Constants

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$ (permittivity of free space)

$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$ (permeability of free space)

$c = 3.00 \times 10^{8}\ \mathrm{m/s}$ (speed of light)

$e = 1.60 \times 10^{-19}\ \mathrm{C}$ (charge of the electron)

$m = 9.11 \times 10^{-31}\mathrm{kg}$ (mass of the electron)

Spherical Coordinates

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\boldsymbol{x}} = \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{y}} = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \end{cases}$$

Cylindrical Coordinates

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\boldsymbol{x}} = \cos\phi\,\hat{\boldsymbol{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{y}} = \sin\phi\,\hat{\boldsymbol{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

Vector Derivatives: Cartesian

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}x\,\hat{\boldsymbol{x}} + \mathrm{d}y\,\hat{\boldsymbol{y}} + \mathrm{d}z\,\hat{\boldsymbol{z}}; \quad \mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial t}{\partial y}\,\hat{\boldsymbol{y}} + \frac{\partial t}{\partial z}\,\hat{\boldsymbol{z}}$

Divergence: $\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \boldsymbol{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\boldsymbol{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}; \quad \mathrm{d}\tau = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\nabla \times \boldsymbol{v} = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Vector Derivatives: Cylindrical

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}s\,\hat{\boldsymbol{s}} + s\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\boldsymbol{z}}; \quad \mathrm{d}\tau = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$$

<u>Gradient</u>: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\boldsymbol{z}}$$

<u>Divergence</u>: $$\nabla \cdot \boldsymbol{v} = \frac{1}{s}\frac{\partial (s v_s)}{\partial s} + \frac{1}{s}\frac{\partial (v_\phi)}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

<u>Curl</u>: $$\nabla \times \boldsymbol{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\boldsymbol{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial (s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial \phi}\right]\hat{\boldsymbol{z}}$$

<u>Laplacian</u>: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Vector Identities

Triple Products

1. $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$
2. $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$

Product Rules

3. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
4. $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}$
5. $\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\nabla f)$
6. $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$
7. $\nabla\times(f\boldsymbol{A}) = f(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\times(\nabla f)$
8. $\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A})$

Second Derivatives

9. $\nabla\cdot(\nabla\times\boldsymbol{A}) = 0$
10. $\nabla\times(\nabla f) = 0$
11. $\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$

Fundamental Theorems

Gradient Theorem: $\int_{\boldsymbol{a}}^{\boldsymbol{b}} (\nabla f) \cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})$

Divergence Theorem: $\int (\nabla \cdot \boldsymbol{A})\, \mathrm{d}\tau = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{a}$

Curl Theorem: $\int (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{a} = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$

Basic Equations of Electrodynamics

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho_f \\ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \\ \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \end{cases}$$

Linear media:

$$\begin{cases} \boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}, \quad \boldsymbol{D} = \epsilon \boldsymbol{E} \\ \boldsymbol{M} = \chi_m \boldsymbol{H}, \quad \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} \end{cases}$$

Potentials

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Lorentz force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Energy, Momentum and Power

$$\text{Energy: } U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \mathrm{d}\tau$$

$$\text{Momentm: } \boldsymbol{P} = \epsilon_0 \int (\boldsymbol{E} \times \boldsymbol{B})\, \mathrm{d}\tau$$

$$\text{Poynting vector: } \boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$$

$$\text{Larmor formula: } P = \frac{\mu_0}{6\pi c} q^2 a^2$$