Electrodynamics and Relativity

The Special Theory of Relativity

The Special Theory of Relativity

Postulates:

- **The principle of relativity**. The laws of physics apply in all inertial reference systems.
- **The universal speed of light**. The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

Special Theory of Relativity (cont'd)

- **Einstein's velocity addition rule**:

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC} / c^2)} \tag{12.1}$$

v_{AC}: speed of A relative to C, v_{AB}: speed of A relative to B, v_{BC}: speed of B relative to C

- **The relativity of simultaneity**: Two events that are simultaneous in one inertial system are not, in general, simultaneous in another.

- Define

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{12.2}$$

Special Theory of Relativity (cont'd)

- **Time dilation**: Moving clocks run slowly than stationary ones

$$\Delta\bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t = \frac{1}{\gamma}\Delta t \tag{12.3}$$

$\Delta\bar{t}$: time interval measured by an observer in his/her own rest frame (proper time)

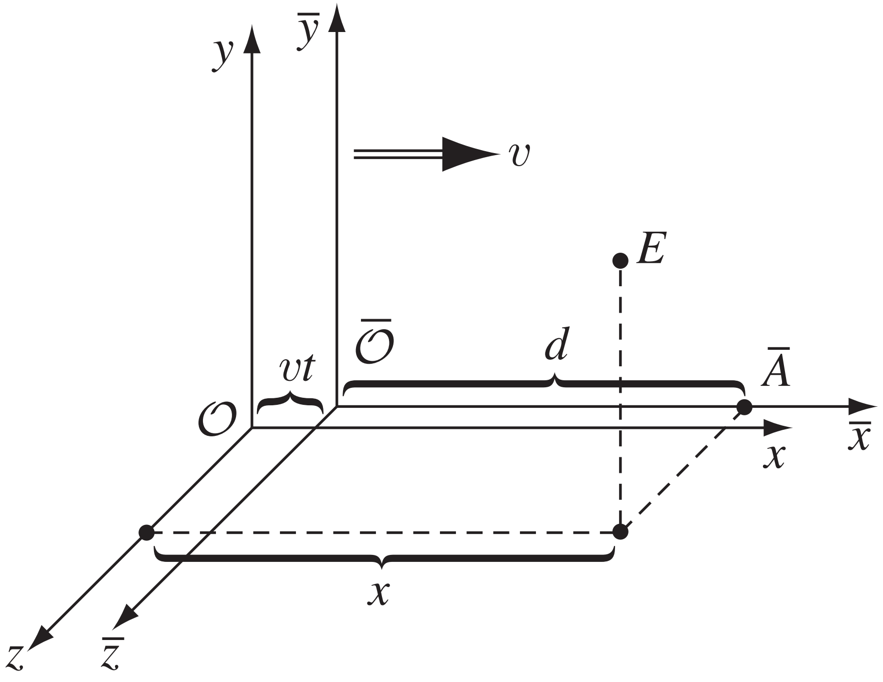
- **Lorentz contraction**: Moving objects are shortened

$$\Delta\bar{x} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x = \gamma\Delta x \tag{12.4}$$

$\Delta\bar{x}$: length measured in the rest frame of the moving object.

- Moving clocks run slow, moving sticks are shortened, and the factor is always γ.
- A moving object is shortened only along the direction of its motion: Dimensions perpendicular to the velocity are *not* contracted.

The Lorentz Transformations



Any physical process consists of one or more events. An "event" is something that takes place at a specific location (x, y, z), at a precise time (t).

Given coordinates (x, y, z, t) of a particular event E in one inertial system $\mathcal{S}$, the coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ of that *same event* in some other inertial system $\bar{\mathcal{S}}$:

$$\begin{aligned} &\text{(i)} \quad \bar{x} = \gamma(x - vt) \\ &\text{(ii)} \quad \bar{y} = y \\ &\text{(iii)} \quad \bar{z} = z \\ &\text{(iv)} \quad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned} \tag{12.5}$$

The Lorentz transformations.

The Lorentz Transformations (cont'd)

We also have

$$
\begin{aligned}
&\text{(i')} \quad x = \gamma(\bar{x} + v\bar{t}) \\
&\text{(ii')} \quad y = \bar{y} \\
&\text{(iii')} \quad z = \bar{z} \\
&\text{(iv')} \quad t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)
\end{aligned}
\tag{12.6}
$$

Four-vectors

Four-vectors

Define the following quantities:

$$x^0 \equiv ct, \quad \beta \equiv \frac{v}{c} \tag{12.7}$$

x, y, z coordinates are numbered, so that

$$x^1 = x, \quad x^2 = y, \quad x^3 = z \tag{12.8}$$

then the Lorentz transformations read

$$\left.\begin{aligned} \bar{x}^0 &= \gamma(x^0 - \beta x^1) \\ \bar{x}^1 &= \gamma(x^1 - \beta x^0) \\ \bar{x}^2 &= x^2 \\ \bar{x}^3 &= x^3 \end{aligned}\right\} \tag{12.9}$$

Four-vectors (cont'd)

Or, in matrix form:

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{12.10}$$

Letting Greek indices run from 0 to 3, it can be written into a single equation:

$$\bar{x}^\mu = \sum_{\nu=0}^{3} (\Lambda^\mu_\nu)\, x^\nu \tag{12.11}$$

where Λ is the Lorentz transformation matrix in Eq. (12.10).

The superscript μ labels the row, the subscript ν labels the column.

Four-vectors (cont'd)

A **4-vector** is defined as any set of four components that transform like (x^0, x^1, x^2, x^3) under Lorentz transformations:

$$\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu \, a^\nu \tag{12.12}$$

For the particular case of a transformation along the x axis:

$$\left.\begin{aligned} \bar{a}^0 &= \gamma(a^0 - \beta a^1) \\ \bar{a}^1 &= \gamma(a^1 - \beta a^0) \\ \bar{a}^2 &= a^2 \\ \bar{a}^3 &= a^3 \end{aligned}\right\} \tag{12.13}$$

Four-vectors (cont'd)

There is a 4-vector analog to the dot product ($\boldsymbol{A} \cdot \boldsymbol{B} \equiv A_x B_x + A_y B_y + A_z B_z$). However, the zeroth components have a minus sign:

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{12.14}$$

This is the **four-dimensional scalar product**. It has the same value in all inertial systems:

$$-\bar{a}^0 \bar{b}^0 + \bar{a}^1 \bar{b}^1 + \bar{a}^2 \bar{b}^2 + \bar{a}^3 \bar{b}^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{12.15}$$

Just as the ordinary dot product is invariant (unchanged) under rotations, this combination is invariant under Lorentz transformations.

Four-vectors (cont'd)

To keep track of the minus sign, we introduce the **covariant** vector a_μ which differs from the **contravariant** a^μ only in the sign of the zeroth component:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3) \tag{12.16}$$

Upper indices designate contravariant vectors; lower indices are for covariant vectors. Raising or lowering the temporal index costs a minus sign ($a_0 = -a^0$); raising or lowering a spatial index changes nothing ($a_1 = a^1$, $a_2 = a^2$, $a_3 = a^3$).

Formally,

$$a_\mu = \sum_{\nu=0}^{3} g_{\mu\nu}\, a^\nu, \quad \text{where} \quad g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12.17}$$

is the **Minkowski metric**.

Four-vectors (cont'd)

The scalar product can now be written with the summation symbol,

$$\sum_{\mu=0}^{3} a_\mu b^\mu \tag{12.18}$$

or

$$a_\mu b^\mu \quad (= a^\mu b_\mu) \tag{12.19}$$

Summation is implied whenever a Greek index is repeated in a product–once as a covariant index and once as contravariant. This is called the **Einstein summation convention**.

The invariant interval

The invariant interval

The scalar product of a 4-vector with itself, $a^\mu a_\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$, can be positive (if the "spatial" terms dominate) or negative (if the "temporal" term dominates) or zero:

If $a^\mu a_\mu > 0$, a^μ is called **spacelike**
If $a^\mu a_\mu < 0$, a^μ is called **timelike**
If $a^\mu a_\mu = 0$, a^μ is called **lightlike**

Suppose event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$, and event B at $(x_B^0, x_B^1, x_B^2, x_B^3)$. The difference,

$$\Delta x^\mu \equiv x_A^\mu - x_B^\mu \tag{12.20}$$

is the **displacement 4-vector**.

The invariant interval (cont'd)

The scalar product of Δx^μ with itself–the **interval** between two events–is a quantity of special importance:

$$I \equiv (\Delta x)_\mu (\Delta x)^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2 \tag{12.21}$$

where t is the time difference between the two events and d is their spatial separation. When you transform to a moving system, the *time* between A and B is altered ($\bar{t} \neq t$), and so is the *spatial separation* ($\bar{d} \neq d$), but the interval I remains the same.

- If interval between two events is timelike, there exists an inertial system (accessible by Lorentz transformation) in which they occur at same point
- If the interval is spacelike, then there exists a system in which the two events occur at the same time
- If the displacement is lightlike, then the two events could be connected by a light signal

Relativistic Mechanics

Proper Velocity

Define

$$\textbf{ordinary velocity } \boldsymbol{u} \equiv \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} \tag{12.22}$$

$$\textbf{proper velocity } \boldsymbol{\eta} \equiv \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}\tau} \tag{12.23}$$

where

$$\mathrm{d}\tau = \sqrt{1 - u^2/c^2}\,\mathrm{d}t \tag{12.24}$$

is the proper time.

We shall reserve v for the relative velocity of two inertial systems.

Proper Velocity (cont'd)

The two velocities are related by Eq. (12.24):

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1 - u^2/c^2}}\boldsymbol{u} \tag{12.25}$$

Proper velocity transforms simply, when you go from one inertial system to another. In fact, η is the spatial part of a 4-vector,

$$\eta^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{12.26}$$

whose zeroth component is

$$\eta^0 = \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{c}{\sqrt{1 - u^2/c^2}} \tag{12.27}$$

Proper Velocity (cont'd)

For the numerator, $\mathrm{d}x^{\mu}$, is a displacement 4-vector, while the denominator, $\mathrm{d}\tau$, is invariant. Thus, when you go from system $\mathcal{S}$ to system $\bar{\mathcal{S}}$, moving at speed v along the common $x\bar{x}$ axis,

$$\left.\begin{aligned} \bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1) \\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0) \\ \bar{\eta}^2 &= \eta^2 \\ \bar{\eta}^3 &= \eta^3 \end{aligned}\right\} \tag{12.28}$$

More generally,

$$\bar{\eta}^{\mu} = \Lambda^{\mu}_{\nu}\, \eta^{\nu} \tag{12.29}$$

η^{μ} is called the **proper velocity 4-vector**, or simply the **4-velocity**.

Proper Velocity (cont'd)

The transformation rule for ordinary velocities is:

$$\left.\begin{aligned} \bar{u}_x &= \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)} \\ \bar{u}_y &= \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)} \\ \bar{u}_z &= \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)} \end{aligned}\right\} \tag{12.30}$$

Relativistic Energy and Momentum

Define **relativistic momentum**

$$\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \qquad (12.31)$$

Relativistic momentum is the spatial part of a 4-vector,

$$p^\mu \equiv m\eta^\mu \qquad (12.32)$$

What does the temporal component,

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-u^2/c^2}} \qquad (12.33)$$

represent?

Einstein identified $p^0 c$ as **relativistic energy**:

$$E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad (12.34)$$

p^μ is called the **energy-momentum 4-vector** (or the **momentum 4-vector**)

Relativistic Energy and Momentum (cont'd)

The relativistic energy is nonzero even when the object is stationary; we call this **rest energy**:

$$E_{\text{rest}} \equiv mc^2 \tag{12.35}$$

The remainder, which is attributable to the motion, we call **kinetic energy**

$$E_{\text{kin}} \equiv E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right) \tag{12.36}$$

In the nonrelativistic regime ($u \ll c$) the square root can be expanded in powers of u^2/c^2, giving

$$E_{\text{kin}} = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots \tag{12.37}$$

E and $\boldsymbol{p}$, as defined by Eqs. (12.31) and (12.34), are conserved:

In every closed system, the total relativistic energy and momentum are conserved.

- Distinction between an invariant quantity (same value in all inertial systems) and a conserved quantity (same value before and after some process).

- Mass is invariant, but not conserved; energy is conserved but not invariant; electric charge is both conserved *and* invariant; velocity is neither conserved nor invariant.

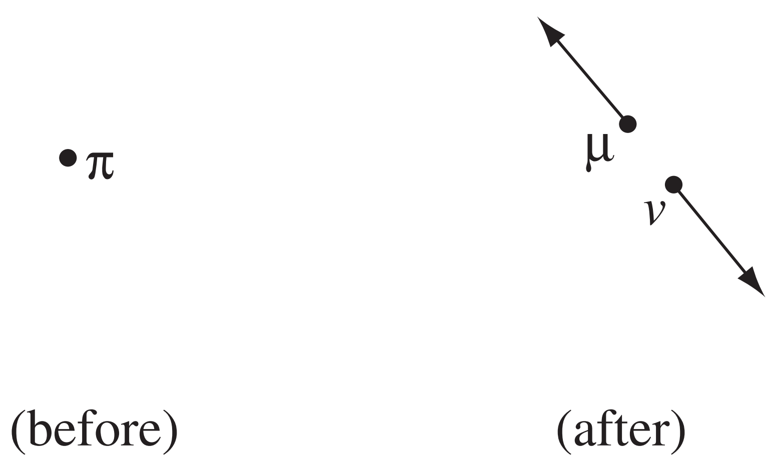
The scalar product of p^μ with itself is

$$p^\mu p_\mu = -(p^0)^2 + (\boldsymbol{p} \cdot \boldsymbol{p}) = -m^2c^2 \qquad (12.38)$$

In terms of the relativistic energy,

$$E^2 - p^2c^2 = m^2c^4 \qquad (12.39)$$

Example



A pion at rest decays into a muon and a neutrino. Find the energy of the outgoing muon, in terms of the two masses, m_π and m_μ (assume $m_\nu = 0$).

Solution

In this case

$$E_{\text{before}} = m_\pi c^2, \qquad \boldsymbol{p}_{\text{before}} = 0$$

$$E_{\text{after}} = E_\mu + E_\nu, \qquad \boldsymbol{p}_{\text{after}} = \boldsymbol{p}_\mu + \boldsymbol{p}_\nu$$

Example (cont'd)

Conservation of momentum requires that $\boldsymbol{p}_\nu = -\boldsymbol{p}_\mu$.

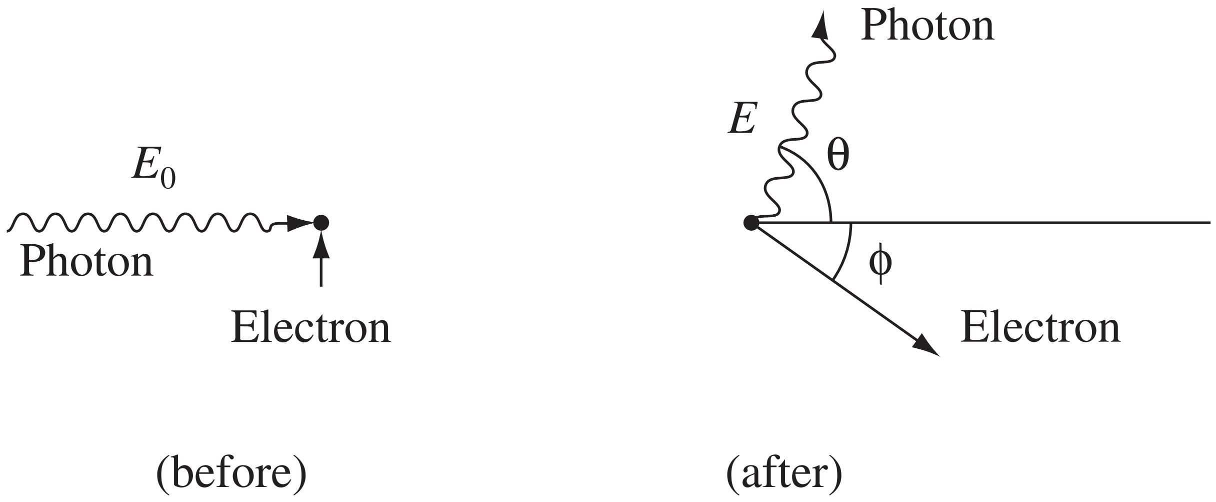
Conservation of energy says that

$$E_\mu + E_\nu = m_\pi c^2$$

Now, $E_\nu = |\boldsymbol{p}_\nu| c$, whereas $|\boldsymbol{p}_\mu| = \sqrt{E_\mu^2 - m_\mu^2 c^4}/c$, by Eq. (12.39), so

$$E_\mu + \sqrt{E_\mu^2 - m_\mu^2 c^4} = m_\pi c^2$$

$$\Rightarrow E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}$$



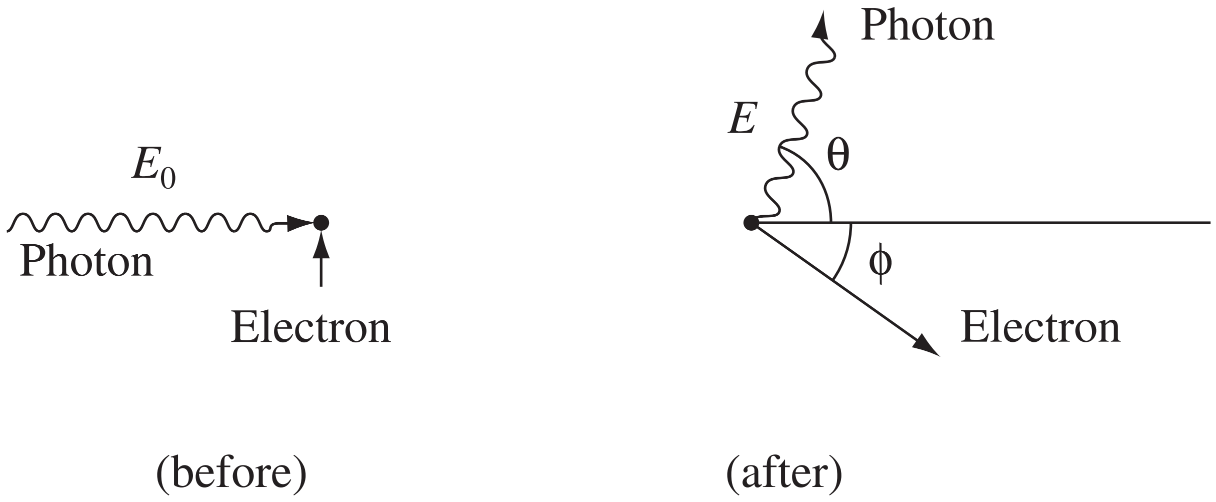
Compton scattering. A photon of energy E_0 “bounces” off an electron, initially at rest. Find the energy E of the outgoing photon, as a function of the scattering angle θ.

Solution

Conservation of momentum in the “vertical” direction gives $p_e \sin\phi = p_p \sin\theta$, or since $p_p = E/c$,

$$\sin\phi = \frac{E}{p_e c}\sin\theta$$

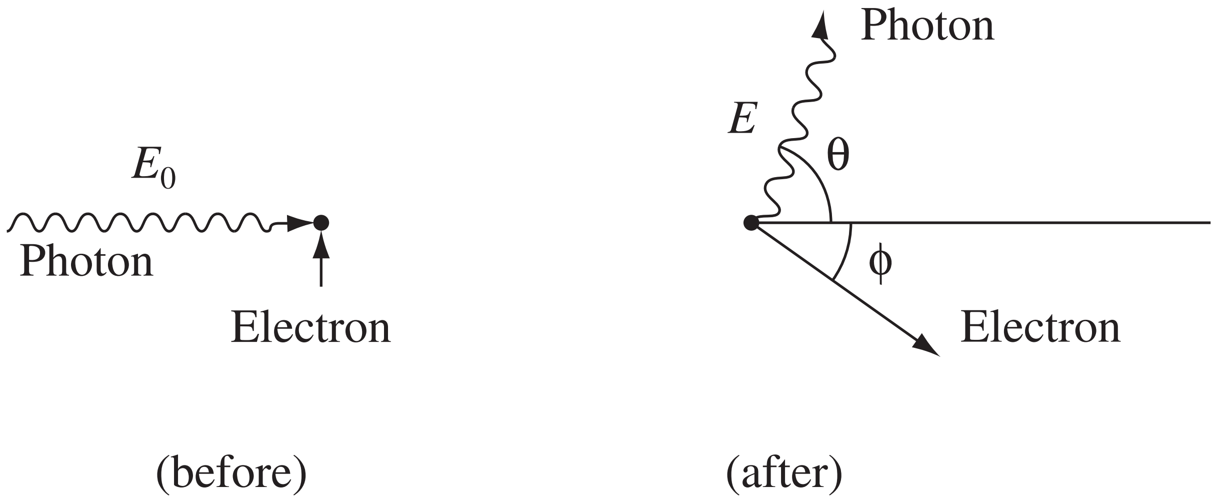
Example 2 (cont'd)



Conservation of momentum in the "horizontal" direction gives

$$\frac{E_0}{c} = p_p \cos\theta + p_e \cos\phi = \frac{E}{c}\cos\theta + p_e\sqrt{1 - \left(\frac{E}{p_e c}\sin\theta\right)^2}$$

or $p_e^2 c^2 = (E_0 - E\cos\theta)^2 + E^2 \sin^2\theta = E_0^2 - 2E_0 E\cos\theta + E^2$

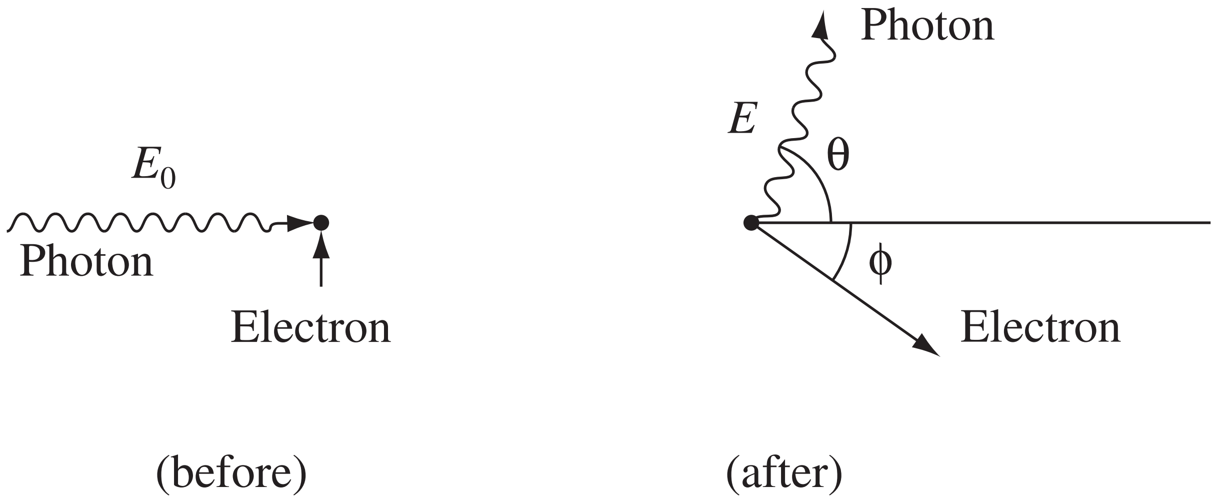


Finally, conservation of energy says that

$$E_0 + mc^2 = E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2} = E + \sqrt{m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2}$$

Solving for E,

$$E = \frac{1}{(1-\cos\theta)/mc^2 + (1/E_0)} \tag{12.40}$$



Expressed in terms of photon wavelength:

$$E = h\nu = \frac{hc}{\lambda}$$

so

$$\lambda = \lambda_0 + \frac{h}{mc}(1 - \cos\theta) \tag{12.41}$$

Relativistic Dynamics

Relativistic Dynamics

Newton's first law is built into the principle of relativity. His second law, in the form

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \tag{12.42}$$

retains its validity in relativistic mechanics, *provided we use the relativistic momentum*.

Example

Motion under a constant force. A particle of mass m is subject to a constant force F. If it starts from rest at the origin, at time $t = 0$, find its position (x), as a function of time.

Solution

$$\frac{\mathrm{d}p}{\mathrm{d}t} = F \Rightarrow p = Ft + \text{constant}$$

But since $p = 0$ at $t = 0$, the constant must be zero, and hence

$$p = \frac{mu}{\sqrt{1 - u^2/c^2}} = Ft$$

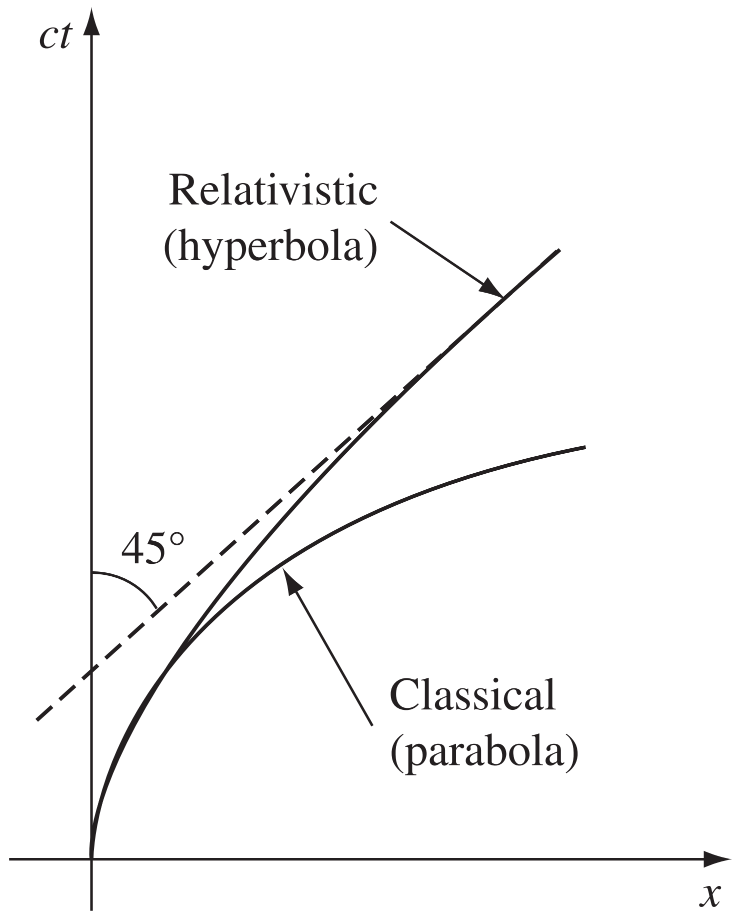
Solving for u, we obtain

$$u = \frac{(F/m)t}{\sqrt{1 + (Ft/mc)^2}} \tag{12.43}$$

Example (cont'd)

$$\therefore x(t) = \frac{F}{m}\int_0^t \frac{t'}{\sqrt{1+(Ft'/mc)^2}}\,\mathrm{d}t'$$
$$= \frac{mc^2}{F}\sqrt{1+(Ft'/mc)^2}\bigg|_0^t$$
$$= \frac{mc^2}{F}\left[\sqrt{1+(Ft/mc)^2}-1\right] \quad (12.44)$$

In place of the classical parabola, $x(t) = (F/2m)t^2$, the graph is a hyperbola; for this reason, motion under a constant force is often called **hyperbolic motion**.



It occurs, for example, when a charged particle is placed in a uniform electric field.

Work is the line integral of the force:

$$W \equiv \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} \tag{12.45}$$

The **work-energy theorem** (“the net work done on a particle equals the increase in its kinetic energy”) holds relativistically:

$$W = \int \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{l} = \int \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \boldsymbol{u}\,\mathrm{d}t$$

while

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \boldsymbol{u} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\boldsymbol{u}}{\sqrt{1-u^2/c^2}}\right) \cdot \boldsymbol{u} \tag{12.46}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{\mathrm{d}E}{\mathrm{d}t} \tag{12.47}$$

Work-Energy Theorem (cont'd)

So

$$W = \int \frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}t = E_{\mathrm{final}} - E_{\mathrm{initial}} \qquad (12.48)$$

Since the rest energy is constant, it does not matter whether we use the total energy, here, or the kinetic energy

Force transformation

Because $\boldsymbol{F}$ is the derivative of momentum with respect to ordinary time, both the numerator and the denominator must be transformed.

Thus,

$$\bar{F}_y = \frac{\mathrm{d}\bar{p}_y}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}p_y}{\gamma\left(\mathrm{d}t - \dfrac{\beta}{c}\,\mathrm{d}x\right)} = \frac{\mathrm{d}p_y/\,\mathrm{d}t}{\gamma\left(1 - \dfrac{\beta}{c}\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{F_y}{\gamma(1-\beta u_x/c)} \tag{12.49}$$

and similarly for the z component:

$$\bar{F}_z = \frac{F_z}{\gamma(1-\beta u_x/c)}$$

Force transformation (cont'd)

Now $\quad \bar{F}_x = \dfrac{\mathrm{d}\bar{p}_x}{\mathrm{d}\bar{t}} = \dfrac{\gamma\left(\mathrm{d}p_x - \beta\,\mathrm{d}p^0\right)}{\gamma\left(\mathrm{d}t - \dfrac{\beta}{c}\,\mathrm{d}x\right)} = \dfrac{\dfrac{\mathrm{d}p_x}{\mathrm{d}t} - \beta\dfrac{\mathrm{d}p^0}{\mathrm{d}t}}{1 - \dfrac{\beta}{c}\dfrac{\mathrm{d}x}{\mathrm{d}t}} = \dfrac{F_x - \dfrac{\beta}{c}\left(\dfrac{\mathrm{d}E}{\mathrm{d}t}\right)}{1 - \beta u_x/c}$

We calculated $\mathrm{d}E/\,\mathrm{d}t$ in Eq. (12.47); putting that in,

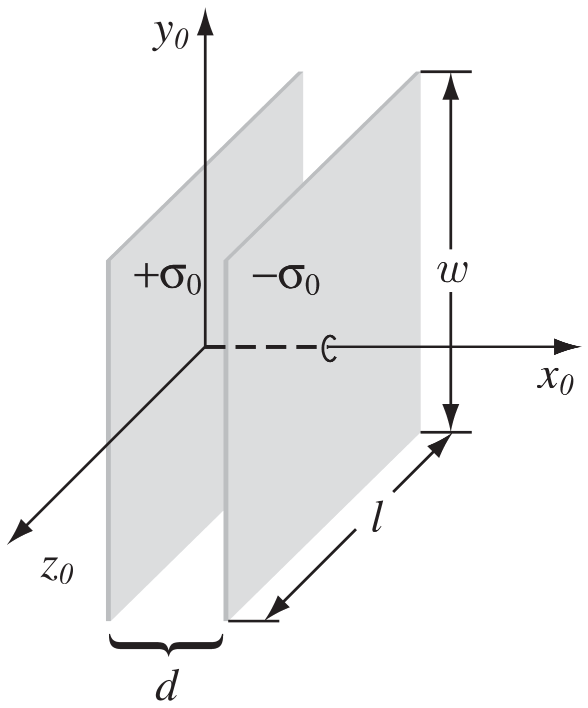
$$\bar{F}_x = \frac{F_x - \beta(\boldsymbol{u}\cdot\boldsymbol{F})/c}{1 - \beta u_x/c} \tag{12.50}$$

If the particle is (instantaneously) at rest in $\mathcal{S}$, so that $\boldsymbol{u} = \boldsymbol{0}$, then

$$\bar{\boldsymbol{F}}_\perp = \frac{1}{\gamma}\boldsymbol{F}_\perp, \quad \bar{F}_\parallel = F_\parallel \tag{12.51}$$

The component of $\boldsymbol{F}$ *parallel* to the motion of $\bar{\mathcal{S}}$ is unchanged, whereas components perpendicular are divided by γ.

Field Transformation



Now, we shall derive the transformation rules for electromagnetic fields: Given the fields in $\mathcal{S}_0$, what are the fields in $\mathcal{S}$?

Consider the simplest possible electric field: the uniform field in the region between the plates of a large parallel-plate capacitor

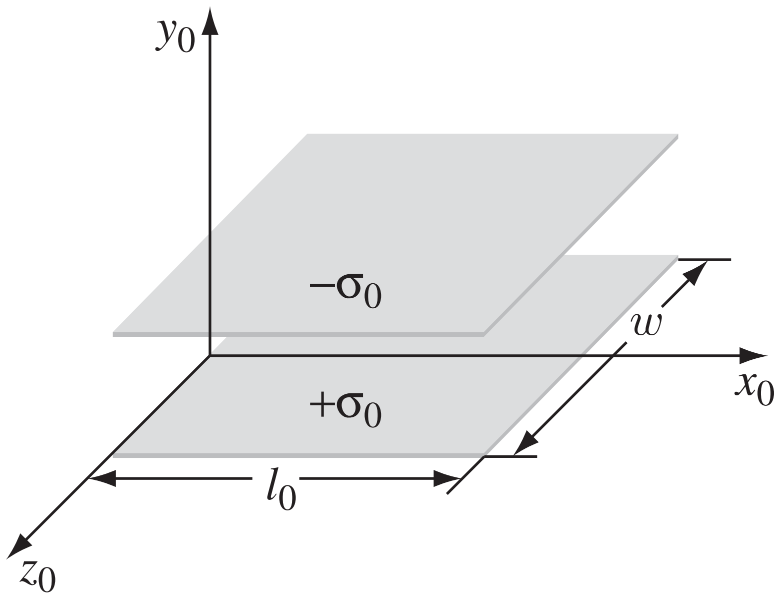
Say the capacitor is at rest in $\mathcal{S}_0$ and carries surface charges $\pm\sigma_0$. Then

$$\boldsymbol{E}_0 = \frac{\sigma_0}{\epsilon_0}\hat{\boldsymbol{x}}$$

But what if we examine this same capacitor from system $\mathcal{S}$, moving to the right at speed v_0?

In this system the plates are moving to the left. The plate separation (d) is Lorentz-contracted, whereas l and w (and σ) are the same in both frames. Since the field does not depend on d, we have

$$E^{\parallel} = E_0^{\parallel} \qquad (12.52)$$

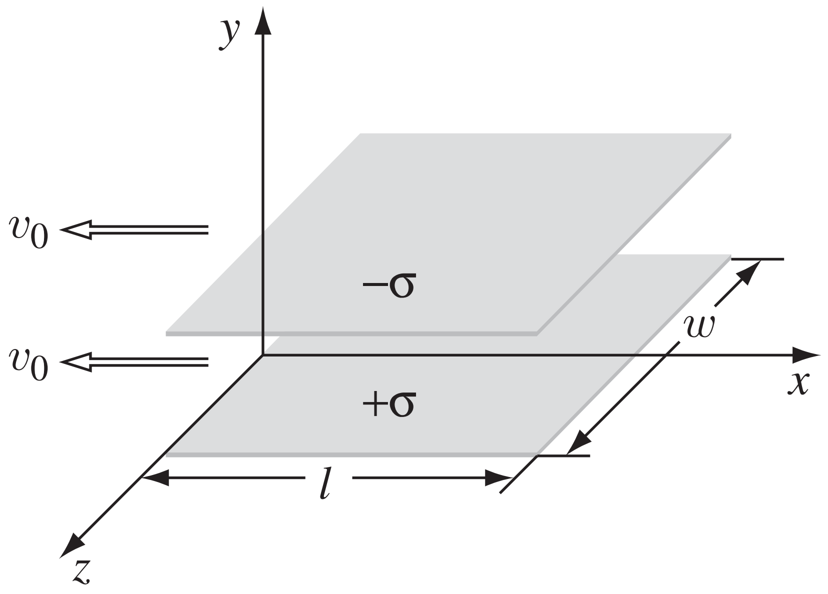


(a)

For perpendicular components, consider the capacitor lined up with the xz plane.

In $\mathcal{S}_0$,

$$\boldsymbol{E}_0 = \frac{\sigma_0}{\epsilon_0}\hat{\boldsymbol{y}} \qquad (12.53)$$



(b)

In $\mathcal{S}$, the field still takes the form

$$\boldsymbol{E} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{y}}, \qquad (12.54)$$

the only difference is the value of the surface charge σ. The total charge on each plate is invariant, and the width (w) is unchanged, but the length (l) is Lorentz-contracted by a factor

$$\frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2} \qquad (12.55)$$

so the charge per unit area is increased by a factor γ_0:

$$\sigma = \gamma_0 \sigma_0 \qquad (12.56)$$

Accordingly,

$$\boldsymbol{E}^{\perp} = \gamma_0 \boldsymbol{E}_0^{\perp} \qquad (12.57)$$

$\perp$: components of $\boldsymbol{E}$ *perpendicular* to the direction of motion of $\mathcal{S}$.

Electric field of a point charge in uniform motion. A point charge q is at rest at the origin in system $\mathcal{S}_0$. What is the electric field of this same charge in system $\mathcal{S}$, which moves to the right at speed v_0 relative to $\mathcal{S}_0$?

Solution

In $\mathcal{S}_0$ the field is

$$\boldsymbol{E}_0 = \frac{1}{4\pi\epsilon_0}\frac{q}{r_0^2}\,\hat{\boldsymbol{r}}$$

$$\begin{cases} E_{x0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q x_0}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} \\[2ex] E_{y0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q y_0}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} \\[2ex] E_{z0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q z_0}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} \end{cases}$$

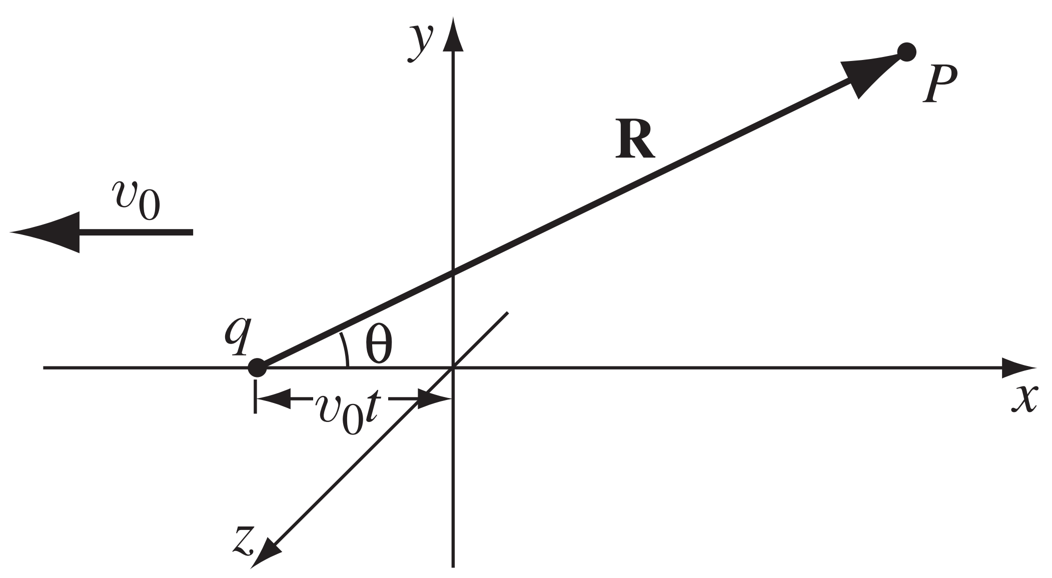
Example (cont'd)

From the transformation rules (Eqs. (12.57) and (12.52)), we have

$$\begin{cases} E_x = E_{x0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\[2ex] E_y = \gamma_0 E_{y0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\[2ex] E_z = \gamma_0 E_{z0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \end{cases}$$

These are still expressed in terms of the $\mathcal{S}_0$ coordinates (x_0, y_0, z_0) of the field point (P).

We shall write them in terms of the $\mathcal{S}$ coordinates of P.



From the Lorentz inverse transformations

$$\begin{cases} x_0 = & \gamma_0(x + v_0 t) & = \gamma_0 R_x \\ y_0 = & y & = R_y \\ z_0 = & z & = R_z \end{cases}$$

where $\boldsymbol{R}$ is the vector from q to P.

Thus

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0} \frac{\gamma_0 q \boldsymbol{R}}{(\gamma_0^2 R^2 \cos^2\theta + R^2 \sin^2\theta)^{3/2}} = \frac{1}{4\pi\epsilon_0} \frac{q(1 - v_0^2/c^2)}{[1 - (v_0^2/c^2)\sin^2\theta]^{3/2}} \frac{\hat{\boldsymbol{R}}}{R^2} \quad (12.58)$$

This is the field of a charge in uniform motion. Notice that the field points away from the instantaneous position of the charge.

How the Fields Transform (cont'd)

The complete set of transformation rules between $\mathcal{S}$ and $\bar{\mathcal{S}}$ frames is given by

$$\begin{aligned}
\bar{E}_x &= E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \qquad \bar{E}_z = \gamma(E_z + vB_y) \\
\bar{B}_x &= B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)
\end{aligned} \tag{12.59}$$

In vector notation, we have

$$\bar{\mathbf{E}}_\parallel = \mathbf{E}_\parallel, \quad \bar{\mathbf{B}}_\parallel = \mathbf{B}_\parallel, \quad \bar{\mathbf{E}}_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp), \quad \bar{\mathbf{B}}_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_\perp\right) \tag{12.60}$$

Example 2

Two special cases:

- If $\boldsymbol{B} = \mathbf{0}$ in $\mathcal{S}$, then

$$\bar{\boldsymbol{B}} = \frac{v}{c^2}(E_z\,\hat{\boldsymbol{y}} - E_y\,\hat{\boldsymbol{z}}) = \frac{v}{c^2}(\bar{E}_z\,\hat{\boldsymbol{y}} - \bar{E}_y\,\hat{\boldsymbol{z}})$$

or, since $\boldsymbol{v} = v\,\hat{\boldsymbol{x}}$,

$$\bar{\boldsymbol{B}} = -\frac{1}{c^2}(\boldsymbol{v} \times \bar{\boldsymbol{E}})$$

- If $\boldsymbol{E} = \mathbf{0}$ in $\mathcal{S}$, then

$$\bar{\boldsymbol{E}} = -\gamma v(B_z\,\hat{\boldsymbol{y}} - B_y\,\hat{\boldsymbol{z}}) = -v(\bar{B}_z\,\hat{\boldsymbol{y}} - \bar{B}_y\,\hat{\boldsymbol{z}})$$

or

$$\bar{\boldsymbol{E}} = \boldsymbol{v} \times \bar{\boldsymbol{B}}$$