Assignment 1 of Quantum Mechanics II [1]

(Due on August 22 2025 at 17:00. Submit your solution to Canvas as a PDF file.)

(You are only required to solve one of problem 3 and problem 4, but you are encouraged to do both.

If you do both, your score from these two problems will be the higher score from one of these two.)

Problem 1 (13 points). In class, we have discussed the free particles in the infinite space, where the position basis states and momentum basis states are not really normalized. Instead, for example, the momentum eigenstates are "normalized" as $\langle \vec{p}_1|\vec{p}_2\rangle = \delta(\vec{p}_1 - \vec{p}_2)$, where $|\vec{p}_{1,2}\rangle$ are states with wave functions $\langle \vec{x}|\vec{p}_{1,2}\rangle = \frac{e^{i\frac{\vec{p}}{\hbar}\cdot\vec{x}}}{(2\pi\hbar)^{d/2}}$. This improper normalization may cause some concerns, because physical states should be normalized properly. The purpose of this problem is to make us more comfortable with the position and momentum eigenstates from two angles. First, we start from a free particle moving in a finite space and take the limit where the size of the space becomes infinity, from which we will see why the normalization in the infinite space makes sense. Second, we will see that although the position and momentum operators do not have properly normalized eigenstate, there exist properly normalized states that look almost the same as their eigenstates. Ultimately, we should keep in mind that states like $|\vec{x}\rangle$ and $|\vec{p}\rangle$ are only formal tricks that allow us to get the results quickly, and as physicists we can be satisfied with using them, although a more mathematically rigorous treatment of them is possible (but much more tedious).

In parts (1-3) of this problem, we will use the periodic boundary condition. Also, for simplicity, we only consider a particle living on a one dimensional space with length L, and the generalization to an arbitrary space dimension is straightforward. In part (4), we will return to the infinite space.

(1) (3 points) Suppose the Hamiltonian of the particle is $H = \frac{p^2}{2m}$. Show that the unnormalized wave functions of the eigenstates of the Hamiltonian can be written as $\psi_n(x) = \langle x|\psi_n\rangle = e^{i\frac{p_n}{\hbar}x}$, where $p_n = \frac{2\pi\hbar n}{L}$, with $n \in \mathbb{Z}$. What is the eigenvalue correponding to the eigenstate $|\psi_n\rangle$?

(2) (3 points) Verify that $\langle \psi_{n_1}|\psi_{n_2}\rangle = L\delta_{n_1,n_2}$.

(3) (4 points) When L becomes large, p_n's become very dense, so it makes sense to consider the Dirac-δ function $\delta(p_{n_1} - p_{n_2})$. For the Dirac-$\delta$ function $\delta(x)$, it is known that $\delta(ax) = \delta(x)/|a|$ for any $a \neq 0$. Use this fact and the previous results to argue that $\langle \psi_{n_1}|\psi_{n_2}\rangle = 2\pi\hbar\delta(p_{n_1} - p_{n_2})$ in the large-L limit.

Remark: Now redefining $|\tilde{\psi}_{1,2}\rangle = \frac{|\psi_{n_{1,2}}\rangle}{\sqrt{2\pi\hbar}}$, whose wave function is $\langle x|\tilde{\psi}_{1,2}\rangle = \frac{e^{i\frac{p_{1,2}}{\hbar}x}}{\sqrt{2\pi\hbar}}$ (where $p_{n_{1,2}}$ is renamed as $p_{1,2}$), we see that $\langle \tilde{\psi}_1|\tilde{\psi}_2\rangle = \delta(p_1 - p_2)$. Identifying $|\tilde{\psi}_{1,2}\rangle$ with the momentum eigenstates in infinite space that are discussed in class, we reproduce the normalization discussed in class.

(4) (3 points) The above analysis shows how the normalization of the momentum eigenstates in infinite space is connected to their finite-space versions. However, we should still keep in mind that physical states should be properly normalized. In this part, we will see that although the position operator $\hat{x}$ has no normalized eigenstate, there are normalized states that are almost its eigenstate. Concretely, given any $\epsilon > 0$ and $x_0 \in \mathbb{R}$, please construct the wave function of a properly normalized state $|\psi_{x_0,\epsilon}\rangle$ such that $\langle \delta|\delta\rangle < \epsilon$, where $|\delta\rangle = \hat{x}|\psi_{x_0,\epsilon}\rangle - x_0|\psi_{x_0,\epsilon}\rangle$. Hint: Consider some Gaussian wave packet. Also, if the wave function of $|\psi_{x_0,\epsilon}\rangle$ is denoted by $\psi_{x_0,\epsilon}(x)$, then $\langle \delta|\delta\rangle = \int_{-\infty}^{\infty} dx\psi^*_{x_0,\epsilon}(x - x_0)^2\psi_{x_0,\epsilon}$.

[1] You are welcome to get back with questions and clarifications if the wording of problems is ambiguous.

Remark: Similar result applies to any Hermitian operator with a continuous spectrum. So in order to be mathematically rigorous, one can use these normalized "almost eigenstates" when dealing with such operators. But for most of our purposes, it suffices to use the improperly normalized states like $|\vec{x}\rangle$ and $|\vec{p}\rangle$.

Problem 2 (13 points). In class, we have discussed four basic principles of quantum mechanics. In this problem, we will apply them to explore quantum trajectories and the quantum Zeno effect.

The setup is as follows. Consider a qubit with Hamiltonian $H = -\mu BY$, and the initial state at time $t = 0$ is $|\uparrow\rangle$. Suppose the qubit first evolves under this Hamiltonian for time δt, then a measurement of Z is performed at $t = \delta t$. After performing the measurement, the qubit evolves under the Hamiltonian $H = -\mu BY$ for another period δt, and then another measurement of Z is performed at $t = 2\delta t$. This process is repeated until the total time passed is $T = N\delta t$, so that there are N periods of evolution governed by the Hamiltonian $H = -\mu BY$ and N measurements of Z. We assume that measurements always happen instantaneously.

(1) (3 points) At $t = \delta t$, what is the probability to find the qubit in state $|\uparrow\rangle$, and what is the probability to find the qubit in state $|\downarrow\rangle$? What is the expectation value $\langle Z\rangle$ measured at this time?

(2) (4 points) This dynamical process can be viewed as an ensemble of 2^N quantum trajectories, where each trajectory is labeled by whether the qubit is in $|\uparrow\rangle$ or $|\downarrow\rangle$ at time $t = n\delta t$, for $n = 1, 2, \cdots, N$. For example, the quantum trajectory denoted by $0000\cdots0$ represents the one where the qubit is in state $|\uparrow\rangle$ after each measurement, and the quantum trajectory denoted by $0100\cdots0$ represents the one where the qubit is in state $|\downarrow\rangle$ after the measurement at $t = 2\delta t$ while in state $|\uparrow\rangle$ after the measurements at other times. What is the probability for each quantum trajectory? Sum up the probabilities of all quantum trajectories to show that the total probability is 1.

(3) (2 points) When $\delta t \to 0$ while T is fixed, what is the probability of the quantum trajectory $0000\cdots0$? First guess the answer and then do the calculation (feel free to use Mathematica for the calculation).

(4) (4 points) At $t = T$, what is the probability to find the qubit in state $|\uparrow\rangle$, and what is the probability to find the qubit in state $|\downarrow\rangle$? What is the expectation value $\langle Z\rangle$ measured at this time? When $\delta t \to 0$ while T is fixed, what do these two probabilities and $\langle Z\rangle$ become? Again, guess first and then do the calculation (and feel free to use Mathematica for the calculation) Hint: It is useful to first consider the relation between p_n and p_{n+1}, with p_n the probability for the qubit to be in state $|\uparrow\rangle$ at time $t = n\delta t$, where $0 \leqslant n \leqslant N-1$. You will find that $p_{n+1} + B = A(p_n + B)$ with appropriate choices of A and B, based on which you can find p_N from p_0.

The phenomenon occuring where $\delta t \to 0$ while T is fixed demonstrated in this example is called the quantum Zeno effect. Look up on internet what it means.

Problem 3 (13 points). In class, we have discussed states and observables in quantum systems. In essence, observables are represented by operators in a Hilbert space, and states are represented by vectors in this space, which encode the expectation values of all observables. If two states are represented by two vectors that are almost the same (up to a phase factor), we naturally expect the expectation values of any operators on these two states are almost the same. In this problem, we give a formal and general proof of this expectation.

In the following, besides solving this problem, please also learn how to translate the above intuitive expectation into precise mathematical statement. This is a process of formulating a problem, which is as important as and sometimes more important than solving a problem.

Let $|\psi_{1,2}\rangle \in \mathcal{H}$ be two normalized states in the Hilbert space $\mathcal{H}$, and suppose O is a Hermitian operator in $\mathcal{H}$. Denoted by λ the eigenvalue of O that has the maximal absolute value among all its eigenvalues. Here λ characterizes the "magnitude" of O, which is assumed to be finite. Assume that $1 - |\langle\psi_1|\psi_2\rangle| \leqslant \epsilon$, where $0 < \epsilon \leqslant 1$. Clearly, a small ϵ indicates that $|\psi_1\rangle$ and $|\psi_2\rangle$ are close to each other.

The goal of this problem is to prove the following inequality:

$$|\langle\psi_1|O|\psi_1\rangle - \langle\psi_2|O|\psi_2\rangle| < 8\sqrt{\epsilon}\lambda. \tag{1}$$

This inequality means that when the two states are getting closer to each other, i.e., when ϵ decreases, the expectation value of a given observable that has a finite magnitude becomes more and more indistinguishable for these two states.

Note: Eq. (1) is not a tight inequality if $\epsilon > 0$, i.e., the right hand side is always strictly greater than the left hand side, and they can never be equal. You are encouraged, but not required, to improve this inequality so that it becomes tight.

(1) (3 points) For two arbitrary unit vectors $|u\rangle$ and $|v\rangle$ in the Hilbert space $\mathcal{H}$, prove that $|\langle u|O|v\rangle| \leqslant \lambda$. This result will be used in part (4).

(2) (3 points) Show that it is always possible to write $|\psi_2\rangle = \cos\theta e^{i\alpha}|\psi_1\rangle + \sin\theta e^{i\beta}|\psi_\perp\rangle$, where $|\psi_\perp\rangle$ is a normalized state that is perpendicular to $|\psi_1\rangle$, i.e., $\langle\psi_\perp|\psi_1\rangle = 0$, and $\theta, \alpha, \beta \in \mathbb{R}$.

(3) (3 points) Use the result of part (2) to write $|\langle\psi_1|O|\psi_1\rangle - \langle\psi_2|O|\psi_2\rangle|$ in terms of $|\psi_1\rangle$, $|\psi_\perp\rangle$, O, α, β and θ.

(4) (4 points) Use the assumption that $1-|\langle\psi_1|\psi_2\rangle| \leqslant \epsilon$ to prove Eq. (1). Hint: Some facts that may be useful include 1) $\cos^2\theta+\sin^2\theta = 1$, $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$, and 2) $|a+b| \leqslant |a|+|b|, \forall a, b \in \mathbb{C}$.

Problem 4 (14 points). Just being a student who can solve problems assigned by the teachers is good, but it is even better if you can propose good problems by yourself. In this class, besides helping you understand the basics of quantum mechanics and how to apply it, I also aim to help you improve your ability to raise good questions, which is useful no matter what you do in the future.

Now you are required to propose a problem that is similarly interesting as problem 3. After proposing it, you need to either give the complete solution to this problem, or argue that this problem is interesting but cannot be solved within a reasonable time frame.

Please use your own taste and opinion to judge what kind of problems are "similarly interesting as problem 3". At the end, the score you get from this problem is determined based on how interesting the grader thinks your proposed problem is, and whether the solution you provide is complete or whether you can convince the grader that the problem is interesting but too difficult.

Good problems may be selected as problems in the midterm and final exams this semester, or assignment problems in the future semesters. Really good problems may be further developed into research projects.