}(\Gamma^2)$ Static (no time dependence of the metric)

$\overset{\text{def}}{=} \frac{1}{2} \partial_i [g^{i\mu}(\cancel{\partial_0 g_{\mu 0}} + \cancel{\partial_0 g_{0\mu}} - \partial_\mu g_{00})]$ static

$= -\frac{1}{2} \partial_i [\eta^{i\mu} \partial_\mu h_{00}] + \mathcal{O}(h^2)$

$R_{00} = -\frac{1}{2} \delta^{ij} \partial_i \partial_j h_{00}$

$= -\frac{1}{2} \nabla^2 h_{00}$

$R_{00} = k(T_{00} - \frac{1}{2} g_{00} T)$

$\rightarrow \nabla^2 h_{00} = -k\rho$

Newtonian equation of correspondence $\rightarrow h_{00} = -2\Phi$

Newtonian field equation $\nabla^2 \Phi = 4\pi G\rho$

All fits provided $k = 8\pi G$

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}}$$ → vacuum-free, cosmological-constant-free Einstein field equation

Add any divergence free tensor onto LHS and still have a good field equation

$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ $\quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

Λ: cosmological constant

Define $T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$

$T^{\Lambda}_{\mu\nu}$ is perfect fluid with $\rho = \frac{\Lambda}{8\pi G}$, $p = -\frac{\Lambda}{8\pi G}$

Represents a form of zero-point energy

Originally noted by Yakov Zeldovich

Often sets $G=1$ as well as $c=1$

G is poorly known. When we measure objects using gravity.

GM is measured much better than M (in many astrophysics / orbital problems)

$G=1$ $c=1$ means that mass. length time have same units

$\frac{G}{c^2}$ converts SI mass to length $[G] = m^3\, kg^{-1}\, s^{-2}$

$\frac{GM_\odot}{c^2} = 1.47\ km$ ← mass of sun

$\frac{GM}{c^3}$ Mass → time

$\frac{GM_\odot}{c^3} = 4.92 \times 10^{-6}\ s$

$\frac{G}{c^4}$ energy → length

$T_{\mu\nu} \equiv$ energy / volume = energy / length^3

$\frac{G}{c^4} T_{\mu\nu} = (\text{length}) / (\text{length})^3 = (\text{length})^{-2} \equiv$ curvature

$$\boxed{G_{\mu\nu} = \frac{8\pi \overset{\text{small}}{G}}{c^4} T_{\mu\nu}}$$

tiny big

Spacetime is hard to bend

Lec 13

Another route to field equation

Field equation via the Einstein Hilbert action

Schematically, define an action as integral of a Lagrange density over all spacetime

$S = \int d^4x\, \mathcal{L}$ → depends on the field you're studying

$= \int d^4x \sqrt{-g}\, \hat{\mathcal{L}}$

Extremization of action amounts to requiring that the action be stationary with respect to the field varying

$$\delta S \equiv \int d^4x \left[\frac{\delta \mathcal{L}}{\delta(\text{fields})} \right] \delta(\text{fields}) = 0$$

$$\frac{\delta \mathcal{L}}{\delta(\text{fields})} = 0$$ → Leads to Euler Lagrange equation for fields

Example $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$

Focus on flat spacetime

$$\phi \to \phi + \delta\phi$$

$$\partial_\mu \phi \to \partial_\mu \phi + \partial_\mu(\delta\phi)$$

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right]$$

Integration by parts (bdry not infinite so generally safe)

$$\rightarrow \delta S = \int d^4x \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right]}_{\text{must vanish}} \delta\phi = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\underset{\text{total}}{\mathcal{L}} = \underbrace{-\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)}_{\text{"kinetic term"}} - \underbrace{\frac{1}{2} m \phi^2}_{\text{"potential energy"}}$$

ϕ: temperature map

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\eta^{\mu\nu} \partial_\nu \phi$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = -\eta^{\mu\nu} \partial_\mu \partial_\nu \phi$$

$$= -\Box \phi$$

E.L → $\boxed{\Box \phi - m^2 \phi = 0}$

Massive Klein-Gordon equation

How do we apply to the theory of gravity?

Apply this dim to gravity

How do we chose $\mathcal{L}$?

- It must yield a scalar S

- It must be built from curvature tensors

Cannot be eliminated by changing frame of reference

We choose $\hat{\mathcal{L}} = R$

Built from metric

$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\; R \qquad R = g^{\alpha\beta}R_{\alpha\beta}(g)$$

$$\delta S = \frac{1}{16\pi G}\int d^4x \frac{\delta}{\delta g^{\alpha\beta}}\left[\sqrt{-g}\, g^{\alpha\beta}R_{\alpha\beta}\right]\delta g^{\alpha\beta} = 0$$

$$\delta\left[\sqrt{-g}\, g^{\alpha\beta}R_{\alpha\beta}\right] = (\delta\sqrt{-g})\, g^{\alpha\beta}R_{\alpha\beta} + \sqrt{-g}\,\delta g^{\alpha\beta}R_{\alpha\beta} + \sqrt{-g}\, g^{\alpha\beta}\,\delta R_{\alpha\beta}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

$$\delta R_{\alpha\beta} = \nabla_\mu(\delta\Gamma^\mu_{\alpha\beta}) - \nabla_\alpha(\delta\Gamma^\mu_{\mu\beta})$$

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\left[\nabla_\nu(g_{\alpha\delta}\, g_{\beta\eta}\, g^{\mu\delta}\,\delta g^{\delta\eta} - \nabla_\alpha(g_{\beta\nu}\,\delta g^{\mu\nu}) - \nabla_\beta(g_{\alpha\nu}\,\delta g^{\mu\nu})\right]$$

$$g^{\alpha\beta}\,\delta R_{\alpha\beta} = \nabla_\alpha\nabla_\beta\left(g^{\alpha\beta}g_{\mu\nu}\,\delta g^{\mu\nu} - \delta g^{\alpha\beta}\right)$$
$$\equiv \nabla_\alpha V^\alpha$$

$$\delta(\sqrt{-g}\, g^{\alpha\beta}R_{\alpha\beta}) = \sqrt{-g}\left[(R_{\alpha\beta} - (\tfrac{1}{2}g^{\mu\nu}R_{\mu\nu})g_{\alpha\beta})\,\delta g^{\alpha\beta} + \nabla_\alpha V^\alpha\right]$$

$$\delta S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\left[G_{\alpha\beta}\,\delta g_{\alpha\beta} + \nabla_\alpha V^\alpha\right]$$

How to get rid of $\nabla_\alpha V^\alpha$?

Carroll says we could invoke divergence theorem

lose contributions at the boundary, only the $G_{\alpha\beta} = 0$ would remain

More rigorous approaches

1. Palatini variation

$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\; g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$$

not g

Vary with both g and Γ

Einstein equation with no source: $G_{\alpha\beta} = 0$

And Γ such that $\nabla_\alpha g_{\beta\nu} = 0$ (connection arises from the metric!)

2. Define boundary associated w. divergence integral carefully

: requires carefully treating curvature in 3 dimensional bdry of 4D spacetime

Find you need to define Lagrangian a bit more carefully

Cancel the $\nabla_\alpha V^\alpha$ term

See Appendix E of Wald ("General Relativity")

* Extremization of Einstein-Hilbert action leads to $G_{\alpha\beta}=0$

"Vacuum Einstein equation"

More general form

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} + \mathcal{L}_M\right] \qquad \left(\text{vs. } \int d^4x\sqrt{-g}\,\hat{\mathcal{L}}\right)$$

Variation $\delta S = 0 \rightarrow \dfrac{\sqrt{-g}}{16\pi G} G_{\alpha\beta} + \dfrac{\partial(\sqrt{-g}\,\mathcal{L}_M)}{\partial g^{\alpha\beta}} = 0$

Define $T_{\alpha\beta} = -\dfrac{2}{\sqrt{-g}} \dfrac{\partial(\sqrt{-g}\,\mathcal{L}_M)}{\partial g^{\alpha\beta}}$ ⇝ get a result

Example $\mathcal{L}_M = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

Faraday tensor: electromagnetic fields

$$S_{EM} = -\frac{1}{4}\int d^4x \sqrt{-g}\, g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}$$

varying the metric

δS_{EM} Do not vary fields $F_{\alpha\beta}, F_{\mu\nu}$

$$\delta S = -\frac{1}{2}\int d^4x \sqrt{-g}\, \underbrace{\left[F_{\mu\alpha}F^{\mu}{}_{\beta} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu}F^{\mu\nu}\right]}_{\text{Stress Energy Tensor for E\&M (Jackson)}} \delta g^{\alpha\beta}$$

$\hat{\mathcal{L}} = R$ yields the Einstein field equations upon variations of the metric

* $\hat{\mathcal{L}} = R$ is the simplest scalar made from curvature tensors

but it is not the only one.

Suppose we want to explore a theory of gravity consistent with Einstein, but differs when R is "small"

$$\hat{L} = R - \frac{\alpha}{R}$$ Expect corrections when $R \lesssim \sqrt{\alpha}$

Vary metric, enforce stationary action

$$G_{\alpha\beta} + \frac{\alpha}{R^2}\left[R_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} R\right] + \alpha\left[g_{\alpha\beta}\nabla_\mu \nabla^\mu - \nabla_{(\alpha}\nabla_{\beta)}\right] R^{-2}$$

$$= 8\pi G T_{\alpha\beta}$$

falsified can't replace cosmological constant

Carroll et al. Phys Rev D, 043528 (2004)

"GR is simplest of all possible relativistic thry of gravity"

Other examples

1. $$S_g = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\left[R + \beta R^2\right]$$

Expect curvatures due to 2^{nd} term to become important when $R \gtrsim \frac{1}{\beta}$

Theoretical prejudice $\beta \sim \hbar^{-1}$

2. Suppose there existed a scalar field that modified how spacetime & source coupled to each other

$$S_g = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R f(\phi)$$

$$S_\phi = \int d^4x \sqrt{-g}\left[g(\phi)\nabla^\mu\nabla_\mu \phi - V(\phi)\right]$$

"Scalar tensor theories" $\rightsquigarrow$ motivated due to certain contradictory contradicted recently

See sec 48 of Carroll

$g_{\alpha\beta} \to$ motion of bodies (geodesics)

$\to$ curvature, tides

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

Solve with 3 Techniques ① "weak field" expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

② Assert Symmetry

③ General solutions : numerical relativity