

The Answer of Assignment 1

Problem 1 Solution

(1) In the position representation, from the eigenvalue equation $\hat{H}\psi(x) = h_n\psi(x)$ and the Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2$, we have:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x) = h_n\psi(x)$$

The general solution is:

$$\psi(x) = c_1 e^{w_1 x} + c_2 e^{w_2 x}, \quad w_1 = i\sqrt{\frac{2mh_n}{\hbar^2}}, \quad w_2 = -i\sqrt{\frac{2mh_n}{\hbar^2}}$$

Given $p_n = \frac{2\pi\hbar n}{L}$, $h_n = \frac{2\pi^2\hbar^2 n^2}{Lm}$. Take $\psi_n(x) = e^{\frac{ip_n x}{\hbar}}$ as an example, the eigenvalue corresponding to $|\psi_n\rangle$ is $\frac{2\pi^2\hbar^2 n^2}{Lm}$.

(2)

$$\begin{aligned} \langle\psi_{n_1}|\psi_{n_2}\rangle &= \int_{-\infty}^{+\infty} \psi_{n_1}^*(x) \psi_{n_2}(x) dx \\ &= \int_{-\infty}^{+\infty} e^{-\frac{ip_{n_1}x}{\hbar}} e^{\frac{ip_{n_2}x}{\hbar}} dx \\ &= \int_{-\infty}^{+\infty} e^{\frac{i(p_{n_2}-p_{n_1})x}{\hbar}} dx \\ &= \int_{-\infty}^{+\infty} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\ &= \lim_{l \rightarrow +\infty} \int_{-l}^{+l} e^{\frac{i2\pi(n_2-n_1)x}{L}} dx \\ &= \lim_{l \rightarrow +\infty} \frac{L \sin\left[\frac{2\pi(n_2-n_1)l}{L}\right]}{\pi(n_2-n_1)} \end{aligned}$$

When $n_2 = n_1$, $\langle\psi_{n_1}|\psi_{n_2}\rangle \rightarrow \infty$ and

$$\lim_{l \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{\sin(lx)}{x} dx = \pi$$

So,

$$\langle\psi_{n_1}|\psi_{n_2}\rangle = L\delta_{n_1 n_2}$$

(3) When $L \rightarrow \infty$, p_n becomes continuous and $\psi_n(x)$ becomes a plane wave.

$$\langle \psi_{p_1} | \psi_{p_2} \rangle = \int_{-\infty}^{+\infty} e^{-\frac{ip_1 x}{\hbar}} e^{\frac{ip_2 x}{\hbar}} dx = \lim_{l \rightarrow +\infty} \frac{2\hbar \sin \left[\frac{(p_2 - p_1)l}{\hbar} \right]}{(p_2 - p_1)} = 2\pi \delta \left(\frac{p_2 - p_1}{\hbar} \right) = 2\pi \hbar \delta(p_2 - p_1)$$

(4) In the position representation, $|\psi_{x_0, \epsilon}\rangle = \int_{-\infty}^{+\infty} \psi_{x_0, \epsilon}(x) |x\rangle dx$

$$\langle \delta | \delta \rangle = \int_{-\infty}^{+\infty} (x\psi_{x_0, \epsilon}(x) - x_0\psi_{x_0, \epsilon}(x))^* (x\psi_{x_0, \epsilon}(x) - x_0\psi_{x_0, \epsilon}(x)) dx$$

Considering the Gaussian wave packet $\psi_{x_0, \epsilon}(x) = \left(\frac{1}{2\pi\epsilon^2}\right)^{1/4} e^{-\frac{(x-x_0)^2}{4\epsilon^2}}$

$$\int_{-\infty}^{+\infty} \psi^* \psi dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1$$

and

$$\int_{-\infty}^{+\infty} (x-x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = -\frac{\partial}{\partial a} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-a(x-x_0)^2} dx \Big|_{a=\frac{1}{2\epsilon^2}}$$

Note that

$$\int_{-\infty}^{+\infty} e^{-a(x-x_0)^2} dx = \sqrt{\frac{\pi}{a}}$$

Therefore,

$$-\frac{\partial}{\partial a} \sqrt{\frac{\pi}{a}} = \frac{1}{2} \sqrt{\pi} a^{-3/2}$$

Substituting $a = \frac{1}{2\epsilon^2}$, we get

$$\int_{-\infty}^{+\infty} (x-x_0)^2 \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{(x-x_0)^2}{2\epsilon^2}} dx = \epsilon^2$$

To ensure $\langle \delta | \delta \rangle < \epsilon$, we can set $\delta = \frac{\sqrt{\epsilon}}{2}$, so $\psi_{x_0, \delta}(x) = \frac{1}{\sqrt{4\pi\delta^2}} e^{-\frac{(x-x_0)^2}{4\delta^2}}$. According to the previous derivation,

$$\langle \delta | \delta \rangle = \int_{-\infty}^{+\infty} (x\psi_{x_0, \delta}(x) - x_0\psi_{x_0, \delta}(x))^* (x\psi_{x_0, \delta}(x) - x_0\psi_{x_0, \delta}(x)) dx = \delta^2 = \frac{\epsilon}{4}$$

this satisfies the condition $\langle \delta | \delta \rangle < \epsilon$.

Problem 2 Solution

(1) Consider a qubit with Hamiltonian

$$H = -\mu BY$$

and the initial state at time $t = 0$ is $|\uparrow\rangle$. We have

$$\hat{H} |\psi\rangle = i\hbar \frac{d}{dt} |\psi\rangle$$

The time evolution of the state is given by:

$$\begin{aligned} |\psi(t)\rangle &= e^{-\frac{i}{\hbar}Ht} |\uparrow\rangle \\ &= e^{-\frac{i}{\hbar}(-\mu BY)t} |\uparrow\rangle \\ &= e^{i\frac{\mu B}{\hbar}Yt} |\uparrow\rangle \end{aligned}$$

Using Taylor expansion, we can express this as:

$$|\psi(t)\rangle = |\uparrow\rangle \left(1 + i\frac{\mu B}{\hbar}Yt - \frac{1}{2} \left(\frac{\mu B}{\hbar}Yt \right)^2 + \dots \right) = \cos(\theta) |\uparrow\rangle - \sin(\theta) |\downarrow\rangle$$

where $\theta = \frac{\mu B}{\hbar}t$. The probability of measuring $|\uparrow\rangle$ at time t is $p_{\uparrow}(t) = \cos^2(\theta)$, and the probability of measuring $|\downarrow\rangle$ is $p_{\downarrow}(t) = \sin^2(\theta)$. The expectation value Z measured at this time is given by:

$$\langle Z \rangle = \langle \psi(t) | Z | \psi(t) \rangle = \cos^2(\theta) \langle \uparrow | Z | \uparrow \rangle + \sin^2(\theta) \langle \downarrow | Z | \downarrow \rangle + 2 \cos(\theta) \sin(\theta) \langle \uparrow | Z | \downarrow \rangle$$

where $\langle \uparrow | Z | \uparrow \rangle = 1$, $\langle \downarrow | Z | \downarrow \rangle = -1$, and $\langle \uparrow | Z | \downarrow \rangle = 0$. Thus, we have:

$$\begin{aligned} \langle Z \rangle &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos(2\theta) \end{aligned}$$

(2) We can record the n -th measurement result as $S_n(0/1)$ and we have:

$$\begin{cases} p(S_0(0)) = 1 \\ p(S_1(0)) = \cos^2(\theta) \cdot 1 \\ \dots \\ p(S_n(0)) = \cos^2(\theta) \cdot p(S_{n-1}(0)) + \sin^2(\theta) \cdot p(S_{n-1}(1)) \end{cases}$$

$$\begin{cases} p(S_0(1)) = 0 \\ p(S_1(1)) = \sin^2(\theta) \cdot 1 \\ \dots \\ p(S_n(1)) = \sin^2(\theta) \cdot p(S_{n-1}(0)) + \cos^2(\theta) \cdot p(S_{n-1}(1)) \end{cases}$$

The number of all the possible sequences at the length n is 2^n . We use the method of induction to testify the sum of the probabilities of all sequences at the length n is 1.

Base case: For $n = 1$, we have:

$$p(S_1(0)) + p(S_1(1)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

Inductive step: Assume it holds for $n = k$, i.e.,

$$\sum_{i=0}^{2^k} p_i = 1.$$

Now consider $n = k + 1$:

$$\sum_{i=0}^{2^{k+1}} p_i = \sum_{i=0}^{2^k} p_i \cdot (\cos^2(\theta) + \sin^2(\theta)) = \sum_{i=0}^{2^k} p_i \cdot 1 = \sum_{i=0}^{2^k} p_i = 1.$$

(3) When $\delta t \rightarrow 0$ while T is fixed:

$$p(|00 \cdots 0\rangle) = \lim_{\delta t \rightarrow 0} p(S_n(0)) = \lim_{\delta t \rightarrow 0} \cos^{2[\frac{T}{\delta t}]}(\frac{\mu B \delta t}{\hbar}) = e^{\frac{2T}{\delta t} \ln(\cos(\frac{\mu B \delta t}{\hbar}))} = 1$$

(4) At $t = T$, what is the probability to find the qubit in state $|\uparrow\rangle$, and what is the probability to find the qubit in state $|\downarrow\rangle$? What is the expectation value $\langle Z \rangle$ measured at this time? we have:

$$p_{n+1}(0) = p_n(0) \cdot \cos^2(\theta) + (1 - p_n(0)) \cdot \sin^2(\theta)$$

$$p_{n+1}(0) - \frac{1}{2} = \cos(2\theta)(p_n(0) - \frac{1}{2})$$

as a result, we have:

$$p_n(0) = \lim_{n \rightarrow \infty} \cos^{\frac{T}{\delta t}}(2\theta) \cdot \frac{1}{2} + \frac{1}{2} = 1$$

$$p_n(1) = \frac{1}{2} - \cos^{\frac{T}{\delta t}}(2\theta) \cdot \frac{1}{2}$$

the expectation value $\langle Z \rangle$ measured at this time is given by:

$$\langle Z \rangle = \cos^{\frac{T}{\delta t}}(2\theta)$$

Problem 3 Solution

(1) A self-adjoint operator O in Hilbert space can be spectrally decomposed. Suppose its eigenvalues are λ_n and the corresponding eigenstates are $|\phi_n\rangle$, then:

$$\langle u|O|v\rangle = \sum_n \lambda_n \langle u|\phi_n\rangle \langle \phi_n|v\rangle \leq \max_n |\lambda_n| \sqrt{\langle u|u\rangle \langle v|v\rangle} = \max_n |\lambda_n| \|u\| \|v\|$$

(2) We try to prove

$$|\psi_2\rangle = \cos(\theta)e^{i\alpha} |\psi_1\rangle + \sin(\theta)e^{i\beta} |\psi_w\rangle$$

In linear algebra, an inner product space can be regarded as the direct sum of a subspace and its orthogonal complement. For $|\psi_2\rangle$, it can be decomposed as $|\psi_2\rangle = \cos(\theta)e^{i\alpha} |\psi_1\rangle + \sin(\theta)e^{i\beta} |\psi_w\rangle$, where $|\psi_1\rangle$ is the projection of $|\psi_2\rangle$ onto $|\psi_1\rangle$, and $|\psi_w\rangle$ is the projection of $|\psi_2\rangle$ onto $|\psi_w\rangle$. Since $|\psi_1\rangle$ and $|\psi_w\rangle$ are orthogonal, their inner product is 0, i.e., $\langle \psi_1|\psi_w\rangle = 0$.