Poynting’s Theorem

Poynting's Theorem

Energy stored in an electric field:

$$\frac{\epsilon_0}{2}\int E^2\,\mathrm{d}\tau$$

Energy stored in a magnetic field:

$$\frac{1}{2\mu_0}\int B^2\,\mathrm{d}\tau$$

Therefore, total energy stored in electromagnetic fields, per unit volume, is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \tag{1}$$

Poynting's Theorem (cont'd)

At time t, some charge and current configuration produces $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. From Lorentz force law, the work done by the electromagnetic forces on a charge q in time $\mathrm{d}t$ is

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\, \mathrm{d}t = q\boldsymbol{E} \cdot \boldsymbol{v}\, \mathrm{d}t.$$

Now, $q = \rho\, \mathrm{d}\tau$ and $\rho\boldsymbol{v} = \boldsymbol{J}$, so the rate at which work is done on all the charges in a volume V is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_V (\boldsymbol{E} \cdot \boldsymbol{J})\, \mathrm{d}\tau \qquad (2)$$

Now $\boldsymbol{E} \cdot \boldsymbol{J}$ is the power delivered per unit volume. Using the Ampere-Maxwell law to eliminate $\boldsymbol{J}$:

$$\boldsymbol{E} \cdot \boldsymbol{J} = \frac{1}{\mu_0}\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t}$$

Poynting's Theorem (cont'd)

From product rule 6,

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B})$$

Invoking Faraday's law ($\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$), it follows that

$$\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) = -\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B})$$

Meanwhile,

$$\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \text{ and } \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2) \tag{3}$$

so

$$\boldsymbol{E} \cdot \boldsymbol{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \frac{1}{\mu_0} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) \tag{4}$$

Poynting's Theorem (cont'd)

Putting this into Eq. (2), and applying the divergence theorem to the second term, we have

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\mathrm{d}\tau - \frac{1}{\mu_0}\oint_S (\boldsymbol{E}\times\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{a} \tag{5}$$

where S is the surface bounding V.

This is **Poynting's theorem**. The first integral on RHS is the total energy stored in the fields. The second term represents the rate at which energy is carried out of V, across its boundary surface, by the electromagnetic fields.

The energy per unit time, per unit area, transported by the fields is called the **Poynting vector**:

$$\boldsymbol{S} \equiv \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B}) \tag{6}$$

Poynting's Theorem (cont'd)

Now, from Eq. (5),

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V u \,\mathrm{d}\tau - \oint_S \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} \tag{7}$$

If no work is done on the charges in V, we have $\mathrm{d}W/\mathrm{d}t = 0$, so

$$\int \frac{\partial u}{\partial t}\,\mathrm{d}\tau = -\oint \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = -\int (\nabla \cdot \boldsymbol{S})\,\mathrm{d}\tau$$

and hence

$$\frac{\partial u}{\partial t} = -\nabla \cdot \boldsymbol{S} \tag{8}$$

This is the continuity equation for the local conservation of electromagnetic energy. u (energy density) plays the role of ρ (charge density) and $\boldsymbol{S}$ takes the part of $\boldsymbol{J}$.

Example

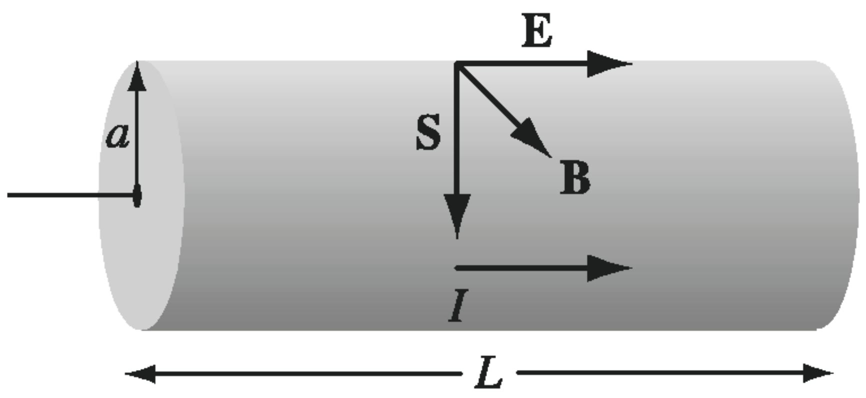
When current flows down a wire, work is done, which shows up as Joule heating of the wire. The energy per unit time delivered to the wire can be calculated using the Poynting vector.

Assuming it's uniform, the electric field parallel to the wire is

$$E = \frac{V}{L}$$

where V is the potential difference between the ends and L is the length of the wire. Magnetic field at the surface is

$$B = \frac{\mu_0 I}{2\pi a}$$



Magnitude of the Poynting vector (which is pointing radially inwards) is

$$S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L}$$

Thus, the power passing in through the surface of the wire is

$$-\int \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} = S(2\pi a L) = VI$$