INTERNATIONAL ECONOMIC REVIEW
Vol. 8, No. 1, February, 1967

THE CONSTRUCTION OF UTILITY FUNCTIONS FROM EXPENDITURE DATA*

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IN CONSIDERING the behavior of the consumer, a market is assumed which offers some n goods for purchase at certain prices and in whatever quantities. A purchase requires an expenditure of money

$$e = \pi_1\xi_1 + \cdots + \pi_n\xi_n = p'x$$

which is determined as the scalar product of the vector $x = \{\xi_1, \cdots, \xi_n\}$ of quantities, which shows the *composition* of the purchase, and the vector $p = \{\pi_1, \cdots, \pi_n\}$ of prevailing prices, where braces { } denote a column vector, and a prime its transposition. The classical assumption about the consumer is that any purchase is such as to give a maximum of utility for the money spent. The consumer is supposed to attach a number $\phi(x)$ to any purchase, according to its composition x, which is the measure of the utility, to the effect that a purchase with composition x made at price p and, therefore, requiring an expenditure, $e = p'x$, is such as to satisfy the maximum utility condition

$$\phi(x) = \max\{\phi(y): p'y \leqq e\} .$$

An equivalent statement of this condition is

$$\phi(x) = \max\{\phi(y): u'y \leqq 1\} ,$$

where $u = p/e$ is the vector of prices divided by expenditure, that is with expenditure taken as the unit of money and is to be called the *balance* vector, corresponding to those prices and that expenditure. The fundamental property required for a *utility function* $\phi(x)$ is that, given a balance u, any composition x which is determined by the condition of maximum utility satisfies $u'x = 1$, so that

$$u'y \leqq 1 \Rightarrow \phi(y) \leqq \phi(x)$$

and

$$\phi(y) \geqq \phi(x) \Rightarrow u'y \geqq 1 .$$

Such an assumption cannot represent necessary deliberations on the part of the consumer. Any actual consumer is quite unaware of the attachment to such a function ϕ, and can even deny by intention and manifest behavior any such attachment. Then if ϕ is to have a proper existence, it would have

* Manuscript received December, 1964.

[1] This work is part of a project on "The Analysis of Consumers' Preferences and the Construction of Index Numbers," conducted at the Econometric Research Program, Princeton University and at Rice University and the Cowles Foundation at Yale University with the support of the National Science Foundation.

to be in the stock of analytical construction of those who entertain the assumption, and based on data of observation. In the earliest form, as the one used by Gossen, Jevons, Menger and Walras, it was assumed that the utility of a composition of goods was the sum of utilities for the separate goods

$$\phi(x) = \phi_1(x_1) + \cdots + \phi_n(x_n) .$$

Edgeworth then considered a general function

$$\phi(x) = \phi(x_1, \cdots, x_n) ,$$

and he also considered the indifference surfaces, the level surfaces $\phi =$ constant of the utility function. But the now familiar approach which is divorced from numerical utility and deals only with indifference surfaces was established by Pareto. Before this the utility analysis in demand theory dealt with utility and utility differences as measurable quantities. By rendering numerical utility inessential, Pareto brought relief to the discomfort of having to assume a measurable utility, the measurability of which was held in doubt.

Here the concern is with the utility function only as a measure of preference for deciding for better or worse between a collection of goods. But never through the long drawn out history of the hypothesis has such a function been generally shown. The revealed preference principle of Samuelson [5], elaborated by Houthakker [4], easily gives a condition for the rejection of the hypothesis of existence. But the principle has been absent by which the hypothesis can be accepted or rejected on the basis of any observed choice of the consumer, supposed to be finite in number; and, in the case of acceptance, a general method is needed for the actual construction of a utility function which will realize the hypothesis for the data.

This problem will be discussed here. For the general problem which arises when the finiteness restriction is removed, one possible approach is by a limiting process, proceeding on the basis of the results which are going to be obtained. It is more general than the problem considered by Samuelson [5], Houthakker [4], Uzawa [6], Afriat [2] and others which involves a demand system and, therefore, quantities for every price situation; that is, a complete system of data. For the data could be assumed infinite but not necessarily complete. Also, even with completeness, the usual assumption of a single valued demand system could be omitted. Or, if a single valued function is assumed, the Lipschitz-type condition assumed by Uzawa [6] and, therefore, also the differentiability assumed by other writers can be dropped. In the familiar investigations, the assumptions have been such as to yield just one functionally independent utility function. In the finite problem, and even in the infinite problem with completeness assumed, there is no such essential uniqueness.

While the results for finite data do not immediately give results for complete data, such as for a demand system, it is also the case that the familiar investigations on demand systems seeking conditions for the existence of a utility function have no scope for the finite problem now to be considered. Those investigations depend on a continuous, even a differentiable structure, which can have no bearing here in the discrete finite case. They do not take into consideration the problem of establishing criteria by which any finite expenditure data can be taken as arising from some complete demand system which satisfies the appropriate conditions.

Let it be supposed that the consumer has been observed on some k occasions of purchase, and the expenditure data obtained for each occasion $r\,(r = 1, \cdots, k)$ provide the pair of vectors (x_r, p_r) which give the composition of purchase and the prevailing prices. Hence the expenditure is $e_r = p_r'x_r$ and the balance vector is $u_r = p_r/e_r$; and, by definition, $u_r'x_r = 1$. Let $E_r = (x_r \mid u_r)$ define the *expenditure figure* for occasion r, and $E = \{E_r \mid r = 1, \cdots, n\}$ the *expenditure configuration* constructed from the data. Only through this configuration does the utility hypothesis have bearing on the data.

The utility hypothesis applied to the configuration E asserts that there exists a utility function φ such that

$$\varphi(x_r) = \max\{\varphi(x) \mid u_r'x \leqq 1\} \qquad (r = 1, \cdots, n)$$

in which case the function φ can be said to exhibit the utility hypothesis for E or to be a utility function *for* E. The data E can be said to have the property of *utility consistency* if the utility hypothesis can be exhibited for it by some function, in other words if it has a utility function.

Now there is the problem of deciding, for any given expenditure configuration E, whether or not it has the property of utility consistency, and, if it has, of constructing a utility function for it.

If utility consistency holds for E, some utility function φ exists for it, and then

$$u_r'x_s \leqq 1 \Rightarrow \varphi(x_r) \geqq \varphi(x_s)$$

and

$$u_r'x_s \leqq 1 \wedge \varphi(x_r) = \varphi(x_s) \Rightarrow u_r'x_s = 1$$

for all $r, s = 1, \cdots, k$. Hence, for all $r, s, \cdots, q = 1, \cdots, k$

$$\begin{aligned} &u_r'x_s \leqq 1 \wedge u_s'x_t \leqq 1 \wedge \cdots \wedge u_q'x_r \leqq 1 \\ &\quad \Rightarrow \varphi(x_r) \geqq \varphi(x_s) \geqq \cdots \geqq \varphi(x_q) \geqq \varphi(x_r) \\ &\quad \Rightarrow \varphi(x_r) = \varphi(x_s) = \cdots = \varphi(x_q)\,. \end{aligned}$$

Hence

$$\begin{aligned} &u_r'x_s \leqq 1 \wedge u_s'x_t \leqq 1 \wedge \cdots \wedge u_q'x_r \leqq 1 \\ &\quad \Rightarrow u_r'x_s = u_s'x_t = \cdots = u_q'x_r = 1\,. \end{aligned}$$

This condition will define the property of *cyclical consistency* for E. It has been shown to be an obviously necessary condition for utility consistency, and it is going to be proved also sufficient. In order to do this, some other consistency conditions will be introduced for E, and finally they will all be proved equivalent. Define $D_{rs} = u_r'x_s - 1$, which may be called the *cross-*

coefficient, from E_r to E_s. The cross-coefficients altogether define the *cross-structure* D for the expenditure configuration E.

The cyclical consistency condition now has the statement

$$D_{rs} \leqq 0 \ , \ D_{st} \leqq 0, \cdots, D_{qr} \leqq 0 \Rightarrow D_{rs} = D_{st} = \cdots = D_{qr} = 0$$

for all $r, s, t, \cdots, q = 1, \cdots, k$. Since a multiple cycle is just a conjunction of simple cycles, and since $D_{rr} = 0$, there is no restriction in assuming $r, s, t, \cdots, q = 1, \cdots, k$ all distinct.

Let a new consistency condition now be defined for E, again through its cross-structure D, by the existence of numbers $\lambda_r (r = 1, \cdots, k)$, to be called *multipliers* for E, satisfying the system of inequalities

$$\lambda_r > 0 \ , \qquad \lambda_r D_{rs} + \lambda_s D_{st} + \cdots \lambda_q D_{qr} \geqq 0 \ ,$$

for all $r, s, t, \cdots, q = 1, \cdots, k$. The consistency of this system of inequalities, in other words the existence of multipliers for E, will define the condition of *multiplier consistency* for E. Again, the same condition is obtained if $r, s, t, \cdots, q = 1, \cdots, k$ are taken to be distinct.

It is obvious that multiplier consistency implies cyclical consistency. For

$$\lambda_r > 0 \wedge D_{rs} \leqq 0 \rightarrow \lambda_r D_{rs} \leqq 0 \ ,$$

and

$$\lambda_r D_{rs} \leqq 0 \wedge \lambda_s D_{st} \leqq 0 \wedge \cdots \wedge \lambda_q D_{qr} \leqq 0$$

with

$$\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qr} \geqq 0$$

implies

$$\lambda_r D_{rs} = \lambda_s D_{st} = \cdots = \lambda_q D_{qr} = 0 \ ,$$

which, with $\lambda_r > 0$, implies

$$D_{rs} = D_{st} = \cdots = D_{qr} = 0 \ .$$

Therefore, multiplier consistency implies cyclical consistency. The converse is also true, as will eventually appear.

Now let still another condition be defined for E through its cross-structure, by the existence of numbers $\lambda_r, \varphi_r (r = 1, \cdots, k)$, to be called *multipliers* and *levels*, satisfying the system of inequalities

$$\lambda_r > 0, \quad \lambda_r D_{rs} \geqq \varphi_s - \varphi_r \qquad\qquad (r, s = 1, \cdots, k) \ .$$

The consistency of this system of inequalities will define the condition of *level consistency* for E.

It is obvious that level consistency implies multiplier consistency, and moreover that any multipliers which realize the level consistency condition also realize the multiplier consistency condition. For, from

$$\lambda_r D_{rs} \geqq \varphi_s - \varphi_r$$

follows

$$\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qr} \geqq \varphi_s - \varphi_r + \varphi_t - \varphi_s + \cdots + \varphi_q - \varphi_r = 0 \ .$$

It will be shown that, conversely, multiplier consistency implies level consistency, and moreover, that any set of multipliers which realized the multiplier consistency condition can be joined with a set of levels to realize the level consistency condition.

THEOREM. *The three conditions of cyclical, multiplier and level consistency on the cross-structure of an expenditure configuration are all equivalent, and are implied by the condition of utility consistency for the configuration.*

It has been seen that utility consistency for the configuration E implies cyclical consistency for its cross-structure D. Also it has been seen that level consistency implies multiplier consistency and that multiplier consistency implies cyclical consistency for D. Hence it remains to be shown that cyclical consistency implies multiplier consistency, and that multiplier consistency implies level consistency, and then the theorem will have been proved.

Introduce the relation W defined by

$$rWs \equiv D_{rs} \leqq 0 \,,$$

it being reflexive, since $D_{rr} = 0$, and then $R = \vec{W}$, the transitive closure of W, this being transitive and such that $W \subset R$, from the form of its definition, and reflexive, since W is reflexive. Then $P = R \cap \bar{R}'$, the antisymmetric part of R, is antisymmetric, from the form of the definition, and transitive, since R is transitive. Hence it is an order. In case it is not a total order, there always exist a total order which is a refinement of it, that is $R \subset T$ where T is a total order, and $T \subset \bar{T}'$, since T is antisymmetric. Without loss in generality, it can be supposed that the occasions are so ordered that $rTs \equiv r < s$.

Now cyclical consistency is equivalent to the condition

$$D_{rs} \leqq 0 \wedge C_{st} \leqq 0 \wedge \cdots \wedge D_{pq} \leqq 0 \Rightarrow D_{qr} \geqq 0$$

which can be stated as

$$R \subset M' \,,$$

where M is the relation defined by

$$rMs \equiv D_{rs} \geqq 0$$

such that

$$rMs \Leftarrow D_{rs} > 0 \Leftrightarrow r\bar{W}s$$

so that

$$\bar{W} \subset M \,.$$

Now cyclical consistency gives $R \subset M'$; and the definition of R gives $W \subset R$, so that $\bar{R}' \subset \bar{W}' \subset M'$. Hence $R \cup \bar{R}' \subset M'$. But $R \cap \bar{R}' \subset T \subset \bar{T}'$ so that $T \subset \bar{R}' \cup R$. Hence $T \subset M'$, or equivalently

$$r < s \Rightarrow D_{sr} \geqq 0 \,.$$

Now assume, as an inductive hypothesis, that, at an $(m-1)$-th stage, multipliers $\lambda_r > 0\,(1 \leqq r < m)$ have been found such that

$$\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qr} \geqq 0 \qquad (1 \leqq r, \cdots, q < m) \,.$$

Then, for the m-th stage to be attained, it is required to find a multiplier $\lambda_m > 0$ such that

$$\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qm} + \lambda_m D_{mr} \geqq 0 \qquad (1 \leqq r, \cdots, q \leqq m-1) .$$

But $D_{mr} \geqq 0$ if $r < m$. Hence let

$$\mu_m = -\min \left\{ \frac{\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qm}}{D_{mr}} \middle| \, 1 \leqq r, \cdots, q < m; D_{mr} > 0 \right\} .$$

Then any $\lambda_m \geqq \max\{0, \mu_m\}$ is as required. Hence the m-th stage is attainable from the $(m-1)$-th. The second stage can obviously be attained, since, with any $\lambda_1 > 0$, there only has to be taken a $\lambda_2 > 0$ such that $\lambda_1 D_{12} + \lambda_2 D_{21} \geqq 0$, which is possible since $D_{12} < 0$ and $D_{21} < 0$ is impossible, by the hypothesis of cyclical consistency. It follows by induction that the k-th stage is attainable, that is, multipliers can be found which realize the multiplier consistency condition. The proof that cyclical consistency implies multiplier consistency is now complete.

To prove that multiplier consistency implies level consistency, assume a set of multiplier λ_r and let $a_{rs} = \lambda_r D_{rs}$. Then

$$a_{rs} + a_{st} + \cdots + a_{qr} \geqq 0 ,$$

for all distinct $r, s, t, \cdots, q$. It is now going to be proved that there exist numbers φ_r such that

$$\text{(a)} \qquad a_{rs} \geqq \varphi_r - \varphi_s \qquad (r \neq s) ,$$

whence the level consistency condition will have been shown. Let

$$a_{rlm\cdots ps} = a_{rl} + a_{lm} + \cdots + a_{ps} ,$$

and let

$$A_{rs} = \min_{l, m, \cdots, p} a_{rlm\cdots ps} .$$

Then

$$a_{rs} \geqq A_{rs} .$$

Also,

$$A_{rs} + A_{sr} \geqq 0 \text{ and } A_{rs} + A_{st} \geqq A_{rt} .$$

Consider the system

$$\text{(A)} \qquad A_{rs} \geqq \varphi_r - \varphi_s \qquad (r \neq s) .$$

Any solution φ_r of (A) is a solution of (a), since $a_{rs} \geqq A_{rs}$. Also, any solution φ_r of (a) is a solution of (A). For

$$a_{rl} \geqq \varphi_r - \varphi_l , \qquad a_{lm} \geqq \varphi_l, \cdots, a_{ps} \geqq \varphi_p - \varphi_s;$$

whence, by addition,

$$\begin{aligned} a_{rlm\cdots ps} &\geqq \varphi_r - \varphi_l - \varphi_m + \cdots + \varphi_p - \varphi_s \\ &= \varphi_r - \varphi_s , \end{aligned}$$

and, therefore,

$$A_{rs} \geqq \varphi_r - \varphi_s .$$

It follows that the consistency of (a), which has to be shown, is equivalent to that of (A), which will be shown now.

The proof depends on an extension property of solution of the subsystems of (A). Thus, assume a solution $\varphi_r (r < m)$ has been found for the subsystem

$$\text{(A, m} - 1) \qquad A_{rs} \geqq \varphi_r - \varphi_s \qquad (r \neq s;\, r, s < m) .$$

It will be shown that it can be extended by an element φ_m to a solution of (A, m).

Thus, there is to be found a number φ_m such that

$$A_{rm} \geqq \varphi_r - \varphi_m, \qquad A_{ms} \geqq \varphi_m - \varphi_s \qquad (r, s < m)$$

that is,

$$A_{ms} + \varphi_s \geqq \varphi_m \geqq \varphi_r - A_{rm} .$$

So the condition that such φ_m can be found is

$$A_{mq} + \varphi_q \geqq \varphi_p - A_{pm} ,$$

where

$$\varphi_p - A_{pm} = \max_r \{\varphi_r - A_{rm}\} , \qquad A_{mq} + \varphi_q = \min_r \{A_{mq} + \varphi_q\} .$$

But if $p = q$, this is equivalent to

$$A_{mq} + A_{qm} \geqq 0 ,$$

which is verified by hypothesis; and if $p \neq q$, it is equivalent to

$$A_{pm} + A_{mq} \geqq \varphi_p - \varphi_q ,$$

which is verified, since by hypothesis,

$$A_{pm} + A_{mq} \geqq A_{pq}, \qquad A_{pq} \geqq \varphi_p - \varphi_q .$$

Since the system (A, 2) trivially has a solution, it follows by induction that the system (A) = (A, k) has solution, and is thus consistent.

THEOREM. *If $E = \{E_r \mid r = 1, \cdots, n\}$ is any expenditure configuration, with figures $E_r = (x_r \mid u_r)(u_r' x_r = 1)$ and cross-coefficients $D_{rs} = u_r' x_s - 1$, and if λ_r, φ_r are any multipliers and levels, being such that*

$$\lambda_r > 0, \qquad \lambda_r D_{rs} \geqq \varphi_s - \varphi_r \qquad (r, s = 1, \cdots, n)$$

and if $g_r = u_r \lambda_r$, and $\varphi_r(x) = \varphi_r + g_r'(x - x_r)$, then

$$\varphi(x) = \min \{\varphi_r(x) \mid r = 1, \cdots, n\}$$

is a function which realizes the utility hypothesis for E.

Now

$$\begin{aligned} \varphi_r(x) &= \varphi_r + g_r'(x - x_r) \\ &= \varphi_r + \lambda_r(u_r' x - 1) \end{aligned}$$

so that

$$\varphi_r(x_s) = \varphi_r + \lambda_r D_{rs} \geqq \varphi_s \ , \qquad \varphi_s(x_s) = \varphi_s \ .$$

Hence

$$\begin{aligned}\varphi(x_s) &= \min\{\varphi_r(x_s) \colon r = 1, \cdots, n\} \\ &= \varphi_s \ .\end{aligned}$$

Also, $\varphi(x) \geqq \varphi_s$ implies $\varphi_s(x) \geqq \varphi_s$, which, since $\lambda_s > 0$, equivalent to $u_s'x \geqq 1$. Therefore, $u_s'x < 1$ implies $\varphi(x) < \varphi_s$. Accordingly,

$$\max\{\varphi(x) \colon u_s'x \leqq 1\} = \varphi_s$$

and

$$u_s'x \leqq 1 \wedge \varphi(x) = \varphi_s \rightarrow u_s'x = 1 \ .$$

The function φ therefore realizes the utility hypothesis for the configuration.

Since level consistency is the condition for the existence of the λ_r, φ_r, there follows

COROLLARY. *For an expenditure configuration to have the property of utility consistency it is sufficient that its cross-structure have the property of level consistency.*

But, by the previous theorem, level consistency is necessary for utility consistency and is equivalent to cyclical consistency, whence

COROLLARY. *The cyclical consistency condition is necessary and sufficient for the utility consistency of a finite expenditure configuration.*

Some comments are now made on the form of the function $\varphi(x)$ which has been constructed. The functions $\varphi_r(x)$ are linear and, therefore, concave and they have gradients $g_r > 0$, so they are increasing functions. Therefore, $\varphi(x)$, since it is the minimum of increasing concave functions, in an increasing concave function. Its level surfaces $\{x \colon \varphi(x) = \varphi\}$ are the convex polyhedral surfaces which are the boundaries of the convex polyhedral regions $\{x \colon \varphi(x) \geqq \varphi\}$ defined by the inequalities $\varphi_r(x) \geqq \varphi$, or equivalently

$$u_r'x \geqq 1 + \frac{\varphi - \varphi_r}{\lambda_r} \qquad (r = 1, \cdots, n) \ .$$

The region $\Omega_s = \{x \colon \varphi(x) = \varphi_s(x)\}$, in which $\varphi(x)$ coincides with $\varphi_r(x)$, is a polyhedral region, which is the projection in x-space of the face in which $\varphi_s(x) = \varphi$ cuts the boundaries of the region in (x, φ)-space defined by these inequalities. Since $\varphi(x_s) = \varphi_s = \varphi_s(x_s)$, as has been seen, it appears that $x_s \in \Omega_s$. Also

$$\Omega_s = \{x \colon \varphi_r(x) \geqq \varphi_s(x); r = 1, \cdots, n\} \ .$$

Hence Ω_r is defined by the inequalities

$$\varphi_r + \lambda_r(u_r'x - 1) \geqq \varphi_s + \lambda_s(u_s'x - 1) \qquad (r = 1, \cdots, n) \ .$$

Thus for a point to belong to two of the cells, say $x \in \Omega_s \cap \Omega_t$, it is required that

$$\varphi_s + \lambda_s(u_s'x - 1) = \varphi_t + \lambda_t(u_t'x - 1) \ .$$

Hence, in a regular case, these cells can only intersect on their boundaries. The regions Ω_r thus constitute a dissection of the x-space into polyhedral cells. In the relative interior of each cell Ω_r, the function $\varphi(x)$ is differentiable and has constant gradient $g(x) = g_r (x \in \Omega_r)$.

Now an index-number formula will be shown which is made intelligible by the construction of this utility function. Given any utility function $\varphi(x)$, the cost of living index with r and s as base and current periods has the determination

$$\rho_{sr} = \min \{u_s'x: \varphi(x) \geqq \varphi_r\} ,$$

where $u_s'x = p_s'x/p_s'x_s$, and $\varphi_r = \varphi(x_r)$. Hence, with determination relative to the function $\varphi(x)$ which has been constructed,

$$\rho_{sr} = \min \left\{u_s'x: u_t'x \geqq 1 + \frac{\varphi_r - \varphi_t}{\lambda_t} ; \quad t = 1, \cdots, k\right\} ,$$

showing a linear program formula which can be evaluated by the usual methods.[2]

It can be seen that the realization of the utility hypothesis by a utility function φ which is concave and has gradient g implies level consistency. For the concavity is equivalent to the condition

$$\varphi(y) - \varphi(x) \leqq g(x)'(y - x) ,$$

and Gossen's Law that preference and price directions coincide in equilibrium, gives $g = u\lambda$, where $\lambda = g'x$ since $u'x = 1$. Hence, with $\varphi(x_r) = \varphi_r$, $g(x_r) = u_r\lambda_r$, there follows

$$\varphi_s - \varphi_r \leqq \lambda_r u_r(x_s - x_r) .$$

Thus

$$\lambda_r D_{rs} \geqq \varphi_s - \varphi_r .$$

By an easy enlargement, the present results can be made to encompass the point of view of Pareto of preference as a relation divorced from a numerical measure.

An expenditure figure $E_r = (x_r \mid u_r)$ is considered as the choice $(x_r \mid W_{u_r})$, of x_r from among all compositions in the set $W_{u_r} = \{x: u_r'x \leqq 1\}$; and the preferences *immediate* in this choice form the set

$$R_r = \{(x_r, x): x \in W_{u_r}\} = (x_r, W_{u_r}) .$$

If these belong to a relation R, for all r, then

$$\bigcup_{r=1,\cdots,k} R_r \subset R ,$$

[2] A further discussion of this and related approaches to index-number construction can be found in S. N. Afriat, "The Cost of Living Index," appearing in *Studies in Mathematical Economics, Essays in Honor of Oskar Morgenstern*, ed. Martin Shubik, chapter 13 (to be published by the Princeton University Press).

and if R is transitive, that is $\vec{R} \subset R$ where $\vec{R}$ is the transitive closure, this is equivalent to

$$R_E \subset R\ ,$$

where

$$R_E = \vec{\bigcup_{r=1,\cdots,n}} R_r$$

can define the preferences *implicit* in the configuration $E = \{E_r \mid r = 1, \cdots, n\}$. Any preference relation which can be a hypothesis for E, in that it is reflexive and transitive and contains all the preferences in the choices shown by E, is *revealed* to the extent of containing R_E.

Now let $\ogreaterthan$ stand for the relation by which one composition is greater than another. That is, $x \ogreaterthan y$ means every quantity in x is at least the corresponding quantity in y, and not all are the same. In any admissible preference hypothesis R, it is to be assumed that the greater is exclusively preferred to the lesser so that

$$xRy \Rightarrow \sim \cdot y \ogreaterthan x\ .$$

That is $R \subset \bar{\ogreaterthan}'$, and, with $R_E \subset R$, this gives

$$R_E \subset \bar{\ogreaterthan}'\ .$$

It will now be seen that this condition, which can be called the *preference consistency* condition and is obviously implied by utility consistency, implies cyclical consistency. For it implies that

$$x_r R x_q \Rightarrow \sim x_q \ogreaterthan x_r$$

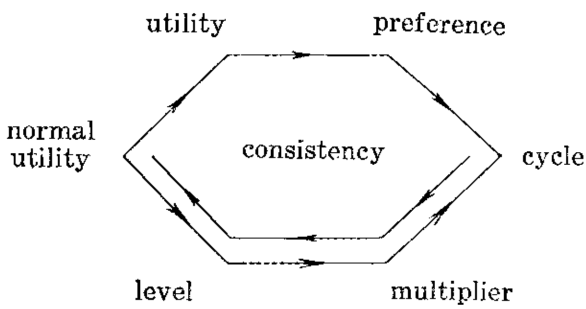
which implies the same as cyclical consistency, namely that

$$D_{rs} \leqq 1 \wedge D_{st} \leqq 1 \wedge \cdots \wedge D_{qr} < 1$$

is impossible, since

$$x_q \ogreaterthan x_r \Rightarrow D_{qr} < 1\ .$$

Now if *normal utility consistency* is defined as utility consistency with realization by a concave utility function, and since, by virtue of the form of the function which has been shown constructible under level consistency and



by the implication of level consistency from the existence of such a function, the following implications are established, those on the outside in the diagram having been quite immediate, and those on the inside having been proved with less immediacy.

It follows therefore that all these six conditions are equivalent. The assumption in the utility consistency condition is weaker, and that in the normal utility consistency condition is stronger than the usual assumption that a utility function be continuous, increasing, and with concave levels. But now, in regard to the finite data, these three conditions appear equivalent. Also seen is the identity of the two approaches involving preference as a relation and utility as a magnitude. The finiteness of the configuration E has been essential for the methods used. Nevertheless it is possible to obtain analogous results without this restriction, though they must be without the present constructiveness.

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