Electrostatic Field in Matter

- Dielectric responds to an external electric field $\boldsymbol{E}_{\text{ext}}(\boldsymbol{r})$ by distorting its charge density to produce a field $\boldsymbol{E}_{\text{self}}(\boldsymbol{r})$ so that the total electric field is the sum of these fields:

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\text{ext}}(\boldsymbol{r}) + \boldsymbol{E}_{\text{self}}(\boldsymbol{r})$$

- Dominant features of dielectrics, on average, are those associated with electric dipole moments
- Polarization: electric dipole moment per unit volume

$$\boldsymbol{p} = \iiint \boldsymbol{P}(\boldsymbol{r}')\,\mathrm{d}v'$$

- Potential of an ideal electric dipole with electric moment $\boldsymbol{p}$: $V(\boldsymbol{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}$

Electrostatic field of dielectric matter

- Electrostatic scalar potential:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \iiint \boldsymbol{P}(\boldsymbol{r}') \cdot \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \mathrm{d}v', \quad \boldsymbol{E} = -\nabla V(\boldsymbol{r})$$

- Integrating by parts:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \oiint \frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \cdot \mathrm{d}\boldsymbol{a}' + \frac{1}{4\pi\epsilon_0} \iiint \frac{-\nabla' \cdot \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}v'$$

- Polarization charge densities (or bound charge densities):

$$\rho_b(\boldsymbol{r}) \equiv -\nabla \cdot \boldsymbol{P}(\boldsymbol{r}), \quad \sigma_b \equiv \boldsymbol{P}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}}$$

Example 1: Uniformly polarized sphere

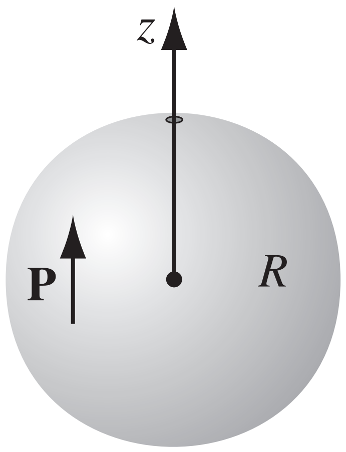
- A sphere of radius R centered at the origin with uniform polarization $\boldsymbol{P}(\boldsymbol{r}) = P\hat{\boldsymbol{z}}$ inside the sphere

- Polarization charge densities:

$$\rho_b(\boldsymbol{r}) = 0, \quad \sigma_b(\boldsymbol{r}) = P\cos\theta$$

- Show that the electrostatic scalar potential and field at an arbitrary point on the z-axis due to the uniformly polarized sphere:

$$V(\boldsymbol{r}) = \begin{cases} \dfrac{Pz}{3\epsilon_0}, & z \le R \\ \dfrac{PR^3}{3\epsilon_0 z^2}, & z \ge R \end{cases} \qquad \boldsymbol{E}(\boldsymbol{r}) = \begin{cases} -\dfrac{P}{3\epsilon_0}\hat{\boldsymbol{z}}, & z < R \\ \dfrac{2PR^3}{3\epsilon_0 z^3}\hat{\boldsymbol{z}}, & z > R \end{cases}$$

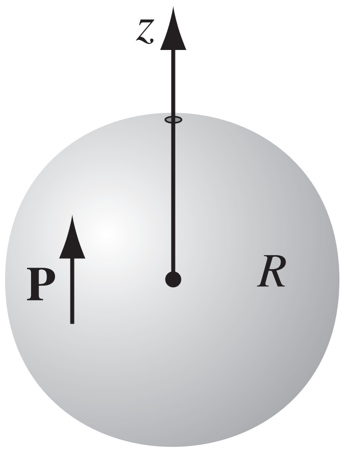


$$\boldsymbol{r} = (0, 0, z), \quad \sigma_b = P\cos\theta$$
$$\boldsymbol{r}' = (R\sin\theta'\cos\phi', R\sin\theta'\sin\phi', R\cos\theta')$$

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\oiint \frac{\sigma(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}a' \\ &= \frac{1}{4\pi\epsilon_0}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} \frac{P\cos\theta'}{\sqrt{R^2\sin^2\theta' + (z-R\cos\theta')^2}} R^2\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi' \end{aligned}$$

But $\displaystyle\int_{\theta'=0}^{\pi} \frac{\cos\theta'\sin\theta'\,\mathrm{d}\theta'}{\sqrt{R^2\sin^2\theta' + (z-R\cos\theta')^2}} = \frac{|R+z|(R^2-Rz+z^2)}{3R^2z^2} - \frac{|R-z|(R^2+Rz+z^2)}{3R^2z^2}$

Example 1 cont'd



$$\therefore V(\boldsymbol{r}) = \begin{cases} \dfrac{Pz}{3\epsilon_0}, & z \leq R \\[2ex] \dfrac{PR^3}{3\epsilon_0 z^2}, & z \geq R \end{cases}$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r}) = \begin{cases} -\dfrac{P}{3\epsilon_0}\hat{\boldsymbol{z}}, & z < R \\[2ex] \dfrac{2PR^3}{3\epsilon_0 z^3}\hat{\boldsymbol{z}}, & z > R \end{cases}$$

Electrostatic field and auxiliary field

- Gauss's law for electrostatic field: $\rho(\boldsymbol{r}) = \rho_f(\boldsymbol{r}) + \rho_b(\boldsymbol{r})$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r})}{\epsilon_0} \quad \Rightarrow \quad \nabla \cdot [\underbrace{\epsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r})}_{\boldsymbol{D}(\boldsymbol{r})}] = \rho_f(\boldsymbol{r})$$

- Electric displacement field:

$$\boldsymbol{D}(\boldsymbol{r}) = \epsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r})$$

- Maxwell's equation for electrostatic auxiliary field:

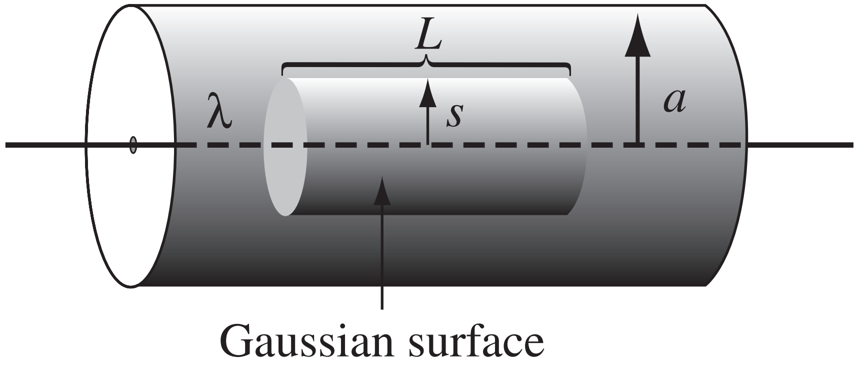
$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho_f(\boldsymbol{r})$$

Gauss's law for electric displacement field

- Gauss's law for electric displacement field in integral form:

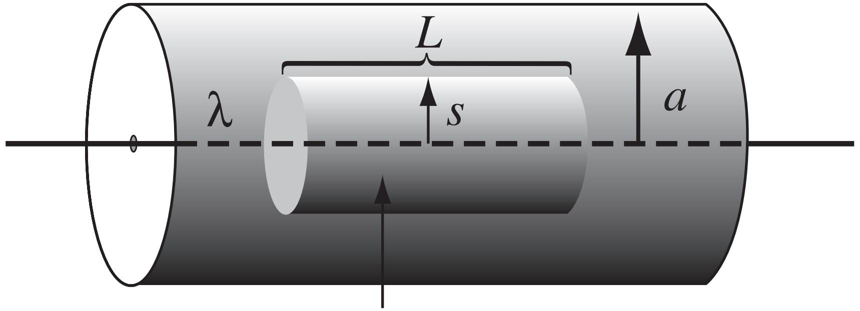
$$\oiint \boldsymbol{D}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{a} = \iiint \nabla \cdot \boldsymbol{D}(\boldsymbol{r})\,\mathrm{d}v = \iiint \rho_f(\boldsymbol{r})\,\mathrm{d}v = Q_{f,\text{ enclosed}}$$

- Gaussian surface can be chosen to match the symmetry of the distribution of free charges to find high-symmetric electric displacement field

- 

Example 2: A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement field.

Example 2



- Drawing a cylindrical Gaussian surface, of radius s and length L.

$$\oiint \boldsymbol{D}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{a} = Q_{f,\,\text{enclosed}}$$

$$D(2\pi s L) = \lambda L$$

$$\therefore \boldsymbol{D} = \frac{\lambda}{2\pi s}\hat{\boldsymbol{s}}$$

This formula holds both within the insulation and outside it. In the latter region, $\boldsymbol{P} = 0$, so

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\boldsymbol{s}}, \quad s > a$$

Inside the rubber, the electric field cannot be determined, since we do not know $\boldsymbol{P}$.

Linear dielectrics

- Linear isotropic dielectrics: electric susceptibility χ_e

$$\boldsymbol{P}(\boldsymbol{r}) = \epsilon_0 \chi_e(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r})$$

- Constitutive equation: dielectric constant κ, permittivity ϵ

$$\begin{aligned}\boldsymbol{D}(\boldsymbol{r}) &= \epsilon_0 \underbrace{[1 + \chi_e(\boldsymbol{r})]}_{\kappa(\boldsymbol{r})} \boldsymbol{E}(\boldsymbol{r}) \\ &= \epsilon_0 \, \kappa(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r}) \\ &= \epsilon(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r})\end{aligned}$$

Linear, isotropic, homogeneous dielectrics

- Homogeneous dielectric: dielectric constant not a function of position

$$\boldsymbol{P}(\boldsymbol{r}) = \epsilon_0 \chi_e \boldsymbol{E}(\boldsymbol{r}), \quad \boldsymbol{D}(\boldsymbol{r}) = \epsilon_0 (1 + \chi_e) \boldsymbol{E}(\boldsymbol{r}) = \epsilon_0 \kappa \boldsymbol{E}(\boldsymbol{r}) = \epsilon \boldsymbol{E}(\boldsymbol{r})$$

- Poisson's equation:

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho_f(\boldsymbol{r}) \quad \Rightarrow \quad \epsilon \nabla \cdot \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{E}(\boldsymbol{r}) \cdot \nabla \epsilon = \rho_f(\boldsymbol{r}) \quad \Rightarrow \quad \nabla^2 V(\boldsymbol{r}) = -\frac{\rho_f(\boldsymbol{r})}{\epsilon}$$

- Polarization charge densities:

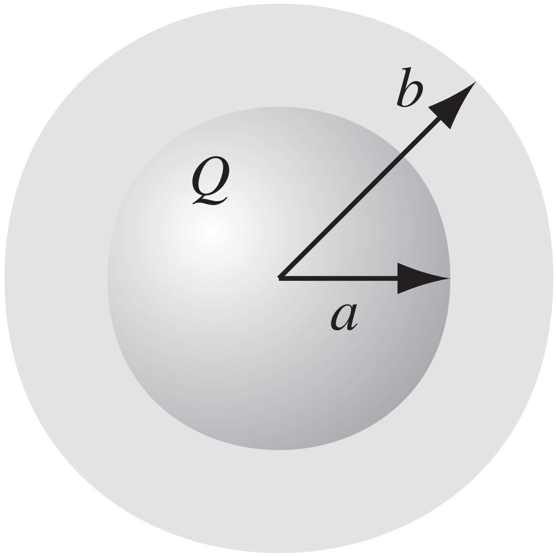
$$\rho_b(\boldsymbol{r}) = -\nabla \cdot \boldsymbol{P}(\boldsymbol{r}) = \left(\frac{1}{\kappa} - 1\right) \rho_f(\boldsymbol{r}), \quad \sigma_b = \boldsymbol{P}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}} = \epsilon_0 (\kappa - 1) \boldsymbol{E}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}}$$

- Total volume charge density: reduced locally by a factor κ

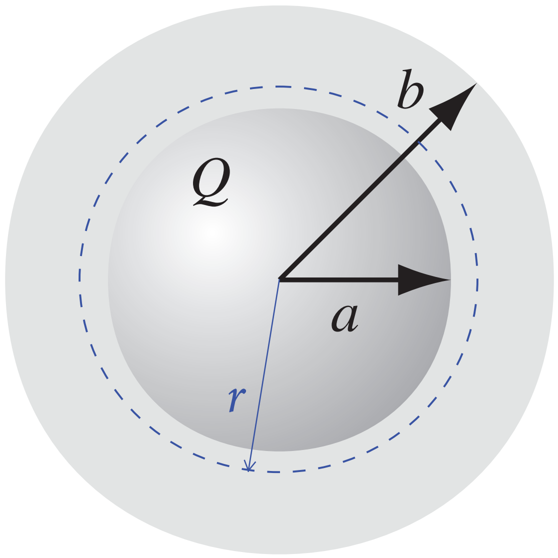
$$\rho(\boldsymbol{r}) = \rho_b(\boldsymbol{r}) + \rho_f(\boldsymbol{r}) = \frac{1}{\kappa} \rho_f(\boldsymbol{r})$$

Example 3

A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ.



Find the potential at the center (relative to infinity).



We shall calculate $\boldsymbol{D}$ using Gauss's law:

$$\oint \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} = Q_{f_\text{enc}}$$

$$D\left(4\pi r^2\right) = Q$$

$$\therefore \boldsymbol{D} = \frac{Q}{4\pi r^2}\hat{\boldsymbol{r}}, \quad \text{for } r > a$$

Inside metal sphere, $\boldsymbol{E} = \boldsymbol{P} = \boldsymbol{D} = \mathbf{0}$.

Potential at center is

$$\begin{aligned} V &= -\int_\infty^0 \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{\ell} \\ &= -\int_\infty^b \left(\frac{Q}{4\pi\epsilon_0 r^2}\right) \mathrm{d}r - \int_b^a \left(\frac{Q}{4\pi\epsilon r^2}\right) \mathrm{d}r \\ &\quad - \int_a^0 (0)\,\mathrm{d}r \\ &= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right) \end{aligned}$$

Energy in dielectric systems

- Work done on the incremental free charge $\Delta\rho_f$: $\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho_f(\boldsymbol{r})$

$$\Delta W = \iiint V(\boldsymbol{r})[\Delta\rho_f(\boldsymbol{r})]\,\mathrm{d}v = \iiint V(\boldsymbol{r})\nabla \cdot [\Delta \boldsymbol{D}(\boldsymbol{r})]\,\mathrm{d}v$$

- Integrating by parts:

$$\Delta W = \iiint \boldsymbol{E}(\boldsymbol{r}) \cdot [\Delta \boldsymbol{D}(\boldsymbol{r})]\,\mathrm{d}v$$

- Linear, homogeneous dielectric: $\boldsymbol{D}(\boldsymbol{r}) = \epsilon \boldsymbol{E}(\boldsymbol{r})$

$$[\Delta \boldsymbol{D}(\boldsymbol{r})] \cdot \boldsymbol{E}(\boldsymbol{r}) = \epsilon[\Delta \boldsymbol{E}(\boldsymbol{r})] \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{2}\Delta[\epsilon \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{E}(\boldsymbol{r})] = \frac{1}{2}\Delta[\boldsymbol{D}(\boldsymbol{r}) \cdot \boldsymbol{E}(\boldsymbol{r})]$$

$$\therefore \Delta W = \Delta \iiint \frac{1}{2}\boldsymbol{D}(\boldsymbol{r}) \cdot \boldsymbol{E}(\boldsymbol{r})\,\mathrm{d}v \quad \Rightarrow \quad U_E = \frac{1}{2}\iiint \boldsymbol{D}(\boldsymbol{r}) \cdot \boldsymbol{E}(\boldsymbol{r})\,\mathrm{d}v$$

- Example 4: A sphere of radius R is filled with material of dielectric constant κ and uniformly embedded free charge ρ_f. What is the energy of this configuration?

From Gauss's law,

$$\boldsymbol{D}(r) = \begin{cases} \frac{\rho_f}{3}\boldsymbol{r}, & r < R \\ \frac{\rho_f}{3}\frac{R^3}{r^2}\hat{\boldsymbol{r}}, & r > R \end{cases}$$

So, electric field is

$$\boldsymbol{E}(r) = \begin{cases} \frac{\rho_f}{3\epsilon_0\kappa}\boldsymbol{r}, & r < R \\ \frac{\rho_f}{3\epsilon_0}\frac{R^3}{r^2}\hat{\boldsymbol{r}}, & r > R \end{cases}$$

$$\begin{aligned}
\text{Energy is } W &= \frac{1}{2}\iiint \boldsymbol{D}\cdot\boldsymbol{E}\,\mathrm{d}v \\
&= \frac{1}{2}\left[\left(\frac{\rho_f}{3}\right)\left(\frac{\rho_f}{3\epsilon_0\kappa}\right)\int_0^R r^2(4\pi r^2\,\mathrm{d}r) + \left(\frac{\rho_f R^3}{3}\right)\left(\frac{\rho_f R^3}{3\epsilon_0}\right)\int_R^\infty \frac{1}{r^4}(4\pi r^2\,\mathrm{d}r)\right] \\
&= \frac{2\pi R^5\rho_f^2}{45\epsilon_0\kappa} + \frac{2\pi R^5\rho_f^2}{9\epsilon_0} \\
&= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\kappa}+1\right)
\end{aligned}$$

Magnetostatic Field in Matter

Magnetostatic field of magnetized matter

- Magnetization: magnetic dipole moment per unit volume

$$\boldsymbol{m} = \iiint \boldsymbol{M}(\boldsymbol{r}')\,\mathrm{d}v'$$

- Magnetostatic vector potential:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint \boldsymbol{M}(\boldsymbol{r}') \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}v', \quad \boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r})$$

- Integrating by parts:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\nabla' \times \boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}v' + \frac{\mu_0}{4\pi}\oiint \frac{\boldsymbol{M}(\boldsymbol{r}') \times \hat{\boldsymbol{n}}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}a'$$

- Magnetization current densities (or bound current densities):

$$\boldsymbol{J}_b(\boldsymbol{r}) = \nabla \times \boldsymbol{M}(\boldsymbol{r}), \quad \boldsymbol{K}_b(\boldsymbol{r}) = \boldsymbol{M}(\boldsymbol{r}) \times \hat{\boldsymbol{n}}$$

Example 5: Uniformly magnetized long cylinder

- An infinitely long cylinder carries a uniform magnetization $\boldsymbol{M}(\boldsymbol{r}) = M\hat{\boldsymbol{z}}$, parallel to its axis.

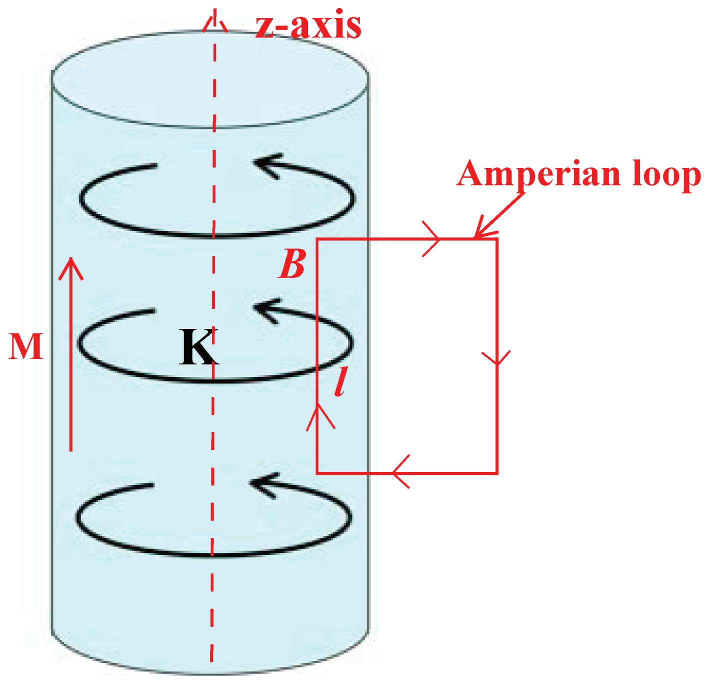
- Magnetization current densities:

$$\boldsymbol{J}_b(\boldsymbol{r}) = \boldsymbol{0}, \quad \boldsymbol{K}_b(\boldsymbol{r}) = M\hat{\boldsymbol{\phi}}$$

- Show that the magnetostatic field due to the uniformly magnetized cylinder:

$$\boldsymbol{B} = \begin{cases} \mu_0 M\hat{\boldsymbol{z}}, & s < a \\ \boldsymbol{0}, & s > a \end{cases}$$

Example 5 cont'd



Using Ampère's law

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_{\text{enclosed}}$$

$$B_z \ell = \mu_0 K \ell$$

$$\therefore \boldsymbol{B} = \mu_0 K \hat{\boldsymbol{z}} = \mu_0 M \hat{\boldsymbol{z}}$$

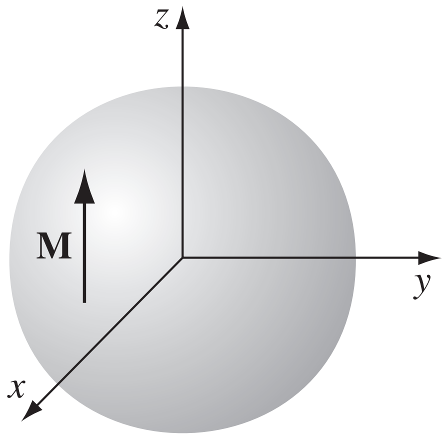
Problem is the same as that of an infinitely long solenoid. Surface current density $\boldsymbol{K}_b = M\hat{\boldsymbol{\phi}}$. Magnetic field outside is zero.

Example 6: Uniformly magnetized sphere

- A sphere of radius R centered at the origin with uniform magnetization $\boldsymbol{M}(\boldsymbol{r}) = M\hat{\boldsymbol{z}}$ inside the sphere
- Magnetization current densities:

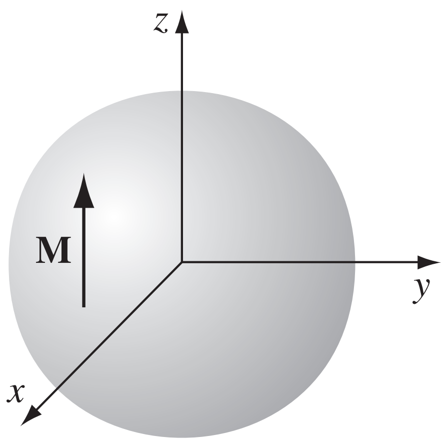
$$\boldsymbol{J}(\boldsymbol{r}) = \boldsymbol{0}, \quad \boldsymbol{K}_b(\boldsymbol{r}) = M\sin\theta\hat{\boldsymbol{\phi}}$$

- Show that the magnetostatic field at an arbitrary point on the z-axis due to the uniformly magnetized sphere (of magnetization $\boldsymbol{M} = M\hat{\boldsymbol{z}}$) is given by



$$\boldsymbol{B}(\boldsymbol{r}) = \begin{cases} \frac{2}{3}\mu_0 M\hat{\boldsymbol{z}}, & z < R \\ \frac{2}{3}\frac{\mu_0 M R^3}{z^3}\hat{\boldsymbol{z}}, & z > R \end{cases}$$

Example 6: Uniformly magnetized sphere cont'd



$$\boldsymbol{K}_b(\boldsymbol{r}) = M\sin\theta\hat{\boldsymbol{\phi}}, \quad \boldsymbol{r} = (0,0,z),$$

$$\boldsymbol{r}' = \left(R\sin\theta'\cos\phi', R\sin\theta'\sin\phi', R\cos\theta'\right)$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iint \boldsymbol{K}_b \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}a'$$

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0 M}{4\pi}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}[\sin\theta'(-\sin\phi'\hat{\boldsymbol{x}} + \cos\phi'\hat{\boldsymbol{y}})] \\ &\quad\times \frac{-R\sin\theta'\cos\phi'\hat{\boldsymbol{x}} - R\sin\theta'\sin\phi'\hat{\boldsymbol{y}} + (z-R\cos\theta')\hat{\boldsymbol{z}}}{\left[R^2\sin^2\theta' + (z-R\cos\theta')^2\right]^{3/2}} R^2\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi' \\ &= \frac{\mu_0 M}{4\pi}\iint \frac{[\sin\theta'\cos\phi'(z-R\cos\theta')]\hat{\boldsymbol{x}} + [\sin\theta'\sin\phi'(z-R\cos\theta')]\hat{\boldsymbol{y}} + R\sin^2\theta'\hat{\boldsymbol{z}}}{\left[R^2\sin^2\theta' + (z-R\cos\theta')^2\right]^{3/2}} R^2\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi'\end{aligned}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 M}{4\pi} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \frac{R \sin^2 \theta' \hat{\boldsymbol{z}}}{\left[R^2 \sin^2 \theta' + (z - R\cos\theta')^2\right]^{3/2}} R^2 \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\phi'$$

$$= \hat{\boldsymbol{z}} \frac{\mu_0 M}{4\pi} R^3 (2\pi) \int_{\theta'=0}^{\pi} \frac{\sin^3 \theta'}{\left[R^2 \sin^2 \theta' + (z - R\cos\theta')^2\right]^{3/2}} \mathrm{d}\theta'$$

$$= \begin{cases} \frac{2}{3}\mu_0 M \hat{\boldsymbol{z}}, & z < R \\ \frac{2}{3}\frac{\mu_0 M R^3}{z^3}\hat{\boldsymbol{z}}, & z > R \end{cases}$$

Note: $\displaystyle\int_{\theta'=0}^{\pi} \frac{\sin^3 \theta'}{\left[R^2 \sin^2 \theta' + (z - R\cos\theta')^2\right]^{3/2}} \mathrm{d}\theta' = \frac{2(z^3 - R^3)\,\mathrm{sign}(R - z) + 2(R^3 + z^3)\,\mathrm{sign}(R + z)}{3R^3 z^3}$

where $\mathrm{sign}(x) =$ -1, 0, +1 if x is negative, zero or positive respectively.

Magnetostatic field and auxiliary field

- Ampère's law for magnetostatic field: $\boldsymbol{J}(\boldsymbol{r}) = \boldsymbol{J}_f(\boldsymbol{r}) + \boldsymbol{J}_b(\boldsymbol{r})$

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}(\boldsymbol{r}) \quad \Rightarrow \quad \nabla \times \left[\frac{\boldsymbol{B}(\boldsymbol{r})}{\mu_0} - \boldsymbol{M}(\boldsymbol{r}) \right] = \boldsymbol{J}_f(\boldsymbol{r})$$

- Magnetostatic auxiliary field:

$$\boldsymbol{H}(\boldsymbol{r}) \equiv \frac{\boldsymbol{B}(\boldsymbol{r})}{\mu_0} - \boldsymbol{M}(\boldsymbol{r})$$

- Maxwell's equation for magnetostatic auxiliary field:

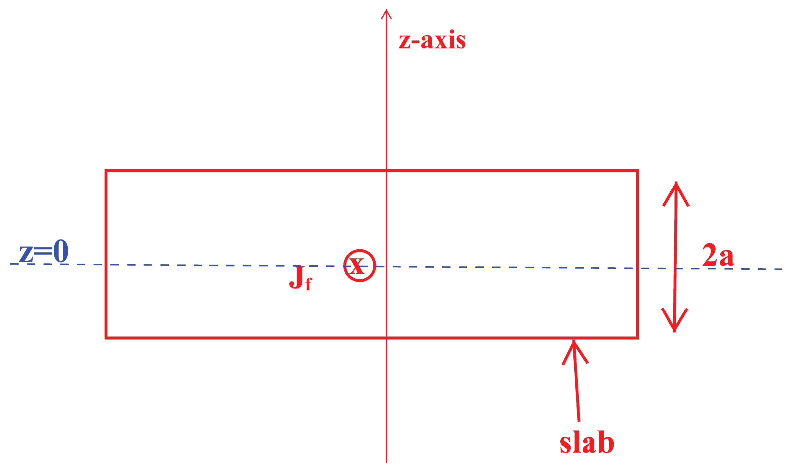
$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}_f(\boldsymbol{r})$$

Ampère’s law for magnetostatic auxiliary field

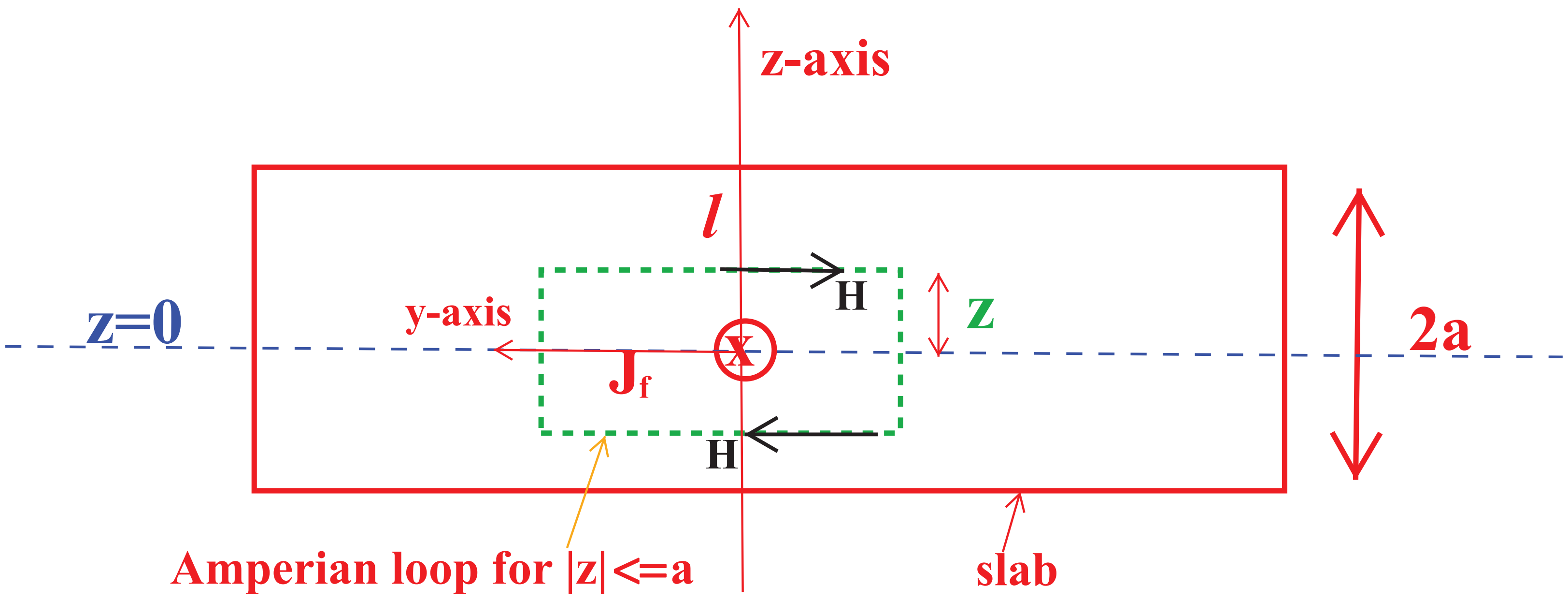
- Ampère’s law for magnetostatic field in integral form:

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{\ell} = \iint [\nabla \times \boldsymbol{H}(\boldsymbol{r})] \cdot \mathrm{d}\boldsymbol{a} = \iint \boldsymbol{J}_f(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{a} = I_{f,\ \text{enclosed}}$$

- Amperian loop can be chosen to match the symmetry of the distribution of free currents to find high-symmetric magnetostatic auxiliary field



Example 7: An infinite slab of magnetic material is parallel to the xy plane lying between $z = -a$ and $z = +a$. The slab carries a uniform free volume current density $\boldsymbol{J}_f(\boldsymbol{r}) = J_0\hat{\boldsymbol{x}}$ where J_0 is a positive constant. Find the magnetostatic auxiliary field.

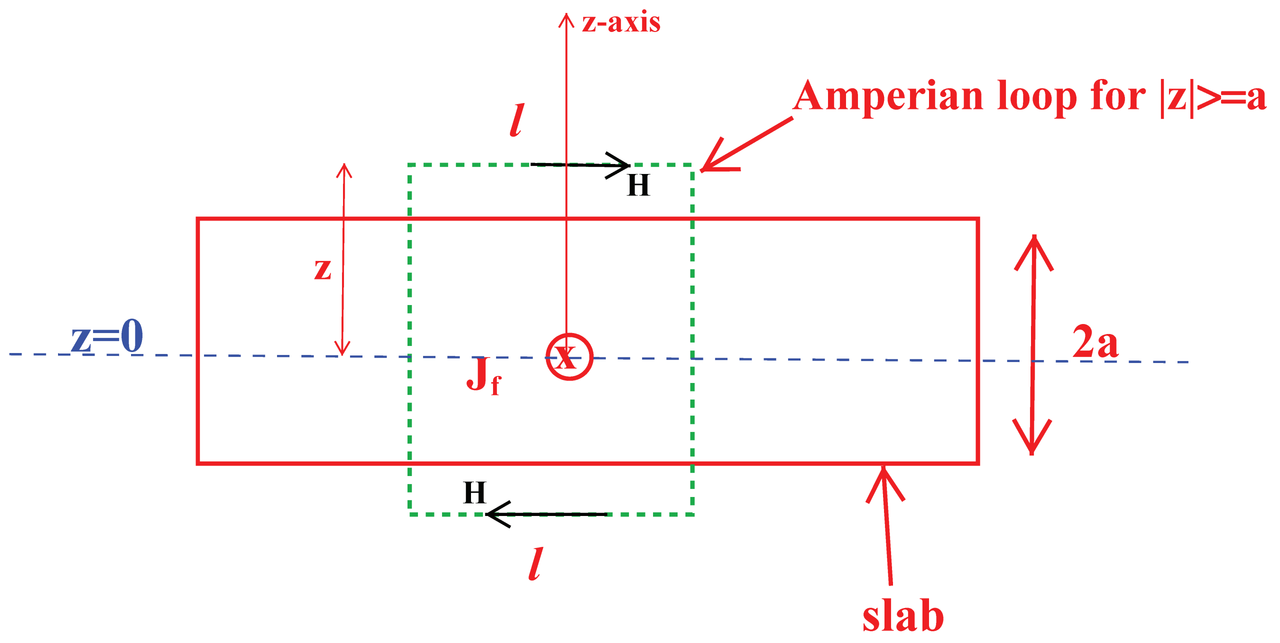


- For $|z| \leq a$,

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{\ell} = I_f$$

$$2Hl = J_0(2zl)$$

$$\therefore \boldsymbol{H} = -J_0 z\,\hat{\boldsymbol{y}}$$



- For $|z| \geq a$,

$$\oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = I_f \quad \Rightarrow \quad 2Hl = J_0(2al)$$

$$\therefore \boldsymbol{H} = -\frac{z}{|z|} J_0 a\, \hat{\boldsymbol{y}}$$

Linear magnetic materials

- Linear isotropic magnetic material:

$$\boldsymbol{M}(\boldsymbol{r}) = \chi_m(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r}), \quad \begin{cases} \chi_m > 0, & \text{paramagnetic} \\ \chi_m < 0, & \text{diamagnetic} \end{cases} \qquad |\chi_m| \ll 1$$

- Constitutive equation: relative permeability μ_r, permeability μ

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{r}) &= \mu_0 \underbrace{[1 + \chi_m(\boldsymbol{r})]}_{\mu_r(\boldsymbol{r})} \boldsymbol{H}(\boldsymbol{r}) \\ &= \mu_0\mu_r(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r}) \\ &= \mu(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r}) \end{aligned}$$

Linear isotropic homogeneous magnetic materials

- Magnetic properties are independent of position:

$$\boldsymbol{M}(\boldsymbol{r}) = \chi_m \boldsymbol{H}(\boldsymbol{r})$$
$$\boldsymbol{B}(\boldsymbol{r}) = \mu_0(1+\chi_m)\boldsymbol{H}(\boldsymbol{r}) = \mu_0\mu_r\boldsymbol{H}(\boldsymbol{r}) = \mu\boldsymbol{H}(\boldsymbol{r})$$

- Poisson's equation: imposing Coulomb gauge

$$\nabla\times\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}_f \Rightarrow \frac{1}{\mu}\nabla\times\boldsymbol{B}(\boldsymbol{r}) - \boldsymbol{B}(\boldsymbol{r})\times\nabla\left(\frac{1}{\mu}\right) = \boldsymbol{J}_f(\boldsymbol{r}) \Rightarrow \nabla^2\boldsymbol{A}(\boldsymbol{r}) = -\mu\boldsymbol{J}_f(\boldsymbol{r})$$

- Magnetization current densities:

$$\boldsymbol{J}_b(\boldsymbol{r}) = \nabla\times\boldsymbol{M}(\boldsymbol{r}) = (\mu_r-1)\boldsymbol{J}_f(\boldsymbol{r})$$
$$\boldsymbol{K}_b(\boldsymbol{r}) = \boldsymbol{M}(\boldsymbol{r})\times\hat{\boldsymbol{n}} = (\mu_r-1)\boldsymbol{H}(\boldsymbol{r})\times\hat{\boldsymbol{n}}$$

Energy in magnetic systems

- Maxwell's equations in matters:

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho_f(\boldsymbol{r},t), \qquad \nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0, \qquad \nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_f(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$$

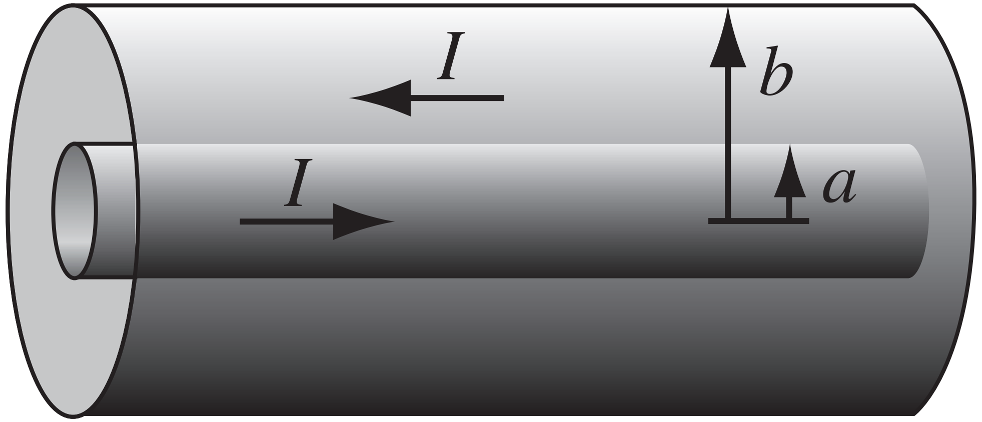
- Energy in linear, isotropic, homogeneous magnetic materials:

$$U_B = \frac{1}{2}\iiint \boldsymbol{H}(\boldsymbol{r}) \cdot \boldsymbol{B}(\boldsymbol{r})\,\mathrm{d}v$$

- Magnetic energy density:

$$u_B = \frac{1}{2}\boldsymbol{H}(\boldsymbol{r}) \cdot \boldsymbol{B}(\boldsymbol{r}) = \frac{1}{2\mu}\boldsymbol{B}(\boldsymbol{r}) \cdot \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu}{2}\boldsymbol{H}(\boldsymbol{r}) \cdot \boldsymbol{H}(\boldsymbol{r})$$

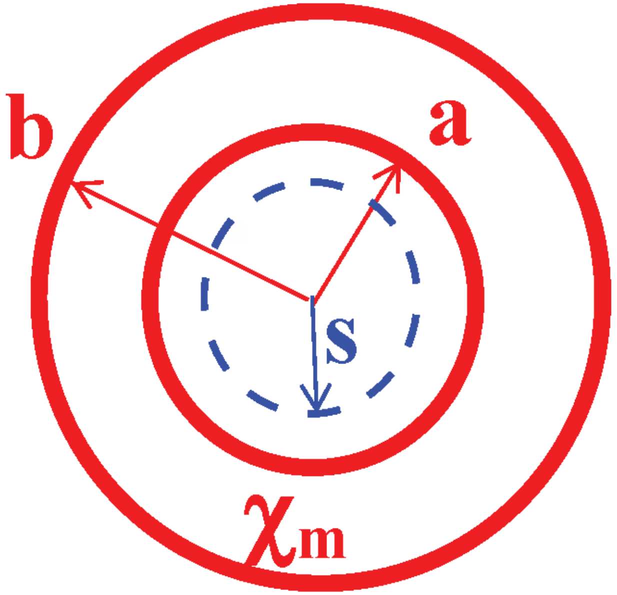
Example 8



- A long coaxial cable consists of an inner cylindrical conductor of radius a and an outer cylindrical shell of radius b. A current I flows along the $+z$ axis uniformly across the inner conductor and returns along the outer one uniformly over its surface.

  The region between the inner conductor and outer shell is filled with a linear magnetic material of susceptibility χ_m.

  Find the energy per unit length of this system.

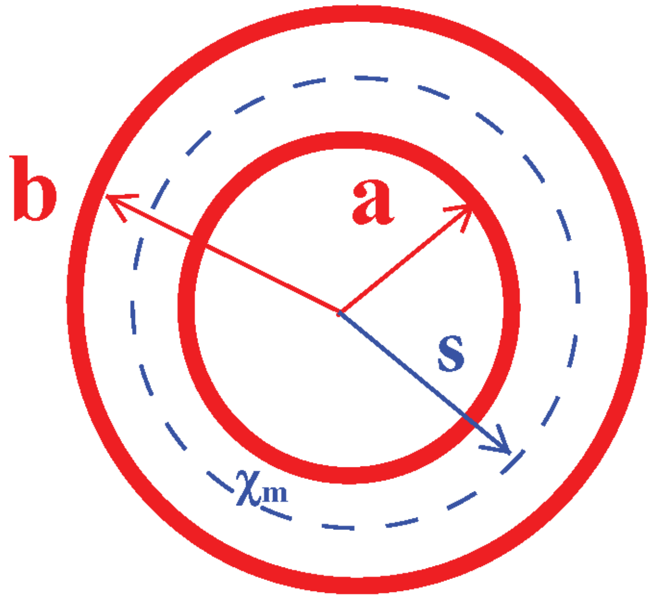


- For $s \leq a$,

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{\ell} = I_{f,\text{enclosed}}$$

$$H(2\pi s) = I\,\frac{\pi s^2}{\pi a^2} = I\,\frac{s^2}{a^2}$$

$$\therefore \boldsymbol{H} = I\,\frac{s}{2\pi a^2}\hat{\boldsymbol{\phi}}$$



- For $a \leq s < b$,

$$\oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = I_{f,\text{enclosed}}$$

$$H(2\pi s) = I$$

$$\therefore \boldsymbol{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}$$

- For $s > b$,

$$\boldsymbol{H} = \boldsymbol{0}$$

Example 8 cont'd

$$\boldsymbol{H} = \begin{cases} \frac{Is}{2\pi a^2}\hat{\boldsymbol{\phi}}, & s \le a \\ \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}, & a \le s < b \\ \mathbf{0}, & s > b \end{cases} \qquad \boldsymbol{B} = \mu_0(1+\chi_m)\boldsymbol{H} = \begin{cases} \frac{\mu_0 Is}{2\pi a^2}\hat{\boldsymbol{\phi}}, & s \le a \\ \frac{\mu_0(1+\chi_m)I}{2\pi s}\hat{\boldsymbol{\phi}}, & a \le s < b \\ \mathbf{0}, & s > b \end{cases}$$

$$\begin{aligned} U_B &= \frac{1}{2}\iiint \boldsymbol{H}\cdot\boldsymbol{B}\,\mathrm{d}v \\ &= \frac{1}{2}\int_{s=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{\ell}\left(\frac{Is}{2\pi a^2}\,\frac{\mu_0 Is}{2\pi a^2}\right)s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z \\ &\quad + \frac{1}{2}\int_{s=a}^{b}\int_{\phi=0}^{2\pi}\int_{z=0}^{\ell}\left(\frac{I}{2\pi s}\,\frac{\mu_0(1+\chi_m)I}{2\pi s}\right)s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z \end{aligned}$$

$$\frac{U_B}{\ell} = \frac{\mu_0 I^2}{16\pi} + \frac{\mu_0(1+\chi_m)I^2}{4\pi}\ln\left(\frac{b}{a}\right)$$

Maxwell's Equations in Matter

Polarization current

- An electric polarization $\boldsymbol{P}$ produces a bound charge density

$$\rho_b = -\nabla \cdot \boldsymbol{P}$$

- Any change in the electric polarization involves a flow of (bound) charge (call it $\boldsymbol{J}_P$), which must be included in the total current. Thus,

$$\frac{\partial \rho_b}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{P}) = -\nabla \cdot \left(\frac{\partial \boldsymbol{P}}{\partial t}\right)$$

- Comparing with the continuity equation, we identify $\partial \boldsymbol{P} / \partial t$ as the polarization current $\boldsymbol{J}_P$.

Maxwell's equations

- Total charge density can be separated into two parts:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \boldsymbol{P}$$

- Current density into three parts:

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b + \boldsymbol{J}_P = \boldsymbol{J}_f + \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}$$

- Gauss's law can now be written as

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \boldsymbol{P})$$

$$\nabla \cdot \boldsymbol{D} = \rho_f$$

where $\boldsymbol{D} \equiv \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$

Maxwell's equations cont'd

- Ampère's-Maxwell law becomes

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J}_f + \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}$$

where $\boldsymbol{H} \equiv \frac{1}{\mu_0} \boldsymbol{B} - \boldsymbol{M}$

- Faraday's law and $\nabla \cdot \boldsymbol{B} = 0$ are not affected by the separation of charge and current into free and bound parts, since they do not involve ρ or $\boldsymbol{J}$.

Maxwell's equations cont'd

- In terms of free charges and currents, then, Maxwell's equations read

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \boldsymbol{D} = \rho_f\,, & \text{(iii)} & \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \text{(ii)} & \nabla \cdot \boldsymbol{B} = 0, & \text{(iv)} & \nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \dfrac{\partial \boldsymbol{D}}{\partial t} \end{array}\right\}$$

- For linear media,

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}\,, \qquad \boldsymbol{M} = \chi_m \boldsymbol{H}$$

so

$$\boldsymbol{D} = \epsilon \boldsymbol{E}\,, \qquad \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$$

where $\epsilon \equiv \epsilon_0(1+\chi_e)$ and $\mu = \mu_0(1+\chi_m)$

Boundary Conditions

- Maxwell's equations in their integral form

(i) $$\oiint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} = Q_{fenc}$$

(ii) $$\oiint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = 0$$

(iii) $$\oint \boldsymbol{E} \cdot \mathrm{d}\ell = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{a}$$

(iv) $$\oint \boldsymbol{H} \cdot \mathrm{d}\ell = I_{fenc} + \frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a}$$

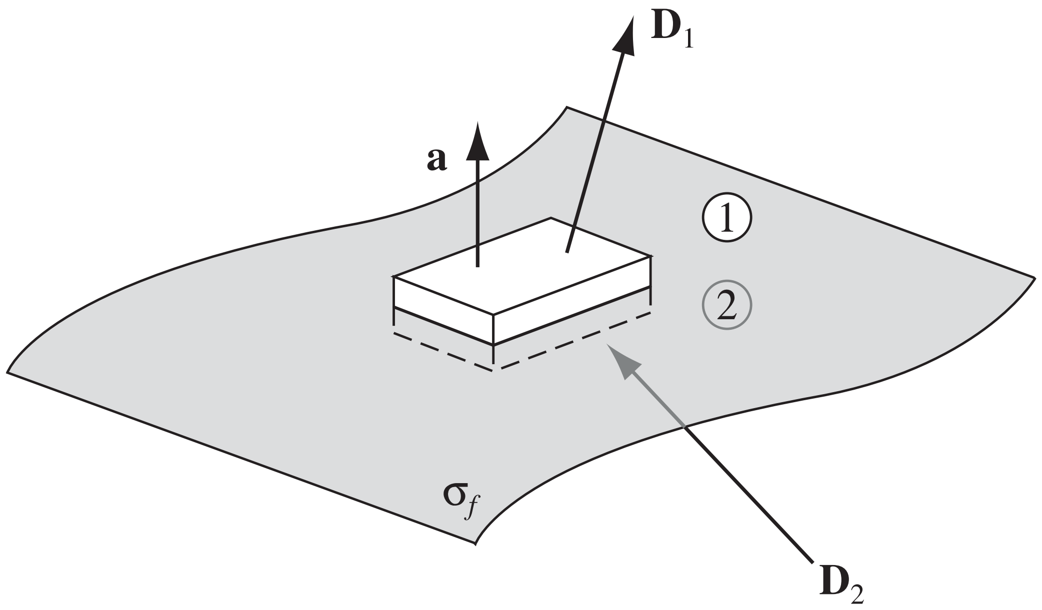
Boundary Conditions cont'd

- Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary, we obtain

$$\boldsymbol{D}_1 \cdot \boldsymbol{a} - \boldsymbol{D}_2 \cdot \boldsymbol{a} = \sigma_f a$$

The positive direction for $\boldsymbol{a}$ is from 2 toward 1. The edge of the wafer contributes nothing in the limit as the thickness goes to zero; nor does any volume charge density. Thus

$$\boxed{D_1^\perp - D_2^\perp = \sigma_f}$$



- Identical reasoning, applied to (ii), gives

$$\boxed{B_1^\perp - B_2^\perp = 0}$$

- Turning to (iii), a very thin Amperian loop straddling the surface gives

$$\boldsymbol{E}_1 \cdot \boldsymbol{\ell} - \boldsymbol{E}_2 \cdot \boldsymbol{\ell} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{a}$$

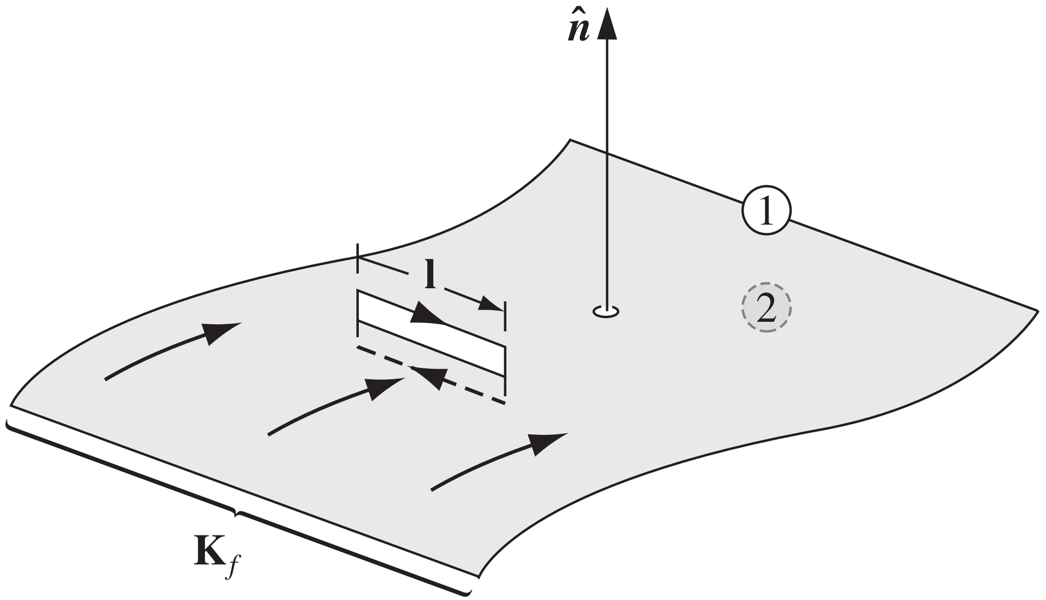
But in the limit as the width of the loop goes to zero, the flux vanishes. Thus

$$\boxed{\boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0}$$

- (iv) implies

$$\boldsymbol{H}_1^{\parallel} \cdot \boldsymbol{\ell} - \boldsymbol{H}_2^{\parallel} \cdot \boldsymbol{\ell} = I_{f_{enc}}$$

If $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the interface (pointing from 2 toward 1),



so that $(\hat{\boldsymbol{n}} \times \boldsymbol{\ell})$ is normal to the Amperian loop, then

$$I_{f_{enc}} = \boldsymbol{K}_f \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{\ell}) = (\boldsymbol{K}_f \times \hat{\boldsymbol{n}}) \cdot \boldsymbol{\ell}$$

$$\therefore \boxed{\boldsymbol{H}_1^{\parallel} - \boldsymbol{H}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}}}$$

Boundary Conditions cont'd

For linear media, the general boundary conditions for electrodynamics can be expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ alone. Thus,

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f\,,$ (iii) $\boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0$

(ii) $B_1^{\perp} - B_2^{\perp} = 0,$ (iv) $\frac{1}{\mu_1}\boldsymbol{B}_1^{\parallel} - \frac{1}{\mu_2}\boldsymbol{B}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}}$