Quantum Mechanics II

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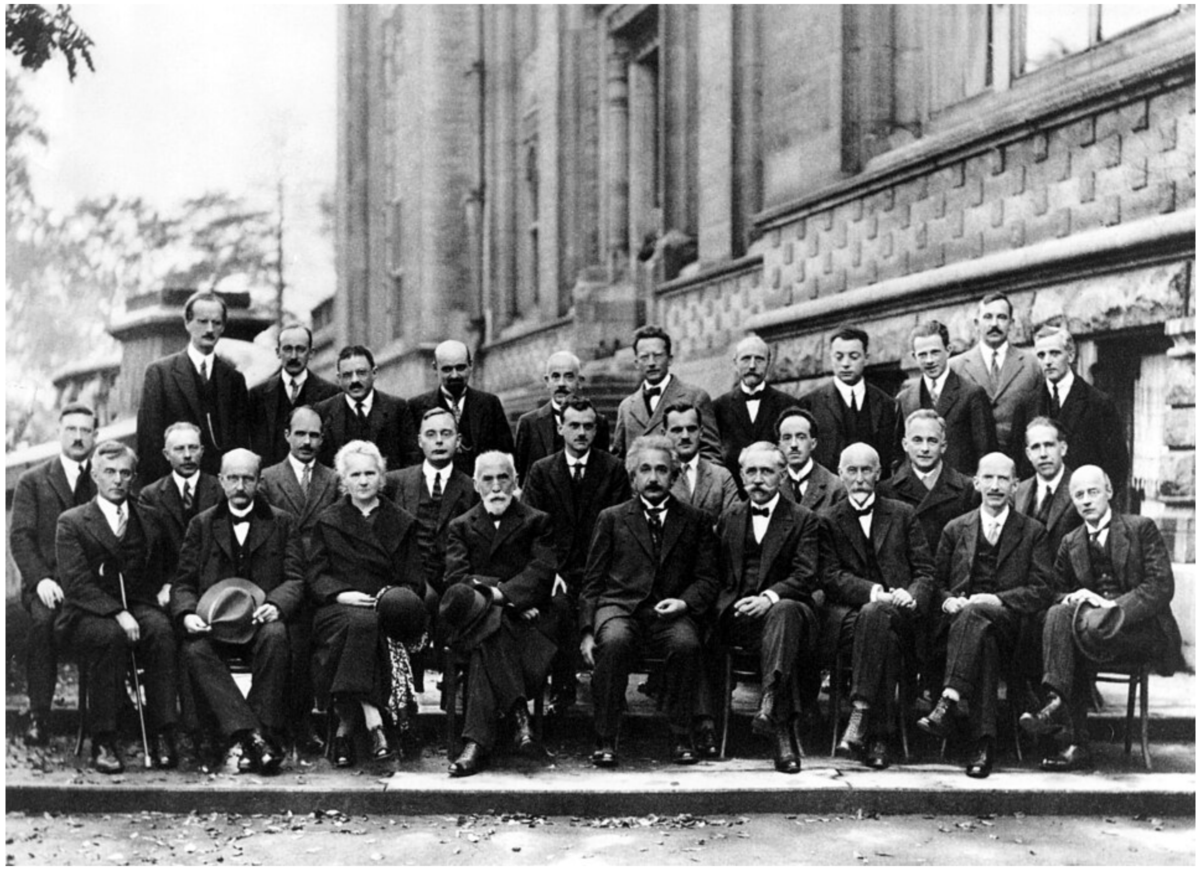
(Picture of the Solvay Conference)



The following basic principles are the rules of our game.

- States are vectors in a Hilbert space.
- States evolve according to the Schrodinger equation.
- Observables are operators in a Hilbert space.
- States collapse after a measurement.
- Bosons (fermions) are symmetric (anti-symmetric) under permutation.

Lectures from the last week (week 2)

- Localized qubits
    - Structure of Hilbert space
    - Dynamics and measurement outcomes
- Bosons and fermions
    - Structure of Hilbert space
    - Creation and annihilation operators

Outline of this week (week 3)

- Writing operators using creation and annihilation operators
- Back to the wave functions
- Causality, locality, and fermion parity
- Localized qubits revisited: Emergent distinguishability from identical particles

By the end of this week, you should be able to:

- Write down operators using creation and annihilation operators
- Explain how to change the single-particle basis in creation operators
- Derive the Schrodinger equation of 2-particle wave functions
- Explain why fermion parity must be preserved at the intuitive level
- Explain the concepts of kinematic and dynamical equivalence intuitively
- Explain why particles' indistinguishability doesn't affect localized qubits

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States can be conveniently described using creation and annihilation operators. What about operators like Hamiltonians?

An example of a typical form of Hamiltonian:

$$H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i,j} V(\vec{r}_i, \vec{r}_j)$$

single-particle kinetic energy

two-particle interaction

To express these operators using creation operators, first describe states using particle numbers.
I. Kinetic energy

Suppose there are $N_{\vec{p}_1}$ particles with momentum $\vec{p}_1$, $N_{\vec{p}_2}$ particles with momentum $\vec{p}_2$, etc.

$$\Rightarrow$$

Total kinetic energy = $\dfrac{p_1^2}{2m}N_{\vec{p}_1} + \dfrac{p_2^2}{2m}N_{\vec{p}_2} + \cdots$

$$\Rightarrow$$

Using creation and annihilation operators,

total kinetic energy = $\displaystyle\sum_{\vec{p}} \frac{p^2}{2m} N_{\vec{p}} = \sum_{\vec{p}} \frac{p^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}}$.

To express these operators using creation operators, first describe states using particle numbers. II. Interaction energy

Suppose there are $N_{\vec{r}_1}$ particles with position $\vec{r}_1$, $N_{\vec{r}_2}$ particles with position $\vec{r}_2$, etc.

$$\Rightarrow$$

Total interaction energy

$$= V(\vec{r}_1, \vec{r}_2)N_{\vec{r}_1}N_{\vec{r}_2} + V(\vec{r}_1, \vec{r}_3)N_{\vec{r}_1}N_{\vec{r}_3} + \cdots + V(\vec{r}_2, \vec{r}_3)N_{\vec{r}_2}N_{\vec{r}_3} + \cdots$$
$$+V(\vec{r}_1, \vec{r}_1)\frac{N_{\vec{r}_1}(N(\vec{r}_1) - 1)}{2} + V(\vec{r}_2, \vec{r}_2)\frac{N_{\vec{r}_2}(N(\vec{r}_2) - 1)}{2} + \cdots$$
$$= \frac{1}{2}\sum_{i \neq j} V(\vec{r}_i, \vec{r}_j)N_{\vec{r}_1}N_{\vec{r}_2} + \frac{1}{2}\sum_{i} V(\vec{r}_i, \vec{r}_i)N_{\vec{r}_i}(N_{\vec{r}_i} - 1)$$

$$\Rightarrow$$

Using creation and annihilation operators,

total interaction energy $= \frac{1}{2}\sum_{\vec{r}_1 \neq \vec{r}_2} V(\vec{r}_1, \vec{r}_2)a^\dagger_{\vec{r}_1}a_{\vec{r}_1}a^\dagger_{\vec{r}_2}a_{\vec{r}_2} + \frac{1}{2}\sum_{\vec{r}} V(\vec{r}, \vec{r})a^\dagger_{\vec{r}}a_{\vec{r}}(a^\dagger_{\vec{r}}a_{\vec{r}} - 1)$.

The interaction has a more compact expression.

$$\text{interaction energy} = \frac{1}{2}\sum_{\vec{r}_1\neq\vec{r}_2} V(\vec{r}_1,\vec{r}_2)a^\dagger_{\vec{r}_1}a_{\vec{r}_1}a^\dagger_{\vec{r}_2}a_{\vec{r}_2} + \frac{1}{2}\sum_{\vec{r}} V(\vec{r},\vec{r})a^\dagger_{\vec{r}}a_{\vec{r}}(a^\dagger_{\vec{r}}a_{\vec{r}} - 1)$$

$$= \frac{1}{2}\sum_{\vec{r}_1\neq\vec{r}_2} V(\vec{r}_1,\vec{r}_2)a^\dagger_{\vec{r}_1}(\pm a^\dagger_{\vec{r}_2}a_{\vec{r}_1})a_{\vec{r}_2} - \frac{1}{2}\sum_{\vec{r}} V(\vec{r},\vec{r})a^\dagger_{\vec{r}}a_{\vec{r}} + \frac{1}{2}\sum_{\vec{r}} V(\vec{r},\vec{r})a^\dagger_{\vec{r}}(1 \pm a^\dagger_{\vec{r}}a_{\vec{r}})a_{\vec{r}}$$

$$= \frac{1}{2}\sum_{\vec{r}_1\neq\vec{r}_2} V(\vec{r}_1,\vec{r}_2)a^\dagger_{\vec{r}_1}a^\dagger_{\vec{r}_2}a_{\vec{r}_2}a_{\vec{r}_1} \pm \frac{1}{2}\sum_{\vec{r}} V(\vec{r},\vec{r})a^\dagger_{\vec{r}}a^\dagger_{\vec{r}}a_{\vec{r}}a_{\vec{r}}$$

$$= \frac{1}{2}\sum_{\vec{r}_1\neq\vec{r}_2} V(\vec{r}_1,\vec{r}_2)a^\dagger_{\vec{r}_1}a^\dagger_{\vec{r}_2}a_{\vec{r}_2}a_{\vec{r}_1} + \frac{1}{2}\sum_{\vec{r}} V(\vec{r},\vec{r})a^\dagger_{\vec{r}}a^\dagger_{\vec{r}}a_{\vec{r}}a_{\vec{r}}$$

$$= \frac{1}{2}\sum_{\vec{r}_1,\vec{r}_2} V(\vec{r}_1,\vec{r}_2)a^\dagger_{\vec{r}_1}a^\dagger_{\vec{r}_2}a_{\vec{r}_2}a_{\vec{r}_1}$$

The kinetic energy is written using $a_{\vec{p}}$, but the interaction is written using $a_{\vec{r}}$.
How can they be unified?

Creation operators in different single-particle basis can be connected by unitary transformations.

$$|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3r\, |\vec{r}\rangle e^{i\frac{\vec{p}}{\hbar}\cdot\vec{r}}$$

single-particle
momentum eigenstate

single-particle
position eigenstate

$$\Updownarrow$$

$$a^\dagger_{\vec{p}}|vac\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3r\, a^\dagger_{\vec{r}} e^{i\frac{\vec{p}}{\hbar}\cdot\vec{r}} |vac\rangle$$

$$\Updownarrow$$

$$a^\dagger_{\vec{p}} = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3r\, a^\dagger_{\vec{r}} e^{i\frac{\vec{p}}{\hbar}\cdot\vec{r}}$$

Kinetic energy can also be expressed using $a_{\vec{r}}$, and interaction energy can be expressed using $a_{\vec{p}}$.

$$\text{total kinetic energy} = \int d^3p \frac{p^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}}$$

$$= \frac{1}{(2\pi\hbar)^3} \int d^3p \frac{p^2}{2m} \int d^3r_1 a^\dagger_{\vec{r}_1} e^{i\frac{\vec{p}}{\hbar}\cdot\vec{r}_1} \int d^3r_2 a_{\vec{r}_2} e^{-i\frac{\vec{p}}{\hbar}\cdot\vec{r}_2}$$

$$= \frac{1}{(2\pi\hbar)^3} \int d^3r_1 d^3r_2 d^3p \frac{1}{2m} a^\dagger_{\vec{r}_1} (-\hbar^2 \nabla^2_{\vec{r}_2} e^{i\frac{\vec{p}}{\hbar}\cdot(\vec{r}_1-\vec{r}_2)}) a_{\vec{r}_2}$$

$$= \frac{1}{(2\pi\hbar)^3} \int d^3r_1 d^3r_2 d^3p e^{i\frac{\vec{p}}{\hbar}\cdot(\vec{r}_1-\vec{r}_2)} a^\dagger_{\vec{r}_1} \frac{-\hbar^2 \nabla^2_{\vec{r}_2}}{2m} a_{\vec{r}_2}$$

$$= \int d^3r_1 d^3r_2 \delta(\vec{r}_1 - \vec{r}_2) a^\dagger_{\vec{r}_1} \frac{-\hbar^2 \nabla^2_{\vec{r}_2}}{2m} a_{\vec{r}_2}$$

$$= \int d^3r a^\dagger_{\vec{r}} \frac{-\hbar^2 \nabla^2}{2m} a_{\vec{r}}$$

Let's go between the position and momentum basis!

- Follow the previous strategy and write the interaction $\dfrac{1}{2}\sum_{\vec{r}_1,\vec{r}_2} V(\vec{r}_1,\vec{r}_2)a^\dagger_{\vec{r}_1}a^\dagger_{\vec{r}_2}a_{\vec{r}_2}a_{\vec{r}_1}$ using $a_{\vec{p}}$.

Remarks

- Not restricted to the position and momentum basis, the connection between the creation operators in two different singlet-particle basis can be derived in a similar way: $a_\alpha^\dagger = \sum_i \langle i \,|\, \alpha \rangle a_i^\dagger$, where $\{\,|\, i\rangle\}$ and $\{\,|\,\alpha\rangle\}$ are two orthonormal basis.

- In assignment 3, you will reproduce the exchange interaction introduced in week 2 using creation and annihilation operators. In this problem with only 2 particles, the approach based on creation and annihilation operators is less convenient than (anti-)symmetrized wave function. But in the presence of many particles, the approach based on creation and annihilation operators is often much more convenient.

- Later in our class, we will consider multiple Hamiltonians of the previous form, but defined on a discrete lattice (instead of continuum space). One example is the so-called Hubbard model, with $H = -t \sum_{\langle i,j \rangle} a_i^\dagger a_j + U \sum_i N_i(N_i - 1)$.

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Restricting to 2-particle states, we expect Schrodinger's wave equation.

Hamiltonian applicable to
any number of particles:

$$H = \int d^3r a^\dagger_{\vec{r}} \frac{-\hbar^2 \nabla^2}{2m} a_{\vec{r}} + \frac{1}{2} \sum_{\vec{r}_1, \vec{r}_2} V(\vec{r}_1, \vec{r}_2) a^\dagger_{\vec{r}_1} a^\dagger_{\vec{r}_2} a_{\vec{r}_2} a_{\vec{r}_1}$$

$$\Big\downarrow \text{???}$$

Schrodinger's wave equation
for 2 particles:

$$i\hbar \frac{\partial \psi(\vec{r}_1, \vec{r}_2, t)}{\partial t} = \left[-\frac{\hbar^2 \nabla_1^2}{2m} - \frac{\hbar^2 \nabla_2^2}{2m} + V(\vec{r}_1, \vec{r}_2) \right] \psi(\vec{r}_1, \vec{r}_2, t)$$

(particles' indistinguishability requires $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_2, \vec{r}_1)$)

The 2-particle wave equation follows from the general Schrodinger equation, $i\hbar \dfrac{d|\psi\rangle}{dt} = H|\psi\rangle$.

A generic normalized 2-particle state: $|\psi\rangle = \int d^3r_1 d^3r_2 \tilde{\psi}(\vec{r}_1, \vec{r}_2) a^\dagger_{\vec{r}_1} a^\dagger_{\vec{r}_2} |vac\rangle$

Action of the kinetic energy operator on $|\psi\rangle$:

$$\int d^3r a^\dagger_{\vec{r}} \frac{-\hbar^2 \nabla^2}{2m} a_{\vec{r}} |\psi\rangle = \int d^3r a^\dagger_{\vec{r}} \frac{-\hbar^2 \nabla^2}{2m} a_{\vec{r}} \int d^3r_1 d^3r_2 \tilde{\psi}(\vec{r}_1, \vec{r}_2) a^\dagger_{\vec{r}_1} a^\dagger_{\vec{r}_2} |vac\rangle$$

$$= \int d^3r d^3r_1 d^3r_2 \tilde{\psi}(\vec{r}_1, \vec{r}_2) \left(\frac{-\hbar^2 \nabla_r^2}{2m} a^\dagger_{\vec{r}} \right) (\delta(\vec{r} - \vec{r}_1) \pm a^\dagger_{\vec{r}_1} a_{\vec{r}}) a^\dagger_{\vec{r}_2} |vac\rangle$$

$$= \int d^3r d^3r_1 d^3r_2 \tilde{\psi}(\vec{r}_1, \vec{r}_2) \left(\frac{-\hbar^2 \nabla_r^2}{2m} a^\dagger_{\vec{r}} \right) (\delta(\vec{r} - \vec{r}_1) a^\dagger_{\vec{r}_2} \pm a^\dagger_{\vec{r}_1} (\delta(\vec{r} - \vec{r}_2) - a^\dagger_{\vec{r}_2} a_{\vec{r}})) |vac\rangle$$

$$= \int d^3r_1 d^3r_2 \tilde{\psi}(\vec{r}_1, \vec{r}_2) \left(\frac{-\hbar^2 \nabla_1^2}{2m} a^\dagger_{\vec{r}_1} a^\dagger_{\vec{r}_2} \pm \frac{-\hbar^2 \nabla_2^2}{2m} a^\dagger_{\vec{r}_2} a^\dagger_{\vec{r}_1} \right) |vac\rangle$$

$$= \int d^3r_1 d^3r_2 \tilde{\psi}(\vec{r}_1, \vec{r}_2) \left(\frac{-\hbar^2 (\nabla_1^2 + \nabla_2^2)}{2m} \right) a^\dagger_{\vec{r}_1} a^\dagger_{\vec{r}_2} |vac\rangle$$

Let’s finish deriving Schrodinger’s wave equation!

- What is the normalization of $\tilde{\psi}(\vec{r}_1, \vec{r}_2)$? Namely, what is $\int d^3 r_1 d^3 r_2 \, |\tilde{\psi}(\vec{r}_1, \vec{r}_2)|^2$?

- Following the similar steps as before, derive the action of the interaction on the 2-particle state.

- Putting the previous results together, finish deriving Schrodinger’s wave equation for 2 particles.

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This understanding of why fermion parity should be conserved is original,

which I didn't realize until I prepared this lecture.

I really like its simplicity and generality.

This part involves complicated calculations that will not be given in class,

but the notes for the calculations are in Canvas.

You only need to understand this topic at the intuitive level.

Fermion parity is always conserved. Why should it be the case?

Each Hamiltonian term contains
an even number of fermionic creation and annihilation operators.

$$H = \int d^3 r f^\dagger_{\vec{r}} \frac{-\hbar^2 \nabla^2}{2m} f_{\vec{r}} + \frac{1}{2} \sum_{\vec{r}_1, \vec{r}_2} V(\vec{r}_1, \vec{r}_2) f^\dagger_{\vec{r}_1} f^\dagger_{\vec{r}_2} f_{\vec{r}_2} f_{\vec{r}_1}$$

$$H = -t \sum_{\langle i,j \rangle} f^\dagger_i f_j + U \sum_{\langle i,j \rangle} f^\dagger_i f_i f^\dagger_j f_j$$

$$H = -t \sum_{\langle i,j \rangle} f^\dagger_i f_j - \Delta \sum_{\langle i,j \rangle} (f^\dagger_i f^\dagger_j + f_j f_i)$$

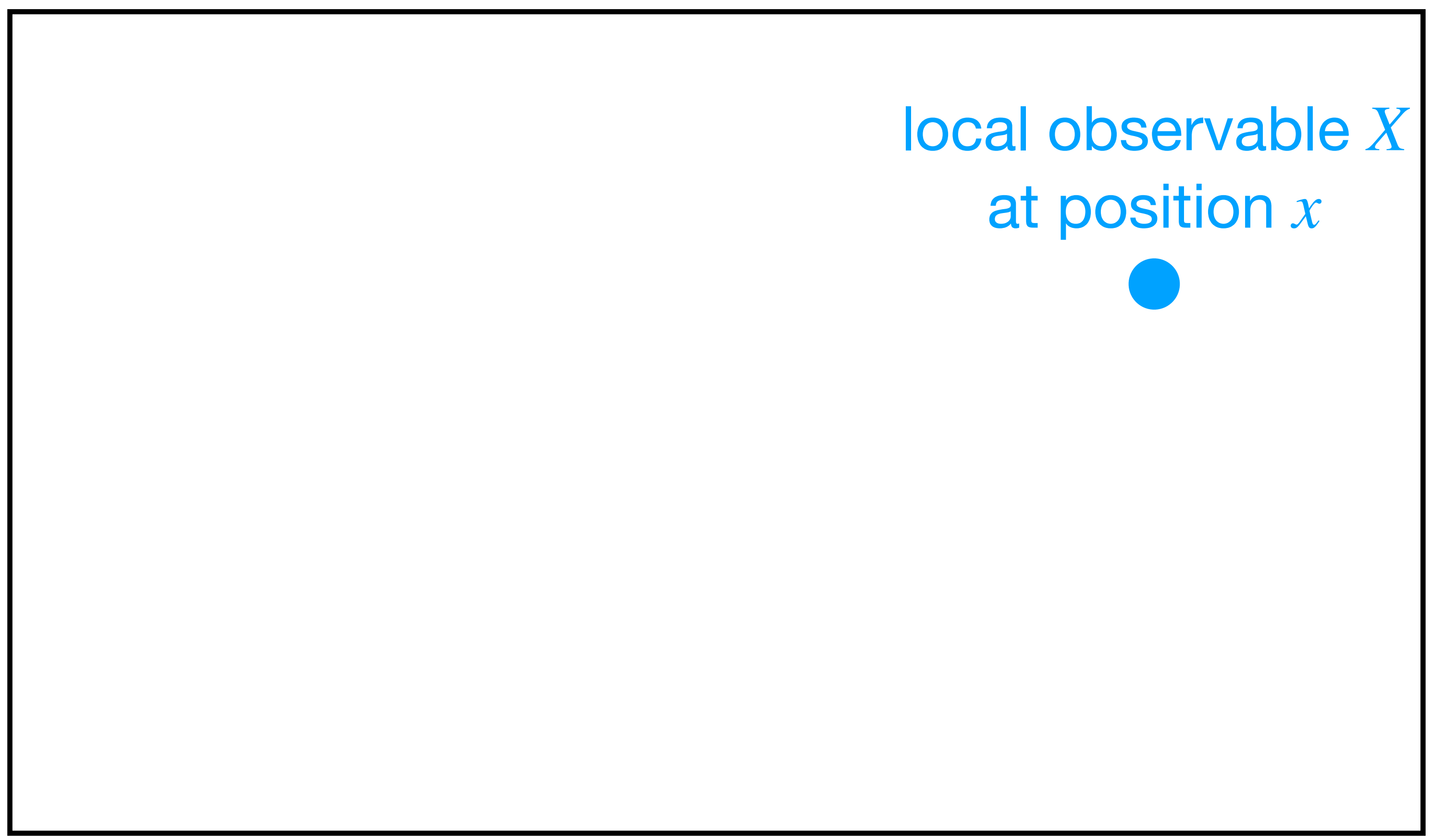
$$\cdots$$

$$\Rightarrow$$

fermion parity is conserved =
fermion number can only be changed by an even number

Short answer:
If fermion parity is not conserved, signals can be transmitted arbitrarily fast.

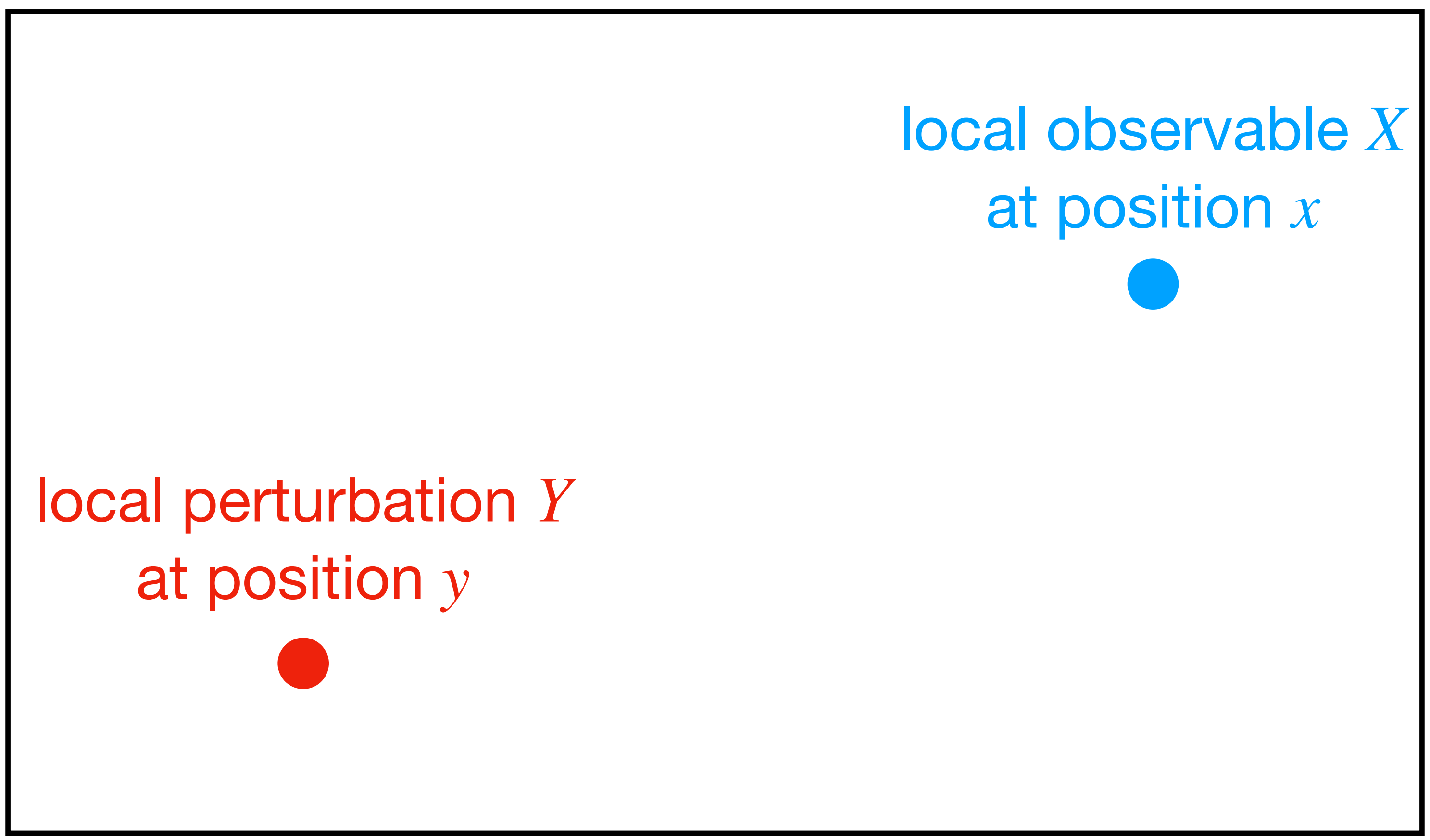
system without fermion parity conservation



Right before the time $t = 0$,
the local observable X has expectation value $\langle X(t=0) \rangle$.

Short answer:
If fermion parity is not conserved, signals can be transmitted arbitrarily fast.

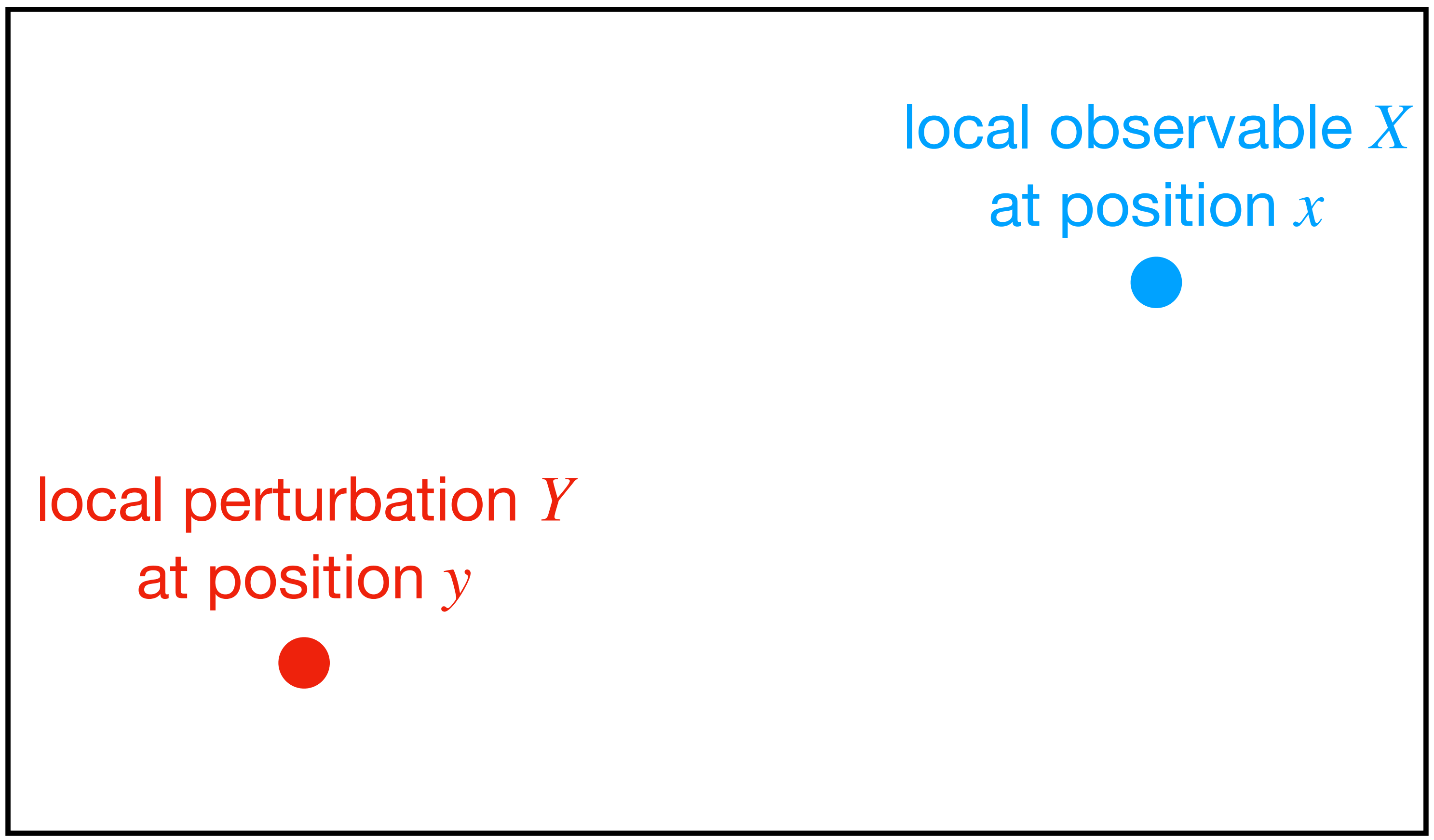
system without fermion parity conservation



At $t = 0$,
the local perturbation Y is added to the Hamiltonian.

Short answer:
If fermion parity is not conserved, signals can be transmitted arbitrarily fast.

system without fermion parity conservation



Immediately after $t = 0$,
the expectation value of X will be changed
by an amount proportional to the perturbation.

This result can be relatively easily obtained using the Heisenberg picture.

<u>Setup</u>

Hamiltonian: $H(t) = H_0 + \lambda(t)Y$ with $\lambda(t<0) = 0$

Time evolution of X without perturbation: $X_0(t) = e^{i\frac{H_0}{\hbar}t} X e^{-i\frac{H_0}{\hbar}t}$

Time evolution of X with perturbation: $X(t) = \mathcal{T} e^{\frac{i}{\hbar}\int_0^t dsH(s)} X \mathcal{T} e^{-\frac{i}{\hbar}\int_0^t dsH(s)}$

<u>Question</u>

What is the change of X due to perturbation, i.e., $X(t) - X_0(t)$?

<u>Answer to first order in λ:</u>

Define $U(t) = e^{\frac{i}{\hbar}H_0 t} \mathcal{T} e^{-\frac{i}{\hbar}\int_0^t dsH(s)}$, then $X(t) = U(t)^\dagger X_0(t) U(t)$.

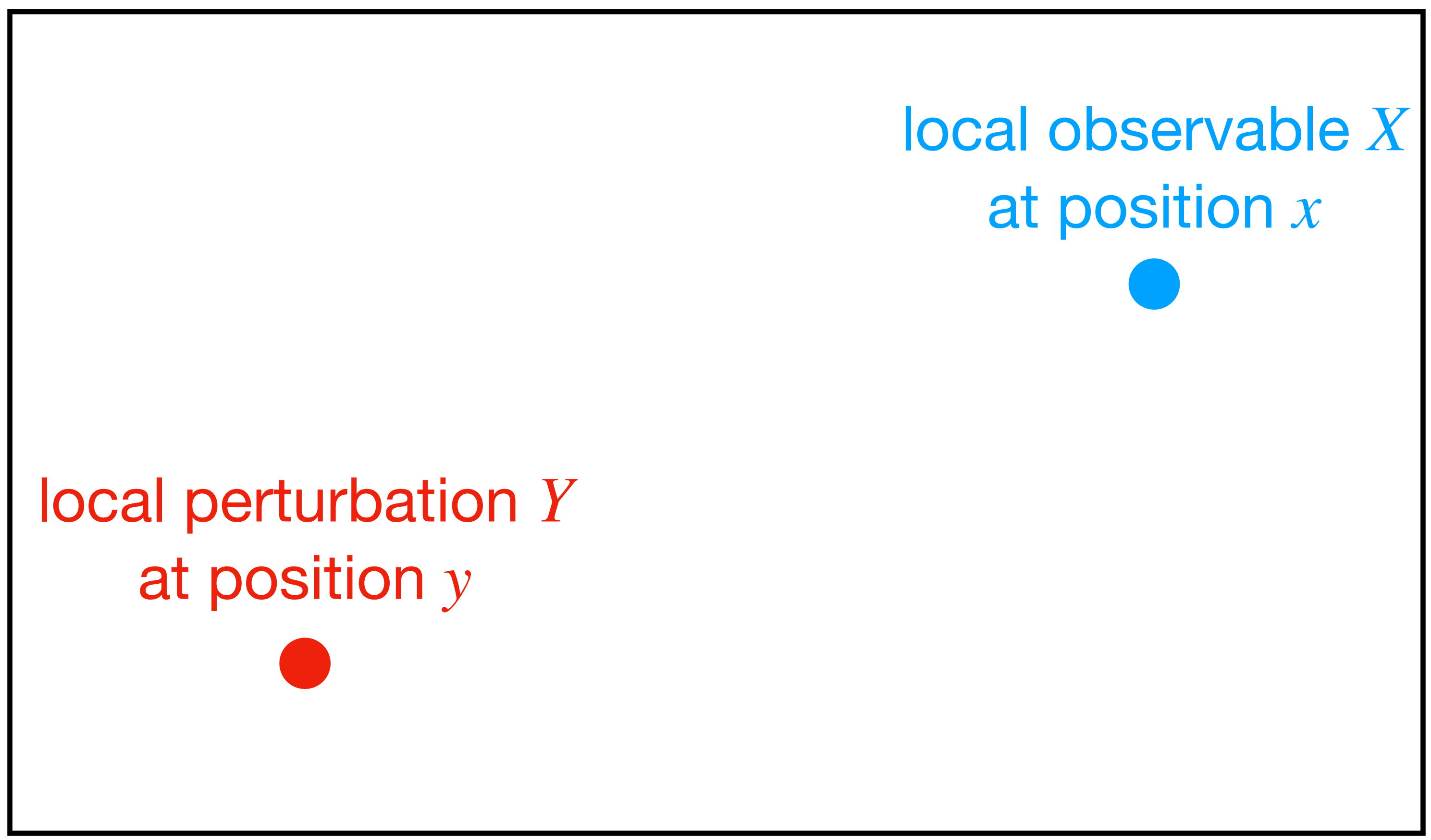
It turns out that $U(t) = \mathcal{T} e^{-\frac{i}{\hbar}\int_0^t ds\lambda(s)Y_0(s)}$, with $Y_0(t) = e^{\frac{i}{\hbar}H_0 t} Y e^{-\frac{i}{\hbar}H_0 t}$.

So $X(t) - X_0(t) = (1 + \frac{i}{\hbar}\int_0^t ds\lambda(s)Y_0(s))X_0(t)(1 - \frac{i}{\hbar}\int_0^t ds\lambda(s)Y_0(s)) - X_0(t) + \mathcal{O}(\lambda^2)$

$$= \frac{i}{\hbar}\int_0^t ds[Y_0(s), X_0(t)]\lambda(s) + \mathcal{O}(\lambda^2).$$

If the fermion parity doesn't have to be conserved, $i[Y_0(s), X_0(t)]$ can be nonzero for arbitrarily small but finite t.

system without fermion parity conservation



Example:

$$X = f_x + f_x^\dagger,\ Y = f_y^\dagger f_y,\ H_0 = g(f_y + f_y^\dagger),$$

$$\Rightarrow i[Y_0(s), X_0(t)] = \frac{1}{4} \sin \frac{2g(t-s)}{\hbar} (f_x - f_x^\dagger)(1 - 2f_x^\dagger f_x)(f_y - f_y^\dagger).$$

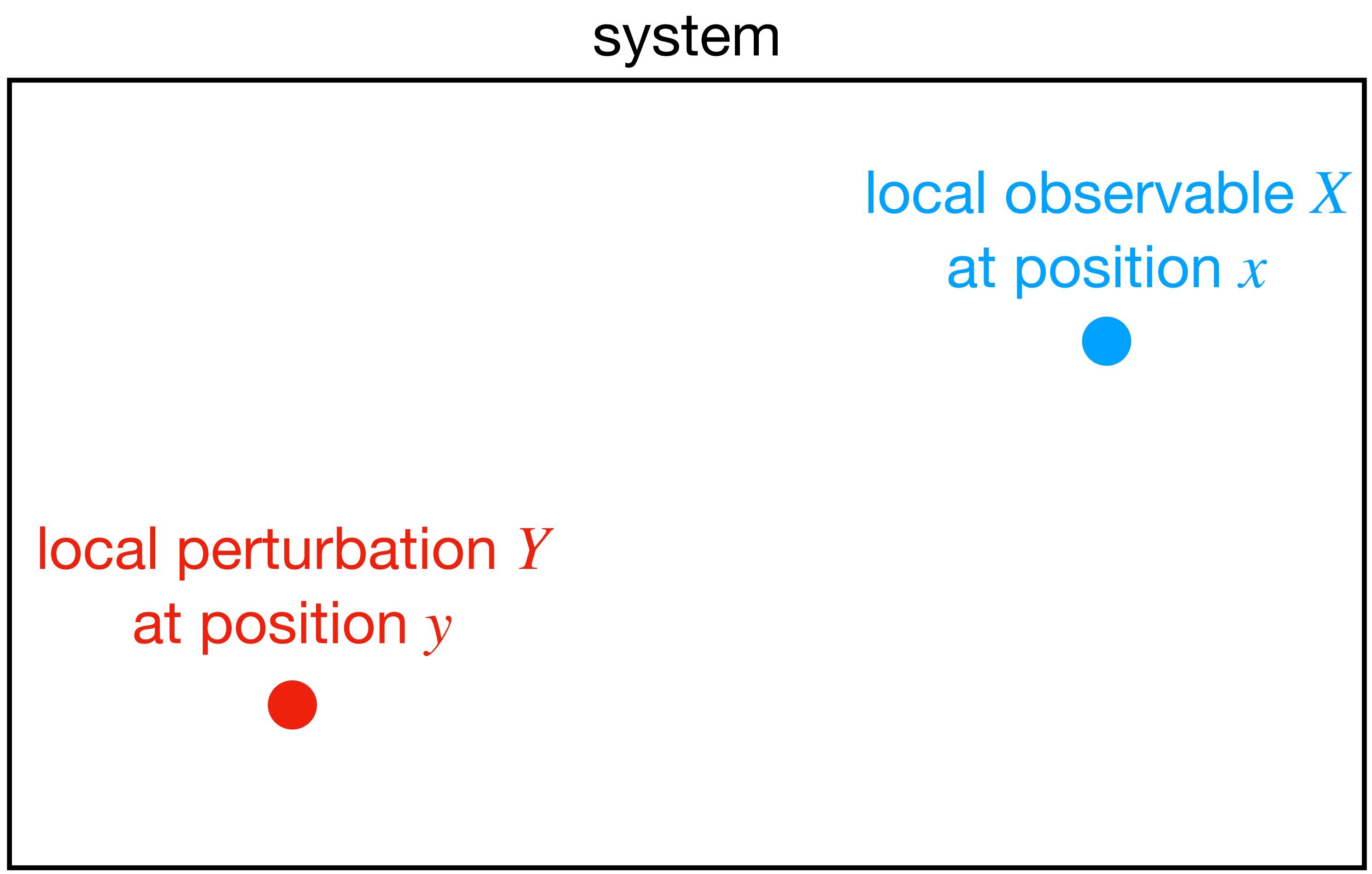
Remarks

- The expectation value of $i[Y_0(s), X_0(t)]$ is called the retarded Green's function. It is only defined when $s \leqslant t$, reflecting causality (i.e., only the past, but not the future, can influence the present).

- The instantaneous signal transmission is due to the fact that local terms in the Hamiltonian do not commute with local operators far away. To forbid instantaneous signal transmission, all local terms must commute with local operators far away.

- Boson parity does not have to be conserved.

Even without going through the calculations, the physical picture should be understood.



Linear response to perturbation:

$$X(t) - X_0(t) = \frac{i}{\hbar}\int_0^t ds[Y_0(s), X_0(t)]\lambda(s) + \mathcal{O}(\lambda^2)$$

Locality:

$[Y_0(s), X_0(t)]$ is negligible for small s and t

see Chen, Lucas, Yin, arXiv: 2303.07386 for a review on
speed limits and locality in many-body quantum dynamics

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This part starts with philosophical and abstract reasoning,

and ends up with mathematically precise and concrete statements.

You only need to understand this part at the intuitive level.

However, the final physical results are important and should be understood,

which justify our previous treatments of localized qubits.

Fundamental particles are all bosons or fermions. Then what are localized qubits?

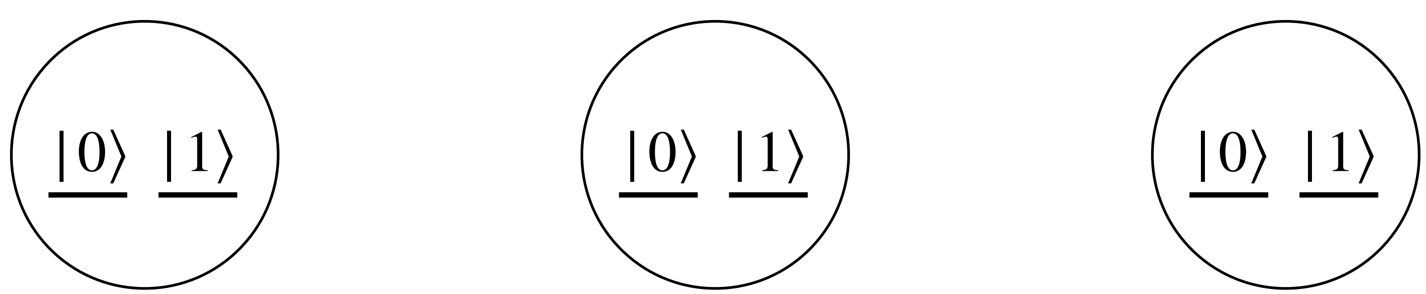
basis states of localized qubits: $|0000\cdots0\rangle,\ |1000\cdots0\rangle,\ |0110\cdots0\rangle,\cdots$

basis states of bosons: $|vac\rangle,\ b_1^\dagger|vac\rangle, b_2^\dagger|vac\rangle,\cdots,(b_1^\dagger)^2|vac\rangle, b_1^\dagger b_2^\dagger|vac\rangle,\cdots$

basis states of fermions: $|vac\rangle,\ f_1^\dagger|vac\rangle, f_2^\dagger|vac\rangle,\cdots, f_1^\dagger f_2^\dagger|vac\rangle,\cdots$

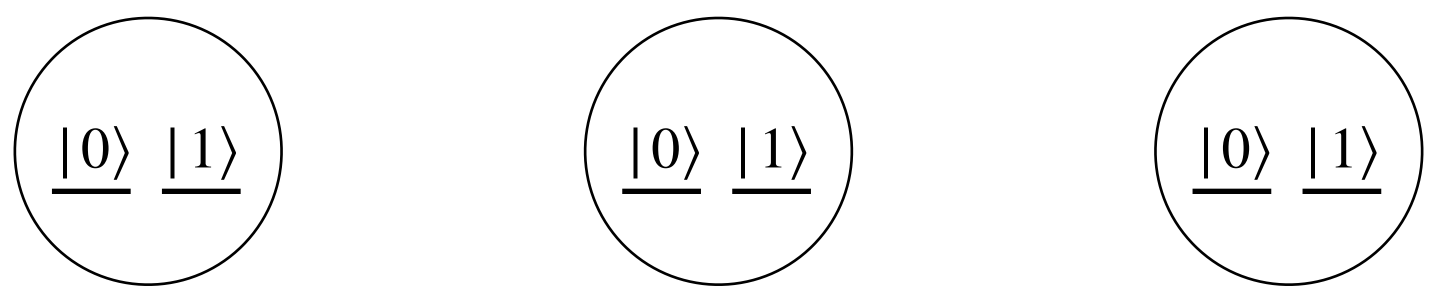
Microscopically, localized qubits come from localized 2-state particles.

localized 2-state particle $\Rightarrow$ qubit	localized 2-state particle $\Rightarrow$ qubit	localized 2-state particle $\Rightarrow$ qubit



Localized qubits are also identical. Should they be symmetrized or anti-symmetrized, depending on nature of the underlying particles?

localized 2-state particle $\Rightarrow$ qubit	localized 2-state particle $\Rightarrow$ qubit	localized 2-state particle $\Rightarrow$ qubit



Short answer:

Bosons, fermions and even distinguishable particles
make no “fundamental difference” if they are localized and form the qubits.
Namely, after the bosons and fermions are localized,
their indistinguishability does not matter any more.

Philosophically,
when can 2 systems or setups
be viewed as equivalent?

What is a system? What is a setup?

Philosophically, when can 2 systems or setups be viewed as equivalent?

What is a system? What is a setup?

a system ≈ a collection of dynamically evolving observables
a setup ≈ a set of systems that can be converted into each other

a quantum system $\approx \{\{O\}, |\psi\rangle, U\}$
$\{O\}$: set of observables, $|\psi\rangle$: state, U: time evolution operator

a quantum setup $\approx \{\{O\}, \{|\psi\rangle\}, \{U\}\}$
$\{O\}$: set of observables, $\{|\psi\rangle\}$: set of states, $\{U\}$: set of time evolution operators

Example:

A qubit in state $|0\rangle$ with Hamiltonian $H = -hX$ is a quantum system.
A qubit without specifying its state or Hamiltonian is a quantum setup.

Two setups can be kinematically equivalent.
Two systems can be dynamically equivalent.

Definition 1:

Quantum setups A with data $\{\{O_A\},\{|\psi_A\rangle\},\{U_A\}\}$ and B with data $\{\{O_B\},\{|\psi_B\rangle\},\{U_B\}\}$ are kinematically equivalent, if there is an invertible map φ, such that for each $O_A, |\psi_A\rangle$, and U_A, the following holds.

- $\langle\psi_A|U_A^\dagger O_A U_A|\psi_A\rangle = (\varphi|\psi_A\rangle)^\dagger(\varphi(U_A))^\dagger\varphi(O_A)(\varphi(U_A))(\varphi|\psi_A\rangle)$: same observables at all times
- $\varphi(O_{A,\vec{r}}) = O_{B,\vec{r}}$: same location

Definition 2:

Quantum systems A and B are dynamically equivalent if they belong to kinematically equivalent quantum setups with the above map φ, such that the following holds.

- $|\psi_B\rangle = \varphi(|\psi_A\rangle)$: same initial state.
- $U_B = \varphi(U_A)$: same Hamiltonian.

Two point-like setups are kinematically equivalent if they have the same Hilbert space dimension.

Example:

Two localized particles, each with d internal states, are kinematically equivalent.

Justification:

observables in particle A: $O_A = \sum_{i,j} O_{ij} |i\rangle\langle j|_A$ observables in particle B: $O_B = \sum_{ij} O'_{ij} |i\rangle\langle j|_B$

- Checking the existence of a invertible map: the "identity" map, i.e.,

$$\varphi(\sum_{ij} O_{ij} |i\rangle\langle j|_A) = \sum_{ij} O_{ij} |i\rangle\langle j|_B, \varphi(\sum_i \psi_i |i_A\rangle) = \sum_i \psi_i |i_B\rangle, \varphi(\sum_{ij} U_{ij} |i_A\rangle\langle j_A|) = \sum_{ij} U_{ij} |i_B\rangle\langle j_B|$$

- Checking the condition on the "same observables at all times":

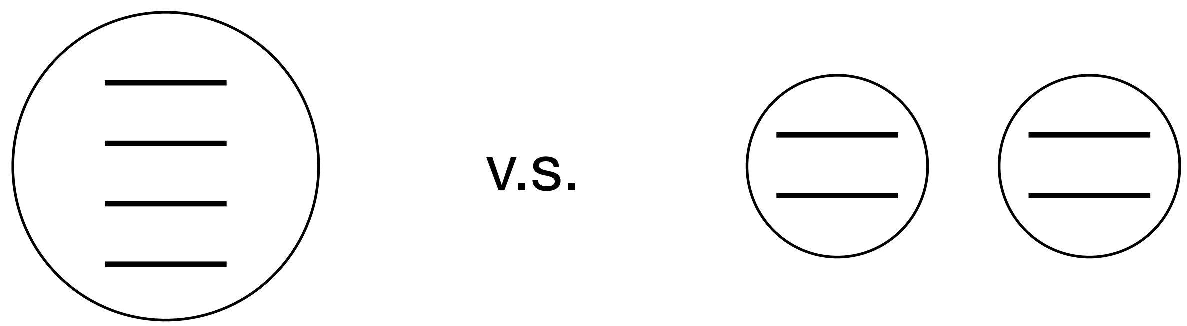
$$(\varphi |\psi_A\rangle)^\dagger(\varphi(U_A))^\dagger \varphi(O_A)(\varphi(U_A))(\varphi |\psi_A\rangle) = \sum_{ijkl} \psi_i^* U_{ij}^* O_{jk} U_{kl} \psi_l = \langle \psi_A | U_A^\dagger O_A U_A | \psi_A\rangle$$

Remark:

Clearly, invertible map φ does not exist for two setups with different Hilbert space dimensions.

Locality is irrelevant for point-like setups, but it is important whenever the locations of degrees of freedom can be resolved.

Question:
Should a particle with 4 internal states and two particles each with 2 internal states be viewed as kinematically equivalent?



Answer:
No! We should be able to resolve the locations of the 2 particles and observe them individually.

Remark:
Whenever the locations of the degrees of freedom can be resolved, we need to check if $\varphi(O_{A,\vec{r}})$ is at location $\vec{r}$.

Next, we will verify that

localized 2-state bosons, localized 2-state fermions,

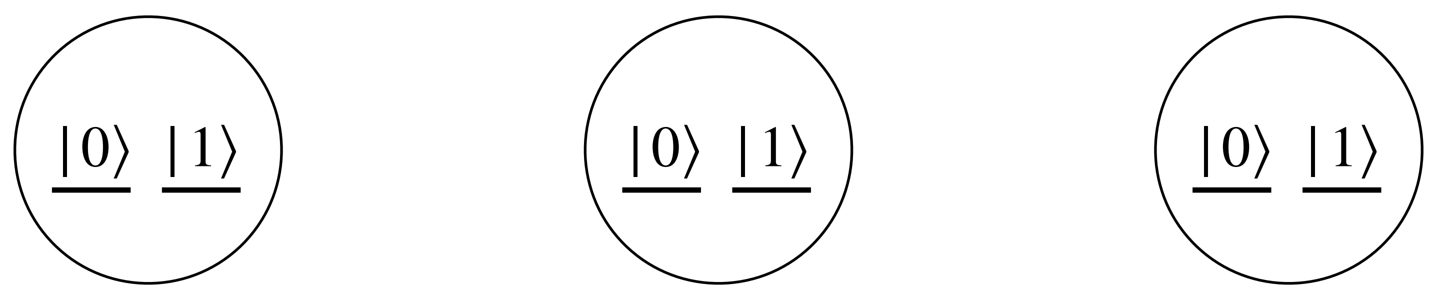
localized 2-state distinguishable particles and localized qubits

are all kinematically equivalent.

So our previous treatments of localized qubits are correct.

If each of the 3 sites hosts a localized boson, some states and operators should be disregarded.

localized 2-state boson $\Rightarrow$ qubit	localized 2-state boson $\Rightarrow$ qubit	localized 2-state boson $\Rightarrow$ qubit



basis of all states: $|vac\rangle, b^\dagger_{x_1,0}|vac\rangle, b^\dagger_{x_2,1}|vac\rangle, \cdots, (b^\dagger_{x_1,0})^2|vac\rangle, b^\dagger_{x_1,0}b^\dagger_{x_2,1}b^\dagger_{x_3,0}|vac\rangle, b^\dagger_{x_1,0}b^\dagger_{x_2,1}b^\dagger_{x_3,1}|vac\rangle\cdots$

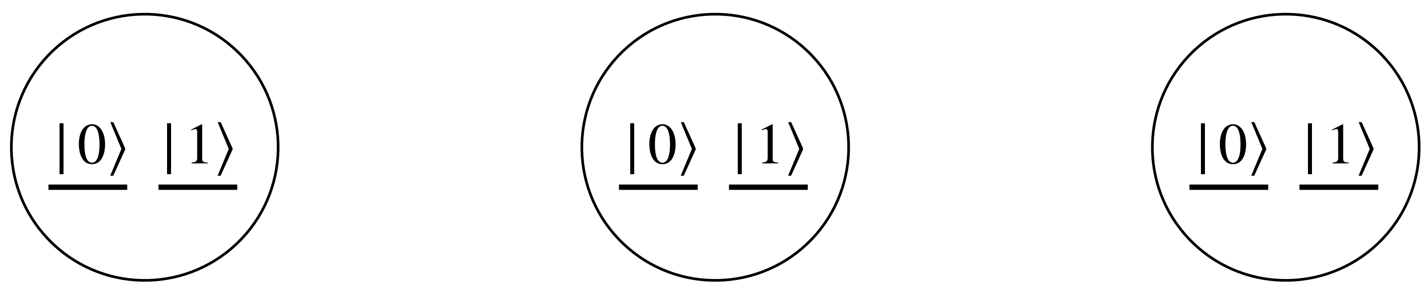
all operators: $b^\dagger_{x_1,0}, b_{x_2,1}, \cdots, b^\dagger_{x_1,0}b^\dagger_{x_2,0}, \cdots, b^\dagger_{x_1,0}b_{x_2,0}, b^\dagger_{x_1,0}b_{x_1,1}, b^\dagger_{x_2,1}b_{x_2,0}, \cdots$

basis of states for localized bosons: $b^\dagger_{x_1,0}b^\dagger_{x_2,1}b^\dagger_{x_3,0}|vac\rangle, b^\dagger_{x_1,0}b^\dagger_{x_2,1}b^\dagger_{x_3,1}|vac\rangle\cdots$

operators for localized bosons: $b^\dagger_{x_1,0}b_{x_1,1}, b^\dagger_{x_2,1}b_{x_2,0}, \cdots$

The remaining basis states and operators can be mapped into basis states and operators of qubits.

localized 2-state boson $\Rightarrow$ qubit	localized 2-state boson $\Rightarrow$ qubit	localized 2-state boson $\Rightarrow$ qubit



basis of states for localized bosons:
$b^{\dagger}_{x_1,0} b^{\dagger}_{x_2,1} b^{\dagger}_{x_3,0} |vac\rangle, b^{\dagger}_{x_1,0} b^{\dagger}_{x_2,1} b^{\dagger}_{x_3,1} |vac\rangle \cdots$ $\longrightarrow$ basis of states for qubits:
$|010\rangle, |011\rangle \cdots$

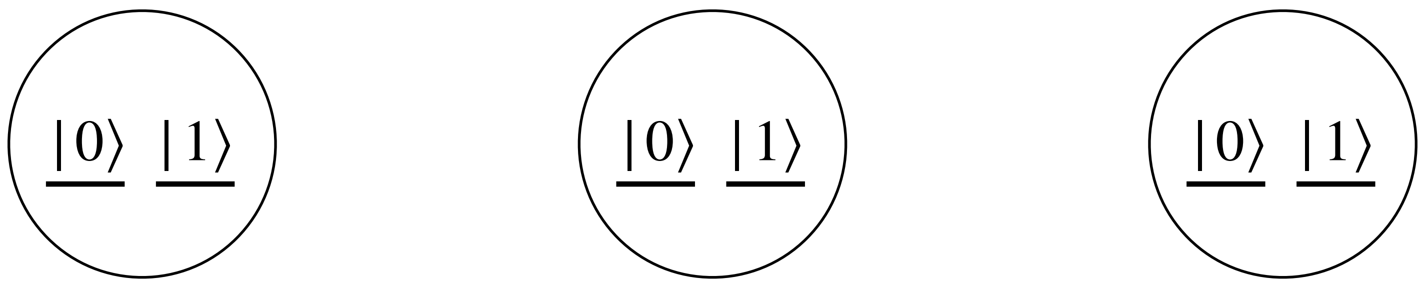
operators for localized bosons:
$b^{\dagger}_{x_1,0} b_{x_1,1}, b^{\dagger}_{x_2,1} b_{x_2,0}, \cdots$ $\longrightarrow$ operators for localized bosons:
$(|0\rangle\langle 1|) \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots, \mathbb{I} \otimes (|1\rangle\langle 0|) \otimes \mathbb{I} \otimes \cdots, \cdots$

Let's figure out the complete map!

- For a general 3-qubit basis state $\sum_{i_1 i_2 i_3} c_{i_1 i_2 i_3} | i_1 i_2 i_3 \rangle$, what is the corresponding bosonic state?
- For a general 3-qubit basis operator $\sum O_{i_1 i_2 i_3, j_1 j_2 j_3} | i_1 i_2 i_3 \rangle \langle j_1 j_2 j_3 |$, what is the corresponding bosonic operator?
- Is this map invertible and locality-preserving?
- Does this map preserve the observables at all times?

If each of the 3 sites hosts a localized fermion, some states and operators should be disregarded.

localized 2-state fermion $\Rightarrow$ qubit | localized 2-state fermion $\Rightarrow$ qubit | localized 2-state fermion $\Rightarrow$ qubit



basis of all states: $|vac\rangle, f^\dagger_{x_1,0}|vac\rangle, f^\dagger_{x_2,1}|vac\rangle, \cdots, f^\dagger_{x_1,0}f^\dagger_{x_2,1}f^\dagger_{x_3,0}|vac\rangle, f^\dagger_{x_1,0}f^\dagger_{x_2,1}f^\dagger_{x_3,1}|vac\rangle\cdots$

all operators: $f^\dagger_{x_1,0}, f_{x_2,1}, \cdots, f^\dagger_{x_1,0}f^\dagger_{x_2,0}, \cdots, f^\dagger_{x_1,0}f_{x_2,0}, f^\dagger_{x_1,0}f_{x_1,1}, f^\dagger_{x_2,1}f_{x_2,0}, \cdots$

basis of states for localized bosons: $f^\dagger_{x_1,0}f^\dagger_{x_2,1}f^\dagger_{x_3,0}|vac\rangle, f^\dagger_{x_1,0}f^\dagger_{x_2,1}f^\dagger_{x_3,1}|vac\rangle\cdots$

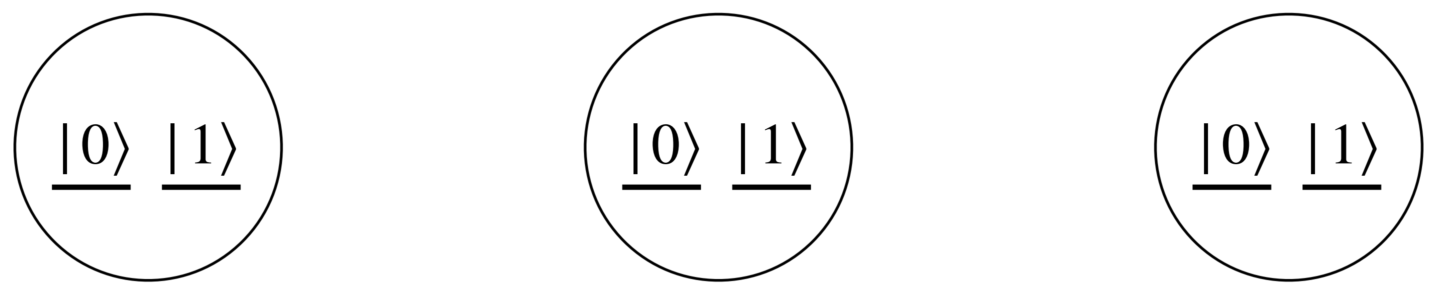
operators for localized bosons: $f^\dagger_{x_1,0}f_{x_1,1}, f^\dagger_{x_2,1}f_{x_2,0}, \cdots$

The basis states and operators can be mapped into basis states and operators of qubits.

localized
2-state fermion
⇒ qubit

localized
2-state fermion
⇒ qubit

localized
2-state fermion
⇒ qubit



basis of states for localized fermions:
$f^\dagger_{x_1,0} f^\dagger_{x_2,1} f^\dagger_{x_3,0} |vac\rangle, f^\dagger_{x_1,0} f^\dagger_{x_2,1} f^\dagger_{x_3,1} |vac\rangle \cdots$

$\longrightarrow$

basis of states for qubits:
$|010\rangle, |011\rangle \cdots$

operators for localized fermions:
$f^\dagger_{x_1,0} f_{x_1,1}, f^\dagger_{x_2,1} f_{x_2,0}, \cdots$

$\longrightarrow$

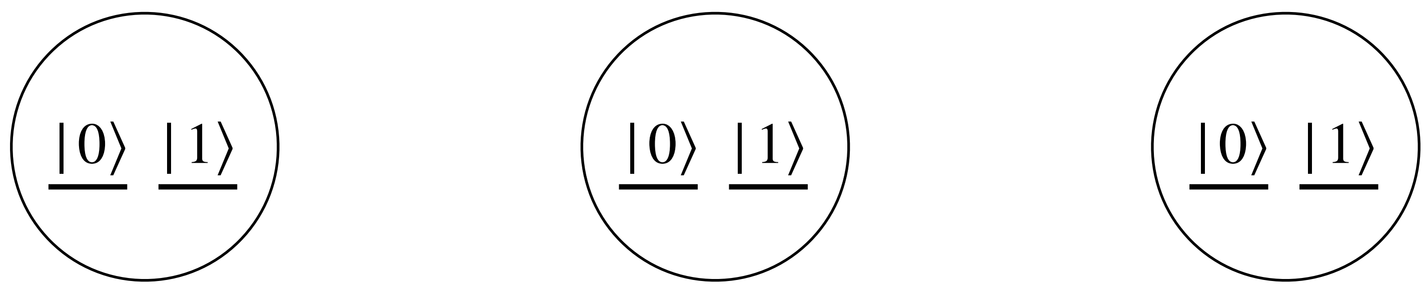
operators for localized bosons:
$(|0\rangle\langle 1|) \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots, \mathbb{1} \otimes (|1\rangle\langle 0|) \otimes \mathbb{1} \otimes \cdots, \cdots$

Let’s figure out the complete map!

- For a general 3-qubit basis state $\sum_{i_1 i_2 i_3} c_{i_1 i_2 i_3} | i_1 i_2 i_3 \rangle$, what is the corresponding fermionic state?
- For a general 3-qubit basis operator $\sum O_{i_1 i_2 i_3, j_1 j_2 j_3} | i_1 i_2 i_3 \rangle \langle j_1 j_2 j_3 |$, what is the corresponding fermionic operator?
- Is this map invertible and locality-preserving?
- Does this map preserve the observables at all times?

If each site hosts a different type of 2-state particle, the mapping to qubits still holds.

localized 2-state particle $\Rightarrow$ qubit	localized 2-state particle $\Rightarrow$ qubit	localized 2-state particle $\Rightarrow$ qubit



basis of states for localized particle:
$a^\dagger_{x_1,0}a^\dagger_{x_2,1}a^\dagger_{x_3,0}|vac\rangle, a^\dagger_{x_1,0}a^\dagger_{x_2,1}a^\dagger_{x_3,1}|vac\rangle\cdots$ $\longrightarrow$ basis of states for qubits:
$|010\rangle, |011\rangle\cdots$

operators for localized fermions:
$a^\dagger_{x_1,0}a_{x_1,1}, a^\dagger_{x_2,1}a_{x_2,0}, \cdots$ $\longrightarrow$ operators for localized bosons:
$(|0\rangle\langle 1|)\otimes\mathbb{I}\otimes\mathbb{I}\otimes\cdots, \mathbb{I}\otimes(|1\rangle\langle 0|)\otimes\mathbb{I}\otimes\cdots, \cdots$

Localized 2-state particles and localized qubits are kinematically equivalent.

- Localized 2-state bosons and fermions are both kinematically equivalent to localized qubits.

- Even if the localized 2-state particles are all distinguishable, this system is kinematically equivalent to localized qubits. So particles' indistinguishability plays no role in this context.

- Our previous treatments of localized qubits are correct. In particular, there is no need to symmetrize or anti-symmetrize the states of localized qubits.

- In general, whether particles' indistinguishability is important should be analyzed based on the context.

Please submit assignment 3 by 17:00 next Friday.

Please read Sections 4.3 and 4.4 of Griffiths

and Sections 3.1-3.3, 3.5, and 3.6 of Sakurai.