Assignment 3 of Quantum Mechanics II [1]

(Due on September 5 2025 at 17:00. Submit your solution to Canvas as a PDF file.)

(You are only required to solve one of problem 3 and problem 4, but you are encouraged to do both.

If you do both, your score from these two problems will be the higher score from one of these two.)

Problem 1 (13 points). In class, we have derived the exchange interactions during week 2 using symmetrized or anti-symmetrized wave functions. We have also learnt how to transform the creation operators in one single-particle basis to creation operators in another single-particle basis. In this problem, we will rederive the exchange interaction using creation and annihilation operators, and you will need to perform some basis change for these operators along the way. The purpose of this problem is to ensure you that the formalism based on creation and annihilation operators can reproduce any results using symmetrized or anti-symmetrized wave functions, and also to familiarize you with the basis change of creation operators.

As our setup, consider a system made of identical particles that can be either bosons or fermions. Denote by $a_{\vec{r}}$ the annihilation operator that annihilates such a particle at position $\vec{r}$. These particles have an interaction energy, taking the form of

$$I = \frac{1}{2}\int d^3r_1 d^3r_2 V(|\vec{r}_1 - \vec{r}_2|) a^\dagger_{\vec{r}_1} a^\dagger_{\vec{r}_2} a_{\vec{r}_2} a_{\vec{r}_1}. \tag{1}$$

(1) (4 points) Suppose there is only a single particle that is in a normalized state with wave function $\psi(\vec{r})$. Express this state in terms of the creation operator $a^\dagger_{\vec{r}}$ and the vacuum state $|vac\rangle$.

(2) (4 points) Suppose two particles are in two orthonormal single-particle states that have single-particle wave functions $\psi_a(\vec{r})$ and $\psi_b(\vec{r})$, respectively. Express this two-particle state in terms of the creation operator $a^\dagger_{\vec{r}}$ and the vacuum state $|vac\rangle$.

(3) (5 points) For the two-particle state in part (2), what is the expectation value of the interaction energy I, in terms of $\psi_a(\vec{r})$, $\psi_b(\vec{r})$ and $V(r)$? Does the result agree with the one obtained using symmetrized or anti-symmetrized wave function in week 2? Please discuss the cases of bosons and fermions separately. Hint: You will need to move the annihilation operators to the right and the creation operators to the left, using the commutators or anti-commutators of the creation and annihilation operators, so that you can utilize the facts that $a_{\vec{r}}|vac\rangle = 0$ and $\langle vac|a^\dagger_{\vec{r}} = 0$.

Problem 2 (13 points). In class, we have discussed how to derive the 2-particle wave equation from states written in terms of creation and annihilation operators. In this problem, we explore a similar problem, but with two types of particles, one of which is a boson and the other is a fermion.

As our setup, denote the creation operators of the boson and fermion at position $\vec{r}$ by $b^\dagger_{\vec{r}}$ and $f^\dagger_{\vec{r}}$, respectively. Besides the usual relations $[b_{\vec{r}}, b_{\vec{r}'}] = 0$, $[b_{\vec{r}}, b^\dagger_{\vec{r}'}] = \delta(\vec{r} - \vec{r}')$, $\{f_{\vec{r}}, f_{\vec{r}'}\} = 0$ and $\{f_{\vec{r}}, f^\dagger_{\vec{r}'}\} = \delta(\vec{r} - \vec{r}')$, we further take $[b_{\vec{r}}, f_{\vec{r}'}] = [b_{\vec{r}}, f^\dagger_{\vec{r}'}] = 0$. Suppose the interaction potential of two bosons at positions $\vec{r}_1$ and $\vec{r}_2$ is $V^{bb}(\vec{r}_1, \vec{r}_2)$, the interaction potential of two fermions at positions $\vec{r}_1$ and $\vec{r}_2$ is $V^{ff}(\vec{r}_1, \vec{r}_2)$, and the interaction potential of a boson at position $\vec{r}_1$ and a fermion at position $\vec{r}_2$ is $V^{bf}(\vec{r}_1, \vec{r}_2)$.

[1] You are welcome to get back with questions and clarifications if the wording of problems is ambiguous.

In the following, you will derive the wave equations for this setup under various conditions. Ultimately, the wave equation is required to be written in the form of $i\hbar\frac{\partial\psi(\vec{r}_1,\vec{r}_2,\vec{r}_3,\cdots,\vec{r}_N,t)}{\partial t} = H\psi(\vec{r}_1,\vec{r}_2,\vec{r}_3,\cdots,\vec{r}_N,t)$, where ψ is a normalized N-particle wave function where the positions of the particles are $\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_N$, and H is an operator acting on wave functions. You need to consider the symmetrization and anti-symmetrization conditions of the wave functions.

(1) (3 points) Denote by $N^b(\vec{r})$ the number of bosons at position $\vec{r}$, and $N^f(\vec{r})$ the number of fermions at position $\vec{r}$. In terms of $N^b(\vec{r})$, $N^f(\vec{r})$, $V^{bb}(\vec{r}_1,\vec{r}_2)$, $V^{ff}(\vec{r}_1,\vec{r}_2)$ and $V^{bf}(\vec{r}_1,\vec{r}_2)$, what is the total interaction energy?

(2) (3 points) Suppose the system contains 1 boson and 1 fermion. Denote the normalized wave function of the system by $\psi(\vec{r}_1,\vec{r}_2)$. What is the symmetrization and/or anti-symmetrization condition of this wave function? Following the procedure in class, derive the wave equation for such a system.

(3) (3 points) Suppose the system contains no boson and 3 fermions. Denote the normalized wave function of the system by $\psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)$. What is the symmetrization and/or anti-symmetrization condition of this wave function? Following the procedure in class, derive the wave equation for such a system.

(4) (4 points) Suppose the system contains in total 2 bosons and 2 fermions. Denote the normalized wave function of the system by $\psi(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4)$. What is the symmetrization and/or anti-symmetrization condition of this wave function? Following the procedure in class, derive the wave equation for such a system.

Problem 3 (13 points). In class, we have discussed identical bosons and fermions. In this problem, we explore a striking phenomenon in fermionic systems, which is the basis for some protocols of quantum computation. In fact, Microsoft has taken a lot of efforts to experimentally realize this protocol.

The setup is shown in Figure 1, which shows a chain with some sites. Denote the number of sites by L ($L = 8$ in Figure 1). There can be fermions living on this chain, and the fermions are assumed to have no internal structure and their positions can only be the sites of the chain (rather than the entire continuum space). Denote by f_i and $f_i^\dagger$ the annihilation and creation operators of the fermions at the i-th site. Namely, $f_i^\dagger|vac\rangle$ is a state with a singel fermion at the i-th site and no fermion at any other site.

Figure 1: A chain with $L = 8$ sites.

(1) (4 points) Define the so-called Majorana operators

$$\gamma_i = f_i + f_i^\dagger, \quad \eta_i = i(f_i^\dagger - f_i). \tag{2}$$

Calculate $\gamma_i - \gamma_i^\dagger$, $\eta_i - \eta_i^\dagger$, $\{\gamma_i, \gamma_j\}$, $\{\eta_i, \eta_j\}$ and $\{\gamma_i, \eta_j\}$, where i and j are the indices of the sites.

(2) (3 points) Suppose the Hamiltonian of these fermions take the following form

$$H = -t\sum_{i=1}^{L-1}(f_{i+1}^\dagger f_i + f_i^\dagger f_{i+1}) - \Delta\sum_{i=1}^{L-1}(f_{i+1}f_i + f_i^\dagger f_{i+1}^\dagger) \tag{3}$$

where t and Δ are real parameters. The first term in the Hamiltonian represents the hopping of fermions from one site to its nearby site, which does not change the total fermion number. The second term is less familiar, which means that two fermions in a pair of adjacent sites can be created or annihilated together.

Such processes change the total fermion numbers, but only by an even number, so they do not change the fermion parity. This model is a valid approximate model for a class of superconductors. Write this Hamiltonian in terms of γ_i and η_i.

(3) (3 points) Consider the special case where $t = \Delta$. Show that the Hamiltonian becomes a sum of commuting Hermitian terms, each of which only involves two nearby sites. For each of these terms, what are its eigenvalues?

(4) (3 points) How many ground states does this Hamiltonian have? If there are multiple ground states, find a set of operators whose expectation values are different in these ground states, so that these operators can be used to distinguish them. Hint: Be careful about what happens at the two ends of the chain.

Remark: You will see that these operators involve sites very far away. This actually means that the ground states of this system can be used to store quantum information in a non-local manner. However, interactions in nature are local, so disturbances to these ground states induced by interactions with the exterior environment will be inefficient in destroying the stored quantum information. This is the key insight in fault-tolerant topological quantum computation.

Problem 4 (14 points). Just being a student who can solve problems assigned by the teachers is good, but it is even better if you can propose good problems by yourself. In this class, besides helping you understand the basics of quantum mechanics and how to apply it, I also aim to help you improve your ability to raise good questions, which is useful no matter what you do in the future.

Now you are required to propose a problem that is similarly interesting as problem 3. After proposing it, you need to either give the complete solution to this problem, or argue that this problem is interesting but cannot be solved within a reasonable time frame.

Please use your own taste and opinion to judge what kind of problems are "similarly interesting as problem 3". At the end, the score you get from this problem is determined based on how interesting the grader thinks your proposed problem is, and whether the solution you provide is complete or whether you can convince the grader that the problem is interesting but too difficult.

Good problems may be selected as problems in the midterm and final exams this semester, or assignment problems in the future semesters. Really good problems may be further developed into research projects.