Economic Theory (2006) 29: 489–504
DOI 10.1007/s00199-005-0026-x

RESEARCH ARTICLE

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Testable implications of Pareto efficiency and individual rationality

Received: 13 August 2003 / Accepted: 24 August 2005 / Published online: 7 October 2005

Abstract This paper investigates the testable implications of Pareto efficiency and individual rationality on finite data sets in exchange economies with finitely many commodities and agents. Efficiency alone provides no restrictions other than a trivial "no waste"-condition. Efficiency together with individual rationality implies robust restrictions.

Keywords Testability · Non-parametric restrictions · Allocation data · Quantifier elimination

JEL Classification Numbers D50 · D51

Introduction

This paper investigates, whether Pareto efficiency and individual rationality impose restrictions on finite data sets in exchange economies, where agents have (strictly) concave and (strictly) monotonic utility functions. A data set consists of a finite number of observations on agents' initial endowments and final allocations in finitely many commodities.

We characterize both concepts in terms of polynomial inequalities, which are parameterized by the data, and show that Pareto efficiency alone does not yield restrictions, except for a trivial "no waste"-condition: with monotonicity efficient

I appreciate the comments of Don Brown, Truman Bewley and Charles Steinhorn. I also thank seminar participants at Yale, Zuerich and Mainz, as well as conference participants at the 12th European General Equilibrium Workshop in Bielefeld. The generous support of the Cowles Foundation is gratefully acknowledged. The paper also benefited greatly from the comments of an anonymous referee.

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allocations are not compatible with underused endowments. If we impose, in addition, individual rationality or require a stable social welfare function across observations, then there exist data sets which refute the combined rationalization concepts. In contrast, individual rationality is a refutable concept on its own.

These results are interesting for two reasons: first, from the characterization result in terms of polynomial inequalities together with the Tarski-Seidenberg theorem, it follows that for any finite data set there is an effective answer to whether this data set can be rationalized as efficient and individually rational. Efficiency and individual rationality can be formulated in terms of observables only. Secondly, Pareto efficiency alone may not be a good criterion for assessing exchange economies. We would not be able to discriminate between efficient and inefficient exchange. Only efficiency as part of a stronger rationalization concept has empirical implications.

Our work is closely related to the work of Bossert and Sprumont (2001, 2002, 2003), who derive necessary and sufficient conditions on choice correspondences to be efficient and individually rational. However, in general their approach fails for finite data sets in structured economic exchange environments. Making use of the powerful techniques of semi-algebraic geometry (see Carvajal, Indrajit, and Synder 2004), the main contribution of our paper consists in deriving testable restrictions of individual rationality and Pareto efficiency in structured economic environments. Moreover, our work complements a recent and extensive literature on testability in a single agent decision or Walrasian context: Bachmann (2004), Bandyopadhyay et al. (2004), Lee and Wong (2005), Matzkin (2005), and McFadden (2005).

The remainder of this paper proceeds as follows: the next section presents the family of polynomial inequalities characterizing individually rational and Pareto optimal (IRPO), where the parameters are given by vectors of initial endowments and final allocations and the unknowns are utility levels and marginal utilities at endowments and consumption allocations. For the case of two observations and two agents the third section derives a linear programming characterization for IRPO in terms of observables only. The fourth section investigates to what extend IRPO is empirically meaningful.

1 Polynomial inequalities for IRPO

Following the usual procedure in the testability literature (see Carvajal, Indrajit, and Synder 2004) we present in this section a system of polynomial inequalities characterizing IRPO. A data set is denoted by $D =< \{\omega_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h}, \{x_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h} >$,[1] initial endowments and final allocations, respectively. For simplicity, we assume that the data are strictly interior. Otherwise we would have to be explicit about the utility functions at the boundary and exclude Inada-conditions: we need marginal utilities at the data to be finite. The datas constitute the parameters of the system.

The interpretations of the unknowns are, respectively: utility levels at the final allocations, utility levels at the endowments, marginal utilities at the final allocations and marginal utilities at the endowments.

[1] As a convention, subscripts, the t-index and letters refer to agents, superscripts, the r-index and numbers to observations.

$\exists \{\bar{V}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in \mathbb{R}$, $\{\bar{W}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in \mathbb{R}$, $\{\hat{\lambda}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in B_{++}^n$ and $\{\tilde{\lambda}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in B_{++}^n$ such that[2]

$$x_t^r \neq x_t^s, r \neq s \Rightarrow \bar{V}_t^r - \bar{V}_t^s - \hat{\lambda}_t^s \cdot (x_t^r - x_t^s) < 0, \tag{1a}$$

$$x_t^r = x_t^s, r \neq s \Rightarrow \bar{V}_t^r = \bar{V}_t^s, \hat{\lambda}_t^r = \hat{\lambda}_t^s, \tag{1b}$$

$$x_t^r \neq \omega_t^s, r \neq s \Rightarrow \bar{V}_t^r - \bar{W}_t^s - \tilde{\lambda}_t^s \cdot (x_t^r - \omega_t^s) < 0 \quad \& \quad \bar{W}_t^s - \bar{V}_t^r - \hat{\lambda}_t^r \cdot (\omega_t^s - x_t^r) < 0, \tag{1c}$$

$$x_t^r = \omega_t^s, r \neq s \Rightarrow \bar{V}_t^r = \bar{W}_t^s, \hat{\lambda}_t^r = \tilde{\lambda}_t^s, \tag{1d}$$

$$x_t^r \neq \omega_t^r \Rightarrow \bar{V}_t^r - \bar{W}_t^r - \tilde{\lambda}_t^r \cdot (x_t^r - \omega_t^r) < 0 \quad \& \quad \bar{W}_t^r - \bar{V}_t^r - \hat{\lambda}_t^r \cdot (\omega_t^r - x_t^r) < 0, \tag{1e}$$

$$x_t^r = \omega_t^r \Rightarrow \bar{V}_t^r = \bar{W}_t^r, \hat{\lambda}_t^r = \tilde{\lambda}_t^r, \tag{1f}$$

$$\omega_t^r \neq \omega_t^s, r \neq s \Rightarrow \bar{W}_t^r - \bar{W}_t^s - \tilde{\lambda}_t^s \cdot (\omega_t^r - \omega_t^s) < 0, \tag{1g}$$

$$\omega_t^r = \omega_t^s, r \neq s \Rightarrow \bar{W}_t^r = \bar{W}_t^s, \tilde{\lambda}_t^r = \tilde{\lambda}_t^s, \tag{1h}$$

$$\bar{W}_t^r \leq \bar{V}_t^r. \tag{1i}$$

This system corresponds to:

IR: Given D, there exists a set of strictly concave, strictly monotonic and C^∞-utility functions[3] $\{U_t\}_{t=a,\dots,m}$ such that $\{x_t^r\}_{t=a,\dots,m}^{r=1,\dots,h}$ is an individually rational allocation of the economy $< \{U_t\}_{t=a,\dots,m}, \{\omega_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} >$, i.e. $x \leq \omega_t^r$ implies $U(x) \leq U(x_t^r), \forall t, r$.

We next consider the above system, where $\hat{\lambda}_t^r$ is replaced by λ_0^r/λ_t^r, $\lambda_0^r \in B_{++}^n$, $\lambda_t^r \in \mathbb{R}_{++}$. We interpret λ_0^r as shadow prices of the aggregate endowments, and λ_t^r as a weight parameter. Moreover, we add a "no waste"-condition (from strict monotonicity):

$$\sum_{t=a}^{m} (\omega_t^r - x_t^r) = 0. \tag{1j}$$

This then corresponds to Pareto efficiency and individual rationality, allowing for changing welfare weights across observations:

IRPO I: Given D, there exists a set of strictly concave, strictly monotonic and C^∞-utility functions $\{U_t\}_{t=a,\dots,m}$ and a weighting scheme $\{\lambda_t\}_{t=a,\dots,m}^{r=1,\dots,h}$, such that: $\forall r$ $\{x_t^r\}_{t=a,\dots,m}^{r=1,\dots,h}$ maximizes the social welfare function $\sum_{t=a}^m \lambda_t^r U_t(x_t^r)$ subject to the feasibility and the individual rationality constraint of the economy $< \{U_t\}_{t=a,\dots,m}, \{\omega_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} >$.

Finally, we require the weighting scheme to be independent of the observation, i.e. not only the same individual utility functions across observations, but also a time-stable welfare function: $\lambda_t^r = \lambda_t$. For simplicity, we here restrict attention to the case, where the Lagrange multipliers for the IR constraints are not strictly positive:

[2] $B_{++}^n \equiv \{\lambda \in \mathbb{R}_{++}^n \mid ||\lambda||_1 \leq 1\}$.

[3] Unless otherwise stated and following Brown and Matzkin (1996), we deal with this "nice" case throughout the paper: it simplifies the analysis and enables us to state and prove the main ideas and results without, for instance, getting lost in the technicalities of weak inequalities.

IRPO II: Given D, there exists a set of strictly concave, strictly monotonic and C^∞-utility functions $\{U_t\}_{t=a,\ldots,m}$ and a weighting scheme $\{\lambda_t\}_{t=a,\ldots,m}$, such that: $\forall r$ $\{x_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h}$ maximizes the social welfare function $\sum_{t=a}^m \lambda_t U_t(x_t^r)$ subject to the feasibility and the individual rationality constraint of the economy $< \{U_t\}_{t=a,\ldots,m}, \{\omega_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h} >$ and the individual rationality constraints are not strictly binding.

To study just Pareto efficiency (PO I, PO II), we can restrict attention to inequalities (1a), (1b) and (1j), where $\hat{\lambda}_t^r$ is replaced by λ_0^r/λ_t^r.

Proposition 1 *System 1 and its variants characterize IR, IRPO I and IRPO II, respectively.*

Proof See Appendix. □

By the Tarski-Seidenberg theorem this means that for any finite data set D, it can be decided whether the family of polynomial inequalities characterizing IRPO has a solution. Rationalizability as efficient and individually rational is a decidable problem.

2 IRPO-restrictions for two observations

Here we derive for two observations a family of polynomial inequalities over the data set such that the IR inequalities for any finite number of agents and the IRPO I inequalities for two agents have a solution if and only if the observables satisfy the derived axioms on the data. Proofs are given in the appendix.

In the following we make use of some λ^*'s and $\hat{\lambda}$'s, to be defined shortly. They result from the maximization of linear objective functions in shadow prices and marginal utilities, parameterized by the data, over restriction sets that are given by linear inequalities again in the data. Hence, the λ^*'s and $\hat{\lambda}$'s are computed exclusively from observations. The definitions we need are the and following:

1. $M_t^{1*} \equiv B^n \cap \{\lambda|\lambda \cdot (x_t^1 - \omega_t^1) \geq 0\}$,[4] $M_t^{2*} \equiv B^n \cap \{\lambda|\lambda \cdot (x_t^2 - \omega_t^2) \geq 0\}$.
2. $M_t^{1**} \equiv M_t^{1*} \cap \{\lambda|\lambda \cdot (\omega_t^2 - \omega_t^1) \geq 0\}$. M_t^{2**} is defined analogously.
3. $\lambda_t^{1x*} \in \arg\max\limits_{\lambda \in M_t^{1*}} \ \lambda \cdot (x_t^2 - \omega_t^1)$. Analogously λ_t^{2x*}.
4. $\lambda_t^{1\omega*} \in \arg\max\limits_{\lambda \in M_t^{1*}} \ \lambda \cdot (\omega_t^2 - \omega_t^1)$. Analogously $\lambda_t^{2\omega*}$.
5. $\lambda_t^{1**} \in \arg\max\limits_{\lambda \in M_t^{1**}} \ \lambda \cdot (x_t^2 - \omega_t^1)$. Analogously λ_t^{2**}.
6. $M_t^{3*} \equiv B^n \cap \{\lambda|\lambda \cdot (\omega_t^2 - x_t^1) \geq 0\}$. Analogously M_t^{4*}, just switched observations.
7. $\lambda_t^{3*} \in \arg\max\limits_{\lambda \in M_t^{3*}} \ \lambda \cdot (x_t^2 - x_t^1)$. Analogously λ_t^{4*}.

With this we get the strong axiom of revealed individual rationality (SARIR):

[4] B^n is the closure of $B_{++}^n \equiv \{\lambda \in \mathbb{R}_{++}^n | ||\lambda||_1 \leq 1\}$ and hence compact.

Proposition 2 (SARIR) *The data for an m-agent, two-observation, n-good economy can be rationalized by IR, if and only if they satisfy the following system of inequalities:*

$$\lambda_t^{2x*} \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow (\omega_t^2 \nleq x_t^1) \;\;\&\;\; \lambda_t^{3*} \cdot (x_t^2 - x_t^1) > 0, \tag{2a}$$

$$\lambda_t^{1x*} \cdot (x_t^2 - \omega_t^1) \leq 0 \Rightarrow (\omega_t^1 \nleq x_t^2) \;\;\&\;\; \lambda_t^{4*} \cdot (x_t^1 - x_t^2) > 0, \tag{2b}$$

$$\begin{aligned}&\left[(\lambda_t^{2**} \cdot (x_t^1 - \omega_t^2) \leq 0) \;\vee\; (\lambda_t^{2\omega*} \cdot (\omega_t^1 - \omega_t^2) \leq 0)\right] \Rightarrow \\ &\left[(\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0) \;\;\&\;\; (\lambda_t^{1\omega*} \cdot (\omega_t^2 - \omega_t^1) > 0)\right],\end{aligned} \tag{2c}$$

$$x_t^1 \nleq \omega_t^1, \qquad x_t^2 \nleq \omega_t^2. \tag{2d}$$

For the following SARIR and Pareto efficiency (SARIRPO) yet some more notations:

1. $M^{5*} \equiv B^n \cap \{\lambda | \lambda \cdot (x_b^2 - x_b^1) \geq 0\}$, and $\lambda^{5*} \in \arg\max_{\lambda \in M^{5*}} \; \lambda \cdot (x_a^2 - x_a^1)$.
2. $M^{6*} \equiv M^{5*} \cap \{\lambda | \lambda \cdot (x_a^2 - x_a^1) \geq 0\}$, and $\lambda^{6b*} \in \arg\max_{\lambda \in M^{6*}} \; \lambda \cdot (\omega_b^2 - x_b^1)$.
3. $M^{7*} \equiv M^{6*} \cap \{\lambda | \lambda \cdot (\omega_b^2 - x_b^1) \geq 0\}$, and $\lambda^{7*} \in \arg\max_{\lambda \in M^{7*}} \; \lambda \cdot (\omega_a^2 - x_a^1)$.

$\lambda_{8*}, \lambda_{9b*}, \lambda_{10*}$ are the analogs (switching observations) to $\lambda_{5*}, \lambda_{6b*}, \lambda_{7*}$, respectively.

Proposition 3 (SARIRPO) *The data for a two-agent, two-observation, n-good economy can be rationalized by IRPO I, if and only if they satisfy SARIR for a and b plus the following two inequalities:*

$$\begin{aligned}&\left[(\lambda_a^{2x*} \cdot (x_a^1 - \omega_a^2) \leq 0) \;\;\&\;\; (\lambda_b^{2x*} \cdot (x_b^1 - \omega_b^2) \leq 0)\right] \Rightarrow \\ &\quad \Big[(x_b^2 \nleq x_b^1) \;\;\&\;\; (\lambda^{5*} \cdot (x_a^2 - x_a^1) > 0) \;\;\& \\ &\qquad (\lambda^{6b*} \cdot (\omega_b^2 - x_b^1) > 0) \;\;\&\;\; (\lambda^{7*} \cdot (\omega_a^2 - x_a^1) > 0)\Big],\end{aligned} \tag{3a}$$

$$\begin{aligned}&\left[(\lambda_a^{1x*} \cdot (x_a^2 - \omega_a^1) \leq 0) \;\;\&\;\; (\lambda_b^{1x*} \cdot (x_b^2 - \omega_b^1) \leq 0)\right] \Rightarrow \\ &\Big[(x_b^1 \nleq x_b^2) \;\;\&\;\; (\lambda^{8*} \cdot (x_a^1 - x_a^2) > 0) \;\;\& \\ &(\lambda^{9b*} \cdot (\omega_b^1 - x_b^2) > 0) \;\;\&\;\; (\lambda^{10*} \cdot (\omega_a^1 - x_a^2) > 0)\Big].\end{aligned} \tag{3b}$$

3 The empirical implications of IRPO

In this section we show the following results: (1) IR imposes restrictions on finite data sets. (2) PO I has no testable restrictions apart from the "no waste"-condition. (3) With strict concavity PO II has testable restrictions. For the special case of two observations and fewer agents than commodities these restrictions are non-generic. (4) With concavity PO II is void. (5) IRPO I is distinguishable from IR, i.e. adding PO I to IR, even though alone it is vacuous, adds restrictions.

The first proposition in this section shows that IR has simple, cross-observational restrictions over and above the fact that per observation the consumption bundle cannot lie in the negative orthant from the endowment vector. These conditions resemble WARP:

Proposition 4 *(Strictly) monotonic, strictly concave individual rationality imposes the following necessary conditions, a weak axiom of individual rationality (WARIR),* $r \neq s, r, s = 1, \ldots, h$*:*

$$x_t^r \leq \omega_t^s \Rightarrow x_t^s \not< \omega_t^r, \tag{4a}$$
$$x_t^r \nleq \omega_t^r. \tag{4b}$$

Proof If, say, $x_t^r \leq \omega_t^s$, then by successive application of the inequalities in system 1: $\bar{W}_t^r \leq \bar{V}_t^r \leq \bar{W}_t^s \leq \bar{V}_t^s$. From (1c) and (1d), we get $\tilde{\lambda}_t^r \cdot (x_t^s - \omega_t^r) > 0$ or $x_t^s = \omega_t^r$. In the first case the conclusion follows from the (strict) positivity of marginal utilities. In the second case, the conclusion is immediate. (4b) is immediate. □

This shows that IR and its characterization – SARIR – are non-vacuous. Moreover, the following example shows that SARIR is a robust strengthening of WARIR:

Example 1

$$x_a^1 = (1, 2)$$
$$x_a^2 = (7, 0.5)$$
$$\omega_a^1 = (6, 1.25)$$
$$\omega_a^2 = (5, 7)$$

It is easy to see that WARIR is satisfied: we have $x_a^1 < \omega_a^2$, but $x_a^2 \not< \omega_a^1$. Moreover, we have $x_a^r \nleq \omega_a^r$ for each observation. But the data does not satisfy (2c): first, we notice that by $x_a^1 < \omega_a^2$, it follows that $\lambda_a^{2**} \cdot (x_a^1 - \omega_a^2) < 0$, so that the antecedence of (2c) is satisfied and hence (2c) relevant. From the requirement that $\lambda \cdot (x_a^1 - \omega_a^1) > 0$, we get that $\lambda(1) < 0.15 * \lambda(2)$. But, when maximizing $\lambda \cdot (x_a^2 - \omega_a^1)$, i.e. $\lambda \cdot (1, -0.75)$, we would like to make $\lambda(1)$ as large as possible. But even when $\lambda(1) = 0.15 * \lambda(2)$, the value of the objective function remains negative, a contradiction to SARIR. Note also that this example is obviously robust to small perturbations.

The next proposition states the trivial testable implications of PO I.

Proposition 5 *The only restriction of Pareto efficiency in an exchange economy with the stated assumptions on preferences and the data set is the "no waste"-condition (1j).*

Proof See Appendix.

Next, we investigate the testable implications of Pareto efficiency with equal weights across observations and non-binding individual rationality constraints (PO II): the following proposition addresses this question in a special case. □

Proposition 6 *If* $h = 2$*,* $m \leq n$*, conditional on satisfying the "no waste"-condition, for almost all data sets rationalization as Pareto efficient with a stable welfare function is possible.*

Proof Since we are seeking for a generic statement, we can WLOG assume that no two final allocations are equal for an agent. Adding up pairwise inequalities 1a for each agent separately, we see that PO II is equivalent to the following system, for all agents t: $\breve{\lambda} \cdot (x_t^1 - x_t^2) > 0$, where $\breve{\lambda} \equiv \lambda_0^2 - \lambda_0^1$. Stacking the $z_t \equiv (x_t^1 - x_t^2)$ as rows into a matrix $Z_{m\times n}$, we look for a solution to the system $Z\breve{\lambda} \gg 0$. By the generalized Farkas' lemma, this has a solution, if and only if the system $\mu' Z = 0$ with $\mu_{m\times 1} > 0$ has none. This is generically the case for $m \leq n$. □

Remark This result is interesting, because for utility maximization with respect to a budget constraint Afriat's theorem shows that two observations are enough for robust non-vacuousness. Geometrically, the proposition tells us that with $h = 2$ we can only rationalize data as Pareto efficient with a stable welfare function, if the vectors of the allocation differences across agents lie in the interior of a halfspace, which they generically do for $m \leq n$. Economically, it should be clear that the vacuousness result, under the requirement of strict concavity, can only hold generically: if the individual rationality constraints are not binding (as assumed in PO II), the unique Pareto-optimal allocation only depends on the aggregate endowment. Hence, any example with equal aggregate endowments across observations and different allocations is a counterexample, as can be seen by adding up the first two inequalities for all the agents.

Our next two examples show that in other cases we can get robust restrictions, and, therefore, that proposition 6 is special. First: two commodities, two observations and three agents ($h = 2$, $m > n$). We make use of the geometrical insight from above: just find an example, where the vectors of allocation differences do not lie in a halfspace.

Example 2

$$\begin{array}{lll} x_a^1 = (1, 1) & x_b^1 = (2, 1) & x_c^1 = (3, 4) \\ x_a^2 = (0.5, 2) & x_b^2 = (4, 0.5) & x_c^2 = (1, 1) \end{array}$$

Adding up inequalities (1a) for each agent separately, we get:

$$\breve{\lambda} \cdot (x_a^1 - x_a^2) > 0 \qquad \breve{\lambda} \cdot (x_b^1 - x_b^2) > 0 \qquad \breve{\lambda} \cdot (x_c^1 - x_c^2) > 0. \tag{5a}$$

The implications – inequality by inequality – are:

$$\breve{\lambda}(1) > 2\breve{\lambda}(2) \qquad \breve{\lambda}(1) < 0.25\breve{\lambda}(2) \qquad \breve{\lambda}(1) > -1.5\breve{\lambda}(2). \tag{5b}$$

This is inconsistent. Moreover, it is obviously robust to small perturbations in the data set.

Second: two goods, three observations and two agents ($h > 2$, $m = n$):

Example 3

$$\begin{array}{lll} x_a^1 = (1, 10) & x_a^2 = (7, 11) & x_a^3 = (3, 1) \\ x_b^1 = (22, 10) & x_b^2 = (1, 2) & x_b^3 = (21, 17) \end{array}$$

From inequalities (1a) for observations 1 and 2, we can infer (just adding them all up across agents): $\lambda_0^2(2) > \lambda_0^1(2)$. Similarly, from observations 1 and 3: $\lambda_0^1(2) > \lambda_0^3(2)$. Hence, $\lambda_0^2(2) > \lambda_0^3(2)$. However, this contradicts observations 2 and 3, using the same type of calculation.

Proposition 7 shows that PO II is vacuous, if agents' utility functions are concave, but not necessarily strictly concave.

Proposition 7 *Under concavity every data set satisfying "no waste" is rationalizable as PO II, for any m and h.*

Proof Adding up inequalities (1a) per agent for each cycle of observations (eliminating utility levels) leads to so-called cyclical monotonicity conditions as found in Brown and Calsamiglia (2003). With concavity, these new inequalities are not strict. Just set $\lambda_0^1 = \cdots = \lambda_0^h = \lambda_0 \gg 0$, and each such cyclical monotonicity condition becomes $0 \geq 0$, a truism. □

The final example shows that IR and IRPO are empirically distinguishable, i.e. that adding a void requirement can yield additional restrictions. We note that the additional "bite" does not come from the "no waste"-condition.

Example 4

$$
\begin{array}{llll}
x_a^1 = (4,4) & x_b^1 = (4,4) & x_a^2 = (6,2) & x_b^2 = (2,6) \\
\omega_a^1 = (5,3) & \omega_b^1 = (3,5) & \omega_a^2 = (7,1) & \omega_b^2 = (1,7)
\end{array}
$$

We will now check system 1 for IRPO I: First, by inequalities 1e and 1i for both agents, we get

$$
\tilde{\lambda}_a^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0 \qquad - \tilde{\lambda}_b^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0. \tag{6a}
$$

The inequalities (1d) for both agents yield:

$$
-\tilde{\lambda}_a^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{\lambda_0^2}{\lambda_a^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0 \qquad \tilde{\lambda}_b^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{\lambda_0^2}{\lambda_b^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0. \tag{6b}
$$

Combining both, we get:

$$
\frac{\lambda_0^2}{\lambda_a^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0 \qquad - \frac{\lambda_0^2}{\lambda_b^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0. \tag{6c}
$$

Multiplying by λ_a^2 and λ_b^2, respectively, we get an immediate contradiction.

We finally illustrate, how example 4 fails to satisfy (3b). First of all, by the requirement that $\lambda \cdot (x_a^1 - \omega_a^1) = \lambda \cdot (-1, 1) \geq 0$, we know that $\lambda(1) \leq \lambda(2)$, but then $\lambda \cdot (x_a^2 - \omega_a^1) = \lambda \cdot (1, -1) \leq 0$ for all those lambda, hence the antecedence in (3b) is satisfied. The same is true, due to the symmetry of the example, for b. Next, we show that $\lambda^{8*} \cdot (x_a^1 - x_a^2) = \lambda^{8*} \cdot (-2, 2) \leq 0$. But this is, because by definition of λ^{8*} we have to have $\lambda^{8*} \cdot (x_b^1 - x_b^2) = \lambda^{8*} \cdot (2, -2) \geq 0$.

We note that this counterexample is not robust to perturbations that "push" x^2 away from the line between x^1 and ω^1.

It is not hard to find a solution to the IR system with parameters given by example 4. Finally, it is not difficult to construct an example with equal aggregate endowments and different final allocations across observations that satisfies IRPO I and not IRPO II. Hence, the two IRPO versions are distinguishable.

Appendix

Proof of Proposition 1 IR is easiest to see: Necessity: (1a)–(1h) is a well-known characterization for C^1 strictly concave functions for each pair of data points. (1i) simply states individual rationality. Strict monotonicity and the interiority assumption allow us to require $\hat{\lambda}_t^r \in B_{++}^n$ and $\tilde{\lambda}_t^r \in B_{++}^n$.

Sufficiency: we have to find smooth, strictly concave and strictly monotonic utility functions that rationalize the data according to IR. Moreover, we want them to attach the right economic meaning to the numbers and vectors in the inequalities. For instance, $\bar{V}_a^1$ should be the utility level of agent a at observation 1, etc. Afriat proposed the following utility function:

$$U_t(x) \equiv \min[\{\bar{V}_t^r + \hat{\lambda}_t^r \cdot (x - x_t^r),\ \bar{W}_t^r + \tilde{\lambda}_t^r \cdot (x - \omega_t^r)\}^{r=1,\ldots,h}]. \quad \text{(A.1a)}$$

This means in particular that $U(x_t^r) = \bar{V}_t^r$, and so for the endowment points, just using the inequalities in system 1. Furthermore,

$$x \leq (<) \omega_t^r \quad \Rightarrow \quad U(x) \leq \bar{W}_t^r + \tilde{\lambda}_t^r \cdot (x - \omega_t^r) \leq (<) \bar{W}_t^r \leq \bar{V}_t^r, \quad \text{(A.1b)}$$

guaranteeing individual rationality. We use that the marginal utility vectors are strictly positive, but the argument would analogously hold true for monotonicity. Due the inequalities in system 1 being strict, we can obviously apply the perturbation and convolution procedure proposed in Chiappori and Rochet (1987), to make this a smooth and strictly concave utility function.

Next, IRPO I: Necessity follows from the well-known characterization of Pareto optima as solutions to the maximization of a weighted additive welfare function in a concave environment,[5] the Kuhn-Tucker conditions for this program and the above mentioned characterization for C^1 strictly concave functions. For sufficiency, we use again Afriat's construction, and hence all the observations on the utility levels at the data points and individual rationality hold. We next have to show that

$$\sum_{t=a}^{m} x_t \leq (<) \sum_{t=a}^{m} \omega_t^r \Rightarrow \sum_{t=a}^{m} \lambda_t^r U(x_t) \leq (<) \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r. \quad \text{(A.1c)}$$

Now,

$$\begin{aligned} \sum_{t=a}^{m} \lambda_t^r U(x_t) &\leq \sum_{t=a}^{m} \lambda_t^r \cdot \left[\bar{V}_t^r + \frac{\lambda_0^r}{\lambda_t^r} \cdot (x_t - x_t^r) \right] \\ &= \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r + \lambda_0^r \cdot \sum_{t=a}^{m} (x_t - x_t^r) \leq (<) \\ &\quad \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r + \lambda_0^r \cdot \sum_{t=a}^{m} (\omega_t^r - x_t^r) = \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r, \end{aligned} \quad \text{(A.1d)}$$

if $\sum_{t=a}^{m} x_t \leq (<) \sum_{t=a}^{m} \omega_t^r$. The first step uses the construction of the utility function as a lower envelope, the second one is just multiplying out, the third step uses

[5] Of course, by (strict) monotonicity, the Pareto frontier is never flat, and thus the separation weights are strictly positive.

the assumption and strict positivity of λ_0, the last one the "no waste"-condition.[6] Finally, it is obvious that the smoothing construction in Chiappori and Rochet (1987) can still be applied, and that the same argument also holds for IRPO II, PO I and PO II. □

For the proof of Propositions 2 and 3 we need a series of lemmas:[7]

Lemma 1 *The following system of polynomial inequalities, having eliminated utility levels and weights, is equivalent to system 1, after specializing to two agents, two observations, and IRPO I:*
$\exists\{\lambda_0^1, \lambda_0^2\} \in B_{++}^n$ *and* $\{\tilde{\lambda}_t^1, \tilde{\lambda}_t^2\}_{t=a,b} \in B_{++}^n$ *such that*

$$\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0 \Rightarrow \lambda_0^1 \cdot (x_t^2 - x_t^1) > 0, \tag{A.2aa}$$
$$\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2ab}$$

$$\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (\omega_t^2 - \omega_t^1) > 0, \tag{A.2ba}$$
$$\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2bb}$$

$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (\omega_t^2 - \omega_t^1) > 0, \tag{A.2ca}$$
$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \lambda_0^1 \cdot (\omega_t^2 - x_t^1) > 0, \tag{A.2cb}$$
$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \lambda_0^1 \cdot (x_t^2 - x_t^1) > 0, \tag{A.2cc}$$
$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2cd}$$

$$\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2d}$$

$$\tilde{\lambda}_t^1 \cdot (x_t^1 - \omega_t^1) > 0, \qquad \tilde{\lambda}_t^2 \cdot (x_t^2 - \omega_t^2) > 0. \tag{A.2e}$$

Proof "⇒": Suppose $\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0$. By (1a) and (1i) $\bar{W}_t^1 \leq \bar{V}_t^1 < \bar{V}_t^2$. By (1a) and (1c) the conclusions follow. Suppose $\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0$. By (1g) and (1i) $\bar{W}_t^1 < \bar{W}_t^2 \leq \bar{V}_t^2$. By (1g) and (1c) the conclusions follow. Suppose $\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0$. By (1c) and (1i) $\bar{W}_t^1 \leq \bar{V}_t^1 < \bar{W}_t^2 \leq \bar{V}_t^2$. By (1a), (1c), (1e) and (1g) the conclusions follow. Suppose $\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0$. By (1c) $\bar{W}_t^1 < \bar{V}_t^2$. By (1c) again the conclusion follows. Finally (A.1e) follows from (1e), (1g) and (1i).

"⇐": This way is somewhat more tedious to show. The following insight will help: recall that the task is to prove from the existence of some numbers that satisfy system (A.2) the existence of numbers that satisfy system (1). That does not mean that for those variables that do not get eliminated, we have to take the same numbers. In other words, when going from one existentially quantified system to another, we have to plug in concrete values for the variables, that can be manipulated in the transition process. Take an example: from the sentence $\exists x, y, z \in \mathbb{R}, x+y+z < 0$ we assume a concrete, but abstract value for x, y, z, i.e. a, b, c, and hence $a+b+c < 0$.

[6] With monotonicity, if some commodity was slack, the shadow price would be zero, and the argument would still go through.

[7] For reasons of specificity we assume each point in the data set for an agent to be different.

But then we also know that $a/1000 + b/1000 + c/1000 < 0$. And hence, we get again $\exists x, y, z \in \mathbb{R}, x + y + z < 0$. Next, we normalize one utility level, say $\bar{W}_t^1 = 0$, and perform Fourier-Motzkin elimination for $\bar{V}_t^1$ and $\bar{W}_t^2$, leaving us with equivalent conditions solely on $\bar{V}_t^2$, which, we show, can always be satisfied with the right choice of scalar multiples of the shadow price and marginal utility vectors, given the conditions in (A.2). To this end, we scale up some vectors and set some arbitrarily close to zero in L_1-norm. Since we deal with each system separately, without taking into account interpersonal restrictions, this proof does not lead to a SARIRPO for IRPO II. Hence, even though we preserve the direction of each λ_0-vector, it might well be that we have to scale them up/down by a different amount in order to guarantee solvability. This is then reflected in a non-stable weighting scheme. Denoting the (strictly positive) scalar multiples associated with $\lambda_0^1, \lambda_0^2, \tilde{\lambda}_t^2, \tilde{\lambda}_t^1$ with u, x, y, z,[8] respectively, and abbreviating the vector products, we can rewrite the specialized system (1) in the following more concise way, which facilitates Fourier-Motzkin elimination.

$$\bar{V}_t^1 - \bar{V}_t^2 - xa_2 < 0, \tag{A.3a}$$
$$\bar{V}_t^2 - \bar{V}_t^1 - ua_1 < 0, \tag{A.3b}$$
$$\bar{V}_t^1 - \bar{W}_t^2 - yb_2 < 0, \tag{A.3c}$$
$$\bar{W}_t^2 - \bar{V}_t^1 - ub_1 < 0, \tag{A.3d}$$
$$\bar{V}_t^2 - zc_2 < 0, \tag{A.3e}$$
$$-\bar{V}_t^2 - xc_1 < 0, \tag{A.3f}$$
$$\bar{V}_t^1 - zd_2 < 0, \tag{A.3g}$$
$$-\bar{V}_t^1 - ud_1 < 0, \tag{A.3h}$$
$$\bar{V}_t^2 - \bar{W}_t^2 - ye_2 < 0, \tag{A.3i}$$
$$\bar{W}_t^2 - \bar{V}_t^2 - xe_1 < 0, \tag{A.3j}$$
$$\bar{W}_t^2 - zf_2 < 0, \tag{A.3k}$$
$$-\bar{W}_t^2 - yf_1 < 0, \tag{A.3l}$$
$$-\bar{V}_t^1 \leq 0, \tag{A.3m}$$
$$\bar{W}_t^2 - \bar{V}_t^2 \leq 0. \tag{A.3n}$$

After some tedious, but straightforward algebra, we get the following hierarchic system of inequalities:

$$\begin{aligned} &-xa_2, -xa_2 - ud_1, -xa_2 - ub_1 - yf_1, -xa_2 - u(b_1 + d_1) - yb_2, \\ &-xa_2 - ub_1 - yb_2, -xe_1 - yf_1, -xe_1 - yb_2 - ud_1, -xe_1 - yb_2, \\ &-xc_1, -yf_1, -yb_2 - ud_1, -yb_2 < \bar{V}_t^2. \end{aligned} \tag{A.4a}$$

$$\begin{aligned} \bar{V}_t^2 &< zc_2, zd_2 + ua_1, zd_2 + ub_1 + ye_2, zd_2 + u(a_1 + b_1) + yb_2, \\ &\quad zf_2 + ye_2, zf_2 + yb_2 + ua_1. \end{aligned} \tag{A.4b}$$

[8] Again, notice that these numbers will depend on the agent we are locking at, which is unproblematic for y and z, but not for u and x, if we would like to cover IRPO II.

$$-zd_2, -zd_2 - ud_1, -zd_2 - ub_1 - yf_1, -zd_2 - u(b_1 + d_1) - yb_2, \quad \text{(A.4c)}$$
$$-zd_2 - ub_1 - yb_2 < 0.$$

$$-zf_2 - yf_1, -zf_2 - yb_2 - ud_1, -zf_2 - yb_2 < 0. \quad \text{(A.4d)}$$

$$-yb_2 - ub_1, -yb_2 - ua_1 - xe_1, -yb_2 - ua_1, -yb_2 - u(a_1 + b_1) - xa_2 < 0. \quad \text{(A.4e)}$$

$$-ye_2, -ye_2 - xe_1, -ye_2 - ub_1 - xa_2 < 0. \quad \text{(A.4f)}$$
$$-xa_2 - ua_1 < 0. \quad \text{(A.4g)}$$

Cumbersome as it might have been, this derivation gives us now a straightforward possibility to finish the proof. We have to consider 16 cases, distinguished by whether the antecedences above are positive or negative: $\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0$ vs. $>$, $\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0$ vs. $>$, $\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0$ vs. $>$, $\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0$ vs. $>$.

1. Case $(\leq, \leq, \leq, \leq)$: We will always write down the following sets: "+" is the set of coefficients that is strictly positive, "$-/0$" is the weakly negative set, and "?" the one, for which we do not know the sign. For this case: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1\}$, "'$-/0$" $= \{a_2, f_1, b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. But now we know that all the coefficients on z are strictly positive, hence A.4b can be always satisfied by raising z a lot, i.e. also $\bar{V}_t^2$ can be made arbitrarily big, so that (A.4a) does not matter. For the same reasons (A.4c) and (A.4d) can be made arbitrarily negative. Furthermore, we notice that the coefficients of u in the rest of the inequalities, i.e. a_1, b_1, are also strictly positive, and so is e_2 for y. Hence by raising u slightly all the inequalities are satisfied.
2. Case $(\leq, \leq, \leq, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, c_1\}$, "$-/0$" $= \{a_2, f_1, b_2\}$ and "?" $= \{d_1, e_1\}$. Same as above, the argument nowhere used the weak negativity of c_1.
3. Case $(\leq, \leq, >, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, b_2\}$, "$-/0$" $= \{a_2, f_1, c_1\}$ and "?" $= \{d_1, e_1, b_1\}$. The arguments about z and $\bar{V}_t^2$ still go through. Since in the rest of the equations all the coefficients on y are strictly positive, we can raise this, make x arbitrarily small, and recall for (A.4g) that $a_1 > 0$.
4. Case $(\leq, >, \leq, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, f_1\}$, "$-/0$" $= \{a_2, b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
5. Case $(>, \leq, \leq, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, a_2\}$, "$-/0$" $= \{f_1, b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
6. Case $(\leq, \leq, >, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_2, c_1\}$, "$-/0$" $= \{a_2, f_1\}$ and "?" $= \{d_1, e_1, b_1\}$. Same as third case.
7. Case $(\leq, >, \leq, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, c_1, f_1\}$, "$-/0$" $= \{a_2, b_2\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
8. Case $(>, \leq, \leq, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, c_1, a_2\}$, "$-/0$" $= \{f_1, b_2\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
9. Case $(\leq, >, >, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_1, b_2\}$, "$-/0$" $= \{c_1, a_2\}$ and "?" $= \{d_1, e_1, f_2, b_1\}$. Here we use the fact that all the coefficients on y, i.e. f_1, b_2, e_2, are strictly positive and make y arbitrarily large. This takes care of (A.4d), (A.4e) and (A.4f). Moreover, we make x and u arbitrarily small, preserving

$(xa_2/a_1) < u$. This takes care of (A.4g) and (A.4c), where we recall that $d_2 > 0$. By both steps (A.4a) is not binding at all, and for (A.4b) we notice that the possible negativity of f_2 does not matter, because whenever it appears it can be offset by making y arbitrarily large. Taking $\bar{V}_t^2 \ll \min[zc_2, zd_2]$, where "$\ll$" this time means somewhat sloppily "enough smaller", which is possible, because u can be made arbitrarily small.

10. Case $(>, \leq, >, \leq)$: "+" $= \{d_2, e_2, a_2, c_2, f_2, b_2\}$, "$-/0$" $= \{f_1, c_1\}$ and "?" $= \{d_1, e_1, b_1, a_1\}$. As before, z and hence $\bar{V}_t^2$ can be made arbitrarily large. Hence (A.4a)–(A.4d) is taken care of. For the next two equations, we make use of the fact that the coefficients on y, b_2, e_2 are strictly positive. Finally (A.4g) can be satisfied by making both x and u small, but in the right proportion.
11. Case $(>, >, \leq, \leq)$: "+" $= \{d_2, e_2, a_2, c_2, f_2, f_1, a_1, b_1\}$, "$-/0$" $= \{b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
12. Case $(>, >, >, \leq)$: "+" $= \{d_2, e_2, a_2, c_2, f_1, b_2\}$, "$-/0$" $= \{c_1\}$ and "?" $= \{d_1, e_1, b_1, a_1, f_2\}$. Same as ninth case, where this time we only have to make sure that for (A.4g) the inequality is right, depending on the sign of a_1.
13. Case $(>, >, \leq, >)$: "+" $= \{d_2, e_2, a_2, c_2, f_1, b_1, f_2, c_1, a_1\}$, "$-/0$" $= \{b_2\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
14. Case $(>, \leq, >, >)$: "+" $= \{d_2, e_2, a_2, c_2, b_2, f_2, c_1\}$, "$-/0$" $= \{f_1\}$ and "?" $= \{d_1, e_1, a_1, b_1\}$. Make z and $\bar{V}_t^2$ arbitrarily large. Then raise y somewhat, which is always offset by z, where the coefficient on y, f_1 is negative, and helps to satisfy the rest of the inequalities. (A.4g) is taken care of in the usual way.
15. Case $(\leq, >, >, >)$: "+" $= \{d_2, e_2, a_1, c_2, b_2, f_1, c_1\}$, "$-/0$" $= \{a_2\}$ and "?" $= \{d_1, e_1, f_2, b_1\}$. Same as ninth case.
16. Case $(>, >, >, >)$: "+" $= \{d_2, e_2, b_2, f_1, c_1, a_2\}$, "$-/0$" $= \emptyset$ and "?" $= \{d_1, e_1, f_2, b_1, a_1, c_2\}$. Make y arbitrarily large, and so with x, whose only possibly negative coefficient, e_1 only appears together with y. Then, as usually, make u small, and take $\bar{V}_t^2 \ll \min[zc_2, zd_2]$, this time possibly negative. And this does not matter in A.4a, because either by x or by y the left-hand side is arbitrarily negative. □

System (A.1) now allows an almost mechanical way to eliminate the quantifiers. To this end we need the following

Lemma 2 *A system of the following form: $\exists \lambda \in B^n_{++}$ such that*

$$\phi_1 \Rightarrow \lambda z_1 > 0, \tag{A.5a}$$

$$\phi_2 \Rightarrow \lambda z_2 > 0, \tag{A.5b}$$

where ϕ_1, ϕ_2 are arbitrary mathematical conditions and z_1, z_2 arbitrary non-zero n-vectors, is equivalent to the following quantifier-free one:[9]

$$\phi_1 \Rightarrow \max_{\lambda \in B^n} \lambda z_1 > 0, \tag{A.6a}$$

$$\phi_2 \Rightarrow \max_{\lambda \in B^n} \lambda z_2 > 0, \tag{A.6b}$$

$$(\phi_1 \wedge \phi_2) \Rightarrow \max_{\lambda \in B^n \cap \{\lambda | \lambda z_2 \geq 0\}} \lambda z_1 > 0. \tag{A.6c}$$

[9] B^n is the closure of B^n_{++} and hence compact.

Proof "⇒": Obvious, if the system (A.5) is true, then there always exists a concrete λ that satisfies the consequence in (A.6) (if the antecedence is true), because it is in the feasibility set of the relevant maximization problem. Then a fortiori the maximum value has to be greater than zero. We also notice that, whenever the feasibility sets are not empty, then the maxima exist. And whenever we need them to be nonempty, they are, as can be seen in the case, when both ϕ_1 and ϕ_2 are true: (A.5b) guarantees that $B^n \cap \{\lambda|\lambda z_2 \geq 0\} \neq \emptyset$.

"⇐": To go back, we consider four cases: firstly, both ϕ_1 and ϕ_2 are false, in which case (A.5) can be trivially satisfied. The next two cases are given by either ϕ_1 or ϕ_2 is true, say ϕ_1. Then an element of the $\arg\max_{\lambda \in B^n} \lambda z_1$ will almost do it, if it were not for it possibly lying on the boundary of B^n. But given that we have a strict inequality (by A.6a) we can always perturb this argumentum maximum in such a way as to preserve the strict inequality and make it an interior element of B^n. Finally, consider the case, when both are true: in that case any strictly convex combination of $\arg\max_{\lambda \in B^n \cap \{\lambda|\lambda z_2 \geq 0\}} \lambda z_1$ with $\arg\max_{\lambda \in B^n} \lambda z_2$ and any interior element of B^n that has just an ϵ-weight on the latter two will satisfy (A.5), given (A.6). □

It is obvious that this lemma holds analogously for any finite number of conditions ϕ. We illustrate it eliminating $\tilde{\lambda}_t^1$ in (A.2) and using the definitions from the main part.

Lemma 3 *Elimination of $\tilde{\lambda}_t^1$ yields the following system (we leave out any inequality, that does not contain $\tilde{\lambda}_t^1$):*

$$\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0 \Rightarrow \lambda_t^{1x*} \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.7a}$$

$$\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0 \Rightarrow \left[(\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0) \ \ \& \ \ (\lambda_t^{1\omega*} \cdot (\omega_t^2 - \omega_t^1) > 0)\right], \tag{A.7b}$$

$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \left[(\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0) \ \ \& \ \ (\lambda_t^{1\omega*} \cdot (\omega_t^2 - \omega_t^1) > 0)\right], \tag{A.7c}$$

$$\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0 \Rightarrow \lambda_t^{1x*} \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.7d}$$

$$x_t^1 \nleq \omega_t^1. \tag{A.7e}$$

Proof "⇒": $\tilde{\lambda}_t^1$ is in the set we are maximizing over and has already a strictly positive value, when the objective function is evaluated at it, hence a fortiori the maximal value must be strictly positive.

"⇐": First of all, we notice that by (A.7e) M_t^{1*} is not empty and in particular has also interior elements. The strategy of the proof is to distinguish the 16 cases, given by the sign of the antecedences, and to argue that in each case either λ_t^{1x*} or λ_t^{1**} or both or any $\lambda \in B_{++}^n$ will do.

For the first case $(\leq, \leq, \leq, \leq)$ we note that $M_t^{1**} \neq \emptyset$, because $\lambda_t^{1\omega*}$ lies (or can be made to lie) in the interior of M_t^{1**}. Furthermore, we know $\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0$,

$\lambda_t^{1**} \cdot (x_t^1 - \omega_t^1) \geq 0$, $\lambda_t^{1**} \cdot (\omega_t^2 - \omega_t^1) \geq 0$, and $\lambda_t^{1**} \in S^{n-1}$. Take a convex combination of $\lambda_t^{1\omega*}$ and λ_t^{1**}, which is close enough to λ_t^{1**}. This convex combination is then our solution to go back, for it continues to lie in the interior of M_t^{1**} for any strictly positive convex coefficient.

The rest of the proof is now straightforward, because cases 2–8, 10, 13, 14[10] are no different from the first case. In cases 9, 12, 15 we take λ_t^{1x*} or rather its perturbation, so that it lies in the interior of M_t^{1*}, as our solution. Finally, in case 16, any element of the interior of M_t^{1*} will do. □

Proof of Propositions 2 and 3 Applying Lemma 2 mechanically, one can derive easily SARIR and SARIRPO. To get SARIRPO in its most concise form, one in addition has to discard some void conditions through a series of lemmas (the details are available from the author upon request). □

Finally, we summarize the entire proof strategy: this proof shows that quantifier elimination after some steps can be carried out almost mechanically by building a hierarchy of constraints that guarantee that one can always go back. And this "algorithm" would have worked also for IRPO I with more agents, because the procedure will start to rely on the assumption of two agents only in the last two steps when we eliminate λ_0^1 and λ_0^2. But then it is obvious that only the hierarchy of restrictions would have been more complicated.

Proof of Proposition 5 For expositional reasons we start with the case $h = 2$. We start with the inequalities 1a for each agent, with $\hat{\lambda}_t^r$ replaced by λ_0^r/λ_t^r. With an argument similar to Brown and Matzkin (1996), Appendix, and using the logical equivalence $[(\phi \Rightarrow \psi) \Leftrightarrow (-\phi \vee \psi)]$, we can require equivalently the existence of λ_0^1, λ_0^2 such that

$$\begin{array}{c}
\lambda_0^2 \cdot (x_a^1 - x_a^2) > 0 \vee \lambda_0^1 \cdot (x_a^2 - x_a^1) > 0, \\
\vdots \qquad\qquad\qquad \vdots \\
\lambda_0^2 \cdot (x_m^1 - x_m^2) > 0 \vee \lambda_0^1 \cdot (x_m^2 - x_m^1) > 0.
\end{array} \tag{A.8a}$$

We suggest the following solution: $\lambda_0^1 = \lambda_0^2 = \lambda_0 \gg 0$. Defining $z_t \equiv (x_t^1 - x_t^2)$, we can rewrite:

$$\begin{array}{c}
\lambda_0 z_a > 0 \vee \lambda_0 z_a < 0, \\
\vdots \qquad\qquad \vdots \\
\lambda_0 z_m > 0 \vee \lambda_0 z_m < 0,
\end{array} \tag{A.8b}$$

which for finite data sets can always be satisfied. If for some agent t, $z_t = 0$, leave the agent out. Any numbers $\bar{V}_t^1 = \bar{V}_t^2$, $\lambda_t^1 = \lambda_t^2 > 0$ will satisfy the Afriat system. Notice that the restriction from equal final allocations to collinear marginal utilities <u>across</u> observations is none, because we can choose the shadow price to be equal across agents and observations.

Next, we allow for arbitrary finite h (for the same reason as above with data equality for an agent t, just take all but one of the observations out): for a generic agent t, the elimination of utility levels and the welfare weights will result in

[10] We use the case numbers from above.

the formal equivalent of the strong axiom of revealed preferences (see Chiappori and Rochet 1987), with observed prices being replaced and quantified over by the unobserved shadow prices. Using the logical formula from above, we get for every $k-$sequence of observations, $(r, r+1, \ldots, r+k-1)$, $k \leq h$, the following disjunctive formula:

$$\lambda_0^{r+1} \cdot (x_t^r - x_t^{r+1}) > 0 \vee \cdots \vee \lambda_0^r \cdot (x_t^{r+k-1} - x_t^r) > 0. \qquad \text{(A.8c)}$$

Again we suggest $\lambda_0^1 = \cdots = \lambda_0^h = \lambda_0 \gg 0$, which is set in such a way that $\lambda_0 z_t^{r,s} \neq 0$, where $r < s$ are arbitrary pairs of observations. As shown, this works for any two-sequence. Now, take an arbitrary k-sequence, $(k > 2)$. We have to show that it is not the case that:

$$\lambda_0 \cdot (x_t^r - x_t^{r+1}) < 0 \text{ AND } \cdots \text{ AND } \lambda_0 \cdot (x_t^{r+k-1} - x_t^r) < 0. \qquad \text{(A.8d)}$$

Notice that we can write strict inequalities, because by construction the equality case has been excluded. Summing up leads to $0 < 0$. □

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