The Answer of Assignment 2

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27/8/2025

Problem 1 Solution

(1) According to the theory of isomorphism of linear spaces, any two linear spaces of the same dimension are isomorphic. Therefore, $\sum\limits_{i,j=0}^{1} c_{ij} |i,j\rangle$ is isomorphic to $\{(x_1, x_2, x_3, x_4)\}, x \in \mathbb{C}$. That is, $v_1 = (c_{00}, c_{01}, c_{10}, c_{11})^T$, $v_2 = (d_{00}, d_{01}, d_{10}, d_{11})^T$.

(2) $\langle\psi_1|\psi_2\rangle = v_1^\dagger v_2$ is right,because: $\langle\psi_1|\psi_2\rangle = (\sum_{i,j=0}^{1} c_{i,j}^* \langle ij|)(\sum_{i,j=0}^{1} c_{i,j} |ij\rangle) = v_1^\dagger v_2$

(3) $O|pq\rangle = O_{ij,kl} |ij\rangle \langle kl| |pq\rangle = O_{ij,kl} |ij\rangle \delta_{kp}\delta_{lq} = O_{ij,pq} |ij\rangle$ if we define $|00\rangle$ as e_1, $|01\rangle$ as e_2, $|10\rangle$ as e_3, $|11\rangle$ as e_4, then we can get the matrix representation of O: $Oe_i = \sum_{j=1}^{4} O_{ji} e_j$ Therefore, the matrix representation of O is:

$$M = \begin{pmatrix} O_{11} & O_{12} & O_{13} & O_{14} \\ O_{21} & O_{22} & O_{23} & O_{24} \\ O_{31} & O_{32} & O_{33} & O_{34} \\ O_{41} & O_{42} & O_{43} & O_{44} \end{pmatrix}$$

(4) $O|\psi_1\rangle$ is the same as Mv_1,because:

$$O|\psi_1\rangle = O_{ij,kl} |ij\rangle \langle kl| \cdot c_{mn} |mn\rangle = O_{ij,kl} |ij\rangle c_{mn}\delta_{km}\delta_{ln} = O_{ij,mn} c_{mn} |ij\rangle$$

this is equivalent to Mv_1 if we define $|00\rangle$ as e_1, $|01\rangle$ as e_2, $|10\rangle$ as e_3, $|11\rangle$ as e_4.:

Problem 2 Solution

(1) Starting from $[b_i^\dagger, b_i^\dagger] = 0$,show that $[b_i, b_j] = 0$. According to the definition of commutation relation, we have:

$$[b_i^\dagger, b_j^\dagger] = b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0$$

Taking the Hermitian conjugate of both sides, we get:

$$(b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger)^\dagger = 0^\dagger$$

$$b_j b_i - b_i b_j = 0$$

so we have:$[b_i, b_j] = 0$

(2)

$$\langle\overline{n_1' n_2' \cdots n_k'}|b_i^\dagger|\overline{n_1 n_2 \cdots n_k}\rangle = \langle\overline{n_1 n_2 \cdots n_k}|b_i|\overline{n_1' n_2' \cdots n_k'}\rangle^* = \delta_{n_1,n_1'}\delta_{n_2,n_2'}\cdots\delta_{n_i,n_i'-1}\cdots\delta_{n_k,n_k'}\sqrt{n_i'}$$

and

$$b_i^\dagger\,|\overline{n_1 n_2 \cdots n_k}\rangle = \sqrt{n_i+1}\,|\overline{n_1 n_2 \cdots (n_i+1)\cdots n_k}\rangle$$

we can get the relation between b_i and $b_i^\dagger$ as follows:

$$b_i\,|\overline{n_1 n_2 \cdots n_k}\rangle = \sqrt{n_i}\,|\overline{n_1 n_2 \cdots (n_i-1)\cdots n_k}\rangle$$

(3)

We have:

$$\begin{aligned}
[b_i, b_j^\dagger]\,|\overline{n_1 n_2 \cdots n_k}\rangle &= b_i b_j^\dagger\,|\overline{n_1 n_2 \cdots n_k}\rangle - b_j^\dagger b_i\,|\overline{n_1 n_2 \cdots n_k}\rangle \\
&= b_i\sqrt{n_j+1}\,|\overline{n_1 n_2 \cdots (n_j+1)\cdots n_k}\rangle \\
&\quad - b_j^\dagger\sqrt{n_i}\,|\overline{n_1 n_2 \cdots (n_i-1)\cdots n_k}\rangle \\
&= \sqrt{n_i(n_j+1)}\,|\overline{n_1 n_2 \cdots (n_i-1)\cdots(n_j+1)\cdots n_k}\rangle \cdot 0
\end{aligned}$$

(4) if $i = j$, we have:

$$[b_i, b_i^\dagger]\,|\overline{n_1 n_2 \cdots n_k}\rangle = (n_i + 1 - n_i)\,|\overline{n_1 n_2 \cdots n_k}\rangle = |\overline{n_1 n_2 \cdots n_k}\rangle$$

(5)

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

When $i = j$, this becomes

$$\{f_i, f_i\} = f_i f_i + f_i f_i = 2f_i^2$$

Since f_i is a fermionic annihilation operator, it satisfies $f_i^2 = 0$. Therefore,

$$\{f_i, f_i\} = 2 \times 0 = 0$$

So, when $i = j$, the anticommutation relation $\{f_i, f_j\} = 0$ holds.

(6)

Suppose $i \neq j$. Calculate $\{f_i, f_j\}\,|n_1 n_2 \cdots\rangle$.

Recall the definition:

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

We need to consider the action of f_i and f_j on the occupation number basis $|n_1 n_2 \cdots\rangle$, where $n_k = 0$ or 1 for fermions.

Consider the following cases:

Case 1: $n_i = 0$ **or** $n_j = 0$

- If $n_i = 0$, then $f_i |n_1 n_2 \cdots\rangle = 0$. - If $n_j = 0$, then $f_j |n_1 n_2 \cdots\rangle = 0$.

Therefore, in either case, both $f_i f_j |n_1 n_2 \cdots\rangle = 0$ and $f_j f_i |n_1 n_2 \cdots\rangle = 0$, so

$$\{f_i, f_j\} |n_1 n_2 \cdots\rangle = 0$$

Case 2: $n_i = 1$ **and** $n_j = 1$

- $f_j |n_1 n_2 \cdots n_j = 1 \cdots n_i = 1 \cdots\rangle = (-1)^{s_1} |n_1 \cdots n_j = 0 \cdots n_i = 1 \cdots\rangle$ - Then f_i acts on this state: $f_i |n_1 \cdots n_j = 0 \cdots n_i = 1 \cdots\rangle = (-1)^{s_2} |n_1 \cdots n_j = 0 \cdots n_i = 0 \cdots\rangle$

Similarly, $f_i f_j |n_1 n_2 \cdots\rangle$ and $f_j f_i |n_1 n_2 \cdots\rangle$ will differ by a sign, but since $i \neq j$, the sum $f_i f_j + f_j f_i$ will always cancel out due to the anticommutation property of fermionic operators.

Therefore,

$$\{f_i, f_j\} |n_1 n_2 \cdots\rangle = 0$$

(7) case 1: $i = j$

$$f_i, f_j^\dagger = f_i f_i^\dagger + f_i^\dagger f_i$$

when $n_i = 0$, we have:

$$\begin{aligned}
&f_i f_i^\dagger |n_1 n_2 \cdots n_i = 0 \cdots\rangle \\
&= f_i (-1)^s |n_1 n_2 \cdots n_i = 1 \cdots\rangle \\
&= (-1)^s (-1)^s |n_1 n_2 \cdots n_i = 0 \cdots\rangle \\
&= |n_1 n_2 \cdots n_i = 0 \cdots\rangle
\end{aligned}$$

when $n_i = 1$, we have:

$$\begin{aligned}
&f_i^\dagger f_i |n_1 n_2 \cdots n_i = 1 \cdots\rangle = f_i^\dagger (-1)^s |n_1 n_2 \cdots n_i = 0 \cdots\rangle \\
&= (-1)^s (-1)^s |n_1 n_2 \cdots n_i = 1 \cdots\rangle = |n_1 n_2 \cdots n_i = 1 \cdots\rangle
\end{aligned}$$

case 2: $i \neq j$

$$\{f_i, f_j^\dagger\} = f_i f_j^\dagger + f_j^\dagger f_i$$

when $n_i = 0$ or $n_j = 0$, we have: $f_i f_j^\dagger \psi = 0 f_j^\dagger f_i \psi = 0$ when $n_i = 1$ and $n_j = 1$, we have the same conclusion as above.

when $n_i = 1$ and $n_j = 0$, we have:

$$\{f_i f_j^\dagger\} |n_1 n_2 \cdots n_i = 1 \cdots n_j = 0 \cdots\rangle = f_i (-1)^{\sum_1^{j-1} n_x} |n_1 n_2 \cdots n_i = 1 \cdots n_j = 1 \cdots\rangle$$
$$+ f_j^\dagger (-1)^{\sum_1^{i-1} n_x} |n_1 n_2 \cdots n_i = 0 \cdots n_j = 0 \cdots\rangle = ((-1)^{s1+s2} + (-1)^{s1+s2+1}) |n_1, \cdots n_i = 0 \cdots n_j = 1 \cdots\rangle$$

Problem 3 Solution

(1) We define $b_I^\dagger$ in

$$b_i^\dagger |n_1, n_2, \cdots, n_k\rangle = |n_1, n_2, \cdots, n_i + 1, \cdots, n_L\rangle = |n_1, n_2, \cdots, 1, n_{i+1}, \cdots, n_L\rangle \text{ if } n_i = 0$$
$$b_i^\dagger |n_1, n_2, \cdots, n_k\rangle = 0 \text{ if } n_i = 1$$

$$b_i |n_1, n_2, \cdots, n_k\rangle = |n_1, n_2, \cdots, n_i - 1, \cdots, n_L\rangle = |n_1, n_2, \cdots, 0, n_{i+1}, \cdots, n_L\rangle \text{ if } n_i = 1$$
$$b_i |n_1, n_2, \cdots, n_k\rangle = 0 \text{ if } n_i = 0$$

$[b_i, b_j] = 0$for any i,jbecause when i=j$(b_ib_j - b_jb_i) |n_1, n_2, \cdots, n_L\rangle = 0ifn_i = 0$ $[b_i, b_j] = 0$if$n_i = 1$because$b_ib_j |n_1, n_2, \cdots, n_L\rangle = |n_1, n_2, \cdots, n_L\rangle$and$b_jb_i |n_1, n_2, \cdots, n_L\rangle = |n_1, n_2, \cdots, n_L\rangle$ when $i \neq j$, $(b_ib_j - b_jb_i) |n_1, n_2, \cdots, n_L\rangle = 0$ because $b_ib_j |n_1, n_2, \cdots, n_L\rangle$ and $b_jb_i |n_1, n_2, \cdots, n_L\rangle$ are the same state. $[b_i, b_j^\dagger] = \delta_{ij}$

case 1: i=j

$$[b_i, b_i^\dagger] = b_ib_i^\dagger - b_i^\dagger b_i$$

if $n_i = 0$, we have:

$$b_ib_i^\dagger |n_1, n_2, \cdots, n_k\rangle = b_i |n_1, n_2, \cdots, 1, n_{i+1}, \cdots, n_L\rangle$$
$$= |n_1, n_2, \cdots, 0, n_{i+1}, \cdots, n_L\rangle = |n_1, n_2, \cdots, n_k\rangle$$

$b_i^\dagger b_i |n_1, n_2, \cdots, n_k\rangle = b_i^\dagger 0 = 0$

case 2: i≠j

$[b_i, b_j^\dagger] = b_ib_j^\dagger - b_j^\dagger b_i = 0$because when$n_i = 0orn_j = 0, b_ib_j^\dagger |n_1, n_2, \cdots, n_L\rangle = 0andb_j^\dagger b_i |n_1, n_2, \cdots, n_L\rangle = 0$

when $n_i = 1$ or $n_j = 1$, $b_ib_j^\dagger |n_1, n_2, \cdots, n_L\rangle$ and $b_j^\dagger b_i |n_1, n_2, \cdots, n_L\rangle$ are the same state. only when $n_i = 1$ and $n_j = 0$, we have:

$$b_ib_j^\dagger |n_1, n_2, \cdots, 1, \cdots, 0, \cdots, n_L\rangle = b_i |n_1, n_2, \cdots, 1, \cdots, 1, \cdots, n_L\rangle$$
$$= |n_1, n_2, \cdots, 0, \cdots, 1, \cdots, n_L\rangle = b_j^\dagger b_i$$

(2) $\sigma_i^\pm = (\sigma_i^x \pm \sigma_i^y \cdot i)/2$if we call

case 1 $[\sigma_i^-, \sigma_j^-] = 0$

when $i = j$, we have:

$$[\sigma_i^-, \sigma_i^-] = \sigma_i^- \sigma_i^- - \sigma_i^- \sigma_i^- = 0$$

case 2 $[\sigma_i^-, \sigma_j^+] = 0$

because σ_i^- and σ_j^+ act on different qubits, they commute with each other. Therefore, when $i \neq j$, we have:

$$[\sigma_i^-, \sigma_j^+] = 0$$

case 3 $\{\sigma_i^-, \sigma_i^+\} = 1$ use matrix representation of the Pauli operators:

$$\sigma_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

so we have:

$$\{\sigma_i^-, \sigma_i^+\} = \sigma_i^- \sigma_i^+ + \sigma_i^+ \sigma_i^- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(3)

Given the definition

$$f_i^\dagger = \left(\prod_{j<i} \sigma_j^z \right) \sigma_i^+,$$

we can express $f_i^\dagger f_i$ in terms of Pauli operators as follows:

$$f_i^\dagger f_i = \left(\prod_{j<i} \sigma_j^z \right) \sigma_i^+ \left(\prod_{j<i} \sigma_j^z \right) \sigma_i^- = \sigma_i^+ \sigma_i^- = \frac{1}{2}(1 + \sigma_i^z).$$

Here, we used the facts that $(\sigma_j^z)^2 = 1$ and σ_j^z commutes with $\sigma_i^\pm$ for $j \neq i$.

Next, we show the anticommutation relations:

$$\{f_i, f_j\} = f_i f_j + f_j f_i = 0,$$

$$\{f_i, f_j^\dagger\} = f_i f_j^\dagger + f_j^\dagger f_i = \delta_{ij}.$$

This follows from the Jordan-Wigner string: for $i \neq j$, the string of σ^z operators ensures the correct anticommutation, while for $i = j$, the local Pauli algebra gives the result.

Therefore,

$$f_i^\dagger f_i = \frac{1}{2}(1 + \sigma_i^z), \quad \{f_i, f_j\} = 0, \quad \{f_i, f_j^\dagger\} = \delta_{ij}.$$

(4) how should the Pauli operators σ_i^+ and σ_i^z can be written in terms of the f operators? we have the definition that:

$$f_i^\dagger = \left(\prod_{j<i} \sigma_j^z \right) \sigma_i^+$$

so from $\langle \overline{n_{1'} n_{2'} \cdot n_{i'}} | f_i^\dagger | \overline{n_1 n_2 n_3 \cdot n_i} \rangle = \langle \overline{n_1 n_2 \cdot n_i} | f_i | \overline{n_{1'} n_{2'} \cdot n_{i'}} \rangle^*$ we can get the relation between f_i and σ_i^+, σ_i^z as follows:

$$f_i = \left(\prod_{j<i} \sigma_j^z \right) \sigma_i^-$$

then try to represent σ_i^z and σ_i^+ in terms of f_i: we use the matrix representation of the Pauli operators:

$$\sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

σ_i^zcan be expressed as:$2\sigma_i^+\sigma_i^- - I = 2f_i^\dagger f_i - I$

like wise:

$$\sigma_i^+ = f_i^\dagger \left(\prod_{j<i} \sigma_j^z\right)^{-1}$$

$$= f_i^\dagger \left(\prod_{j<i} (2f_j^\dagger f_j - I)\right)$$