The Non-Existence of Representative Agents

Matthew O. Jackson* and Leeat Yariv†‡

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Abstract

We characterize environments in which there exists a representative agent: an agent who inherits the structure of preferences of the population that she represents. The existence of such a representative agent imposes strong restrictions on individual utility functions—requiring them to be *linear* in the allocation and additively separable in any parameter that characterizes agents' preferences (e.g., a risk aversion parameter, a discount factor, etc.). Commonly used classes of utility functions (exponentially discounted utility functions, CRRA or CARA utility functions, logarithmic functions, etc.) do not admit a representative agent.

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*Department of Economics, Stanford University, the Santa Fe Institute, and CIFAR. http://www.stanford.edu/∼jacksonm e-mail: jacksonm@stanford.edu

†Department of Economics, Princeton University. http://www.princeton.edu/yariv e-mail: lyariv@princeton.edu

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1 Introduction

1.1 Overview

Groups of individuals, in aggregate, can behave quite differently from the individuals themselves. For example, the classic Sonnenschein-Mantel-Debreu Theorem (Sonnenschein, 1973; Mantel, 1974; Debreu, 1974) illustrated that even if individuals' demand functions each satisfy standard conditions, some of the most vital of those conditions—e.g., the weak axiom of revealed preference—are lost when those demands are aggregated.

This can be problematic, a model that admits arbitrary aggregate behavior is hard to work with. Thus, it is common to assume well-behaved aggregate behavior, implicitly presuming agents in the underlying economy satisfy certain restrictions for the model to be consistent. In particular, the literature has often assumed the existence of a well-behaved *representative agent*, one whose choices or preferences reflect those aggregated across society. The notion itself can be traced back to Edgeworth (1881) and Marshall (1890).[1] Since the publication of the Lucas Critique (1976), micro-founding economic models has become pervasive and the use of a representative agent as a modeling tool has become standard practice.

The existence of one sort of representative agent was theoretically founded in the mid-twentieth century by Gorman (1953, 1961). Gorman showed that in order to have a representative Marshallian demand function for an economy, such that the representative demand at the aggregate income level is equal to the sum of individual demands, agents' indirect utility functions have to take a restrictive form, termed the "Gorman Form," and have identical dependence on income.

Specifically, let $D(p, y)$ denote a Marshallian demand as a function of a vector of prices p and an income level y. Gorman's (1953) results imply that in order for there to exist a representative D such that

$$D(p, \sum_i y_i) = \sum_i D_i(p, y_i)$$

for all vectors of individual income levels y_i, it must be that the agents have linear and identical Engel curves, up to a parallel shift. As Gorman (1961) showed later, this imposes strong restrictions on the preferences in society—essentially requiring that they either be quasi-linear in income, or identical (up to a normalization) and homothetic.

Although Gorman's results are discouraging, most of the settings that researchers have analyzed with representative agents are not modeled through Marshallian demand functions nor do they require that a representation hold for all distributions of income. Most models involve decisions that are far more constrained. For instance, representative agents have been used to analyze how agents make consumption and savings decisions in the face of returns to savings that are impacted by various policies (e.g., Lucas, 1978), how agents choose their

[1] Edgeworth (1881) referred to a "representative particular," while Marshall (1890) referred to a "representative firm."

labor supply in the face of a tax schedule (e.g., Chamley, 1986), and how agents select public goods (e.g., Rogoff, 1990). Even though these decision problems involve maximizing a utility function with respect to some resource constraints, none of them fit into the Gorman setting.

Instead of presuming a common demand function, researchers assume there is a single agent in the economy and specify that agent's *preferences*. This allows a derivation of the agent's behavior in reaction to various influences and policies, as well as the analysis of inefficiencies and welfare. Ultimately, models often specify an agent with some characteristics a, who has a utility function of the form $V(x, a)$, where x can correspond to one dimension of consumption, a consumption stream, etc. The agent's choice of x can then be subject to various feasibility constraints. If, for example, a captures the agent's income and x stands for a bundle of goods, this formalization can be translated back into the Gorman form. But if, instead, a captures the level of risk aversion, or discounting, or a political ideology, this specification no longer fits the Gorman framework.

Although researchers are generally careful not to claim that a representative-agent formulation is a valid substitute for the analysis of a heterogeneous population, that hope is implicit. Such results are clearly of limited interest if there does not exist *any* population in which individual agents' preferences are represented by $V(x, a_i)$, with heterogeneous characteristics a_i, such that there exists some agent with preferences represented by $V(x, a)$, for some characteristics a, who could proxy for the population.

Thus, we ask whether there exists *at least one* possible set of weights λ_i—e.g., representing the relative fractions of different groups in the population—such that if the population has a fraction λ_i of agents with preference parameter a_i, then there exists some representative agent with preference parameter a for whom the utility of the average outcome is a proxy for the average utility. For private goods, this restriction takes the form:

$$V(\sum \lambda_i x_i, a) = \sum_i \lambda_i V(x_i, a_i), \tag{1}$$

while for common consumption, or public goods, this restriction takes the form:

$$V(x, a) = \sum_i \lambda_i V(x, a_i).$$

For instance, this formulation is required when looking for a policy that maximizes society's utilitarian welfare (as in, e.g., Chamley, 1986). We show that the classes of utility functions that admit a representative agent, for either private or public goods, are extremely restrictive.

The following example illustrates the difficulty with this representative-agent construct.

Example (CRRA Utility Functions): Consider a population of n agents with CRRA (isoelastic) utility functions. Each agent i is identified by a CRRA parameter $a_i \in (0, 1)$ and gets a utility from a commonlshared reward x given by

$$V(x; a_i) = \frac{x^{1-a_i} - 1}{1 - a_i}.$$

A representative agent would have utility proportional to some convex combination of the population. Namely, for a profile of coefficients of relative risk aversion $\mathbf{a} \equiv (a_1, ..., a_n)$, up to an affine transformation, her utility of the common reward x would be given by:

$$U(x, \mathbf{a}) = \sum_i \lambda_i V(x; a_i) = \sum_i \lambda_i \frac{x^{1-a_i} - 1}{1 - a_i},$$

for some positive weights λ_i.

Straightforward calculations show that, whenever the a_i's are not all identical, the resulting coefficient of relative risk aversion,

$$-\frac{xU''(x, \mathbf{a})}{U'(x, \mathbf{a})} = \frac{\sum_i \lambda_i a_i x^{-a_i}}{\sum_i \lambda_i x^{-a_i}},$$

changes with x.

This means that the representative agent cannot be characterized by a utility function $V(x, a)$ that satisfies the same property (constant relative risk aversion) satisfied by the utility functions of all members of the population she represents.

If we move to a setting with private allocations, the problem becomes even starker. In this case, the weighted sum of agents' utilities is

$$\sum_i \lambda_i V(x_i; a_i) = \sum_i \lambda_i \frac{x_i^{1-a_i} - 1}{1 - a_i}.$$

This cannot be represented by any function of $\sum_i \lambda_i x_i$ when the x_i's differ, unless all the a_i's are 0—so all agents have to be risk neutral and evaluating a linear function.

Our main results below prove that this example's conclusions are typical. We fully characterize the classes of preferences for which representative agents exist. When consumption is common, we show that only parameterized classes of utility functions that are *separable in agents' utility parameters* admit representative agents. The assumption that consumption is common applies to environments corresponding to members of a household sharing consumption and savings, or a community—a neighborhood, a state, or a country—benefiting from a common public good, etc. When consumption is private, corresponding to settings of consumer behavior, and encompassing the original examples offered by Lucas (1978), the existence of a representative agent turns out to be even more demanding. In this case, we show that *only utility functions that are linear in consumption and additively separable in parameters* admit representative agents.

The literature using representative agents almost never uses linear utility functions, nor utility functions that are additively separable in agents' preference parameters. Our results then suggest that commonly used classes of utility functions (logarithmic, weighting mean and variance, prospect theoretic, etc., in addition to exponentially-discounted utilities and CRRA or CARA utilities as described above) cannot be aggregated to generate a representative agent who is characterized by preferences from the same class.

1.2 Related Literature

We are certainly not the first to point out issues with the use of representative agents. Beyond Gorman's contributions, the notion has endured scrutiny practically since its inception, and actively since the beginning of the twentieth century, see Robbins (1928) and surveys in Kirman (1992) and Hartley (1996).

Nonetheless, as mentioned, the Lucas Critique (1976) brought new life to micro-founding economic models using the representative-agent construct. Representative-agent models have been used to explain observed aggregate fluctuations of an economy in classical business cycle theories (e.g., Kydland and Prescott, 1982; King, Plosser, and Rebelo, 1988), to design tax systems (e.g., see Chamley, 1986; Judd, 1985; and literature that followed), to estimate tax rates on factor incomes and consumption (e.g., Mendoza, Razin, and Tesar, 1994), to assess moral hazard and adverse selection constraints in insurance markets (see Prescott and Townsend, 1984 and literature that followed), etc.

Although much of this literature mentions the assumption of population homogeneity, the fragility of some of its conclusions to heterogeneity have been inspected only fairly recently and in particular contexts, see e.g. An, Chang, and Kim (2009), Constantinides (1982), Gollier (2001), Kaplan, Moll, and Violante (2018), Mazzocco (2004), Mongin (1998), and Golman (2011). Our contribution is in highlighting a basic and general principle that drives all such observations.

As discussed, there is also a literature characterizing conditions under which aggregate demand, or aggregate behavior, features similar properties to underlying demands (for references, see Chiappori and Ekelend, 1999). In contrast, we focus on whether any modeler who assumes some properties of a representative agent's preferences must be making errors when presuming these preferences reflect preferences of a heterogeneous population satisfying similar properties. This question is not answered by the demand-based literature, and yet it covers many, if not most, of the settings in which representative agents are used.

Our insights are in the spirit of Jackson and Yariv (2015) and several papers cited there, which showed that there is no utilitarian aggregation of exponentially-discounted preferences that satisfies time consistency.[2] We show that such impossibilities are a pervasive phenomenon—applying to many preference formulations and general sources of heterogeneity—and can be argued directly.

Our results also provide insight into observed differences between individual and group decision making. It is well-documented in the experimental and empirical literature that groups exhibit different behavioral patterns than individuals in various environments, ranging from choices between risky alternatives (chronicled since Wallach, Kogan, and Bem, 1962), to choices of timing (see Ibanez, Czermak, and Sutter, 2009; Schaner, 2015; and references therein), allocation decisions (Cason and Mui, 1997; Ambrus, Greiner, and Pathak, 2015), etc. Our conclusions are in line with these observations when groups behave according to some convex combination of their members' preferences. Indeed, experimental evidence

[2] See also Apesteguia and Ballester (2016), who consider related stochastic models of choice.

suggests that group members place substantial weight on utilitarian motives (e.g., Charness and Rabin, 2002; Jackson and Yariv, 2014). We show that if individual behavior is inconsistent with linear utilities, there is no reason to expect groups to echo choices of well-behaved individuals.

2 Representative Agents with Private Allocations

We first consider the case in which individuals each have their own allocation and the representative agent evaluates the aggregate/average allocation. For example, the allocation could stand for consumption, investment, and/or savings levels. Agents may exhibit heterogeneity in their discount factors, risk aversion parameters, or other preference parameters, as well as their endowments of human capital, wealth, and so on.

Formally, $n \geq 2$ agents evaluate allocations, generically denoted by x that come from some set D_x, which is a closed and convex subset of $\mathbb{R}_+^\ell$ for some ℓ. Also, we assume that there exists some $x \in D_x$ for which x is positive in all dimensions and $0 \leq y \leq x$ implies that $y \in D_x$.[3]

The heterogeneity of agents' preferences is captured by an index $a \in D_a$, where D_a is some index set. Depending on the application, the parameter a would represent an agent's risk aversion parameter, discount factor, endowment of human capital or wealth, etc.

Utility functions are functions $V : D_x \times D_a \to \mathbb{R}$ that are continuous in the allocation (the first variable).[4]

We say that there exists *a representative agent* with private allocations if there exists *some* $(\lambda_1, ..., \lambda_n) \in [0,1)^n$, where $\sum_{i=1}^n \lambda_i = 1$,[5] such that for *some* $(a_1, \ldots, a_n) \in D_a^n$, there exists $\bar{a} \in D_a$ for which for all $(x_1, \ldots, x_n) \in D_x^n$:[6]

$$\sum_{i=1}^n \lambda_i V(x_i; a_i) = V(\sum_{i=1}^n \lambda_i x_i; \bar{a}).$$

[3]The necessary assumption for our results amounts to D_x containing an open ball. By translating the domain and adding points of closure (given continuous utility functions), one gets inclusion of the origin and closure.

[4]Using techniques from Corollary 3 of Rado and Baker (1987), one can extend a key lemma in our proof to hold for Lebesgue measurable functions, but the proof is more transparent with continuous functions, a standard assumption.

[5]The proofs of Theorems 1 and 2 below can be extended to the case in which the $\sum_{i=1}^n \lambda_i$ is not required to be one, provided that D_x is unbounded above. It is then then still required that $\lambda_i > 0$ for at least two agents (which is implied above since $\lambda_i < 1$ for all i and the sum is 1). Otherwise, the setting boils down to one with a single agent and representation is trivial.

[6]We have not placed any restrictions on how V depends on a, and so this allows $V(\cdot : \bar{a})$ to be any function that is continuous in x.

When utility functions are concave, a Pareto optimal allocation is a solution to the maximization $\sum_{i=1}^{n} \lambda_i V(x_i; a_i)$ for some weights. The utilitarian social-welfare function corresponds to the special case in which each coefficient λ_i, $i = 1, ..., n$, is the fraction of the population characterized by preference parameter a_i and allocation x_i per person.

One could also contemplate welfare that is assessed with a set of weights that do not coincide with the respective fractions of "types" in the population. One can show that such an assumption requires utilities to be *independent of the allocation altogether*. We maintain the definition above since it corresponds to most applications of representative agents in the literature, and since it provides for more "conservative" conclusions as it places weaker restrictions on preferences.

The existence of a representative agent is a type of convexity requirement on the space of utility functions. Indeed, consider the special case in which the x_i's are all equal. The existence of a representative agent requires that a convex combination of individuals' utility functions is in the same class to which those individual utility functions belong.

In our formulation, $\bar{a}$ is the representative agent's preference parameter. The representative agent's utility function is often assumed to take the form $be^{c-\bar{a}x}$, $(c + bx)^{\bar{a}}/\bar{a}$, $\bar{a}\ log(x)$, etc.[7] This setting fits a classic example from Lucas (1978), who considers individuals making consumption versus savings decisions each period.

The following is the characterization of utility functions that admit a representative agent when allocations are private.

THEOREM **1** *There exists a representative agent $\bar{a}$ in the case of private allocations, relative to some $\lambda \in [0, 1)^n$ and some $(a_1, \ldots, a_n) \in D_a^n$, if and only if $V(x; a) = c \cdot x + h(a)$ for all $x \in D_x$ and $a \in \{a_i \mid \lambda_i > 0\} \cup \{\bar{a}\}$, where $c \in \mathbb{R}^\ell$ and $h : D_a \to \mathbb{R}$ satisfies $h(\bar{a}) = \sum_i \lambda_i h(a_i)$.*

The structure characterized by Theorem 1 requires linearity in the allocation x and additive separability in the type parameter a. It is clearly not satisfied by utility functions that are commonly used in economic modeling—strictly concave utility functions, CRRA or CARA utility functions,or exponentially-discounted utilities do not satisfy the restriction. In such cases, assuming a representative agent whose utility is taken from the same class of heterogeneous individuals' preferences would generate inaccurate estimates of aggregate behavior and welfare.

If we additionally require the representative-agent restriction hold for *all* preference profiles, the structural implications of Theorem 1 apply to all preference parameters. Specifically, if we assume that $D_a = [0, 1]$ and that $V(x; a)$ is continuous in a, the existence of a representative agent is tantamount to $V(x; a) = c \cdot x + h(a)$ for all $(x, a) \in D_x \times D_a$, where $c \in \mathbb{R}^\ell$ and $h : D_a \to \mathbb{R}$ is a continuous function.

[7] In these formulations, b and c are taken as constants. For instance, the form $be^{c-\bar{a}x}$ with $b = 1$ and $c = 0$ would correspond to a representative agent with a CARA utility function and the form $(c + bx)^{\bar{a}}/\bar{a}$ with $c = 0$ and $b = 1$ would correspond to a representative agent with a CRRA utility function.

Detailed proofs appear in Section 5 and rely on analysis of Pexider's equation (see, e.g., Azcél, 1966). Intuitively, if a representative agent exists, a marginal change in the private allocation x_i of any agent i has a proportional effect on the allocation the representative agent considers, where the proportional factor corresponds to the individual's weight in society. The only way to get marginal utility calculations line up for all agents is to have linearity in x.

3 Representative Agents with Common Alternatives

The case of private allocations applies to most of the work built upon representative agents in macroeconomics and finance. We now expand the analysis to admit alternatives that are jointly evaluated. For example, in household decision making, expenditures and savings are often common across household members. Furthermore, common consumption is central to many models of political economy and public finance. In these models, agents make decisions over the level of some public good. As we now show, the existence of a representative agent in such environments is still very restrictive, but entails a different sort of separability.

We maintain the same basic structure of preference heterogeneity as above. For all that follows, we add the conditions that D_a is $[0, 1]$, that utility functions $V(x; a)$ are continuous in a, and that there exists at least one $x^* \in D_x$ for which $V(x^*; a)$ is strictly monotone (increasing or decreasing) in a.[8]

The restriction that $V(x^*; a)$ is monotone in a for some $x^* \in D_x$ is weak and satisfied for many classes of commonly used utility functions. For instance, exponential discounting, CRRA, and CARA satisfy the condition. Although we maintain this condition for presentation simplicity and since it allows for most cases covered in the literature, we note that the proofs imply that this condition can, in fact, be weakened to a requirement that $V(x^*; a)$ be piece-wise monotonic for some $x^* \in D_x$, which is satisfied for practically all preference specifications appearing in the literature.[9]

We say that there exists *a representative agent* with common alternatives if there exists some $(\lambda_1, ..., \lambda_n) \in [0, 1)^n$, where $\sum_{i=1}^{n} \lambda_i = 1$, such that for any $(a_1, ..., a_n) \in D_a^n$, there exists $\bar{a} \in [0, 1]$ for which:

$$\sum_{i=1}^{n} \lambda_i V(x; a_i) = V(x; \bar{a})$$

[8]The assumption that $V(x; a)$ is continuous in a simplifies our proof presentation, but is not necessary. The continuity of $V(x^*; a)$ in a is implied by montonicity combined with the existence of a representative agent, and is all that is required for our main result.

[9]Without some such assumption, one admits the possibility that all preferences are type-independent, in which case there is no meaningful heterogeneity in the population. In such cases, a representative agent exists trivially. This requirement is not needed in the case of private allocations since there agents can differ in their consumption. That potential variation imposes a stronger requirement on a representative agent, even with identical preferences.

for all x.

In the case of common alternatives, a definition that requires our restriction to hold for only one preference-parameter profile $(a_1, ..., a_n)$ could be trivially satisfied in a mechanical fashion. For instance, for $f(x), g(x)$ such that $f(x) > g(x)$ for all x, defining $V(x; a) = f(x)$ for low values of a, $V(x; a) = g(x)$ for high values of a, and $V(x; a) = \frac{1}{2}f(x) + \frac{1}{2}g(x)$ for intermediate values of a would suffice for the existence of a representative agent, but appears rather unnatural. This is why we require the restriction to hold for *all* preference parameters.

Theorem 2 characterizes the class of utility functions that admit a representative agent.

THEOREM **2** *There exists a representative agent with common alternatives if and only if $V(x; a) = h(a)f(x) + g(x)$ for all $(x, a) \in D_x \times D_a$, for some continuous functions $h(a)$, $f(x)$, and $g(x)$ such that $h(\cdot)$ is monotone, and $f(x^*) \neq 0$.*

Although the restrictions implied by the existence of a representative agent for common alternatives are weaker than those for private allocations, they are still sufficiently strong as to rule out nearly all commonly assumed utility functions. From the examples mentioned so far, exponential discounting, CARA and CRRA utility functions with risk-aversion parameters do not satisfy the restrictions of Theorem 2, nor do concave loss functions with bliss points serving as parameters—e.g., single-peaked preferences.

One contrast between the common-alternative and private-allocation cases pertains to the class of concave utility functions. Certainly, a mixture of concave functions is concave. Thus, when considering the full class of concave functions, with common consumption, a representative agent does exist, and is characterized by the convex combination of agents' utility functions, though that function may look quite different from the functions that are being aggregated. This does not violate the theorem since there is no representation of the class of all concave functions that satisfies the monotonicity requirement.

4 Strongly Representative Agents

We now consider a more demanding notion of representative agents. Under this variant, the representative agent's preference parameter $\bar{a}$ is the weighted average of individual agents' preference parameters. For instance, suppose an empiricist observes individual preference parameters with noise and erroneously assumes the population is homogenous. A natural estimate for the preference parameter corresponding to that population, as well as its legitimate representative agent under the homogeneity assumption, would be the observed parameters' average. For examples relating to discount-factor estimations, see the survey by Frederick, Loewenstein, and O'Donoghue (2002). With a large population of individuals, the estimated average parameter may not be biased. However, as we now show, welfare assessments based on the estimated utility function may be inaccurate. In fact, the classes of utility functions admitting such strongly representative agents are even more restrictive than those identified above and are *linear* in the preference parameter.

As before, we start with the case of private allocations, maintaining our assumptions that $D_a = [0,1]$, $V(x;a)$ is continuous in a, and that $V(x^*;a)$ is strictly monotone in a for at least one $x^* \in D_x$.

There exists a *strongly representative agent* with private allocations if there exists some $(\lambda_1, ..., \lambda_n) \in [0,1)^n$, where $\sum_{i=1}^{n} \lambda_i = 1$, such that for all $(a_1, \ldots, a_n) \in D_a^n$ and $(x_1, \ldots, x_n) \in D_x^n$:

$$\sum_{i=1}^{n} \lambda_i V(x_i; a_i) = V\left(\sum_{i=1}^{n} \lambda_i x_i; \sum_{i=1}^{n} \lambda_i a_i\right). \tag{2}$$

PROPOSITION **1** *There exists a strongly representative agent when allocations are private if and only if there exist constants b_1, b_2, and $c \in \mathbb{R}^\ell$ such that $V(x;a) = c \cdot x + b_1 a + b_2$ for all $(x,a) \in D_x \times D_a$.*

Analogously, with common alternatives, there exists *a strongly representative agent* with common alternatives if there exists some $(\lambda_1, ..., \lambda_n) \in [0,1)^n$, where $\sum_{i=1}^{n} \lambda_i = 1$, such that for any $a_1, ..., a_n \in D_a^n$ and $x \in D_x$:

$$\sum_{i=1}^{n} \lambda_i V(x; a_i) = V\left(x; \sum_{i=1}^{n} \lambda_i a_i\right). \tag{3}$$

PROPOSITION **2** *There exists a strongly representative agent when alternatives are common if and only if there exist continuous functions $f(x), g(x)$ such that $V(x;a) = af(x) + g(x)$ for all $(x,a) \in D_x \times D_a$.*

5 Proofs

We begin with some lemmas that provide the key structure behind the proofs. The lemmas provide a variation on the analysis of Pexider's equation. The proofs use techniques developed in Azcél (1966, 1969), Eichhorn (1978), and Diewert (2011).

LEMMA **1** *Let $f(x)$ be a continuous function on $[0,t]$. Suppose that, for some $\lambda \in (0,1)$,*

$$f(\lambda x + (1-\lambda)y) = f(\lambda x) + f((1-\lambda)y) \text{ for all } x, y \in [0,t]$$

then $f(x) = cx$ for all $x \in [0,t]$, where c is a scalar.

Proof of Lemma 1: Let $t' = \min\{\lambda, 1-\lambda\}t$.

We first show that for any positive integer k and any $z \in [0, t']$, $f(z) = kf(z/k)$.

Let $x = \frac{z}{\lambda k}$ and $y = \frac{(k-1)z}{(1-\lambda)k}$. Note that, since $z \leq t'$ then by construction, $x, y \in [0, t]$. Then, $f(z) = f\left(\frac{z}{k}\right) + f\left(\frac{(k-1)z}{k}\right)$. For $k = 1$ this establishes the claim. For $k \geq 2$, writing $\frac{(k-1)z}{k} = \lambda \frac{z}{\lambda k} + (1-\lambda)\frac{(k-2)z}{(1-\lambda)k}$, it follows that

$$f(z) = f\left(\frac{z}{k}\right) + f\left(\frac{z}{k}\right) + f\left(\frac{(k-1)z}{k}\right) = 2f\left(\frac{z}{k}\right) + f\left(\frac{(k-2)z}{k}\right).$$

Continuing recursively establishes that $f(z) = kf(z/k)$ for all $z \in [0, t']$ and for all positive integers k.

Next, we show this implies that $f(x) = cx$ for all $x \in [0, t']$. Let $c = \frac{f(t')}{t'}$. For any $x = \frac{m}{n}t'$, where m and n are integers such that $m < n$, we have $\frac{x}{m} = \frac{t'}{n}$ and, from above, $f(x) = \frac{m}{n}f(t') = cx$. From continuity, $f(x) = cx$ for all $x \in [0, t']$.[10]

Now, suppose $\min\{\lambda, 1-\lambda\} = \lambda$ so that $\lambda z \in [0, t']$ for all $z \in [0, t]$. Then:

$$\begin{aligned} f(z) &= f(\lambda z) + f((1-\lambda)z) = c\lambda z + f((1-\lambda)z) = \\ &= c\lambda z + f(\lambda(1-\lambda)z) + f((1-\lambda)^2 z) = c(\lambda z + \lambda(1-\lambda))z + f((1-\lambda)^2 z) = \\ &= cz\lambda \sum_{i=0}^{\infty}(1-\lambda)^i + \lim_{n\to\infty} f((1-\lambda)^n z) = cz + \lim_{n\to\infty} f((1-\lambda)^n z). \end{aligned}$$

Since $f(0) = 0$ (as $f(0) = kf(0)$) and f is continuous, $f(z) = cz$ for all $z \in [0, t]$. Similar arguments follow for $\min\{\lambda, 1-\lambda\} = 1-\lambda$. ▮

LEMMA **2** *Let $f(x)$ be a continuous function on D_x such that there exists some $\lambda \in (0, 1)$ for which*

$$f(\lambda x + (1-\lambda)y) = f(\lambda x) + f((1-\lambda)y) \text{ for all } x, y \in D_x.$$

Then there exists $c \in \Re^\ell$ such that $f(x) = c \cdot x$ for all $x \in D_x$.

Proof of Lemma 2: First, note that $f(0) = 0$, since $\lambda \times 0 + (1-\lambda) \times 0 = 0$ and so $f(0) = f(0) + f(0) = 2f(0)$.

Next, note that, by assumption, there exists $x \in D_x$ that is positive in all dimensions such that $0 \leq y \leq x$ implies $y \in D_x$. Let $D' = \{y : 0 \leq y \leq x\}$. Let D'_j be the subset of D' such that $y \in D'_j$ implies $y_k = 0$ for all $k \neq j$.

Applying Lemma 1 to each D'_j implies that for each dimension j, there exists c_j for which $f(y) = c_j y_j$ whenever $y \in D'_j$.

Next, let $d = \min\{\lambda, 1-\lambda\}$ and $D'' = \{y : 0 \leq \frac{y}{d} \leq x\}$. For $y \in D''$, abuse notation and write y_j to denote the vector that is the projection of y onto its j-th dimension, and y_{-j} to be

[10] Note that were D_x unbounded, e.g. $D_x = [0, \infty)$, the proof would be completed here.

$y - y_j$. Then, for any $y \in D''$, by the definition of D'', it follows that $(1-\lambda)y_j + (1-\lambda)\frac{y_{-j}}{1-\lambda} \in D'$. Therefore,

$$f(y) = f\left(\lambda y_j + (1-\lambda)y_j + (1-\lambda)\frac{y_{-j}}{1-\lambda}\right) = \lambda c_j y_j + f\left((1-\lambda)y_j + (1-\lambda)\frac{y_{-j}}{1-\lambda}\right).$$

Repeating the argument for the second term, we get $f(y) = c_j y_j + f(y_{-j})$. Then, iterating on the remaining dimensions,

$$f(y) = c \cdot y.$$

for any $y \in D''$.

Next, let $d' = \max\{\lambda, 1-\lambda\}$ and consider any $z \in D_x$ such that $d'z \in D''$. Then,

$$f(z) = f(\lambda z) + f((1-\lambda)z) = \lambda c \cdot z + (1-\lambda)c \cdot z = c \cdot z.$$

For any $z \in D_x$, there exists some t for which $d'^t z \in D''$, and so by iterating on the above argument, the result follows. ∎

Lemma 3 *Let $f_1(x)$, $f_2(x)$, and $f_3(x)$ be continuous functions on D_x. If, for some $\lambda \in (0,1)$,*

$$f_1(\lambda x + (1-\lambda)y) = \lambda f_2(x) + (1-\lambda)f_3(y) \text{ for all } x, y \in D_x,$$

then there exist constants $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^\ell$ such that

$$\begin{aligned} f_1(x) &= c \cdot x + \lambda a + (1-\lambda)b \\ f_2(x) &= c \cdot x + a, \\ f_3(x) &= c \cdot x + b. \end{aligned}$$

Proof of Lemma 3: Let $x = 0$. Then

$$f_1((1-\lambda)y) = \lambda f_2(0) + (1-\lambda)f_3(y) \text{ for all } y \in D_x.$$

Define $a \equiv f_2(0)$. Then,

$$f_3(y) = \frac{1}{1-\lambda}\left[f_1((1-\lambda)y) - \lambda a\right].$$

Similarly, if we define $b \equiv f_3(0)$, we get:

$$f_2(x) = \frac{1}{\lambda}\left[f_1(\lambda x) - (1-\lambda)b\right].$$

Plugging into the assumed equality, we have:

$$\begin{aligned} f_1(\lambda x + (1-\lambda)y) &= \lambda f_2(x) + (1-\lambda)f_3(y) = \\ &= f_1(\lambda x) + f_1((1-\lambda)y) - \lambda a - (1-\lambda)b. \end{aligned}$$

Define $f(x) \equiv f_1(x) - \lambda a - (1-\lambda)b$. Then, from the last equality we have

$$f(\lambda x + (1-\lambda)y) = f(\lambda x) + f((1-\lambda)y).$$

Lemma 2 then implies that $f(x) = c \cdot x$ and the result follows. ∎

5.1 Proofs Pertaining to Private Allocations

Proof of Theorem 1: Suppose that for some $\lambda_1, ..., \lambda_n \in [0, 1)$ for which $\sum_{i=1}^{n} \lambda_i = 1$, and some $(a_1, \ldots, a_n) \in D_a^n$, there exists $\bar{a} \in D_a$ such that for all $(x_1, \ldots, x_n) \in D_x^n$:

$$\sum_{i=1}^{n} \lambda_i V(x_i; a_i) = V(\sum_{i=1}^{n} \lambda_i x_i; \bar{a}).$$

There must exist some i for which $0 < \lambda_i < 1$. Let $x_i = x$, $x_j = y$ for all $j \neq i$. It follows that

$$V(\lambda_i x + (1-\lambda_i)y; \bar{a}) = \lambda_i V(x; a_i) + (1-\lambda_i) \sum_{j \neq i} V(y; a_j),$$

for any $x \in D_x$ and $y \in D_x$ and the characterization of V follows from Lemma 3. In particular, the first application of the lemma uses $f_1(x) = V(x; \bar{a})$, $f_2(x) = V(x; a_i)$, and $f_3(x) = \sum_{j \neq i} V(\cdot; a_j)$ and thus

$$\begin{aligned} &V(x; \bar{a}) = c \cdot x + h(\bar{a}), \\ &V(x; a_i) = c \cdot x + h(a_i), \\ &\textstyle\sum_{j \neq i} V(\cdot; a_j) = c \cdot x + b_i, \end{aligned}$$

where $h(\bar{a}) = \lambda_i h(a_i) + (1 - \lambda_1) b_i$. Iterating to apply the lemma to any j for which $\lambda_j > 0$, one similarly gets that $h(\bar{a}) = \lambda_j h(a_j) + (1 - \lambda_j) b_j$. These iterations imply that for any j for which $\lambda_j > 0$,

$$V(x; a_j) = c \cdot x + h(a_j)$$

and

$$\sum_{i \neq j} V(x; a_i) = c \cdot x + b_j.$$

Thus, combining these, it follows that $b_j = \sum_{i \neq j: \lambda_i > 0} \lambda_i h(a_i)$. Therefore,

$$h(\bar{a}) = \sum_{i: \lambda_i > 0} \lambda_i h(a_i) = \sum_{i} \lambda_i h(a_i),$$

as claimed and any extension of $h(\cdot)$ to D_a would do. The converse follows directly. ▮

Proof of Proposition 1: The proof follows combining the implications of the functional forms from Theorem 1 together with Proposition 2, as both representations hold by either simply fixing any profile of a_is, or working with all agents having the same allocation. ▮

5.2 Proofs Pertaining to Common Alternatives

We start with the proof of Proposition 2, which is useful for proving Theorem 2.

Proof of Proposition 2: We show that (3) implies that there exist continuous functions $f(x), g(x)$ such that $V(x;a) = af(x) + g(x)$ for all x, a, as the converse is straightforward. Let $a_1 = r$, $a_2 = a_3 = ... = a_n = s$, and $x_1 = x_2 = ... = x_n = x$. Then, the existence of a strongly representative agent, for some $\lambda_i \in (0,1)$, implies that:

$$V(x;\lambda_i r + (1-\lambda_i)s) = \lambda_i V(x;r) + (1-\lambda_i)V(x;s)$$

and Lemma 3 (now applied on the a dimension), together with the continuity of V in a, imply the result. ▮

Proof of Theorem 2: Let

$$h(a) \equiv V(x^*;a).$$

Notice that $h(\cdot)$ is monotone in a and, therefore, from continuity, $\mathrm{Im}_h\, D_a = \tilde{D}_{h,a}$ is a compact set and $h^{-1} : \tilde{D}_{h,a} \to D_a$ is continuous and monotone as well. Now let

$$G(x;a) = V(x;h^{-1}(a)).$$

By our assumption on V, for some $\lambda_1, ..., \lambda_n \geq 0, \sum_{i=1}^{n} \lambda_i = 1$, and for any $a_1, ..., a_n \in D_a^n$, there exists $\bar{a}$ such that for all x,

$$\sum_{i=1}^{n} \lambda_i G(x;a_i) = G(x;\bar{a}),$$

and:

$$\sum_{i=1}^{n} \lambda_i G(x^*;a_i) = \sum_{i=1}^{n} \lambda_i a_i = G(x^*;\bar{a}) = \bar{a}.$$

Therefore, G satisfies the assumptions of Proposition 2, so that there exist continuous functions $f(x), g(x)$ such that

$$G(x;a) = af(x) + g(x),$$

which, in turn, implies that

$$V(x;a) = h(a)f(x) + g(x).$$

The converse is immediate. ▮

6 References

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