The Answer of Assignment 2

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Problem 1 Solution

(1) According to the theory of isomorphism of linear spaces, any two linear spaces of the same dimension are isomorphic. Therefore, $\sum\limits_{i,j=0}^{1} c_{ij} |i,j\rangle$ is isomorphic to $\{(x_1, x_2, x_3, x_4)\}, x \in \mathbb{C}$. That is, $v_1 = (c_{00}, c_{01}, c_{10}, c_{11})^T$, $v_2 = (d_{00}, d_{01}, d_{10}, d_{11})^T$.

(2) $\langle \psi_1 | \psi_2 \rangle = v_1^\dagger v_2$ 成立, 这是因为 $\langle \psi_1 | \psi_2 \rangle = (\sum_{i,j=0}^{1} c_{i,j}^* \langle ij|)(\sum i, , j = 0^1 c_{i,j} |ij\rangle) = v_1^\dagger v_2$

(3) $O|pq\rangle = O_{ij,kl} |ij\rangle \langle kl| |pq\rangle = O_{ij,kl} |ij\rangle \delta_{kp} \delta_{lq} = O_{ij,pq} |ij\rangle$ if we define $|00\rangle$ as e_1, $|01\rangle$ as e_2, $|10\rangle$ as e_3, $|11\rangle$ as e_4, then we can get the matrix representation of O: $Oe_i = \sum_{j=1}^{4} O_{ji} e_j$ Therefore, the matrix representation of O is:

$$M = \begin{pmatrix} O_{11} & O_{12} & O_{13} & O_{14} \\ O_{21} & O_{22} & O_{23} & O_{24} \\ O_{31} & O_{32} & O_{33} & O_{34} \\ O_{41} & O_{42} & O_{43} & O_{44} \end{pmatrix}$$

(4) $O|\psi_1\rangle$ is the same as Mv_1,because:

$$O|\psi_1\rangle = O_{ij,kl} |ij\rangle \langle kl| \cdot c_{mn} |mn\rangle = O_{ij,kl} |ij\rangle c_{mn} \delta_{km} \delta_{ln} = O_{ij,mn} c_{mn} |ij\rangle$$

this is equivalent to Mv_1 if we define $|00\rangle$ as e_1, $|01\rangle$ as e_2, $|10\rangle$ as e_3, $|11\rangle$ as e_4.:

Problem 2 Solution

(1) Starting from $[b_i^\dagger, b_i^\dagger] = 0$,show that $[b_i, b_j] = 0$. According to the definition of commutation relation, we have:

$$[b_i^\dagger, b_j^\dagger] = b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0$$

Taking the Hermitian conjugate of both sides, we get:

$$(b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger)^\dagger = 0^\dagger$$

$$b_j b_i - b_i b_j = 0$$

so we have:$[b_i, b_j] = 0$

(2)

$$\langle\overline{n'_1 n'_2 \cdots n'_k}|b_i^\dagger|\overline{n_1 n_2 \cdots n_k}\rangle = \langle\overline{n_1 n_2 \cdots n_k}|b_i|\overline{n'_1 n'_2 \cdots n'_k}\rangle^* = \delta_{n_1,n'_1}\delta_{n_2,n'_2}\cdots\delta_{n_i,n'_i-1}\cdots\delta_{n_k,n'_k}\sqrt{n'_i}$$

and

$$b_i^\dagger\left|\overline{n_1 n_2 \cdots n_k}\right\rangle = \sqrt{n_i+1}\left|\overline{n_1 n_2 \cdots (n_i+1) \cdots n_k}\right\rangle$$

we can get the relation between b_i and $b_i^\dagger$ as follows:

$$b_i\left|\overline{n_1 n_2 \cdots n_k}\right\rangle = \sqrt{n_i}\left|\overline{n_1 n_2 \cdots (n_i-1) \cdots n_k}\right\rangle$$

(3)

We have:

$$\begin{aligned}
[b_i, b_j^\dagger]\left|\overline{n_1 n_2 \cdots n_k}\right\rangle &= b_i b_j^\dagger\left|\overline{n_1 n_2 \cdots n_k}\right\rangle - b_j^\dagger b_i\left|\overline{n_1 n_2 \cdots n_k}\right\rangle \\
&= b_i\sqrt{n_j+1}\left|\overline{n_1 n_2 \cdots (n_j+1) \cdots n_k}\right\rangle \\
&\quad - b_j^\dagger\sqrt{n_i}\left|\overline{n_1 n_2 \cdots (n_i-1) \cdots n_k}\right\rangle \\
&= \sqrt{n_i(n_j+1)}\left|\overline{n_1 n_2 \cdots (n_i-1) \cdots (n_j+1) \cdots n_k}\right\rangle \cdot 0
\end{aligned}$$

(4) if i $\neq j, we have:$

Problem 3 Solution

(1)

(2)

(3)

(4) how should the Pauli operators σ_i^+ and σ_i^z can be written in terms of the f operators? we have the definition that:

$$f_i^\dagger = \left(\prod_{j<i}\sigma_j^z\right)\sigma_i^+$$

so from $\langle\overline{n_{1'} n_{2'} \cdot n_{i'}}|f_i^\dagger|\overline{n_1 n_2 n_3 \cdot n_i}\rangle$=$\langle\overline{n_1 n_2 \cdot n_i}|f_i|\overline{n_{1'} n_{2'} \cdot n_{i'}}\rangle^*$ we can get the relation between f_i and σ_i^+, σ_i^z as follows:

$$f_i = \left(\prod_{j<i}\sigma_j^z\right)\sigma_i^-$$

then try to represent σ_i^z and σ_i^+ in terms of f_i: we use the matrix representation of the Pauli operators:

$$\sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

σ_i^zcan be expressed as:$2\sigma_i^+\sigma_i^- - I = 2f_i^\dagger f_i - I$

like wise:

$$\sigma_i^+ = f_i^\dagger \left(\prod_{j<i} \sigma_j^z \right)^{-1}$$

$$= f_i^\dagger \left(\prod_{j<i} (2f_j^\dagger f_j - I) \right)$$