The Answer of Assignment 1

Problem 1 Solution

(1) In the position representation, from the eigenvalue equation $\hat{H}\psi(x) = h_n\psi(x)$ and the Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2$, we have:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) = h_n\psi(x)$$

The general solution is:

$$\psi(x) = c_1 e^{w_1 x} + c_2 e^{w_2 x}, \quad w_1 = i\sqrt{\frac{2mh_n}{\hbar^2}}, \quad w_2 = -i\sqrt{\frac{2mh_n}{\hbar^2}}$$

Given $p_n = \frac{2\pi\hbar n}{L}$, $h_n = \frac{2\pi^2\hbar^2 n^2}{Lm}$. Take $\psi_n(x) = e^{\frac{ip_n x}{\hbar}}$ as an example, the eigenvalue corresponding to $|\psi_n\rangle$ is $\frac{2\pi^2\hbar^2 n^2}{Lm}$.

(2)

$$\begin{aligned}
\langle\psi_{n_1}|\psi_{n_2}\rangle &= \int_{-\infty}^{+\infty} \psi_{n_1}^*(x)\,\psi_{n_2}(x)\,dx \\
&= \int_{-\infty}^{+\infty} e^{-\frac{ip_{n_1}x}{\hbar}} e^{\frac{ip_{n_2}x}{\hbar}} dx \\
&= \int_{-\infty}^{+\infty} e^{\frac{i(p_{n_2}-p_{n_1})x}{\hbar}} dx \\
&= \int_{-\infty}^{+\infty} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\
&= \lim_{l\to+\infty} \int_{-l}^{+l} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\
&= \lim_{l\to+\infty} \frac{L\sin\left[\frac{2\pi(n_2-n_1)l}{L}\right]}{\pi(n_2-n_1)}
\end{aligned}$$

When $n_2 = n_1$, $\langle\psi_{n_1}|\psi_{n_2}\rangle \to \infty$ and

$$\lim_{l\to+\infty}\int_{-\infty}^{+\infty} \frac{\sin(lx)}{x}dx = \pi$$

So,

$$\langle\psi_{n_1}|\psi_{n_2}\rangle = L\delta_{n_1 n_2}$$

(3) When $L \to \infty$, p_n becomes continuous and $\psi_n(x)$ becomes a plane wave.

$$\langle \psi_{p_1} | \psi_{p_2} \rangle = \int_{-\infty}^{+\infty} e^{-\frac{ip_1 x}{\hbar}} e^{\frac{ip_2 x}{\hbar}} dx = \lim_{l \to +\infty} \frac{2\hbar \sin\left[\frac{(p_2 - p_1)l}{\hbar}\right]}{(p_2 - p_1)} = 2\pi\delta(\frac{p_2 - p_1}{\hbar}) = 2\pi\hbar\delta(p_2 - p_1)$$

(4) In the position representation, $|\psi_{x_0,\epsilon}\rangle = \int_{-\infty}^{+\infty} \psi_{x_0,\epsilon}(x) |x\rangle \, dx$

$$\langle \delta | \delta \rangle = \int_{-\infty}^{+\infty} (x\psi_{x_0,\epsilon}(x) - x_0\psi_{x_0,\epsilon}(x))^*(x\psi_{x_0,\epsilon}(x) - x_0\psi_{x_0,\epsilon}(x))dx$$

Considering the Gaussian wave packet $\psi_{x_0,\epsilon}(x) = (\frac{1}{2\pi\epsilon^2})^{1/4} e^{\frac{-(x-x_0)^2}{4\epsilon^2}}$

$$\int_{-\infty}^{+\infty} \psi^*\psi dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy = 1$$

and

$$\int_{-\infty}^{+\infty} (x - x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = -\frac{\partial}{\partial a} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-a(x-x_0)^2} dx \bigg|_{a=\frac{1}{2\epsilon^2}}$$

Note that

$$\int_{-\infty}^{+\infty} e^{-a(x-x_0)^2} dx = \sqrt{\frac{\pi}{a}}$$

Therefore,

$$-\frac{\partial}{\partial a}\sqrt{\frac{\pi}{a}} = \frac{1}{2}\sqrt{\pi} a^{-3/2}$$

Substituting $a = \frac{1}{2\epsilon^2}$, we get

$$\int_{-\infty}^{+\infty} (x - x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \epsilon^2$$

To ensure $\langle \delta | \delta \rangle < \epsilon$, we can set $\delta = \frac{\sqrt{\epsilon}}{2}$, so $\psi_{x_0,\delta}(x) = \frac{1}{\sqrt[4]{2\pi\delta^2}} e^{-\frac{(x-x_0)^2}{4\delta^2}}$. According to the previous derivation,

$$\langle \delta | \delta \rangle = \int_{-\infty}^{+\infty} (x\psi_{x_0,\delta}(x) - x_0\psi_{x_0,\delta}(x))^* (x\psi_{x_0,\delta}(x) - x_0\psi_{x_0,\delta}(x)) \, dx = \delta^2 = \frac{\epsilon}{4}$$

this satisfies the condition $\langle \delta | \delta \rangle < \epsilon$.

Problem 2 Solution

(1) Consider a qubit with Hamiltonian

$$H = -\mu BY$$

and and the initial state at time t = 0 is $|\uparrow\rangle$. We have

$$\hat{H} |\psi\rangle = i\hbar \frac{d}{dt} |\psi\rangle$$

The time evolution of the state is given by:

$$\begin{aligned}|\psi(t)\rangle &= e^{-\frac{i}{\hbar}Ht}\,|\uparrow\rangle \\ &= e^{-\frac{i}{\hbar}(-\mu BY)t}\,|\uparrow\rangle \\ &= e^{i\frac{\mu B}{\hbar}Yt}\,|\uparrow\rangle\end{aligned}$$

Using Taylor expansion, we can express this as:

$$|\psi(t)\rangle = |\uparrow\rangle\left(1 + i\frac{\mu B}{\hbar}Yt - \frac{1}{2}\left(\frac{\mu B}{\hbar}Yt\right)^2 + \cdots\right) = \cos(\theta)\,|\uparrow\rangle - \sin(\theta)\,|\downarrow\rangle$$

where $\theta = \frac{\mu B}{\hbar}t$. The probability of measuring $|\uparrow\rangle$ at time t is $p_\uparrow(t) = \cos^2(\theta)$, and the probability of measuring $|\downarrow\rangle$ is $p_\downarrow(t) = \sin^2(\theta)$. The expectation value Z measured at this time is given by:

$$\langle Z\rangle = \langle\psi(t)|Z|\psi(t)\rangle = \cos^2(\theta)\,\langle\uparrow|Z|\uparrow\rangle + \sin^2(\theta)\,\langle\downarrow|Z|\downarrow\rangle + 2\cos(\theta)\sin(\theta)\,\langle\uparrow|Z|\downarrow\rangle$$

where $\langle\uparrow|Z|\uparrow\rangle = 1$, $\langle\downarrow|Z|\downarrow\rangle = -1$, and $\langle\uparrow|Z|\downarrow\rangle = 0$. Thus, we have:

$$\begin{aligned}\langle Z\rangle &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos(2\theta)\end{aligned}$$

likewise, we can record the n-th measure result as $S_n(0/1)$ and we have:

$$\begin{cases} p(S_0(0)) = 1 \\ p(S_1(0)) = \cos^2(\theta)\cdot 1 \\ \cdots \\ p(S_n(0)) = \cos^2(\theta)\cdot p(S_{n-1}(0)) + \sin^2(\theta)\cdot p(S_{n-1}(1)) \end{cases}$$

$$\begin{cases} p(S_0(1)) = 0 \\ p(S_1(1)) = \sin^2(\theta)\cdot 1 \\ \cdots \\ p(S_n(1)) = \sin^2(\theta)\cdot p(S_{n-1}(0)) + \cos^2(\theta)\cdot p(S_{n-1}(1)) \end{cases}$$

The number of all the possible sequences at the length n is 2^n. We use the method of induction to testify the sum of the probabilities of all sequences at the length n is 1.

Base case: For $n = 1$, we have:

$$p(S_1(0)) + p(S_1(1)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

Inductive step: Assume it holds for $n = k$, i.e.,

$$\text{when} n = k+1 \sum_{i=0}^{2^n} p = \sum_{i=0}^{2^{n-1}} p\cdot(cos^2(\theta) + sin^2(\theta)) + (\sum_{i=0}^{2^{n-1}} p\cdot\sin^2(\theta)) + \cos^2(\theta)\cdot(\sum_{i=0}^{2^{n-1}} p) = 1$$