Implications of Pareto efficiency for two-agent (household) choice

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ABSTRACT

We study when two-member household choice behavior is compatible with Pareto optimality. We ask when an external observer of household choices, who does not know the individuals' preferences, can rationalize the choices as being Pareto-optimal. Our main contribution is to reduce the problem of rationalization to a graph-coloring problem. As a result, we obtain simple tests for Pareto optimal choice behavior. In addition to the tests, and using our graph-theoretic representation, we show that Pareto rationalization is equivalent to a system of quadratic equations being solvable.

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1. Introduction

We study the implications of *efficiency* for *choice behavior*. Concretely, for a given collective of agents, we want to describe the choices that are consistent with Pareto optimality, when we are ignorant about the individual agents' preferences. We develop a series of simple necessary conditions (tests) for choice to be consistent with efficiency; the tests lead up to a necessary and sufficient condition.

Consider a household (a two-person collective) that has to select an alternative from a finite set. For instance, a couple may have to decide how to spend their income, or two researchers who work at the same laboratory may have to choose which projects to fund with their common grants. We focus on the two agent case here because it is the simplest and most primitive environment in which to study Pareto efficient choice. The two-person problem entails a significant simplification because if, say, alternatives a and b are Pareto efficient, and we determine that one agent ranks a over b, then we pin down the other agent's preferences to ranking b over a. This is not the case in the general n person problem. In a sense, there is a "dimension reduction" that is very helpful in the two-agent case. We imagine that the main applications of our results are to household choice and, as we explain below, to the study of partial orders of dimension 2.

Suppose we are given the households' choice behavior. That is, we know what the household would choose from each possible subset of alternatives. We ask when the observed choice can be *Pareto rationalized*: we want to know when we can find two preference relations, one for each household member, such that for any given subset of alternatives, the choices are exactly the Pareto optimal alternatives within the set. We are interested in what the choice behavior of such a household looks like, and how we can test if a given household's choices can be rationalized.

Pareto rationalization depends on certain conditions that the collective choices must satisfy. Some conditions are classical: For example, if an alternative x is selected from a set A, and we shrink A to $B \subset A$ while maintaining x as still available in B, then x must be chosen from B; this condition is usually called Chernoff's Axiom. The classical conditions allow us to work with two binary relations: revealed Pareto domination R and Pareto indifference O. Pareto rationalization requires that there be two individual preferences $>_1$ and $>_2$ such that xRy if and only if x Pareto dominates y, and xOy if and only if x and y are Pareto incomparable. We present new necessary and sufficient conditions for Pareto rationalization.

The problem is motivated by revealed preference theory. In revealed preference theory, one wants to know when the choice behavior of an individual agent is rational. By observing his choice from pairs of alternatives, we can infer the agent's preferences and then judge whether his behavior is rational or not according to some external criterion (e.g. whether his preferences are acyclic, transitive, etc.)

We consider instead the collective behavior of two agents. Their choices reveal their collective preferences, but we cannot immediately reconstruct their individual preferences, and therefore cannot judge whether they are rational or not. To make it harder, the observed group preferences, R and O, need not satisfy some classical rationality conditions.

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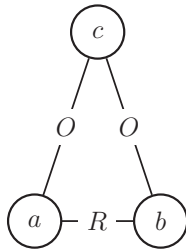E-mail address: fede@hss.caltech.edu (F. Echenique).

Fig. 1. (ac, bc) aRb or bRa, aOc and cOb.

Our main results are for the case where choices from all possible budgets are observable. One can use our results in the case when that is not possible; in Section 6 we comment on how they can be used, and present a simple example with partial observability.

We present the paper in terms of rationalizing choice, but one can also phrase the results as a contribution to dimension theory: a partial order $\succ$ on a set X has dimension N if there are n linear orders $>_i$, $i = 1, \ldots n$ such that

$$x \succ y \Leftrightarrow \forall i (x >_i y).$$

We focus on the case $n = 2$. The problem was first studied by Dushnik and Miller (1941), who provide a characterization of the two-agent rationalizable orders. The problems of dimensions $n > 2$ are open.[1] The Dushnik–Miller result is a deep result about partial orders, but it does not constitute a useful test for rationalizability because it is non-constructive. The characterization is in terms of a property of the partial order that cannot be verified constructively.

In the economic literature, Sprumont (2001) has studied two-agent Pareto rationalizability when one is interested in preferences with a particular structure.[2] Sprumont works with a continuum of alternatives, and studies Pareto two-agent rationalizations with "regular" preferences. We focus instead on the discrete case, where we put no structure on agents' preferences (other than rationality). We shall work with strict preferences in the main discussion of our results, but in the final section of the paper we show that household choices are rationalizable in strict preferences if and only if they are rationalizable in nonstrict preferences.

The following observation is key in our analysis. Consider a pair of alternatives, (x, y). One of two things can happen: either one alternative is revealed Pareto preferred to the other, or the two alternatives are not Pareto ranked. In the first case, if x is chosen out of the set $\{x, y\}$, then we write xRy, and in the second case we write xOy. The first case is simple: there is no ambiguity as any rationalizing preferences must coincide with the Pareto order; if xRy and $>_1$, $>_2$ are rationalizing preferences, then $x >_1 y$ and $x >_2 y$. The second case presents us with a choice. If xOy then either $x >_1 y$ and $y >_2 x$ or $y >_1 x$ and $x >_2 y$. In the first case, the direction of individual preference is determinate. In the second case we have a degree of freedom and we cannot infer individual preferences from the Pareto ranking. This degree of freedom makes the problem of Pareto rationalization substantially different from classical individual revealed preference theory.

We represent the problem in graph-theoretic terms. We think of each alternative as a vertex in a complete graph. For any two alternatives x and y, the edge xy is labeled with R or O. The key building block of our analysis is in Fig. 1, where aRb, aOc and cOb.

It is easy to see that if $(>_1, >_2)$ is a rationalization then either c is the *best* or the *worst* alternative for $>_1$ out of the set $\{a, b, c\}$. Furthermore, if c is the best for $>_1$, then it is the worst for $>_2$, and vice versa. So one agent's preference points towards c and the other agent's away from c.

A graph contains (in principle) many configurations like the one in Fig. 1. If we are trying to build a rationalization $(>_1, >_2)$ then the decision we make on one such configuration, in terms of whether agent 1 or agent 2 prefers c, affects the decision we make on others. We reduce this problem to one of *graph bi-coloring*: one color for the triplet (ac, bc) represents agent 1 preferring c out of $\{a, b, c\}$; the other color represents agent 2 preferring c out of $\{a, b, c\}$.

This graph coloring approach allows us to formulate a simple test (a necessary condition) for Pareto rationalizability. The test is based on the two-coloring rationality condition being equivalent to the absence of certain odd cycles; this test is easy to implement. Passing the test is, unfortunately, not sufficient to guarantee Pareto rationalizability. We present another necessary condition for choice behavior to be Pareto rationalizable. The condition is based on solving a particular quadratic system of equations where each variable can only take the values 1 and -1. This condition, together with the test and two classical revealed preference axioms, is sufficient for choice behavior to be Pareto rationalizable.

The rest of the paper is organized as follows. Section 2 presents some preliminary results. In Section 3 we present the graph-theoretic notions we employ. Section 4 contains our main results. We remark on the extension of our results to non-strict preferences in Section 5. We present some conclusions in Section 7.

2. Preliminaries

2.1. Definitions

Let X be a nonempty, finite set of alternatives. We call a nonempty subset $B \subset X$ a *budget*. Denote by $\mathcal{P}(X)$ the set of all budgets $B \subset X$. A *choice function* is a function

$$g : \mathcal{P}(X) \to \mathcal{P}(X)$$

such that $g(B) \subset B$ for all $B \in \mathcal{P}(X)$.

A *strict preference relation* on X is a total, antisymmetric and transitive binary relation on X (a linear order).

Given two preference relations, $>_1$ and $>_2$, and a budget $B \in \mathcal{P}(X)$, say that $a \in B$ is *Pareto dominated in B with respect to $>_1$ and $>_2$* if there is some $b \in B$ such that $b >_1 a$ and $b >_2 a$; in this case we say that b *Pareto dominates* a (with respect to the two preferences $>_1$ and $>_2$). We call $a \in B$ *Pareto efficient in B* if it is not Pareto dominated in B. Observe that if a and b are Pareto efficient in some set $B \supseteq \{a, b\}$, then either $a >_1 b$ and $b >_2 a$, or $b >_1 a$ and $a >_2 b$

A choice function g is *two-agent Pareto rationalizable* if there are two strict preference relations $>_1$ and $>_2$, such that for all $B \in \mathcal{P}(X)$, $f(B)$ is the set of all Pareto efficient alternatives in B with respect to $>_1$ and $>_2$. In this case, we say that $(>_1, >_2)$ is a *Pareto rationalization* of f, or a *rationalizing pair*.

A pair of sets (V, E) is a (directed) *graph* whenever $E \subset V \times V$. We say that the elements of V are the vertexes of the graph, and that $(v, v') \in E$ means that there is an edge pointing from v to v'.

2.2. Preliminary results

We present two axioms that are necessary for Pareto rationalization. They are standard axioms in the literature, and they are known to be necessary for single-agent rationalization as well (see, e.g. Moulin (1991)). The results in Lemmas (1) and (2) are also standard.

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[1] Dimension theory was introduced by Dushnik and Miller in their 1941 article. A large literature on dimension theory has been developed: the book by Trotter (2001) is a recent exposition.

[2] See also Sprumont (2000), who works out the relationship between Pareto and Nash rationalizability.

Axiom 1. For all $B_1, B_2 \in \mathcal{P}(X)$, if $B_1 \subset B_2$ and $a \in B_1 \cap f(B_2)$, then $a \in f(B_1)$.

The interpretation of Axiom 1 is that if f is Pareto rationalizable, and a is a Pareto efficient choice among the alternatives in a given budget, it must remain Pareto efficient among the alternatives of any smaller budget that contains a. Axiom 1 is a standard axiom in choice theory, usually called Chernoff's axiom (Moulin, 1991). We obtain the following version of this standard result

Lemma 1. *Axiom 1 is necessary for Pareto rationalizability.*

Proof. Suppose that f is Pareto rationalizable and that $(>_1, >_2)$ is a rationalization. By definition, we have that if $a \in B_1 \cap f(B_2)$ for some $B_1 \subset B_2$, then a is not dominated by any $b \in B_2$ (with respect to $(>_1, >_2)$). Since $B_1 \subset B_2$ it follows that a is not dominated by any $b \in B_1$. Thus, a is Pareto efficient in B_1 and $a \in f(B_1)$, as $f(B_1)$ is the set of all Pareto efficient alternatives in B_1 with respect to the rationalization $(>_1, >_2)$. This proves the lemma. □

Axiom 2. For all $B_1 \in \mathcal{P}(X)$, if $a \in B_1 \setminus f(B_1)$, then for some $b \in f(B_1)$ [3]

$$f(ab) = b.$$

The interpretation of Axiom 2 is that if f is Pareto rationalizable, and a is not Pareto efficient among the alternatives in a given budget, then there must exist an efficient alternative in this budget which Pareto dominates it. The following lemma is essentially in Moulin (1991), page 306:

Lemma 2. *Axiom 2 is necessary for Pareto rationalizability.*

Proof. Suppose that $(>_1, >_2)$ is a rationalization and $a \in B_1 \setminus f(B_1)$ for some $B_1 \in \mathcal{P}(X)$. Then a is not Pareto efficient in B_1 with respect to $(>_1, >_2)$. If follows that a is Pareto dominated in B_1 by some other alternative $a_1 \in B_1$. If $a_1 \notin f(B_1)$ then $a_1 \in B_1 \setminus f(B_1)$ and by the same argument a_1 is Pareto dominated in B_1 by some other alternative $a_2 \in B_1$. And so on, since the set B_1 is finite and rationalizing preferences are transitive, we will eventually find an alternative $b \in f(B_1)$ that Pareto dominates a, and hence b is the only Pareto efficient alternative in the budget ab: $f(ab) = b$, $b \in f(B_1)$. □

We next introduce two binary relations. Given is a choice function f. The *strict revealed preference relation* associated to f is the binary relation R defined as aRb if $f(ab) = a$.[4] The *indifference relation* associated to f is the binary relation O defined as aOb and bOa if $f(ab) = ab$. Note that O is symmetric.

The following well-known (Moulin, 1991) observation illustrates the importance of R.

Lemma 3. *If f satisfies Axioms 1 and 2, R is transitive.*

Proof. Assume that f satisfies both axioms and aRb, bRc. Using Axiom 1 twice we get $b \notin f(ab) \Rightarrow b \notin f(abc)$ and $c \notin f(bc) \Rightarrow c \notin f(abc)$. Since $f(abc)$ is nonempty it follows that $f(abc) = a$. Now Axiom 2 implies that $f(ac) = a$ as a is the only alternative in $f(abc)$. Thus aRc and the relation R is transitive. □

Remark. Note that O may not be transitive, as the following example with $X = \{a, b, c\}$ illustrates:

$$c >_1 a >_1 b,$$
$$b >_2 c >_2 a;$$

where aOb and bOc but cRa.

We henceforward assume that the choice function f satisfies Axioms 1 and 2 and so R is transitive. We are interested in when a pair of linear orders $(>_1, >_2)$ is a rationalization of the choice function g, given that it satisfies both axioms. Suppose first that $(>_1, >_2)$ is a rationalization. Note $f(ab)$ is either a singleton or $f(ab) = ab$. If $f(ab) = a$ is a singleton then $a >_1 b$ and $a >_2 b$ as a is the only efficient element in ab with respect to $(>_1, >_2)$. If $f(ab) = ab$, then both a and b must be efficient, meaning that

$$((a >_1 b) \text{ and } (b >_2 a)) \quad \text{or} \quad ((a >_2 b) \text{ and } (b >_1 a)). \tag{1}$$

We arrive at the following important observation.

Lemma 4. *Suppose the choice function f satisfies Axioms 1 and 2. Then a pair of binary relations $(>_1, >_2)$, over the universal set X, is a rationalization of f if and only if $a >_1 b$ and $a >_2 b$ whenever aRb, (1) holds whenever aOb, and both relations are acyclic.*

Proof. We already proved the forward direction of the claim. Conversely, suppose that $(>_1, >_2)$ is a pair of acyclic relations such that $a >_1 b$ and $a >_2 b$ whenever aRb, and (1) holds whenever aOb. By these assumptions, $>_1$ and $>_2$ are total and antisymmetric as well. Since both relations are acyclic, it follows that they must also be transitive and hence linear orders.

Consider a budget $B \in \mathcal{P}(X)$ and an alternative $b \in B \setminus f(B)$. By Axiom 2, $f(ab) = a$ for some $a \in f(B)$. That is, aRb. Then, by assumption, $a >_1 b$ and $a >_2 b$, and so b is Pareto dominated in B with respect to the pair $(>_1, >_2)$. On the other hand, take any $a \in f(B)$ and suppose that for some $c \in B$, $c >_1 a$ and $c >_2 a$. Since both relations agree over the pair of alternatives (a, c), our assumptions imply that cRa. That is, $a \notin f(ac)$. But $a \in f(B) \cap \{a, c\}$ and by Axiom 1 $a \in f(ac)$, a contradiction. Thus, $c >_1 a$ and $c >_2 a$ cannot simultaneously hold for any $c \in B$, which implies that a is not Pareto dominated in B, i.e., a is Pareto efficient in B with respect to the pair of relations $(>_1, >_2)$. Since $a \in f(B)$ and $b \in B \setminus f(B)$ are generic elements of the corresponding sets, we conclude that $f(B)$ contains the Pareto efficient alternatives in B and only those. Hence, $(>_1, >_2)$ is a Pareto rationalization of the choice function f. □

Axioms 1 and 2 are necessary. We present an example below to the effect that they are not sufficient. By Lemma 4, then, it should be clear that property (1) captures the restrictions in Pareto rationalizability, in addition to the standard properties in Axioms 1 and 2. In the next section we shall use a graph-theoretic structure to understand property (1) better.

3. A graph coloring problem

We proceed to illustrate how we translate the problem of characterizing Pareto rationalizability into a graph-coloring problem. Suppose that $(>_1, >_2)$ is a rationalization of the choice function f. By Lemma 4, we know this is equivalent to $(>_1, >_2)$ being a pair of acyclic relations such that $a >_1 b$ and $a >_2 b$ whenever aRb, and (1) holds whenever aOb. We can rewrite these two properties of $(>_1, >_2)$ as

if $a >_1 b$ and aRb, then $a >_2 b$; if $a >_1 b$ and aOb, then $b >_2 a$. (2)

As a consequence, a rationalizing relation $>_1$ defines a rationalization $(>_1, >_2)$. A choice function g defines R and O; so $>_1$ and (2) gives us $>_2$. This simple observation will allow us to translate characterizing Pareto rationalizability into a graph-coloring problem.

Let G be the directed graph

$$G = (X, \{ab | aRb \text{ or } aOb\}).$$

The vertices of G are all the alternatives from the universal set X and the edges of G represent all the revealed binary comparisons of alternatives in X: we label the edge (a, b) by R if aRb or bRa, and we

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[3] We are going to abuse notation and write $a_1 a_2 \ldots a_t$ instead of $\{a_1, a_2, \ldots, a_t\}$.
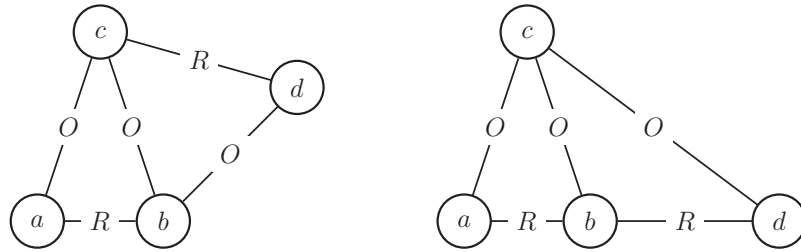[4] The relation R is also called the base relation.

Fig. 2. (Left) (ac, bc) and (cb, db) are dichromatic and (right) (ac, bc) and (bc, cd) are monochromatic.

label the edge (a, b) by O if aOb or bOa. For example, imagine three alternatives a, b and c for which aOc, cOb and aRb. The situation is represented in Fig. 1 in Section 1.

The configuration in Fig. 1 is crucial. This configuration would not be possible if R described a single agent's strict preference relation, and O described her indifference relation. With a Pareto rationalization $(>_1, >_2)$, we see that $>_1$ must either point *away from c* or point *towards c*:

Lemma 5. *Suppose that* $(>_1, >_2)$ *is a rationalization and $a, b, c \in X$ are three alternatives for which aOc, cOb, and aRb or bRa. Then the following is true*

$$(a >_1 c \text{ and } b >_1 c) \text{ or } (c >_1 a \text{ and } c >_1 b). \tag{3}$$

Proof. Let aRb. We cannot have $b >_1 c$ and $c >_1 a$, as this results in the cycle $a >_1 b >_1 c >_1 a$, contradicting the assumption that $>_1$ is acyclic (we have $a >_1 b$ and $a >_2 b$ since aRb). Similarly, we cannot have $a >_1 c$ and $c >_1 b$ because this implies $b >_2 c$ and $c >_2 a$, which contradicts the acyclicity of $>_2$. Two possibilities remain: either $a >_1 c$ and $b >_1 c$, or $c >_1 a$ and $c >_1 b$. That is, (3) holds. We readily see that, after relabeling a and b, (3) also holds if we have bRa instead of aRb. □

Remark. Statement (3) in Lemma 5 clearly holds for $>_2$ as well; and if one of the alternatives in (3) holds for $>_1$, the other holds for $>_2$.

Given that (3) holds for any rationalization $(>_1, >_2)$ and any three alternatives $a, b, c \in X$ with aOc, cOb, and aRb or bRa, we seek to understand the structure of the graph G. Any necessary condition for G that we hope to derive is indirectly a condition for the choice function f, as G depends on the revealed preference relations R and O, which, in turn, correspond to choice functions satisfying Axioms 1 and 2.

We can express our observation (3) using a second graph, which is undirected:

$$F = (\{ab | a \neq b\}, \{(ac, bc) | aRb, aOc, bOc\}).$$

Note that the vertexes of F are edges of G; there is an edge between ac and bc if they are in a relation like the one in Fig. 1: they are elements of O which are related by R.

We say that the edge $f = (ac, bc) \in F$ is *colored* 1 if $(a >_1 c$ and $b >_1 c)$, and that f is *colored* -1 if $(c >_1 a$ and $c >_1 b)$. Consider Fig. 1: the edge (ac, bc) is colored 1 when $>_1$ points *away* from the common vertex c; the edge is colored -1 when $>_1$ points *t*oward the common vertex. Since the assumed rationalization $(>_1, >_2)$ verifies equation (3) by Lemma 5, it follows that it induces a coloring of every edge of the graph F. We now obtain a simple necessary condition for rationalizability based on this coloring.

Consider two adjacent edges $f_1, f_2 \in F$; let $f_1 = (ac, bc)$, for alternatives a, b, and c such that the vertex bc of F is an endpoint of the edge f_2. We have $f_2 = (cb, db)$ or $f_2 = (bc, dc)$ for some other alternative d. The first possibility is represented on the left in Fig. 2 while the second possibility is on the right. When two adjacent edges

$f_1 = (ac, bc)$, $f_2 = (cb, db)$ are related as in Fig. 2 on the left, we say that they are *dichromatic* as they must have different colors in any rationalization. When $f_1 = (ac, bc)$, $f_2 = (bc, dc)$ are related as in Fig. 2 on the right, we say they are *monochromatic* as they must have the same color.

Formally, we say that a pair of edges (f_1, f_2), of the graph F is *monochromatic* when $f_1 = (ac, bc)$ and $f_2 = (bc, dc)$ for distinct alternatives $a, b, c, d \in X$. When $f_1 = (ac, bc)$ and $f_2 = (cb, db)$ for distinct alternatives $a, b, c, d \in X$, we call the pair of edges (f_1, f_2) *dichromatic*.

Lemma 6. *Let* $(>_1, >_2)$ *be a rationalization and f_1, f_2 be two adjacent edges in the graph F. If they are monochromatic* (*related as on the left in Fig. 2*) *then the two edges have different colors. If f_1, f_2 are dichromatic* (*related as on the right*), *then they have the same color.*

Proof. Consider first the case $f_1 = (ac, bc)$, $f_2 = (cb, db)$. If the edge f_1 is colored -1 then $c >_1 b$, and by equation (3) $d >_1 b$ so that f_2 is colored 1. Similarly, if f_1 is colored 1 then $b >_1 c$, and by (3) $b >_1 d$: f_2 is colored -1. Either way, f_1 and f_2 have different colors. The second case $f_1 = (ac, bc)$, $f_2 = (bc, dc)$ is treated the same way. □

These ideas allow us to formulate a simple condition.

Axiom 3. Every cycle in F has an even number of dichromatic pairs.

We have argued that a rationalization implies a coloring of the edges of F in which monochromatic pairs have the same color, and dichromatic pairs have opposite colors. Coloring using two colors (bicoloring) is equivalent to the absence of a cycle in which there is an odd number of vertexes forcing a switch in the color of adjacent edges (a simple result in graph theory). In our particular case, the vertex linking two adjacent edges of the graph F forces a switch in color if they are dichromatic and forces them to have the same color if they are monochromatic. We get for free the following

Lemma 7. *If f is Pareto rationalizable, it must satisfy Axiom 3.*

We finish this section with an example of a choice function that satisfies Axioms 1 and 2, but not 3. Such a function is not two-agent Pareto rationalizable.

Example. Let $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be the set of alternatives and define the choice function g by the following table; the entry corresponding to row a_i and column a_j is $g(\{a_i, a_j\})$.

	a_1	a_2	a_3	a_4	a_5	a_6
a_1		a_1	$a_3 a_1$	$a_1 a_4$	a_1	$a_1 a_6$
a_2			$a_2 a_3$	a_4	$a_2 a_5$	$a_2 a_6$
a_3				a_4	$a_3 a_5$	a_6
a_4					$a_4 a_5$	$a_4 a_6$
a_5						a_6

The table defines the relations R and O. It is then easy to extend g to all of $\mathcal{P}(X)$ so that it satisfies Axioms 1 and 2.

Notice next that $f_1 = (a_1 a_3, a_2 a_3)$, $f_2 = (a_3 a_2, a_6 a_2)$, $f_3 = (a_6 a_2, a_5 a_2)$, $f_4 = (a_2 a_5, a_4 a_5)$, $f_5 = (a_4 a_5, a_3 a_5)$, and $f_6 = (a_5 a_3, a_1 a_3)$ are edges of the graph F such that f_1, f_2 are dichromatic, f_2, f_3 are monochromatic, f_3, f_4 are dichromatic, f_4, f_5 are monochromatic, f_5, f_6 are dichromatic, and f_6, f_1 are monochromatic. Then $f_1 f_2 f_3 f_4 f_5 f_6 f_1$
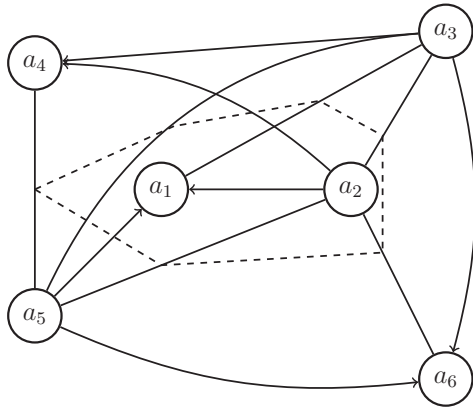
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Fig. 3. A choice function satisfying Axioms 1 and 2 but not 3.

is a cycle in F which has an odd number (three) of dichromatic pairs. See Fig. 3, in which the edges of G are drawn with continuous lines and the edges of F are drawn as dotted lines. In Fig. 3, an arrow at the end of an edge indicates the direction of R. The absence of any edge or arrow indicates that the edge corresponds to O.

The choice function g then satisfies Axioms 1 and 2, but not Axiom 3.

4. Main results

Our main result is a characterization of Pareto rationalizability. We express the characterization as a graph coloring problem in Theorem 1. We present the same characterization in Theorem 2, in terms of a system of quadratic equations having a solution. The version of the result in Theorem 2 may be computationally the most convenient.

In Section 3, we reduced the problem of Pareto rationalization to a problem of bi-coloring the graph F. Our characterization of Pareto rationalization requires paying special attention to certain configurations of edges in F.

We say that a triple of edges (f_1, f_2, f_3) of the graph F is *3-cyclic* provided

$$f_1 = (ca, ea), \qquad f_2 = (cb, eb), \qquad f_3 = (ba, da),$$

for some distinct alternatives a, b, c, d, $e \in X$ with dOe, dOc. Notice that $(f_1, (cd, ed), f_3)$, $(f_3, (bc.dc), f_1)$ and $(f_3, (be, de), f_1)$ are 3-cyclic triples as well and that the ordering of the edges in a 3-cyclic triple matters.

The role of 3-cyclic edges becomes apparent in the following

Theorem 1. *A choice function is Pareto rationalizable if and only if it satisfies Axioms 1–3, and there is a coloring of the edges of the graph F (in 1 and -1) such that two edges in a monochromatic pair have the same color, two edges in a dichromatic pair have different colors, and for every 3-cyclic triple (f_1, f_2, f_3) either f_1 and f_2 have the same color or f_1 and f_3 have the same color.*

Proof. Let g be a choice function that satisfies the hypotheses. We are going to construct a Pareto rationalization $(>_1, >_2)$ of g. The construction starts by defining the two binary relations $>_1$ and $>_2$ over only some pairs of alternatives from X at first, and then extending them to the entire set X. For each pair of alternatives (a, b) with aRb, define $a >_1 b$ and $a >_2 b$. For each pair (a, b) define $a >_1 b$, $b >_2 a$ if for some $c \in X$ it is the case that $(ab, cb) \in F$ is colored 1 or $(ba, ca) \in F$ is colored -1. The pair $(>_1, >_2)$ is currently only partially defined. □

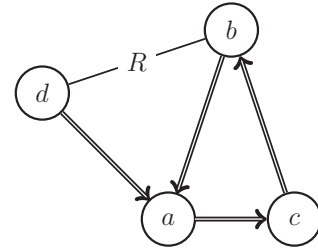We present a trivial fact that will be invoked often in the sequel:

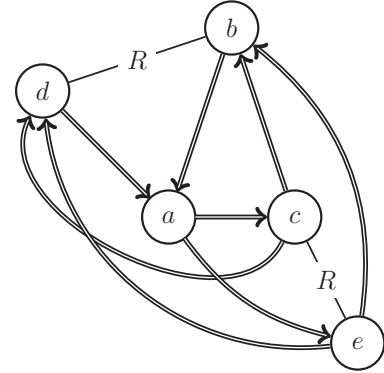**Fig. 4.** A cycle for $>_1$, indicated with directed edges.

Fig. 5. Implied existence of d and e driving the cycle.

Lemma 8. *Let x, y, $z \in X$ be three distinct alternatives. If $x >_1 y >_1 z$ and $xOyOz$, then xOz.*

We illustrate the rest of the argument using Figs. 4–6. Suppose there is a $>_1$-3-cycle $a >_1 b >_1 c >_1 a$ for some a, b, $c \in X$. It is easy to see that we must have $aObOcOa$: if all three of the revealed relations belong to R then R is not acyclic, a contradiction; if exactly two of the relations belong to R then the third must also since R is transitive, a contradiction; if only one of the relations belongs to R then we have not colored each edge of the graph F.

We represent the situation in Fig. 4; we omit the label on O edges to simplify the figures. The bold edges with arrows indicate the preference $>_1$; for example, the preference $a >_1 b$ mandates an arrow $\Longrightarrow$ on the ab edge pointing in the direction of a.

Now, aOb and $a >_1 b$ implies that there is $d \in X$ such that either $(ba, da) \in F$ or $(ab, db) \in F$. Suppose, without loss of generality, that $(ba, da) \in F$ is the case. Then aOd and $a >_1 d$ as (ba, da) is colored -1; see Fig. 4. Now consider the edge cd. By Lemma 4 we must have cOd. Hence (bc, dc) must be an edge in F colored in 1. This means that $d >_1 c$. (We have assumed that $(ba, da) \in F$, but if we instead assume
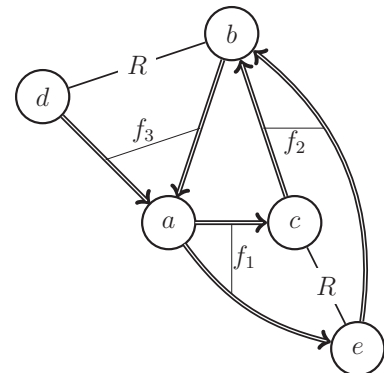

Fig. 6. $f_1 = (ca, ea)$, $f_2 = (cb, eb)$ and $f_3 = (ba, da)$.

that $(ab, db) \in F$, we end up with the same picture after relabeling a, b and c.)

Next, cOa and $c >_1 a$ imply the existence of some some $e \in X$ such that either $(ca, ea) \in F$ or $(ac, ec) \in F$. Suppose without loss of generality that $f_1 = (ca, ea) \in F$. Then eOa and $e >_1 a$, as (ca, ea) is colored with 1. See Fig. 5.

Note that $(ca, ea) \in F$ implies that cRe or eRc. We invoke Lemma 4 with eOa, aOb, $e >_1 a$, and $a >_1 b$ to get eOb and bOe. Hence, $f_2 = (cb, eb) \in F$; in addition, $b >_1 c$ implies that f_2 is colored -1 and thus $b >_1 e$. See Fig. 5. Now, dOe as an R relation is not possible by Lemma 4. Then, $(dRb) \vee (bRd))$ implies that $(de, be) \in F$, and $b >_1 e$ implies that (de, be) is colored 1. Hence, $d >_1 e$. The case $(ac, ec) \in F$ leads to the same picture after relabeling a, b and c.

Consider the edges $f_1 = (ca, ea), f_2 = (cb, eb)$ and $f_3 = (ba, da)$ which all belong to the graph F. Notice that f_3 colored -1, f_2 colored -1, and f_1 colored 1. In addition, we have dOe, dOc. This means that the triple of edges (f_1, f_2, f_3) is 3-cyclic with f_1 and f_2 having opposite colors, and f_1 and f_3 having opposite colors. We have reached a contradiction with the assumptions of the lemma. Hence, there is no 3-cycle belonging to the (not yet total) relation $>_1$. The situation is represented in Fig. 6.

Suppose there is a 3-cycle $a >_2 b >_2 c >_2 a$ which belongs to $>_2$. As for $>_1$, it must be the case that $aObOcOa$. But then $a >_1 c >_1 b >_1 a$, a contradiction.

Suppose next that $>_1$ is not acyclic: for some $a_1, a_2, \ldots, a_t \in X$, $t > 4$ we have $a_1 >_1 a_2 >_1 \ldots >_1 a_t >_1 a_1$. We may assume that this is the shortest cycle, meaning that t is minimal. If $a_1 R a_3$ then there is a shorter cycle $a_1, a_3, a_4, \ldots, a_t, a_1$, a contradiction; if $a_3 R a_1$, then there is a 3-cycle belonging to $>_1$, again a contradiction. If $a_3 R a_1$ then there is a 3-cycle that belongs to $>_1$, a contradiction. Hence, $a_1 O a_3$. Now suppose that $a_1 R a_2$. If $a_2 R a_3$ then $a_1 R a_3$ as R is transitive, a contradiction. Then $a_2 O a_3$ and $(a_1 a_3, a_2 a_3)$ must be an edge of the graph F that is colored in 1. We conclude that $a_1 >_1 a_3$ and there is a shorter cycle, a contradiction. So $a_1 O a_2$.

This allows us to conclude that $a_1 O a_2 O \ldots O a_t O a_1$. Because $a_1 O a_2$ and $a_1 >_1 a_2$, for some $b \in X$ either $(a_2 a_1, ba_1) \in F$ or $(a_1 a_2, ba_2) \in F$. Suppose that $(a_2 a_1, ba_1) \in F$. Then $a_1 >_1 b$. Now $a_t >_1 a_1$, $a_1 >_1 a_2$ and $a_1 >_1 b$ imply that $a_t O a_2$ and $a_t O b$, by Lemma 4. So $(a_2 a_t, ba_t)$ is an edge in F. If it is colored in -1 we get a 3-cycle $a_1 >_1 a_2 >_1 a_t >_1 a_1$, a contradiction. It should be colored in 1, so $a_t >_1 a_2$, and we once again have a shorter cycle, a contradiction. The case $(a_1 a_2, ba_2) \in F$ leads to a contradiction in a similar fashion.

Therefore, $>_1$ is acyclic and can be extended to a total, antisymmetric and transitive relation (a preference relation) on X. In light of observation (2) in Section 3, the (extended) preference relation $>_1$ extends $>_2$ to a total and antisymmetric binary relation on X which does not possess any 3-cycles. On the other hand, if a total relation has a cycle, it is easy to see that it has a 3-cycle. It follows that $>_2$ is also acyclic and Lemma 4 implies that $(>_1, >_2)$ is a rationalization of the choice function g.

We finish the proof by proving the converse statement. Let $(>_1, >_2)$ be a rationalization of the choice function g. We have already shown that g must satisfy Axioms 1–3. We have also seen in Lemma 6 that the rationalization $(>_1, >_2)$ induces a coloring of the edges of the graph F that respects di- and monochromatic pairs. Finally, if (f_1, f_2, f_3) is a 3-cyclic triple and we suppose that f_1 and f_2 have opposite colors and f_1 and f_3 have opposite colors, we must conclude that either $a >_1 b >_1 c >_1 a$ or $a >_1 c >_1 b >_1 a$. A contradiction of the transitivity of $>_1$. Hence, f_1 and f_2 have the same color or f_1 and f_3 have the same color.

Remark. The proof of Theorem 1 is constructive. Given a coloring of the edges of the graph F, we described a simple procedure for constructing a rationalization $(>_1, >_2)$ of the given choice function: the rationalization must agree with the revealed relation R, as well

as the coloring of the graph F over pairs of alternatives ab which represent an endpoint of an edge in the graph F. Next, we extend $>_1$ to the entire set X, which also determines $>_2$ as a linear order on X. We showed that the pair $(>_1, >_2)$ constructed in this way is indeed a rationalization.

Theorem 1 is a characterization of Pareto rationalizability. From the results in Section 3, we know that, if a choice function satisfies Axioms 1–3 there is a coloring of the edges of the induced graph F which assigns opposite colors to edges from a dichromatic pair and equal colors to edges from a monochromatic pair. Pareto rationalizability amounts, over and above Axioms 1–3, to finding a coloring of the edges of F, which in addition to respecting di- and monochromatic pairs also has the property that for every 3-cyclic triple (f_1, f_2, f_3) either f_1 and f_2 have the same color or f_1 and f_3 have the same color.

Notice that if there is a coloring of the edges of F that satisfies the conditions in Theorem 1, we can reverse the color of every edge and still have a coloring that satisfies those conditions: the reversed coloring leads to the same rationalizing pair as the original one, with $>_2$ in place of $>_1$. To avoid dealing with this duality, and to simplify our notation, we introduce a third graph H. The vertexes of H are the edges of the graph F, and two vertexes f_1, f_2 in H are joined by an edge in H if and only if there is a path in F connecting f_1 with f_2.

In the graph F, the notions of monochromatic or dichromatic pairs of edges only applied to adjacent edges; that is, edges with a common vertex. We can easily extend this definition to pairs of edges that are connected by a path in F, calling a pair of edges monochromatic if there is a path with an even number of dichromatic pairs joining them, and dichromatic if there is a path with an odd number of dichromatic pairs joining them. The more general notion of mono- and di-chromatic is well-defined under Axiom 3, as Axiom 3 then insures that no pair of edges from F is both dichromatic and monochromatic.

Under Axiom 3, then, we can extend our definitions of dichromatic and monochromatic edges in F to give a bi-coloring of H. We say that the edge $f_1 f_2$ (in H) is colored 1 if the corresponding path (in F) has an even number of dichromatic pairs, and is colored -1 if the path has an odd number of dichromatic pairs.

Now, H is the dual graph of F. It is well-known and easy to see that Axiom 3 is equivalent to

Axiom 3′. Every cycle in H has an even number of edges colored -1.

Observe that a choice function induces a coloring of the *edges* in H just like it labels some pairs of edges in F as di- or monochromatic. We cannot choose the coloring of the edges of H, in contrast to the coloring of F we have discussed above.

We now express the consequences of Theorem 1 for H using a system of equations. A solution to the system of equations will be a coloring of the vertexes of H.

For the graph H, Theorem 1 implies the following. A choice function that satisfies Axioms 1–3 is Pareto rationalizable if and only if there is a coloring of the *vertexes* of H in 1 and -1, such that two vertexes $h_1, h_2 \in H$, such that $h_1 h_2$ is and edge in H, have equal colors if $h_1 h_2$ is colored 1 and opposite colors if $h_1 h_2$ is colored -1; for every 3-cyclic triple of *vertexes* (h_1, h_2, h_3) either h_1 and h_2 have the same color or h_1 and h_3 have the same color.[5] We introduce one variable for each vertex of H; these variables can take the values 1 or -1.

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[5] A triple of vertexes (h_1, h_2, h_3) in H is called *3-cyclic* if and only if that same triple viewed as a triple of edges in F, is 3-cyclic.

Let k be the number of 3-cyclic triples:

$$T_1 = (h_{1,1}, h_{1,2}, h_{1,3}), T_2 = (h_{2,1}, h_{2,2}, h_{2,3}), \ldots, T_k$$
$$= (h_{k,1}, h_{k,2}, h_{k,3}).$$

Remark. As we observed before, 3-cyclic triples come in groups of four. It is sufficient to satisfy the requirement in Theorem 1 only about one 3-cyclic triple out of those four and we may choose to ignore the other three. For theoretical purposes, the redundancy plays no role. For computationally constructing a Pareto rationalization, it would be important to only include one representative 3-cyclic triple.

We proceed to describe a system of equations, named (*), by enumerating all the component equations.

First, include the $3k$ equations

$$h_{i,j}^2 = 1$$

in the system (*). These must be included because $h_{i,j}$ must take the value 1 or -1.

Next, if the same vertex belongs to two different 3-cyclic triples, we have that

$$h_{i_1,j_1} = h_{i_2,j_2}$$

belongs to (*), where h_{i_1,j_1} and h_{i_2,j_2} are the same vertex in H. If h_{i_1,j_1} and h_{i_2,j_2} are two vertexes for which $h_{i_1,j_1}h_{i_2,j_2}$ is an edge in H colored 1, then

$$h_{i_1,j_1} = h_{i_2,j_2}$$

is an equation in the system; if $h_{i_1,j_1}h_{i_2,j_2}$ is an edge in H colored -1, then

$$h_{i_1,j_1} = -h_{i_2,j_2}$$

must be included in the system of equations (*).

Finally, for every $i = 1, 2, \ldots, k$, the equation

$$(h_{i,1}h_{i,2} + h_{i,1}h_{i,3} - 1)^2 = 1$$

belongs to the system (*). Notice that $h_{i,1}$ and $h_{i,2}$ having opposite colors or $h_{i,1}$ and $h_{i,3}$ having opposite colors is equivalent to $h_{i,1}h_{i,2} + h_{i,1}h_{i,3} \neq -2$, which is equivalent to $h_{i,1}h_{i,2} + h_{i,1}h_{i,3} \in \{0, 2\}$. This, in turn, is equivalent to $(h_{i,1}h_{i,2} + h_{i,1}h_{i,3} - 1)^2 = 1$.

Axiom 4. The system of equations (*) has a solution.

Evidently, Axiom 4 is simply a reformulation of the conditions in Theorem 1. So Axiom 4 is both a necessary and sufficient condition for the existence of a coloring of the vertexes of H in the hypotheses of Theorem 1. We thus obtain:

Theorem 2. *The choice function f is Pareto rationalizable if and only if it satisfies* Axioms 1–4.

Unfortunately, the system (*) is not linear, and its solution may present significant computational problems. That said, the construction of the system should be computationally feasible (all the graphs involved can be constructed in polynomial time). One computational advantage of the system is that it does not involve equations of order higher than 2, as $h_{i,j}^2 = 1$ implies that

$$(h_{i,1}h_{i,2} + h_{i,1}h_{i,3} - 1)^2 = 3 + 2h_{i,2}h_{i,3} - 2h_{i,1}h_{i,2} - 2h_{i,1}h_{i,3}.$$

5. Rationalizability in non-strict preferences

We proceed to discuss rationalization by non-strict preferences. We observe that our characterization still holds in this case, when we define Pareto efficiency in the weak sense that an alternative a in B is Pareto efficient if there is no $b \in B$ that both agents strictly prefer.

A choice function g is *two-agent Pareto rationalizable in non-strict preferences* if there are two total, reflexive and transitive binary relations $>_1$ and $>_2$ on the universal set of alternatives X, such that for all $B \in \mathcal{P}(X)$, $f(B)$ is the set of Pareto efficient alternatives in B with respect to $(>_1, >_2)$. In this case, we say that $(>_1, >_2)$ is a *Pareto rationalization of g in non-strict preferences*.

Theorem 3. *The choice function g is Pareto rationalizable in non-strict preferences if and only if it satisfies* Axioms 1–4.

Proof. Suppose g satisfies Axioms 1–4. Then Theorem 2 implies that g is rationalizable by some strict preferences $(>_1, >_2)$, and therefore g is also rationalizable by non-strict preferences. On the other hand, suppose g is rationalizable by nonstrict $(>_1, >_2)$. It is readily seen that Axioms 1–2 must still hold. Also, if aRb, bOc, cOa, then $>_1$ and $>_2$ must be strict on (a, b) and (b, c): if $b =_1 c$ then $b =_2 c$ and it must be the case that aRc since aRb. This contradicts aOc. The remaining Axioms 3 and 4 only deal with these tractable pairs bOc and cOa, over which there are can be no indifferences belonging to rationalizing relations, and it follows that they are also necessary. □

6. Rationalizability with partial observations

We have followed the classical approach in choice theory, and worked with a choice function that is defined on every budget. The results are, however, applicable to situations where there is partial observability of budgets; as is typically the case in empirical applications of revealed preference theory.

With partial observability, one obtains the relations R and O from the data, and one needs to check that R is acyclic (this is the counterpart to checking the strong axiom of revealed preference in classical revealed preference theory). We can then use the construction in the proof of Theorem 1. Specifically, the construction of graphs G and F only require R and O as inputs.

For example, if $X = \{a, b, c, d\}$. Suppose that a Pareto dominates b and d Pareto dominates c; but that we observe that a and c, b and c, and b and d are Pareto unrelated. Notice that we do not observe anything about how a and d are related: hence we face partial observability.

Now the graph G is (X, E) with $E = \{ab, ac, bc, bd, dc\}$. The graph F is (E, P), with $P = \{(ac, bc), (bc, bd)\}$. Note that the pair (ac, bc) and (bc, bd) is dichromatic. If we set (ac, bc) to have color 1, so $a >_1 c$ and $b >_1 c$, we must set (bc, bd) to have color -1, so that $b >_1 c$ and $b >_1 d$. This information, together with the information about R, implies that

$$a >_1 b >_1 d >_1 c.$$

For agent 2 we have the opposite direction of preferences for O edges of G. So we have that $c >_2 a$, $c >_2 b$, and $d >_1 b$; while it is still the case that $a >_2 b$ and $d >_2 c$. So we obtain

$$d >_2 c >_2 a >_2 b.$$

7. Conclusion

We characterize two-agent Pareto rationalizability of choice functions. We present several simple tests for Pareto rationalizabil-

ity, most notably the verification of Axiom 3. This axiom is a new consequence of Pareto rationalizability, and its verification seems to be computationally simple.

Our characterization, on the other hand, essentially involves solving a system of quadratic equations. Thus, applying our characterization may be computationally hard; it may not present a substantial advantage over exhaustively searching over all possible rationalizing linear orderings. For problems of a given size, though, it seems intuitive that the constructions we have discovered (the graph H) simplifies the problem.

Finally, we have worked on the two-agent case. We do not know if our graph-theoretic approach extends to the case when we have more than two agents. The difficulty is that there is no two-coloring which allows us to formulate meaningful, simple graph tests, which seem to lead in the direction of a characterization. The general problem with n agents remains open.

Acknowledgements

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