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ON A SYSTEM OF INEQUALITIES IN DEMAND ANALYSIS: AN EXTENSION OF THE CLASSICAL METHOD*

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1. INTRODUCTION

AN EXPENDITURE CORRESPONDENCE being defined as a correspondence between budget constraints and admitted commodity bundles, a demand system, or its associated expenditure system, appears as one special case, and another case is that of a finite correspondence. A theorem will be proved on the existence of a positive solution for a certain system of homogeneous linear inequalities. Such a system can be associated with any finite expenditure correspondence, together with a number e between 0 and 1 interpreted as a level of cost-efficiency. The existence of a solution is equivalent to the admissibility of the hypothesis that the consumer, whose behavior is represented by the correspondence, (i) *has a definite structure of wants*, represented by an order in the commodity space, and (ii) *programs at a level of cost-efficiency e*. Any solution permits the immediate construction of a semi-increasing polyhedral concave utility function which realizes the hypothesis. The method for this construction is shown. When $e = 1$ the utility function fits the data exactly, in the usual sense that its maximum under any budget constraint is at the corresponding commodity point, and when $e < 1$ it can be considered to fit it approximately, to an extent indicated by e. A formula is given for the critical cost-efficiency, defined as the upper limit of possible e. Standard demand analysis which involves a strict maximum under the budget constraint, expressed also by the "revealed preference" idea, is put in perspective with this approach.

2. EXPENDITURE CORRESPONDENCE

Let Ω_n, Ω^n denote the spaces of non-negative row and column vectors, Ω being the non-negative numbers. Then any $p \in \Omega_n$, $x \in \Omega^n$ have a product $px \in \Omega$, which can signify the cost of quantities x at prices p.

A *demand* is any price-quantity pair $(p, x) \in \Omega_n \times \Omega^n$ with $px > 0$. Then $u = M^{-1}p$, where $M = px$ is the expenditure, defines the associated *exchange vector*. It forms with x a pair (u, x) such that $ux = 1$ which can be called the *budget* associated with the demand.

A *utility relation* is any order in Ω^n, that is any $R \subset \Omega^n \times \Omega^n$ which is reflexive and transitive,

$$(2.1) \qquad xRx, \qquad xRyR\ldots Rz \Longrightarrow xRz.$$

A *utility function* is any $\phi : \Omega^n \rightarrow \Omega$. It represents a utility relation R if

$$(2.2) \qquad xRy \Longleftrightarrow \phi(x) \geqq \phi(y).$$

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Such representation for R implies it is complete

(2.3) $$xRy \vee yRx\,.$$

A *demand function* $F(p, M) \in \Omega^n$ ($p \in \Omega_n$, $M > 0$), by definition with the properties

(2.4) $$pF(p, M) = M\,, \quad F(\lambda p, \lambda M) = F(p, M)\,,$$

determines the *expenditure function*

(2.5) $$f(u) = F(u, 1)$$

with the property

(2.6) $$uf(u) = 1$$

from which it is again determined as

(2.7) $$F(p, M) = f(M^{-1}p)\,.$$

An *expenditure correspondence* is any $E \subset \Omega^n \times \Omega_n$ with the property

(2.8) $$xEu \Longrightarrow ux = 1\,.$$

Then an expenditure function corresponds to the case of a correspondence E with the properties

(2.9) $$Eu \neq 0\,, \quad x, yEu \Longrightarrow x = y\,,$$

that is, for all $u \in \Omega_n$, Eu has just one element x, which can be denoted $x = f(u)$, f being an expenditure function for which

(2.10) $$xEu \Longleftrightarrow x = f(u)\,.$$

Questions asked about a demand function are expressible in terms of the associated expenditure function, and can be applied just as well to a general expenditure correspondence, and in particular to a finite expenditure correspondence

(2.11) $$E = [(x_r, u_r) : r = 1, \ldots, k]$$

such as will be considered.

3. UTILITY—COST EFFICIENCIES

Consider a utility relation R and a demand (p, x) with $px > 0$. A relation $H^* = H^*(R; p, x)$ between them is defined by the condition

(3.1) (H^*) $$py \leqq px\,, \quad y \neq x \Longrightarrow xRy, y\bar{R}x\,,$$

which is to say x is strictly preferred to every other y which costs no more at the prices p. If R is represented by a utility function this condition is equivalent to

$$(3.2)\qquad (H^*)\qquad py \leqq px,\quad y = x \Longrightarrow \phi(x) > \phi(y)\,.$$

With $u = M^{-1}p$ where $M = px$, an equivalent statement in terms of the associated budget (u, x) is

$$(3.3)\qquad (H^*)\qquad uy \leqq 1\,,\quad y \neq x \Longrightarrow xRy,\quad y\bar{R}x\,.$$

This can be called the relation of *strict compatibility* between a utility relation, or function, and a demand, or its associated budget. An expenditure correspondence E being a set of budgets, strict compatibility $H_E^*(R)$ of R with E is definite by simultaneous compatibility of R with all the budgets of E:

$$(3.4)\qquad H_E^*(R) \equiv xEu\,,\quad uy \leqq 1\,,\quad y \neq x \Longrightarrow xRy\,,\quad y\bar{R}x\,.$$

The existence of an order R such that this holds is denoted H_E^*, and defines the *strict consistency* of E.

Now let further relations between a utility relation R and an expenditure correspondence E be defined by

$$(3.5)\qquad \begin{aligned} &H_E'(R) \equiv xEu\,,\quad uy \leqq 1 \Longrightarrow xRy \\ &H_E''(R) \equiv xEu\,,\quad yRx \Longrightarrow uy \geqq 1 \end{aligned}$$

with conjunction

$$(3.6)\qquad H_E(R) \equiv H_E'(R) \wedge H_E''(R)\,,$$

by which R and E can be said to be *compatible*. Thus H' signifies that x is as good as any y which costs no more at the prices p, or that maximum utility is obtained for the cost, and H'' signifies any y which is as good as x costs as much, or that the utility has been obtained at minimum cost. In the language of cost-benefit analysis, these are conditions of *cost-efficiency* and *cost-efficacy*. It is evident that

$$(3.7)\qquad H_E^*(R) \Longrightarrow H_E(R)\,,$$

that is, compatibility is implied by strict compatibility. Let H_E' be defined for H' in the same way as the similar conditions for H^*, and similarly with H' and H. Then H_E asserts the *consistency* of E.

It is noticed that $H_E'(R)$ derives from $H_E^*(R)$ just by replacing the requirement for an absolute maximum by a requirement for a maximum. But while H_E^*, and similarly H_E, is a proper condition, that is there exist E for which it can be asserted and other E for which it can be denied, H_E' is vacuous, since it is always validated by a constant utility function.

It can be remarked, incidentally, that if R is semi-increasing, $x > y \Longrightarrow xRy$ then $H' \Longrightarrow H''$. Also if R is lower-continuous, that is the sets $xR = [y : xRy]$ are closed, then $H'' \Longrightarrow H'$; this agrees with an observation of Debreu [3]. Accordingly if, for instance, R is represented by a continuous increasing utility function then H' and H'' are equivalent, so in their conjunction one is redundant, that is mathematically but not economically. But there is no need here to make any assumptions whatsoever about the order R.

It can be granted that as a basic principle H^* requiring an absolute maximum is unwarranted in place of the more standard H' which requires just a maximum. However, while H^* produces the well-known discussion of Samuelson [6] and Houthakker [5], described as revealed preference theory, that discussion is not generalized but its entire basis evaporates when H^* becomes H'. From this circumstance there is a hint that the nature of that theory is not properly gathered in its usual description. The critical feature of it is not that it deals with maxima under budget constraints but that it deals especially with absolute maxima. This might have intrinsic suitability, by mathematical accident, for dealing with continuous demand functions. But it is not a direct expression of normal economic principles, which recognize significance only for a maximum. If the matter is to be reinitiated, then H' is admitted as such a principle and so equally is H'', so their conjunction H comes into view as an inevitable basis required by normal economic principles. The question of H_E for an expenditure correspondence is proper, that is, capable of being true and false, unlike H'_E which is always true. Also, since $H^* \Longrightarrow H$, this provides a generalization of the theory with H^*. It happens, as the mathematical accident just mentioned, that if E is a continuous demand function then $H^*_E \Longleftrightarrow H_E$. Thus the distinctive revealed preference theory is not lost in this generalization but it just receives a reformulation which puts it in perspective with a normal and broader economic theory not admitting description as revealed preference theory, which moreover is capable of a further simple and necessary extension now to be considered.

4. PARTIAL EFFICIENCY

With an expenditure correspondence E interpreted as representing the behavior of the consumer, there is the hypothesis that the consumer (i) *has a definite structure of wants*, represented by a utility relation R, and (ii) *is an efficient programmer*. Then H_E is the condition of the consistency of the data E with that hypothesis. If it is not satisfied, so the data reject the hypothesis, the hypothesis can be modified. If (i) is not to be modified, either because there is no way of doing this systematically or because it is a necessary basic assumption, as it is for instance in economic index number theory, then (ii) must be modified. Instead of requiring exact efficiency, a form of partial efficiency, signified by a certain level of *cost-efficiency* e where $0 \leqq e \leqq 1$, will be considered. When $e = 1$ there is return to the original, exact efficiency model.

Thus consider a relation $H(R, e; p, x)$ between a demand (p, x) and a utility relation R together with a number e given by the conjunction H of conditions

$$(4.1) \qquad \begin{array}{ll} (H') & py \leqq Me \Longrightarrow xRy \\ (H'') & yRx \Longrightarrow py \geqq Me \end{array}$$

where $M = px$. They assert x is as good as any y which costs no more than the fraction eM of the cost M of x, at the prices p, and also any y as good as x costs at least that fraction. In the language of cost-benefit analysis these are conditions of cost-efficacy and cost-efficiency, but modified to allow a margin of waste, which is the fraction $(1 - e)M$ of the outlay M. It is noticed that if H is not to be satisfied vacuously then $e \geqq 0$; and then from H'', with R reflexive necessarily $e \leqq 1$.

With R given, for simplicity of illustration say by a continuous increasing strictly quasiconcave function ϕ, and with $p > 0$ and M fixed, it can be seen what varying tolerance this condition gives to x as e increases from 0 to 1. When $e = 0, x$ is permitted to be any point in the budget simplex B described by $px = M$, $x \geqq 0$. When $e = 1$, x is required to be the unique point x on B for which

$$\phi(x) = \max [\phi(y) : py = M]. \tag{4.2}$$

For $0 \leqq e \leqq 1$ let x_e be the unique point in the set B_e described by $px = Me$ for which

$$\phi(x_e) = \max [\phi(y) : py = Me]. \tag{4.3}$$

Then x is required to be in the convex set $S_e \subset B$ defined by

$$\phi(x) \geqq \phi(x_e), \quad px = M. \tag{4.4}$$

Evidently, if

$$0 \leqq e \leqq e' \leqq 1 \tag{4.5}$$

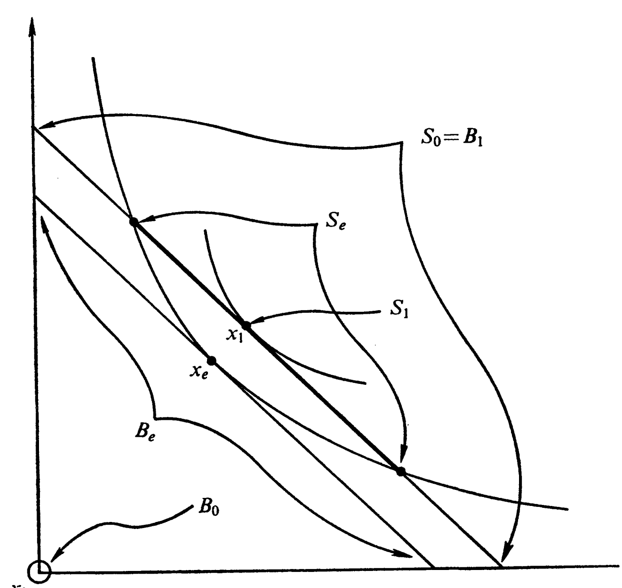

FIGURE 1

then

$$B = S_0 \supset S_e \supset S_{e'} \supset S_1 = \{x_1\} . \tag{4.6}$$

That is, the tolerance regions S_e for x for M a nested family of convex sets, starting at the entire budget simplex B when $e = 0$ and, as e increases to 1, shrinking to the single point x_1 attained when $e = 1$. The higher the level of cost-efficiency the less the tolerance, and when cost-efficiency is at its maximum 1 all tolerance is removed: the consumer is required, as usual, to purchase just that point which gives the absolute maximum of utility.

For an expenditure correspondence E, now define

$$H_E(R, e) \equiv (\wedge\, xEu)H(R, e; u, x) \tag{4.7}$$

as *compatibility of E with R at the level of cost-efficiency e.* Then

$$H_E(e) = (\vee\, R)H_E(R, e) \tag{4.8}$$

will define e-consistency of E, or consistency at the level of cost-efficiency e. Immediately

$$H_E(1) \Longleftrightarrow H_E , \tag{4.9}$$

so 1-consistency of E is identical with the formerly defined consistency. Also

$$H_E(0) \Longleftrightarrow T , \tag{4.10}$$

in other words 0-consistency is valid for every E. Further

$$H_E(e) , \quad e' \leqq e \Longrightarrow H_E(e') , \tag{4.11}$$

that is, consistency at any level of cost-efficiency implies it at every lower level. Hence with

$$e_E = \sup\, [e : H_E(e)] , \tag{4.12}$$

defining the *critical cost-efficiency* of any expenditure correspondence E it follows from (4.9), (4.10) and (4.11) that

$$0 \leqq e_E \leqq 1 , \tag{4.13}$$

$$e < e_E \Longrightarrow H_E(e) , \quad e > e_E \Longrightarrow \bar{H}_E(e) .$$

5. CONSISTENCY TEST

The condition $H_E(e)$ will now be investigated on the basis of a finite correspondence E with elements (u_t, x_t), $t = 1, \ldots, k$ derived from a set of demands (p_t, x_t), with $u_t = M_t^{-1}p_t$ where $M_t = p_t x_t$. Let $D_{rs} = u_r x_s - 1$, and for any w with $0 \leqq w \leqq 1$ let $D_{rs}^w = D_{rs} + w$. If e is a cost-efficiency, so $0 \leqq e \leqq 1$, then $1 - e = w$ is a cost-innefficiency, and $0 \leqq w \leqq 1$.

With E now fixed, $H_E(e)$ will be denoted H^w. A further condition on E, defined for every w, is

(5.1) $\qquad K^w \equiv (D^w_{rs}, D^w_{st}, \ldots, D^w_{qr}) \leq 0$ impossible for all $r, s, \ldots, q$.

An alternative statement is

(5.2) $\qquad D^w_{rs} \leqq 0,\ D^w_{st} \leqq 0, \ldots, D^w_{qr} \leqq 0 \Longrightarrow D^w_{rs} = \ldots = D^w_{qr} = 0.$

In the following K^w appears as a necessary condition for H^w, and later it will be proved also sufficient.

THEOREM 1. $H^w \Longrightarrow K^w$.

By definition H^w means there exists an order R such that

(5.3)
$$\begin{array}{ll} (H') & u_r x \leqq 1\text{-}w \Longrightarrow x_r R x \\ (H'') & x R x_r \Longrightarrow u_r x \geqq 1\text{-}w \end{array}$$

for $r = 1, \ldots, k$. Therefore, with $x = x_s$, and

(5.4)
$$D^w_{rs} = D_{rs} + w = u_r x_s - 1 + w\,,$$

it follows that

(5.5)
$$\begin{array}{ll} (H') & D^w_{rs} \leqq 0 \Longrightarrow x_r R x_s \\ (H'') & x_s R x_r \Longrightarrow D^w_{rs} \geqq 0\,. \end{array}$$

Hence

(5.6)
$$D^w_{rs} \leqq 0,\ D^w_{st} \leqq 0, \ldots, D^w_{qr} \leqq 0$$

by (H') in (5.5) implies

(5.7)
$$x_r R x_s R x_r R \ldots x_q R x_r$$

which by transitivity implies

(5.8)
$$x_r R x_q \ldots R x_t R x_s R x_r$$

which by (H'') in (5.6) implies

(5.9)
$$D^w_{qr} \geqq 0, \ldots, D^w_{st} \geqq 0,\ D^w_{rs} \geqq 0\,,$$

which with (5.6) implies

(5.10)
$$D^w_{rs} = D^w_{st} = \ldots = D^w_{qr} = 0\,.$$

Thus H^w implies (5.6) implies (5.9), that is H^w implies K^w, as required.

6. CONSTRUCTION ALGORITHM

For any $\lambda_r > 0$ and $\phi_r > 0$ consider the condition

(6.1)
$$C^w(\lambda, \phi) \equiv \lambda_r S^w_{rs} \geqq \phi_s - \phi_r$$

and let C^w be the assertion that there exist such λ_r, ϕ_r. Also introduce the function

(6.2) $$\phi(x) = \min_r [\phi_r + \lambda_r(u_r x - 1 + w)],$$

which is polyhedral concave, from its form of definition, and semi-increasing, since $u_r \geq 0$, $\lambda_r > 0$, so that $\lambda_r u_r \geq 0$. The relation R it represents is such that

(6.3) $$xRy \Longleftrightarrow \phi(x) \geqq \phi(y).$$

THEOREM 2. $C^w(\lambda, \phi) \Longrightarrow H^w(R)$. *That is, if λ_r, ϕ_r satisfy the above inequalities then E is compatible with R and a level of cost-efficiency $e = 1-w$.*

Thus, a restatement of $C^w(\lambda, \phi)$ is

(6.4) $$\phi_r + \lambda_r(u_r x_s - e) \geqq \phi_s$$

which shows that

(6.5) $$\phi(x_s) \geqq \phi_s.$$

In any case

(6.6) $$\phi_s + \lambda_s(u_s x - e) \geqq \phi(x)$$

so now, since $\lambda_r > 0$,

(6.7) $$\begin{aligned} u_s x < e &\Longrightarrow \phi_s > \phi_s + \lambda_s(u_s x - e) \\ &\Longrightarrow \phi_s > \phi(x) \\ &\Longrightarrow \phi(x_s) > \phi(x). \end{aligned}$$

Hence

(6.8) $$u_s x < e \Longrightarrow \phi(x_s) > \phi(x),$$

and similarly, or just because ϕ here is continuous,

(6.9) $$u_s x \leqq e \Longrightarrow \phi(x_s) \geqq \phi(x).$$

This demonstrates $H^w(R)$ with $w = 1-e$. Thus the theorem is proved.

7. CRITICAL EFFICIENCY

It is now proved that

(7.1) $$H^w \Longrightarrow K^w, \quad C^w \Longrightarrow H^w,$$

so in order to prove that $K^w \Longrightarrow H^w$, and hence that

(7.2) $$H^w \Longleftrightarrow K^w,$$

it suffices to prove that

(7.3) $$K^w \Longrightarrow C^w.$$

The foregoing discussion shows a motive for the main theorem which is going to be proved in Section 8, which is the equivalence of K^w and C^w. It is noticed also that the consequent equivalence of H^w and K^w provides a simple procedure for determining the critical cost efficiency e_E of the given finite correspondence E. Denoting this by $\bar{e}$, and with $\bar{w} = 1 - \bar{e}$, so that

$$(7.4) \qquad \bar{w} = \inf[w : H^w]$$

the equivalence (7.2) shows that also

$$(7.5) \qquad \bar{w} = \inf[w : K^w]$$

and with this expression $\bar{w}$ is easy to evaluate.

Thus, let

$$(7.6) \qquad d = \min_{rs\ldots q} \max[D_{rs}, D_{st}, \ldots, D_{qr}]$$

so, the condition K^0 being denoted K, clearly

$$(7.7) \qquad d > 0 \Longrightarrow K, \quad d < 0 \Longrightarrow \bar{K}$$

though $d = 0$ leaves K undecided. Moreover

$$(7.8) \qquad d \leqq 0 \Longrightarrow d + w > 0 \Longleftrightarrow K^w .$$

This shows that

$$(7.9) \qquad \begin{aligned} d \geqq 0 &\Longrightarrow \bar{w} = 0 \\ d < 0 &\Longrightarrow \bar{w} = -d . \end{aligned}$$

THEOREM 3. $\bar{w} = \max[0, -d]$.

In the following, the set of numbers D_{rs} can, for simplicity, stand for the former set of numbers $D_{rs}^w = D_{rs} + w$ for any w. Since the earlier $D_{rr} = 0$, so that $D_{rr}^w \geqq 0$ for all $w \geqq 0$, it is a set of numbers D_{rs} with $D_{rr} \geqq 0$ that will be considered.

8. THEOREM ON INEQUALITIES

For any numbers $D_{rs}(r, s = 1, \ldots, k)$ with $D_{rr} \geqq 0$, the following conditions are defined:

(8.1) (C) *There exist* $\lambda_r > 0, \ \phi_r > 0$
such that $\lambda_r D_{rs} \geqq \phi_s - \phi_r .$

(8.2) (S) *There exist* $\theta_{rs} \geqq 0$
such that $\sum_s \theta_{rs} = \sum_s \theta_{sr}$
and $\sum_s D_{rs}\theta_{rs} \leqq 0 .$

(8.3) (K) $(D_{rs}, D_{st}, \ldots, D_{qr}) \leqq 0$
is impossible for all (distinct) $r, s, t, \ldots, q$.

The purpose is to prove the equivalence of C and K. The procedure will be to show equivalences between C and $\bar{S}$ and between S and $\bar{K}$, $\bar{S}$ and $\bar{K}$ denoting the denials of S and K.

Here $x \leq 0$ for a vector $x = (x_1, \ldots, x_m)$, or the statement $x_r \leq 0$, means $x_r \leqq 0$ for all r and $x_r < 0$ for some r.

Evidently in (K), with the proviso $D_{rr} \geqq 0$, no different condition results when $r, s. \ldots$ are restricted to be distinct, but with them distinct the condition is in a finitely testable form.

The following is required.

LEMMA. *For any matrix a, either $ax \geqq 0$ for some $x > 0$ or $ua \leq 0$ for some $u \geqq 0$, and not both.*

This can be deduced from the standard result: for any matrix a and vector q, either $ax \geqq q$ for some x or $ua = 0$, $uq = 1$ for some $u \geqq 0$ and not both (see e.g., Gale [4]).

Thus, $ax \geqq 0$ for some $x > 0$ if and only if $ax \geqq 0$ for some $x \geqq I$, where I is the vector with all elements 1, that is

$$\begin{pmatrix} a \\ 1 \end{pmatrix} x \geqq \begin{pmatrix} 0 \\ I \end{pmatrix} \quad \text{for some} \quad x\,,$$

1 being the unit matrix. But, by the standard result, this is false if and only if

$$(u\ v)\begin{pmatrix} a \\ 1 \end{pmatrix} = 0, \quad (u\ v)\begin{pmatrix} 0 \\ I \end{pmatrix} = 1 \quad \text{for some} \quad (u\ v) \geq 0\,.$$

This is

$$ua + v = 0,\ vI = 1 \quad \text{for some} \quad u \geqq 0,\ v \geqq 0\,,$$

equivalently $ua \leq 0$ for some $u \geqq 0$.

THEOREM 4. *C is equivalent to $\bar{S}$.*

Let $a = (b\ c)$ be the partitioned matrix of order $k^2 \times 2k$ where in the row with index $i = (r\ s)$ the elements are

$$\begin{aligned} b_{ij} &= D_{rs} && \text{if } j = r \\ &= 0 && \text{otherwise}. \\ c_{ik} &= k && \text{if } k = r \\ &= -1 && \text{if } k = s \\ &= 0 && \text{otherwise}. \end{aligned}$$

Let $x = \begin{pmatrix} \lambda \\ \phi \end{pmatrix}$ be the partitioned vector of order $2k \times 1$ with elements λ_j and ϕ_k. Then

$$ax = b\lambda + c\phi$$

is the vector of order $k^2 \times 1$ whose element with index $i = (r, s)$ is

$$\begin{aligned}(ax)_i &= (b\lambda)_i + (c\phi)_i \\ &= \sum_j b_{ij}\lambda_j + \sum_k c_{ik}\phi_k \\ &= D_{rs}\lambda_r + \phi_r - \phi_s\,.\end{aligned}$$

Hence $ax \geqq 0$, with $x > 0$, is equivalent to the system (C).

Let u be the vector of order $1 \times k^2$ whose element with index $i = (r, s)$ is $u_i = \theta_{rs}$. Then

$$ua = (ub\ uc)$$

is the partitioned vector of order $1 \times 2k$ with elements

$$\begin{aligned}(ub)_j &= \sum_i u_i b_{ij} \\ &= \sum_{rs} \theta_{rs} b_{rs\,,\,j} \\ &= \sum_s \theta_{js} D_{js} \\ (uc)_k &= \sum_i u_i c_{ik} \\ &= \sum_{rs} \theta_{rs} c_{rs\,,\,k} \\ &= \sum_s \theta_{ks} - \sum_r \theta_{rk}\,.\end{aligned}$$

Now $ua \leq 0$ is equivalent to $ub \leqq 0$ and $uc \leqq 0$ together with the requirement that $ub \leq 0$ or $uc \leq 0$. However,

$$\sum_k (uc)_k = \sum_{ks} \theta_{ks} - \sum_{rk} \theta_{rk} = 0$$

so that $uc \leqq 0$ is equivalent to $uc = 0$, since a sum of non-positive terms is zero if and only if each term is zero. It follows that $ua \leq 0$ is equivalent to $ub \leq 0$ and $uc = 0$. Hence $ua \leq 0$ with $u \geqq 0$ is equivalent to the system (S). It follows now by the Lemma that (C) and (S) are exclusive exhaustive possibilities, which was to be proved.

A matrix θ whose elements in any corresponding row and column have the same sum can be called *sum-symmetric.* Doubly-stochasitic matrices are a particular example. Any cyclic permutation σ on a subset of $1, \ldots, k$ is represented by a cyclic sequence of distinct elements where each element corresponds to its successor, and the last corresponds to the first. Thus, with $\sigma = (r, s, \ldots, q)$, let $1\sigma = 1_{rs\ldots q}$ be the matrix with elements $(1_\sigma)_{ij} = 1$ if $\sigma i = j$ and otherwise 0, that is, the elements in positions $rs, st, \ldots, qr$ are 1 and all others are zero. Such a matrix is non-negative sum-symmetric. It can be shown that any sum-symmetric $\theta \geqq 0$ is expressible in the form

$$\theta = \sum \lambda_\sigma 1_\sigma$$

for $\lambda_\sigma \geqq 0$ defined on cycles σ (Afriat [**2**]).

THEOREM 5. *S is equivalent to $\bar{K}$.*

Assume, $\bar{K}$, that is

$$D_{rs} < 0\,,\ D_{st} \leqq 0\,,\ \ldots,\ D_{qr} \leqq 0$$

for some distinct $r, s, \ldots, q$. Then S is verified with $\theta = 1_{rs,,,q}$.

Now assume S, say

(i) $$\sum_j D_{ij}\theta_{ij} \leqq 0 \quad \text{for all } i\,,$$

while

(ii) $$\sum_j D_{rj}\theta_{rj} < 0\,,$$

for some $\theta \geqq 0$ such that

(iii) $$\sum_j \theta_{ij} = \sum_j \theta_{ij} \qquad \text{for all } i\,.$$

From (ii), with $\theta \geqq 0$, it follows that

$$D_{rs} < 0,\ \theta_{rs} > 0$$

for some s. Then by (iii), $\theta_{st} > 0$ for some t. But then (i) with $i = s$ implies $\theta_{st} > 0$, $D_{st} \leq 0$ for some t. Similarly t has a successor, and the sequence $r, s, t, \ldots$ can continue until it arrives at an element q whose successor repeats some earlier element r', since the number of elements is finite.

Then a sequence

$$r, s, t, \ldots, r', s', t', \ldots, q$$

has been constructed, where q is followed by r', so r' and its successors form a cycle. If $r = r'$, then

$$D_{rs} < 0,\ D_{st} \leqq 0, \ldots, D_{qr} \leqq 0\,,$$

so $\bar{K}$ is demonstrated. If $r \neq r'$ but

$$(D_{r's'}, D_{s't'}, \ldots, D_{qr'}) \leq 0$$

then again $\bar{K}$ is demonstrated. Otherwise $r \neq r'$ and

(iv) $$D_{r's'} = D_{s't'} = \ldots = D_{qr'} = 0\,.$$

while

(v) $$\lambda = \min\,(\theta_{r's'}, \theta_{s't'}, \ldots, \theta_{qr'}) > 0\,.$$

Let

$$\bar{\theta} = \theta - \lambda 1_{r's'\ldots q}\,.$$

Then because of (iv) and (v) it appears that (i), (ii) and (iii) are still satisfied with $\bar{\theta}$ in place of θ. Then the sequence $r, s, \ldots, r'$ can be continued, but the first continuation to q and back to r' cannot be repeated since some one of $\bar{\theta}_{r's'}, \ldots, \bar{\theta}_{qr'}$ is zero. By eliminating in this way every continuation which does not return to r, finally a sequence will be obtained which returns to r, and $\bar{K}$ will be demonstrated. Thus S implies $\bar{K}$ and the proof of equivalence is complete.

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