Deriving Risk Aversion Premium

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Dec. 19, 2016

Equation (5) $\sim$ (9) is derived by following Choi et al.(2007) pp.1931 $\sim$ 1932

Using definition of risk premium,

$$u(w_0(1-r)) = \alpha u(w_0(1-h) + (1-\alpha)u(w_0(1+h) \tag{5}$$

CASE 1. CARA

$$\frac{-exp(-\rho w_0(1-r))}{\rho} = \alpha\frac{-exp(-\rho w_0(1-h))}{\rho} + (1-\alpha)\frac{-exp(-\rho w_0(1+h))}{\rho} \tag{6}$$

Rearranging terms. Then,

$$r(h) = 1 + \frac{1}{pw_0}log\{\alpha exp(-\rho w_0(1-h)) + (1-\alpha)exp(-\rho w_0(1+h))\} \tag{7}$$

To use Talyor expansion, computing first and second derivatives

$$\begin{aligned}
r'(h) &= \frac{1}{\rho w_0}\frac{\alpha exp(-\rho w_0(1-h))\rho w_0 + (1-\alpha)exp(-\rho w_0(1+h))(-\rho w_0)}{\alpha exp(-\rho w_0(1-h)) + (1-\alpha)exp(-\rho w_0(1+h))} \\
&= \frac{\alpha exp(-\rho w_0(1-h)) - (1-\alpha)exp(-\rho w_0(1+h))}{\alpha exp(-\rho w_0(1-h)) + (1-\alpha)exp(-\rho w_0(1+h))} \\
r''(h) &= \frac{\rho w_0 K - \rho w_0\{\alpha exp(-\rho w_0(1-h)) - (1-\alpha)exp(-\rho w_0(1+h))\}^2}{K} \\
K &= \{\alpha exp(-\rho w_0(1-h)) + (1-\alpha)exp(-\rho w_0(1+h))\}^2
\end{aligned}$$

Then,

$$\begin{aligned}
r'(0) &= 2\alpha - 1 \\
r''(0) &= 4\rho w_0\alpha(1-\alpha)
\end{aligned}$$

$$r(h) \approx (2\alpha-1)h + (2\rho w_0\alpha(1-\alpha))h^2 \tag{8}$$

$$r(1) \approx (2\alpha-1) + (2\rho w_0\alpha(1-\alpha)) \tag{9}$$

CASE 2. CRRA

$$\frac{(w_0(1-r))^{(1-\rho)}}{1-\rho} = \alpha\frac{(w_0(1-h))^{(1-\rho)}}{1-\rho} + (1-\alpha)\frac{(w_0(1+h))^{(1-\rho)}}{1-\rho} \tag{6}$$

Rearranging terms. Then,

$$r(h) = 1 - \frac{1}{w_o}\{\alpha(w_0(1-h))^{(1-\rho)} + (1-\alpha)(w_0(1+h))^{(1-\rho)}\}^{\frac{1}{1-\rho}} \tag{7}$$

To use Talyor expansion, computing first and second derivatives

$$\begin{aligned}
r'(h) &= -\frac{1}{w_0(1-\rho)}K^{\frac{\rho}{1-\rho}}w_0^{1-\rho}(1-\rho)(-\alpha(1-h)^{-\rho} + (1-\alpha)(1+h)^{-\rho}) \\
r''(h) &= -\frac{1}{w_0(1-\rho)}\{\frac{\rho}{1-\rho}K^{\frac{2\rho-1}{1-\rho}}w_0^{2-2\rho}(1-\rho)^2(-\alpha(1-h)^{-\rho} + (1-\alpha)(1+h)^{-\rho})^2 \\
&\quad + K^{\frac{\rho}{1-\rho}}w_0^{1-\rho}(1-\rho)(-\alpha(1-h)^{-\rho-1}\rho + (1-\alpha)(1+h)^{-\rho-1}(-\rho))\} \\
K &= \alpha(w_0(1-h))^{(1-\rho)} + (1-\alpha)(w_0(1+h))^{(1-\rho)}
\end{aligned}$$

Then,

$$\begin{aligned}
r'(0) &= 2\alpha - 1 \\
r''(0) &= 4\rho\alpha(1-\alpha)
\end{aligned}$$

$$r(h) \approx (2\alpha-1)h + (2\rho\alpha(1-\alpha))h^2 \tag{8}$$

$$r(1) \approx (2\alpha-1) + (2\rho\alpha(1-\alpha)) \tag{9}$$