# Maxwell's Stress Tensor

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The total electromagnetic force on the charges in volume $V$:

$$\boldsymbol{F} = \int_V (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\rho \, \mathrm{d}\tau = \int_V (\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}) \, \mathrm{d}\tau. \tag{1}$$

The force per unit volume is

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \tag{2}$$

Writing this in terms of fields alone, we eliminate $\rho$ and $\boldsymbol{J}$ by using Maxwell's equations (i) and (iv):

$$\boldsymbol{f} = \epsilon_0 (\nabla \cdot \boldsymbol{E})\boldsymbol{E} + \left( \frac{1}{\mu_0} \nabla \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \times \boldsymbol{B}.$$

Now

$$\frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) = \left( \frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} \right) + \left( \boldsymbol{E} \times \frac{\partial \boldsymbol{B}}{\partial t} \right)$$

# Maxwell's Stress Tensor (cont'd)



so

$$\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} = \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) + \boldsymbol{E} \times (\nabla \times \boldsymbol{E})$$

Thus

$$\boldsymbol{f} = \epsilon_0[(\nabla \cdot \boldsymbol{E})\boldsymbol{E} - \boldsymbol{E} \times (\nabla \times \boldsymbol{E})] - \frac{1}{\mu_0}[\boldsymbol{B} \times (\nabla \times \boldsymbol{B})] - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$$

To make things look more symmetrical, we include a term $(\nabla \cdot \boldsymbol{B})\boldsymbol{B}$, which is zero $\because \nabla \cdot \boldsymbol{B} = 0$.

Now, product rule 4 gives

$$\nabla(E^2) = 2(\boldsymbol{E} \cdot \nabla)\boldsymbol{E} + 2\boldsymbol{E} \times (\nabla \times \boldsymbol{E}),$$

so

$$\boldsymbol{E} \times (\nabla \times \boldsymbol{E}) = \frac{1}{2}\nabla(E^2) - (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}$$

and the same goes for $\boldsymbol{B}$.

# Maxwell's Stress Tensor (cont'd)



Therefore,

$$\begin{aligned} \boldsymbol{f} &= \epsilon_0[(\nabla \cdot \boldsymbol{E})\boldsymbol{E} + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}] + \frac{1}{\mu_0}[(\nabla \cdot \boldsymbol{B})\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}] \\ &\quad - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) \end{aligned} \tag{3}$$

It can be simplified by introducing the **Maxwell stress tensor**,

$$T_{ij} \equiv \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right) \tag{4}$$

The indices $i$ and $j$ refer to the coordinates $x$, $y$, and $z$, so the stress tensor has a total of nine components ($T_{xx}$, $T_{yy}$, $T_{xz}$, $T_{yx}$, etc).

# Maxwell's Stress Tensor (cont'd)



Thus, for instance,

$$T_{xx} = \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2)$$

$$T_{xy} = \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y)$$

Because it carries two indices, where a vector has only one, $T_{ij}$ is sometimes written with a double arrow: $\overleftrightarrow{\boldsymbol{T}}$. One can form the dot product of $\overleftrightarrow{\boldsymbol{T}}$ with a vector $\boldsymbol{a}$:

$$(\boldsymbol{a} \cdot \overleftrightarrow{\boldsymbol{T}})_j = \sum_{i=x,y,z} a_i T_{ij}, \quad (\overleftrightarrow{\boldsymbol{T}} \cdot \boldsymbol{a})_j = \sum_{i=x,y,z} T_{ji} a_i \tag{5}$$

The resulting object, which has one remaining index, is itself a vector.

# Maxwell's Stress Tensor (cont'd)



In particular, the divergence of $\overleftrightarrow{\boldsymbol{T}}$ has as its $j$th component

$$
\begin{aligned}
(\nabla \cdot \overleftrightarrow{\boldsymbol{T}})_j = &\epsilon_0 \left[ (\nabla \cdot \boldsymbol{E}) E_j + (\boldsymbol{E} \cdot \nabla) E_j - \frac{1}{2} \nabla_j E^2 \right] \\
&+ \frac{1}{\mu_0} \left[ (\nabla \cdot \boldsymbol{B}) B_j + (\boldsymbol{B} \cdot \nabla) B_j - \frac{1}{2} \nabla_j B^2 \right]
\end{aligned}
$$

Thus the force per unit volume (Eq. (3)) can be written in a simpler form

$$
\boldsymbol{f} = \nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \epsilon_0 \mu_0 \frac{\partial \boldsymbol{S}}{\partial t} \tag{6}
$$

where $\boldsymbol{S}$ is the Poynting vector.

The total force on the charges in $V$ (Eq. (1)) is

$$
\boldsymbol{F} = \oint_S \overleftrightarrow{\boldsymbol{T}} \cdot \mathrm{d}\boldsymbol{a} - \epsilon_0 \mu_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_V \boldsymbol{S} \, \mathrm{d}\tau \tag{7}
$$

# Conservation of Momentum

# Conservation of Momentum



From Newton's second law,

$$\boldsymbol{F} = \frac{\mathrm{d}\,\boldsymbol{p}_{\mathrm{mech}}}{\mathrm{d}t}$$

Therefore, Eq. (7) can be written as

$$\frac{\mathrm{d}\,\boldsymbol{p}_{\mathrm{mech}}}{\mathrm{d}t} = -\epsilon_0\mu_0\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{S}\;\mathrm{d}\tau + \oint_S \overleftrightarrow{\boldsymbol{T}}\cdot\,\mathrm{d}\boldsymbol{a} \tag{8}$$

where $\boldsymbol{p}_{\mathrm{mech}}$ is the (mechanical) momentum of the particles contained in the volume $V$.

The first integral represents momentum stored in the fields:

$$\boldsymbol{p} = \mu_0\epsilon_0\int_V \boldsymbol{S}\;\mathrm{d}\tau \tag{9}$$

while the second integral is the momentum per unit time flowing in through the surface.

# Conservation of Momentum (cont'd)

Eq. (8) states the conservation of momentum in electrodynamics.

The field momentum density is

$$\boldsymbol{g} = \mu_0 \epsilon_0 \boldsymbol{S} = \epsilon_0 (\boldsymbol{E} \times \boldsymbol{B}) \tag{10}$$

and the momentum flux transported by the fields is $-\overleftrightarrow{\boldsymbol{T}}$. If the mechanical momentum in $V$ is not changing, then

$$\int \frac{\partial \boldsymbol{g}}{\partial t}\,\mathrm{d}\tau = \oint \overleftrightarrow{\boldsymbol{T}} \cdot \mathrm{d}\boldsymbol{a} = \int \nabla \cdot \overleftrightarrow{\boldsymbol{T}}\,\mathrm{d}\tau$$

and hence

$$\frac{\partial \boldsymbol{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\boldsymbol{T}} \tag{11}$$

"Continuity equation" for electromagnetic momentum, with $\boldsymbol{g}$ in the role of $\rho$ and $-\overleftrightarrow{\boldsymbol{T}}$ playing the part of $\boldsymbol{J}$.