Assignment 2 of Quantum Mechanics II [1]

(Due on August 29 2025 at 17:00. Submit your solution to Canvas as a PDF file.)

(You are only required to solve one of problem 3 and problem 4, but you are encouraged to do both.

If you do both, your score from these two problems will be the higher score from one of these two.)

Problem 1 (13 points). In class, we have discussed the matrix representation of the states and operators in a Hilbert space that is the tensor product of the Hilbert spaces of two qubits. It is claimed that if the matrix representation of a state $|\psi\rangle$ is acted by the matrix representation of an operator O, the result is the same as the matrix representation of $O|\psi\rangle$. In this problem, we justify the matrix representation discussed in class rigorously.

Concretely, our goal is to prove two statements.

1. If the matrix representation of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are the vectors v_1 and v_2, respectively, then the inner product $\langle\psi_1|\psi_2\rangle = v_1^\dagger v_2$.
2. If the matrix representation of the state $|\psi\rangle$ is the vector v, and the matrix representation of the operator O is M, then the matrix representation of $O|\psi\rangle$ is Mv.

If these two statements hold, then the matrix representation correctly captures structure of the states and operators in the tensor product Hilbert space, so that in the future, we can use this type of matrix representations to do calculations in systems made of multiple qubits. This matrix representation is essential in numerical computations relevant to these systems.

(1) (3 points) Suppose $|\psi_1\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ and $|\psi_2\rangle = d_{00}|00\rangle + d_{01}|01\rangle + d_{10}|10\rangle + d_{11}|11\rangle$. What are their matrix representations, v_1 and v_2?

(2) (3 points) Does the relation $\langle\psi_1|\psi_2\rangle = v_1^\dagger v_2$ hold? Why?

(3) (4 points) Suppose $O = \sum_{i,j,k,l=0}^{1} O_{ij,kl}|ij\rangle\langle kl|$. What is M, the matrix representation of O?

(4) (3 points) What is the matrix representation of $O|\psi_1\rangle$? Is it the same as Mv_1?

Remark 1: The calculations above might appear a bit tedious, so please think if there is a simpler way to understand the above results.

Remark 2: One should not take the validity of the matrix representation discussed in class for granted, and one should really check it as in this problem. To have an example of wrong matrix representation, consider writing the matrices for states following the "from left to right" recipe, while writing the matrices for operators following the "from right to left" recipe. These matrices are not valid representations because they do not preserve the structure of the Hilbert space.

Problem 2 (13 points). In class, we have discussed the creation and annihilation operators of bosons and fermions. In particular, we have seen the commutation relations of the bosonic creation and annihilation operators, i.e., $[b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0$ and $[b_i, b_j^\dagger] = \delta_{ij}$, and the anti-commutation relations of the fermionic

[1] You are welcome to get back with questions and clarifications if the wording of problems is ambiguous.

creation and annihilation operators, i.e., $\{f_i, f_j\} = \{f_i^\dagger, f_j^\dagger\} = 0$ and $\{f_i, f_j^\dagger\} = \delta_{ij}$. We have only proved that $[b_i^\dagger, b_j^\dagger] = 0$ in class. In this problem, we prove the rest of them.

(1) (1 points) Starting from $[b_i^\dagger, b_j^\dagger] = 0$, show that $[b_i, b_j] = 0$.

(2) (2 points) Next, we turn to $[b_i, b_j^\dagger] = \delta_{ij}$. To this end, we need to know the result obtained by acting b_i on the basis state $\overline{|n_1 n_2 \cdots\rangle}$. Based on the definition of $b_i^\dagger$ and the fact that b_i is the Hermitian conjugate of $b_i^\dagger$, calculate $b_i\overline{|n_1 n_2 \cdots\rangle}$.

(3) (2 points) As discussed in class, to show $[b_i, b_j^\dagger] = \delta_{ij}$, it suffices to show this relation for a basis. Suppose $i \neq j$, calculate $[b_i, b_j^\dagger]\overline{|n_1 n_2 \cdots\rangle}$.

(4) (2 points) Suppose $i = j$. Calculate $[b_i, b_j^\dagger]\overline{|n_1 n_2 \cdots\rangle}$. Combining part (3) and part (4), what is $[b_i, b_j^\dagger]$?

(5) (1 points) Now we move to the relation $\{f_i^\dagger, f_j^\dagger\} = 0$. Suppose $i = j$. Explain why $\{f_i, f_j\} = 0$.

(6) (2 points) Suppose $i \neq j$. Calculate $\{f_i, f_j\}\overline{|n_1 n_2 \cdots\rangle}$. Hint: You need to discuss a few cases separately.

(7) (3 points) Calculate $\{f_i, f_j^\dagger\}\overline{|n_1 n_2 \cdots\rangle}$. Hint: You need to discuss a few cases separately.

Problem 3 (13 points). In class, we have discussed three types of degrees of freedom, localized qubits, bosons and fermions. Naively, they look pretty different. In this problem, we will see that they can be related, although sometimes the relation can be subtle.

Figure 1: A chain with $L = 8$ sites.

We begin by considering a set of L qubits living on a one dimensional chain. Each location of a qubit is called a "site" (see Figure 1). The basis state of these qubits can be taken to be of the form $|n_1, n_2, \cdots, n_L\rangle$, where n_i can be 0 or 1, for $i = 1, 2, \cdots, L$.

Next, consider a fermionic system, where the locations of the fermions are only allowed to be taken from the sites of the chain in Figure 1 (as opposed to the continuum space). The basis of the Fock space of the fermions would also be written as $|n_1, n_2, \cdots, n_L\rangle$, where $n_i = 0$ means there is no fermion at the i-th site, and $n_i = 1$ means there is a fermion at the i-th site.

We can further consider a bosonic system, where the locations of the bosons are only allowed to be taken from the sites in Figure 1. Moreover, we imagine for some reason at each site there can be at most 1 boson at each site. Then the basis of the Fock space of the bosons could also be written as $|n_1, n_2, \cdots, n_L\rangle$, where $n_i = 0$ means there is no boson at the i-th site, and $n_i = 1$ means there is a boson at the i-th site.

The above observation already shows that the states in the qubit chain, the fermionic chain and bosonic chain have 1-to-1 correspondence. Below we will explore how the operators in these systems are related.

(1) (4 points) This time we start with the bosonic system, where the locations of the bosons are only allowed to be at the sites of the chain. Moreover, suppose the bosons are so strongly interacting, so that at each site there can be at most one boson. Such bosons are known as "hard-core bosons". Define the operator $b_i^\dagger$ via $b_i^\dagger|n_1, n_2, \cdots, n_{i-1}, n_i, n_{i+1}, \cdots, n_L\rangle = |n_1 n_2 \cdots n_{i-1}, 1, n_{i+1}, \cdots, n_L\rangle$ if $n_i = 0$ and $b_i^\dagger|n_1, n_2, \cdots, n_{i-1}, n_i, n_{i+1}, \cdots, n_L\rangle = 0$ if $n_i = 1$. Show that $[b_i, b_j] = 0$ for all i and j, $[b_i, b_j^\dagger] = 0$ if $i \neq j$, and $\{b_i, b_i^\dagger\} = 1$.

Remark: These relations are a mixture of the standard commutation and anti-commutation relations of bosons and fermions discussed in problem 2. The origin of these relations is of course the definition of $b_i^\dagger$, which differs from our definition of fermionic creation operator only by the prefactor.

(2) (4 points) Next, we consider the qubit system. Based on the standard Pauli operators, we define $\sigma_i^\pm = \sigma_i^x \pm i\sigma_i^y$. Show that $[\sigma_i^-, \sigma_j^-] = 0$ for all i and j, $[\sigma_i^-, \sigma_j^+] = 0$ for $i \neq j$ and $\{\sigma_i^-, \sigma_i^+\} = 1$.

Remark: Combining the above two parts, we can conclude that $b_i^\dagger$ of the hard-core bosons can be identified with σ_i^+ of the qubits.

(3) (2 points) Finally, we turn to the fermionic chain. Define $f_i^\dagger = \left(\prod_{j<i} \sigma_j^z\right) \sigma_i^+$. How should $f_i^\dagger f_i$ be expressed in terms of the Pauli operators? Show that for all i and j, $\{f_i, f_j\} = 0$ and $\{f_i, f_j^\dagger\} = \delta_{ij}$. Hint: Comparing this definition of $f_i^\dagger$ and the definition of the fermionic creation operator in class, these anti-commutation relations should be pretty obvious.

Remark: This transformation that relates the f's and the Pauli operators is called the Jordan-Wigner transformation. Since the f's can be viewed as the fermionic operators, the transformation relates the operators in the qubit chain and the fermionic chain.

(4) (3 points) How should the Pauli operators σ_i^+ and σ_i^z be written in terms of the f operators?

Remark: From this problem, it may be tempting to say that localized qubits, hard-core bosons and fermions are all equivalent. However, this is misleading. It is indeed appropriate to say that localized qubits and hard-core bosons are equivalent. However, these two are not really equivalent to fermions, because the Jordan-Wigner transformation maps local operators into non-local ones due to the string of operators multiplied together. In technical terms, this transformation does not preserve the local operator algebra, and the fermionic system should only be viewed as a system closely related (but not equivalent) to the localized qubits and hard-core bosons.

Problem 4 (14 points). Just being a student who can solve problems assigned by the teachers is good, but it is even better if you can propose good problems by yourself. In this class, besides helping you understand the basics of quantum mechanics and how to apply it, I also aim to help you improve your ability to raise good questions, which is useful no matter what you do in the future.

Now you are required to propose a problem that is similarly interesting as problem 3. After proposing it, you need to either give the complete solution to this problem, or argue that this problem is interesting but cannot be solved within a reasonable time frame.

Please use your own taste and opinion to judge what kind of problems are "similarly interesting as problem 3". At the end, the score you get from this problem is determined based on how interesting the grader thinks your proposed problem is, and whether the solution you provide is complete or whether you can convince the grader that the problem is interesting but too difficult.

Good problems may be selected as problems in the midterm and final exams this semester, or assignment problems in the future semesters. Really good problems may be further developed into research projects.