

# The Answer of Assignment 1

## Problem 1 Solution

(1) In the position representation, from the eigenvalue equation  $\hat{H}\psi(x) = h_n\psi(x)$  and the Hamiltonian operator  $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2$ , we have:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) = h_n\psi(x)$$

The general solution is:

$$\psi(x) = c_1 e^{w_1 x} + c_2 e^{w_2 x}, \quad w_1 = i\sqrt{\frac{2mh_n}{\hbar^2}}, \quad w_2 = -i\sqrt{\frac{2mh_n}{\hbar^2}}$$

Given  $p_n = \frac{2\pi\hbar n}{L}$ ,  $h_n = \frac{2\pi^2\hbar^2 n^2}{Lm}$ . Take  $\psi_n(x) = e^{\frac{ip_n x}{\hbar}}$  as an example, the eigenvalue corresponding to  $|\psi_n\rangle$  is  $\frac{2\pi^2\hbar^2 n^2}{Lm}$ .

(2)

$$\begin{aligned} \langle\psi_{n_1}|\psi_{n_2}\rangle &= \int_{-\infty}^{+\infty} \psi_{n_1}^*(x) \psi_{n_2}(x) dx \\ &= \int_{-\infty}^{+\infty} e^{-\frac{ip_{n_1}x}{\hbar}} e^{\frac{ip_{n_2}x}{\hbar}} dx \\ &= \int_{-\infty}^{+\infty} e^{\frac{i(p_{n_2}-p_{n_1})x}{\hbar}} dx \\ &= \int_{-\infty}^{+\infty} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\ &= \lim_{l \rightarrow +\infty} \int_{-l}^{+l} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\ &= \lim_{l \rightarrow +\infty} \frac{L \sin\left[\frac{2\pi(n_2-n_1)l}{L}\right]}{\pi(n_2-n_1)} \end{aligned}$$

When  $n_2 = n_1$ ,  $\langle\psi_{n_1}|\psi_{n_2}\rangle \rightarrow \infty$  and

$$\lim_{l \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{\sin(lx)}{x} dx = \pi$$

So,

$$\langle\psi_{n_1}|\psi_{n_2}\rangle = L\delta_{n_1 n_2}$$

(3) When  $L \rightarrow \infty$ ,  $p_n$  becomes continuous and  $\psi_n(x)$  becomes a plane wave.

$$\langle \psi_{p_1} | \psi_{p_2} \rangle = \int_{-\infty}^{+\infty} e^{-\frac{ip_1 x}{\hbar}} e^{\frac{ip_2 x}{\hbar}} dx = \lim_{l \rightarrow +\infty} \frac{2\hbar \sin \left[ \frac{(p_2 - p_1)l}{\hbar} \right]}{(p_2 - p_1)} = 2\pi \delta \left( \frac{p_2 - p_1}{\hbar} \right) = 2\pi \hbar \delta(p_2 - p_1)$$

(4) In the position representation,  $|\psi_{x_0, \epsilon}\rangle = \int_{-\infty}^{+\infty} \psi_{x_0, \epsilon}(x) |x\rangle dx$

$$\langle \delta | \delta \rangle = \int_{-\infty}^{+\infty} (x\psi_{x_0, \epsilon}(x) - x_0\psi_{x_0, \epsilon}(x))^* (x\psi_{x_0, \epsilon}(x) - x_0\psi_{x_0, \epsilon}(x)) dx$$

Considering the Gaussian wave packet  $\psi_{x_0, \epsilon}(x) = \left(\frac{1}{2\pi\epsilon^2}\right)^{1/4} e^{-\frac{(x-x_0)^2}{4\epsilon^2}}$

$$\int_{-\infty}^{+\infty} \psi^* \psi dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1$$

and

$$\int_{-\infty}^{+\infty} (x-x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = -\frac{\partial}{\partial a} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-a(x-x_0)^2} dx \Big|_{a=\frac{1}{2\epsilon^2}}$$

Note that

$$\int_{-\infty}^{+\infty} e^{-a(x-x_0)^2} dx = \sqrt{\frac{\pi}{a}}$$

Therefore,

$$-\frac{\partial}{\partial a} \sqrt{\frac{\pi}{a}} = \frac{1}{2} \sqrt{\pi} a^{-3/2}$$

Substituting  $a = \frac{1}{2\epsilon^2}$ , we get

$$\int_{-\infty}^{+\infty} (x-x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \epsilon^2$$

To ensure  $\langle \delta | \delta \rangle < \epsilon$ , we can set  $\delta = \frac{\sqrt{\epsilon}}{2}$ , so  $\psi_{x_0, \delta}(x) = \frac{1}{\sqrt{4\pi\delta^2}} e^{-\frac{(x-x_0)^2}{4\delta^2}}$ . According to the previous derivation,

$$\langle \delta | \delta \rangle = \int_{-\infty}^{+\infty} (x\psi_{x_0, \delta}(x) - x_0\psi_{x_0, \delta}(x))^* (x\psi_{x_0, \delta}(x) - x_0\psi_{x_0, \delta}(x)) dx = \delta^2 = \frac{\epsilon}{4}$$

this satisfies the condition  $\langle \delta | \delta \rangle < \epsilon$ .

## Problem 2 Solution

(1) Consider a qubit with Hamiltonian

$$H = -\mu BY$$

and the initial state at time  $t = 0$  is  $|\uparrow\rangle$ . We have

$$\hat{H} |\psi\rangle = i\hbar \frac{d}{dt} |\psi\rangle$$

The time evolution of the state is given by:

$$\begin{aligned} |\psi(t)\rangle &= e^{-\frac{i}{\hbar}Ht} |\uparrow\rangle \\ &= e^{-\frac{i}{\hbar}(-\mu BY)t} |\uparrow\rangle \\ &= e^{i\frac{\mu B}{\hbar}Yt} |\uparrow\rangle \end{aligned}$$

Using Taylor expansion, we can express this as:

$$|\psi(t)\rangle = |\uparrow\rangle \left( 1 + i\frac{\mu B}{\hbar}Yt - \frac{1}{2} \left( \frac{\mu B}{\hbar}Yt \right)^2 + \dots \right) = \cos(\theta) |\uparrow\rangle - \sin(\theta) |\downarrow\rangle$$

where  $\theta = \frac{\mu B}{\hbar}t$ . The probability of measuring  $|\uparrow\rangle$  at time  $t$  is  $p_{\uparrow}(t) = \cos^2(\theta)$ , and the probability of measuring  $|\downarrow\rangle$  is  $p_{\downarrow}(t) = \sin^2(\theta)$ . The expectation value  $Z$  measured at this time is given by:

$$\langle Z \rangle = \langle \psi(t) | Z | \psi(t) \rangle = \cos^2(\theta) \langle \uparrow | Z | \uparrow \rangle + \sin^2(\theta) \langle \downarrow | Z | \downarrow \rangle + 2 \cos(\theta) \sin(\theta) \langle \uparrow | Z | \downarrow \rangle$$

where  $\langle \uparrow | Z | \uparrow \rangle = 1$ ,  $\langle \downarrow | Z | \downarrow \rangle = -1$ , and  $\langle \uparrow | Z | \downarrow \rangle = 0$ . Thus, we have:

$$\begin{aligned} \langle Z \rangle &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos(2\theta) \end{aligned}$$

(2) We can record the  $n$ -th measurement result as  $S_n(0/1)$  and we have:

$$\begin{cases} p(S_0(0)) = 1 \\ p(S_1(0)) = \cos^2(\theta) \cdot 1 \\ \dots \\ p(S_n(0)) = \cos^2(\theta) \cdot p(S_{n-1}(0)) + \sin^2(\theta) \cdot p(S_{n-1}(1)) \end{cases}$$

$$\begin{cases} p(S_0(1)) = 0 \\ p(S_1(1)) = \sin^2(\theta) \cdot 1 \\ \dots \\ p(S_n(1)) = \sin^2(\theta) \cdot p(S_{n-1}(0)) + \cos^2(\theta) \cdot p(S_{n-1}(1)) \end{cases}$$

The number of all the possible sequences at the length  $n$  is  $2^n$ . We use the method of induction to testify the sum of the probabilities of all sequences at the length  $n$  is 1.

**Base case:** For  $n = 1$ , we have:

$$p(S_1(0)) + p(S_1(1)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

**Inductive step:** Assume it holds for  $n = k$ , i.e.,

$$\sum_{i=0}^{2^k} p_i = 1.$$

Now consider  $n = k + 1$ :

$$\sum_{i=0}^{2^{k+1}} p_i = \sum_{i=0}^{2^k} p_i \cdot (\cos^2(\theta) + \sin^2(\theta)) = \sum_{i=0}^{2^k} p_i \cdot 1 = \sum_{i=0}^{2^k} p_i = 1.$$

(3) When  $\delta t \rightarrow 0$  while  $T$  is fixed:

$$p(|00 \cdots 0\rangle) = \lim_{\delta t \rightarrow 0} p(S_n(0)) = \lim_{\delta t \rightarrow 0} \cos^{2[\frac{T}{\delta t}]}(\frac{\mu B \delta t}{\hbar}) = e^{\frac{2T}{\delta t} \ln(\cos(\frac{\mu B \delta t}{\hbar}))} = 1$$

(4) At  $t = T$ , what is the probability to find the qubit in state  $|\uparrow\rangle$ , and what is the probability to find the qubit in state  $|\downarrow\rangle$ ? What is the expectation value  $\langle Z \rangle$  measured at this time? we have:

$$p_{n+1}(0) = p_n(0) \cdot \cos^2(\theta) + (1 - p_n(0)) \cdot \sin^2(\theta)$$

$$p_{n+1}(0) - \frac{1}{2} = \cos(2\theta)(p_n(0) - \frac{1}{2})$$

as a result, we have:

$$p_n(0) = \lim_{n \rightarrow \infty} \cos^{\frac{T}{\delta t}}(2\theta) \cdot \frac{1}{2} + \frac{1}{2} = 1$$

$$p_n(1) = \frac{1}{2} - \cos^{\frac{T}{\delta t}}(2\theta) \cdot \frac{1}{2}$$

the expectation value  $\langle Z \rangle$  measured at this time is given by:

$$\langle Z \rangle = \cos^{\frac{T}{\delta t}}(2\theta)$$

### Problem 3 Solution

(1) A self-adjoint operator  $O$  in Hilbert space can be spectrally decomposed. Suppose its eigenvalues are  $\lambda_n$  and the corresponding eigenstates are  $|\phi_n\rangle$ , then:

$$\langle u|O|v\rangle = \sum_n \lambda_n \langle u|\phi_n\rangle \langle \phi_n|v\rangle \leq \max_n |\lambda_n| \sqrt{\langle u|u\rangle \langle v|v\rangle} = \max_n |\lambda_n| \|u\| \|v\|$$

(2) We try to prove

$$|\psi_2\rangle = \cos(\theta)e^{i\alpha}|\psi_1\rangle + \sin(\theta)e^{i\beta}|\psi_w\rangle$$

In linear algebra, an inner product space can be regarded as the direct sum of a subspace and its orthogonal complement. For  $|\psi_2\rangle$ , it can be decomposed as  $|\psi_2\rangle = \cos(\theta)e^{i\alpha}|\psi_1\rangle + \sin(\theta)e^{i\beta}|\psi_w\rangle$ , where  $|\psi_1\rangle$  is the projection of  $|\psi_2\rangle$  onto  $|\psi_1\rangle$ , and  $|\psi_w\rangle$  is the projection of  $|\psi_2\rangle$  onto  $|\psi_w\rangle$ . Since  $|\psi_1\rangle$  and  $|\psi_w\rangle$  are orthogonal, their inner product is 0, i.e.,  $\langle \psi_1|\psi_w\rangle = 0$ .

(3)

$$|\langle \psi_1|O|\psi_1\rangle - \langle \psi_2|O|\psi_2\rangle| = |(1 - \cos^2(\theta))\langle \psi_1|O|\psi_1\rangle - \cos(\beta - \alpha)\sin(2\theta)\langle \psi_1|O|\psi_w\rangle|$$

(4)

$$\begin{cases} 1 - |\langle \psi_1|\psi_2\rangle| \leq \epsilon \\ \langle \psi_1|O|\psi_2\rangle \leq \lambda \end{cases}$$

$$|\langle \psi_1|O|\psi_1\rangle - \langle \psi_2|O|\psi_2\rangle| = |\langle \psi_1|O|\psi_1\rangle - \langle \psi_1|O|\psi_2\rangle + \langle \psi_1|O|\psi_2\rangle - \langle \psi_2|O|\psi_2\rangle| = |\langle \psi_1|O||\psi_1\rangle - |\psi_2\rangle| + |$$