The Answer of Assignment 1

Problem 1 Solution

(1) In the position representation, from the eigenvalue equation $\hat{H}\psi(x) = h_n\psi(x)$ and the Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2$, we have:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) = h_n\psi(x)$$

The general solution is:

$$\psi(x) = c_1 e^{w_1 x} + c_2 e^{w_2 x}, \quad w_1 = i\sqrt{\frac{2mh_n}{\hbar^2}}, \quad w_2 = -i\sqrt{\frac{2mh_n}{\hbar^2}}$$

Given $p_n = \frac{2\pi\hbar n}{L}$, $h_n = \frac{2\pi^2\hbar^2 n^2}{Lm}$. Take $\psi_n(x) = e^{\frac{ip_n x}{\hbar}}$ as an example, the eigenvalue corresponding to $|\psi_n\rangle$ is $\frac{2\pi^2\hbar^2 n^2}{Lm}$.

(2)

$$\begin{aligned}
\langle\psi_{n_1}|\psi_{n_2}\rangle &= \int_{-\infty}^{+\infty} \psi_{n_1}^*(x)\,\psi_{n_2}(x)\,dx \\
&= \int_{-\infty}^{+\infty} e^{-\frac{ip_{n_1}x}{\hbar}} e^{\frac{ip_{n_2}x}{\hbar}} dx \\
&= \int_{-\infty}^{+\infty} e^{\frac{i(p_{n_2}-p_{n_1})x}{\hbar}} dx \\
&= \int_{-\infty}^{+\infty} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\
&= \lim_{l\to+\infty} \int_{-l}^{+l} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\
&= \lim_{l\to+\infty} \frac{L\sin\left[\frac{2\pi(n_2-n_1)l}{L}\right]}{\pi(n_2-n_1)}
\end{aligned}$$

When $n_2 = n_1$, $\langle\psi_{n_1}|\psi_{n_2}\rangle \to \infty$ and

$$\lim_{l\to+\infty} \int_{-\infty}^{+\infty} \frac{\sin(lx)}{x} dx = \pi$$

So,

$$\langle\psi_{n_1}|\psi_{n_2}\rangle = L\delta_{n_1 n_2}$$

(3) When $L \to \infty$, p_n becomes continuous and $\psi_n(x)$ becomes a plane wave.

$$\langle \psi_{p_1} | \psi_{p_2} \rangle = \int_{-\infty}^{+\infty} e^{-\frac{ip_1 x}{\hbar}} e^{\frac{ip_2 x}{\hbar}} dx = \lim_{l \to +\infty} \frac{2\hbar \sin\left[\frac{(p_2 - p_1)l}{\hbar}\right]}{(p_2 - p_1)} = 2\pi\delta(\frac{p_2 - p_1}{\hbar}) = 2\pi\hbar\delta(p_2 - p_1)$$

(4) In the position representation, $|\psi_{x_0,\epsilon}\rangle = \int_{-\infty}^{+\infty} \psi_{x_0,\epsilon}(x) \, |x\rangle \, dx$

$$\langle \delta | \delta \rangle = \int_{-\infty}^{+\infty} (x\psi_{x_0,\epsilon}(x) - x_0\psi_{x_0,\epsilon}(x))^* (x\psi_{x_0,\epsilon}(x) - x_0\psi_{x_0,\epsilon}(x)) dx$$

Considering the Gaussian wave packet $\psi_{x_0,\epsilon}(x) = (\frac{1}{2\pi\epsilon^2})^{1/4} e^{\frac{-(x-x_0)^2}{4\epsilon^2}}$

$$\int_{-\infty}^{+\infty} \psi^* \psi dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy = 1$$

and

$$\int_{-\infty}^{+\infty} (x - x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = -\frac{\partial}{\partial a} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-a(x-x_0)^2} dx \Big|_{a=\frac{1}{2\epsilon^2}}$$

$$\int_{-\infty}^{+\infty} e^{-a(x-x_0)^2} dx = \sqrt{\frac{\pi}{a}}$$

$$-\frac{\partial}{\partial a} \sqrt{\frac{\pi}{a}} = \frac{1}{2} \sqrt{\pi} a^{-3/2}$$

$a = \frac{1}{2\epsilon^2}$

$$\int_{-\infty}^{+\infty} (x - x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \epsilon^2$$

for the purpose of comstrict the value of $\langle \delta | \delta \rangle < \epsilon$, we have