

# The Answer of Assignment 2

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## Problem 1 Solution

(1) According to the theory of isomorphism of linear spaces, any two linear spaces of the same dimension are isomorphic. Therefore,  $\sum_{i,j=0}^1 c_{ij} |i, j\rangle$  is isomorphic to  $\{(x_1, x_2, x_3, x_4)\}, x \in \mathbb{C}$ . That is,  $v_1 = (c_{00}, c_{01}, c_{10}, c_{11})^T$ ,  $v_2 = (d_{00}, d_{01}, d_{10}, d_{11})^T$ .

(2)  $\langle \psi_1 | \psi_2 \rangle = v_1^\dagger v_2$  成立, 这是因为  $\langle \psi_1 | \psi_2 \rangle = (\sum_{i,j=0}^1 c_{i,j}^* \langle ij|) (\sum_{i,j=0}^1 c_{i,j} |ij\rangle) = v_1^\dagger v_2$

(3)  $O |pq\rangle = O_{ij,kl} |ij\rangle \langle kl| |pq\rangle = O_{ij,kl} |ij\rangle \delta_{kp} \delta_{lq} = O_{ij,pq} |ij\rangle$  if we define  $|00\rangle$  as  $e_1$ ,  $|01\rangle$  as  $e_2$ ,  $|10\rangle$  as  $e_3$ ,  $|11\rangle$  as  $e_4$ , then we can get the matrix representation of  $O$ :  $Oe_i = \sum_{j=1}^4 O_{ji} e_j$  Therefore, the matrix representation of  $O$  is:

$$M = \begin{pmatrix} O_{11} & O_{12} & O_{13} & O_{14} \\ O_{21} & O_{22} & O_{23} & O_{24} \\ O_{31} & O_{32} & O_{33} & O_{34} \\ O_{41} & O_{42} & O_{43} & O_{44} \end{pmatrix}$$

(4)  $O |\psi_1\rangle$  is the same as  $Mv_1$ , because:

$$O |\psi_1\rangle = O_{ij,kl} |ij\rangle \langle kl| \cdot c_{mn} |mn\rangle = O_{ij,kl} |ij\rangle c_{mn} \delta_{km} \delta_{ln} = O_{ij,mn} c_{mn} |ij\rangle$$

this is equivalent to  $Mv_1$  if we define  $|00\rangle$  as  $e_1$ ,  $|01\rangle$  as  $e_2$ ,  $|10\rangle$  as  $e_3$ ,  $|11\rangle$  as  $e_4$ .

## Problem 2 Solution

(1) Starting from  $[b_i^\dagger, b_i^\dagger] = 0$ , show that  $[b_i, b_j] = 0$ . According to the definition of commutation relation, we have:

$$[b_i^\dagger, b_j^\dagger] = b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0$$

Taking the Hermitian conjugate of both sides, we get:

$$(b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger)^\dagger = 0^\dagger$$

$$b_j b_i - b_i b_j = 0$$

so we have:  $[b_i, b_j] = 0$

(2)

$$\langle \overline{n'_1 n'_2 \cdots n'_k} | b_i^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle = \langle \overline{n_1 n_2 \cdots n_k} | b_i | \overline{n'_1 n'_2 \cdots n'_k} \rangle^* = \delta_{n_1, n'_1} \delta_{n_2, n'_2} \cdots \delta_{n_i, n'_i-1} \cdots \delta_{n_k, n'_k} \sqrt{n'_i}$$

and

$$b_i^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle = \sqrt{n_i + 1} | \overline{n_1 n_2 \cdots (n_i + 1) \cdots n_k} \rangle$$

we can get the relation between  $b_i$  and  $b_i^\dagger$  as follows:

$$b_i | \overline{n_1 n_2 \cdots n_k} \rangle = \sqrt{n_i} | \overline{n_1 n_2 \cdots (n_i - 1) \cdots n_k} \rangle$$

(3)

We have:

$$\begin{aligned} [b_i, b_j^\dagger] | \overline{n_1 n_2 \cdots n_k} \rangle &= b_i b_j^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle - b_j^\dagger b_i | \overline{n_1 n_2 \cdots n_k} \rangle \\ &= b_i \sqrt{n_j + 1} | \overline{n_1 n_2 \cdots (n_j + 1) \cdots n_k} \rangle \\ &\quad - b_j^\dagger \sqrt{n_i} | \overline{n_1 n_2 \cdots (n_i - 1) \cdots n_k} \rangle \\ &= \sqrt{n_i(n_j + 1)} | \overline{n_1 n_2 \cdots (n_i - 1) \cdots (n_j + 1) \cdots n_k} \rangle - 0 \end{aligned}$$

(4) if  $i = j$ , we have:

$$[b_i, b_i^\dagger] | \overline{n_1 n_2 \cdots n_k} \rangle = (n_i + 1 - n_i) | \overline{n_1 n_2 \cdots n_k} \rangle = | \overline{n_1 n_2 \cdots n_k} \rangle$$

(5)

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

When  $i = j$ , this becomes

$$\{f_i, f_i\} = f_i f_i + f_i f_i = 2f_i^2$$

Since  $f_i$  is a fermionic annihilation operator, it satisfies  $f_i^2 = 0$ . Therefore,

$$\{f_i, f_i\} = 2 \times 0 = 0$$

So, when  $i = j$ , the anticommutation relation  $\{f_i, f_j\} = 0$  holds.

(6)

Suppose  $i \neq j$ . Calculate  $\{f_i, f_j\} | n_1 n_2 \cdots \rangle$ .

Recall the definition:

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

We need to consider the action of  $f_i$  and  $f_j$  on the occupation number basis  $|n_1 n_2 \dots\rangle$ , where  $n_k = 0$  or  $1$  for fermions.

Consider the following cases:

**Case 1:**  $n_i = 0$  or  $n_j = 0$

- If  $n_i = 0$ , then  $f_i |n_1 n_2 \dots\rangle = 0$ . - If  $n_j = 0$ , then  $f_j |n_1 n_2 \dots\rangle = 0$ .

Therefore, in either case, both  $f_i f_j |n_1 n_2 \dots\rangle = 0$  and  $f_j f_i |n_1 n_2 \dots\rangle = 0$ , so

$$\{f_i, f_j\} |n_1 n_2 \dots\rangle = 0$$

**Case 2:**  $n_i = 1$  and  $n_j = 1$

-  $f_j |n_1 n_2 \dots n_j = 1 \dots n_i = 1 \dots\rangle = (-1)^{s_1} |n_1 \dots n_j = 0 \dots n_i = 1 \dots\rangle$  - Then  $f_i$  acts on this state:  $f_i |n_1 \dots n_j = 0 \dots n_i = 1 \dots\rangle = (-1)^{s_2} |n_1 \dots n_j = 0 \dots n_i = 0 \dots\rangle$

Similarly,  $f_i f_j |n_1 n_2 \dots\rangle$  and  $f_j f_i |n_1 n_2 \dots\rangle$  will differ by a sign, but since  $i \neq j$ , the sum  $f_i f_j + f_j f_i$  will always cancel out due to the anticommutation property of fermionic operators.

Therefore,

$$\{f_i, f_j\} |n_1 n_2 \dots\rangle = 0$$

**(7) case 1:**  $i = j$

$$f_i, f_j^\dagger = f_i f_i^\dagger + f_i^\dagger f_i$$

when  $n_i = 0$ , we have:

$$\begin{aligned} f_i f_i^\dagger |n_1 n_2 \dots n_i = 0 \dots\rangle \\ = f_i (-1)^s |n_1 n_2 \dots n_i = 1 \dots\rangle \\ = (-1)^s (-1)^s |n_1 n_2 \dots n_i = 0 \dots\rangle \\ = |n_1 n_2 \dots n_i = 0 \dots\rangle \end{aligned}$$

when  $n_i = 1$ , we have:

$$\begin{aligned} f_i^\dagger f_i |n_1 n_2 \dots n_i = 1 \dots\rangle &= f_i^\dagger (-1)^s |n_1 n_2 \dots n_i = 0 \dots\rangle \\ &= (-1)^s (-1)^s |n_1 n_2 \dots n_i = 1 \dots\rangle = |n_1 n_2 \dots n_i = 1 \dots\rangle \end{aligned}$$

**case 2:**  $i \neq j$

$$f_i, f_j^\dagger = f_i f_j^\dagger + f_j^\dagger f_i$$

when  $n_i = 0$  or  $n_j = 0$ , we have:

### Problem 3 Solution

(1) We define  $b_i^\dagger$  in

$$b_i^\dagger |n_1, n_2, \dots, n_k\rangle = |n_1, n_2, \dots, n_i + 1, \dots, n_L\rangle = |n_1, n_2, \dots, 1, n_{i+1}, \dots, n_L\rangle \text{ if } n_i = 0$$

$$b_i^\dagger |n_1, n_2, \dots, n_k\rangle = 0 \text{ if } n_i = 1$$

$$b_i |n_1, n_2, \dots, n_k\rangle = |n_1, n_2, \dots, n_i - 1, \dots, n_L\rangle = |n_1, n_2, \dots, 0, n_{i+1}, \dots, n_L\rangle \text{ if } n_i = 1$$

$$b_i |n_1, n_2, \dots, n_k\rangle = 0 \text{ if } n_i = 0$$

$[b_i, b_j] = 0$  for any  $i, j$  because when  $i=j$   $(b_i b_j - b_j b_i) |n_1, n_2, \dots, n_L\rangle = 0$  if  $n_i = 0$   $[b_i, b_j] = 0$  if  $n_i = 1$  because  $b_i b_j |n_1, n_2, \dots, n_L\rangle = |n_1, n_2, \dots, n_L\rangle$  and  $b_j b_i |n_1, n_2, \dots, n_L\rangle = |n_1, n_2, \dots, n_L\rangle$  when  $i \neq j$ ,  $(b_i b_j - b_j b_i) |n_1, n_2, \dots, n_L\rangle = 0$  because  $b_i b_j |n_1, n_2, \dots, n_L\rangle$  and  $b_j b_i |n_1, n_2, \dots, n_L\rangle$  are the same state. (2)  $\sigma_i^{+/-} = (\sigma_i^x + / - \sigma_i^y \cdot i)/2$  (3)

(4) how should the Pauli operators  $\sigma_i^+$  and  $\sigma_i^z$  can be written in terms of the  $f$  operators? we have the definition that:

$$f_i^\dagger = \left( \prod_{j < i} \sigma_j^z \right) \sigma_i^+$$

so from  $\langle \overline{n_1 n_2 \dots n_i} | f_i^\dagger | \overline{n_1 n_2 n_3 \dots n_i} \rangle = \langle \overline{n_1 n_2 \dots n_i} | f_i | \overline{n_1 n_2 \dots n_i} \rangle^*$  we can get the relation between  $f_i$  and  $\sigma_i^+$ ,  $\sigma_i^z$  as follows:

$$f_i = \left( \prod_{j < i} \sigma_j^z \right) \sigma_i^-$$

then try to represent  $\sigma_i^z$  and  $\sigma_i^+$  in terms of  $f_i$ : we use the matrix representation of the Pauli operators:

$$\sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\sigma_i^z$  can be expressed as:  $2\sigma_i^+ \sigma_i^- - I = 2f_i^\dagger f_i - I$

like wise:

$$\begin{aligned} \sigma_i^+ &= f_i^\dagger \left( \prod_{j < i} \sigma_j^z \right)^{-1} \\ &= f_i^\dagger \left( \prod_{j < i} (2f_j^\dagger f_j - I) \right) \end{aligned}$$