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$w = d+1$

$y = n$

$Aw = y$

$w = (d+1) \times 1$

$y = n \times 1$

so size of matrix A

is $n \times (d+1)$

Let determinant of A is V_n.

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$$

By multiple of Row added to Row of Determinant.

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 0 & x_2-x_1 & x_2^2-x_1 & \cdots & x_2^{n-1}-x_1^{n-1} \\ \vdots & \vdots & & & \vdots \\ 0 & x_n-x_1 & x_n^2-x_1^2 & \cdots & x_n^{n-1}-x_n^{n-1} \end{vmatrix}$$

x_1 times column $n-1$ from column n,
x_1 times column $n-2$ from column $n-1$.

$$a_{ij} = (x_i^{j-1} - x_1^{j-1}) - (x_1 x_i^{j-2} - x_1^{j-1}) = (x_i - x_1)x_i^{j-2}$$

$$V_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & x_2-x_1 & (x_2-x_1)x_2 & & (x_2-x_1)x_2^{n-2} \\ 0 & x_3-x_1 & (x_3-x_1)x_3 & & (x_3-x_1)x_3^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_n-x_1 & (x_n-x_1)x_n & & (x_n-x_1)x_n^{n-2} \end{vmatrix}$$

$$V_n = \prod_{k=2}^{n}(x_k - x_1)\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & x_2 & & x_2^{n-2} \\ 0 & 1 & x_3 & & x_3^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & x_n & & x_n^{n-2} \end{vmatrix} = \prod_{k=2}^{n}(x_k - 1)\begin{vmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{vmatrix}$$

$$V_n = \prod_{k=2}^{n}(x_k - x_1) V_{n-1}$$

V_2, by the time we get to it (it will concern elements x_{n-1} and x_n)

$$V_2 = \begin{vmatrix} 1 & x_{n-1} \\ 1 & x_n \end{vmatrix} = x_n - x_{n-1}$$

So, $|A| = \prod_{1 \le i < j \le n}(x_j - x_i)$

x_i $(1 \le i \le n)$ are all different.

If $\det A \neq 0$, there is matrix B that $BA = I$ and $AB = I$ which is A^{-1}.

so, $Aw = y$

$A^{-1}Aw = A^{-1}y$

$w = A^{-1}y$

Let A^+ be a Pseudo inverse of A and A is $n\times(d+1)$ vandermonde matrix $(n>d+1)$

By the definition of Pseudo inverse, following properties hold.

- $AA^+A=A$
- $A^+AA^+=A^+$
- $A^+=(A^TA)^{-1}A^T$

So, if column vectors of A are linearly independent, w can be compute $w=A^+y$

$$\begin{pmatrix} Aw=y \\ A^+Aw=A^+y \\ (A^TA)^{-1}A^TAw=A^+y \\ w=A^+y \end{pmatrix}$$

A is $n\times(d+1)$ vandermonde matrix, so

By the SVD. $A=U\Sigma V^T$ (U is $n\times n$ orthogonal matrix,
V is $(d+1)\times(d+1)$ orthogonal matrix
Σ is $n\times(d+1)$ diagonal matrix)

$$\Sigma=\underbrace{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & \vdots & \ddots & \sigma_{d+1} \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{n}\Bigg\}n$$ (σ_i are singular value)

$$A^+=V\Sigma^+U^T, \quad \Sigma^+=\begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & & \vdots \\ \vdots & & \ddots & 1/\sigma_{d+1} \\ 0 & \cdots & & 0 \end{bmatrix}$$

and $$A=\begin{bmatrix} x_1^0 & x_1 & x_1^2 & \cdots & x_1^n \\ x_2^0 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ x_n^0 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

So let assume that $c_0v_0+c_1v_1+\cdots+c_nv_n=\vec{0}$, where $v_j=(x_0^j, x_1^j, \cdots, x_n^j)$ is the j-th column written as a vector and $c_0,\cdots,c_n\in R$

Then we can get the k-th coordinate

$$c_0+c_1x_k+c_2x_k^2+\cdots+c_nx_k^n=0$$

which means that x_k is a root of the polynomial $p(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n$

Now if the polynomial $p(x)$ of degree at most n has $(n+1)$ different roots $x_0, x_1, \cdots, x_n$, it must be the zero polynomial and we get that $c_0=c_1=\cdots=c_n=0$.

So, the vectors $v_0, v_1, v_2, \cdots, v_n$ are linearly independent.

Therefore, $w=A^+y$