

2.2 Power Utility Function

$$u(c|w) = \frac{(c+w)^{(1-\rho)}}{1-\rho}$$

Case 1. $c_t > 0$ and $c_{t+s} = 0$, which means $\lambda_{t+s} > 0$
F.O.Cs are

$$(c_t + w)^{-\rho} - \lambda = 0 \quad (1)$$

$$\beta(c_{t+s} + w)^{-\rho} - \frac{1}{1+r}\lambda + \lambda_{t+s} = 0 \quad (2)$$

$$\lambda_{t+s} > 0$$

$$\iff -\beta w^{-\rho} + \frac{1}{1+r}(m+w)^{-\rho} > 0$$

$$\iff \frac{1}{1+r} > \frac{\beta(m+w)^{\rho}}{w^{\rho}}$$

$$\iff 1+r < \frac{w^{\rho}}{\beta(m+w)^{\rho}}$$

Case 2. $c_t = 0$ and $c_{t+s} > 0$, which means $\lambda_t > 0$
F.O.Cs are

$$(c_t + w)^{-\rho} - \lambda + \lambda_t = 0 \quad (3)$$

$$\beta(c_{t+s} + w)^{-\rho} - \frac{1}{1+r}\lambda = 0 \quad (4)$$

As in case 1, we can get

$$(1+r) > \frac{((1+r)m+w)^{\rho}}{w^{\rho}\beta} \quad (5)$$

Case 3. F.O.Cs are

$$(c_t + w)^{-\rho} - \lambda = 0 \quad (6)$$

$$\beta(c_{t+s} + w)^{-\rho} - \frac{1}{1+r}\lambda = 0 \quad (7)$$

(6) and (7) yields

$$c_t + w = ((1+r)\beta)^{-\frac{1}{\rho}}(c_{t+s} + w) \quad (8)$$

Using the budget constraint,

$$c_{t+s}^* = \frac{K(1+r)}{K+(1+r)}\{m + (1 - \frac{1}{K})w\}$$

$$c_t^* = \frac{(1+r)}{K+(1+r)}m + \frac{1-K}{K+(1+r)}w$$

where $K = ((1+r)\beta)^{\frac{1}{\rho}}$