

The Answer of Assignment 2

WEI SHUANG

31/8/2025

Problem 1 Solution

(1) According to the theory of isomorphism of linear spaces, any two linear spaces of the same dimension are isomorphic. Therefore, $\sum_{i,j=0}^1 c_{ij} |i, j\rangle$ is isomorphic to $\{(x_1, x_2, x_3, x_4)\}, x \in \mathbb{C}$. That is, $v_1 = (c_{00}, c_{01}, c_{10}, c_{11})^T$, $v_2 = (d_{00}, d_{01}, d_{10}, d_{11})^T$.

(2) $\langle \psi_1 | \psi_2 \rangle = v_1^\dagger v_2$ 成立, 这是因为 $\langle \psi_1 | \psi_2 \rangle = (\sum_{i,j=0}^1 c_{i,j}^* \langle ij|) (\sum_{i,j=0}^1 c_{i,j} |ij\rangle) = v_1^\dagger v_2$

(3) $O |pq\rangle = O_{ij,kl} |ij\rangle \langle kl| |pq\rangle = O_{ij,kl} |ij\rangle \delta_{kp} \delta_{lq} = O_{ij,pq} |ij\rangle$ if we define $|00\rangle$ as e_1 , $|01\rangle$ as e_2 , $|10\rangle$ as e_3 , $|11\rangle$ as e_4 , then we can get the matrix representation of O : $Oe_i = \sum_{j=1}^4 O_{ji} e_j$ Therefore, the matrix representation of O is:

$$M = \begin{pmatrix} O_{11} & O_{12} & O_{13} & O_{14} \\ O_{21} & O_{22} & O_{23} & O_{24} \\ O_{31} & O_{32} & O_{33} & O_{34} \\ O_{41} & O_{42} & O_{43} & O_{44} \end{pmatrix}$$

(4) $O |\psi_1\rangle$ is the same as Mv_1 , because:

$$O |\psi_1\rangle = O_{ij,kl} |ij\rangle \langle kl| \cdot c_{mn} |mn\rangle = O_{ij,kl} |ij\rangle c_{mn} \delta_{km} \delta_{ln} = O_{ij,mn} c_{mn} |ij\rangle$$

this is equivalent to Mv_1 if we define $|00\rangle$ as e_1 , $|01\rangle$ as e_2 , $|10\rangle$ as e_3 , $|11\rangle$ as e_4 .

Problem 2 Solution

(1) Starting from $[b_i^\dagger, b_i^\dagger] = 0$, show that $[b_i, b_j] = 0$. According to the definition of commutation relation, we have:

$$[b_i^\dagger, b_j^\dagger] = b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0$$

Taking the Hermitian conjugate of both sides, we get:

$$(b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger)^\dagger = 0^\dagger$$

$$b_j b_i - b_i b_j = 0$$

so we have: $[b_i, b_j] = 0$

(2)

$$\langle \overline{n'_1 n'_2 \cdots n'_k} | b_i^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle = \langle \overline{n_1 n_2 \cdots n_k} | b_i | \overline{n'_1 n'_2 \cdots n'_k} \rangle^* = \delta_{n_1, n'_1} \delta_{n_2, n'_2} \cdots \delta_{n_i, n'_i-1} \cdots \delta_{n_k, n'_k} \sqrt{n'_i}$$

and

$$b_i^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle = \sqrt{n_i + 1} | \overline{n_1 n_2 \cdots (n_i + 1) \cdots n_k} \rangle$$

we can get the relation between b_i and b_i^\dagger as follows:

$$b_i | \overline{n_1 n_2 \cdots n_k} \rangle = \sqrt{n_i} | \overline{n_1 n_2 \cdots (n_i - 1) \cdots n_k} \rangle$$

(3)

We have:

$$\begin{aligned} [b_i, b_j^\dagger] | \overline{n_1 n_2 \cdots n_k} \rangle &= b_i b_j^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle - b_j^\dagger b_i | \overline{n_1 n_2 \cdots n_k} \rangle \\ &= b_i \sqrt{n_j + 1} | \overline{n_1 n_2 \cdots (n_j + 1) \cdots n_k} \rangle \\ &\quad - b_j^\dagger \sqrt{n_i} | \overline{n_1 n_2 \cdots (n_i - 1) \cdots n_k} \rangle \\ &= \sqrt{n_i(n_j + 1)} | \overline{n_1 n_2 \cdots (n_i - 1) \cdots (n_j + 1) \cdots n_k} \rangle - 0 \end{aligned}$$

(4) if $i = j$, we have:

$$[b_i, b_i^\dagger] | \overline{n_1 n_2 \cdots n_k} \rangle = (n_i + 1 - n_i) | \overline{n_1 n_2 \cdots n_k} \rangle = | \overline{n_1 n_2 \cdots n_k} \rangle$$

(5)

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

When $i = j$, this becomes

$$\{f_i, f_i\} = f_i f_i + f_i f_i = 2f_i^2$$

Since f_i is a fermionic annihilation operator, it satisfies $f_i^2 = 0$. Therefore,

$$\{f_i, f_i\} = 2 \times 0 = 0$$

So, when $i = j$, the anticommutation relation $\{f_i, f_j\} = 0$ holds.

(6)

Suppose $i \neq j$. Calculate $\{f_i, f_j\} | n_1 n_2 \cdots \rangle$.

Recall the definition:

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

We need to consider the action of f_i and f_j on the occupation number basis $|n_1 n_2 \dots\rangle$, where $n_k = 0$ or 1 for fermions.

Consider the following cases:

Case 1: $n_i = 0$ or $n_j = 0$

- If $n_i = 0$, then $f_i |n_1 n_2 \dots\rangle = 0$. - If $n_j = 0$, then $f_j |n_1 n_2 \dots\rangle = 0$.

Therefore, in either case, both $f_i f_j |n_1 n_2 \dots\rangle = 0$ and $f_j f_i |n_1 n_2 \dots\rangle = 0$, so

$$\{f_i, f_j\} |n_1 n_2 \dots\rangle = 0$$

Case 2: $n_i = 1$ and $n_j = 1$

- $f_j |n_1 n_2 \dots n_j = 1 \dots n_i = 1 \dots\rangle = (-1)^{s_1} |n_1 \dots n_j = 0 \dots n_i = 1 \dots\rangle$ - Then f_i acts on this state: $f_i |n_1 \dots n_j = 0 \dots n_i = 1 \dots\rangle = (-1)^{s_2} |n_1 \dots n_j = 0 \dots n_i = 0 \dots\rangle$

Similarly, $f_i f_j |n_1 n_2 \dots\rangle$ and $f_j f_i |n_1 n_2 \dots\rangle$ will differ by a sign, but since $i \neq j$, the sum $f_i f_j + f_j f_i$ will always cancel out due to the anticommutation property of fermionic operators.

Therefore,

$$\{f_i, f_j\} |n_1 n_2 \dots\rangle = 0$$

(7) case 1: $i = j$

$$f_i, f_j^\dagger = f_i f_i^\dagger + f_i^\dagger f_i$$

when $n_i = 0$, we have:

$$\begin{aligned} f_i f_i^\dagger |n_1 n_2 \dots n_i = 0 \dots\rangle \\ = f_i (-1)^s |n_1 n_2 \dots n_i = 1 \dots\rangle \\ = (-1)^s (-1)^s |n_1 n_2 \dots n_i = 0 \dots\rangle \\ = |n_1 n_2 \dots n_i = 0 \dots\rangle \end{aligned}$$

when $n_i = 1$, we have:

$$\begin{aligned} f_i^\dagger f_i |n_1 n_2 \dots n_i = 1 \dots\rangle &= f_i^\dagger (-1)^s |n_1 n_2 \dots n_i = 0 \dots\rangle \\ &= (-1)^s (-1)^s |n_1 n_2 \dots n_i = 1 \dots\rangle = |n_1 n_2 \dots n_i = 1 \dots\rangle \end{aligned}$$

case 2: $i \neq j$

$$\{f_i, f_j^\dagger\} = f_i f_j^\dagger + f_j^\dagger f_i$$

when $n_i = 0$ or $n_j = 0$, we have: $f_i f_j^\dagger \psi = 0$ $f_j^\dagger f_i \psi = 0$ when $n_i = 1$ and $n_j = 1$, we have the same conclusion as above.

when $n_i = 1$ and $n_j = 0$, we have:

$$\begin{aligned} \{f_i f_j^\dagger\} |n_1 n_2 \dots n_i = 1 \dots n_j = 0 \dots\rangle &= f_i (-1)^{\sum_{1}^{j-1} n_x} |n_1 n_2 \dots n_i = 1 \dots n_j = 1 \dots\rangle \\ + f_j^\dagger (-1)^{\sum_{1}^{i-1} n_x} |n_1 n_2 \dots n_i = 0 \dots n_j = 0 \dots\rangle &= (-1)^{s_1+s_2} + (-1)^{s_1+s_2+1} |n_1, \dots n_i = 0 \dots n_j = 1 \dots\rangle \end{aligned}$$

Problem 3 Solution

(1) We define b_i^\dagger in

$$b_i^\dagger |n_1, n_2, \dots, n_k\rangle = |n_1, n_2, \dots, n_i + 1, \dots, n_L\rangle = |n_1, n_2, \dots, 1, n_{i+1}, \dots, n_L\rangle \text{ if } n_i = 0$$

$$b_i^\dagger |n_1, n_2, \dots, n_k\rangle = 0 \text{ if } n_i = 1$$

$$b_i |n_1, n_2, \dots, n_k\rangle = |n_1, n_2, \dots, n_i - 1, \dots, n_L\rangle = |n_1, n_2, \dots, 0, n_{i+1}, \dots, n_L\rangle \text{ if } n_i = 1$$

$$b_i |n_1, n_2, \dots, n_k\rangle = 0 \text{ if } n_i = 0$$

$[b_i, b_j] = 0$ for any i, j because when $i=j$ $(b_i b_j - b_j b_i) |n_1, n_2, \dots, n_L\rangle = 0$ if $n_i = 0$ $[b_i, b_j] = 0$ if $n_i = 1$ because $b_i b_j |n_1, n_2, \dots, n_L\rangle = |n_1, n_2, \dots, n_L\rangle$ and $b_j b_i |n_1, n_2, \dots, n_L\rangle = |n_1, n_2, \dots, n_L\rangle$ when $i \neq j$, $(b_i b_j - b_j b_i) |n_1, n_2, \dots, n_L\rangle = 0$ because $b_i b_j |n_1, n_2, \dots, n_L\rangle$ and $b_j b_i |n_1, n_2, \dots, n_L\rangle$ are the same state. $[b_i, b_j^\dagger] = \delta_{ij}$

case 1: $i=j$

$$[b_i, b_i^\dagger] = b_i b_i^\dagger - b_i^\dagger b_i$$

if $n_i = 0$, we have:

$$\begin{aligned} b_i b_i^\dagger |n_1, n_2, \dots, n_k\rangle &= b_i |n_1, n_2, \dots, 1, n_{i+1}, \dots, n_L\rangle \\ &= |n_1, n_2, \dots, 0, n_{i+1}, \dots, n_L\rangle = |n_1, n_2, \dots, n_k\rangle \end{aligned}$$

$$b_i^\dagger b_i |n_1, n_2, \dots, n_k\rangle = b_i^\dagger 0 = 0$$

case 2: $i \neq j$

$$[b_i, b_j^\dagger] = b_i b_j^\dagger - b_j^\dagger b_i = 0 \text{ because when } n_i = 0 \text{ or } n_j = 0, b_i b_j^\dagger |n_1, n_2, \dots, n_L\rangle = 0 \text{ and } b_j^\dagger b_i |n_1, n_2, \dots, n_L\rangle = 0$$

when $n_i = 1$ or $n_j = 1$, $b_i b_j^\dagger |n_1, n_2, \dots, n_L\rangle$ and $b_j^\dagger b_i |n_1, n_2, \dots, n_L\rangle$ are the same state.

only when $n_i = 1$ and $n_j = 0$, we have:

$$\begin{aligned} b_i b_j^\dagger |n_1, n_2, \dots, 1, \dots, 0, \dots, n_L\rangle &= b_i |n_1, n_2, \dots, 1, \dots, 1, \dots, n_L\rangle \\ &= |n_1, n_2, \dots, 0, \dots, 1, \dots, n_L\rangle = b_j^\dagger b_i |n_1, n_2, \dots, n_L\rangle \end{aligned}$$

(2) $\sigma_i^\pm = (\sigma_i^x \pm \sigma_i^y \cdot i)/2$ if we call **case 1** $[\sigma_i^-, \sigma_j^-] = 0$ when $i = j$, we have:

$$[\sigma_i^-, \sigma_i^-] = \sigma_i^- \sigma_i^- - \sigma_i^- \sigma_i^- = 0 \quad (3)$$

when $i \neq j$, we have: (4) how should the Pauli operators σ_i^+ and σ_i^z can be written in terms of the f operators? we have the definition that:

$$f_i^\dagger = \left(\prod_{j < i} \sigma_j^z \right) \sigma_i^+$$

so from $\langle \overline{n_1 n_2 \cdot n_{i'}} | f_i^\dagger | \overline{n_1 n_2 n_3 \cdot n_i} \rangle = \langle \overline{n_1 n_2 \cdot n_i} | f_i | \overline{n_1 n_2 \cdot n_{i'}} \rangle^*$ we can get the relation between f_i and σ_i^+ , σ_i^z as follows:

$$f_i = \left(\prod_{j < i} \sigma_j^z \right) \sigma_i^-$$

then try to represent σ_i^z and σ_i^+ in terms of f_i : we use the matrix representation of the Pauli operators:

$$\sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

σ_i^z can be expressed as: $2\sigma_i^+ \sigma_i^- - I = 2f_i^\dagger f_i - I$

like wise:

$$\begin{aligned} \sigma_i^+ &= f_i^\dagger \left(\prod_{j < i} \sigma_j^z \right)^{-1} \\ &= f_i^\dagger \left(\prod_{j < i} (2f_j^\dagger f_j - I) \right) \end{aligned}$$