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Collective risk aversion

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Abstract In this article we analyze the risk attitude of a group of heterogeneous agents and we develop a theory of comparative collective risk tolerance. In particular, we characterize how shifts in the distribution of individual levels of risk tolerance affect the group's attitude towards risk. In a model with efficient risk-sharing and two agents an increase in the level of risk tolerance of one or of both agents might have an ambiguous impact on the collective level of risk tolerance; the latter increases for some levels of aggregate wealth while it decreases for other levels of aggregate wealth. For more general populations we characterize the effect of first-order like shifts (individual levels of risk tolerance more concentrated on high values) and second-order like shifts (more dispersion on individual levels of risk tolerance) on the collective level of risk tolerance. We also evaluate how shifts in the distribution of individual levels of risk tolerance impact the collective level of risk tolerance in a framework with exogenous egalitarian sharing rules. Our results permit to better characterize differences in risk taking behavior between groups and individuals and among groups with different distributions of risk preferences.

1 Introduction

Many decisions to undertake risks are made by groups. A priori, one would expect that the theory of comparative risk aversion developed by Pratt (1964) and Arrow (1971),

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which characterizes the proclivity of individuals to undertake risks, would easily translate into a theory of group risk taking. Consider, for instance, three individuals, *A, B,* and *C*. Suppose that *C* is more risk averse than *B* and *B* is more risk averse than *A*. Intuition strongly suggests that, when acting together, *A* and *C* would be less willing to undertake risks than *A* and *B*. Paradoxically, Mazzocco (2004) showed that such intuition is not always correct. For some levels of wealth an increase in the degree of risk aversion of the most risk averse individual in a group may decrease the collective level of risk aversion.[1] Mazzocco (2004) presented this paradoxical result through a numerical example with two individuals and isoelastic preferences. Our objective in this article is to extend this line of inquiry by establishing precisely the conditions for this phenomenon to occur and, more generally, by evaluating how changes in the distribution of individual preferences affect a group's attitudes towards risk.

To be perfectly clear, the Arrow–Pratt theory of comparative risk aversion does apply to utility functions of groups. So, for example, if a group is more risk averse than another in the Arrow–Pratt sense then this group will also require a larger risk premium to eliminate a fair risk. The interpretation of such comparative statics result, however, is clouded by the following fact. The degree of risk aversion of the group depends upon both the distribution of preferences among the agents and their optimal allocations. So a change in the distribution of preferences impacts the collective level of risk aversion through two channels. A direct one as well as an indirect one due to the fact that changes in the distribution of preferences lead, in turn, to changes in the efficient allocation of wealth. Therefore, if risk is shared efficiently, collective risk aversion has to be determined endogenously.

There is one special case in which the problem greatly simplifies: given an efficient allocation of wealth, if all individuals in the group have a constant and common absolute cautiousness (the derivative of the reciprocal of absolute risk aversion), e.g., under CARA or CRRA utility functions with a common level of relative risk aversion, the group has the same absolute cautiousness (Wilson 1968). Comparative statics of risk aversion at the aggregate level is then not different from comparative statics at the individual level. The assumption of homogeneity in individual preferences, however, does not have empirical support (e.g., Barsky et al. 1997) and, in fact, defeats the purpose of Arrow–Pratt's theory of comparative risk aversion. Therefore, in this article we tackle the problem of comparing attitudes towards risk among groups composed by individuals with heterogeneous risk preferences.

We show, in the setting of Mazzocco's (2004) paper, that the collective level of risk tolerance is a wealth share weighted average of the individual levels of risk tolerance and increasing the risk tolerance level of one agent has two effects: an increase of one of the terms of the average but a possible decrease of its relative weight in the average. As a result, there are two possible shapes for the collective risk tolerance as a function of the risk tolerance level of one of the agents: increasing curve or increasing then decreasing curve.

[1] The fact that efficient groups may behave in a complex manner is well known. Pratt and Zeckhauser (1989) showed, for example, that a group may be willing to accept a gamble which combines two individually unacceptable lotteries.

In fact, we establish the possibility of an even more perplexing situation: An increase in the degree of risk tolerance of *both* members of a couple may decrease their collective degree of risk tolerance.[2] We clearly characterize these different situations in terms of the size of the aggregate endowment relative to the endowment that corresponds to the fair efficient allocation. We also characterize, for the two-agent case and for more general populations, first-order like shifts (individual levels of risk tolerance more concentrated on high values) that have an unambiguous impact on the collective level of risk tolerance.

Since the key aspect of our analysis is preference heterogeneity we also evaluate how more dispersion on the individual levels of risk tolerance (second-order shifts) affects the collective risk preferences. We show that, for high levels of wealth (relative to the level that corresponds to the fair efficient allocation), more heterogeneity tends to increase collective risk tolerance, while the opposite is true for low levels of wealth.

Finally, we extend our analysis to a framework in which all members of a group receive the same endowment (egalitarian groups). This setup is appropriate to analyze situations in which the members of a group derive utility from a public good and situations in which a private good is simultaneously consumed by many individuals. For example, many goods within a household are simultaneously consumed by all the members of a family. Within this framework, and under very general individual preferences, we establish the impact of first- and second-order shifts on the collective level of risk tolerance.

In addition to the work of Mazzocco (2004), our article is closely related to the work of Hara et al. (2007), who studied the properties of collective preferences for a given distribution of individual risk preferences. We extend their analysis by evaluating how changes in the distribution of individual preferences affect the collective attitudes towards risk. In this way, our analysis also complements the work of Gollier (2001, 2007), who explored how heterogeneity in the initial endowment of wealth and how heterogeneity in beliefs affect a group's attitude towards risk. At a more general level, we believe that our results may shed light into the empirical literature on 'choice shifts', which compares decisions made by groups relative to decisions made by the members of the group in situations of uncertainty (e.g., Baker et al. 2008; Shupp and Williams 2008; Masclet et al. 2009), a topic which we further discuss in Sect. 6.[3]

The article proceeds as follows: In Sect. 2 we present the model with efficient risk sharing and we establish a number of useful results about the efficient allocations of endowments and the collective risk preferences. In Sect. 3 we briefly evaluate the case of CARA preferences, which serves as a useful benchmark. In Sect. 4 we analyze the case of isoelastic heterogeneous preferences. After presenting general properties of collective preferences we evaluate shifts in the distribution of individual preferences, first in the case of two agents and then under more general populations. In Sect. 5

[2] We also show, however, that a uniform increase in the degree of risk tolerance of both individuals unambiguously increases the risk tolerance of the group.

[3] In this literature the objective is to elicit the risk attitude of groups as compared to the members of the group. Another strand of related empirical literature evaluates whether, under uncertainty, groups behave in a more consistent manner than individuals (see e.g., Bone et al. 1999; Charness et al. 2007).

we evaluate collective risk preferences for the case of exogenous egalitarian sharing rules, while Sect. 6 concludes. All the proofs are provided in Appendix.

2 The model

We consider a standard static model in which a group of heterogeneous agents consume a single good. The endowment per person in the consumption good is defined by a random variable x on the probability space (Ω, F, P). Agents have a common belief over the probability space. In order to take into account finite as well as infinite sets of agents, the agent space is described by (I, ι, Q), where $I = [0, \infty)$ and Q is a probability measure on I. Individuals are indexed by $i \in I$ and we denote by E^Q the expectation with respect to Q.

We consider a 'consensus' group *à la* Samuelson (1956). That is, the group acts as if there was a social planner who wants to reach a Pareto efficient allocation of risks and solves the following maximization program

$$U(x) = \max_{\int x_i \mathrm{d}Q(i)=x} \int \lambda_i u_i(x_i) \mathrm{d}Q(i). \tag{1}$$

where u_i is the utility function of agent i, x_i is the consumption of agent i, and λ_i is the weight (e.g., decision power) granted to agent i. The utility function $U(x)$ corresponds to the highest social utility level among all possible endowment distributions across agents.

Throughout the article, we make the following assumption on the utility functions.

Assumption *(U)* For all i, the utility function $u_i : [d_i, \infty) \to \mathbb{R} \cup \{-\infty\}$ is assumed to be infinitely differentiable on (d_i, ∞) with $u_i' > 0$ and $u_i'' < 0$ and satisfies Inada's conditions, i.e., $\lim_{x\to d_i} u_i'(x) = \infty$ and $\lim_{x\to\infty} u_i'(x) = 0$.

For a given agent i and a given consumption level x, the absolute (resp. relative) risk aversion $A_i(x)$ (resp. $R_i(x)$), the absolute (resp. relative) risk tolerance $t_i(x)$ (resp. $s_i(x)$) are given by

$$A_i(x) = -\frac{u_i''(x)}{u_i'(x)}, \qquad R_i(x) = -x\frac{u_i''(x)}{u_i'(x)} = xA_i(x)$$
$$t_i(x) = -\frac{u_i'(x)}{u_i''(x)} = \frac{1}{A_i(x)}, \qquad s_i(x) = -\frac{u_i'(x)}{xu_i''(x)} = \frac{1}{R_i(x)} = \frac{t_i(x)}{x}.$$

Note that CARA and CRRA utility funtions clearly satisfy Assumption (U).

If we denote by v the function defined by $v(x, i) = u_i'(x)$, we will also make the following assumption.

Assumption *(LSPM)* The function v is log-supermodular in (x, i) , i.e., $\frac{\partial \log v}{\partial x}(x, i)$ is nondecreasing in i.

Remark that the log-supermodularity of $v(x, i)$ means that $A(x, i) = A_i(x)$ is non-increasing in i or that agent i is less risk averse (and more risk tolerant) than agent j when $i \geq j$.

We have then the following classical result

Proposition 1 *Under Assumption (U), there exists a family of functions $(f_i)_{i\in[0,\infty]}$ such that*

- $f_i : [d, \infty) \to [d_i, \infty)$ *with* $d = \int d_i \mathrm{d}Q(i)$ *is infinitely differentiable and increasing*
- $\int f_i(x)\mathrm{d}Q(i) = x$ *for all* $x \in [d, \infty)$
- $U(x) = \int \lambda_i u_i(f_i(x))\mathrm{d}Q(i)$.

We will say that $(f_i)_{i\in[0,\infty]}$ is an efficient sharing rule associated with the maximization program of Eq. 1.

We recall the following well known results that relate the collective risk aversion and risk tolerance to the individual ones through the efficient sharing rule.

Proposition 2 (Wilson 1968; Hara et al. 2007) *Let us assume that (U) is satisfied. Let x be a given aggregate wealth and let $(f_i)_{i\in I}$ be the efficient sharing rules associated with the maximization program of Eq. 1. The collective absolute risk tolerance $t(x) = -\frac{U'(x)}{U''(x)}$ and the collective relative risk tolerance $s(x) = -\frac{U'(x)}{xU''(x)}$ are given by*

$$t(x) = \int t_i(f_i(x))\mathrm{d}Q(i),$$
$$s(x) = \int \frac{f_i(x)}{x} s_i(f_i(x))\mathrm{d}Q(i).$$

The relative risk tolerance $s(x)$ of the group is then an average of the individual levels of relative risk tolerances $s_i(f_i(x))$ weighted by the optimal individual shares of consumption. Analogously, the degree of relative risk aversion of the group is an average of the individual degrees of relative risk aversion.The group is then less risk averse than the most risk averse agent and more risk averse than the least risk-averse one. In terms of the example given in the introduction this implies, in particular, that a group composed by B and C will always be less willing to undertake risks than a group composed by A and B.

It is easy to show that $s'(x)$ is positive and then that the collective relative risk aversion $R(x) = -x\frac{U''(x)}{U'(x)}$ is decreasing in x. This fact has been underlined by Hara et al. (2007, Proposition 6). They further show (Corollary 7) that $R(x)$ approaches the degree of relative risk aversion of the most (least) risk averse agent as x converges to zero (infinity).

At this stage we consider very general utility functions and we may assume, without loss of generality, that all the members of the group are granted the same weight (it suffices to replace the utility function u_i by $\lambda_i u_i$, note that the LSPM property is not impacted by this modification). In the next we consider then the equally weighted Pareto optimum. We also assume that there exists an efficient fair allocation. In other words, there exists x^* such that $(x_i)_{i\in I}$, with $x_i = x^*$ for all i, is efficient.

The following proposition provides an analysis of how the aggregate consumption x is shared among the agents depending on the position of x relatively to the fair efficient allocation x^*

Proposition 3 *Under the Assumptions (U) and (LSPM), we have the following results.*

1. *For $x \geq x^*$, the optimal allocation $(x_i)_{i \in I}$ associated to the aggregate wealth x is such that $x_i \geq x^*$, for all i, and x_i increases with i. Furthermore, if all the utility functions are DARA then $t_i(x_i)$ increases with i.*
2. *For $x \leq x^*$, the optimal allocation $(x_i)_{i \in I}$ associated to the aggregate wealth x is such that $x_i \leq x^*$, for all i, and x_i decreases with i.*

Although of some interest by itself, this Proposition will also play an important role in the analysis that follows.

3 CARA utility functions

Let us consider constant absolute risk-aversion/tolerance utility functions of the form

$$u_i(x) = -\theta_i \exp\left(-\frac{x}{\theta_i}\right). \tag{2}$$

We have $t_i(x) = \theta_i$ and $t(x) = \int \theta_i \mathrm{d}Q(i)$. If the agents are indexed by their absolute levels of risk tolerance we have $\theta_i = i$ and the log-supermodularity assumption is satisfied. We have then $t(x) = E^Q\left[\tilde{\theta}\right]$ and the collective level of risk tolerance does not depend on the wealth allocation among the agents. It is immediate that FSD shifts on the distribution of the individual levels of risk tolerance lead to an increase of the collective level of absolute (and relative) risk tolerance. More heterogeneity, in the sense of shifts in the distribution of preferences that preserve the mean, have no effect on the group's risk tolerance. These results will serve as a useful benchmark.

4 CRRA utility functions

Let us consider constant relative risk-aversion/tolerance utility functions of the form

$$u_i(x) = \frac{1}{1 - \frac{1}{b_i}} x^{1 - \frac{1}{b_i}}. \tag{3}$$

where b_i is the level of relative risk tolerance of individual i and $\frac{1}{b_i}$ is his level of relative risk aversion. In such a setting, we have

$$A_i = \frac{1}{b_i x}, \quad R_i = \frac{1}{b_i}, \quad t_i = b_i x, \quad \text{and} \quad s_i = b_i.$$

Since the utility functions are no more defined up to a multiplicative constant, we do not assume anymore that the $\lambda_i = 1$ for all i. However, we still assume that there exists a wealth level x^* for which the fair allocation $(x_i)_{i\in I}$ with $x_i = x^*$ for all i, is efficient. Note that the existence of such a fair allocation can always be granted through a judicious choice of the weights $(\lambda_i)_{i\in I}$. The first-order conditions for Pareto optimality give then that $\lambda_i\,(x^*)^{1-\frac{1}{b_i}}$ is independent of i. The Pareto problem can be rewritten as follows $U(x) = \max_{\int \frac{x_i}{x^*}\mathrm{d}Q(i)=\frac{x}{x^*}} \int \frac{1}{1-\frac{1}{b_i}}\left(\frac{x_i}{x^*}\right)^{1-\frac{1}{b_i}}\mathrm{d}Q(i)$, up to a multiplicative constant. If we renormalize the individual consumptions to measure them in terms of multiples of x^*, we are led to analyze the situation where $u_i(x) = \frac{1}{1-\frac{1}{b_i}}x^{1-\frac{1}{b_i}}$ and all the weights λ_i are equal to 1. Note that with this renormalization, $x = 1$ corresponds to the efficient fair allocation.

In the next we consider then the equally weighted Pareto optimum. Since the agents differ by only one characteristic, namely their level b_i of relative risk tolerance, we might index them by this characteristic or we may, in other words, assume that $b_i = i$. For a given function h, we may then write indifferently $\int h(b_i)\mathrm{d}Q(i)$ or $\int h(b)\mathrm{d}Q(b)$ or $E^Q[h(\tilde{b})]$. The level of relative risk-aversion is then decreasing with i and the log-supermodularity condition is immediately satisfied.

The following Proposition uses these assumptions to characterize precisely the functions defining collective preferences and the collective level of risk aversion.

Proposition 4 *In a group made of agents with constant but heterogeneous levels of relative risk aversion, we have at the equally weighted Pareto optimum*

$$U(x) = \int \frac{b_i}{b_i - 1} e^{(1-b_i)\Phi^{-1}(x)}\mathrm{d}Q(i)$$

with

$$\Phi\,(t) \equiv \int e^{-b_i t}\mathrm{d}Q(i).$$

The collective degree of relative risk aversion $R(x)$ is given by

$$R(x) \equiv -x\frac{U''(x)}{U'(x)} = \frac{x}{\int b_i \mathrm{d}\widetilde{Q}(i)} = \frac{1}{s(x)}. \tag{4}$$

with $\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q} \equiv \frac{e^{-b\Phi^{-1}(x)}}{\int e^{-b_i\Phi^{-1}(x)}\mathrm{d}Q(i)}$.

As seen in the Proof in Appendix, the Lagrange multiplier of the Pareto optimum problem is given by $q = \exp(\Phi^{-1}(x))$ and q is then the shadow price associated to the constraint $\sum_{i\in I} x_i = x$. We clearly have $\Phi(0) = 1$, which means that $\Phi^{-1}(x^*) = 0$ and $q(x^*) = 1$ for the efficient fair allocation $x^* = 1$. Since the agents are risk averse, high levels of aggregate wealth have a low shadow price and low levels of aggregate wealth have high shadow price and we can easily derive that $q(x) < 1$ for $x > x^*$and $q(x) > 1$ for $x < x^*$. This means, in particular, that we have $\Phi^{-1}(x) < 0$ for $x > 1$ and $\Phi^{-1}(x) > 0$ for $x < 1$.

4.1 A model with two agents

Mazzocco (2004) shows that, in a model with two agents, an increase in the level of risk tolerance of one of the agents might have an ambiguous impact on the collective level of risk tolerance. It increases for some levels of aggregate wealth while it decreases for other levels of aggregate wealth. Since this is only stated on a numerical example in Mazzocco (2004), let us clearly express this result.

Proposition 5 *In a model with two agents with $b_1 < b_2$, there exists $\overline{x} \leq 1$ such that a small increase of b_2 leads to an increase of the collective level of risk tolerance $t(x)$ for all $x \geq \overline{x}$ and to a decrease of the collective level of risk tolerance $t(x)$ for all $x \leq \overline{x}$.*

Recall that after our normalization, $x^* = 1$ corresponds to the fair efficient allocation. Proposition 5 means then that an increase of the risk tolerance level of the most risk tolerant agent increases (decreases) the collective level of risk tolerance for levels of wealth above (below) a given threshold that is below the fair efficient allocation. Note that the threshold $\overline{x}$ depends on b_1 and b_2. This means that for x above the fair allocation, any increase of b_2 increases the collective risk tolerance. In fact, above the fair allocation, an increase of b_2 also increases the weight granted to b_2 leading to an increase of the collective level of risk tolerance. For a wealth level x below the fair allocation, the impact of an increase of b_2 is less clear. Indeed, we showed in Proposition 3 that for low levels of aggregate wealth the least risk tolerant agent (the most risk averse) has a larger share of the total wealth and an increase of b_2 leads to an increase of the weight granted to b_1 (the share of total wealth of agent 1). The increase of b_2 has then two effects in opposite directions: an increase of one of the terms of the average (namely the greatest one) and an increase of the weight of the smallest one. Since the second effect does not exist for $x = 1$, the first effect continues to dominate for x above a given threshold $\overline{x} \leq 1$ while the second effect dominates for $x \leq \overline{x}$.

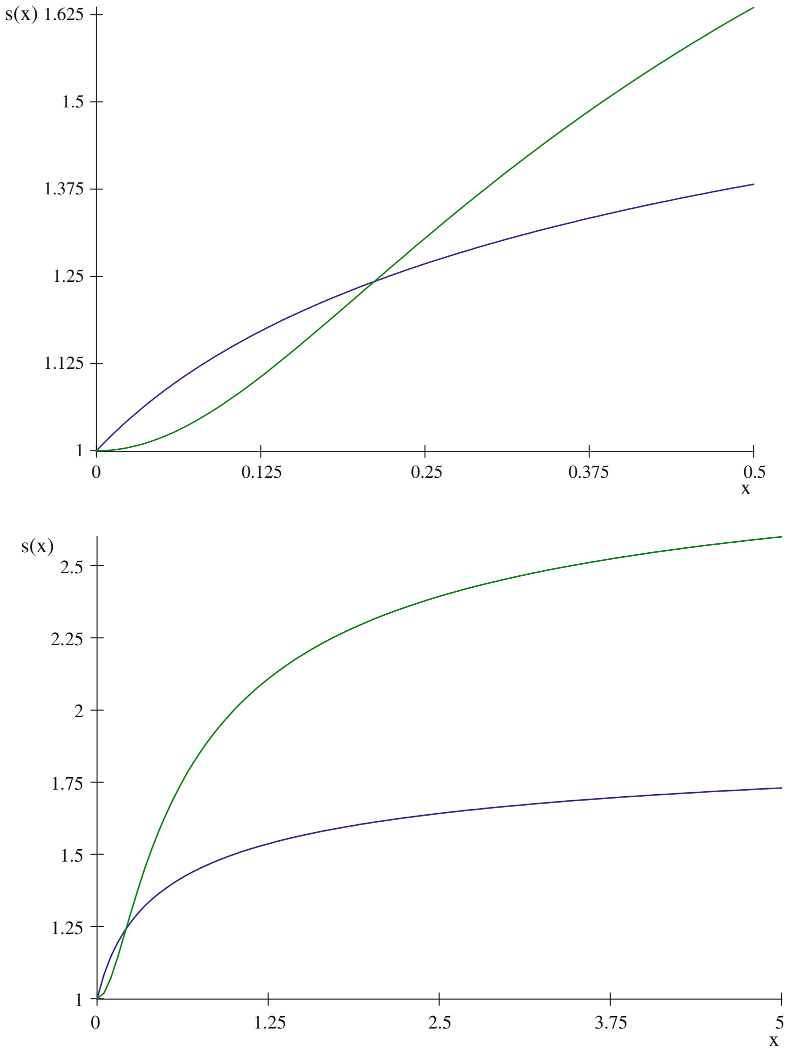
Figures 1–2 illustrate the impact of an increase of b_2.

The next Proposition analyzes more in detail how the collective level of risk tolerance evolves as a function of b_2.

Proposition 6 *In a model with two agents with $b_1 < b_2$, b_1 being given, for $x \geq \frac{1}{2}$, the function $b_2 \to t_x(b_2)$ is increasing on (b_1, ∞) with $\lim_{b_2 \to b_1} t_x(b_2) = xb_1$ and $t_x(b_2) \sim_{b_2 \to \infty} \left(x - \frac{1}{2}\right) b_2$. For $x < \frac{1}{2}$, there exists $b^*(x, b_1) > b_1$ such that the function $b_2 \to t_x(b_2)$ is increasing on $(b_1, b^*(x, b_1))$ and decreasing on $(b^*(x, b_1), \infty)$ with $\lim_{b_2 \to b_1} t_x(b_2) = xb_1$ and $\lim_{b_2 \to \infty} t_x(b_2) = xb_1$.*

In summary, for (very) low levels of wealth, increasing the risk tolerance of the more risk tolerant agent has an ambiguous impact on the collective attitude towards risk. Proposition 6 characterizes precisely the conditions for this paradoxical result to occur. Figures 3, 4, and 5 illustrate the different possible shapes for $b_2 \to t_x(b_2)$ (or equivalently of $b_2 \to s_x(b_2)$). The asymptotic behavior of $t_x(b_2)/b_2$ is illustrated in Figs. 6 and 7.

Propositions 5 and 6 establish the behavior of collective preferences as a function of one of the agent's risk tolerance. Another important question is what happens



Figs. 1–2 At two different scales, we represent the collective relative risk tolerance as a function of the total wealth, with $b_1 = 1$ and $b_2 = 2$ for the *curve* with a higher (lower) level of collective relative risk tolerance for low (high) levels of wealth and $b'_1 = 1$ and $b'_2 = 3$ for the other *curve*. *Both curves* converge slowly to the associated level of relative risk tolerance as can be shown on the second figure. An increase of the risk tolerance level of the most risk tolerant agent leads to an increase (decrease) of the collective level of risk tolerance above (below) a given threshold $x^* \leq 1$. With our parameters, we have $x^* = 0.21$. When $b_2 = 2$ and b'_2 is in the neighborhood of b_2, we have $x^* = 0.14$. When $b'_2 = 3$ and b_2 is in the neighborhood of b'_2, we have $x^* = 0.27$

to the collective level of risk tolerance when *both* agents become more risk tolerant. In the next Proposition we show that the ambiguous impact disappears when we consider a uniform increase of risk tolerance across the agents, but that for non-uniform increases in risk tolerance the collective degree of risk tolerance may still be lower.

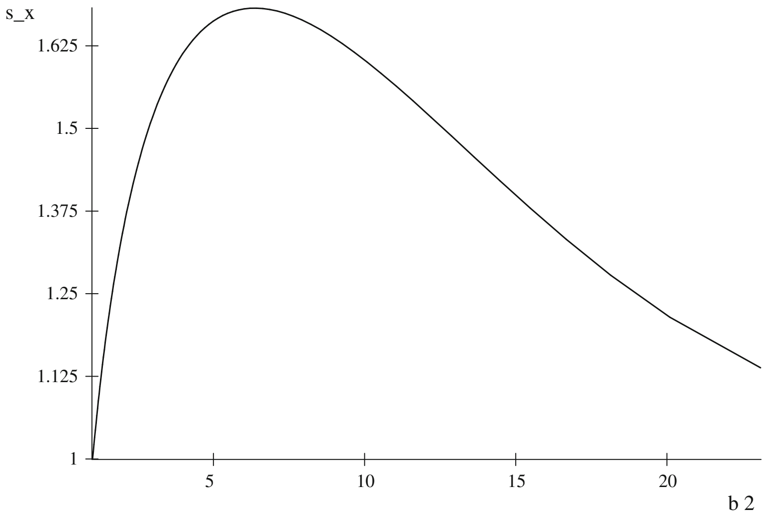


Fig. 3 In a setting with two agents with levels of relative risk tolerance b_1 and b_2, we represent the collective relative risk tolerance $s_x(b_2)$ as a function of b_2 for $b_1 = 1$ and for $x = 0.4$. For $b_2 = b_1 = 1$, s_x is equal to 1. The collective relative risk tolerance increases then decreases with b_2 and $\lim_{b_2 \to \infty} s_x(b_2) = b_1$

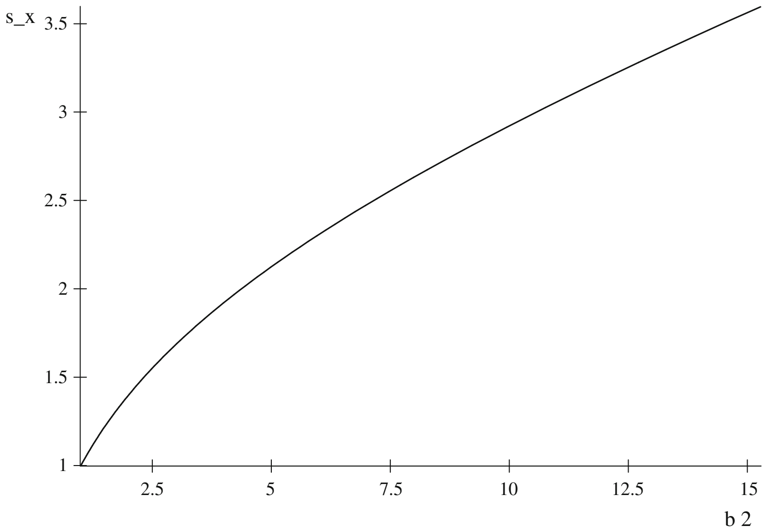


Fig. 4 In a setting with two agents with levels of relative risk tolerance b_1 and b_2, we represent the collective relative risk tolerance $s_x(b_2)$ as a function of b_2 for $b_1 = 1$ and for $x = 0.55$. For $b_2 = b_1 = 1$, s_x is equal to 1. The collective relative risk tolerance increases with b_2

Proposition 7 *Let $b_1 < b_2$ be given.*

1. *Let us consider a uniform increase of the individual levels of risk tolerance of the form $b_1 + h$ and $b_2 + h$ with $h > 0$. The associated level of collective risk tolerance $t_x(h)$ increases with h for all x.*

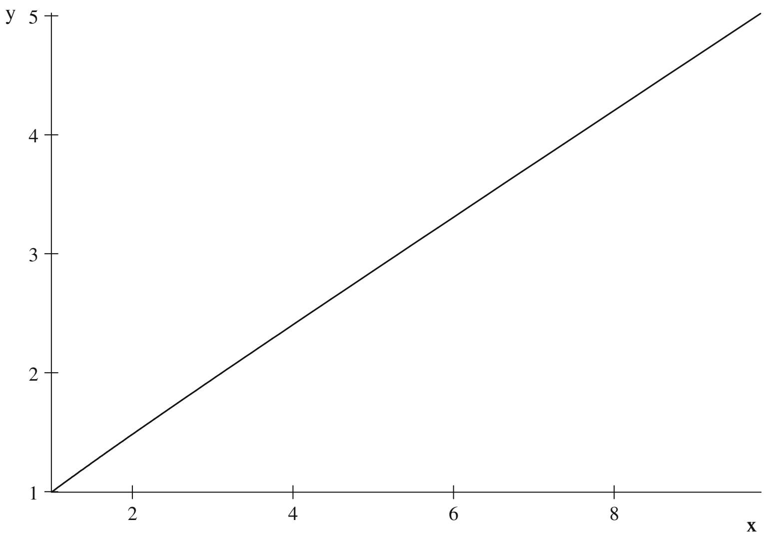


Fig. 5 In a setting with two agents with levels of relative risk tolerance b_1 and b_2, we represent the aggregate relative risk tolerance $s_x(b_2)$ as a function of b_2 for $b_1 = 1$ and for $x = 0.9$. For $b_2 = b_1 = 1$, s_x is equal to 1. The aggregate relative risk tolerance increases with b_2 and is almost linear

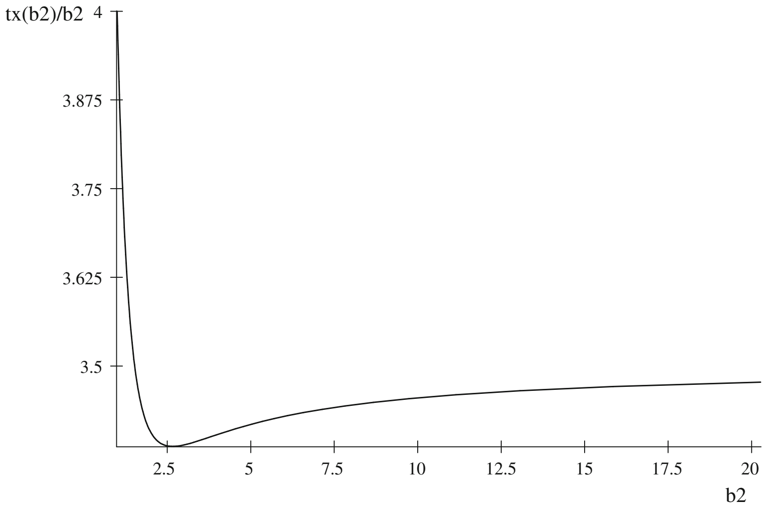


Fig. 6 In a setting with two agents with levels of relative risk tolerance b_1 and b_2, we represent the ratio $t_x(b_2)/b_2$ for $x = 4$ and $b_1 = 1$. For $b_2 = b_1 = 1$ it is immediate that $t_x(b_2)/b_2 = t_x(b_2) = x$. The ratio converges to $x - 0.5 = 3.5$ when b_2 converges to ∞

2. *Let $k > 0$ be given and let us consider an increase of the individual levels of risk tolerance of the form $b_1 + kh$ and $b_2 + h$ with $h > 0$. The associated level of collective risk tolerance $t_x(h)$ increases with h for $x \leq 1$ if $k \geq \frac{b_1}{b_2}$ and increases with h for $x \geq 1$ if $k \leq 1$. In particular, $t_x(h)$ increases with h for all x if $k \in [\frac{b_1}{b_2}, 1]$.*

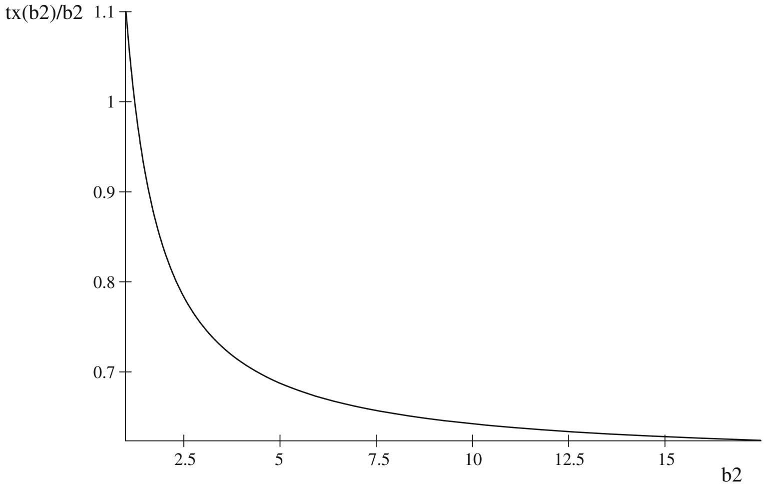


Fig. 7 In a setting with two agents with levels of relative risk tolerance b_1 and b_2, we represent the ratio $t_x(b_2)/b_2$ for $x = 1.1$ and $b_1 = 1$. For $b_2 = b_1 = 1$ it is immediate that $t_x(b_2)/b_2 = t_x(b_2) = x$. The ratio converges to $x - 0.5 = 0.6$ when b_2 converges to ∞

Let us illustrate the second point by two extreme situations. For k very small (near to 0), the shifts we are considering are almost of the form $(b_1, b_2) \to (b_1, b_2 + \varepsilon)$ that have already been considered in Proposition 5. These shifts increase the risk tolerance level of the second agent and also increase its weight for $x \geq 1$. These shifts lead then to an unambiguous increase of the aggregate level of risk tolerance for $x \geq 1$. For k near infinity (and h very small), the shifts we are considering are almost of the form $(b_1, b_2) \to (b_1 + \varepsilon, b_2)$ and such shifts increase the risk tolerance level of the first agent and also increase its weight when $x \leq 1$. These shifts lead then to an unambiguous increase of the aggregate level of risk tolerance for $x \leq 1$. The proposition shows that there is a range for k for which the shifts have an unambiguous impact without restrictions on x. However, for small levels of k we cannot conclude that the collective level of risk tolerance is higher.[4]

We are also interested in the impact of more heterogeneity among our two agents. The next result shows that more heterogeneity leads to a higher collective risk tolerance level for high wealth levels (above the fair efficient allocation) and to a lower collective risk tolerance level for low wealth levels (below the fair efficient allocation).

Proposition 8 *Let b_1 and b_2 be given with $b_1 < b_2$ and let us consider a shift of the form $b_1 - h$ and $b_2 + h$ with $h > 0$. The associated level of collective risk tolerance $t_x(h)$ increases (resp. decreases) with h for $x \geq 1$ (for $x \leq 1$).*

This result is very intuitive. We have already seen that the collective level of risk tolerance is near the risk tolerance level of the most risk tolerant agent for high levels of wealth and is near the risk tolerance level of the least risk tolerant level agent for low

[4] This should not be 'too' surprising given our previous results since for $k = 0$ we are back in the situation in which only b_2 increases.

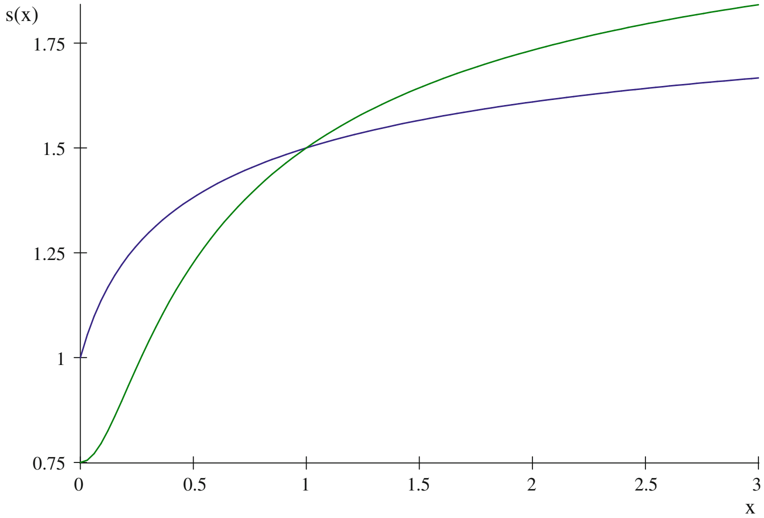


Fig. 8 In a setting with two agents with levels of relative risk tolerance b_1 and b_2, we represent the collective relative risk tolerance as a function of the total wealth with $b_1 = 1$ and $b_2 = 2$ for the *curve* with a higher (lower) level of collective relative risk tolerance for low (high) levels of wealth and $b_1' = \frac{3}{4}$ and $b_2' = \frac{9}{4}$ for the other *curve*. The shift from $(b_{1,} b_2)$ to $(b'_{1,} b'_2)$ is a Mean Preserving Spread as in Proposition 8 and the *two curves* cross at $x^* = 1$

levels of wealth. More heterogeneity leads to an increase of the risk tolerance level of the most risk tolerant agent and to a decrease of the risk tolerance level of the least risk tolerant agent. This leads then to an increase of the collective level of risk tolerance for high levels of wealth and to a decrease of the collective level of risk tolerance for low levels of wealth. Proposition 8 permits to give a precise meaning to high and low levels of wealth since the fair efficient allocation appears to be the relevant threshold. Figure 8 illustrates this result.

4.2 General populations

The different results of the previous section permit to see that first-order stochastic dominance shifts do not guarantee an increase in the collective degree of risk tolerance. In this section we initially consider whether a stronger notion of first-order stochastic dominance leads to an unambiguous impact on the group's degree of risk tolerance. Then we evaluate the effect of more heterogeneity within a group. In order to treat the problem in a quite general setting, we consider from now on general populations described by a distribution on $I = [0, \infty)$. To relate the results in this more general setting to those obtained in the 2-agent framework, we will attach a specific attention to distributions with a 2-point support.

Since FSD is not a good candidate to obtain comparative static results, let us recall the following definition corresponding to a stronger notion of first-order dominance.

Definition 1 Monotone Likelihood Ratio Dominance (MLR). Let P and Q denote two probability measures on $I = [0, \infty)$. We say that P dominates Q in the sense of MLR

($P \succcurlyeq_{\mathrm{MLR}} Q$) if there exist numbers $0 \le \alpha \le \beta \le \infty$ and a nondecreasing function $h : [\alpha, \beta] \to [0, \infty]$ such that $P([0, \alpha)) = Q((\beta, \infty)) = 0$ and $\mathrm{d}P(i) = h(i)\mathrm{d}Q(i)$.

In other words, an MLR-dominated shift for a given probability measure puts less weight for higher values of i. This concept is widely used in the statistical literature and was first introduced in the context of portfolio problems by Landsberger and Meilijson (1990). MLR dominance is stronger than FSD and, in particular, an MLR-dominated shift for a given distribution reduces the mean.

When the supports of P and Q are reduced to two points, (b_1^1, b_2^1) with $b_1^1 < b_2^1$ for P and (b_1^2, b_2^2) with $b_1^2 < b_2^2$ for Q, we necessarily have $b_1^2 < b_2^2 \le b_1^1 < b_2^1$ or $b_1^2 = b_1^1$ and $b_2^2 = b_2^1$. In the first case, any average of the risk tolerance levels in the support of Q is smaller than any average of the risk tolerance levels in the support of P and the collective risk tolerance level is higher under P. The (most) interesting case is when both probability measures have the same support. We then have two populations with the same set of possible levels of individual level of risk tolerance b_1 and b_2 but with different proportions of agents in each category: a proportion p_1 (resp. $p_2 = 1 - p_1$) of agents that have an individual level b_1 (resp. b_2) of risk tolerance under P and a proportion q_1 (resp. $q_2 = 1 - q_1$) of agents that have an individual level b_1 (resp. b_2) of risk tolerance under Q. The MLR dominance is characterized in this setting by $\frac{p_2}{p_1} \ge \frac{q_2}{q_1}$ (or equivalently $q_1 \ge p_1$ or $q_2 \le p_2$).

Proposition 9 *Let us consider two populations. In the first one, we have proportions p_1 (resp. $p_2 = 1 - p_1$) of agents with an individual level b_1 of risk tolerance (resp. b_2). In the second one, we have proportions q_1 (resp. $q_2 = 1 - q_1$) of agents with an individual level b_1 of risk tolerance (resp. b_2). If we assume that $\frac{p_2}{p_1} \ge \frac{q_2}{q_1}$, then the collective level of risk tolerance is higher in the first population.*

In summary, when the support of the population is reduced to two points, an MLR dominant shift in the degree of risk tolerance of the members of the group increases the group's risk tolerance, so such shift clearly characterizes the notion of a "more risk tolerant group". The following proposition generalizes the impact of MLR shifts for distributions with more general supports.

Proposition 10 *Let us consider two populations characterized by two distributions P and Q of individual levels of risk tolerance. If $P \succcurlyeq_{\mathrm{MLR}} Q$ then for all $x \le 1$, we have $t_x^P \ge t_x^Q$. However, $P \succcurlyeq_{\mathrm{MLR}} Q$ does not guarantee an increase of the collective level of risk tolerance when $x > 1$.*

MLR provides then a satisfying answer to the impact of shifts for low levels of wealth (when $x \le 1$), which corresponds to the case where the unilateral increase of one of the individual levels of risk tolerance failed to guarantee an increase of the aggregate level of risk tolerance. In the following Proposition we show that when the density function (introduced in Definition 1) $h = \frac{\mathrm{d}P}{\mathrm{d}Q}$ has an exponential growth rate, then we do have an unambiguous impact on collective risk tolerance.

Proposition 11 *Let us consider two populations characterized by two distributions P and Q on $[0, \infty)$of individual levels of risk tolerance such that $P \succcurlyeq_{\mathrm{MLR}} Q$ with $\frac{\mathrm{d}P}{\mathrm{d}Q}(b) = \lambda \exp(kb)$ for some positive k and λ. For all x, we have $t_x^P \ge t_x^Q$.*

We have seen in the 2-agent setting (Proposition 8) that more heterogeneity has a clear impact on the collective level of risk tolerance depending on the relative position of the aggregate wealth with respect to the fair efficient allocation. We are now interested in establishing the effect of "more heterogeneity" in the general setting. For this purpose, we introduce the following definition.

Definition 2 Portfolio Dominance (PD). Let Q_1 and Q_2 denote two probability measures on $I = [0, \infty)$. We say that Q_1 dominates Q_2 in the sense of PD ($Q_1 \succcurlyeq_{\text{PD}} Q_2$) if we have $\int v(i-a)\mathrm{d}Q_1(i) = 0 \Rightarrow \int v(i-a)\mathrm{d}Q_2(i) \leq 0$ for any real number a and any nonnegative and nonincreasing function v.

This concept has been introduced in the context of portfolio problems by Landsberger and Meilijson (1990) and further studied by Gollier (1997). In the portfolio context it is related to the degree of riskiness of the asset returns. In our context, it is related to the level of individual heterogeneity in relative risk tolerance. In particular, a mean preserving PD-dominated shift for a given distribution increases the variance (Jouini and Napp 2008, Proposition 3).

The following proposition uses this concept to characterize the impact of "more heterogeneity" in the distribution of individual preferences.

Proposition 12 *Let us consider two populations, respectively, characterized by distributions P and Q of individual levels of risk tolerance. If P and Q are symmetric with respect to some b^* with $\frac{\mathrm{d}Q}{\mathrm{d}P}$ nonincreasing before b^* and nondecreasing after b^* then $P \succcurlyeq_{\text{PD}} Q$ and $P \succcurlyeq_{\text{SSD}} Q$ and the collective level of risk tolerance t_x^Q under Q is higher than (resp. lower than) the aggregate level of risk tolerance t_x^P under P for $x \geq 1$ (for $x \leq 1$).*

The intuition is, in essence, the same as that in Proposition 8. At high wealth levels those individuals that have a high tolerance for risk are more representative of the collective level of risk tolerance, but the opposite is true at low wealth levels. More dispersion in the levels of risk tolerance of the group then leads to a higher (lower) level of collective risk tolerance for high (low) wealth levels.

5 The case of egalitarian groups

It is interesting to analyze the aggregate behavior in a model where all the agents consume the total consumption x. This is the case when x is a public good. This is also the case when x is a private good but simultaneously consumed by all the agents in the group. We may consider that both agents in a couple get utility from saving money or from holding consumption goods and consider these goods as owned by the couple and not shared among them through an efficient sharing rule. This is also the setup used in a number of recent experiments that compare the degree of risk aversion of groups with that of individuals (e.g., Shupp and Williams 2008; Baker et al. 2008; Masclet et al. 2009) and where the rewards of the group are exogenously divided equally among its members. We may imagine, for example, that such experiments reflect the widely observed regularity of partnerships with equal sharing rules.

In the next, the endowment in the consumption good is defined by a random variable x on the probability space (Ω, F, P) and the social utility function is given by

$$U(x) = \int \lambda_i u_i(x)\mathrm{d}Q(i). \tag{5}$$

where u_i is the utility function of agent i and λ_i is the weight granted to agent i.

We have then

$$R(x) \equiv -x\frac{U''(x)}{U'(x)} = -x\frac{\int \lambda_i u_i''(x)\mathrm{d}Q(i)}{\int \lambda_i u_i'(x)\mathrm{d}Q(i)} = \int R_i(x)\mathrm{d}P^u(i) \tag{6}$$

where P^u is the probability measure defined by $\frac{\mathrm{d}P^u}{\mathrm{d}Q}(i) = \frac{\lambda_i u_i'(x)}{\int \lambda_i u_i'(x)\mathrm{d}Q(i)}$.

Let us first analyze the case with two agents and CRRA functions. We have then $u_i(x) = \frac{1}{1-\frac{1}{b_i}}x^{1-\frac{1}{b_i}}, i = 1, 2,$ and we take $\lambda_i = 1, i = 1, 2,$ as in the previous section. Equation 6 can be rewritten as follows

$$R(x) = \frac{\frac{1}{b_1}x^{-\frac{1}{b_1}} + \frac{1}{b_2}x^{-\frac{1}{b_2}}}{x^{-\frac{1}{b_1}} + x^{-\frac{1}{b_2}}} = \frac{R_1 x^{-R_1} + R_2 x^{-R_2}}{x^{-R_1} + x^{-R_2}}$$

and the aggregate relative risk aversion is a weighted arithmetic average of the individual levels of relative risk aversion. As in the private good case, the collective level of relative risk aversion decreases with x and it approaches the degree of relative risk aversion of the most (least) risk averse agent in the economy as x converges to zero (infinity). Notice also that the weights are given by the individual marginal utilities and the highest weight is granted to the lowest (highest) individual level relative risk aversion for $x > 1$ (for $x < 1$). An increase of the individual level of risk tolerance of the most risk tolerant agent might then have an ambiguous impact. We have the following result:

Proposition 13 *In a model with a public good, two agents and CRRA functions with $b_1 < b_2$, there exists $\overline{x} \leq 1$ such that a small increase of b_2 leads to an increase of the collective level of risk tolerance $t(x)$ for all $x \geq \overline{x}$ and to a decrease of the collective level of risk tolerance $t(x)$ for all $x \leq \overline{x}$.*

In particular, this means that FSD shifts of the individual levels of risk tolerance are not sufficient to increase the collective level of risk tolerance. The next result illustrates the impact of a mean preserving spread on the individual levels of risk aversion in a 2-agent setting.

Proposition 14 *In a model with a public good, two agents and CRRA functions with $b_1 < b_2$ (or equivalently $R_1 > R_2$), a shift of the form $R_1 - h$ and $R_2 + h$ with $h > 0$ increases (decreases) the aggregate level of relative risk aversion $R(x)$ for $x \leq 1$ (for $x \geq 1$). It increases (decreases) the collective level of risk tolerance $t(x)$ for $x \geq 1$ (for $x \leq 1$).*

In the next we characterize, in a general distribution setting, the impact of MLR shifts on the collective level of risk tolerance/aversion. Note that the following result is obtained for very general utility functions.

Proposition 15 *Let us consider two populations, respectively, characterized by distributions P_1 and P_2 on (I, ι). Under Assumptions (U) and (LSPM) and if $P_2 \succcurlyeq_{\mathrm{MLR}} P_1$ then the collective level of risk aversion (risk tolerance) under P_1 is higher (lower) than under P_2.*

The result in Proposition 15 is quite powerful. Under the assumption that the members of the group consume the same endowment, and under weak restrictions on the individual utility functions, if the individual levels of risk aversion are more concentrated on high values (in the sense of MLR dominance) then the collective level of risk aversion will be higher for all endowment levels. In this case MLR dominance provides a clear characterization of comparative collective risk aversion.[5]

The next proposition analyzes the impact of more heterogeneity on the individual levels of risk aversion. For a given distribution P on (I, ι) and for a given x, we denote by P^x the image measure of P by $i \to -\frac{u_i''(x)}{u_i'(x)}$. The measure P^x describes the distribution of the individual levels of risk aversion at a given wealth level x.

Proposition 16 *Let us consider two populations, respectively, characterized by distributions P_1 and P_2 on (I, ι) and let us assume that (U) is satisfied. If, for a given x, $u_i'(x)$ is nondecreasing in i, $-\frac{u_i''(x)}{u_i'(x)}$ is decreasing with i*[6] *and $P_2^x \succcurlyeq_{\mathrm{PD}} P_1^x$, then the collective level of risk aversion (risk tolerance), at x, under P_2 is higher (lower) than under P_1.*

In particular, if there exists x^* such that all the individual marginal utilities $u_i'(x^*)$ are equal, then $u_i'(x)$ is nondecreasing in i for $x \le x^*$. It suffices then to have that $-\frac{u_i''(x)}{u_i'(x)}$ is decreasing with i and $P_2^x \succcurlyeq_{\mathrm{PD}} P_1^x$ to conclude that the aggregate level of risk aversion, at x, under P_2 is higher than under P_1. In other words, under weak assumptions on the utility function, less heterogeneity in risk aversion, in the sense of PD dominance, implies a higher level of collective risk aversion when wealth is sufficiently low.

6 Conclusion

Mazzocco (2004) established the counter-intuitive result that an increase in the level of risk tolerance of one of the individuals in a couple may reduce their collective degree of risk tolerance. We studied precisely the conditions for this phenomenon to occur. More generally, we established conditions under which groups with individual

[5] Note that Proposition 15 can easily be extended for higher order collective preferences towards risk. For example, if we assume that $u''(x, i)$ is LSPM in (x, i), then an MLR-dominant shift decreases the collective degree of absolute (and relative) prudence.

[6] Note that this last condition is just a little bit stronger than the LSPM condition.

levels of risk tolerance more concentrated on high values and groups that are more heterogeneous will display higher risk tolerance, both with efficient risk-sharing and with an exogenous egalitarian sharing rule. Our results permit to better characterize differences in risk taking behavior between groups and individuals and among groups with different distributions of risk preferences.

It should be possible to design experiments to evaluate if our results are consistent with elicited risk attitudes of groups and individuals. Shupp and Williams (2008) compare the willingness to pay for lotteries of small groups and individuals in a setup similar to that of Sect. 5. They conclude that, for most lotteries, group choices are significantly different from the mean of the individual choices (groups tend to be more risk averse than individuals for low-expected-value lotteries but less risk averse than individuals for high-expected-value lotteries). These results are consistent with a large number of studies in social psychology that show "risky" and "cautious" shifts in group risk-taking behavior relative to the mean of the individual choices (see e.g., Clark 1971). We have seen that there is no reason to believe that the group's willingness to pay (derived from the collective preferences), and more generally the willingness to take risks, should be the same as the mean of the individual members' willingness to pay. In particular, even with CRRA individual preferences, the fact that the group's relative risk aversion decreases with x implies that "cautious" shifts should be more prevalent in low-expected-value (low-stakes) lotteries while "risky" shifts should be more prevalent in high-expected-value (high-stakes) lotteries, precisely what Shupp and Williams (2008) found.[7] It would be interesting to further explore the experimental relevance of our results by proceeding to inter-groups comparisons. For instance, to analyze the difference in risk attitudes between two couples that only differ by the risk aversion level of one of the members, e.g., the man.

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Appendix

Proof of Proposition 1 Let us denote by φ the function defined by $\varphi(q) = \int \left(u_i'\right)^{-1}(\frac{q}{\lambda_i})\mathrm{d}Q(i)$. By Inada's conditions and since u_i is increasing and strictly concave for all i, φ is well defined on $(0,\infty)$ and decreasing. Furthermore, from the monotone convergence Theorem we have $\lim_{x\to 0}\varphi(q) = \infty$ and $\lim_{x\to\infty}\varphi(q) = d$. Let us then define $f_i : [d,\infty) \to [d_i,\infty)$ by $f_i(x) = \left(u_i'\right)^{-1}(\frac{\varphi^{-1}(x)}{\lambda_i})$, we clearly have $x = \int f_i(x)\mathrm{d}Q(i)$ and $\lambda_i u_i'(f_i(x)) = \varphi^{-1}(x)$ for all i and is independent of i. The family $(f_i(x))$ satisfies then the first-order conditions of the maximization program defined by Eq. 1 and since this program is concave we have $U(x) = \int \lambda_i u_i(f_i(x))\mathrm{d}Q(i)$. $\square$

[7] Eliaz et al. (2006) show that risky and cautious shifts in groups can be seen as a failure of expected utility theory.

Proof of Proposition 3 Since the fair allocation $x_i = x^*, i \in I$, is efficient and since we granted the same weight to all the agents, the first-order conditions for Pareto optimality give us that $u_i'(x)$ is independent of i. Since $\frac{\partial u}{\partial x}(x, i)$ is LSPM, integrating $\frac{\partial}{\partial x} \log u_i'(x)$ between $x \geq x^*$ and x^*, gives us that $u_i'(x)$ increases with i. Let us consider the Pareto allocation $(x_i)_{i \in I}$ associated to the aggregate wealth $x \geq x^*$. We have that $u_i'(x_i)$ is independent of i. We also necessarily have $x_{i_0} \geq x^*$ for some i_0 and then $u_{i_0}'(x_{i_0}) \leq u_{i_0}'(x^*)$ by concavity of the utility functions. We have then $u_i'(x_i) \leq u_i'(x^*)$ for all i and consequently $x_i \geq x^*$ for all i. Since $u_i'(x)$ increases with i and $u_i'(x)$ is independent of i and since $u_i'(x)$ is decreasing in x, we have that x_i increases with i. By the LSPM property $t(x, i)$ is increasing in i and by the DARA property, it is increasing in x. We have then that $t(x_i, i)$ is increasing in i.

For $x \leq x^*$, integrating $\frac{\partial}{\partial x} \log u_i'(x)$ between x and x^*, gives us that $u_i'(x)$ decreases with i. The same kind of arguments as above give that x_i decreases with i. □

Proof of Proposition 4 The first-order condition gives us the optimal allocation of agent i, $x_i = q^{-b_i}$, where q is the Lagrange multiplier. Using the resource constraint we obtain $\int q^{-b_i} \mathrm{d}Q(i) = x$ or $\int \exp(-b_i \ln q) \mathrm{d}Q(i) = x$. We obtain that $q = \exp(\Phi^{-1}(x))$ hence $x_i = \mathrm{e}^{-b_i \Phi^{-1}(x)}$. Plugging this back into the utility function we obtain the representative agent's utility.

The degree of relative risk aversion of the representative agent can be derived directly from the expression of the representative agent utility function or using Proposition 2. We have $s = \int s_i \frac{x_i}{x} \mathrm{d}Q(i)$. Since $x_i = \mathrm{e}^{-b_i \Phi^{-1}(x)}$ the result follows. □

Proof of Proposition 5 Let us consider b_1 as given and denote by $t_x(b_2)$ the aggregate risk tolerance level when the risk tolerance level of agent 2 is given by $b_2 \geq b_1$. We have

$$t_x(b_2) = \frac{1}{2} b_1 \exp(-b_1 a_x(b_2)) + \frac{1}{2} b_2 \exp(-b_2 a_x(b_2))$$

where $a_x(b_2)$ is the solution of

$$\psi(a_x(b_2), b_1, b_2) = x$$

with

$$\psi(a, b_1, b_2) = \frac{1}{2} \exp(-b_1 a) + \frac{1}{2} \exp(-b_2 a).$$

We get after computations

$$\frac{\mathrm{d}}{\mathrm{d}b_2} t_x(b_2) = \frac{1}{2} \frac{\exp(-(b_1 + b_2) a_x(b_2))}{b_1 \exp(-b_1 a_x(b_2)) + b_2 \exp(-b_2 a_x(b_2))} \varphi(a_x(b_2), b_1, b_2)$$

with

$$\varphi(a, b_1, b_2) = b_1 + b_2 \exp((b_1 - b_2) a) - a (b_2 - b_1) b_1.$$

For $x \geq 1$, we have $a_x(b_2) \leq 0$ and $\frac{\mathrm{d}}{\mathrm{d}b_2} t_x(b_2) > 0$. The aggregate level of risk tolerance is an increasing function of b_2.

Let us now focus on the case $x \leq 1$. It is easy to check that $a_x(b_2)$ is always positive for $x \in (0, 1)$ and decreases with x from ∞ to 0. It is also easy to see that $\varphi(a, b_1, b_2)$

is decreasing in a, positive for $a=0$ and converges to $-\infty$ when a converges to ∞. There exists then a level $x^*<1$ such that $\frac{\mathrm{d}}{\mathrm{d}b_2}t_x(b_2)=0$. For $x<x^*$ (resp. $x>x^*$), we have $\frac{\mathrm{d}}{\mathrm{d}b_2}t_x(b_2)<0$ (resp. $\frac{\mathrm{d}}{\mathrm{d}b_2}t_x(b_2)>0$). □

Proof of Proposition 6 We have

$$t_x(b_2)=\frac{1}{2}b_1\exp(-b_1a_x(b_2))+\frac{1}{2}b_2\exp(-b_2a_x(b_2))$$

where $a_x(b_2)$ is the solution of

$$\frac{1}{2}\exp(-b_1a_x(b_2))+\frac{1}{2}\exp(-b_2a_x(b_2))=x.$$

We know that

$$\frac{\mathrm{d}}{\mathrm{d}b_2}t_x(b_2)=\varphi(a_x(b_2),b_1,b_2)=b_1+b_2\exp((b_1-b_2)\,a_x(b_2))-a_x(b_2)\,(b_2-b_1)\,b_1.$$

We have already seen that $\varphi(a,b_1,b_2)$ is decreasing in a, positive for $a=0$ and converges to $-\infty$ when a converges to ∞. Let us denote by $a(b_1,b_2)$ the solution of $\varphi(a,b_1,b_2)=0$. Let us first consider the case $x\leq\frac{1}{2}$. The function $a_x(b_2)$ is clearly decreasing with b_2 and we have $\lim_{b_2\to b_1}a_x(b_2)=-\frac{\ln x}{b_1}$ and $\lim_{b_2\to\infty}a_x(b_2)=-\frac{\ln 2x}{b_1}$. Furthermore, since $\frac{\partial\varphi}{\partial a}$ is decreasing, $\frac{\partial a}{\partial b_2}$ has the same sign as $\frac{\partial\varphi}{\partial b_2}$. Direct computations give $\frac{\partial\varphi}{\partial b_2}=u-ab_1-ab_2u$ with $u=\exp((b_1-b_2)\,a)$ and a such that $b_1+b_2u-b_1a(b_2-b_1)=0$. Substituting a in $\frac{\partial\varphi}{\partial b_2}$ gives $\frac{\partial\varphi}{\partial b_2}=(b_1-b_2)^{-1}\,b_1^{-1}\left(b_2b_1u+b_1^2+b_1^2u+b_2^2u^2\right)<0$. The function $a(b_1,b_2)$ decreases then with b_2 and we have $\lim_{b_2\to b_1}a(b_1,b_2)=\infty$ and $\lim_{b_2\to\infty}a_x(b_2)=0$. There exists then b_2^* such that $a_x(b_2)=a(b_1,b_2)$ and $\frac{d}{db_2}t_x(b_2)=0$.

It is immediate that for $a_x(b_2)>a(b_1,b_2)$ we have $\frac{\mathrm{d}}{\mathrm{d}b_2}t_x(b_2)<0$ and for $a_x(b_2)<a(b_1,b_2)$ we have $\frac{\mathrm{d}}{\mathrm{d}b_2}t_x(b_2)>0$. It suffices to show that $a_x(b_2)$ and $a(b_1,b_2)$ cross only once to establish the result. Let us consider b_2^* such that $a_x(b_2)=a(b_1,b_2)$ and let us compute $\frac{\mathrm{d}}{\mathrm{d}b_2}\left(a_x(b_2)-a(b_1,b_2)\right)$ at b_2^*. By definition, we have $a(b_1,b_2^*)=a_x(b_2^*)$ and we denote it by a^*. Direct computations give

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}b_2}\left(a_x(b_2^*)-a(b_1,b_2^*)\right)&=\frac{\frac{\partial\varphi}{\partial b_2}}{\frac{\partial\varphi}{\partial a}}(a^*,b_{1,}b_2^*)-\frac{\frac{\partial\psi}{\partial b_2}}{\frac{\partial\psi}{\partial a}}(a^*,b_{1,}b_2^*)\\&=\frac{A}{\left(b_1\mathrm{e}^{-a^*b_1}+b_2^*\mathrm{e}^{-a^*b_2^*}\right)\left(b_1+b_2^*\mathrm{e}^{a^*(b_1-b_2^*)}\right)\left(b_2^*-b_1\right)}.\end{aligned}$$

with

$$\begin{aligned}A=\Big(&a^*b_1^2\mathrm{e}^{-a^*b_1}+a^*b_1^2\mathrm{e}^{-a^*b_2^*}+(a^*b_1b_2^*\mathrm{e}^{-a^*b_1}+a^*b_1b_2^*\mathrm{e}^{-a^*b_2^*}-b_1\mathrm{e}^{-a^*b_1}\\&-b_2^*\mathrm{e}^{-a^*b_2^*})\mathrm{e}^{a^*(b_1-b_2^*)}\Big).\end{aligned}$$

By definition, we have $\varphi(a^*, b_1, b_2^*) = 0$. Replacing $b_2^* \mathrm{e}^{a^*(b_1 - b_2^*)}$ by $a^* \left(b_2^* - b_1\right) b_1 - b_1$, we get

$$\frac{\mathrm{d}}{\mathrm{d}b_2}\left(a_x(b_2^*) - a(b_1, b_2^*)\right) = \frac{b_1^2 a^* \left(\mathrm{e}^{-a^* b_1} + \mathrm{e}^{-a^* b_2^*} + a^*(b_2^* - b_1)\mathrm{e}^{-a^* b_2^*}\right)}{\left(b_1 \mathrm{e}^{-a^* b_1} + b_2^* \mathrm{e}^{-a^* b_2^*}\right)\left(b_1 + b_2^* \mathrm{e}^{a^*(b_1 - b_2^*)}\right)\left(b_2^* - b_1\right)} > 0.$$

We have then that $\frac{\mathrm{d}}{\mathrm{d}b_2} a_x(b_2) > \frac{\mathrm{d}}{\mathrm{d}b_2} a(b_1, b_2)$ each time these two functions cross. This means that they can cross only once.

For $x \geq 1$, we have already shown that $\frac{\mathrm{d}}{\mathrm{d}b_2} t_x(b_2)$ is positive for all b_2 and $t_x(b_2)$ is then increasing.

For $1 \geq x \geq \frac{1}{2}$, let us show that there is no crossing between $a_x(b_2)$ and $a(b_1, b_2)$. For that purpose let us consider $a = a_x(b_2) = a(b_1, b_2)$. We have then

$$\frac{1}{2}\exp(-b_1 a) + \frac{1}{2}\exp(-b_2 a) = x$$
$$\text{and } b_1 + b_2 \exp((b_1 - b_2)\, a) - a\,(b_2 - b_1)\, b_1 = 0$$

which can be rewritten as follows

$$\exp((b_1 - b_2)\, a) = 2x \exp(b_1 a) - 1$$
$$\text{and} \exp((b_1 - b_2)\, a) = \frac{a\,(b_2 - b_1)\, b_1 - b_1}{b_2}$$

which gives

$$2x \exp(b_1 a) - 1 = \frac{a\,(b_2 - b_1)\, b_1 - b_1}{b_2}$$
$$\text{or } 2x b_2 \exp(b_1 a) = (a b_1 + 1)\,(b_2 - b_1)$$

Remark that $\exp(b_1 a) \geq 1 + b_1 a$ and we have then $2x b_2 \leq (b_2 - b_1)$ or $(2x - 1)\, b_2 \leq -b_1$ which is impossible since we assumed $2x - 1 \geq 0$.

For $x \geq \frac{1}{2}$, it is easy to check that $\lim_{b_2 \to b_1} a_x(b_2) = -\frac{\ln x}{b_1}$ and $a_x(b_2) \sim_{b_2 \to \infty} -\frac{\ln(2x-1)}{b_2}$. The limits of t_x derive from there. □

Proof of Proposition 7 It suffices to prove directly the second point. We have

$$t_x(h) = \frac{1}{2}(b_1 + kh)\exp(-(b_1 + kh)a_x(h)) + \frac{1}{2}(b_2 + h)\exp(-(b_2 + h)a_x(h))$$

where $a_x(h)$ is the solution of $\psi(a_x(h), b_1, b_1 + kh) = x$.

We get after computations

$$\frac{\mathrm{d}}{\mathrm{d}h} t_x(0) \equiv \phi(a_x(0), b_1, b_2)$$

where

$$\phi(a, b_1, b_2) = kb_1 \exp((b_2 - b_1)\,a) + b_2 \exp((b_1 - b_2)\,a) + b_1 + kb_2 \\ + a\,(b_2 - b_1)\,(kb_2 - b_1).$$

For $x \leq 1$, we have $a_x(0) \geq 0$ and it suffices to impose $k \geq \frac{b_1}{b_2}$ to have $\frac{\mathrm{d}}{\mathrm{d}h} t_x(0) \geq 0$ for all x.

For $x \geq 1$, we have $a_x(0) \leq 0$ and it suffices to remark that $\frac{\partial^2 \phi}{\partial a^2}$ is positive, that $\frac{\partial \phi}{\partial a}(0, b_1, b_2) = (b_2 - b_1)(b_2 + b_1)(k-1)$ and $\phi\,(0, b_1, b_2) = (k+1)\,(b_1 + b_2) > 0$. It suffices to impose $k \leq 1$ to have $\frac{\partial \phi}{\partial a}(0, b_1, b_2) \leq 0$ and then $\phi(a, b_1, b_2) \geq 0$ for $a \leq 0$ and hence $\frac{\mathrm{d}}{\mathrm{d}h} t_x(0) \geq 0$. □

Proof of Proposition 8 We have

$$t_x(h) = \frac{1}{2}\,(b_1 - h) \exp\left(-(b_1 - h)a_x(h)\right) + \frac{1}{2}\,(b_2 + h) \exp\left(-(b_2 + h)a_x(h)\right)$$

where $a_x(h)$ is the solution of $\psi(a_x(h), b_1 - h, b_2 + h) = x$. We want to show that for $x < 1$, we have $\frac{\mathrm{d}t_x}{\mathrm{d}h}(0) < 0$ and for $x > 1$ we have $\frac{\mathrm{d}t_x}{\mathrm{d}h}(0) < 0$.

We have $\frac{\mathrm{d}a_x}{\mathrm{d}h}(0) = \frac{a(\exp(-ab_1) - \exp(-ab_2))}{b_1 \exp(-ab_1) + b_2 \exp(-ab_2)}$ where $a = a_x(0)$, and

$$\frac{\mathrm{d}t_x}{\mathrm{d}h}(0) = -\frac{1}{2} \exp\left(-ab_1\right) + \frac{1}{2} \exp\left(-ab_2\right) + \frac{1}{2} ab_1 \exp\left(-ab_1\right) - \frac{1}{2} ab_2 \exp\left(-ab_2\right) \\ - \left(\frac{1}{2} b_1^2 \exp\left(-ab_1\right) + \frac{1}{2} b_2^2 \exp\left(-ab_2\right)\right) \left(\frac{a\,(\exp(-ab_1) - \exp(-ab_2))}{b_1 \exp(-ab_1) + b_2 \exp(-ab_2)}\right)$$

which is of the same sign as $g(a)$ with

$$\begin{aligned} g(a) &= (b_1 \exp\left(-ab_1\right) + b_2 \exp\left(-ab_2\right)) \left(\exp\left(-ab_2\right) - \exp\left(-ab_1\right)\right. \\ &\quad \left. + ab_1 \exp\left(-ab_1\right) - ab_2 \exp\left(-ab_2\right)\right) \\ &\quad - (a\,(\exp(-ab_1) - \exp(-ab_2))\,(b_1^2 \exp\left(-ab_1\right) + b_2^2 \exp\left(-ab_2\right)) \\ &= -\,b_1 \exp\left(-2ab_1\right) + b_2 \exp\left(-2ab_2\right) - (b_2 - b_1 - ab_1^2 + ab_2^2) \\ &\quad \times\ \exp\left(-a\,(b_1 + b_2)\right) \end{aligned}$$

which has the same sign as $-\ell(a)$ with

$$\ell(a) = b_1 \exp\left(-a(b_1 - b_2)\right) - b_2 \exp\left(-a(b_2 - b_1)\right) + (b_2 - b_1 - ab_1^2 + ab_2^2).$$

We have $\lim_{a \to \infty} \ell(a) = \infty$, $\ell(0) = 0$ and $\lim_{a \to -\infty} \ell(a) = -\infty$. We also have

$$\ell'(a) = (b_1 - b_2)(-b_2 - b_1 - b_1 \exp\left(a(b_2 - b_1)\right) - b_2 \exp\left(a(b_1 - b_2)\right) > 0.$$

The function ℓ is then negative on $\mathbb{R}_-$ and positive on $\mathbb{R}_+$. Since $a > 0$ for $x < 1$ and $a < 0$ for $x > 1$ this gives the result. □

Proof of Proposition 9 We denote by t_x^P (resp. t_x^Q) the aggregate level of risk tolerance in the first (resp. second) population when the aggregate wealth is x. We have

$$t_x^P = p_1 b_1 \exp(-b_1 a_x^P) + p_2 b_2 \exp(-b_2 a_x^P)$$

where a_x^P is the solution of

$$p_1 \exp(-b_1 a_x^P) + p_2 \exp(-b_2 a_x^P) = x.$$

We have similar formulas for t_x^Q and we have that $t_x^P \geq t_x^Q$ if and only

$$\frac{q_1 \exp(-b_1 a_x^Q)}{p_1 \exp(-b_1 a_x^P)} \geq \frac{q_2 \exp(-b_2 a_x^Q)}{p_2 \exp(-b_2 a_x^P)} \text{or equivalently} p_1 \exp(-b_1 a_x^P) \leq q_1 \exp(-b_1 a_x^Q) \tag{7}$$

If $x \leq 1$, we have $a_x^P \leq a_x^Q$ and the result is immediate. Let us now consider the case $x \geq 1$. Note that $\exp(-b_1 a_x^Q)$ and $\exp(-b_2 a_x^Q)$ correspond to the Pareto optimal allocations x_1^Q and x_2^Q in population Q while $\exp(-b_1 a_x^P)$ and $\exp(-b_2 a_x^P)$ correspond to the Pareto optimal allocation x_1^P and x_2^P in population P. We want to prove that $\frac{p_1}{q_1} x_1^P \leq x_1^Q$. Remark that the allocation $(\frac{p_1}{q_1} x_1^P, \frac{p_2}{q_2} x_2^P)$ is feasible in population Q. Let us compare $u_1'\left(\frac{p_1}{q_1} x_1^P\right)$ and $u_2'\left(\frac{p_2}{q_2} x_2^P\right)$ or $\left(\frac{p_1}{q_1} x_1^P\right)^{-\frac{1}{b_1}}$ and $\left(\frac{p_2}{q_2} x_2^P\right)^{-\frac{1}{b_2}}$. By construction we have $\left(x_1^P\right)^{-\frac{1}{b_1}} = \left(x_2^P\right)^{-\frac{1}{b_2}}$ and since $\frac{p_1}{q_1} \leq 1$ and $\frac{p_2}{q_2} \geq 1$ we have $u_1'\left(\frac{p_1}{q_1} x_1^P\right) \geq u_2'\left(\frac{p_2}{q_2} x_2^P\right)$. Since Pareto optimal allocations are characterized by the condition $u_1'\left(x_1^Q\right) = u_2'\left(x_2^Q\right)$ and since the allocation $(\frac{p_1}{q_1} x_1^P, \frac{p_2}{q_2} x_2^P)$ is feasible, we necessarily have, by concavity of u_1 and u_2, $\frac{p_1}{q_1} x_1^P \leq x_1^Q$. □

Proof of Proposition 10 We first prove that for $x \leq 1$ an MLR shift is suficient to increase the aggregate level of risk tolerance. Suppose that $P \succcurlyeq_{\text{MLR}} Q$. Since the MLR order is stronger than the FSD order, we have for all nondecreasing function h, $E^P\left[h\left(\tilde{b}\right)\right] \geq E^Q\left[h\left(\tilde{b}\right)\right]$, hence $\Phi_Q(t) \geq \Phi_P(t)$ for $t \geq 0$. Since Φ_Q and Φ_P are decreasing, then for all $x \leq 1, \Phi_P^{-1}(x) \leq \Phi_Q^{-1}(x)$. Since $\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}\widetilde{P}} = \frac{\mathrm{d}Q}{\mathrm{d}P} \mathrm{e}^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))} \frac{E^P\left[\mathrm{e}^{-b\Phi_P^{-1}(x)}\right]}{E^Q\left[\mathrm{e}^{-b\Phi_Q^{-1}(x)}\right]}$, we obtain that $\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}\widetilde{P}}$ is the product (modulo a constant) of the decreasing function $\mathrm{e}^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))}$ with the decreasing function $\frac{\mathrm{d}Q}{\mathrm{d}P}$ (both of them being positive) and is then decreasing and $\widetilde{P} \succeq_{\text{MLR}} \widetilde{Q}$ which gives $E^{\tilde{Q}}\left[\tilde{b}\right] \leq E^{\tilde{P}}\left[\tilde{b}\right]$ and $t_x^P \geq t_x^Q$ or $R_x^P \leq R_x^Q$.

However, MLR does not guarantee an increase of the aggregate level of risk tolerance when $x > 1$ as shown in the next counter-example.

Let us consider a model in which the distribution of the risk tolerances is given by $\exp(-b) 1_{b \geq 0}$ and let us consider a shift obtained through multiplication of the exponential density by $1 + \varepsilon 1_{b > b^*}$ (for some $b^* > 0$ and some $\varepsilon > 0$) and by renormalization. The function $b \to 1 + \varepsilon 1_{b>b^*}$ is nondecreasing and the shift is MLR. The aggregate level of risk tolerance in the initial population is given by $t(x) = \int_0^\infty b \exp(-b) \exp(-ab) \mathrm{d}b$ where a solves $\int_0^\infty \exp(-b) \exp(-ab) \mathrm{d}b = x$. We obtain $a = \frac{1}{x} - 1$ and $t(x) = \frac{1}{(a+1)^2} = x^2$.

After the shift, the aggregate level of risk tolerance is given by

$$t_\varepsilon(x) = \frac{\int_0^\infty b \exp(-b) \exp(-a_\varepsilon b) \mathrm{d}b + \varepsilon \int_{b^*}^\infty b \exp(-b) \exp(-a_\varepsilon b) \mathrm{d}b}{\int_0^\infty \exp(-b) \mathrm{d}b + \varepsilon \int_{b^*}^\infty \exp(-b) \mathrm{d}b} \tag{8}$$

$$= \frac{\frac{1}{(a_\varepsilon+1)^2} + \varepsilon \left(\frac{1}{a_\varepsilon+1} b^* \exp\left(-(a_\varepsilon+1)b^*\right) + \frac{1}{(a_\varepsilon+1)^2} \exp\left(-(a_\varepsilon+1)b^*\right) \right)}{1 + \varepsilon \exp\left(-b^*\right)}$$

where a_ε solves $\frac{\int_0^\infty \exp(-b) \exp(-a_\varepsilon b) \mathrm{d}b + \varepsilon \int_{b^*}^\infty \exp(-b) \exp(-a_\varepsilon b) \mathrm{d}b}{\int_0^\infty \exp(-b) \mathrm{d}b + \varepsilon \int_{b^*}^\infty \exp(-b) \mathrm{d}b} = x$ or $\frac{\frac{1}{a_\varepsilon+1} + \varepsilon \frac{1}{a_\varepsilon+1} \exp(-(a_\varepsilon+1)b^*)}{1+\varepsilon \exp(-b^*)} = x$. We have then $\frac{1}{a+1} = \frac{\frac{1}{a_\varepsilon+1} + \varepsilon \frac{1}{a_\varepsilon+1} \exp(-(a_\varepsilon+1)b^*)}{1+\varepsilon \exp(-b^*)}$. Let us consider the difference $t_\varepsilon(x) - t(x)$. It is positively proportional to

$$\Delta = \left(\frac{1}{\beta^2} + \varepsilon \left(\frac{1}{\beta} y^* \exp\left(-\beta y^*\right) + \frac{1}{\beta^2} \exp\left(-\beta y^*\right) \right) \right) \left(1 + \varepsilon \exp\left(-y^*\right)\right) \tag{9}$$
$$- \left(\frac{1}{\beta} + \varepsilon \frac{1}{\beta} \exp\left(-\beta y^*\right) \right)^2$$

where $\beta = a_\varepsilon + 1$ and where $\frac{1}{a+1}$ has been replaced by its value as a function of β. This quantity is of the form $\mu\varepsilon + \nu\varepsilon^2$ and since we want it to be positive for all $\varepsilon > 0$ we have to check if μ is positive. But $\mu = \frac{(\exp(\beta b^*) - \exp(b^*) + \beta b^* \exp(b^*))}{\exp(\beta b^*) \exp(b^*) \beta^2}$ and is positively proportional to $\exp(\beta b^*) - \exp(b^*) + \beta b^* \exp(b^*)$. It is easy to remark that for $\beta = \frac{1}{b^{*2}}$ and b^* sufficiently large, this quantity is negative. Let us chose then a pair (β, b^*) for which this quantity is negative and let us take ε sufficiently small such that the quantity Δ itself is negative and let us finally take $x = \frac{\frac{1}{(a_\varepsilon+1)^2} + \varepsilon \left(\frac{1}{a_\varepsilon+1} b^* \exp(-(a_\varepsilon+1)b^*) + \frac{1}{(a_\varepsilon+1)^2} \exp(-(a_\varepsilon+1)b^*) \right)}{1+\varepsilon \exp(-b^*)}$. The resulting shift leads then to a decrease of the collective level of risk tolerance at x. □

Proof of Proposition 11 We just have to consider the case where $x \geq 1$. We have

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}\widetilde{P}} = \frac{\mathrm{d}Q}{\mathrm{d}P} \mathrm{e}^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))} \frac{E^P[\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^Q[\mathrm{e}^{-b\Phi_Q^{-1}(x)}]} \tag{10}$$
$$= \exp(-kb - b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))).$$

To conclude, it is sufficient to show that $k + (\Phi_Q^{-1}(x) - \Phi_P^{-1}(x)) > 0$. We have $E^P[\mathrm{e}^{-b(\Phi_Q^{-1}(x)+k)}] = \frac{E^Q[\mathrm{e}^{-b\Phi_Q^{-1}(x)}]}{E^Q[\exp(kb)]} < x$ and since $\Phi_P^{-1}(x)$ is characterized by $E^P[\mathrm{e}^{-b\Phi_P^{-1}(x)}] = x$ we have $k + (\Phi_Q^{-1}(x) - \Phi_P^{-1}(x)) > 0$. □

Proof of Proposition 12 Assume that P and Q are symmetric with respect to some b^* with $\frac{\mathrm{d}Q}{\mathrm{d}P}$ nonincreasing before b^* and nondecreasing after b^* then $P \succcurlyeq_{\mathrm{PD}} Q$ and $P \succcurlyeq_{\mathrm{SSD}} Q$ (Jouini and Napp 2008). Let us denote by Φ_P and Φ_Q the functions, respectively, defined by $\Phi_P(t) = \int \mathrm{e}^{-bt}\mathrm{d}P(b)$ and $\Phi_Q(t) = \int \mathrm{e}^{-bt}\mathrm{d}Q(b)$. Since e^{-bt} is decreasing and convex for $t \geq 0$, we have by SSD, $\Phi_Q(t) \geq \Phi_P(t)$ for all $t \geq 0$. For $x \leq 1, \Phi_P^{-1}(x)$ and $\Phi_Q^{-1}(x)$ are positive. Furthermore, both Φ_Q and Φ_P are decreasing and we have then $\Phi_Q^{-1}(x) \geq \Phi_P^{-1}(x)$ for all $x \leq 1$. Since $b \to \mathrm{e}^{-b\Phi_P^{-1}(x)}$ is decreasing and positive, we have by PD, $\frac{E^P[b\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^P[\mathrm{e}^{-b\Phi_P^{-1}(x)}]} \geq \frac{E^Q[b\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^Q[\mathrm{e}^{-b\Phi_P^{-1}(x)}]} = E^{P_1}[b]$ with $\frac{\mathrm{d}P_1}{\mathrm{d}Q} = \frac{\mathrm{e}^{-b\Phi_P^{-1}(x)}}{E^Q[\mathrm{e}^{-b\Phi_P^{-1}(x)}]}$. Let us consider Q_1 defined by $\frac{\mathrm{d}Q_1}{\mathrm{d}Q} = \frac{\mathrm{e}^{-b\Phi_Q^{-1}(x)}}{E^Q[\mathrm{e}^{-b\Phi_Q^{-1}(x)}]}$. We have $\frac{\mathrm{d}Q_1}{\mathrm{d}P_1} = \frac{E^Q[\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^Q[\mathrm{e}^{-b\Phi_Q^{-1}(x)}]}\mathrm{e}^{-b\left(\Phi_Q^{-1}(x)-\Phi_P^{-1}(x)\right)}$ and is decreasing. Hence $E^{P_1}[b] \geq E^{Q_1}[b] = \frac{E^Q[b\mathrm{e}^{-b\Phi_Q^{-1}(x)}]}{E^Q[\mathrm{e}^{-b\Phi_Q^{-1}(x)}]}$. We have then the result for $x \leq 1$. For $x \geq 1$, since both distributions are symmetric with respect to b^*, we have $\frac{E^P[b\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^P[\mathrm{e}^{-b\Phi_P^{-1}(x)}]} = \frac{E^P[(2b^*-b)\mathrm{e}^{-(2b^*-b)\Phi_P^{-1}(x)}]}{E^P[\mathrm{e}^{-(2b^*-b)\Phi_P^{-1}(x)}]} = 2b^* - \frac{E^P[b\mathrm{e}^{b\Phi_P^{-1}(x)}]}{E^P[\mathrm{e}^{b\Phi_P^{-1}(x)}]}$ and $\frac{E^Q[b\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^Q[\mathrm{e}^{-b\Phi_P^{-1}(x)}]} = 2b^* - \frac{E^Q[b\mathrm{e}^{b\Phi_P^{-1}(x)}]}{E^Q[\mathrm{e}^{b\Phi_P^{-1}(x)}]}$. Since $\Phi_P^{-1}(x)$ is negative for $x \geq 1$, the function $b \to \mathrm{e}^{b\Phi_P^{-1}(x)}$ is decreasing and positive, and we have by PD that $\frac{E^P[b\mathrm{e}^{b\Phi_P^{-1}(x)}]}{E^P[\mathrm{e}^{b\Phi_P^{-1}(x)}]} \geq \frac{E^Q[b\mathrm{e}^{b\Phi_Q^{-1}(x)}]}{E^Q[\mathrm{e}^{b\Phi_Q^{-1}(x)}]}$ or $E^{P_1}[b] = \frac{E^Q[b\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^Q[\mathrm{e}^{-b\Phi_P^{-1}(x)}]} \geq \frac{E^P[b\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^P[\mathrm{e}^{-b\Phi_P^{-1}(x)}]}$. Since $\frac{\mathrm{d}Q_1}{\mathrm{d}P_1} = \frac{E^Q[\mathrm{e}^{-b\Phi_P^{-1}(x)}]}{E^Q[\mathrm{e}^{-b\Phi_Q^{-1}(x)}]}\mathrm{e}^{-b\left(\Phi_Q^{-1}(x)-\Phi_P^{-1}(x)\right)}$, it is now increasing, we have then $E^{P_1}[b] \leq E^{Q_1}[b]$ which gives the result. □

Proof of Proposition 13 Let us denote by $t(x, h)$ the aggregate level of risk tolerance at x when the individual levels of risk tolerance are given by b_1 and $b_2 + h$, we have

$$t(x,h) = \frac{\exp(-\frac{\ln x}{b_1}) + \exp(-\frac{\ln x}{b_2+h})}{\frac{1}{b_1}\exp(-\frac{\ln x}{b_1}) + \frac{1}{b_2+h}\exp(-\frac{\ln x}{b_2+h})}$$

and

$$\frac{\partial t}{\partial h}(x,0) = \frac{b_1\left((b_2-b_1)(\ln x)\,\mathrm{e}^{-\frac{\ln x}{b_1}} + b_1b_2\left(\mathrm{e}^{-\frac{\ln x}{b_1}} + \mathrm{e}^{-\frac{\ln x}{b_2}}\right)\right)\mathrm{e}^{-\frac{\ln x}{b_2}}}{b_2\left(b_1\mathrm{e}^{-\frac{\ln x}{b_2}} + b_2\mathrm{e}^{-\frac{\ln x}{b_1}}\right)^2}$$

and we clearly have $\frac{\partial t}{\partial h}(x,0) > 0$ for $x > 1$. Furthermore, if we denote by $L(x)$ the quantity $L(x) = (b_2 - b_1)(\ln x) + b_1b_2\left(1 + \mathrm{e}^{\ln x\left(\frac{1}{b_1}-\frac{1}{b_2}\right)}\right)$, we have $\frac{\mathrm{d}L}{\mathrm{d}\ln x} =$

$(b_2 - b_1)\left(1 + e^{\ln x\left(\frac{1}{b_1} - \frac{1}{b_2}\right)}\right) > 0$ and $\lim_{x\to 0} L(x) = -\infty$ and $L(1) = 2b_1b_2 > 0$. There exists then $x^* < 1$ such that $\frac{\partial t}{\partial h}(x, 0) < 0$ for $x < x^*$ and $\frac{\partial t}{\partial h}(x, 0) > 0$ for $x > x^*$. □

Proof of Proposition 14 Let us denote by $R(x, h)$ the aggregate level of risk aversion at x when the individual levels of risk aversion are given by $R_1 - h$ and $R_2 + h$, we have $R(x, h) = \frac{(R_1-h)\exp(-\ln x(R_1-h))+(R_2+h)\exp(-\ln x(R_2+h))}{\exp(-\ln x(R_1-h))+\exp(-\ln x(R_2+h))}$ and $\frac{\partial R}{\partial h}(x, 0) = -\frac{e^{-2\ln xR_1} - e^{-2\ln xR_2} + 2\ln x(R_2-R_1)e^{-\ln xR_1}e^{-\ln xR_2}}{(e^{-\ln xR_1}+e^{-\ln xR_2})^2}$ which is clearly negative for $x > 1$ and positive for $x < 1$. □

Proof of Proposition 15 Let us denote by $U(x, 1)$ and $U(x, 2)$ the social utility functions, respectively, associated to P_1 and P_2. We denote, respectively, by $F(i, 1)$ and $F(i, 2)$ the cumulative distributions of P_1 and P_2. We have

$$U'(x, j) = \int \frac{\partial u}{\partial x}(x, i)F'(i, j)\mathrm{d}i, \quad j = 1, 2.$$

By assumption, $\frac{\partial u}{\partial x}(x, i)$ is log-supermodular. Furthermore, since $P_2 \succcurlyeq_{\mathrm{MLR}} P_1$, F' (i, j) is also LSPM. By Karlin's Theorem $U'(x, j)$ is log-supermodular. Therefore, $\frac{\partial \ln U'(x,j)}{\partial x}$ increases with j or in other words

$$-\frac{U''(x, 1)}{U'(x, 1)} \geq -\frac{U''(x, 2)}{U'(x, 2)}$$

which gives $R_1(x) \geq R_2(x)$. □

Proof of Proposition 16 Let us denote by $U(x, 1)$ and $U(x, 2)$ the social utility functions, respectively, associated to P_1 and P_2. Since $\frac{\partial u}{\partial x}(x, i)$ is increasing in i and $-\frac{u_i''(x)}{u_i'(x)}$ is decreasing in i, there exists a nonnegative and decreasing function Ψ^x such that $E\left[u_i'(x)\middle| -\frac{u_i''(x)}{u_i'(x)}\right] = \Psi^x\left(-\frac{u_i''(x)}{u_i'(x)}\right)$ for $i \in I$. We have then

$$-\frac{U''(x, j)}{U'(x, j)} = \frac{\int -\frac{u_i''(x)}{u_i'(x)}u_i'(x)\mathrm{d}P_j(i)}{\int u_i'(x, j)\mathrm{d}P_j(i)}, j = 1, 2$$
$$= \frac{E^{P_j^x}\left[X\Psi^x(X)\right]}{E^{P_j^x}\left[\Psi^x(X)\right]}.$$

By Portfolio Dominance we immediately have $\frac{E^{P_2^x}[X\Psi^x(X)]}{E^{P_2^x}[\Psi^x(X)]} \geq \frac{E^{P_1^x}[X\Psi^x(X)]}{E^{P_1^x}[\Psi^x(X)]}$. □

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