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RESEARCH ARTICLE

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Testable implications of Pareto efficiency and individual rationality

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Abstract This paper investigates the testable implications of Pareto efficiency and individual rationality on finite data sets in exchange economies with finitely many commodities and agents. Efficiency alone provides no restrictions other than a trivial "no waste"-condition. Efficiency together with individual rationality implies robust restrictions.

Keywords Testability · Non-parametric restrictions · Allocation data · Quantifier elimination

JEL Classification Numbers D50 · D51

Introduction

This paper investigates, whether Pareto efficiency and individual rationality impose restrictions on finite data sets in exchange economies, where agents have (strictly) concave and (strictly) monotonic utility functions. A data set consists of a finite number of observations on agents' initial endowments and final allocations in finitely many commodities.

We characterize both concepts in terms of polynomial inequalities, which are parameterized by the data, and show that Pareto efficiency alone does not yield restrictions, except for a trivial "no waste"-condition: with monotonicity efficient

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allocations are not compatible with underused endowments. If we impose, in addition, individual rationality or require a stable social welfare function across observations, then there exist data sets which refute the combined rationalization concepts. In contrast, individual rationality is a refutable concept on its own.

These results are interesting for two reasons: first, from the characterization result in terms of polynomial inequalities together with the Tarski-Seidenberg theorem, it follows that for any finite data set there is an effective answer to whether this data set can be rationalized as efficient and individually rational. Efficiency and individual rationality can be formulated in terms of observables only. Secondly, Pareto efficiency alone may not be a good criterion for assessing exchange economies. We would not be able to discriminate between efficient and inefficient exchange. Only efficiency as part of a stronger rationalization concept has empirical implications.

Our work is closely related to the work of Bossert and Sprumont (2001, 2002, 2003), who derive necessary and sufficient conditions on choice correspondences to be efficient and individually rational. However, in general their approach fails for finite data sets in structured economic exchange environments. Making use of the powerful techniques of semi-algebraic geometry (see Carvajal, Indrajit, and Synder 2004), the main contribution of our paper consists in deriving testable restrictions of individual rationality and Pareto efficiency in structured economic environments. Moreover, our work complements a recent and extensive literature on testability in a single agent decision or Walrasian context: Bachmann (2004), Bandyopadhyay et al. (2004), Lee and Wong (2005), Matzkin (2005), and McFadden (2005).

The remainder of this paper proceeds as follows: the next section presents the family of polynomial inequalities characterizing individually rational and Pareto optimal (IRPO), where the parameters are given by vectors of initial endowments and final allocations and the unknowns are utility levels and marginal utilities at endowments and consumption allocations. For the case of two observations and two agents the third section derives a linear programming characterization for IRPO in terms of observables only. The fourth section investigates to what extend IRPO is empirically meaningful.

1 Polynomial inequalities for IRPO

Following the usual procedure in the testability literature (see Carvajal, Indrajit, and Synder 2004) we present in this section a system of polynomial inequalities characterizing IRPO. A data set is denoted by $D =< \{\omega_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h}, \{x_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h} >$,[1] initial endowments and final allocations, respectively. For simplicity, we assume that the data are strictly interior. Otherwise we would have to be explicit about the utility functions at the boundary and exclude Inada-conditions: we need marginal utilities at the data to be finite. The datas constitute the parameters of the system.

The interpretations of the unknowns are, respectively: utility levels at the final allocations, utility levels at the endowments, marginal utilities at the final allocations and marginal utilities at the endowments.

[1] As a convention, subscripts, the t-index and letters refer to agents, superscripts, the r-index and numbers to observations.

$\exists\{\bar{V}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in \mathbb{R}$, $\{\bar{W}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in \mathbb{R}$, $\{\hat{\lambda}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in B_{++}^n$ and $\{\tilde{\lambda}_t^r\}_{t=a,\dots,m}^{r=1,\dots,h} \in B_{++}^n$ such that[2]

$$
\begin{aligned}
x_t^r \neq x_t^s, r \neq s &\Rightarrow \bar{V}_t^r - \bar{V}_t^s - \hat{\lambda}_t^s \cdot (x_t^r - x_t^s) < 0, && \text{(1a)}\\
x_t^r = x_t^s, r \neq s &\Rightarrow \bar{V}_t^r = \bar{V}_t^s, \hat{\lambda}_t^r = \hat{\lambda}_t^s, && \text{(1b)}\\
x_t^r \neq \omega_t^s, r \neq s &\Rightarrow \bar{V}_t^r - \bar{W}_t^s - \tilde{\lambda}_t^s \cdot (x_t^r - \omega_t^s) < 0 \quad \& && \text{(1c)}\\
&\bar{W}_t^s - \bar{V}_t^r - \hat{\lambda}_t^r \cdot (\omega_t^s - x_t^r) < 0,\\
x_t^r = \omega_t^s, r \neq s &\Rightarrow \bar{V}_t^r = \bar{W}_t^s, \hat{\lambda}_t^r = \tilde{\lambda}_t^s, && \text{(1d)}\\
x_t^r \neq \omega_t^r &\Rightarrow \bar{V}_t^r - \bar{W}_t^r - \tilde{\lambda}_t^r \cdot (x_t^r - \omega_t^r) < 0 \quad \& && \text{(1e)}\\
&\bar{W}_t^r - \bar{V}_t^r - \hat{\lambda}_t^r \cdot (\omega_t^r - x_t^r) < 0,\\
x_t^r = \omega_t^r &\Rightarrow \bar{V}_t^r = \bar{W}_t^r, \hat{\lambda}_t^r = \tilde{\lambda}_t^r, && \text{(1f)}\\
\omega_t^r \neq \omega_t^s, r \neq s &\Rightarrow \bar{W}_t^r - \bar{W}_t^s - \tilde{\lambda}_t^s \cdot (\omega_t^r - \omega_t^s) < 0, && \text{(1g)}\\
\omega_t^r = \omega_t^s, r \neq s &\Rightarrow \bar{W}_t^r = \bar{W}_t^s, \tilde{\lambda}_t^r = \tilde{\lambda}_t^s, && \text{(1h)}\\
\bar{W}_t^r &\leq \bar{V}_t^r. && \text{(1i)}
\end{aligned}
$$

This system corresponds to:

IR: Given D, there exists a set of strictly concave, strictly monotonic and C^∞-utility functions[3] $\{U_t\}_{t=a,\dots,m}$ such that $\{x_t^r\}_{t=a,\dots,m}^{r=1,\dots,h}$ is an individually rational allocation of the economy $< \{U_t\}_{t=a,\dots,m}, \{\omega_t^r\}_{t=a,\dots,m}^{r=1,\dots,h}\} >$, i.e. $x \leq \omega_t^r$ implies $U(x) \leq U(x_t^r), \forall t, r$.

We next consider the above system, where $\hat{\lambda}_t^r$ is replaced by λ_0^r/λ_t^r, $\lambda_0^r \in B_{++}^n$, $\lambda_t^r \in \mathbb{R}_{++}$. We interpret λ_0^r as shadow prices of the aggregate endowments, and λ_t^r as a weight parameter. Moreover, we add a "no waste"-condition (from strict monotonicity):

$$\sum_{t=a}^{m}(\omega_t^r - x_t^r) = 0. \tag{1j}$$

This then corresponds to Pareto efficiency and individual rationality, allowing for changing welfare weights across observations:

IRPO I: Given D, there exists a set of strictly concave, strictly monotonic and C^∞-utility functions $\{U_t\}_{t=a,\dots,m}$ and a weighting scheme $\{\lambda_t\}_{t=a,\dots,m}^{r=1,\dots,h}$, such that: $\forall r$ $\{x_t^r\}_{t=a,\dots,m}^{r=1,\dots,h}$ maximizes the social welfare function $\sum_{t=a}^m \lambda_t^r U_t(x_t^r)$ subject to the feasibility and the individual rationality constraint of the economy $< \{U_t\}_{t=a,\dots,m}, \{\omega_t^r\}_{t=a,\dots,m}^{r=1,\dots,h}\} >$.

Finally, we require the weighting scheme to be independent of the observation, i.e. not only the same individual utility functions across observations, but also a time-stable welfare function: $\lambda_t^r = \lambda_t$. For simplicity, we here restrict attention to the case, where the Lagrange multipliers for the IR constraints are not strictly positive:

[2] $B_{++}^n \equiv \{\lambda \in \mathbb{R}_{++}^n | ||\lambda||_1 \leq 1\}$.

[3] Unless otherwise stated and following Brown and Matzkin (1996), we deal with this "nice" case throughout the paper: it simplifies the analysis and enables us to state and prove the main ideas and results without, for instance, getting lost in the technicalities of weak inequalities.

IRPO II: Given D, there exists a set of strictly concave, strictly monotonic and C^∞-utility functions $\{U_t\}_{t=a,\ldots,m}$ and a weighting scheme $\{\lambda_t\}_{t=a,\ldots,m}$, such that: $\forall r$ $\{x_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h}$ maximizes the social welfare function $\sum_{t=a}^m \lambda_t U_t(x_t^r)$ subject to the feasibility and the individual rationality constraint of the economy $< \{U_t\}_{t=a,\ldots,m}, \{\omega_t^r\}_{t=a,\ldots,m}^{r=1,\ldots,h}\} >$ and the individual rationality constraints are not strictly binding.

To study just Pareto efficiency (PO I, PO II), we can restrict attention to inequalities (1a), (1b) and (1j), where $\hat{\lambda}_t^r$ is replaced by λ_0^r/λ_t^r.

Proposition 1 *System 1 and its variants characterize IR, IRPO I and IRPO II, respectively.*

Proof See Appendix. □

By the Tarski-Seidenberg theorem this means that for any finite data set D, it can be decided whether the family of polynomial inequalities characterizing IRPO has a solution. Rationalizability as efficient and individually rational is a decidable problem.

2 IRPO-restrictions for two observations

Here we derive for two observations a family of polynomial inequalities over the data set such that the IR inequalities for any finite number of agents and the IRPO I inequalities for two agents have a solution if and only if the observables satisfy the derived axioms on the data. Proofs are given in the appendix.

In the following we make use of some λ^*'s and $\hat{\lambda}$'s, to be defined shortly. They result from the maximization of linear objective functions in shadow prices and marginal utilities, parameterized by the data, over restriction sets that are given by linear inequalities again in the data. Hence, the λ^*'s and $\hat{\lambda}$'s are computed exclusively from observations. The definitions we need are the and following:

1. $M_t^{1*} \equiv B^n \cap \{\lambda | \lambda \cdot (x_t^1 - \omega_t^1) \geq 0\}$,[4] $M_t^{2*} \equiv B^n \cap \{\lambda | \lambda \cdot (x_t^2 - \omega_t^2) \geq 0\}$.
2. $M_t^{1**} \equiv M_t^{1*} \cap \{\lambda | \lambda \cdot (\omega_t^2 - \omega_t^1) \geq 0\}$. M_t^{2**} is defined analogously.
3. $\lambda_t^{1x*} \in \arg\max_{\lambda \in M_t^{1*}} \lambda \cdot (x_t^2 - \omega_t^1)$. Analogously λ_t^{2x*}.
4. $\lambda_t^{1\omega*} \in \arg\max_{\lambda \in M_t^{1*}} \lambda \cdot (\omega_t^2 - \omega_t^1)$. Analogously $\lambda_t^{2\omega*}$.
5. $\lambda_t^{1**} \in \arg\max_{\lambda \in M_t^{1**}} \lambda \cdot (x_t^2 - \omega_t^1)$. Analogously λ_t^{2**}.
6. $M_t^{3*} \equiv B^n \cap \{\lambda | \lambda \cdot (\omega_t^2 - x_t^1) \geq 0\}$. Analogously M_t^{4*}, just switched observations.
7. $\lambda_t^{3*} \in \arg\max_{\lambda \in M_t^{3*}} \lambda \cdot (x_t^2 - x_t^1)$. Analogously λ_t^{4*}.

With this we get the strong axiom of revealed individual rationality (SARIR):

[4] B^n is the closure of $B_{++}^n \equiv \{\lambda \in \mathbb{R}_{++}^n | ||\lambda||_1 \leq 1\}$ and hence compact.

Proposition 2 (SARIR) *The data for an m-agent, two-observation, n-good economy can be rationalized by IR, if and only if they satisfy the following system of inequalities:*

$$\lambda_t^{2x*} \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow (\omega_t^2 \not\leq x_t^1) \;\;\&\;\; \lambda_t^{3*} \cdot (x_t^2 - x_t^1) > 0, \tag{2a}$$

$$\lambda_t^{1x*} \cdot (x_t^2 - \omega_t^1) \leq 0 \Rightarrow (\omega_t^1 \not\leq x_t^2) \;\;\&\;\; \lambda_t^{4*} \cdot (x_t^1 - x_t^2) > 0, \tag{2b}$$

$$\begin{aligned}&\left[(\lambda_t^{2**} \cdot (x_t^1 - \omega_t^2) \leq 0) \;\vee\; (\lambda_t^{2\omega*} \cdot (\omega_t^1 - \omega_t^2) \leq 0)\right] \Rightarrow \\ &\left[(\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0) \;\;\&\;\; (\lambda_t^{1\omega*} \cdot (\omega_t^2 - \omega_t^1) > 0)\right],\end{aligned} \tag{2c}$$

$$x_t^1 \not\leq \omega_t^1, \qquad x_t^2 \not\leq \omega_t^2. \tag{2d}$$

For the following SARIR and Pareto efficiency (SARIRPO) yet some more notations:

1. $M^{5*} \equiv B^n \cap \{\lambda | \lambda \cdot (x_b^2 - x_b^1) \geq 0\}$, and $\lambda^{5*} \in \arg\max_{\lambda \in M^{5*}} \lambda \cdot (x_a^2 - x_a^1)$.
2. $M^{6*} \equiv M^{5*} \cap \{\lambda | \lambda \cdot (x_a^2 - x_a^1) \geq 0\}$, and $\lambda^{6b*} \in \arg\max_{\lambda \in M^{6*}} \lambda \cdot (\omega_b^2 - x_b^1)$.
3. $M^{7*} \equiv M^{6*} \cap \{\lambda | \lambda \cdot (\omega_b^2 - x_b^1) \geq 0\}$, and $\lambda^{7*} \in \arg\max_{\lambda \in M^{7*}} \lambda \cdot (\omega_a^2 - x_a^1)$.

$\lambda_{8*}, \lambda_{9b*}, \lambda_{10*}$ are the analogs (switching observations) to $\lambda_{5*}, \lambda_{6b*}, \lambda_{7*}$, respectively.

Proposition 3 (SARIRPO) *The data for a two-agent, two-observation, n-good economy can be rationalized by IRPO I, if and only if they satisfy SARIR for a and b plus the following two inequalities:*

$$\begin{aligned}&\left[(\lambda_a^{2x*} \cdot (x_a^1 - \omega_a^2) \leq 0) \;\;\&\;\; (\lambda_b^{2x*} \cdot (x_b^1 - \omega_b^2) \leq 0)\right] \Rightarrow \\ &\quad\Big[(x_b^2 \not\leq x_b^1) \;\;\&\;\; (\lambda^{5*} \cdot (x_a^2 - x_a^1) > 0) \;\;\&\\ &\qquad (\lambda^{6b*} \cdot (\omega_b^2 - x_b^1) > 0) \;\;\&\;\; (\lambda^{7*} \cdot (\omega_a^2 - x_a^1) > 0)\Big],\end{aligned} \tag{3a}$$

$$\begin{aligned}&\left[(\lambda_a^{1x*} \cdot (x_a^2 - \omega_a^1) \leq 0) \;\;\&\;\; (\lambda_b^{1x*} \cdot (x_b^2 - \omega_b^1) \leq 0)\right] \Rightarrow \\ &\Big[(x_b^1 \not\leq x_b^2) \;\;\&\;\; (\lambda^{8*} \cdot (x_a^1 - x_a^2) > 0) \;\;\&\\ &(\lambda^{9b*} \cdot (\omega_b^1 - x_b^2) > 0) \;\;\&\;\; (\lambda^{10*} \cdot (\omega_a^1 - x_a^2) > 0)\Big].\end{aligned} \tag{3b}$$

3 The empirical implications of IRPO

In this section we show the following results: (1) IR imposes restrictions on finite data sets. (2) PO I has no testable restrictions apart from the "no waste"-condition. (3) With strict concavity PO II has testable restrictions. For the special case of two observations and fewer agents than commodities these restrictions are non-generic. (4) With concavity PO II is void. (5) IRPO I is distinguishable from IR, i.e. adding PO I to IR, even though alone it is vacuous, adds restrictions.

The first proposition in this section shows that IR has simple, cross-observational restrictions over and above the fact that per observation the consumption bundle cannot lie in the negative orthant from the endowment vector. These conditions resemble WARP:

Proposition 4 *(Strictly) monotonic, strictly concave individual rationality imposes the following necessary conditions, a weak axiom of individual rationality (WARIR), $r \neq s, r, s = 1, \ldots, h$:*

$$x_t^r \leq \omega_t^s \Rightarrow x_t^s \not< \omega_t^r, \tag{4a}$$
$$x_t^r \not\leq \omega_t^r. \tag{4b}$$

Proof If, say, $x_t^r \leq \omega_t^s$, then by successive application of the inequalities in system 1: $\bar{W}_t^r \leq \bar{V}_t^r \leq \bar{W}_t^s \leq \bar{V}_t^s$. From (1c) and (1d), we get $\tilde{\lambda}_t^r \cdot (x_t^s - \omega_t^r) > 0$ or $x_t^s = \omega_t^r$. In the first case the conclusion follows from the (strict) positivity of marginal utilities. In the second case, the conclusion is immediate. (4b) is immediate. □

This shows that IR and its characterization – SARIR – are non-vacuous. Moreover, the following example shows that SARIR is a robust strengthening of WARIR:

Example 1

$$x_a^1 = (1, 2)$$
$$x_a^2 = (7, 0.5)$$
$$\omega_a^1 = (6, 1.25)$$
$$\omega_a^2 = (5, 7)$$

It is easy to see that WARIR is satisfied: we have $x_a^1 < \omega_a^2$, but $x_a^2 \not< \omega_a^1$. Moreover, we have $x_a^r \not\leq \omega_a^r$ for each observation. But the data does not satisfy (2c): first, we notice that by $x_a^1 < \omega_a^2$, it follows that $\lambda_a^{2**} \cdot (x_a^1 - \omega_a^2) < 0$, so that the antecedence of (2c) is satisfied and hence (2c) relevant. From the requirement that $\lambda \cdot (x_a^1 - \omega_a^1) > 0$, we get that $\lambda(1) < 0.15 * \lambda(2)$. But, when maximizing $\lambda \cdot (x_a^2 - \omega_a^1)$, i.e. $\lambda \cdot (1, -0.75)$, we would like to make $\lambda(1)$ as large as possible. But even when $\lambda(1) = 0.15 * \lambda(2)$, the value of the objective function remains negative, a contradiction to SARIR. Note also that this example is obviously robust to small perturbations.

The next proposition states the trivial testable implications of PO I.

Proposition 5 *The only restriction of Pareto efficiency in an exchange economy with the stated assumptions on preferences and the data set is the "no waste"-condition (1j).*

Proof See Appendix.

Next, we investigate the testable implications of Pareto efficiency with equal weights across observations and non-binding individual rationality constraints (PO II): the following proposition addresses this question in a special case. □

Proposition 6 *If $h = 2$, $m \leq n$, conditional on satisfying the "no waste"-condition, for almost all data sets rationalization as Pareto efficient with a stable welfare function is possible.*

Proof Since we are seeking for a generic statement, we can WLOG assume that no two final allocations are equal for an agent. Adding up pairwise inequalities 1a for each agent separately, we see that PO II is equivalent to the following system, for all agents t: $\breve{\lambda} \cdot (x_t^1 - x_t^2) > 0$, where $\breve{\lambda} \equiv \lambda_0^2 - \lambda_0^1$. Stacking the $z_t \equiv (x_t^1 - x_t^2)$ as rows into a matrix $Z_{m \times n}$, we look for a solution to the system $Z\breve{\lambda} \gg 0$. By the generalized Farkas' lemma, this has a solution, if and only if the system $\mu' Z = 0$ with $\mu_{m \times 1} > 0$ has none. This is generically the case for $m \leq n$. □

Remark This result is interesting, because for utility maximization with respect to a budget constraint Afriat's theorem shows that two observations are enough for robust non-vacuousness. Geometrically, the proposition tells us that with $h = 2$ we can only rationalize data as Pareto efficient with a stable welfare function, if the vectors of the allocation differences across agents lie in the interior of a halfspace, which they generically do for $m \leq n$. Economically, it should be clear that the vacuousness result, under the requirement of strict concavity, can only hold generically: if the individual rationality constraints are not binding (as assumed in PO II), the unique Pareto-optimal allocation only depends on the aggregate endowment. Hence, any example with equal aggregate endowments across observations and different allocations is a counterexample, as can be seen by adding up the first two inequalities for all the agents.

Our next two examples show that in other cases we can get robust restrictions, and, therefore, that proposition 6 is special. First: two commodities, two observations and three agents ($h = 2$, $m > n$). We make use of the geometrical insight from above: just find an example, where the vectors of allocation differences do not lie in a halfspace.

Example 2

$$\begin{array}{lll} x_a^1 = (1, 1) & x_b^1 = (2, 1) & x_c^1 = (3, 4) \\ x_a^2 = (0.5, 2) & x_b^2 = (4, 0.5) & x_c^2 = (1, 1) \end{array}$$

Adding up inequalities (1a) for each agent separately, we get:

$$\breve{\lambda} \cdot (x_a^1 - x_a^2) > 0 \qquad \breve{\lambda} \cdot (x_b^1 - x_b^2) > 0 \qquad \breve{\lambda} \cdot (x_c^1 - x_c^2) > 0. \tag{5a}$$

The implications – inequality by inequality – are:

$$\breve{\lambda}(1) > 2\breve{\lambda}(2) \qquad \breve{\lambda}(1) < 0.25\breve{\lambda}(2) \qquad \breve{\lambda}(1) > -1.5\breve{\lambda}(2). \tag{5b}$$

This is inconsistent. Moreover, it is obviously robust to small perturbations in the data set.

Second: two goods, three observations and two agents ($h > 2$, $m = n$):

Example 3

$$\begin{array}{lll} x_a^1 = (1, 10) & x_a^2 = (7, 11) & x_a^3 = (3, 1) \\ x_b^1 = (22, 10) & x_b^2 = (1, 2) & x_b^3 = (21, 17) \end{array}$$

From inequalities (1a) for observations 1 and 2, we can infer (just adding them all up across agents): $\lambda_0^2(2) > \lambda_0^1(2)$. Similarly, from observations 1 and 3: $\lambda_0^1(2) > \lambda_0^3(2)$. Hence, $\lambda_0^2(2) > \lambda_0^3(2)$. However, this contradicts observations 2 and 3, using the same type of calculation.

Proposition 7 shows that PO II is vacuous, if agents' utility functions are concave, but not necessarily strictly concave.

Proposition 7 *Under concavity every data set satisfying "no waste" is rationalizable as PO II, for any m and h.*

Proof Adding up inequalities (1a) per agent for each cycle of observations (eliminating utility levels) leads to so-called cyclical monotonicity conditions as found in Brown and Calsamiglia (2003). With concavity, these new inequalities are not strict. Just set $\lambda_0^1 = \cdots = \lambda_0^h = \lambda_0 \gg 0$, and each such cyclical monotonicity condition becomes $0 \geq 0$, a truism. □

The final example shows that IR and IRPO are empirically distinguishable, i.e. that adding a void requirement can yield additional restrictions. We note that the additional "bite" does not come from the "no waste"-condition.

Example 4

$$x_a^1 = (4,4)\ x_b^1 = (4,4)\ x_a^2 = (6,2)\ x_b^2 = (2,6)$$
$$\omega_a^1 = (5,3)\ \omega_b^1 = (3,5)\ \omega_a^2 = (7,1)\ \omega_b^2 = (1,7)$$

We will now check system 1 for IRPO I: First, by inequalities 1e and 1i for both agents, we get

$$\tilde{\lambda}_a^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0 \qquad -\tilde{\lambda}_b^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0. \tag{6a}$$

The inequalities (1d) for both agents yield:

$$-\tilde{\lambda}_a^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{\lambda_0^2}{\lambda_a^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0 \qquad \tilde{\lambda}_b^1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{\lambda_0^2}{\lambda_b^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0. \tag{6b}$$

Combining both, we get:

$$\frac{\lambda_0^2}{\lambda_a^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0 \qquad -\frac{\lambda_0^2}{\lambda_b^2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0. \tag{6c}$$

Multiplying by λ_a^2 and λ_b^2, respectively, we get an immediate contradiction.

We finally illustrate, how example 4 fails to satisfy (3b). First of all, by the requirement that $\lambda \cdot (x_a^1 - \omega_a^1) = \lambda \cdot (-1, 1) \geq 0$, we know that $\lambda(1) \leq \lambda(2)$, but then $\lambda \cdot (x_a^2 - \omega_a^1) = \lambda \cdot (1, -1) \leq 0$ for all those lambda, hence the antecedence in (3b) is satisfied. The same is true, due to the symmetry of the example, for b. Next, we show that $\lambda^{8*} \cdot (x_a^1 - x_a^2) = \lambda^{8*} \cdot (-2, 2) \leq 0$. But this is, because by definition of λ^{8*} we have to have $\lambda^{8*} \cdot (x_b^1 - x_b^2) = \lambda^{8*} \cdot (2, -2) \geq 0$.

We note that this counterexample is not robust to perturbations that "push" x^2 away from the line between x^1 and ω^1.

It is not hard to find a solution to the IR system with parameters given by example 4. Finally, it is not difficult to construct an example with equal aggregate endowments and different final allocations across observations that satisfies IRPO I and not IRPO II. Hence, the two IRPO versions are distinguishable.

Appendix

Proof of Proposition 1 IR is easiest to see: Necessity: (1a)–(1h) is a well-known characterization for C^1 strictly concave functions for each pair of data points. (1i) simply states individual rationality. Strict monotonicity and the interiority assumption allow us to require $\hat{\lambda}_t^r \in B_{++}^n$ and $\tilde{\lambda}_t^r \in B_{++}^n$.

Sufficiency: we have to find smooth, strictly concave and strictly monotonic utility functions that rationalize the data according to IR. Moreover, we want them to attach the right economic meaning to the numbers and vectors in the inequalities. For instance, $\bar{V}_a^1$ should be the utility level of agent a at observation 1, etc. Afriat proposed the following utility function:

$$U_t(x) \equiv \min[\{\bar{V}_t^r + \hat{\lambda}_t^r \cdot (x - x_t^r), \bar{W}_t^r + \tilde{\lambda}_t^r \cdot (x - \omega_t^r)\}^{r=1,\ldots,h}]. \quad \text{(A.1a)}$$

This means in particular that $U(x_t^r) = \bar{V}_t^r$, and so for the endowment points, just using the inequalities in system 1. Furthermore,

$$x \leq (<)\omega_t^r \quad \Rightarrow \quad U(x) \leq \bar{W}_t^r + \tilde{\lambda}_t^r \cdot (x - \omega_t^r) \leq (<)\bar{W}_t^r \leq \bar{V}_t^r, \quad \text{(A.1b)}$$

guaranteeing individual rationality. We use that the marginal utility vectors are strictly positive, but the argument would analogously hold true for monotonicity. Due the inequalities in system 1 being strict, we can obviously apply the perturbation and convolution procedure proposed in Chiappori and Rochet (1987), to make this a smooth and strictly concave utility function.

Next, IRPO I: Necessity follows from the well-known characterization of Pareto optima as solutions to the maximization of a weighted additive welfare function in a concave environment,[5] the Kuhn-Tucker conditions for this program and the above mentioned characterization for C^1 strictly concave functions. For sufficiency, we use again Afriat's construction, and hence all the observations on the utility levels at the data points and individual rationality hold. We next have to show that

$$\sum_{t=a}^{m} x_t \leq (<) \sum_{t=a}^{m} \omega_t^r \Rightarrow \sum_{t=a}^{m} \lambda_t^r U(x_t) \leq (<) \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r. \quad \text{(A.1c)}$$

Now,

$$\begin{aligned} \sum_{t=a}^{m} \lambda_t^r U(x_t) &\leq \sum_{t=a}^{m} \lambda_t^r \cdot \left[\bar{V}_t^r + \frac{\lambda_0^r}{\lambda_t^r} \cdot (x_t - x_t^r)\right] \\ &= \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r + \lambda_0^r \cdot \sum_{t=a}^{m} (x_t - x_t^r) \leq (<) \\ &\quad \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r + \lambda_0^r \cdot \sum_{t=a}^{m} (\omega_t^r - x_t^r) = \sum_{t=a}^{m} \lambda_t^r \bar{V}_t^r, \end{aligned} \quad \text{(A.1d)}$$

if $\sum_{t=a}^{m} x_t \leq (<) \sum_{t=a}^{m} \omega_t^r$. The first step uses the construction of the utility function as a lower envelope, the second one is just multiplying out, the third step uses

[5] Of course, by (strict) monotonicity, the Pareto frontier is never flat, and thus the separation weights are strictly positive.

the assumption and strict positivity of λ_0, the last one the "no waste"-condition.[6] Finally, it is obvious that the smoothing construction in Chiappori and Rochet (1987) can still be applied, and that the same argument also holds for IRPO II, PO I and PO II. □

For the proof of Propositions 2 and 3 we need a series of lemmas:[7]

Lemma 1 *The following system of polynomial inequalities, having eliminated utility levels and weights, is equivalent to system 1, after specializing to two agents, two observations, and IRPO I:*
$\exists\{\lambda_0^1, \lambda_0^2\} \in B_{++}^n$ *and* $\{\tilde{\lambda}_t^1, \tilde{\lambda}_t^2\}_{t=a,b} \in B_{++}^n$ *such that*

$$\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0 \Rightarrow \lambda_0^1 \cdot (x_t^2 - x_t^1) > 0, \tag{A.2aa}$$
$$\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2ab}$$

$$\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (\omega_t^2 - \omega_t^1) > 0, \tag{A.2ba}$$
$$\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2bb}$$

$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (\omega_t^2 - \omega_t^1) > 0, \tag{A.2ca}$$
$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \lambda_0^1 \cdot (\omega_t^2 - x_t^1) > 0, \tag{A.2cb}$$
$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \lambda_0^1 \cdot (x_t^2 - x_t^1) > 0, \tag{A.2cc}$$
$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2cd}$$

$$\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0 \Rightarrow \tilde{\lambda}_t^1 \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.2d}$$

$$\tilde{\lambda}_t^1 \cdot (x_t^1 - \omega_t^1) > 0, \qquad \tilde{\lambda}_t^2 \cdot (x_t^2 - \omega_t^2) > 0. \tag{A.2e}$$

Proof "⇒": Suppose $\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0$. By (1a) and (1i) $\bar{W}_t^1 \leq \bar{V}_t^1 < \bar{V}_t^2$. By (1a) and (1c) the conclusions follow. Suppose $\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0$. By (1g) and (1i) $\bar{W}_t^1 < \bar{W}_t^2 \leq \bar{V}_t^2$. By (1g) and (1c) the conclusions follow. Suppose $\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0$. By (1c) and (1i) $\bar{W}_t^1 \leq \bar{V}_t^1 < \bar{W}_t^2 \leq \bar{V}_t^2$. By (1a), (1c), (1e) and (1g) the conclusions follow. Suppose $\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0$. By (1c) $\bar{W}_t^1 < \bar{V}_t^2$. By (1c) again the conclusion follows. Finally (A.1e) follows from (1e), (1g) and (1i).

"⇐": This way is somewhat more tedious to show. The following insight will help: recall that the task is to prove from the existence of some numbers that satisfy system (A.2) the existence of numbers that satisfy system (1). That does not mean that for those variables that do not get eliminated, we have to take the same numbers. In other words, when going from one existentially quantified system to another, we have to plug in concrete values for the variables, that can be manipulated in the transition process. Take an example: from the sentence $\exists x, y, z \in \mathbb{R}, x + y + z < 0$ we assume a concrete, but abstract value for x, y, z, i.e. a, b, c, and hence $a+b+c < 0$.

[6] With monotonicity, if some commodity was slack, the shadow price would be zero, and the argument would still go through.

[7] For reasons of specificity we assume each point in the data set for an agent to be different.

But then we also know that $a/1000 + b/1000 + c/1000 < 0$. And hence, we get again $\exists x, y, z \in \mathbb{R}, x + y + z < 0$. Next, we normalize one utility level, say $\bar{W}_t^1 = 0$, and perform Fourier-Motzkin elimination for $\bar{V}_t^1$ and $\bar{W}_t^2$, leaving us with equivalent conditions solely on $\bar{V}_t^2$, which, we show, can always be satisfied with the right choice of scalar multiples of the shadow price and marginal utility vectors, given the conditions in (A.2). To this end, we scale up some vectors and set some arbitrarily close to zero in L_1-norm. Since we deal with each system separately, without taking into account interpersonal restrictions, this proof does not lead to a SARIRPO for IRPO II. Hence, even though we preserve the direction of each λ_0-vector, it might well be that we have to scale them up/down by a different amount in order to guarantee solvability. This is then reflected in a non-stable weighting scheme. Denoting the (strictly positive) scalar multiples associated with $\lambda_0^1, \lambda_0^2, \tilde{\lambda}_t^2, \tilde{\lambda}_t^1$ with u, x, y, z,[8] respectively, and abbreviating the vector products, we can rewrite the specialized system (1) in the following more concise way, which facilitates Fourier-Motzkin elimination.

$$\bar{V}_t^1 - \bar{V}_t^2 - xa_2 < 0, \tag{A.3a}$$
$$\bar{V}_t^2 - \bar{V}_t^1 - ua_1 < 0, \tag{A.3b}$$
$$\bar{V}_t^1 - \bar{W}_t^2 - yb_2 < 0, \tag{A.3c}$$
$$\bar{W}_t^2 - \bar{V}_t^1 - ub_1 < 0, \tag{A.3d}$$
$$\bar{V}_t^2 - zc_2 < 0, \tag{A.3e}$$
$$-\bar{V}_t^2 - xc_1 < 0, \tag{A.3f}$$
$$\bar{V}_t^1 - zd_2 < 0, \tag{A.3g}$$
$$-\bar{V}_t^1 - ud_1 < 0, \tag{A.3h}$$
$$\bar{V}_t^2 - \bar{W}_t^2 - ye_2 < 0, \tag{A.3i}$$
$$\bar{W}_t^2 - \bar{V}_t^2 - xe_1 < 0, \tag{A.3j}$$
$$\bar{W}_t^2 - zf_2 < 0, \tag{A.3k}$$
$$-\bar{W}_t^2 - yf_1 < 0, \tag{A.3l}$$
$$-\bar{V}_t^1 \leq 0, \tag{A.3m}$$
$$\bar{W}_t^2 - \bar{V}_t^2 \leq 0. \tag{A.3n}$$

After some tedious, but straightforward algebra, we get the following hierarchic system of inequalities:

$$\begin{aligned}&-xa_2, -xa_2 - ud_1, -xa_2 - ub_1 - yf_1, -xa_2 - u(b_1 + d_1) - yb_2,\\ &-xa_2 - ub_1 - yb_2, -xe_1 - yf_1, -xe_1 - yb_2 - ud_1, -xe_1 - yb_2,\\ &-xc_1, -yf_1, -yb_2 - ud_1, -yb_2 < \bar{V}_t^2.\end{aligned} \tag{A.4a}$$

$$\begin{aligned}\bar{V}_t^2 &< zc_2, zd_2 + ua_1, zd_2 + ub_1 + ye_2, zd_2 + u(a_1 + b_1) + yb_2,\\ &\quad zf_2 + ye_2, zf_2 + yb_2 + ua_1.\end{aligned} \tag{A.4b}$$

[8] Again, notice that these numbers will depend on the agent we are locking at, which is unproblematic for y and z, but not for u and x, if we would like to cover IRPO II.

$$-zd_2, -zd_2 - ud_1, -zd_2 - ub_1 - yf_1, -zd_2 - u(b_1 + d_1) - yb_2, \quad \text{(A.4c)}$$
$$-zd_2 - ub_1 - yb_2 < 0.$$

$$-zf_2 - yf_1, -zf_2 - yb_2 - ud_1, -zf_2 - yb_2 < 0. \quad \text{(A.4d)}$$

$$-yb_2 - ub_1, -yb_2 - ua_1 - xe_1, -yb_2 - ua_1, -yb_2 - u(a_1 + b_1) - xa_2 < 0. \quad \text{(A.4e)}$$

$$-ye_2, -ye_2 - xe_1, -ye_2 - ub_1 - xa_2 < 0. \quad \text{(A.4f)}$$
$$-xa_2 - ua_1 < 0. \quad \text{(A.4g)}$$

Cumbersome as it might have been, this derivation gives us now a straightforward possibility to finish the proof. We have to consider 16 cases, distinguished by whether the antecedences above are positive or negative: $\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0$ vs. $>$, $\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0$ vs. $>$, $\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0$ vs. $>$, $\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0$ vs. $>$.

1. Case $(\leq, \leq, \leq, \leq)$: We will always write down the following sets: "+" is the set of coefficients that is strictly positive, "$-/0$" is the weakly negative set, and "?" the one, for which we do not know the sign. For this case: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1\}$, "'$-/0$" $= \{a_2, f_1, b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. But now we know that all the coefficients on z are strictly positive, hence A.4b can be always satisfied by raising z a lot, i.e. also $\bar{V}_t^2$ can be made arbitrarily big, so that (A.4a) does not matter. For the same reasons (A.4c) and (A.4d) can be made arbitrarily negative. Furthermore, we notice that the coefficients of u in the rest of the inequalities, i.e. a_1, b_1, are also strictly positive, and so is e_2 for y. Hence by raising u slightly all the inequalities are satisfied.
2. Case $(\leq, \leq, \leq, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, c_1\}$, "$-/0$" $= \{a_2, f_1, b_2\}$ and "?" $= \{d_1, e_1\}$. Same as above, the argument nowhere used the weak negativity of c_1.
3. Case $(\leq, \leq, >, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, b_2\}$, "$-/0$" $= \{a_2, f_1, c_1\}$ and "?" $= \{d_1, e_1, b_1\}$. The arguments about z and $\bar{V}_t^2$ still go through. Since in the rest of the equations all the coefficients on y are strictly positive, we can raise this, make x arbitrarily small, and recall for (A.4g) that $a_1 > 0$.
4. Case $(\leq, >, \leq, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, f_1\}$, "$-/0$" $= \{a_2, b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
5. Case $(>, \leq, \leq, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, a_2\}$, "$-/0$" $= \{f_1, b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
6. Case $(\leq, \leq, >, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_2, c_1\}$, "$-/0$" $= \{a_2, f_1\}$ and "?" $= \{d_1, e_1, b_1\}$. Same as third case.
7. Case $(\leq, >, \leq, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, c_1, f_1\}$, "$-/0$" $= \{a_2, b_2\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
8. Case $(>, \leq, \leq, >)$: "+" $= \{d_2, e_2, a_1, c_2, f_2, b_1, c_1, a_2\}$, "$-/0$" $= \{f_1, b_2\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
9. Case $(\leq, >, >, \leq)$: "+" $= \{d_2, e_2, a_1, c_2, f_1, b_2\}$, "$-/0$" $= \{c_1, a_2\}$ and "?" $= \{d_1, e_1, f_2, b_1\}$. Here we use the fact that all the coefficients on y, i.e. f_1, b_2, e_2, are strictly positive and make y arbitrarily large. This takes care of (A.4d), (A.4e) and (A.4f). Moreover, we make x and u arbitrarily small, preserving

$(xa_2/a_1) < u$. This takes care of (A.4g) and (A.4c), where we recall that $d_2 > 0$. By both steps (A.4a) is not binding at all, and for (A.4b) we notice that the possible negativity of f_2 does not matter, because whenever it appears it can be offset by making y arbitrarily large. Taking $\bar{V}_t^2 \ll \min[zc_2, zd_2]$, where "$\ll$" this time means somewhat sloppily "enough smaller", which is possible, because u can be made arbitrarily small.

10. Case $(>, \leq, >, \leq)$: "+" $= \{d_2, e_2, a_2, c_2, f_2, b_2\}$, "$-/0$" $= \{f_1, c_1\}$ and "?" $= \{d_1, e_1, b_1, a_1\}$. As before, z and hence $\bar{V}_t^2$ can be made arbitrarily large. Hence (A.4a)–(A.4d) is taken care of. For the next two equations, we make use of the fact that the coefficients on y, b_2, e_2 are strictly positive. Finally (A.4g) can be satisfied by making both x and u small, but in the right proportion.
11. Case $(>, >, \leq, \leq)$: "+" $= \{d_2, e_2, a_2, c_2, f_2, f_1, a_1, b_1\}$, "$-/0$" $= \{b_2, c_1\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
12. Case $(>, >, >, \leq)$: "+" $= \{d_2, e_2, a_2, c_2, f_1, b_2\}$, "$-/0$" $= \{c_1\}$ and "?" $= \{d_1, e_1, b_1, a_1, f_2\}$. Same as ninth case, where this time we only have to make sure that for (A.4g) the inequality is right, depending on the sign of a_1.
13. Case $(>, >, \leq, >)$: "+" $= \{d_2, e_2, a_2, c_2, f_1, b_1, f_2, c_1, a_1\}$, "$-/0$" $= \{b_2\}$ and "?" $= \{d_1, e_1\}$. Same as first case.
14. Case $(>, \leq, >, >)$: "+" $= \{d_2, e_2, a_2, c_2, b_2, f_2, c_1\}$, "$-/0$" $= \{f_1\}$ and "?" $= \{d_1, e_1, a_1, b_1\}$. Make z and $\bar{V}_t^2$ arbitrarily large. Then raise y somewhat, which is always offset by z, where the coefficient on y, f_1 is negative, and helps to satisfy the rest of the inequalities. (A.4g) is taken care of in the usual way.
15. Case $(\leq, >, >, >)$: "+" $= \{d_2, e_2, a_1, c_2, b_2, f_1, c_1\}$, "$-/0$" $= \{a_2\}$ and "?" $= \{d_1, e_1, f_2, b_1\}$. Same as ninth case.
16. Case $(>, >, >, >)$: "+" $= \{d_2, e_2, b_2, f_1, c_1, a_2\}$, "$-/0$" $= \emptyset$ and "?" $= \{d_1, e_1, f_2, b_1, a_1, c_2\}$. Make y arbitrarily large, and so with x, whose only possibly negative coefficient, e_1 only appears together with y. Then, as usually, make u small, and take $\bar{V}_t^2 \ll \min[zc_2, zd_2]$, this time possibly negative. And this does not matter in A.4a, because either by x or by y the left-hand side is arbitrarily negative. □

System (A.1) now allows an almost mechanical way to eliminate the quantifiers. To this end we need the following

Lemma 2 *A system of the following form: $\exists \lambda \in B^n_{++}$ such that*

$$\phi_1 \Rightarrow \lambda z_1 > 0, \tag{A.5a}$$

$$\phi_2 \Rightarrow \lambda z_2 > 0, \tag{A.5b}$$

where ϕ_1, ϕ_2 are arbitrary mathematical conditions and z_1, z_2 arbitrary non-zero n-vectors, is equivalent to the following quantifier-free one:[9]

$$\phi_1 \Rightarrow \max_{\lambda \in B^n} \lambda z_1 > 0, \tag{A.6a}$$

$$\phi_2 \Rightarrow \max_{\lambda \in B^n} \lambda z_2 > 0, \tag{A.6b}$$

$$(\phi_1 \wedge \phi_2) \Rightarrow \max_{\lambda \in B^n \cap \{\lambda | \lambda z_2 \geq 0\}} \lambda z_1 > 0. \tag{A.6c}$$

9 B^n is the closure of B^n_{++} and hence compact.

Proof "⇒": Obvious, if the system (A.5) is true, then there always exists a concrete λ that satisfies the consequence in (A.6) (if the antecedence is true), because it is in the feasibility set of the relevant maximization problem. Then a fortiori the maximum value has to be greater than zero. We also notice that, whenever the feasibility sets are not empty, then the maxima exist. And whenever we need them to be nonempty, they are, as can be seen in the case, when both ϕ_1 and ϕ_2 are true: (A.5b) guarantees that $B^n \cap \{\lambda|\lambda z_2 \geq 0\} \neq \emptyset$.

"⇐": To go back, we consider four cases: firstly, both ϕ_1 and ϕ_2 are false, in which case (A.5) can be trivially satisfied. The next two cases are given by either ϕ_1 or ϕ_2 is true, say ϕ_1. Then an element of the $\arg\max_{\lambda \in B^n} \lambda z_1$ will almost do it, if it were not for it possibly lying on the boundary of B^n. But given that we have a strict inequality (by A.6a) we can always perturb this argumentum maximum in such a way as to preserve the strict inequality and make it an interior element of B^n. Finally, consider the case, when both are true: in that case any strictly convex combination of $\arg\max_{\lambda \in B^n \cap \{\lambda|\lambda z_2 \geq 0\}} \lambda z_1$ with $\arg\max_{\lambda \in B^n} \lambda z_2$ and any interior element of B^n that has just an ϵ-weight on the latter two will satisfy (A.5), given (A.6). □

It is obvious that this lemma holds analogously for any finite number of conditions ϕ. We illustrate it eliminating $\tilde{\lambda}_t^1$ in (A.2) and using the definitions from the main part.

Lemma 3 *Elimination of $\tilde{\lambda}_t^1$ yields the following system (we leave out any inequality, that does not contain $\tilde{\lambda}_t^1$):*

$$\lambda_0^2 \cdot (x_t^1 - x_t^2) \leq 0 \Rightarrow \lambda_t^{1x*} \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.7a}$$

$$\tilde{\lambda}_t^2 \cdot (\omega_t^1 - \omega_t^2) \leq 0 \Rightarrow \left[(\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0) \;\;\&\;\; (\lambda_t^{1\omega*} \cdot (\omega_t^2 - \omega_t^1) > 0)\right], \tag{A.7b}$$

$$\tilde{\lambda}_t^2 \cdot (x_t^1 - \omega_t^2) \leq 0 \Rightarrow \left[(\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0) \;\;\&\;\; (\lambda_t^{1\omega*} \cdot (\omega_t^2 - \omega_t^1) > 0)\right], \tag{A.7c}$$

$$\lambda_0^2 \cdot (\omega_t^1 - x_t^2) \leq 0 \Rightarrow \lambda_t^{1x*} \cdot (x_t^2 - \omega_t^1) > 0, \tag{A.7d}$$

$$x_t^1 \not\leq \omega_t^1. \tag{A.7e}$$

Proof "⇒": $\tilde{\lambda}_t^1$ is in the set we are maximizing over and has already a strictly positive value, when the objective function is evaluated at it, hence a fortiori the maximal value must be strictly positive.

"⇐": First of all, we notice that by (A.7e) M_t^{1*} is not empty and in particular has also interior elements. The strategy of the proof is to distinguish the 16 cases, given by the sign of the antecedences, and to argue that in each case either λ_t^{1x*} or λ_t^{1**} or both or any $\lambda \in B_{++}^n$ will do.

For the first case $(\leq, \leq, \leq, \leq)$ we note that $M_t^{1**} \neq \emptyset$, because $\lambda_t^{1\omega*}$ lies (or can be made to lie) in the interior of M_t^{1**}. Furthermore, we know $\lambda_t^{1**} \cdot (x_t^2 - \omega_t^1) > 0$,

$\lambda_t^{1**} \cdot (x_t^1 - \omega_t^1) \geq 0$, $\lambda_t^{1**} \cdot (\omega_t^2 - \omega_t^1) \geq 0$, and $\lambda_t^{1**} \in S^{n-1}$. Take a convex combination of $\lambda_t^{1\omega*}$ and λ_t^{1**}, which is close enough to λ_t^{1**}. This convex combination is then our solution to go back, for it continues to lie in the interior of M_t^{1**} for any strictly positive convex coefficient.

The rest of the proof is now straightforward, because cases 2–8, 10, 13, 14[10] are no different from the first case. In cases 9, 12, 15 we take λ_t^{1x*} or rather its perturbation, so that it lies in the interior of M_t^{1*}, as our solution. Finally, in case 16, any element of the interior of M_t^{1*} will do. □

Proof of Propositions 2 and 3 Applying Lemma 2 mechanically, one can derive easily SARIR and SARIRPO. To get SARIRPO in its most concise form, one in addition has to discard some void conditions through a series of lemmas (the details are available from the author upon request). □

Finally, we summarize the entire proof strategy: this proof shows that quantifier elimination after some steps can be carried out almost mechanically by building a hierarchy of constraints that guarantee that one can always go back. And this "algorithm" would have worked also for IRPO I with more agents, because the procedure will start to rely on the assumption of two agents only in the last two steps when we eliminate λ_0^1 and λ_0^2. But then it is obvious that only the hierarchy of restrictions would have been more complicated.

Proof of Proposition 5 For expositional reasons we start with the case $h = 2$. We start with the inequalities 1a for each agent, with $\hat{\lambda}_t^r$ replaced by λ_0^r/λ_t^r. With an argument similar to Brown and Matzkin (1996), Appendix, and using the logical equivalence $[(\phi \Rightarrow \psi) \Leftrightarrow (-\phi \vee \psi)]$, we can require equivalently the existence of λ_0^1, λ_0^2 such that

$$\begin{array}{c}
\lambda_0^2 \cdot (x_a^1 - x_a^2) > 0 \vee \lambda_0^1 \cdot (x_a^2 - x_a^1) > 0, \\
\vdots \qquad\qquad\qquad \vdots \\
\lambda_0^2 \cdot (x_m^1 - x_m^2) > 0 \vee \lambda_0^1 \cdot (x_m^2 - x_m^1) > 0.
\end{array} \tag{A.8a}$$

We suggest the following solution: $\lambda_0^1 = \lambda_0^2 = \lambda_0 \gg 0$. Defining $z_t \equiv (x_t^1 - x_t^2)$, we can rewrite:

$$\begin{array}{c}
\lambda_0 z_a > 0 \vee \lambda_0 z_a < 0, \\
\vdots \qquad\qquad \vdots \\
\lambda_0 z_m > 0 \vee \lambda_0 z_m < 0,
\end{array} \tag{A.8b}$$

which for finite data sets can always be satisfied. If for some agent t, $z_t = 0$, leave the agent out. Any numbers $\bar{V}_t^1 = \bar{V}_t^2$, $\lambda_t^1 = \lambda_t^2 > 0$ will satisfy the Afriat system. Notice that the restriction from equal final allocations to collinear marginal utilities across observations is none, because we can choose the shadow price to be equal across agents and observations.

Next, we allow for arbitrary finite h (for the same reason as above with data equality for an agent t, just take all but one of the observations out): for a generic agent t, the elimination of utility levels and the welfare weights will result in

[10] We use the case numbers from above.

the formal equivalent of the strong axiom of revealed preferences (see Chiappori and Rochet 1987), with observed prices being replaced and quantified over by the unobserved shadow prices. Using the logical formula from above, we get for every $k-$sequence of observations, $(r, r+1, \ldots, r+k-1)$, $k \leq h$, the following disjunctive formula:

$$\lambda_0^{r+1} \cdot (x_t^r - x_t^{r+1}) > 0 \vee \cdots \vee \lambda_0^r \cdot (x_t^{r+k-1} - x_t^r) > 0. \qquad \text{(A.8c)}$$

Again we suggest $\lambda_0^1 = \cdots = \lambda_0^h = \lambda_0 \gg 0$, which is set in such a way that $\lambda_0 z_t^{r,s} \neq 0$, where $r < s$ are arbitrary pairs of observations. As shown, this works for any two-sequence. Now, take an arbitrary k-sequence, $(k > 2)$. We have to show that it is not the case that:

$$\lambda_0 \cdot (x_t^r - x_t^{r+1}) < 0 \text{ AND } \cdots \text{ AND } \lambda_0 \cdot (x_t^{r+k-1} - x_t^r) < 0. \qquad \text{(A.8d)}$$

Notice that we can write strict inequalities, because by construction the equality case has been excluded. Summing up leads to $0 < 0$. □

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An Afriat Theorem for the collective model of household consumption

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Abstract

We provide a nonparametric 'revealed preference' characterization of rational household behavior in terms of the collective consumption model, while accounting for general (possibly non-convex) individual preferences. We establish a *Collective Axiom of Revealed Preference* (*CARP*), which provides a necessary and sufficient condition for data consistency with collective rationality. Our main result takes the form of a 'collective' version of the *Afriat Theorem* for rational behavior in terms of the unitary model. This theorem has some interesting implications. With only a finite set of observations, the nature of consumption externalities (positive or negative) in the intra-household allocation process is non-testable. The same non-testability conclusion holds for privateness (with or without externalities) or publicness of consumption. By contrast, concavity of individual utility functions (representing convex preferences) turns out to be testable. In addition, monotonicity is testable for the model that assumes all household consumption is public.

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1 Introduction

The collective model has become increasingly popular to analyze household consumption behavior. Chiappori (1988, 1992) introduced this model as a valuable alternative for the standard unitary model, which describes household behavior as if the household were a single decision maker, who maximizes 'household' preferences subject to the household budget constraint. The collective model explicitly recognizes that the household consists of multiple decision makers (household members) with own rational preferences. It only assumes that the observed household consumption decisions are Pareto efficient outcomes of an intra-household allocation process.

Browning and Chiappori (1998) suggested a most general collective consumption model, which accounts for externalities and public consumption within the household. In addition, they make the minimalistic assumption that the empirical analyst does not observe which consumption quantities are privately consumed (possibly characterized by externalities) and which quantities are publicly consumed. Focusing on a parametric characterization of this general model, they establish that for two-person households collectively rational household behavior requires a pseudo-Slutsky matrix that can be written as the sum of a symmetric negative semi-definite matrix and a rank one matrix. Browning and Chiappori show necessity of this condition; Chiappori and Ekeland (2006) address the associated sufficiency question.

For this general collective consumption model, Cherchye, De Rock and Vermeulen (2007) recently presented a nonparametric 'revealed preference' characterization in the tradition of Afriat (1967) and Varian (1982).[1] They established a necessary nonparametric condition and a complementary sufficient nonparametric condition that allow for testing whether a finite number of observations can be rationalized in terms of the collective model. They argue that, in general, the necessary condition and the sufficient condition do not coincide. For deriving their results, these authors assume concave utility functions (representing convex preferences) of the individual household

[1] See also Samuelson (1938), Houthakker (1950) and Diewert (1973) for seminal contributions on the revealed preference approach to analyzing consumption behavior.

members; this implies a convex utility possibility set and a corresponding characterization of Pareto efficient intra-household allocations. In addition, they exclude negative externalities. Note that a similar convexity assumption is made by Browning and Chiappori (1998) and Chiappori and Ekeland (2006) in their parametric setting.

We complete the results of Cherchye, De Rock and Vermeulen (2007). More specifically, we address the same questions but we drop the prior assumptions that members' preferences are convex and that there are no negative externalities. Relaxing convexity implies that the household utility possibility set can be non-convex (even if the budget set is linear). Indeed, it has been argued that convexity assumptions are problematic in the presence of (positive or negative) externalities; see for example Starr (1969), Starret (1972) and, more recently, Mas-Colell, Whinston and Green (1995). In addition, the fact that we do not impose convexity *a priori* is consistent with the nonparametric 'revealed preference' approach for analyzing the unitary model, which equally does not assume convex (*in casu* unitary 'household') preferences *a priori* but only maintains *local non-satiation* as a minimal assumption (see, e.g., Varian, 1982). In our approach, we only maintain a *local collective non-satiation* assumption which, as we will argue, provides a natural 'collective' version of the non-satiation assumption that is used in the context of the unitary model.

Our main result is that we derive a nonparametric 'revealed preference' condition in terms of the observed aggregate quantity and price data that is both *necessary and sufficient* for household behavior to be consistent with Pareto efficiency under general preferences of the household members. Specifically, we show that the condition which Cherchye, De Rock and Vermeulen (2007) identified as necessary (but not sufficient) for a collective rationalization of the data under convex individual preferences, becomes both necessary and sufficient when dropping the convex preferences assumption. The condition has a formally similar structure as the *Generalized Axiom of Revealed Preference* (*GARP*) that provides a necessary and sufficient condition for consistency of observed household behavior with the unitary model (see Varian, 1982); and, therefore, we call it the *Collective Axiom of Revealed Preference* (*CARP*). This characterization of collectively rational behavior in terms of data consistency with *CARP* takes the form of a 'collective' version of the well-known *Afriat Theorem* for the unitary model.

Apart from the general collective consumption model, we also consider two special cases of this general model: (i) the model that excludes public consumption and externalities (also known as the 'egoistic' model); and (ii) the model that excludes private consumption (and, thus, in which all consumption is public).[2] Chiappori and Ekeland (2006) use the same distinction between a general case and special cases to structure their parametric results on empirical characterizations of collective consumption models. Moreover, the special cases are mostly considered in empirical applications of the collective model. Interestingly, we find that data consistency with *CARP* is also necessary and sufficient for a data rationalization in terms of these special cases. This implies that assumptions regarding privateness or publicness of consumption are non-testable under the abovementioned assumptions. In fact, as we will discuss, our results yield some additional testability conclusions regarding properties (concavity and monotonicity) of the data rationalizing utility functions. We will contrast these results with existing findings on the (non)parametric characterization of collective consumption models under convex preferences, and on the nonparametric characterization of the unitary model.

From a practical point of view, because *CARP* only includes observable price and quantity information, our results directly imply a necessary and sufficient 'revealed preference' condition for collective rationality that can be tested on a finite number of observations. Because of our minimal prior assumptions, this *CARP* test can be conceived as a 'pure test' of Pareto efficient collective consumption behavior. Interestingly, this test applies to a general number of observations and has direct practical applicability. It follows from Proposition 3 of Cherchye, De Rock and Vermeulen (2007) that collectively rational behavior (summarized in terms of *CARP*) can be rejected if and only if there are at least three observations and three goods. Cherchye, De Rock, Sabbe and Vermeulen (2008) present an integer programming version of the test, including a MATLAB code and an empirical application to real-life data.

At this point, it is worth indicating that our results can also be instru-

[2] The 'egoistic' model actually encompasses a wider class of member-specific utilities, which model altruism in a specific way: it also includes so-called *caring preferences* of the individual household members, which depend not only on the member's own (egoistic) utility but also on the other member's utility. Chiappori (1992) argues that, given Pareto efficiency, the empirical implications of caring preferences are indistinguishable from those of egoistic preferences. As such, while we will not indicate this explicitly in the following discussion, our conclusions for the egoistic model carry over to the (more general) caring model.

mental in alternative contexts. For example, they readily extend to the general case of multi-person group consumption. See Chiappori and Ekeland (2006, 2008) for discussion on the relevance of the collective model within the context of group consumption. To ease our exposition, the theoretical discussion in the following sections focuses on two-person households. Generalizations for M-member groups ($M \geq 2$) can be obtained along the lines of Cherchye, De Rock and Vermeulen (2007; supplemental material). Next, the nonparametric approach to analyzing collective consumption behavior is closely related to the literature on testable nonparametric restrictions of general equilibrium models. See Brown and Matzkin (1996), Brown and Shannon (2000) and, for a more recent survey, Carvajal, Ray and Snyder (2004). As such, our insights can be useful to conceive nonparametric general equilibrium restrictions in the case of non-convex preferences. Lastly, our results for the collective consumption model can also be relevant for nonparametric production analysis. See Cherchye De Rock and Vermeulen (2008), who adopt a formally similar collective model for analyzing economies of scope in the context of multi-output production (in casu under convex output possibility sets).

Before entering our analysis, two final remarks are in order. First, parametric applications of the collective model often use so-called 'assignable' quantity information, which means that the empirical analyst observes how much a group member consumes of the corresponding goods; see, for example, Browning, Bourguignon, Chiappori and Lechene (1994) and Bourguignon, Browning and Chiappori (2008). Such information is often partly, but not fully, available in practical applications (through budget surveys). To keep our discussion focused, we will abstract from such assignable quantity information in what follows. Still, it is worth emphasizing that including such information in our following results is relatively easy; it can proceed analogously as in Cherchye, De Rock and Vermeulen (2008), who focus on nonparametric testing and recovery (or 'identifiability') of collective consumption models under the maintained assumption of convex preferences, while accounting for assignable quantity information.

The second introductory remark pertains to the precise interpretation of our following testability conclusions regarding concave utility functions; see in particular our discussion in Section 6. In this respect, it is worth to briefly recall the subtle relation between 'convex preferences' and 'concave utility functions'. Concave utility functions always imply a convex utility possibility set (when the budget set is linear). In other words, if we reject collective rationality for a convex utility possibility set, then we also reject collective rationality for concave utility functions. Thus, testability of a convex utility possibility set relates unambiguously to testability of concave utility functions. But it is also well-known that, under some mild technical conditions, convex preferences can be represented by a concave utility function; see Kannai (1977) and Richter and Wong (2004) for discussion. As such, when assuming these mild conditions, our testability results regarding concave utility functions directly extend to convex preferences. Given this, we will sometimes refer to convex preferences instead of concave utility functions in the following.

The remainder of this study is structured as follows. Section 2 sets the stage by briefly recapturing the unitary *GARP* condition and the corresponding *Afriat Theorem*. Section 3 defines collective rationality. Section 4 provides the associated nonparametric *CARP* condition and the *Collective Afriat Theorem*. Section 5 focuses on two special cases of the collective model. Section 6 relates our findings to existing results in the literature. Finally, Section 7 concludes. The Appendix contains the proofs of our main results.

2 Unitary rationality

We first recapture the nonparametric conditions for unitary rationality, as they have been presented by Varian (1982). This will ease our following discussion, as it enables comparing our main results for the collective rationality model with those for the unitary rationality model.

We consider a household that purchases the (non-zero) N-vector of quantities $\mathbf{q} \in \mathbb{R}_+^N$ when confronted with the prices $\mathbf{p} \in \mathbb{R}_{++}^N$. Suppose T observations of the household consumption behavior. For each observation t, we use $\mathbf{p}_t$ and $\mathbf{q}_t$ to denote the (observed) aggregate prices and quantities, respectively; while $S = \{(\mathbf{p}_t; \mathbf{q}_t), t = 1, ..., T\}$ represents the set of observations. We can then define the condition for a unitary rationalization of a set of observations S, which -to recall- models household behavior as if the household were a single decision maker (i.e. the household maximizes a single utility function).

Definition 1 (unitary rationalization) *Let $S = \{(\mathbf{p}_t; \mathbf{q}_t); t = 1, ..., T\}$ be a set of observations. A utility function U provides a* unitary rationalization *of S if for each observation t we have $U(\mathbf{q}_t) \geq U(\mathbf{q}_r)$ for all $\mathbf{q}_r$ with $\mathbf{p}_t'\mathbf{q}_r \leq \mathbf{p}_t'\mathbf{q}_t$.*

The nonparametric condition for unitary rationality only assumes *local non-satiation* of the utility function U. As argued by Varian (1982), local non-satiation avoids trivial rationalizations of the data for the unitary model: without this additional assumption, any observed household consumption behavior can be rationalized in terms of this model.

Definition 2 (local non-satiation) *A utility function U satisfies* local non-satiation *if the following holds. Suppose quantities $\mathbf{q}_r$. Then for any $\epsilon > 0$ there exist quantities $\mathbf{q}$ with $\|\mathbf{q} - \mathbf{q}_r\| < \epsilon$ such that $U(\mathbf{q}) > U(\mathbf{q}_r)$.*

Varian (1982) established that there exists a locally non-satiated utility function that provides a unitary rationalization of the set of observations S if and only if the data satisfy the *Generalized Axiom of Revealed Preference* (*GARP*).

Definition 3 (GARP) *Let $S = \{(\mathbf{p}_t; \mathbf{q}_t); t = 1, ..., T\}$ be a set of observations. The set S satisfies the* Generalized Axiom of Revealed Preference (GARP) *if there exist relations R_0, R that meet:*

(i) if $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'\mathbf{q}_t$ then $\mathbf{q}_s \ R_0 \ \mathbf{q}_t$;

(ii) if $\mathbf{q}_s \ R_0 \ \mathbf{q}_u$, $\mathbf{q}_u \ R_0 \ \mathbf{q}_v$, ..., $\mathbf{q}_z \ R_0 \ \mathbf{q}_t$ for some (possibly empty) sequence $(u, v, ..., z)$ then $\mathbf{q}_s \ R \ \mathbf{q}_t$;

(iii) if $\mathbf{q}_s \ R \ \mathbf{q}_t$ then $\mathbf{p}_t'\mathbf{q}_t \leq \mathbf{p}_t'\mathbf{q}_s$.

In words, the quantities $\mathbf{q}_s$ are 'directly revealed preferred' over the quantities $\mathbf{q}_t$ (i.e. $\mathbf{q}_s \ R_0 \ \mathbf{q}_t$) if $\mathbf{q}_s$ were chosen when $\mathbf{q}_t$ were equally attainable (i.e. $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'\mathbf{q}_t$); see rule (i). Next, the 'revealed preference' relation R exploits transitivity of preferences; see rule (ii). Finally, rule (iii) imposes that the quantities $\mathbf{q}_t$ cannot be more expensive than revealed preferred quantities $\mathbf{q}_s$.

The following *Afriat Theorem* (Varian, 1982; based on Afriat, 1967) gives a nonparametric characterization of rational consumption behavior in terms of the unitary model.

Theorem 1 (Afriat Theorem) *Let $S = \{(\mathbf{p}_t; \mathbf{q}_t); t = 1, ..., T\}$ be a set of observations. The following statements are equivalent:*

(i) There exists a utility function U that satisfies local non-satiation *and that provides a* unitary rationalization *of S;*

(ii) The set S satisfies GARP*;*

(iii) For all $t, r \in \{1, ..., T\}$, there exist numbers U_t, $\lambda_t \in \mathbb{R}_{++}$ that meet the Afriat inequalities

$$U_r - U_t \leq \lambda_t \mathbf{p}_t' \left(\mathbf{q}_r - \mathbf{q}_t\right);$$

(iv) There exists a continuous, monotonically increasing and concave utility function U that satisfies local non-satiation *and that provides a* unitary rationalization *of S.*

In this result, condition (ii) implies that data consistency with *GARP* is necessary and sufficient for a unitary rationalization of the data. Condition (iii) provides an equivalent characterization in terms of the *Afriat inequalities*, which allow an explicit construction of the utility levels associated with each observation t (i.e. utility level U_t for observed quantities $\mathbf{q}_t$). Finally, condition (iv) states that, if there exists a utility function that provides a unitary rationalization of the set S, then there exists a continuous, monotone and concave utility function that provides such a rationalization. This also implies that continuity, monotonicity and concavity of the data rationalizing utility function is non-testable for the unitary model; i.e., violations of continuity, monotonicity or concavity cannot be detected with a finite number of observations.

3 Collective rationality

We consider a two-member (1 and 2) household. Like before, the household purchases the (non-zero) N-vector of quantities $\mathbf{q} \in \mathbb{R}_+^N$ with corresponding prices $\mathbf{p} \in \mathbb{R}_{++}^N$. All goods can be consumed privately, publicly or both. Generally, we have $\mathbf{q} = \mathbf{q}^1 + \mathbf{q}^2 + \mathbf{q}^h$ for $\mathbf{q}$ the (observed) aggregate quantities, $\mathbf{q}^1$ and $\mathbf{q}^2$ the (unobserved) private quantities of each household member, and $\mathbf{q}^h$ the (unobserved) public quantities.

Following Browning and Chiappori (1998), we consider general preferences for the household members that may depend not only on the own private and public quantities, but also on the other individual's private quantities; this allows for (positive or negative) externalities. Formally, this means that the preferences of each household member m $(m = 1, 2)$ can be represented by a utility function of the form U^m that is defined in the arguments $\mathbf{q}^1$, $\mathbf{q}^2$ and $\mathbf{q}^h$.

For aggregate quantities $\mathbf{q}$, we define *feasible personalized quantities* $\widehat{\mathbf{q}}$ as

$$\widehat{\mathbf{q}} = \left(\mathfrak{q}^1, \mathfrak{q}^2, \mathfrak{q}^h\right) \text{ with } \mathfrak{q}^1, \mathfrak{q}^2, \mathfrak{q}^h \in \mathbb{R}^n_+ \text{ and } \mathfrak{q}^1 + \mathfrak{q}^2 + \mathfrak{q}^h = \mathbf{q}.$$

Each $\widehat{\mathbf{q}}$ captures a feasible decomposition of the aggregate quantities $\mathbf{q}$ into private quantities ($\mathfrak{q}^1$ and $\mathfrak{q}^2$) and public quantities ($\mathfrak{q}^h$). This reflects that the general model allows for both private and public consumption; for the two special cases mentioned in the introduction, some of these components of $\widehat{\mathbf{q}}$ are zero by construction (see also Section 5). In the following, we consider feasible personalized quantities because we assume that the 'true' personalized quantities are not observed. Throughout, we will use that each $\widehat{\mathbf{q}}$ defines a unique $\mathbf{q}$.

Given this, a collective rationalization of S requires the existence of utility functions U^1 and U^2 such that each observed consumption bundle can be characterized as Pareto efficient.

Definition 4 (collective rationalization) *Let $S = \{(\mathbf{p}_t; \mathbf{q}_t); t = 1, ..., T\}$ be a set of observations. A pair of utility functions U^1 and U^2 provides a* collective rationalization *of S if for each observation t there exist feasible personalized quantities $\widehat{\mathbf{q}}_t$ such that $U^m(\widehat{\mathbf{q}}_r) > U^m(\widehat{\mathbf{q}}_t)$ implies $U^l(\widehat{\mathbf{q}}_r) < U^l(\widehat{\mathbf{q}}_t)$ $(m \neq l)$ for all feasible personalized quantities $\widehat{\mathbf{q}}_r$ with $\mathbf{p}_t'\mathbf{q}_r \leq \mathbf{p}_t'\mathbf{q}_t$.*

Just like the nonparametric condition for unitary rationality assumes local non-satiation, we will assume *local collective non-satiation* of the individual utility functions U^1 and U^2. Because it has a formally similar structure, this local collective non-satiation concept can be interpreted as the 'collective' analogue of the local non-satiation concept in Definition 2. In what follows, we will *only* use local collective non-satiation as a maintained assumption. As such, and in contrast with Cherchye, De Rock and Vermeulen (2007), we do *not* maintain concavity and (positive) monotonicity of the individual utility functions. *Inter alia*, this implies that we do not *a priori* exclude negative consumption externalities.

Definition 5 (local collective non-satiation) *A pair of utility functions U^1 and U^2 satisfies* local collective non-satiation *if the following holds. Suppose feasible personalized quantities $\widehat{\mathbf{q}}_{r_1}$ and $\widehat{\mathbf{q}}_{r_2}$, and let $\mathbf{r} = \{r_1\} \cup \{r_2\}$.*

Then for any $\epsilon \geq 0$ there exist quantities $\mathbf{q}$ with $\left\|\sum_{r\in\mathbf{r}}\mathbf{q}_r - \mathbf{q}\right\| \leq \epsilon$ such that $U^1(\widehat{\mathbf{q}}) \geq U^1(\widehat{\mathbf{q}}_{r_1})$ and $U^2(\widehat{\mathbf{q}}) \geq U^2(\widehat{\mathbf{q}}_{r_2})$, with at least one strict inequality if $\epsilon > 0$, for feasible personalized quantities $\widehat{\mathbf{q}}$.

In words, suppose an initial situation in which member 1 evaluates feasible personalized quantities $\widehat{\mathbf{q}}_{r_1}$ (corresponding to $\mathbf{q}_{r_1}$) and member 2 evaluates feasible personalized quantities $\widehat{\mathbf{q}}_{r_2}$ (corresponding to $\mathbf{q}_{r_2}$). Then for $\epsilon > 0$ and $\mathbf{q}$ close to the 'summed quantities' $\sum_{r\in\mathbf{r}}\mathbf{q}_r$ there exist personalized quantities $\widehat{\mathbf{q}}$ that entail a Pareto improvement as compared to this initial situation.[3] As for the specification of $\sum_{r\in\mathbf{r}}\mathbf{q}_r$, we need to consider two cases. In the first case, $r_1 = r_2$ and thus $\sum_{r\in\mathbf{r}}\mathbf{q}_r = \mathbf{q}_{r_1} = \mathbf{q}_{r_2}$. This pertains to an initial situation in which both household members evaluate the same feasible personalized quantities $\widehat{\mathbf{q}}_{r_1} = \widehat{\mathbf{q}}_{r_2}$; and local collective non-satiation implies that a Pareto improvement is possible for (aggregate) quantities $\mathbf{q}$ close to $\mathbf{q}_{r_1} = \mathbf{q}_{r_2}$. In the second case, $r_2 \neq r_1$ and thus $\sum_{r\in\mathbf{r}}\mathbf{q}_r = \mathbf{q}_{r_1} + \mathbf{q}_{r_2}$. This pertains to an initial situation in which both household members evaluate different feasible personalized quantities $\widehat{\mathbf{q}}_{r_1}$ and $\widehat{\mathbf{q}}_{r_2}$; and local collective non-satiation implies that a Pareto improvement is possible for (aggregate) quantities $\mathbf{q}$ close to the sum $\mathbf{q}_{r_1} + \mathbf{q}_{r_2}$.

This assumption of local collective non-satiation avoids trivial collective rationalizations of a set of observations S: without this assumption, any observed household consumption behavior can be rationalized in terms of the collective consumption model (i.e. any set S of quantity choices $\mathbf{q}$ under alternative prices $\mathbf{p}$ can be characterized as Pareto efficient intra-household allocations).[4] As such, this maintained assumption plays exactly the same role for the collective model as the standard local non-satiation assumption for the unitary model.

[3] As compared to Definition 2, we also consider $\epsilon = 0$ in Definition 5. The condition for $\epsilon = 0$ is a technical one, which is only required in a limiting case, i.e. $\mathbf{r} = \{t_1, t_2\}$ and $\mathbf{p}'_s\mathbf{q}_s = \mathbf{p}'_s(\mathbf{q}_{t_1} + \mathbf{q}_{t_2})$, to obtain rule (iii) in Definition 6 (*Collective Axiom of Revealed Preference; CARP*) as a necessary condition for a collective rationalization of a set of observations S. See Step 1 in the proof of Theorem 2.

[4] More specifically, local collective non-satiation in Definition 5 is crucial to obtain rules (iii) (for $\mathbf{r} = \{t_1, t_2\}$ with $t_1 \neq t_2$) and (iv) in Definition 6 (*CARP*) as necessary conditions for a collective rationalization of a set of observations S; see Step 1 in the proof of Theorem 2. Without these rules, any set S trivially satisfies the remaining rules in the definition of *CARP*. For example, it can be verified that, in this case, consistency of any set S with *CARP* is achieved for a transitive specification of the relations H^1, H^2 which satisfies, for all $\mathbf{q}_s$ and $\mathbf{q}_t$, that $\mathbf{q}_s H^1 \mathbf{q}_t$ implies $\mathbf{q}_t H^2 \mathbf{q}_s$.

To conclude this section, we introduce an equivalent characterization of Pareto efficiency in Definition 4, which will be useful for our following discussion:

Lemma 1 *Let $S = \{(\mathbf{p}_t; \mathbf{q}_t); t = 1, ..., T\}$ be a set of observations. A pair of utility functions U^1 and U^2 provides a* collective rationalization *of S if and only if for each observation t there exist feasible personalized quantities $\widehat{\mathbf{q}}_t$ such that, for all $\widehat{\mathbf{q}}_r$ with $\mathbf{p}_t'\mathbf{q}_r \leq \mathbf{p}_t'\mathbf{q}_t$, there exist μ_{tr}^1 and $\mu_{tr}^2 \in \mathbb{R}_{++}$ that imply*

$$\mu_{tr}^1 U^1(\widehat{\mathbf{q}}_t) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_t) \geq \mu_{tr}^1 U^1(\widehat{\mathbf{q}}_r) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_r).$$

This alternative characterization of collective rationality (or Pareto efficiency) will return in the *collective Afriat inequalities* that we introduce below (Theorem 2). Essentially, it requires that for observation t to correspond to a Pareto efficient intrahousehold allocation there must exist $\widehat{\mathbf{q}}_t$ such that, for any $\widehat{\mathbf{q}}_r$ that was equally attainable under the given prices (i.e. with $\mathbf{p}_t'\mathbf{q}_r \leq \mathbf{p}_t'\mathbf{q}_t$), we can define strictly positive weights μ_{tr}^1 and μ_{tr}^2 for which the corresponding weighted sum of utilities for $\widehat{\mathbf{q}}_t$ exceeds the one for $\widehat{\mathbf{q}}_r$ (i.ė. $\mu_{tr}^1 U^1(\widehat{\mathbf{q}}_t) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_t) \geq \mu_{tr}^1 U^1(\widehat{\mathbf{q}}_r) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_r)$). We note that, for each t, the weights μ_{tr}^1 and μ_{tr}^2 depend on the identity of $\widehat{\mathbf{q}}_r$. This follows from the fact we do not assume concave utility functions for the individual members, which -to recall- implies that we may have a non-convex utility possibility set (even if the budget set is linear). If we would have assumed a convex utility possibility set (e.g. because of concave individual utility functions representing convex preferences), then we could specify $\mu_{tr}^m = \mu_t^m$ $(m = 1, 2)$ for all $\widehat{\mathbf{q}}_r$; this case is usually considered in the literature on collective consumption models.

We illustrate by Figure 1, which presents a non-convex utility possibility set corresponding to some given budget. In that figure, $\widehat{\mathbf{q}}_s$ (corresponding to point 1 in Figure 1) is Pareto inefficient: the intersection of the utility possibility set with the light shaded area is non-empty. For example, $\widehat{\mathbf{q}}_t$ (point 2) implies a Pareto improvement over $\widehat{\mathbf{q}}_s$, because $U^1(\widehat{\mathbf{q}}_t) > U^1(\widehat{\mathbf{q}}_s)$ and $U^2(\widehat{\mathbf{q}}_t) > U^2(\widehat{\mathbf{q}}_s)$. In terms of the characterization in Lemma 1, we have $\mu_{st}^1 U^1(\widehat{\mathbf{q}}_s) + \mu_{st}^2 U^2(\widehat{\mathbf{q}}_s) < \mu_{st}^1 U^1(\widehat{\mathbf{q}}_t) + \mu_{st}^2 U^2(\widehat{\mathbf{q}}_t)$ for each possible specification of μ_{st}^1 and μ_{st}^2. By contrast, $\widehat{\mathbf{q}}_t$ is Pareto efficient: the intersection of the utility possibility set with the dark shaded area is empty. Correspondingly, $\widehat{\mathbf{q}}_t$ is consistent with the Pareto efficiency characterization in Lemma 1.

The figure also allows us to illustrate the difference between the characterization of Pareto efficiency for non-convex utility possibility sets with

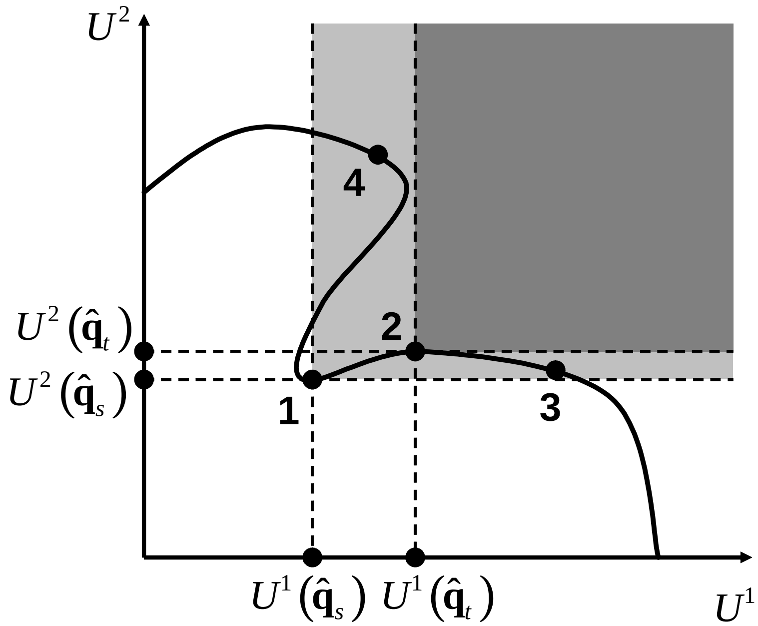

Figure 1: Pareto efficiency

the one for convex utility possibility sets: for the given (non-convex) utility possibility set, $\widehat{\mathbf{q}}_t$ is Pareto efficient when compared to the member-specific utilities that correspond to points 3 and 4, while there exist convex combinations of these member-specific utilities that imply a Pareto improvement over $\widehat{\mathbf{q}}_t$. This difference pertains to the fact that in Lemma 1, for each t, the Pareto weights μ_{tr}^1 and μ_{tr}^2 can vary with the identity of r.

4 Nonparametric 'revealed preference' characterization

We first present the *Collective Axiom of Revealed Preference* ($CARP$), which we can interpret as a natural 'collective' extension of the unitary $GARP$ condition. $CARP$ imposes empirical restrictions on *hypothetical* preference relations H_0^m and H^m, which capture 'feasible' specifications of the individual preference relations given the information that is revealed by the set of observations S: $\mathbf{q}_s\ H^m\ \mathbf{q}_t$ ($\mathbf{q}_s\ H_0^m\ \mathbf{q}_t$) means that we 'hypothesize' that member m (directly) prefers the quantities $\mathbf{q}_s$ over the quantities $\mathbf{q}_t$. Note that, while the 'true' preferences are expressed in terms of the feasible personalized quantities $\widehat{\mathbf{q}}$ (i.e. member m prefers $\mathbf{q}_s$ over $\mathbf{q}_t$ only if $U^m(\widehat{\mathbf{q}}_s) \geq U^m(\widehat{\mathbf{q}}_t)$), the hypothetical preferences only use observable information (captured by the observed aggregate prices $\mathbf{p}$ and quantities $\mathbf{q}$ in the set S). This naturally complies with the assumption that in the general model we have no information concerning the feasible personalized quantities.

Definition 6 (CARP) *Let* $S = \{(\mathbf{p}_t; \mathbf{q}_t)\,; t = 1, ..., T\}$ *be a set of observations. The set* S *satisfies the* Collective Axiom of Revealed Preference (CARP) *if there exist* hypothetical *relations* H_0^m, H^m *for each member* $m \in \{1, 2\}$ *that meet:*

(i) if $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'\mathbf{q}_t$ *then* $\mathbf{q}_s\ H_0^1\ \mathbf{q}_t$ *or* $\mathbf{q}_s\ H_0^2\ \mathbf{q}_t$;

(ii) if $\mathbf{q}_s\ H_0^m\ \mathbf{q}_u$, $\mathbf{q}_u\ H_0^m\ \mathbf{q}_v$, ..., $\mathbf{q}_z\ H_0^m\ \mathbf{q}_t$ *for some (possibly empty) sequence* $(u, v, ..., z)$ *then* $\mathbf{q}_s\ H^m\ \mathbf{q}_t$;

(iii) for $\mathbf{r} = \{t_1\} \cup \{t_2\}$: *if* $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s' \sum_{r \in \mathbf{r}} \mathbf{q}_r$ *and* $\mathbf{q}_{t_1}\ H^m\ \mathbf{q}_s$ *then* $\mathbf{q}_s\ H_0^l\ \mathbf{q}_{t_2}$ *(with* $m \neq l$*)*;

(iv) if $\mathbf{q}_{s_1}\ H^1\ \mathbf{q}_t$ *and* $\mathbf{q}_{s_2}\ H^2\ \mathbf{q}_t$ *then* $\mathbf{p}_t'\mathbf{q}_t \leq \mathbf{p}_t' \sum_{r \in \mathbf{r}} \mathbf{q}_r$ *for* $\mathbf{r} = \{s_1\} \cup \{s_2\}$.

This *CARP* axiom has a direct interpretation in terms of the Pareto efficiency requirement that underlies collective rationality. Rule (i) states that, if the quantities $\mathbf{q}_s$ were chosen while the quantities $\mathbf{q}_t$ were equally attainable (under the prices $\mathbf{p}_s$), then it must be that at least one member prefers the quantities $\mathbf{q}_s$ over the quantities $\mathbf{q}_t$ (i.e. $\mathbf{q}_s\ H_0^1\ \mathbf{q}_t$ or $\mathbf{q}_s\ H_0^2\ \mathbf{q}_t$). Rule (ii) captures transitivity. As for rule (iii), we note that the 'summed quantities' $\sum_{r \in \mathbf{r}} \mathbf{q}_r = \mathbf{q}_{t_1} + \mathbf{q}_{t_2}$ if $t_1 \neq t_2$ and $\sum_{r \in \mathbf{r}} \mathbf{q}_r = \mathbf{q}_t$ if $t_1 = t_2 = t$. Given this, rule (iii) can again be interpreted in terms of Pareto efficiency. Specifically, it states that, if member m prefers $\mathbf{q}_{t_1}$ over $\mathbf{q}_s$ for the bundle $\sum_{r \in \mathbf{r}} \mathbf{q}_r$ not more expensive than $\mathbf{q}_s$, then the choice of $\mathbf{q}_s$ can be rationalized only if the other member l prefers $\mathbf{q}_s$ over $\mathbf{q}_{t_2}$. Indeed, if this last condition were not satisfied, then the bundle $\sum_{r \in \mathbf{r}} \mathbf{q}_r$ (under the given prices $\mathbf{p}_s$ and outlay $\mathbf{p}_s'\mathbf{q}_s$) would imply a Pareto improvement over the chosen bundle $\mathbf{q}_s$. Rule (iv), finally, complements rule (iii). It states that, if member m prefers $\mathbf{q}_{s_m}$ over $\mathbf{q}_t$, then the choice of $\mathbf{q}_t$ can be rationalized only if it is not more expensive than the (newly defined) 'summed quantities' $\sum_{r \in \mathbf{r}} \mathbf{q}_r$. Indeed, if this last condition were not met, then for the given prices $\mathbf{p}_t$ and outlay $\mathbf{p}_t'\mathbf{q}_t$ both members would be better off by buying the quantities $\sum_{r \in \mathbf{r}} \mathbf{q}_r$

rather than the chosen quantities $\mathbf{q}_t$, which of course conflicts with collective rationality.

At this point, it is interesting to note that the *CARP* axiom has an analogous structure as the *GARP* axiom that applies to the unitary model. Specifically, *GARP* states (*in casu* unitary) rationality conditions in terms of the preference information that is revealed by the observed price and quantity data. Essentially, *CARP* does the same, but now the revealed preference information is understood in terms of the collective model of household consumption and, thus, pertains to the individual household members.[5]

We also recall that *GARP* provides a necessary and sufficient condition for rational consumption behavior in the context of the unitary model. The next theorem, which contains our core result, shows that *CARP* equally provides a necessary and sufficient condition for collectively rational consumption behavior. It provides a 'collective' version of the *Afriat Theorem* for unitary rationality.

Theorem 2 (Collective Afriat Theorem) *Let $S = \{(\mathbf{p}_t; \mathbf{q}_t)\,; t = 1, ..., T\}$ be a set of observations. The following statements are equivalent:*

(i) There exists a pair of utility functions U^1 and U^2 that satisfy local collective non-satiation *and that provide a* collective rationalization *of S;*

(ii) The set S satisfies CARP*;*

(iii) For all $t, r_1, r_2 \in \{1, ..., T\}$, with $\mathbf{r} = \{r_1\} \cup \{r_2\}$, there exist numbers U_t^1, U_t^2, $\mu_{t\mathbf{r}}^1$, $\mu_{t\mathbf{r}}^2 \in \mathbb{R}_{++}$ that meet the collective Afriat inequalities

$$\left[\mu_{t\mathbf{r}}^1 U_{r_1}^1 + \mu_{t\mathbf{r}}^2 U_{r_2}^2\right] - \left[\mu_{t\mathbf{r}}^1 U_t^1 + \mu_{t\mathbf{r}}^2 U_t^2\right] \leq \mathbf{p}_t' \left(\sum_{r\in\mathbf{r}} \mathbf{q}_r - \mathbf{q}_t\right);$$

(iv) There exists a pair of continuous and monotonically increasing utility functions U^1 and U^2 that satisfy local collective non-satiation *and that provide a* collective rationalization *of S.*

This result implies that data consistency with *CARP* is *necessary and sufficient* for a collective rationalization of the data; see condition (ii). In

[5]In this respect, we can also refer to the discussion in Cherchye, De Rock, Vermeulen (2007, supplemental material). When presenting the generalized version of *CARP* for M-member households, the authors argue that *CARP* concides with *GARP* when $M = 1$ (i.e. the household consists of a single decision maker/household member).

turn, this institutes a test for collective rationality that can be tested on the basis of observable price and quantity information. Next, condition (iii) provides an equivalent characterization in terms of *collective Afriat inequalities*; this complies with the Pareto efficiency characterization in Lemma 1 (using 'summed quantities' $\sum_{r \in \mathbf{r}} \mathbf{q}_r$). Finally, condition (iv) implies that, as soon as there exists a pair of utility functions that provide a rationalization, there exist continuous and monotone utility functions that provide a rationalization; in the current context, monotonicity means that $\widehat{\mathbf{q}}_{r_1} \geq \widehat{\mathbf{q}}_{r_2}$ and $\widehat{\mathbf{q}}_{r_1} \neq \widehat{\mathbf{q}}_{r_2}$ implies $U^1(\widehat{\mathbf{q}}_{r_1}) \geq U^1(\widehat{\mathbf{q}}_{r_2})$ and $U^2(\widehat{\mathbf{q}}_{r_1}) \geq U^2(\widehat{\mathbf{q}}_{r_2})$ with at least one strict inequality. *Inter alia*, this implies that for the general collective consumption model continuity and monotonicity are non-testable for a finite number of observations. Because a violation of monotonicity cannot be detected with a finite number of observations, we also conclude that the nature (positive or negative) of the consumption externalities is non-testable for the general consumption model. Section 6, which relates our findings to existing results in the literature, will discuss additional interesting implications of the *Collective Afriat Theorem*. For example, it will turn out that concavity of the individual utility functions (representing convex preferences) is testable for the collective consumption model.

5 Special cases

In this section we discuss two special cases of the collective model: (i) the model with all consumption private and no externalities, which is also known as the 'egoistic' model, and (ii) the model in which all consumption is public. The next result states that, as soon as there exists a rationalization of the data in terms of the general collective consumption model, we can provide a rationalization of the same data in terms of the egoistic model.

Proposition 1 *Suppose there exists a pair of utility functions U^1 and U^2 that satisfy* local collective non-satiation *and that provide a* collective rationalization *of S. Then, for $\mathbf{q} = \mathfrak{q}^1 + \mathfrak{q}^2$, there exists a pair of continuous and monotonically increasing utility functions $U^1(\widehat{\mathbf{q}}) = V^1(\mathfrak{q}^1)$ and $U^2(\widehat{\mathbf{q}}) = V^2(\mathfrak{q}^2)$, which exclude consumption externalities and public consumption, that satisfy* local collective non-satiation *and that provide a* collective rationalization *of S.*

We conclude that *CARP* consistency is also necessary and sufficient for a data rationalization in terms of the egoistic model, and that monotonicity is non-testable for this model. In addition, just like for the general model (see also our discussion in the next section), we also obtain for this special model that concavity of the individual utility functions is testable.

It is interesting to compare the result in Proposition 1 with the following result, which pertains to the model with all consumption public.

Proposition 2 *Suppose there exists a pair of utility functions U^1 and U^2 that satisfy* local collective non-satiation *and that provide a* collective rationalization *of S. Then, for $\mathbf{q} = \mathfrak{q}^h$, there exists a pair of continuous utility functions $U^1(\widehat{\mathbf{q}}) = W^1(\mathfrak{q}^h)$ and $U^2(\widehat{\mathbf{q}}) = W^2(\mathfrak{q}^h)$, which exclude private consumption, that satisfy* local collective non-satiation *and that provide a* collective rationalization *of S.*

Thus, we again obtain that *CARP* provides a necessary and sufficient condition for data consistency with this special collective consumption model. The fact that *CARP* characterizes the general model as well as the two special models provides the general conclusion that, for the given setting that allows for possibly non-convex preferences, publicness or privateness of consumption does not yield testable implications; we will discuss this more elaborately in the next section. In addition, and analogous to before, we can conclude that concavity is testable for this special case with all consumption public. Still, contrary to before, it turns out that also monotonicity is testable for this special model. In other words, we may need utility functions with negative marginal utilities of the publicly consumed quantities to obtain a collective rationalization.

Example 1 in the Appendix illustrates the result on monotonicity: it presents data that require non-monotone individual utility functions for a rationalization in terms of the special model with all consumption public. In fact, the construction of the example also suggests an empirical test for the monotonicity property in this special case of the collective model. More specifically, there exist a pair of monotone utility functions W^1 and W^2 that provide a collective rationalization of the data (with only public consumption) if and only if the data satisfy *CARP* and, in addition to rules (i) to (iv) in Definition 6, the rule

$$\text{(v) if } \mathbf{q}_s \geq \mathbf{q}_t \text{ and } \mathbf{q}_s \neq \mathbf{q}_t \text{ then not } \mathbf{q}_t H^1 \mathbf{q}_s \text{ and not } \mathbf{q}_t H^2 \mathbf{q}_s.$$

Intuitively, we have $\mathbf{q}_s = \mathfrak{q}_s^h$ and $\mathbf{q}_t = \mathfrak{q}_t^h$ when all consumption is public, which means that all personalized quantity information is observed. And, thus, monotonicity excludes that any of the members prefers $\mathbf{q}_t$ over $\mathbf{q}_s$ when $\mathbf{q}_s \geq \mathbf{q}_t$ and $\mathbf{q}_s \neq \mathbf{q}_t$; this is captured by rule (v). We conclude that, if there do not exist relations H_0^m and H^m $(m = 1, 2)$ that satisfy rules (i) to (iv) in Definition 6 and the above rule (v), then no monotonically increasing utility functions exist with only publicly consumed quantities as arguments.

6 Relation to existing results

Our results developed in the previous sections bear interesting relations with existing results. We can relate our nonparametric findings for the collective model to those on the (non)parametric characterization of collective consumption models under convex preferences. In addition, we can contrast our findings with those on the nonparametric characterization of the unitary model.

First, we can establish a relationship between our results and Chiappori and Ekeland's (2006) findings on the parametric characterization of collective consumption models while maintaining the assumption of convex preferences. As already indicated, we find that the *same CARP* condition is (necessary and sufficient) for data consistency with the general collective model as well as with the two special cases defined above. This parallels the conclusion of Chiappori and Ekeland that 'locally' (i.e. in a sufficiently small neighborhood of a given point) "an assumption like privateness (or publicness) of individual consumptions is not testable from data on group behavior" (Chiappori and Ekeland, 2006, p.4). In other words, Chiappori and Ekeland's 'local' parametric result for convex preferences complies with our, by construction 'global', nonparametric result when dropping the assumption of convex preferences. For completeness, we must add that the conclusion changes when requiring monotonicity of the data rationalizing utility functions: it follows from our results in the previous section that monotonicity is testable for the model with all consumption public, while it is non-testable for the other models under consideration (i.e. the general model and the model with all consumption private and no externalities).

It is also interesting to relate these conclusions to those of Cherchye, De Rock and Vermeulen (2008), who maintain the assumption of convex preferences, and who do obtain *different* nonparametric data consistency conditions for the different (i.e. general and special) collective consumption models. Notably, this finding of different 'global' (nonparametric) characterizations for different models contrasts with Chiappori and Ekeland's 'local' (parametric) conclusion cited above (-to recall- under the same convexity assumption). Cherchye, De Rock and Vermeulen interpret that these diverging findings confirm Chiappori and Ekeland's 'strong suspicion' that their local results must not hold globally. Following this interpretation, the results in the current paper add that Chiappori and Ekeland's conclusion does hold globally if convexity of the individual preferences is no longer maintained as an assumption.

Next, we can compare our results with those on the nonparametric characterization of the unitary consumption model (summarized by Varian, 1982 and 2006). In this respect, we first recall that the nonparametric characterization for the unitary model implies that monotonicity and concavity of a data rationalizing household utility function is not testable in the standard case with a linear budget set and positive prices.[6] More specifically, it is well-known that, if there exists a household utility function that rationalizes the data in terms of the unitary model (because the data are consistent with $GARP$), then there always exists a monotone and concave household utility function that provides such a rationalization. See also Theorem 1 above.

In contrast to this conclusion for the unitary model, we find for the collective model that concavity of the individual household members' utility functions (or, convexity of the individual preferences) is testable. In particular, there may exist utility functions for the individual household members that rationalize the data in terms of the collective model (because the data are consistent with $CARP$), while there do not exist concave individual utility functions that provide such a rationalization. This conclusion follows from contrasting the $CARP$ condition, which characterizes the collective model when convex preferences are not imposed, with the condition in Proposition 1 of Cherchye, De Rock and Vermeulen (2007), which characterizes the collective model when convex preferences (implying a convex utility possibility set) are imposed. In general, the two conditions do not coincide. For instance, Example 2 of Cherchye, De Rock and Vermeulen (2007) provides

[6] In the case of non-linear budget sets, Forges and Minelli (2006) argue that concavity is testable for the unitary model. Given our following argument, this *a fortiori* also holds for the collective model. In fact, our insights on testing data consistency with the collective model for linear budget sets could serve as a useful basis when considering non-linear budget sets in a collective setting.

data that pass *CARP* (which -to recall- these authors present as a necessary condition) but reject collective rationality under concave utility functions; these data can be rationalized in the collective consumption model, but not with concave individual utility functions.

Finally, as for monotonicity of the data rationalizing individual utility functions, we conclude that testability of monotonicity depends on the specific collective model under consideration. For the general model and the special model with all consumption private and no externalities (i.e. the egoistic model), we have demonstrated that monotonicity is non-testable when the data satisfy *CARP*: if there exist individual utility functions that rationalize the data in terms of these collective models, then there always exist monotone utility functions that provide such a rationalization. This implies that negative marginal utilities (including negative externalities) cannot be detected for these models. By contrast, we find that monotonicity of the individual utility functions can be rejected for the special model with no private consumption (i.e. all consumption is public). We conclude that, in this case, it is possible to detect negative marginal utilities of the publicly consumed goods. The intuition of the diverging results for the two special cases is the following. In the egoistic model, negative marginal utilities are pretty useless in the absence of externalities: individual rationality implies that goods with a negative impact on own utility are simply not chosen by the individual household member. This is not the case when all consumption is public. In that case, it cannot be ruled out that a certain public good has a positive impact on the utility of only one of the members, while it affects the other member negatively.

7 Summary and conclusions

We have presented a nonparametric 'revealed preference' characterization of the general model of collectively rational (i.e. Pareto efficient) household consumption behavior, which accounts for (positive or negative) externalities and public consumption in the household. Our distinguishing feature is that we allow for non-convex individual preferences. Given this, we have derived a necessary and sufficient condition for collective rationality. Because this condition provides a natural extension of Varian's *Generalized Axiom of Revealed Preference* (*GARP*) for individually rational consumption behavior, we have called it the *Collective Axiom of Revealed Preference* (*CARP*).

Our main result provides a collective version of the *Afriat Theorem* for unitary rational behavior. We also obtain that, when accounting for possibly non-convex individual preferences, *CARP* characterizes not only the general model but also special cases of this general model (i.e. the case with all consumption private and no externalities, and the case with all consumption public).

In turn, our results provide some interesting testability conclusions, which bear relation to existing results on the (non)parametric characterization of collective consumption models under convex preferences, and on the nonparametric characterization of the unitary model. We find that, with only a finite set of observations, privateness (with or without externalities) or publicness of consumption is non-testable. For the general model and the special (egoistic) model with all consumption private and no externalities, the same non-testability conclusion holds for monotonicity of the individual utility functions; this also implies that, in the general model, the nature (positive or negative) of the consumption externalities cannot be tested. Still, monotonicity turns out to be testable for the special model with all consumption public. In addition, concavity of the individual utility functions (representing convex preferences) is testable for all collective consumption models under consideration.

Finally, our results suggest operational tests of collective rationality, and of concavity and monotonicity of the individual utility functions. More specifically, such tests can use the integer programming (IP) formulations in Cherchye, De Rock and Vermeulen (2008): these authors provide an IP test of the *CARP* condition, which -to recall- applies to collective rationality under possibly non-concave utility functions, as well as IP tests for collective rationality under concave utility functions. Comparison of the different test results allows us to conclude whether or not concave utility functions (representing convex preferences) rationalize the observed collective consumption behavior. An analogous IP formulation applies to the monotonicity test for the special model with all consumption public (suggested in Section 5). See Cherchye, De Rock, Sabbe and Vermeulen (2008) on the practical implementation of the IP tests.

Appendix: proofs

Proof of Lemma 1

Necessity. Suppose the utility functions U^1 and U^2 provide a collective rationalization of S. Because the result follows trivially if $U^1(\widehat{\mathbf{q}}_t) \geq U^1(\widehat{\mathbf{q}}_r)$ and $U^2(\widehat{\mathbf{q}}_t) \geq U^2(\widehat{\mathbf{q}}_r)$, we focus on $U^m(\widehat{\mathbf{q}}_t) > U^m(\widehat{\mathbf{q}}_r)$ and $U^l(\widehat{\mathbf{q}}_t) < U^l(\widehat{\mathbf{q}}_r)$ $(m \neq l)$. Then there exists μ_{tr}^1, $\mu_{tr}^2 \in \mathbb{R}_{++}$ with $\frac{\mu_{tr}^m}{\mu_{tr}^l} \geq \frac{U^l(\widehat{\mathbf{q}}_r) - U^l(\widehat{\mathbf{q}}_t)}{U^m(\widehat{\mathbf{q}}_t) - U^m(\widehat{\mathbf{q}}_r)} > 0$, which implies $\mu_{tr}^1 U^1(\widehat{\mathbf{q}}_t) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_t) \geq \mu_{tr}^1 U^1(\widehat{\mathbf{q}}_r) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_r)$.

Sufficiency. Suppose there exist utility functions U^1 and U^2 and μ_{tr}^1, $\mu_{tr}^2 \in \mathbb{R}_{++}$ that imply, for each t, $\mu_{tr}^1 U^1(\widehat{\mathbf{q}}_t) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_t) \geq \mu_{tr}^1 U^1(\widehat{\mathbf{q}}_r) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_r)$ for some feasible personalized quantities $\widehat{\mathbf{q}}_t$ and all $\widehat{\mathbf{q}}_r$ with $\mathbf{p}_t'\mathbf{q}_r \leq \mathbf{p}_t'\mathbf{q}_t$. We prove *ad absurdum*. Suppose the functions U^1 and U^2 do not provide a collective rationalization of S. That is, for some t we have for all $\widehat{\mathbf{q}}_t$ that there exists $\widehat{\mathbf{q}}_r$ such that $U^m(\widehat{\mathbf{q}}_t) < U^m(\widehat{\mathbf{q}}_r)$ and $U^l(\widehat{\mathbf{q}}_t) \leq U^l(\widehat{\mathbf{q}}_r)$. But then, for the given t, we have for all $\widehat{\mathbf{q}}_t$ that there exists $\widehat{\mathbf{q}}_r$ such that $\mu_{tr}^1 U^1(\widehat{\mathbf{q}}_t) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_t) < \mu_{tr}^1 U^1(\widehat{\mathbf{q}}_r) + \mu_{tr}^2 U^2(\widehat{\mathbf{q}}_r)$ for all μ_{tr}^1, $\mu_{tr}^2 \in \mathbb{R}_{++}$; and this implies a contradiction. ■

Proof of Theorem 2

Step 1: (i) implies (ii). Suppose there exists a pair of collectively non-satiated utility functions U^1 and U^2 that provide a *collective rationalization* of S. This implies that for all $t \in \{1, ..., T\}$ there exists a specification of the feasible personalized quantities $\widehat{\mathbf{q}}_t$ such that $U^m(\widehat{\mathbf{q}}_r) > U^m(\widehat{\mathbf{q}}_t)$ implies $U^l(\widehat{\mathbf{q}}_r) < U^l(\widehat{\mathbf{q}}_t)$ $(m \neq l)$ for all $\widehat{\mathbf{q}}_r$ with $\mathbf{p}_t'\mathbf{q}_r \leq \mathbf{p}_t'\mathbf{q}_t$.

For this specification of the feasible personalized quantities, we can specify hypothetical relations H^m for all $s \in \{1, ..., T\}$ and $m \in \{1, 2\}$ as follows:

$$\mathbf{q}_s H^m \mathbf{q}_t \ \Leftrightarrow U^m(\widehat{\mathbf{q}}_s) \geq U^m(\widehat{\mathbf{q}}_t). \tag{1}$$

We next have to verify whether this specification of the hypothetical relations satisfies the rules (i)-(iv) in the *CARP* Definition 6. Note that we do not distinguish between the relations H^m and H_0^m because the specification of the relation H^m is obtained by using real numbers. This also makes that rule (ii) is automatically satisfied.

As for rule (i), a collective rationalization of the data requires that $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'\mathbf{q}_t$ implies that for any $\widehat{\mathbf{q}}_s$ and $\widehat{\mathbf{q}}_t$ we have $U^1(\widehat{\mathbf{q}}_s) \geq U^1(\widehat{\mathbf{q}}_t)$ or $U^2(\widehat{\mathbf{q}}_s) \geq U^2(\widehat{\mathbf{q}}_t)$, which necessarily obtains $\mathbf{q}_s H^1 \mathbf{q}_t$ or $\mathbf{q}_s H^2 \mathbf{q}_t$.

As for rule (iii), we make the distinction between $t_1 = t_2 = t$, which implies $\mathbf{r} = \{t\}$, and $t_1 \neq t_2$, which implies $\mathbf{r} = \{t_1, t_2\}$. We first consider $\mathbf{r} = \{t\}$, and we prove rule (iii) *ad absurdum*. Suppose $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'\mathbf{q}_t$ in combination with $\mathbf{q}_t H^m \mathbf{q}_s$ and not $\mathbf{q}_s H^l \mathbf{q}_t$ (with $l \neq m$). Given our specification of the hypothetical relations, this implies that $U^m(\widehat{\mathbf{q}}_t) \geq U^m(\widehat{\mathbf{q}}_s)$ and $U^l(\widehat{\mathbf{q}}_s) < U^l(\widehat{\mathbf{q}}_t)$. But then $\widehat{\mathbf{q}}_s$ is not Pareto efficient, which conflicts with a collective rationalization of the data.

We use local collective non-satiation to obtain that rule (iii) is satisfied for $\mathbf{r} = \{t_1, t_2\}$. To see this, suppose rule (iii) does not hold, i.e. $\mathbf{q}_{t_1} H^m \mathbf{q}_s$ and not $\mathbf{q}_s H^l \mathbf{q}_{t_2}$ (with $l \neq m$) for $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'(\mathbf{q}_{t_1} + \mathbf{q}_{t_2})$. Note that $\mathbf{q}_{t_1} H^m \mathbf{q}_s$ and not $\mathbf{q}_s H^l \mathbf{q}_{t_2}$ implies that $U^m(\widehat{\mathbf{q}}_{t_1}) \geq U^m(\widehat{\mathbf{q}}_s)$ and $U^l(\widehat{\mathbf{q}}_s) < U^l(\widehat{\mathbf{q}}_{t_2})$. Local collective non-satiation implies that for any $\epsilon \geq 0$ there exist quantities $\mathbf{q}$ with $\|\mathbf{q}_{t_1} + \mathbf{q}_{t_2} - \mathbf{q}\| \leq \epsilon$ for which $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'\mathbf{q}$ and such that $U^m(\widehat{\mathbf{q}}) \geq U^m(\widehat{\mathbf{q}}_{t_1})$ and $U^l(\widehat{\mathbf{q}}) \geq U^l(\widehat{\mathbf{q}}_{t_2})$. But then, given the above, we also have that $U^m(\widehat{\mathbf{q}}) \geq U^m(\widehat{\mathbf{q}}_s)$ and $U^l(\widehat{\mathbf{q}}) > U^l(\widehat{\mathbf{q}}_s)$, and $\mathbf{p}_s'\mathbf{q}_s \geq \mathbf{p}_s'\mathbf{q}$; this means that $\widehat{\mathbf{q}}_s$ is not Pareto efficient, which conflicts with a collective rationalization of the set S.

As for rule (iv), we again make the distinction between $s_1 = s_2 = s$, which implies $\mathbf{r} = \{s\}$, and $s_1 \neq s_2$, which implies $\mathbf{r} = \{s_1, s_2\}$. We first consider $\mathbf{r} = \{s\}$, and we prove rule (iv) *ad absurdum*. Suppose $\mathbf{q}_s\ H^1\ \mathbf{q}_t$ and $\mathbf{q}_s\ H^2\ \mathbf{q}_t$ in combination with $\mathbf{p}_t'\mathbf{q}_t > \mathbf{p}_t'\mathbf{q}_s$. On the one hand, $\mathbf{q}_s\ H^1\ \mathbf{q}_t$ and $\mathbf{q}_s\ H^2\ \mathbf{q}_t$ implies $U^1(\widehat{\mathbf{q}}_s) \geq U^1(\widehat{\mathbf{q}}_t)$ and $U^2(\widehat{\mathbf{q}}_s) \geq U^2(\widehat{\mathbf{q}}_t)$. On the other hand, because $\mathbf{p}_t'\mathbf{q}_t > \mathbf{p}_t'\mathbf{q}_s$, local collective non-satiation implies that there exists $\mathbf{q}$ close enough to the quantities $\mathbf{q}_s$ with $\mathbf{p}_t'\mathbf{q}_t > \mathbf{p}_t'\mathbf{q}$ so that $U^m(\widehat{\mathbf{q}}) \geq U^m(\widehat{\mathbf{q}}_s)$ and $U^l(\widehat{\mathbf{q}}) \geq U^l(\widehat{\mathbf{q}}_s)$ with at least one strict inequality. But then $U^m(\widehat{\mathbf{q}}) \geq U^m(\widehat{\mathbf{q}}_t)$ and $U^l(\widehat{\mathbf{q}}) \geq U^l(\widehat{\mathbf{q}}_t)$, with at least one strict inequality, and $\mathbf{p}_t'\mathbf{q}_t > \mathbf{p}_t'\mathbf{q}$; this means that $\widehat{\mathbf{q}}_t$ is not Pareto efficient, which conflicts with a collective rationalization of the set S.

We analogously prove that rule (iv) is satisfied for $\mathbf{r} = \{s_1, s_2\}$. For $s_1 \neq s_2$, let $\mathbf{q}_{s_1}\ H^1\ \mathbf{q}_t$ and $\mathbf{q}_{s_2}\ H^2\ \mathbf{q}_t$ in combination with $\mathbf{p}_t'\mathbf{q}_t > \mathbf{p}_t'(\mathbf{q}_{s_1} + \mathbf{q}_{s_2})$. On the one hand, $\mathbf{q}_{s_1}\ H^1\ \mathbf{q}_t$ and $\mathbf{q}_{s_2}\ H^2\ \mathbf{q}_t$ implies $U^1(\widehat{\mathbf{q}}_{s_1}) \geq U^1(\widehat{\mathbf{q}}_t)$ and $U^2(\widehat{\mathbf{q}}_{s_2}) \geq U^2(\widehat{\mathbf{q}}_t)$. On the other hand, because $\mathbf{p}_t'\mathbf{q}_t > \mathbf{p}_t'(\mathbf{q}_{s_1} + \mathbf{q}_{s_2})$, one can analogously as in rule (iii) show that local collective non-satiation implies that $\widehat{\mathbf{q}}_t$ is not Pareto efficient, which conflicts with a collective rationalization of the set S.

Step 2: (ii) implies (iii). We must show that, if the set S satisfies *CARP*, then for all $t, r_1, r_2 \in \{1, ..., T\}$, with $\mathbf{r} = \{r_1\} \cup \{r_2\}$, there exist numbers U_t^1, U_t^2, $\mu_{t\mathbf{r}}^1$, $\mu_{t\mathbf{r}}^2 \in \mathbb{R}_{++}$ that meet the *collective Afriat inequalities.*

First, for the given specification of the relations H^m consistent with *CARP*, we specify the numbers $U_t^m \in \mathbb{R}_{++}$ that satisfy for each $\mathbf{q}_{t_1}$, $\mathbf{q}_{t_2}$:

$$\begin{aligned} \text{if } \mathbf{q}_{t_1} H^m \mathbf{q}_{t_2} \text{ and not } \mathbf{q}_{t_2} H^m \mathbf{q}_{t_1} \text{ then } U_{t_1}^m &> U_{t_2}^m; \quad (2) \\ \text{and if } \mathbf{q}_{t_1} H^m \mathbf{q}_{t_2} \text{ and } \mathbf{q}_{t_2} H^m \mathbf{q}_{t_1} \text{ then } U_{t_1}^m &= U_{t_2}^m. \quad (3) \end{aligned}$$

Note that, if not $\mathbf{q}_{t_1} H^m \mathbf{q}_{t_2}$ and not $\mathbf{q}_{t_2} H^m \mathbf{q}_{t_1}$, then there is no restriction on the corresponding values of $U_{t_1}^m$ and $U_{t_2}^m$. Generally, the specific values of $U_{t_1}^m$ and $U_{t_2}^m$ are irrelevant for our following argument.

Given this, for each $t, r_1, r_2 \in \{1, ..., T\}$, with $\mathbf{r} = \{r_1\} \cup \{r_2\}$, we must specify numbers $\mu_{t\mathbf{r}}^1$, $\mu_{t\mathbf{r}}^2 \in \mathbb{R}_{++}$ such that the corresponding *collective Afriat inequality* $\left[\mu_{t\mathbf{r}}^1 U_{r_1}^1 + \mu_{t\mathbf{r}}^2 U_{r_2}^2\right] - \left[\mu_{t\mathbf{r}}^1 U_t^1 + \mu_{t\mathbf{r}}^2 U_t^2\right] \leq \mathbf{p}_t' \left(\sum_{r \in \mathbf{r}} \mathbf{q}_r - \mathbf{q}_t\right)$ is met. We distinguish two possible cases:

CASE 1: $\mathbf{q}_{r_1} H^1 \mathbf{q}_t$ and $\mathbf{q}_{r_2} H^2 \mathbf{q}_t$. Given the specification of the numbers U_t^m in (2) and (3), we then have that $U_{r_1}^1 \geq U_t^1$ and $U_{r_2}^2 \geq U_t^2$. Since $\mathbf{q}_{r_1} H^1 \mathbf{q}_t$ and $\mathbf{q}_{r_2} H^2 \mathbf{q}_t$, rule (iv) of Definition 6 implies that we must have $\mathbf{p}_t'(\sum_{r \in \mathbf{r}} \mathbf{q}_r - \mathbf{q}_t) \geq 0$. For $\mathbf{p}_t'(\sum_{r \in \mathbf{r}} \mathbf{q}_r - \mathbf{q}_t) > 0$ the corresponding *collective Afriat inequality* is satisfied by setting $\mu_{t\mathbf{r}}^1$ and $\mu_{t\mathbf{r}}^2$ sufficiently small. Next, for $\mathbf{p}_t' \left(\sum_{r \in \mathbf{r}} \mathbf{q}_r - \mathbf{q}_t\right) = 0$ we note that $\mathbf{q}_{r_1} H^1 \mathbf{q}_t$ and $\mathbf{q}_{r_2} H^2 \mathbf{q}_t$ imply, respectively, $\mathbf{q}_t H^2 \mathbf{q}_{r_2}$ and $\mathbf{q}_t H^1 \mathbf{q}_{r_1}$ (see rule (iii) in Definition 6). As a result, we obtain $U_{r_1}^1 = U_t^1$ and $U_{r_2}^2 = U_t^2$ because of (3), and the corresponding *collective Afriat inequality* is satisfied for any $\mu_{t\mathbf{r}}^1$ and $\mu_{t\mathbf{r}}^2$.

CASE 2: Not $\mathbf{q}_{r_1} H^1 \mathbf{q}_t$ or not $\mathbf{q}_{r_2} H^2 \mathbf{q}_t$. Without losing generality, let us assume that we have not $\mathbf{q}_{r_1} H^1 \mathbf{q}_t$. Our specification in (2) and (3) then implies that we exclude $U_{r_1}^1 \geq U_t^1$. So we can specify that $U_{r_1}^1 < U_t^1$. For the given value of $\mathbf{p}_t'(\sum_{r \in \mathbf{r}} \mathbf{q}_r - \mathbf{q}_t)$, we then set $\mu_{t\mathbf{r}}^1$ sufficiently large and $\mu_{t\mathbf{r}}^2$ sufficiently small such that $\mu_{t\mathbf{r}}^1 \left[U_{r_1}^1 - U_t^1\right] \leq \mathbf{p}_t' \left(\sum_{r \in \mathbf{r}} \mathbf{q}_r - \mathbf{q}_t\right) - \mu_{t\mathbf{r}}^2 \left[U_{r_2}^2 - U_t^2\right]$, i.e. the corresponding *collective Afriat inequality* is met.

Step 3: (iii) implies (iv). Our proof contains two steps. We address data rationalization in Step 3a, and continuity and monotonicity in Step 3b.

Step 3a: data rationalization. Suppose for all $t, r_1, r_2 \in \{1, ..., T\}$, with $\mathbf{r} = \{r_1\} \cup \{r_2\}$, there exist numbers U_t^1, U_t^2, $\mu_{t\mathbf{r}}^1$, $\mu_{t\mathbf{r}}^2 \in \mathbb{R}_{++}$ that meet the *collective Afriat inequalities.* We have to prove that we can then specify utility functions U^1 and U^2 such that for each observation t there exist feasible personalized quantities $\widehat{\mathbf{q}}_t$ and, for all $\widehat{\mathbf{q}}_r$ with $\mathbf{p}_t'\mathbf{q}_r \leq \mathbf{p}_t'\mathbf{q}_t$, μ^1, $\mu^2 \in \mathbb{R}_{++}$ that imply

$$\mu^1 U^1(\widehat{\mathbf{q}}_t) + \mu^2 U^2(\widehat{\mathbf{q}}_t) \geq \mu^1 U^1(\widehat{\mathbf{q}}_r) + \mu^2 U^2(\widehat{\mathbf{q}}_r).$$

To obtain the result, we define $U^1(\widehat{\mathbf{q}}_r)$ and $U^2(\widehat{\mathbf{q}}_r)$ such that

$$U^1(\widehat{\mathbf{q}}_r) = \max_t \min_s \left[U_t^1 + \frac{\mathbf{p}_s'(\mathbf{q}_r - \mathbf{q}_t)}{\mu^1_{s\{t\}}} \text{ subject to } U_t^2 \geq \overline{U}_r^2 \right], \text{ and} \quad (4)$$

$$U^2(\widehat{\mathbf{q}}_r) = U_{t^*}^2, \quad (5)$$

with t^* the observation that solves the max problem in (4). As for the specification of $\overline{U}_r^2$, we use:

$$\text{if } r \text{ is observed then } \overline{U}_r^2 = U_r^2,$$

$$\text{if } r \text{ is unobserved then } \overline{U}_r^2 = \begin{cases} \max_{t \in \{1,\dots,T\}: \mathbf{q}_t \leq \mathbf{q}_r} U_t^2 \text{ if } \mathbf{q}_t \leq \mathbf{q}_r \text{ for some } t, \\ 0 \text{ otherwise.} \end{cases}$$

As a first step, we show that we have $U^1(\widehat{\mathbf{q}}_t) = U_t^1$ and $U^2(\widehat{\mathbf{q}}_t) = U_t^2$ for each observed t. We prove *ad absurdum*. If the result does not hold, then there exists $t^* \neq t$ such that

$$U_t^1 < U_{t^*}^1 + \min_s \frac{\mathbf{p}_s'(\mathbf{q}_t - \mathbf{q}_{t^*})}{\mu^1_{s\{t^*\}}} \text{ and } U_t^2 \leq U_{t^*}^2$$

In that case, we have for all $\mu^1 \in \mathbb{R}_{++}$ and $\mu^2 \in \mathbb{R}_{++}$:

$$\mu^1 U_t^1 + \mu^2 U_t^2 < \mu^1 \left[U_{t^*}^1 + \min_s \frac{\mathbf{p}_s'(\mathbf{q}_t - \mathbf{q}_{t^*})}{\mu^1_{s\{t^*\}}} \right] + \mu^2 U_{t^*}^2,$$

and thus

$$\mu^1 U_t^1 + \mu^2 U_t^2 < \mu^1 U_{t^*}^1 + \mu^2 U_{t^*}^2 + \mu^1 \left[\frac{\mathbf{p}_t'(\mathbf{q}_t - \mathbf{q}_{t^*})}{\mu^1_{t\{t^*\}}} \right].$$

Now consider $\mu^1 = \mu^1_{t\{t^*\}}$, $\mu^2 = \mu^2_{t\{t^*\}}$. We obtain

$$\mu^1_{t\{t^*\}} U_t^1 + \mu^2_{t\{t^*\}} U_t^2 < \mu^1_{t\{t^*\}} U_{t^*}^1 + \mu^2_{t\{t^*\}} U_{t^*}^2 + \mathbf{p}_t'(\mathbf{q}_t - \mathbf{q}_{t^*}),$$

or

$$\mathbf{p}_t'(\mathbf{q}_{t^*}-\mathbf{q}_t) < \left(\mu^1_{t\{t^*\}}U^1_{t^*}+\mu^2_{t\{t^*\}}U^2_{t^*}\right) - \left(\mu^1_{t\{t^*\}}U^1_{t}+\mu^2_{t\{t^*\}}U^2_{t}\right),$$

which violates the collective Afriat inequalities.

Next, we show that our specification of U^1 and U^2 provides a collective rationalization of the set S. Consider $\mathbf{q}_r$ and $\mathbf{p}_t'\mathbf{q}_t \geq \mathbf{p}_t'\mathbf{q}_r$ for some observed t. Then we must show that there exist μ^1, $\mu^2 \in \mathbb{R}_{++}$ such that $\mu^1U^1(\widehat{\mathbf{q}}_t)+\mu^2U^2(\widehat{\mathbf{q}}_t) \geq \mu^1U^1(\widehat{\mathbf{q}}_r)+\mu^2U^2(\widehat{\mathbf{q}}_r)$, with $U^1(\widehat{\mathbf{q}}_r)$ and $U^2(\widehat{\mathbf{q}}_r)$ defined in (4) and (5). We prove the result for $\mu^1=\mu^1_{t\{t^*\}}$ and $\mu^2=\mu^2_{t\{t^*\}}$. First, because of the *collective Afriat inequalities* we have

$$\left[\mu^1_{t\{t^*\}}U^1_{t^*}+\mu^2_{t\{t^*\}}U^2_{t^*}\right] - \left[\mu^1_{t\{t^*\}}U^1_{t}+\mu^2_{t\{t^*\}}U^2_{t}\right] \leq \mathbf{p}_t'(\mathbf{q}_{t^*}-\mathbf{q}_t),$$

which implies

$$\begin{aligned}
\mu^1_{t\{t^*\}}U^1_{t^*}+\mu^2_{t\{t^*\}}U^2_{t^*}+\mathbf{p}_t'(\mathbf{q}_r-\mathbf{q}_{t^*}) &\leq \mu^1_{t\{t^*\}}U^1_{t}+\mu^2_{t\{t^*\}}U^2_{t}+\mathbf{p}_t'(\mathbf{q}_r-\mathbf{q}_t)\\
&\leq \mu^1_{t\{t^*\}}U^1_{t}+\mu^2_{t\{t^*\}}U^2_{t}\\
&= \mu^1_{t\{t^*\}}U^1(\widehat{\mathbf{q}}_t)+\mu^2_{t\{t^*\}}U^2(\widehat{\mathbf{q}}_t).
\end{aligned}$$

It suffices then to show that

$$\mu^1_{t\{t^*\}}U^1(\widehat{\mathbf{q}}_r)+\mu^2_{t\{t^*\}}U^2(\widehat{\mathbf{q}}_r) \leq \mu^1_{t\{t^*\}}U^1_{t^*}+\mu^2_{t\{t^*\}}U^2_{t^*}+\mathbf{p}_t'(\mathbf{q}_r-\mathbf{q}_{t^*}),$$

or, using (4) and (5),

$$\mu^1_{t\{t^*\}}U^1_{t^*}+\mu^2_{t\{t^*\}}U^2_{t^*}+\mu^1_{t\{t^*\}}\left[\frac{\mathbf{p}_{s^*}'(\mathbf{q}_r-\mathbf{q}_{t^*})}{\mu^1_{s^*\{t^*\}}}\right] \leq \mu^1_{t\{t^*\}}U^1_{t^*}+\mu^2_{t\{t^*\}}U^2_{t^*}+\mathbf{p}_t'(\mathbf{q}_r-\mathbf{q}_{t^*}),$$

with s^* the observation that solves the min problem in (4). The resulting condition

$$\frac{\mathbf{p}_{s^*}'(\mathbf{q}_r-\mathbf{q}_{t^*})}{\mu^1_{s^*\{t^*\}}} \leq \frac{\mathbf{p}_t'(\mathbf{q}_r-\mathbf{q}_{t^*})}{\mu^1_{t\{t^*\}}}$$

is satisfied because s^* solves the min problem in (4). We obtain that there exists a pair of utility functions U^1 and U^2 that provide a collective rationalization of S.

Step 3b: monotonicity and continuity. Given the construction of U^1 and U^2 in (4) and (5), we can always construct $\widehat{\mathbf{q}}$ that obtains monotonicity and continuity. Let us first consider monotonicity. For observed t_1 and

t_2, monotonicity means that $\widehat{\mathbf{q}}_{t_1} \geq \widehat{\mathbf{q}}_{t_2}$ and $\widehat{\mathbf{q}}_{t_1} \neq \widehat{\mathbf{q}}_{t_2}$ implies $U^1(\widehat{\mathbf{q}}_{t_1}) \geq U^1(\widehat{\mathbf{q}}_{t_2})$ and $U^2(\widehat{\mathbf{q}}_{t_1}) \geq U^2(\widehat{\mathbf{q}}_{t_2})$, with at least one strict inequality. This condition is easily met for a given set S. For example, suppose that there are 2 goods e_1, $e_2 \in \{1, ..., N\}$ with strictly positive quantities for all observed t, i.e. $(\mathbf{q}_t)_{e_1} > 0$ and $(\mathbf{q}_t)_{e_2} > 0$ for all t (with $(\mathbf{x})_e$ the e-th entry of the vector $\mathbf{x}$). In that case, it suffices to specify $\left(\mathfrak{q}^m_{t1}\right)_{e_m} > \left(\mathfrak{q}^m_{t2}\right)_{e_m}$ if $U^m(\widehat{\mathbf{q}}_{t_1}) > U^m(\widehat{\mathbf{q}}_{t_2})$ $(m = 1, 2)$. (Empirically, the existence of e_1 and e_2 with $(\mathbf{q}_t)_{e_1} > 0$ and $(\mathbf{q}_t)_{e_2} > 0$ for all observed t is a very mild assumption. Formally similar constructions are possible if this assumption is not met, but they are mathematically less elegant.) Using (4) and (5), monotonicity in terms of $\widehat{\mathbf{q}}_t$ and $U^m(\widehat{\mathbf{q}}_t)$ for observed t implies monotonicity in terms of $\widehat{\mathbf{q}}_r$ with $\mathbf{q}_r$ observed or unobserved.

Let us then consider continuity. If the functions U^1 and U^2 in (4) and (5) have any discontinuities, their number will be finite by construction, because the number of observations T is finite. As such, the discontinuities can be 'fixed' by linear interpolation without interfering with the rationalization argument in Step 3a.

Step 4: (iv) implies (i). This is trivial. ■

Proof of Propositions 1 and 2

The results follow from the construction of the proof of Theorem 2. Specifically, the data rationalization argument (Step 3a) and the continuity argument (using linear interpolation; Step 3b) apply for general utility functions U^1 and U^2 defined in general $\widehat{\mathbf{q}}$. These arguments directly extend to the special cases with utility functions V^m defined in $\mathfrak{q}^m$ $(m = 1, 2)$, for Proposition 1, and utility functions W^m defined in $\mathfrak{q}^h$ $(m = 1, 2)$, for Proposition 2. Finally, the argument for monotonicity (Step 3b of the proof of Theorem 2) directly extends to the case with utility functions V^m defined in $\mathfrak{q}^m$ $(m = 1, 2)$, which completes the proof of Proposition 1. ■

Example 1

This example illustrates that monotonicity of the individual utility functions is testable when private consumption is excluded (i.e. all consumption is public). As a preliminary remark, we recall that $W^m(\mathbf{q}_t) = U^m_t$ for observed t. We thus need a set S that satisfies *CARP* only if $\mathbf{q}_{t_2} H^m \mathbf{q}_{t_1}$

(and not $\mathbf{q}_{t_1} H^m \mathbf{q}_{t_2}$) for $\mathbf{q}_{t_1} \geq \mathbf{q}_{t_2}$ and $\mathbf{q}_{t_1} \neq \mathbf{q}_{t_2}$: using (1), this corresponds to $W^m(\mathbf{q}_{t_2}) > W^m(\mathbf{q}_{t_1})$ for $\mathbf{q}_{t_1} \geq \mathbf{q}_{t_2}$ and $\mathbf{q}_{t_1} \neq \mathbf{q}_{t_2}$, which violates monotonicity. We complete the argument by considering the set $S = \{(\mathbf{p}_t; \mathbf{q}_t), t = 1, 2, 3, 4\}$ with

$$\begin{aligned}
\mathbf{q}_1 &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{q}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{q}_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \\
\mathbf{p}_1 &= \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 2 \\ 2 \\ 9 \end{pmatrix}, \mathbf{p}_3 = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}, \mathbf{p}_4 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.
\end{aligned}$$

It can be verified that this set satisfies *CARP*. However, for $t = 1$ *or* 3 we necessarily obtain $\mathbf{q}_4 H^1 \mathbf{q}_t$ or $\mathbf{q}_4 H^2 \mathbf{q}_t$, while $\mathbf{q}_t \geq \mathbf{q}_4$ and $\mathbf{q}_t \neq \mathbf{q}_4$. We obtain this last result in two steps:

Step 1: CARP consistency requires (for $m \neq l$ and $s, t \in \{1, 2, 3\}$)

$$\mathbf{q}_1 \ H_0^m \ \mathbf{q}_2, \ \mathbf{q}_2 \ H_0^m \ \mathbf{q}_3 \text{ and } \mathbf{q}_3 \ H_0^l \ \mathbf{q}_2, \ \mathbf{q}_2 \ H_0^l \ \mathbf{q}_1, \tag{6}$$

$$\text{while not } \mathbf{q}_s H_0^1 \mathbf{q}_t \text{ and } \mathbf{q}_s H_0^2 \mathbf{q}_t. \tag{7}$$

The reasoning goes as follows. First, because for all $s, t \in \{1, 2, 3\}$ we have $\mathbf{p}_s' \mathbf{q}_s > \mathbf{p}_s' \mathbf{q}_t$, rule (i) of Definition 6 implies $\mathbf{q}_s \ H_0^1 \ \mathbf{q}_t$ or $\mathbf{q}_s \ H_0^2 \ \mathbf{q}_t$ and rule (iv) excludes $\mathbf{q}_s \ H_0^1 \ \mathbf{q}_t$ and $\mathbf{q}_s \ H_0^2 \ \mathbf{q}_t$. Next, because $\mathbf{p}_1' \mathbf{q}_1 > \mathbf{p}_1' (\mathbf{q}_2 + \mathbf{q}_3)$, rule (iv) of Definition 6 excludes $\mathbf{q}_2 \ H_0^1 \ \mathbf{q}_1$ and $\mathbf{q}_3 \ H_0^2 \ \mathbf{q}_1$ and, conversely, $\mathbf{q}_2 \ H_0^2 \ \mathbf{q}_1$ and $\mathbf{q}_3 \ H_0^1 \ \mathbf{q}_1$. Similarly, $\mathbf{p}_3' \mathbf{q}_3 > \mathbf{p}_3' (\mathbf{q}_1 + \mathbf{q}_2)$ excludes $\mathbf{q}_2 \ H_0^1 \ \mathbf{q}_3$ and $\mathbf{q}_1 \ H_0^2 \ \mathbf{q}_3$ and, conversely, $\mathbf{q}_2 \ H_0^2 \ \mathbf{q}_3$ and $\mathbf{q}_1 \ H_0^1 \ \mathbf{q}_3$. We conclude that *CARP* consistency requires (6) and (7).

Step 2: because $\mathbf{p}_4' \mathbf{q}_4 > \mathbf{p}_4' \mathbf{q}_2$, rule (i) of Definition 6 implies $\mathbf{q}_4 \ H_0^1 \ \mathbf{q}_2$ or $\mathbf{q}_4 \ H_0^2 \ \mathbf{q}_2$. Together with (6) and (7), this implies $\mathbf{q}_4 H^m \mathbf{q}_3$ (and not $\mathbf{q}_3 H^m \mathbf{q}_4$) or $\mathbf{q}_4 H^l \mathbf{q}_1$ (and not $\mathbf{q}_1 H^l \mathbf{q}_4$), which gives the result.

We conclude that a data rationalization of the given set requires non-monotone individual utility functions: for $t = 1$ or 3, we have $W^1(\mathbf{q}_4) > W^1(\mathbf{q}_t)$ or $W^2(\mathbf{q}_4) > W^2(\mathbf{q}_t)$, while $\mathbf{q}_t \geq \mathbf{q}_4$ and $\mathbf{q}_t \neq \mathbf{q}_4$.

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Implications of Pareto efficiency for two-agent (household) choice

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ABSTRACT

We study when two-member household choice behavior is compatible with Pareto optimality. We ask when an external observer of household choices, who does not know the individuals' preferences, can rationalize the choices as being Pareto-optimal. Our main contribution is to reduce the problem of rationalization to a graph-coloring problem. As a result, we obtain simple tests for Pareto optimal choice behavior. In addition to the tests, and using our graph-theoretic representation, we show that Pareto rationalization is equivalent to a system of quadratic equations being solvable.

1. Introduction

We study the implications of *efficiency* for *choice behavior*. Concretely, for a given collective of agents, we want to describe the choices that are consistent with Pareto optimality, when we are ignorant about the individual agents' preferences. We develop a series of simple necessary conditions (tests) for choice to be consistent with efficiency; the tests lead up to a necessary and sufficient condition.

Consider a household (a two-person collective) that has to select an alternative from a finite set. For instance, a couple may have to decide how to spend their income, or two researchers who work at the same laboratory may have to choose which projects to fund with their common grants. We focus on the two agent case here because it is the simplest and most primitive environment in which to study Pareto efficient choice. The two-person problem entails a significant simplification because if, say, alternatives a and b are Pareto efficient, and we determine that one agent ranks a over b, then we pin down the other agent's preferences to ranking b over a. This is not the case in the general n person problem. In a sense, there is a "dimension reduction" that is very helpful in the two-agent case. We imagine that the main applications of our results are to household choice and, as we explain below, to the study of partial orders of dimension 2.

Suppose we are given the households' choice behavior. That is, we know what the household would choose from each possible subset of alternatives. We ask when the observed choice can be *Pareto rationalized*: we want to know when we can find two preference relations, one for each household member, such that for any given subset of alternatives, the choices are exactly the Pareto optimal alternatives within the set. We are interested in what the choice behavior of such a household looks like, and how we can test if a given household's choices can be rationalized.

Pareto rationalization depends on certain conditions that the collective choices must satisfy. Some conditions are classical: For example, if an alternative x is selected from a set A, and we shrink A to $B \subset A$ while maintaining x as still available in B, then x must be chosen from B; this condition is usually called Chernoff's Axiom. The classical conditions allow us to work with two binary relations: revealed Pareto domination R and Pareto indifference O. Pareto rationalization requires that there be two individual preferences $>_1$ and $>_2$ such that xRy if and only if x Pareto dominates y, and xOy if and only if x and y are Pareto incomparable. We present new necessary and sufficient conditions for Pareto rationalization.

The problem is motivated by revealed preference theory. In revealed preference theory, one wants to know when the choice behavior of an individual agent is rational. By observing his choice from pairs of alternatives, we can infer the agent's preferences and then judge whether his behavior is rational or not according to some external criterion (e.g. whether his preferences are acyclic, transitive, etc.)

We consider instead the collective behavior of two agents. Their choices reveal their collective preferences, but we cannot immediately reconstruct their individual preferences, and therefore cannot judge whether they are rational or not. To make it harder, the observed group preferences, R and O, need not satisfy some classical rationality conditions.

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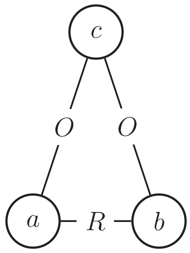

Fig. 1. (ac, bc) aRb or bRa, aOc and cOb.

Our main results are for the case where choices from all possible budgets are observable. One can use our results in the case when that is not possible; in Section 6 we comment on how they can be used, and present a simple example with partial observability.

We present the paper in terms of rationalizing choice, but one can also phrase the results as a contribution to dimension theory: a partial order $\succ$ on a set X has dimension N if there are n linear orders $>_i$, $i = 1, \ldots n$ such that

$$x \succ y \Leftrightarrow \forall i (x >_i y).$$

We focus on the case $n = 2$. The problem was first studied by Dushnik and Miller (1941), who provide a characterization of the two-agent rationalizable orders. The problems of dimensions $n > 2$ are open.[1] The Dushnik–Miller result is a deep result about partial orders, but it does not constitute a useful test for rationalizability because it is non-constructive. The characterization is in terms of a property of the partial order that cannot be verified constructively.

In the economic literature, Sprumont (2001) has studied two-agent Pareto rationalizability when one is interested in preferences with a particular structure.[2] Sprumont works with a continuum of alternatives, and studies Pareto two-agent rationalizations with "regular" preferences. We focus instead on the discrete case, where we put no structure on agents' preferences (other than rationality). We shall work with strict preferences in the main discussion of our results, but in the final section of the paper we show that household choices are rationalizable in strict preferences if and only if they are rationalizable in nonstrict preferences.

The following observation is key in our analysis. Consider a pair of alternatives, (x, y). One of two things can happen: either one alternative is revealed Pareto preferred to the other, or the two alternatives are not Pareto ranked. In the first case, if x is chosen out of the set $\{x, y\}$, then we write xRy, and in the second case we write xOy. The first case is simple: there is no ambiguity as any rationalizing preferences must coincide with the Pareto order; if xRy and $>_1, >_2$ are rationalizing preferences, then $x >_1 y$ and $x >_2 y$. The second case presents us with a choice. If xOy then either $x >_1 y$ and $y >_2 x$ or $y >_1 x$ and $x >_2 y$. In the first case, the direction of individual preference is determinate. In the second case we have a degree of freedom and we cannot infer individual preferences from the Pareto ranking. This degree of freedom makes the problem of Pareto rationalization substantially different from classical individual revealed preference theory.

We represent the problem in graph-theoretic terms. We think of each alternative as a vertex in a complete graph. For any two alternatives x and y, the edge xy is labeled with R or O. The key building block of our analysis is in Fig. 1, where aRb, aOc and cOb.

[1] Dimension theory was introduced by Dushnik and Miller in their 1941 article. A large literature on dimension theory has been developed: the book by Trotter (2001) is a recent exposition.

[2] See also Sprumont (2000), who works out the relationship between Pareto and Nash rationalizability.

It is easy to see that if $(>_1, >_2)$ is a rationalization then either c is the *best* or the *worst* alternative for $>_1$ out of the set $\{a, b, c\}$. Furthermore, if c is the best for $>_1$, then it is the worst for $>_2$, and vice versa. So one agent's preference points towards c and the other agent's away from c.

A graph contains (in principle) many configurations like the one in Fig. 1. If we are trying to build a rationalization $(>_1, >_2)$ then the decision we make on one such configuration, in terms of whether agent 1 or agent 2 prefers c, affects the decision we make on others. We reduce this problem to one of *graph bi-coloring*: one color for the triplet (ac, bc) represents agent 1 preferring c out of $\{a, b, c\}$; the other color represents agent 2 preferring c out of $\{a, b, c\}$.

This graph coloring approach allows us to formulate a simple test (a necessary condition) for Pareto rationalizability. The test is based on the two-coloring rationality condition being equivalent to the absence of certain odd cycles; this test is easy to implement. Passing the test is, unfortunately, not sufficient to guarantee Pareto rationalizability. We present another necessary condition for choice behavior to be Pareto rationalizable. The condition is based on solving a particular quadratic system of equations where each variable can only take the values 1 and -1. This condition, together with the test and two classical revealed preference axioms, is sufficient for choice behavior to be Pareto rationalizable.

The rest of the paper is organized as follows. Section 2 presents some preliminary results. In Section 3 we present the graph-theoretic notions we employ. Section 4 contains our main results. We remark on the extension of our results to non-strict preferences in Section 5. We present some conclusions in Section 7.

2. Preliminaries

2.1. Definitions

Let X be a nonempty, finite set of alternatives. We call a nonempty subset $B \subset X$ a *budget*. Denote by $\mathcal{P}(X)$ the set of all budgets $B \subset X$. A *choice function* is a function

$$g : \mathcal{P}(X) \to \mathcal{P}(X)$$

such that $g(B) \subset B$ for all $B \in \mathcal{P}(X)$.

A *strict preference relation* on X is a total, antisymmetric and transitive binary relation on X (a linear order).

Given two preference relations, $>_1$ and $>_2$, and a budget $B \in \mathcal{P}(X)$, say that $a \in B$ is *Pareto dominated in B with respect to* $>_1$ *and* $>_2$ if there is some $b \in B$ such that $b >_1 a$ and $b >_2 a$; in this case we say that b *Pareto dominates* a (with respect to the two preferences $>_1$ and $>_2$). We call $a \in B$ *Pareto efficient inB* if it is not Pareto dominated in B. Observe that if a and b are Pareto efficient in some set $B \supseteq \{a, b\}$, then either $a >_1 b$ and $b >_2 a$, or $b >_1 a$ and $a >_2 b$

A choice function g is *two-agent Pareto rationalizable* if there are two strict preference relations $>_1$ and $>_2$, such that for all $B \in \mathcal{P}(X)$, $f(B)$ is the set of all Pareto efficient alternatives in B with respect to $>_1$ and $>_2$. In this case, we say that $(>_1, >_2)$ is a *Pareto rationalization* of f, or a *rationalizing pair*.

A pair of sets (V, E) is a (directed) *graph* whenever $E \subset V \times V$. We say that the elements of V are the vertexes of the graph, and that $(v, v') \in E$ means that there is an edge pointing from v to v'.

2.2. Preliminary results

We present two axioms that are necessary for Pareto rationalization. They are standard axioms in the literature, and they are known to be necessary for single-agent rationalization as well (see, e.g. Moulin (1991)). The results in Lemmas (1) and (2) are also standard.

Axiom 1. For all $B_1, B_2 \in \mathcal{P}(X)$, if $B_1 \subset B_2$ and $a \in B_1 \cap f(B_2)$, then $a \in f(B_1)$.

The interpretation of Axiom 1 is that if f is Pareto rationalizable, and a is a Pareto efficient choice among the alternatives in a given budget, it must remain Pareto efficient among the alternatives of any smaller budget that contains a. Axiom 1 is a standard axiom in choice theory, usually called Chernoff's axiom (Moulin, 1991). We obtain the following version of this standard result

Lemma 1. Axiom 1 *is necessary for Pareto rationalizability*.

Proof. Suppose that f is Pareto rationalizable and that $(>_1, >_2)$ is a rationalization. By definition, we have that if $a \in B_1 \cap f(B_2)$ for some $B_1 \subset B_2$, then a is not dominated by any $b \in B_2$ (with respect to $(>_1, >_2)$). Since $B_1 \subset B_2$ it follows that a is not dominated by any $b \in B_1$. Thus, a is Pareto efficient in B_1 and $a \in f(B_1)$, as $f(B_1)$ is the set of all Pareto efficient alternatives in B_1 with respect to the rationalization $(>_1, >_2)$. This proves the lemma. □

Axiom 2. For all $B_1 \in \mathcal{P}(X)$, if $a \in B_1 \setminus f(B_1)$, then for some $b \in f(B_1)$ [3]

$f(ab) = b$.

The interpretation of Axiom 2 is that if f is Pareto rationalizable, and a is not Pareto efficient among the alternatives in a given budget, then there must exist an efficient alternative in this budget which Pareto dominates it. The following lemma is essentially in Moulin (1991), page 306:

Lemma 2. Axiom 2 *is necessary for Pareto rationalizability*.

Proof. Suppose that $(>_1, >_2)$ is a rationalization and $a \in B_1 \setminus f(B_1)$ for some $B_1 \in \mathcal{P}(X)$. Then a is not Pareto efficient in B_1 with respect to $(>_1, >_2)$. If follows that a is Pareto dominated in B_1 by some other alternative $a_1 \in B_1$. If $a_1 \notin f(B_1)$ then $a_1 \in B_1 \setminus f(B_1)$ and by the same argument a_1 is Pareto dominated in B_1 by some other alternative $a_2 \in B_1$. And so on, since the set B_1 is finite and rationalizing preferences are transitive, we will eventually find an alternative $b \in f(B_1)$ that Pareto dominates a, and hence b is the only Pareto efficient alternative in the budget ab: $f(ab) = b$, $b \in f(B_1)$. □

We next introduce two binary relations. Given is a choice function f. The *strict revealed preference relation* associated to f is the binary relation R defined as aRb if $f(ab) = a$.[4] The *indifference relation* associated to f is the binary relation O defined as aOb and bOa if $f(ab) = ab$. Note that O is symmetric.

The following well-known (Moulin, 1991) observation illustrates the importance of R.

Lemma 3. *If f satisfies* Axioms 1 and 2, *R is transitive*.

Proof. Assume that f satisfies both axioms and aRb, bRc. Using Axiom 1 twice we get $b \notin f(ab) \Rightarrow b \notin f(abc)$ and $c \notin f(bc) \Rightarrow c \notin f(abc)$. Since $f(abc)$ is nonempty it follows that $f(abc) = a$. Now Axiom 2 implies that $f(ac) = a$ as a is the only alternative in $f(abc)$. Thus aRc and the relation R is transitive. □

Remark. Note that O may not be transitive, as the following example with $X = \{a, b, c\}$ illustrates:

$c >_1 a >_1 b$,
$b >_2 c >_2 a$;

where aOb and bOc but cRa.

[3] We are going to abuse notation and write $a_1 a_2 \ldots a_t$ instead of $\{a_1, a_2, \ldots, a_t\}$.
[4] The relation R is also called the base relation.

We henceforward assume that the choice function f satisfies Axioms 1 and 2 and so R is transitive. We are interested in when a pair of linear orders $(>_1, >_2)$ is a rationalization of the choice function g, given that it satisfies both axioms. Suppose first that $(>_1, >_2)$ is a rationalization. Note $f(ab)$ is either a singleton or $f(ab) = ab$. If $f(ab) = a$ is a singleton then $a >_1 b$ and $a >_2 b$ as a is the only efficient element in ab with respect to $(>_1, >_2)$. If $f(ab) = ab$, then both a and b must be efficient, meaning that

$$((a >_1 b) \text{ and } (b >_2 a)) \quad \text{or} \quad ((a >_2 b) \text{ and } (b >_1 a)). \tag{1}$$

We arrive at the following important observation.

Lemma 4. *Suppose the choice function f satisfies* Axioms 1 and 2. *Then a pair of binary relations $(>_1, >_2)$, over the universal set X, is a rationalization of f if and only if $a >_1 b$ and $a >_2 b$ whenever aRb,* (1) *holds whenever aOb, and both relations are acyclic.*

Proof. We already proved the forward direction of the claim. Conversely, suppose that $(>_1, >_2)$ is a pair of acyclic relations such that $a >_1 b$ and $a >_2 b$ whenever aRb, and (1) holds whenever aOb. By these assumptions, $>_1$ and $>_2$ are total and antisymmetric as well. Since both relations are acyclic, it follows that they must also be transitive and hence linear orders.

Consider a budget $B \in \mathcal{P}(X)$ and an alternative $b \in B \setminus f(B)$. By Axiom 2, $f(ab) = a$ for some $a \in f(B)$. That is, aRb. Then, by assumption, $a >_1 b$ and $a >_2 b$, and so b is Pareto dominated in B with respect to the pair $(>_1, >_2)$. On the other hand, take any $a \in f(B)$ and suppose that for some $c \in B$, $c >_1 a$ and $c >_2 a$. Since both relations agree over the pair of alternatives (a, c), our assumptions imply that cRa. That is, $a \notin f(ac)$. But $a \in f(B) \cap \{a, c\}$ and by Axiom 1 $a \in f(ac)$, a contradiction. Thus, $c >_1 a$ and $c >_2 a$ cannot simultaneously hold for any $c \in B$, which implies that a is not Pareto dominated in B, i.e., a is Pareto efficient in B with respect to the pair of relations $(>_1, >_2)$. Since $a \in f(B)$ and $b \in B \setminus f(B)$ are generic elements of the corresponding sets, we conclude that $f(B)$ contains the Pareto efficient alternatives in B and only those. Hence, $(>_1, >_2)$ is a Pareto rationalization of the choice function f. □

Axioms 1 and 2 are necessary. We present an example below to the effect that they are not sufficient. By Lemma 4, then, it should be clear that property (1) captures the restrictions in Pareto rationalizability, in addition to the standard properties in Axioms 1 and 2. In the next section we shall use a graph-theoretic structure to understand property (1) better.

3. A graph coloring problem

We proceed to illustrate how we translate the problem of characterizing Pareto rationalizability into a graph-coloring problem. Suppose that $(>_1, >_2)$ is a rationalization of the choice function f. By Lemma 4, we know this is equivalent to $(>_1, >_2)$ being a pair of acyclic relations such that $a >_1 b$ and $a >_2 b$ whenever aRb, and (1) holds whenever aOb. We can rewrite these two properties of $(>_1, >_2)$ as

$$\text{if } a >_1 b \text{ and } aRb, \text{ then } a >_2 b; \text{ if } a >_1 b \text{ and } aOb, \text{ then } b >_2 a. \tag{2}$$

As a consequence, a rationalizing relation $>_1$ defines a rationalization $(>_1, >_2)$. A choice function g defines R and O; so $>_1$ and (2) gives us $>_2$. This simple observation will allow us to translate characterizing Pareto rationalizability into a graph-coloring problem.

Let G be the directed graph

$$G = (X, \{ab | aRb \text{ or } aOb\}).$$

The vertices of G are all the alternatives from the universal set X and the edges of G represent all the revealed binary comparisons of alternatives in X: we label the edge (a, b) by R if aRb or bRa, and we

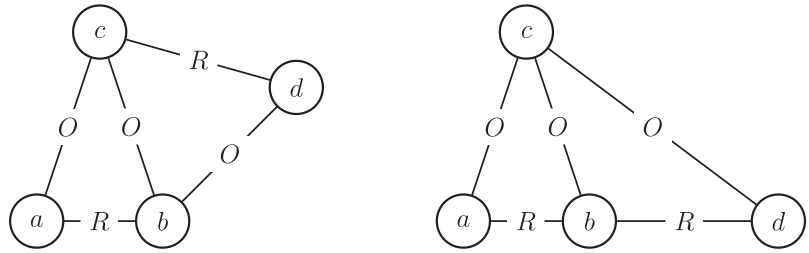

Fig. 2. (Left) (ac, bc) and (cb, db) are dichromatic and (right) (ac, bc) and (bc, cd) are monochromatic.

label the edge (a, b) by O if aOb or bOa. For example, imagine three alternatives a,b and c for which aOc, cOb and aRb. The situation is represented in Fig. 1 in Section 1.

The configuration in Fig. 1 is crucial. This configuration would not be possible if R described a single agent's strict preference relation, and O described her indifference relation. With a Pareto rationalization $(>_1, >_2)$, we see that $>_1$ must either point *away fromc or point towardsc*:

Lemma 5. *Suppose that $(>_1, >_2)$ is a rationalization and $a, b, c \in X$ are three alternatives for which aOc, cOb, and aRb or bRa. Then the following is true*

$$(a>_1 c \text{ and } b>_1 c) \text{ or } (c>_1 a \text{ and } c>_1 b). \tag{3}$$

Proof. Let aRb. We cannot have $b>_1 c$ and $c>_1 a$, as this results in the cycle $a>_1 b>_1 c>_1 a$, contradicting the assumption that $>_1$ is acyclic (we have $a>_1 b$ and $a>_2 b$ since aRb). Similarly, we cannot have $a>_1 c$ and $c>_1 b$ because this implies $b>_2 c$ and $c>_2 a$, which contradicts the acyclicity of $>_2$. Two possibilities remain: either $a>_1 c$ and $b>_1 c$, or $c>_1 a$ and $c>_1 b$. That is, (3) holds. We readily see that, after relabeling a and b, (3) also holds if we have bRa instead of aRb. □

Remark. Statement (3) in Lemma 5 clearly holds for $>_2$ as well; and if one of the alternatives in (3) holds for $>_1$, the other holds for $>_2$.

Given that (3) holds for any rationalization $(>_1, >_2)$ and any three alternatives $a, b, c \in X$ with aOc, cOb, and aRb or bRa, we seek to understand the structure of the graph G. Any necessary condition for G that we hope to derive is indirectly a condition for the choice function f, as G depends on the revealed preference relations R and O, which, in turn, correspond to choice functions satisfying Axioms 1 and 2.

We can express our observation (3) using a second graph, which is undirected:

$$F = (\{ab|a \neq b\}, \{(ac, bc)|aRb, aOc, bOc\}).$$

Note that the vertexes of F are edges of G; there is an edge between ac and bc if they are in a relation like the one in Fig. 1: they are elements of O which are related by R.

We say that the edge $f = (ac, bc) \in F$ is *colored*1 if $(a>_1 c$ and $b>_1 c)$, and that f is *colored*−1 if $(c>_1 a$ and $c>_1 b)$. Consider Fig. 1: the edge (ac, bc) is colored 1 when $>_1$ points *away* from the common vertex c; the edge is colored −1 when $>_1$ points *t* oward the common vertex. Since the assumed rationalization $(>_1, >_2)$ verifies equation (3) by Lemma 5, it follows that it induces a coloring of every edge of the graph F. We now obtain a simple necessary condition for rationalizability based on this coloring.

Consider two adjacent edges $f_1, f_2 \in F$; let $f_1 = (ac, bc)$, for alternatives a, b, and c such that the vertex bc of F is an endpoint of the edge f_2. We have $f_2 = (cb, db)$ or $f_2 = (bc, dc)$ for some other alternative d. The first possibility is represented on the left in Fig. 2 while the second possibility is on the right. When two adjacent edges $f_1 = (ac, bc)$, $f_2 = (cb, db)$ are related as in Fig. 2 on the left, we say that they are *dichromatic* as they must have different colors in any rationalization. When $f_1 = (ac, bc)$, $f_2 = (bc, dc)$ are related as in Fig. 2 on the right, we say they are *monochromatic* as they must have the same color.

Formally, we say that a pair of edges (f_1, f_2), of the graph F is *monochromatic* when $f_1 = (ac, bc)$ and $f_2 = (bc, dc)$ for distinct alternatives $a, b, c, d \in X$. When $f_1 = (ac, bc)$ and $f_2 = (cb, db)$ for distinct alternatives $a, b, c, d \in X$, we call the pair of edges (f_1, f_2) *dichromatic*.

Lemma 6. *Let $(>_1, >_2)$ be a rationalization and f_1, f_2 be two adjacent edges in the graph F. If they are monochromatic (related as on the left in Fig. 2) then the two edges have different colors. If f_1, f_2 are dichromatic (related as on the right), then they have the same color.*

Proof. Consider first the case $f_1 = (ac, bc)$, $f_2 = (cb, db)$. If the edge f_1 is colored −1 then $c>_1 b$, and by equation (3) $d>_1 b$ so that f_2 is colored 1. Similarly, if f_1 is colored 1 then $b>_1 c$, and by (3) $b>_1 d$: f_2 is colored −1. Either way, f_1 and f_2 have different colors. The second case $f_1 = (ac, bc)$, $f_2 = (bc, dc)$ is treated the same way. □

These ideas allow us to formulate a simple condition.

Axiom 3. Every cycle in F has an even number of dichromatic pairs.

We have argued that a rationalization implies a coloring of the edges of F in which monochromatic pairs have the same color, and dichromatic pairs have opposite colors. Coloring using two colors (bicoloring) is equivalent to the absence of a cycle in which there is an odd number of vertexes forcing a switch in the color of adjacent edges (a simple result in graph theory). In our particular case, the vertex linking two adjacent edges of the graph F forces a switch in color if they are dichromatic and forces them to have the same color if they are monochromatic. We get for free the following

Lemma 7. *If f is Pareto rationalizable, it must satisfy Axiom 3.*

We finish this section with an example of a choice function that satisfies Axioms 1 and 2, but not 3. Such a function is not two-agent Pareto rationalizable.

Example. Let $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be the set of alternatives and define the choice function g by the following table; the entry corresponding to row a_i and column a_j is $g(\{a_i, a_j\})$.

	a_1	a_2	a_3	a_4	a_5	a_6
a_1		a_1	a_3a_1	a_1a_4	a_1	a_1a_6
a_2			a_2a_3	a_4	a_2a_5	a_2a_6
a_3				a_4	a_3a_5	a_6
a_4					a_4a_5	a_4a_6
a_5						a_6

The table defines the relations R and O. It is then easy to extend g to all of $\mathcal{P}(X)$ so that it satisfies Axioms 1 and 2.

Notice next that $f_1 = (a_1a_3, a_2a_3)$, $f_2 = (a_3a_2, a_6a_2)$, $f_3 = (a_6a_2, a_5a_2)$, $f_4 = (a_2a_5, a_4a_5)$, $f_5 = (a_4a_5, a_3a_5)$, and $f_6 = (a_5a_3, a_1a_3)$ are edges of the graph F such that f_1, f_2 are dichromatic, f_2, f_3 are monochromatic, f_3, f_4 are dichromatic, f_4, f_5 are monochromatic, f_5, f_6 are dichromatic, and f_6, f_1 are monochromatic. Then $f_1f_2f_3f_4f_5f_6f_1$

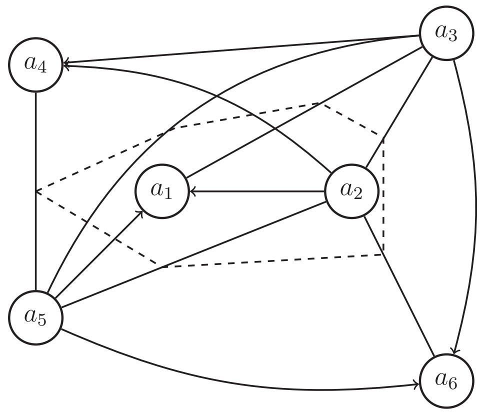

Fig. 3. A choice function satisfying Axioms 1 and 2 but not 3.

is a cycle in F which has an odd number (three) of dichromatic pairs. See Fig. 3, in which the edges of G are drawn with continuous lines and the edges of F are drawn as dotted lines. In Fig. 3, an arrow at the end of an edge indicates the direction of R. The absence of any edge or arrow indicates that the edge corresponds to O.

The choice function g then satisfies Axioms 1 and 2, but not Axiom 3.

4. Main results

Our main result is a characterization of Pareto rationalizability. We express the characterization as a graph coloring problem in Theorem 1. We present the same characterization in Theorem 2, in terms of a system of quadratic equations having a solution. The version of the result in Theorem 2 may be computationally the most convenient.

In Section 3, we reduced the problem of Pareto rationalization to a problem of bi-coloring the graph F. Our characterization of Pareto rationalization requires paying special attention to certain configurations of edges in F.

We say that a triple of edges (f_1, f_2, f_3) of the graph F is *3-cyclic* provided

$$f_1 = (ca, ea), \qquad f_2 = (cb, eb), \qquad f_3 = (ba, da),$$

for some distinct alternatives $a, b, c, d, e \in X$ with dOe, dOc. Notice that $(f_1, (cd, ed), f_3)$, $(f_3, (bc.dc), f_1)$ and $(f_3, (be, de), f_1)$ are 3-cyclic triples as well and that the ordering of the edges in a 3-cyclic triple matters.

The role of 3-cyclic edges becomes apparent in the following

Theorem 1. *A choice function is Pareto rationalizable if and only if it satisfies Axioms 1–3, and there is a coloring of the edges of the graph F (in 1 and -1) such that two edges in a monochromatic pair have the same color, two edges in a dichromatic pair have different colors, and for every 3-cyclic triple (f_1, f_2, f_3) either f_1 and f_2 have the same color or f_1 and f_3 have the same color.*

Proof. Let g be a choice function that satisfies the hypotheses. We are going to construct a Pareto rationalization $(>_1, >_2)$ of g. The construction starts by defining the two binary relations $>_1$ and $>_2$ over only some pairs of alternatives from X at first, and then extending them to the entire set X. For each pair of alternatives (a, b) with aRb, define $a >_1 b$ and $a >_2 b$. For each pair (a, b) define $a >_1 b$, $b >_2 a$ if for some $c \in X$ it is the case that $(ab, cb) \in F$ is colored 1 or $(ba, ca) \in F$ is colored -1. The pair $(>_1, >_2)$ is currently only partially defined. □

We present a trivial fact that will be invoked often in the sequel:

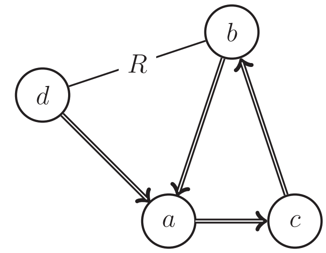

Fig. 4. A cycle for $>_1$, indicated with directed edges.

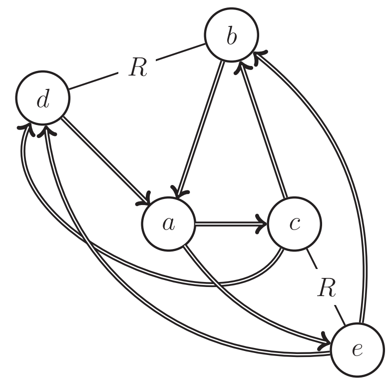

Fig. 5. Implied existence of d and e driving the cycle.

Lemma 8. *Let $x, y, z \in X$ be three distinct alternatives. If $x >_1 y >_1 z$ and $xOyOz$, then xOz.*

We illustrate the rest of the argument using Figs. 4–6. Suppose there is a $>_1$-3-cycle $a >_1 b >_1 c >_1 a$ for some $a, b, c \in X$. It is easy to see that we must have $aObOcOa$: if all three of the revealed relations belong to R then R is not acyclic, a contradiction; if exactly two of the relations belong to R then the third must also since R is transitive, a contradiction; if only one of the relations belongs to R then we have not colored each edge of the graph F.

We represent the situation in Fig. 4; we omit the label on O edges to simplify the figures. The bold edges with arrows indicate the preference $>_1$; for example, the preference $a >_1 b$ mandates an arrow $\Longrightarrow$ on the ab edge pointing in the direction of a.

Now, aOb and $a >_1 b$ implies that there is $d \in X$ such that either $(ba, da) \in F$ or $(ab, db) \in F$. Suppose, without loss of generality, that $(ba, da) \in F$ is the case. Then aOd and $a >_1 d$ as (ba, da) is colored -1; see Fig. 4. Now consider the edge cd. By Lemma 4 we must have cOd. Hence (bc, dc) must be an edge in F colored in 1. This means that $d >_1 c$. (We have assumed that $(ba, da) \in F$, but if we instead assume

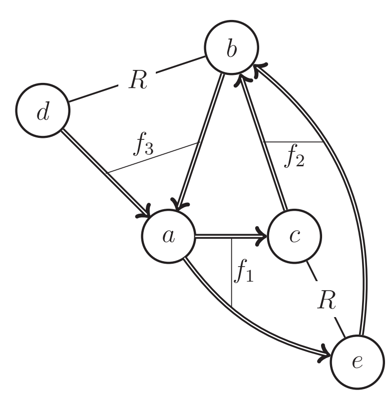

Fig. 6. $f_1 = (ca, ea)$, $f_2 = (cb, eb)$ and $f_3 = (ba, da)$.

that $(ab, db) \in F$, we end up with the same picture after relabeling a, b and c.)

Next, cOa and $c >_1 a$ imply the existence of some some $e \in X$ such that either $(ca, ea) \in F$ or $(ac, ec) \in F$. Suppose without loss of generality that $f_1 = (ca, ea) \in F$. Then eOa and $e >_1 a$, as (ca, ea) is colored with 1. See Fig. 5.

Note that $(ca, ea) \in F$ implies that cRe or eRc. We invoke Lemma 4 with eOa, aOb, $e >_1 a$, and $a >_1 b$ to get eOb and bOe. Hence, $f_2 = (cb, eb) \in F$; in addition, $b >_1 c$ implies that f_2 is colored -1 and thus $b >_1 e$. See Fig. 5. Now, dOe as an R relation is not possible by Lemma 4. Then, $(dRb) \vee (bRd))$ implies that $(de, be) \in F$, and $b >_1 e$ implies that (de, be) is colored 1. Hence, $d >_1 e$. The case $(ac, ec) \in F$ leads to the same picture after relabeling a, b and c.

Consider the edges $f_1 = (ca, ea)$, $f_2 = (cb, eb)$ and $f_3 = (ba, da)$ which all belong to the graph F. Notice that f_3 colored -1, f_2 colored -1, and f_1 colored 1. In addition, we have dOe, dOc. This means that the triple of edges (f_1, f_2, f_3) is 3-cyclic with f_1 and f_2 having opposite colors, and f_1 and f_3 having opposite colors. We have reached a contradiction with the assumptions of the lemma. Hence, there is no 3-cycle belonging to the (not yet total) relation $>_1$. The situation is represented in Fig. 6.

Suppose there is a 3-cycle $a >_2 b >_2 c >_2 a$ which belongs to $>_2$. As for $>_1$, it must be the case that $aObOcOa$. But then $a >_1 c >_1 b >_1 a$, a contradiction.

Suppose next that $>_1$ is not acyclic: for some $a_1, a_2, \ldots, a_t \in X$, $t > 4$ we have $a_1 >_1 a_2 >_1 \ldots >_1 a_t >_1 a_1$. We may assume that this is the shortest cycle, meaning that t is minimal. If $a_1 R a_3$ then there is a shorter cycle $a_1, a_3, a_4, \ldots, a_t, a_1$, a contradiction; if $a_3 R a_1$, then there is a 3-cycle belonging to $>_1$, again a contradiction. If $a_3 R a_1$ then there is a 3-cycle that belongs to $>_1$, a contradiction. Hence, $a_1 O a_3$. Now suppose that $a_1 R a_2$. If $a_2 R a_3$ then $a_1 R a_3$ as R is transitive, a contradiction. Then $a_2 O a_3$ and $(a_1 a_3, a_2 a_3)$ must be an edge of the graph F that is colored in 1. We conclude that $a_1 >_1 a_3$ and there is a shorter cycle, a contradiction. So $a_1 O a_2$.

This allows us to conclude that $a_1 O a_2 O \ldots O a_t O a_1$. Because $a_1 O a_2$ and $a_1 >_1 a_2$, for some $b \in X$ either $(a_2 a_1, b a_1) \in F$ or $(a_1 a_2, b a_2) \in F$. Suppose that $(a_2 a_1, b a_1) \in F$. Then $a_1 >_1 b$. Now $a_t >_1 a_1$, $a_1 >_1 a_2$ and $a_1 >_1 b$ imply that $a_t O a_2$ and $a_t O b$, by Lemma 4. So $(a_2 a_t, b a_t)$ is an edge in F. If it is colored in -1 we get a 3-cycle $a_1 >_1 a_2 >_1 a_t >_1 a_1$, a contradiction. It should be colored in 1, so $a_t >_1 a_2$, and we once again have a shorter cycle, a contradiction. The case $(a_1 a_2, b a_2) \in F$ leads to a contradiction in a similar fashion.

Therefore, $>_1$ is acyclic and can be extended to a total, antisymmetric and transitive relation (a preference relation) on X. In light of observation (2) in Section 3, the (extended) preference relation $>_1$ extends $>_2$ to a total and antisymmetric binary relation on X which does not possess any 3-cycles. On the other hand, if a total relation has a cycle, it is easy to see that it has a 3-cycle. It follows that $>_2$ is also acyclic and Lemma 4 implies that $(>_1, >_2)$ is a rationalization of the choice function g.

We finish the proof by proving the converse statement. Let $(>_1, >_2)$ be a rationalization of the choice function g. We have already shown that g must satisfy Axioms 1–3. We have also seen in Lemma 6 that the rationalization $(>_1, >_2)$ induces a coloring of the edges of the graph F that respects di- and monochromatic pairs. Finally, if (f_1, f_2, f_3) is a 3-cyclic triple and we suppose that f_1 and f_2 have opposite colors and f_1 and f_3 have opposite colors, we must conclude that either $a >_1 b >_1 c >_1 a$ or $a >_1 c >_1 b >_1 a$. A contradiction of the transitivity of $>_1$. Hence, f_1 and f_2 have the same color or f_1 and f_3 have the same color.

Remark. The proof of Theorem 1 is constructive. Given a coloring of the edges of the graph F, we described a simple procedure for constructing a rationalization $(>_1, >_2)$ of the given choice function: the rationalization must agree with the revealed relation R, as well as the coloring of the graph F over pairs of alternatives ab which represent an endpoint of an edge in the graph F. Next, we extend $>_1$ to the entire set X, which also determines $>_2$ as a linear order on X. We showed that the pair $(>_1, >_2)$ constructed in this way is indeed a rationalization.

Theorem 1 is a characterization of Pareto rationalizability. From the results in Section 3, we know that, if a choice function satisfies Axioms 1–3 there is a coloring of the edges of the induced graph F which assigns opposite colors to edges from a dichromatic pair and equal colors to edges from a monochromatic pair. Pareto rationalizability amounts, over and above Axioms 1–3, to finding a coloring of the edges of F, which in addition to respecting di- and monochromatic pairs also has the property that for every 3-cyclic triple (f_1, f_2, f_3) either f_1 and f_2 have the same color or f_1 and f_3 have the same color.

Notice that if there is a coloring of the edges of F that satisfies the conditions in Theorem 1, we can reverse the color of every edge and still have a coloring that satisfies those conditions: the reversed coloring leads to the same rationalizing pair as the original one, with $>_2$ in place of $>_1$. To avoid dealing with this duality, and to simplify our notation, we introduce a third graph H. The vertexes of H are the edges of the graph F, and two vertexes f_1, f_2 in H are joined by an edge in H if and only if there is a path in F connecting f_1 with f_2.

In the graph F, the notions of monochromatic or dichromatic pairs of edges only applied to adjacent edges; that is, edges with a common vertex. We can easily extend this definition to pairs of edges that are connected by a path in F, calling a pair of edges monochromatic if there is a path with an even number of dichromatic pairs joining them, and dichromatic if there is a path with an odd number of dichromatic pairs joining them. The more general notion of mono- and di-chromatic is well-defined under Axiom 3, as Axiom 3 then insures that no pair of edges from F is both dichromatic and monochromatic.

Under Axiom 3, then, we can extend our definitions of dichromatic and monochromatic edges in F to give a bi-coloring of H. We say that the edge $f_1 f_2$ (in H) is colored 1 if the corresponding path (in F) has an even number of dichromatic pairs, and is colored -1 if the path has an odd number of dichromatic pairs.

Now, H is the dual graph of F. It is well-known and easy to see that Axiom 3 is equivalent to

Axiom 3′. Every cycle in H has an even number of edges colored -1.

Observe that a choice function induces a coloring of the *edges* in H just like it labels some pairs of edges in F as di- or monochromatic. We cannot choose the coloring of the edges of H, in contrast to the coloring of F we have discussed above.

We now express the consequences of Theorem 1 for H using a system of equations. A solution to the system of equations will be a coloring of the vertexes of H.

For the graph H, Theorem 1 implies the following. A choice function that satisfies Axioms 1–3 is Pareto rationalizable if and only if there is a coloring of the *vertexes* of H in 1 and -1, such that two vertexes $h_1, h_2 \in H$, such that $h_1 h_2$ is and edge in H, have equal colors if $h_1 h_2$ is colored 1 and opposite colors if $h_1 h_2$ is colored -1; for every 3-cyclic triple of *vertexes* (h_1, h_2, h_3) either h_1 and h_2 have the same color or h_1 and h_3 have the same color.[5] We introduce one variable for each vertex of H; these variables can take the values 1 or -1.

[5] A triple of vertexes (h_1, h_2, h_3) in H is called *3-cyclic* if and only if that same triple viewed as a triple of edges in F, is 3-cyclic.

Let k be the number of 3-cyclic triples:

$$T_1 = (h_{1,1}, h_{1,2}, h_{1,3}), T_2 = (h_{2,1}, h_{2,2}, h_{2,3}), \ldots, T_k = (h_{k,1}, h_{k,2}, h_{k,3}).$$

Remark. As we observed before, 3-cyclic triples come in groups of four. It is sufficient to satisfy the requirement in Theorem 1 only about one 3-cyclic triple out of those four and we may choose to ignore the other three. For theoretical purposes, the redundancy plays no role. For computationally constructing a Pareto rationalization, it would be important to only include one representative 3-cyclic triple.

We proceed to describe a system of equations, named (*), by enumerating all the component equations.

First, include the $3k$ equations

$$h_{i,j}^2 = 1$$

in the system (*). These must be included because $h_{i,j}$ must take the value 1 or -1.

Next, if the same vertex belongs to two different 3-cyclic triples, we have that

$$h_{i_1,j_1} = h_{i_2,j_2}$$

belongs to (*), where h_{i_1,j_1} and h_{i_2,j_2} are the same vertex in H. If h_{i_1,j_1} and h_{i_2,j_2} are two vertexes for which $h_{i_1,j_1}h_{i_2,j_2}$ is an edge in H colored 1, then

$$h_{i_1,j_1} = h_{i_2,j_2}$$

is an equation in the system; if $h_{i_1,j_1}h_{i_2,j_2}$ is an edge in H colored -1, then

$$h_{i_1,j_1} = -h_{i_2,j_2}$$

must be included in the system of equations (*).

Finally, for every $i = 1, 2, \ldots, k$, the equation

$$(h_{i,1}h_{i,2} + h_{i,1}h_{i,3} - 1)^2 = 1$$

belongs to the system (*). Notice that $h_{i,1}$ and $h_{i,2}$ having opposite colors or $h_{i,1}$ and $h_{i,3}$ having opposite colors is equivalent to $h_{i,1}h_{i,2} + h_{i,1}h_{i,3} \neq -2$, which is equivalent to $h_{i,1}h_{i,2} + h_{i,1}h_{i,3} \in \{0, 2\}$. This, in turn, is equivalent to $(h_{i,1}h_{i,2} + h_{i,1}h_{i,3} - 1)^2 = 1$.

Axiom 4. The system of equations (*) has a solution.

Evidently, Axiom 4 is simply a reformulation of the conditions in Theorem 1. So Axiom 4 is both a necessary and sufficient condition for the existence of a coloring of the vertexes of H in the hypotheses of Theorem 1. We thus obtain:

Theorem 2. *The choice function f is Pareto rationalizable if and only if it satisfies Axioms 1–4.*

Unfortunately, the system (*) is not linear, and its solution may present significant computational problems. That said, the construction of the system should be computationally feasible (all the graphs involved can be constructed in polynomial time). One computational advantage of the system is that it does not involve equations of order higher than 2, as $h_{i,j}^2 = 1$ implies that

$$(h_{i,1}h_{i,2} + h_{i,1}h_{i,3} - 1)^2 = 3 + 2h_{i,2}h_{i,3} - 2h_{i,1}h_{i,2} - 2h_{i,1}h_{i,3}.$$

5. Rationalizability in non-strict preferences

We proceed to discuss rationalization by non-strict preferences. We observe that our characterization still holds in this case, when we define Pareto efficiency in the weak sense that an alternative a in B is Pareto efficient if there is no $b \in B$ that both agents strictly prefer.

A choice function g is *two-agent Pareto rationalizable in non-strict preferences* if there are two total, reflexive and transitive binary relations $\succ_1$ and $\succ_2$ on the universal set of alternatives X, such that for all $B \in \mathcal{P}(X)$, $f(B)$ is the set of Pareto efficient alternatives in B with respect to $(\succ_1, \succ_2)$. In this case, we say that $(\succ_1, \succ_2)$ is a *Pareto rationalization ofg in non-strict preferences.*

Theorem 3. *The choice function g is Pareto rationalizable in non-strict preferences if and only if it satisfies Axioms 1–4.*

Proof. Suppose g satisfies Axioms 1–4. Then Theorem 2 implies that g is rationalizable by some strict preferences $(\succ_1, \succ_2)$, and therefore g is also rationalizable by non-strict preferences. On the other hand, suppose g is rationalizable by nonstrict $(\succ_1, \succ_2)$. It is readily seen that Axioms 1–2 must still hold. Also, if aRb, bOc, cOa, then $\succ_1$ and $\succ_2$ must be strict on (a, b) and (b, c): if $b =_1 c$ then $b =_2 c$ and it must be the case that aRc since aRb. This contradicts aOc. The remaining Axioms 3 and 4 only deal with these tractable pairs bOc and cOa, over which there are can be no indifferences belonging to rationalizing relations, and it follows that they are also necessary. □

6. Rationalizability with partial observations

We have followed the classical approach in choice theory, and worked with a choice function that is defined on every budget. The results are, however, applicable to situations where there is partial observability of budgets; as is typically the case in empirical applications of revealed preference theory.

With partial observability, one obtains the relations R and O from the data, and one needs to check that R is acyclic (this is the counterpart to checking the strong axiom of revealed preference in classical revealed preference theory). We can then use the construction in the proof of Theorem 1. Specifically, the construction of graphs G and F only require R and O as inputs.

For example, if $X = \{a, b, c, d\}$. Suppose that a Pareto dominates b and d Pareto dominates c; but that we observe that a and c, b and c, and b and d are Pareto unrelated. Notice that we do not observe anything about how a and d are related: hence we face partial observability.

Now the graph G is (X, E) with $E = \{ab, ac, bc, bd, dc\}$. The graph F is (E, P), with $P = \{(ac, bc), (bc, bd)\}$. Note that the pair (ac, bc) and (bc, bd) is dichromatic. If we set (ac, bc) to have color 1, so $a \succ_1 c$ and $b \succ_1 c$, we must set (bc, bd) to have color -1, so that $b \succ_1 c$ and $b \succ_1 d$. This information, together with the information about R, implies that

$$a \succ_1 b \succ_1 d \succ_1 c.$$

For agent 2 we have the opposite direction of preferences for O edges of G. So we have that $c \succ_2 a$, $c \succ_2 b$, and $d \succ_1 b$; while it is still the case that $a \succ_2 b$ and $d \succ_2 c$. So we obtain

$$d \succ_2 c \succ_2 a \succ_2 b.$$

7. Conclusion

We characterize two-agent Pareto rationalizability of choice functions. We present several simple tests for Pareto rationalizability, most notably the verification of Axiom 3. This axiom is a new consequence of Pareto rationalizability, and its verification seems to be computationally simple.

Our characterization, on the other hand, essentially involves solving a system of quadratic equations. Thus, applying our characterization may be computationally hard; it may not present a substantial advantage over exhaustively searching over all possible rationalizing linear orderings. For problems of a given size, though, it seems intuitive that the constructions we have discovered (the graph H) simplifies the problem.

Finally, we have worked on the two-agent case. We do not know if our graph-theoretic approach extends to the case when we have more than two agents. The difficulty is that there is no two-coloring which allows us to formulate meaningful, simple graph tests, which seem to lead in the direction of a characterization. The general problem with n agents remains open.

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