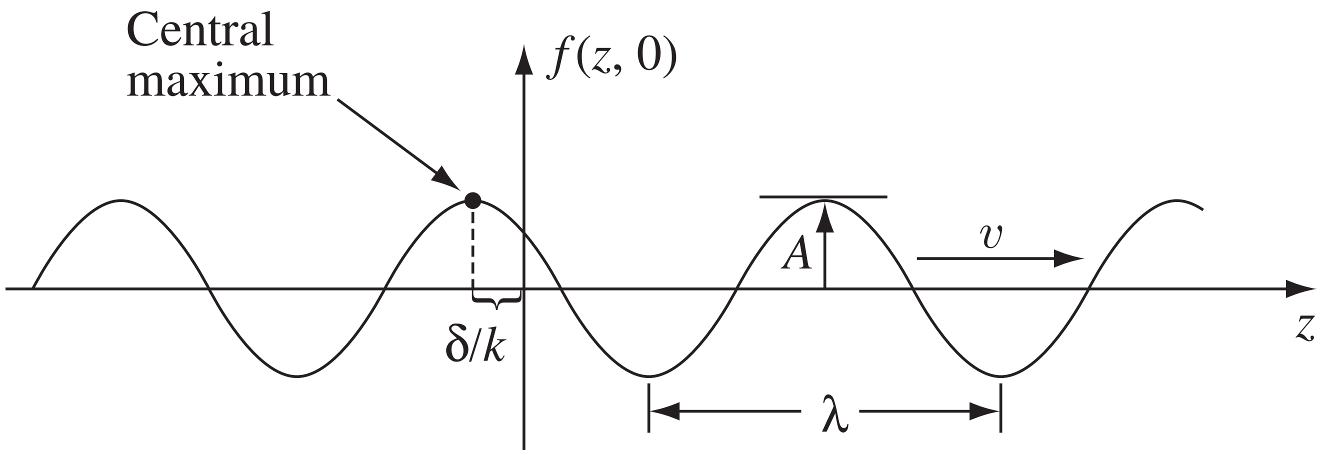
Electromagnetic Waves

Sinusoidal Waves



The function is

$$f(z,t) = A\cos[k(z - vt) + \delta] \qquad (9.1)$$

at $t = 0$.

- A is the amplitude of the wave. Argument of the cosine is called the phase, and δ is the phase constant, and $-\pi \leq \delta \leq \pi$.
- At $z = vt - \delta/k$, the phase is zero and call this the “central maximum”. If $\delta = 0$, the central maximum passes the origin at time $t = 0$.
- Generally, δ/k is the distance by which the central maximum (and hence the entire wave) is “delayed”.

Sinusoidal Waves (cont'd)

k is the wave number; it is related to the wavelength λ by the equation

$$\lambda = \frac{2\pi}{k} \tag{9.2}$$

The frequency ν (number of oscillations per unit time) is

$$\nu = \frac{v}{\lambda} \tag{9.3}$$

The angular frequency ω is

$$\omega = 2\pi\nu = kv \tag{9.4}$$

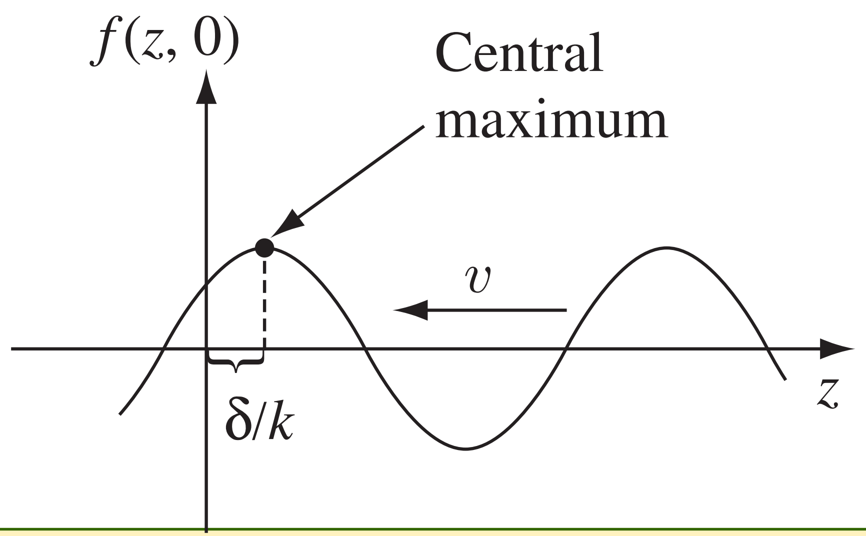
Sinusoidal Waves (cont'd)

Eq. (9.1) can be rewritten as

$$f(z,t) = A\cos(kz - \omega t + \delta). \tag{9.5}$$

Similarly, a sinusoidal oscillation of wave number k and (angular) frequency ω traveling to the *left* can be written as

$$f(z,t) = A\cos(-kz - \omega t + \delta). \tag{9.6}$$



Complex notation

From Euler's formula,

$$e^{i\theta} = \cos\theta + i\,\sin\theta \tag{9.7}$$

the sinusoidal wave (Eq. (9.5) can be written

$$f(z,t) = \mathrm{Re}[Ae^{i(kz-\omega t+\delta)}] \tag{9.8}$$

where $\mathrm{Re}(\xi)$ denotes the real part of the complex number ξ.

We introduce the complex wave function

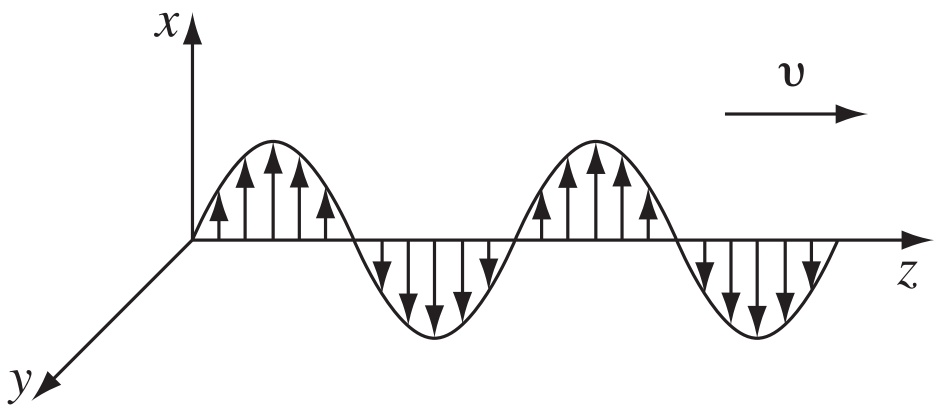
$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} \tag{9.9}$$

with the complex amplitude $\tilde{A} \equiv Ae^{i\delta}$ absorbing the phase constant. The actual wave function is the real part of $\tilde{f}$:

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)] \tag{9.10}$$

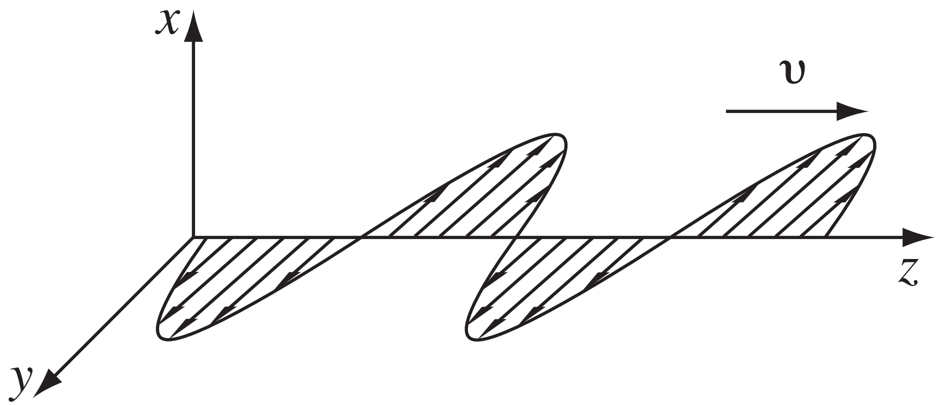
Polarization

Polarization



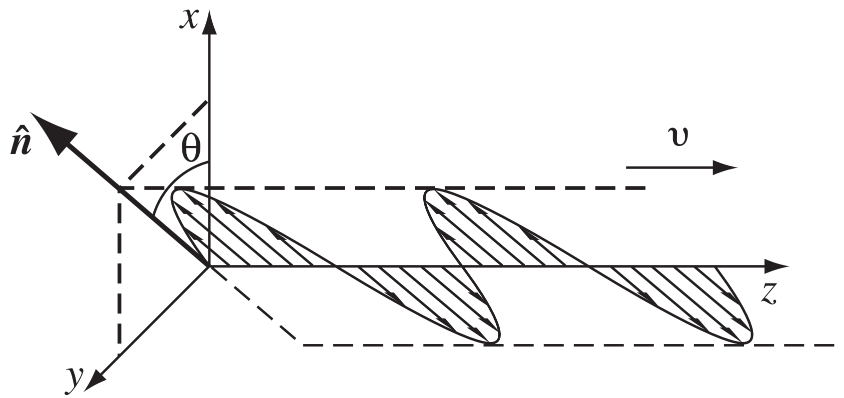
(a) Vertical polarization

$$\tilde{\boldsymbol{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\boldsymbol{x}} \tag{9.11}$$



(b) Horizontal polarization

$$\tilde{\boldsymbol{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\boldsymbol{y}} \tag{9.12}$$



(c) Polarization vector

$$\tilde{\boldsymbol{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\boldsymbol{n}} \qquad (9.13)$$

The **polarization vector** $\hat{\boldsymbol{n}}$ defines the plane of vibration.

Because the waves are transverse, $\hat{\boldsymbol{n}}$ is perpendicular to the direction of propagation:

$$\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}} = 0 \qquad (9.14)$$

In terms of the **polarization angle** θ,

$$\hat{\boldsymbol{n}} = \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}} \qquad (9.15)$$

The wave can be considered a superposition of two waves

$$\tilde{\boldsymbol{f}} = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\hat{\boldsymbol{x}} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\hat{\boldsymbol{y}} \qquad (9.16)$$

Electromagnetic Waves in Vacuum

Wave Equation for E and B

In regions of space where there is no charge or current, Maxwell's equations give

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\boldsymbol{E} = 0, & \text{(iii)} & \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \\ \text{(ii)} & \nabla\cdot\boldsymbol{B} = 0, & \text{(iv)} & \nabla\times\boldsymbol{B} = \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \end{array}\right\} \tag{9.17}$$

They constitute a set of coupled, first-order, partial differential equations for $\boldsymbol{E}$ and $\boldsymbol{B}$. They can be decoupled by applying the curl to (iii) and (iv):

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{E}) &= \nabla(\nabla\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E} = \nabla\times\left(-\frac{\partial\boldsymbol{B}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}(\nabla\times\boldsymbol{B}) = -\mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2} \\ \nabla\times(\nabla\times\boldsymbol{B}) &= \nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B} = \nabla\times\left(\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right) \\ &= \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla\times\boldsymbol{E}) = -\mu_0\epsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2} \end{aligned}$$

Wave Equation for $\boldsymbol{E}$ and $\boldsymbol{B}$ (cont'd)

Or, since $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$,

$$\nabla^2 \boldsymbol{E} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \quad \nabla^2 \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{9.18}$$

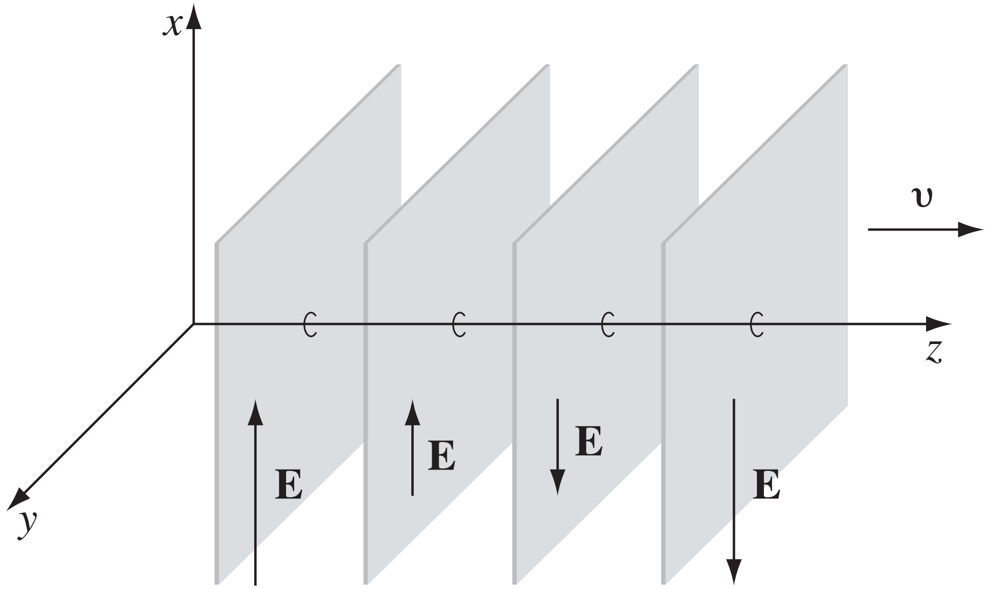
We now have separate equations for $\boldsymbol{E}$ and $\boldsymbol{B}$, but they are of second order.

In vacuum, then, each Cartesian component of $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfies the three-dimensional wave equation with

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s} \tag{9.19}$$

Monochromatic Plane Waves

If the waves are traveling in the z direction and have no x or y dependence, they are called plane waves because the fields are uniform over every plane perpendicular to the direction of propagation



$$\tilde{\boldsymbol{E}}(z,t) = \tilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\boldsymbol{B}}(z,t) = \tilde{\boldsymbol{B}}_0 e^{i(kz-\omega t)} \tag{9.20}$$

Monochromatic Plane Waves (cont'd)

Maxwell's equations impose constraints on $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$. In particular, since $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$, it follows that

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0 \tag{9.21}$$

Electromagnetic waves are transverse: the electric and magnetic fields are perpendicular to the direction of propagation.

Faraday's law, $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, gives

$$-k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x, \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y \tag{9.22}$$

That is,

$$\tilde{\boldsymbol{B}}_0 = \frac{k}{\omega}(\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{E}}_0) \tag{9.23}$$

$\boldsymbol{E}$ and $\boldsymbol{B}$ are in phase and mutually perpendicular, and

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0 \tag{9.24}$$

The fourth of Maxwell's equations, $\nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 (\partial \boldsymbol{E}/\partial t)$, reproduces Eq. 9.23.

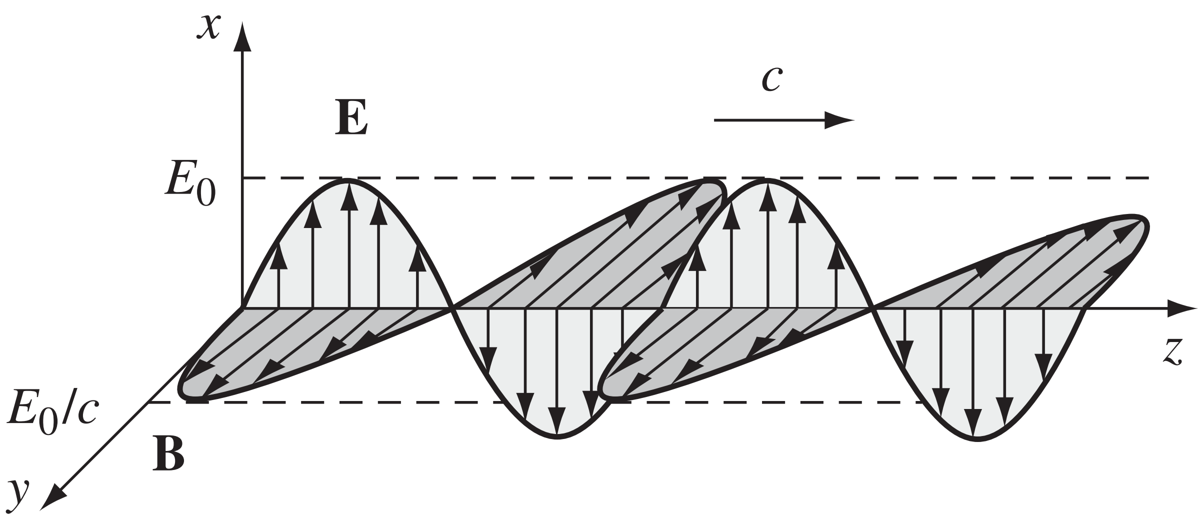
Example

If $\boldsymbol{E}$ points in the x direction, then $\boldsymbol{B}$ points in the y direction (Eq. (9.23)):

$$\tilde{\boldsymbol{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\,\hat{\boldsymbol{x}}, \quad \tilde{\boldsymbol{B}}(z,t) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}\,\hat{\boldsymbol{y}}$$

or (taking the real part)

$$\boldsymbol{E} = E_0 \cos(kz - \omega t + \delta)\,\hat{\boldsymbol{x}}, \qquad \boldsymbol{B} = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{\boldsymbol{y}} \tag{9.25}$$



For a monochromatic plane waves traveling in an arbitrary direction, we have

$$\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\,\hat{\boldsymbol{n}}$$
$$\tilde{\boldsymbol{B}}(\boldsymbol{r}, t) = \frac{1}{c}\hat{\boldsymbol{k}} \times \tilde{\boldsymbol{E}} = \frac{1}{c}\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}) \tag{9.26}$$

where $\boldsymbol{k}$ is the **propagation** (or **wave**) **vector**.

Because $\boldsymbol{E}$ is transverse,

$$\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} = 0 \tag{9.27}$$

The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector $\boldsymbol{k}$ and polarization $\hat{\boldsymbol{n}}$ are

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta)\,\hat{\boldsymbol{n}} \tag{9.28}$$
$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} E_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta)\,(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}) \tag{9.29}$$

Energy and Momentum in EM Waves

The energy per unit volume stored in electromagnetic fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \tag{9.30}$$

In the case of a monochromatic plane wave (Eq. (9.25))

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2 \tag{9.31}$$

so the electric and magnetic contributions are equal:

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \tag{9.32}$$

The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector:

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}) \tag{9.33}$$

Energy and Momentum in EM Waves (cont'd)

For monochromatic plane waves propagating in the z direction,

$$\boldsymbol{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\boldsymbol{z}} = cu\,\hat{\boldsymbol{z}} \tag{9.34}$$

Electromagnetic fields not only carry energy, they also carry momentum. The momentum density stored in the fields is

$$\boldsymbol{g} = \frac{1}{c^2}\boldsymbol{S} \tag{9.35}$$

For monochromatic plane waves, then,

$$\boldsymbol{g} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\boldsymbol{z}} = \frac{1}{c}u\,\hat{\boldsymbol{z}} \tag{9.36}$$

Energy and Momentum in EM Waves (cont'd)

The average of cosine-squared over a complete cycle is $1/2$, so

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2 \tag{9.37}$$

$$\langle \boldsymbol{S} \rangle = \frac{1}{2} c\epsilon_0 E_0^2 \,\hat{\boldsymbol{z}} \tag{9.38}$$

$$\langle \boldsymbol{g} \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \,\hat{\boldsymbol{z}} \tag{9.39}$$

where $\langle\ \rangle$ denotes the (time) average over a complete cycle. The average power per unit area transported by an electromagnetic wave is called the **intensity**:

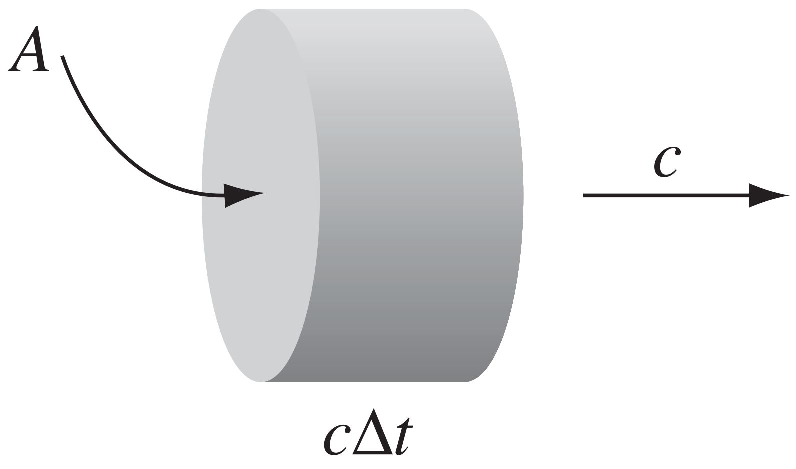
$$I \equiv \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2 \tag{9.40}$$

Energy and Momentum in EM Waves (cont'd)

When light falls on a perfect absorber it delivers its momentum to the surface.

In a time Δt the momentum transfer is $\Delta \boldsymbol{p} = \langle\, \boldsymbol{g} \rangle A c\, \Delta t$, so the **radiation pressure** (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{I}{c} \qquad (9.41)$$



Electromagnetic Waves in Matter

Propagation in Linear Media

Inside matter, but in regions where there is no free charge or free current, Maxwell's equations become

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\boldsymbol{D} = 0, & \text{(iii)} & \nabla\times\boldsymbol{E} = -\dfrac{\partial\boldsymbol{B}}{\partial t} \\ \text{(ii)} & \nabla\cdot\boldsymbol{B} = 0, & \text{(iv)} & \nabla\times\boldsymbol{H} = \dfrac{\partial\boldsymbol{D}}{\partial t} \end{array}\right\} \tag{9.42}$$

If the medium is *linear*,

$$\boldsymbol{D} = \epsilon\boldsymbol{E}, \quad \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} \tag{9.43}$$

and *homogeneous* (so ϵ and μ do not vary from point to point), Maxwell's equations reduce to

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\boldsymbol{E} = 0, & \text{(iii)} & \nabla\times\boldsymbol{E} = -\dfrac{\partial\boldsymbol{B}}{\partial t} \\ \text{(ii)} & \nabla\cdot\boldsymbol{B} = 0, & \text{(iv)} & \nabla\times\boldsymbol{B} = \mu\epsilon\dfrac{\partial\boldsymbol{E}}{\partial t} \end{array}\right\} \tag{9.44}$$

Propagation in Linear Media (cont'd)

Eqs. (9.44) differ from the vacuum analogs (Eqs. (9.17)) only in the replacement of $\mu_0\epsilon_0$ by $\mu\epsilon$. Electromagnetic waves propagate through a linear homogeneous medium at a speed

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n} \tag{9.45}$$

where

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \tag{9.46}$$

is the **index of refraction** of the material.

For most materials, μ is very close to μ_0, so

$$n \cong \sqrt{\epsilon_r} \tag{9.47}$$

where ϵ_r is the dielectric constant.

Propagation in Linear Media (cont'd)

All previous results carryover, with the simple transcription $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$ and hence $c \to v$. The energy density is

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) \tag{9.48}$$

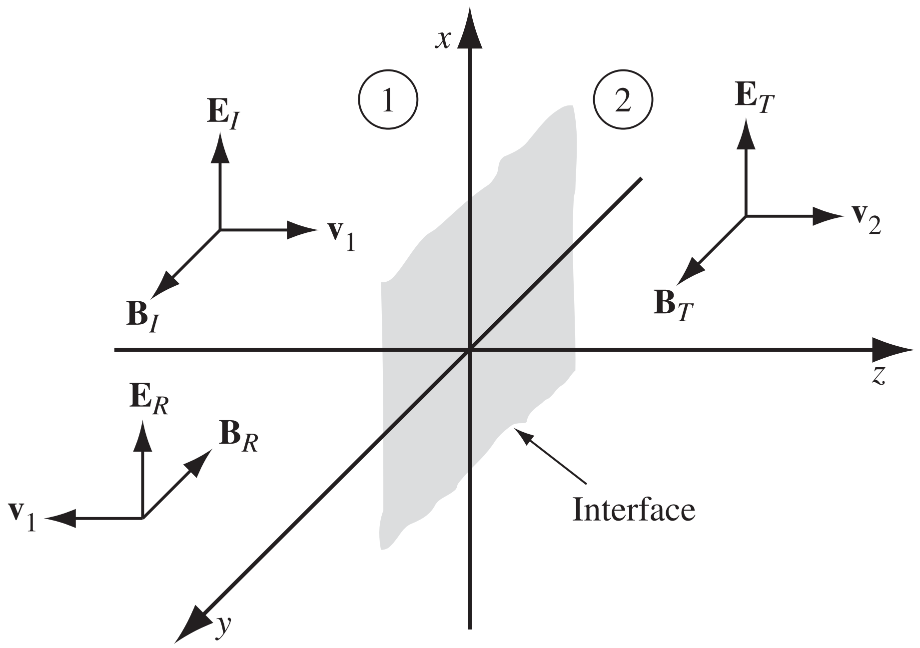
and the Poynting vector is

$$\boldsymbol{S} = \frac{1}{\mu}(\boldsymbol{E} \times \boldsymbol{B}) \tag{9.49}$$

For monochromatic plane waves, the frequency and wave number are related by $\omega = kv$, the amplitude of $\boldsymbol{B}$ is $1/v$ times the amplitude of $\boldsymbol{E}$ (Eq. (9.24)), and the intensity is

$$I = \frac{1}{2}\epsilon v E_0^2 \tag{9.50}$$

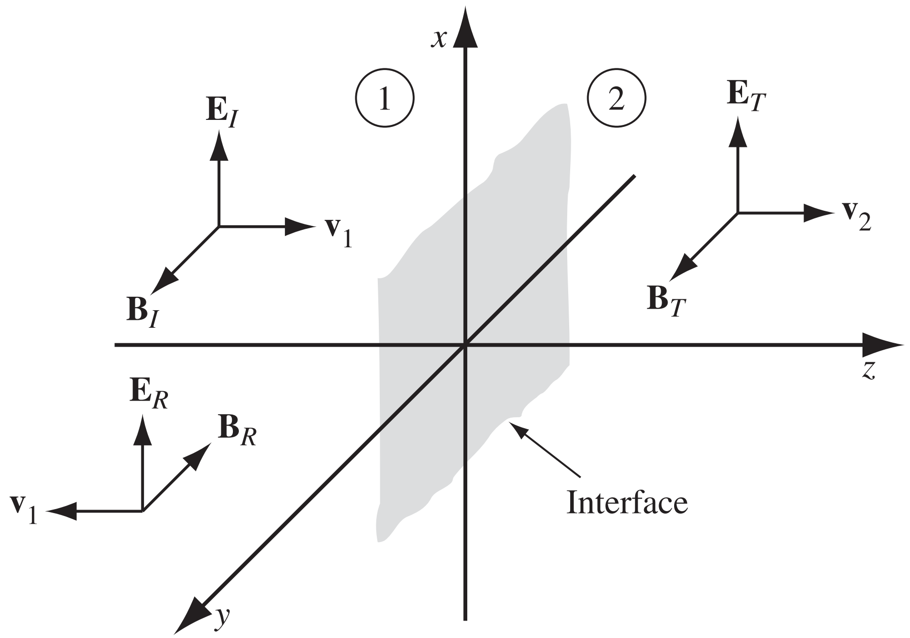
Reflection and Transmission at Normal Incidence



When a wave passes from one transparent medium into another, the following electrodynamic boundary conditions have to be satisfied:

$$\left.\begin{array}{llll} \text{(i)} & \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, & \text{(iii)} & \boldsymbol{E}_1^{\parallel} = \boldsymbol{E}_2^{\parallel} \\ \text{(ii)} & B_1^{\perp} = B_2^{\perp}, & \text{(iv)} & \frac{1}{\mu_1}\boldsymbol{B}_1^{\parallel} = \frac{1}{\mu_2}\boldsymbol{B}_2^{\parallel} \end{array}\right\} \tag{9.51}$$

Reflection and Transmission at Normal Incidence (cont'd)



Suppose the xy plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches interface from the left:

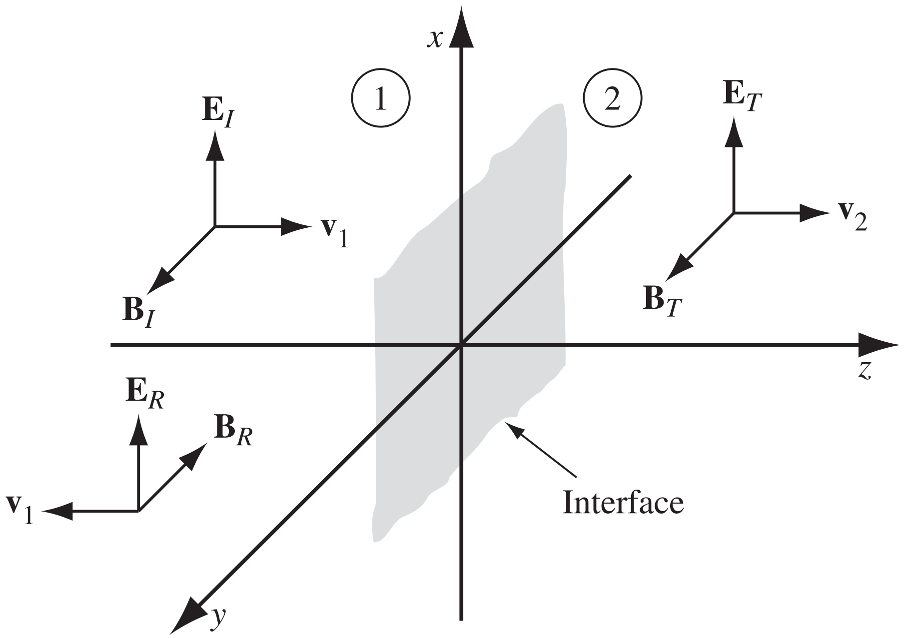
$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_I(z,t) &= \tilde{E}_{0_I} e^{i(k_1 z - \omega t)} \hat{\boldsymbol{x}} \\ \tilde{\boldsymbol{B}}_I(z,t) &= \frac{1}{v_1} \tilde{E}_{0_I} e^{i(k_1 z - \omega t)} \hat{\boldsymbol{y}} \end{aligned}\right\} \qquad (9.52)$$

It gives rise to a reflected wave

$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_R(z,t) &= \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{\boldsymbol{x}} \\ \tilde{\boldsymbol{B}}_R(z,t) &= -\frac{1}{v_1} \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{\boldsymbol{y}} \end{aligned}\right\} \qquad (9.53)$$

and a transmitted wave in medium (2)

$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_T(z,t) &= \tilde{E}_{0_T} e^{i(k_2 z - \omega t)} \hat{\boldsymbol{x}} \\ \tilde{\boldsymbol{B}}_T(z,t) &= \frac{1}{v_2} \tilde{E}_{0_T} e^{i(k_2 z - \omega t)} \hat{\boldsymbol{y}} \end{aligned}\right\} \qquad (9.54)$$



At $z = 0$, the combined fields on the left, $\tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R$ and $\tilde{\boldsymbol{B}}_I + \tilde{\boldsymbol{B}}_R$, must join the fields on the right, $\tilde{\boldsymbol{E}}_T + \tilde{\boldsymbol{B}}_T$, in accordance with the boundary conditions Eqs. (9.51).

In this case there are no components perpendicular to surface, so (i) and (ii) are trivial. However, (iii) requires

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T} \tag{9.55}$$

while (iv) says

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0_I} - \frac{1}{v_1}\tilde{E}_{0_R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0_T}\right) \tag{9.56}$$

or

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T} \tag{9.57}$$

$$\text{where,} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{9.58}$$

Eqs. (9.55) and (9.57) are solved for the outgoing amplitudes, in terms of the incident amplitude:

$$\tilde{E}_{0_R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0_I}\,, \qquad \tilde{E}_{0_T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0_I} \tag{9.59}$$

If the μ are close to their values in vacuum, then $\beta = v_1/v_2$, and we have

$$\tilde{E}_{0_R} = \left(\frac{v_2-v_1}{v_2+v_1}\right)\tilde{E}_{0_I}\,, \qquad \tilde{E}_{0_T} = \left(\frac{2v_2}{v_2+v_1}\right)\tilde{E}_{0_I} \tag{9.60}$$

The reflected wave is in phase (right side up) if $v_2 > v_1$ and out of phase (upside down) if $v_2 < v_1$; the real amplitudes are related by

$$E_{0_R} = \left|\frac{v_2-v_1}{v_2+v_1}\right| E_{0_I}\,, \qquad E_{0_T} = \left(\frac{2v_2}{v_2+v_1}\right) E_{0_I} \tag{9.61}$$

Reflection and Transmission at Normal Incidence (cont'd)

or, in terms of the indices of refraction,

$$E_{0_R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_{0_I}, \quad E_{0_T} = \left(\frac{2n_1}{n_1 + n_2} \right) E_{0_I} \tag{9.62}$$

Now, from Eq. (9.50), the intensity (average power per unit area) is

$$I = \frac{1}{2} \epsilon v E_0^2$$

If $\mu_1 = \mu_2 = \mu_0$, then the ratio of the reflected intensity to the incident intensity is

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}} \right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \tag{9.63}$$

whereas the ratio of the transmitted intensity to the incident intensity is

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left(\frac{E_{0_T}}{E_{0_I}} \right)^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2} \tag{9.64}$$

Reflection and Transmission at Normal Incidence (cont'd)

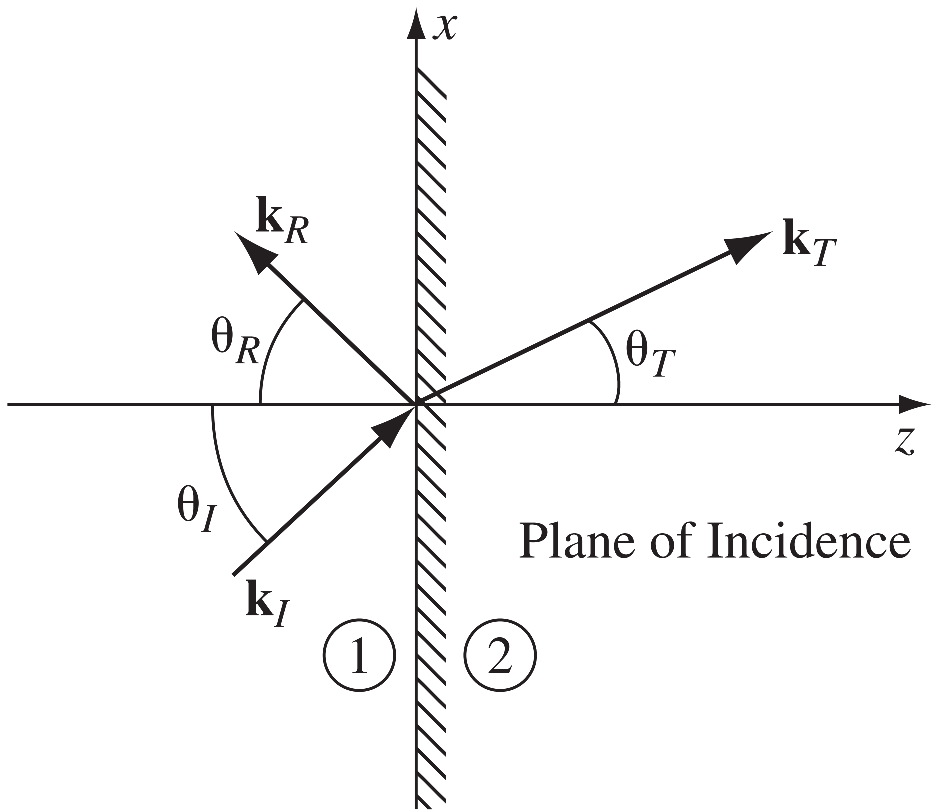
R is called the **reflection coefficient** and T the **transmission coefficient**; they measure the fraction of the incident energy that is reflected and transmitted, respectively.

From conservation of energy, we have

$$R + T = 1 \tag{9.65}$$

Reflection and Transmission at Oblique Incidence

Laws of Geometric Optics



Suppose that a monochromatic plane wave, approaching from the left, meets the boundary at an arbitrary angle θ_I

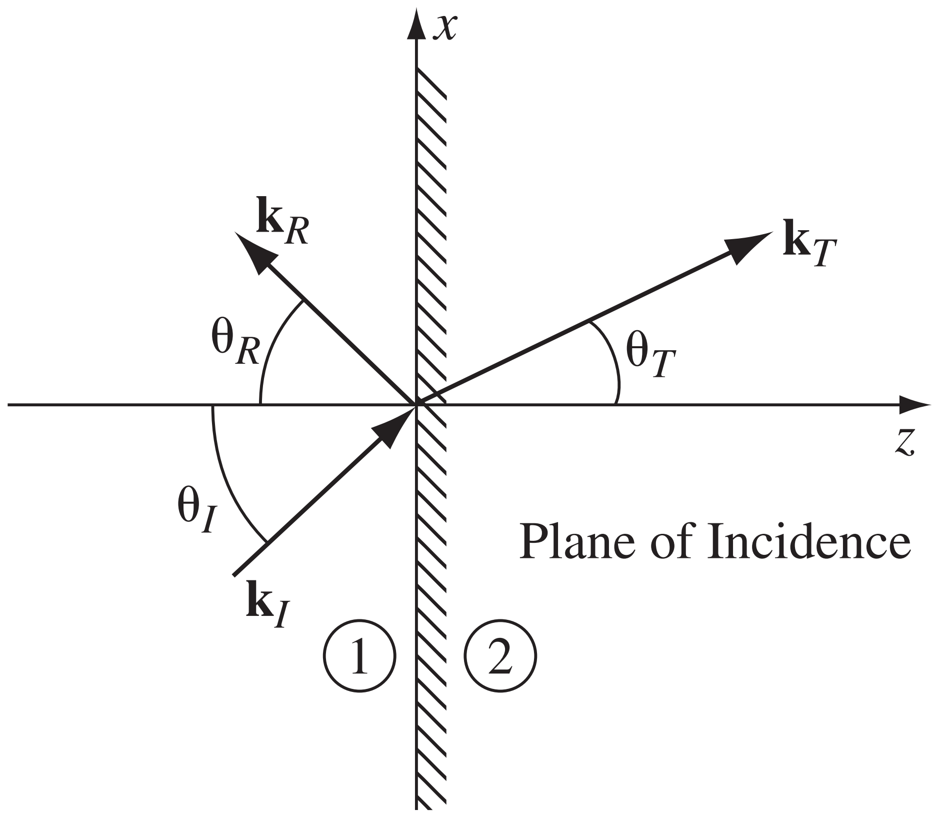
$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_I(\boldsymbol{r},t) &= \tilde{\boldsymbol{E}}_{0_I} e^{i(\boldsymbol{k}_I\cdot\boldsymbol{r}-\omega t)} \\ \tilde{\boldsymbol{B}}_I(\boldsymbol{r},t) &= \frac{1}{v_1}(\hat{\boldsymbol{k}}_I \times \tilde{\boldsymbol{E}}_I) \end{aligned}\right\} \tag{9.66}$$

resulting in a reflected wave

$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_R(\boldsymbol{r},t) &= \tilde{\boldsymbol{E}}_{0_R} e^{i(\boldsymbol{k}_R\cdot\boldsymbol{r}-\omega t)} \\ \tilde{\boldsymbol{B}}_R(\boldsymbol{r},t) &= \frac{1}{v_1}(\hat{\boldsymbol{k}}_R \times \tilde{\boldsymbol{E}}_R) \end{aligned}\right\} \tag{9.67}$$

and a transmitted wave in medium (2)

$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_T(\boldsymbol{r},t) &= \tilde{\boldsymbol{E}}_{0_T} e^{i(\boldsymbol{k}_T\cdot\boldsymbol{r}-\omega t)} \\ \tilde{\boldsymbol{B}}_T(\boldsymbol{r},t) &= \frac{1}{v_2}(\hat{\boldsymbol{k}}_T \times \tilde{\boldsymbol{E}}_T) \end{aligned}\right\} \tag{9.68}$$



All three waves have the same frequency ω. The three wave numbers are related by Eq. (9.4):

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega$$

$$\text{or} \quad k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T \qquad (9.69)$$

The combined fields in medium (1), $\tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R$ and $\tilde{\boldsymbol{B}}_I + \tilde{\boldsymbol{B}}_R$, must be joined to the fields $\tilde{\boldsymbol{E}}_T$ and $\tilde{\boldsymbol{B}}_T$ in medium (2), using BCs Eqs. (9.51).

These all share the generic structure

$$(\ldots)e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)} + (\ldots)e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t)} = (\ldots)e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t)}, \quad \text{at} \ \ z = 0 \qquad (9.70)$$

$$(...)e^{i(\boldsymbol{k}_I\cdot\boldsymbol{r}-\omega t)} + (...)e^{i(\boldsymbol{k}_R\cdot\boldsymbol{r}-\omega t)}$$
$$= (...)e^{i(\boldsymbol{k}_T\cdot\boldsymbol{r}-\omega t)}, \quad \text{at } z = 0$$

Because the boundary conditions must hold at all points on the plane, and for all times, these exponential factors must be equal.

The time factors are already equal. As for the spatial terms

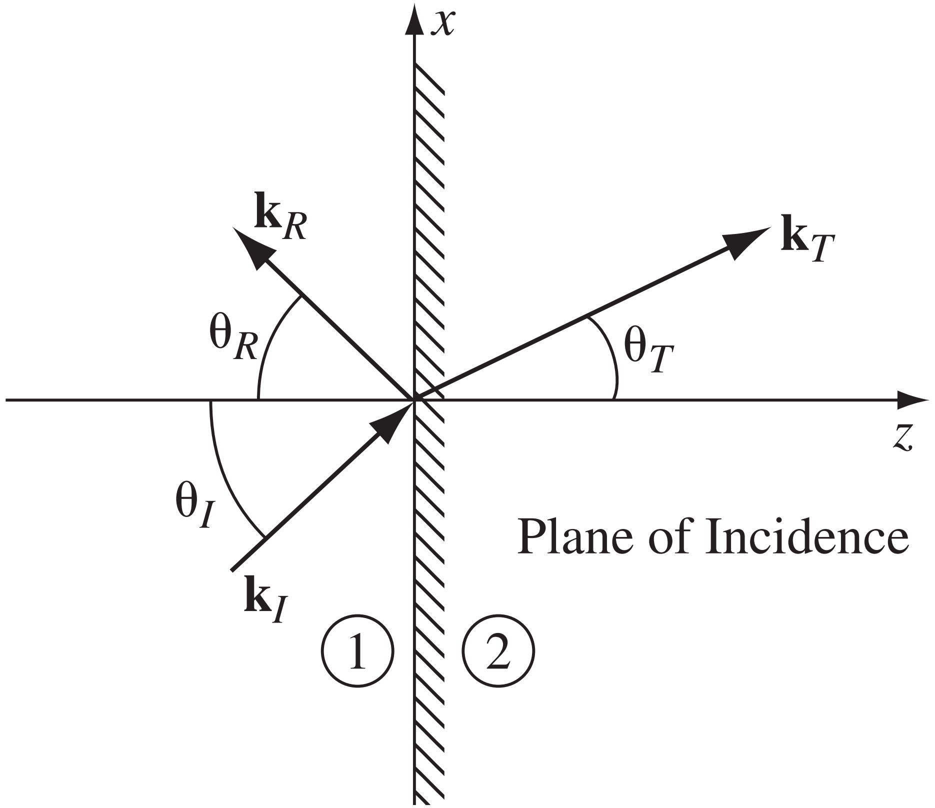
$$\boldsymbol{k}_I\cdot\boldsymbol{r} = \boldsymbol{k}_R\cdot\boldsymbol{r} = \boldsymbol{k}_T\cdot\boldsymbol{r}, \quad \text{when} \quad z = 0 \quad (9.71)$$

That is,

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y$$
$$= x(k_T)_x + y(k_T)_y \quad (9.72)$$

for all x and all y. But Eq. (9.72) can only hold if the components are separately equal, for if $x = 0$, we get

$$(k_I)_y = (k_R)_y = (k_T)_y \quad (9.73)$$



while $y = 0$ gives

$$(k_I)_x = (k_R)_x = (k_T)_x \tag{9.74}$$

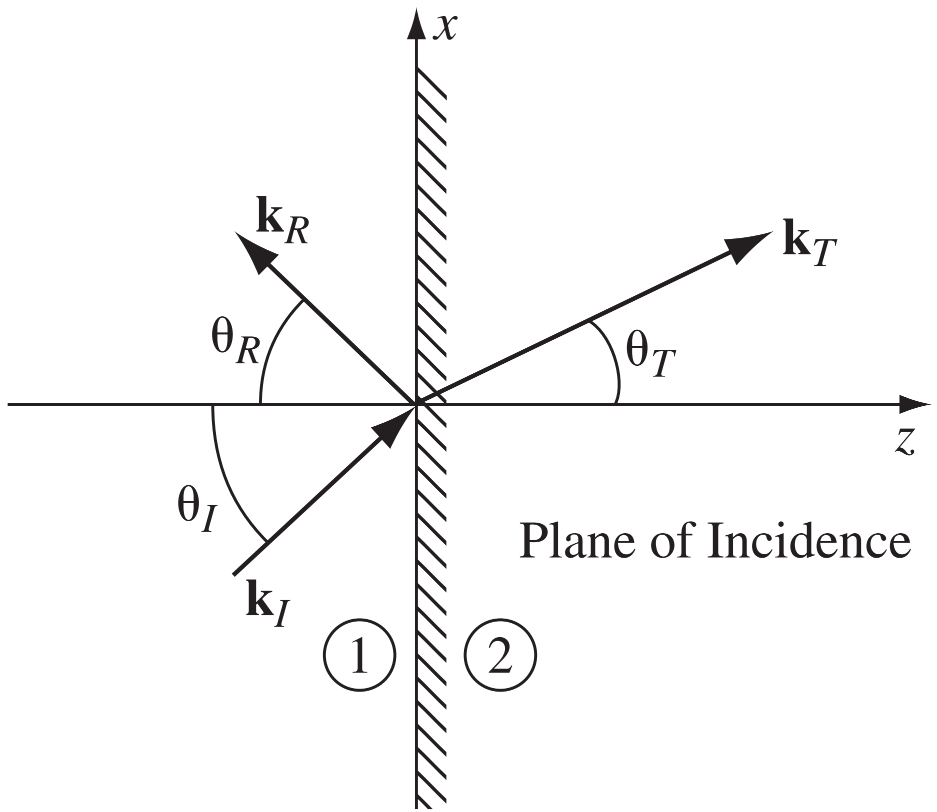
We may as well orient our axes so that $\boldsymbol{k}_I$ lies in the xz plane (i.e. $(k_I)_y = 0$); according to Eq. (9.73), so too will $\boldsymbol{k}_R$ and $\boldsymbol{k}_T$.

First Law: The incident, reflected, and transmitted wave vectors form a plane (called the **plane of incidence**), which also includes the normal to the surface (here, the z axis). Eq. (9.74) implies

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \tag{9.75}$$

θ_I: **angle of incidence**, θ_R: **angle of reflection**, θ_T: **angle of refraction**.

Laws of Geometric Optics (cont'd)



In view of Eq. (9.69), then,

Second Law: The angle of incidence is equal to the angle of reflection,

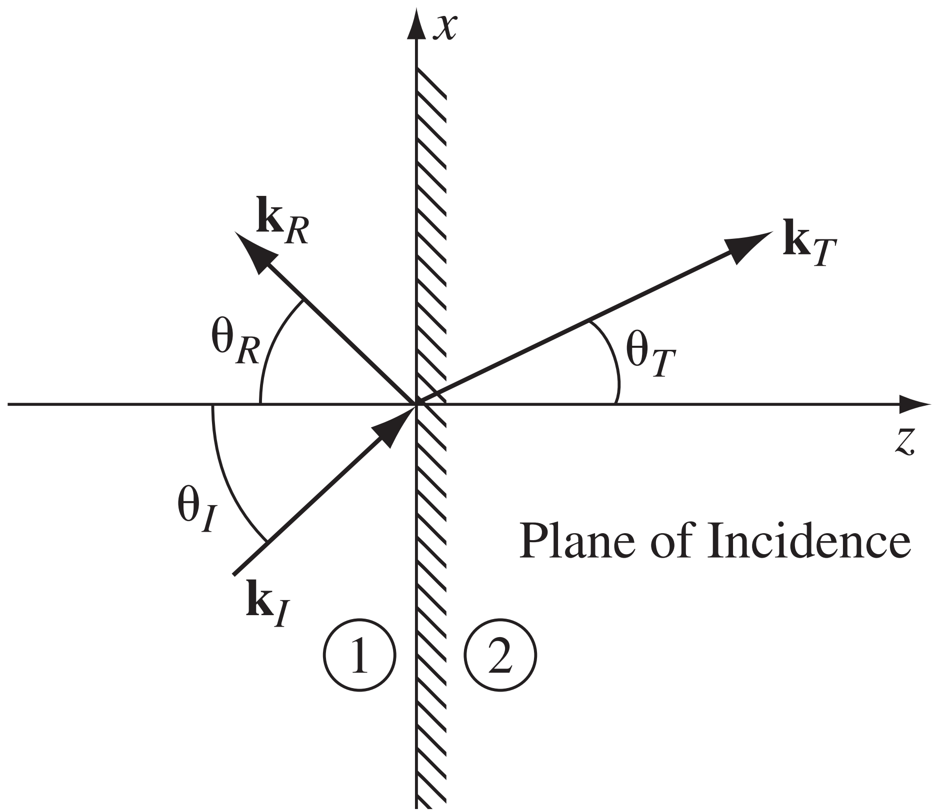
$$\theta_I = \theta_R \tag{9.76}$$

This is the **law of reflection**.

Third Law:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \tag{9.77}$$

This is the **law of refraction**, or **Snell's law**.

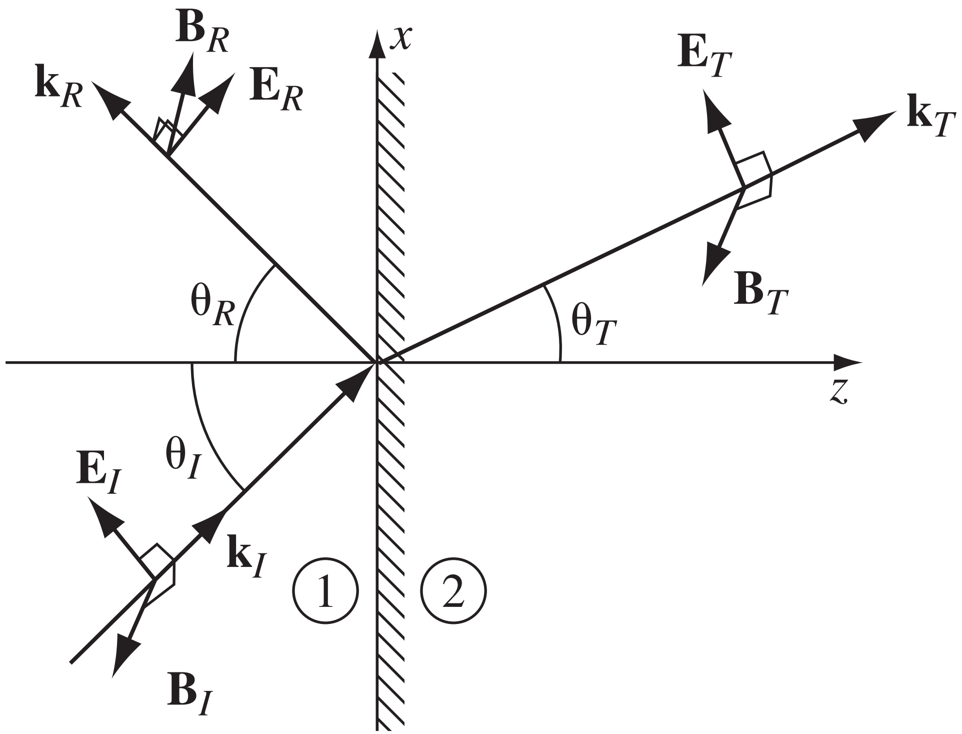


Having taken care of the exponential factors, the boundary conditions Eq. (9.51) become:

$$\begin{cases} \text{(i)} \ \epsilon_1(\tilde{\boldsymbol{E}}_{0_I} + \tilde{\boldsymbol{E}}_{0_R})_z = \epsilon_2(\tilde{\boldsymbol{E}}_{0_T})_z \\ \text{(ii)} \ (\tilde{\boldsymbol{B}}_{0_I} + \tilde{\boldsymbol{B}}_{0_R})_z = (\tilde{\boldsymbol{B}}_{0_T})_z \\ \text{(iii)} \ (\tilde{\boldsymbol{E}}_{0_I} + \tilde{\boldsymbol{E}}_{0_R})_{x,y} = (\tilde{\boldsymbol{E}}_{0_T})_{x,y} \\ \text{(iv)} \ \frac{1}{\mu_1}(\tilde{\boldsymbol{B}}_{0_I} + \tilde{\boldsymbol{B}}_{0_R})_{x,y} = \frac{1}{\mu_2}(\tilde{\boldsymbol{B}}_{0_T})_{x,y} \end{cases} \tag{9.78}$$

where $\tilde{\boldsymbol{B}}_0 = (1/v)\hat{\boldsymbol{k}} \times \tilde{\boldsymbol{E}}_0$ in each case. The last two represent pairs of equations, one for the x-component and one for the y-component.

Fresnel Equations (cont'd)



Suppose that the polarization of the incident wave is parallel to the plane of incidence (i.e. xz plane). Then the reflected and transmitted waves are also polarized in this plane.

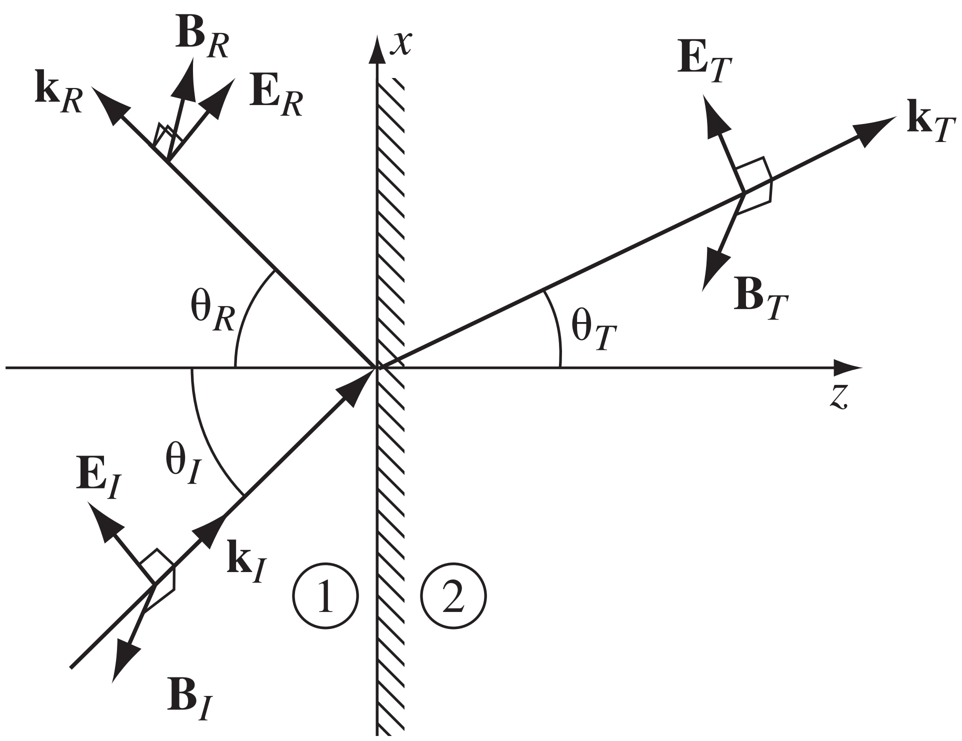
Then (i): $[\epsilon_1(\tilde{\boldsymbol{E}}_{0_I} + \tilde{\boldsymbol{E}}_{0_R})_z = \epsilon_2(\tilde{\boldsymbol{E}}_{0_T})_z]$ reads

$$\begin{aligned}\epsilon_1(-\tilde{E}_{0_I}\sin\theta_I + \tilde{E}_{0_R}\sin\theta_R) \\ = \epsilon_2(-\tilde{E}_{0_T}\sin\theta_T)\end{aligned} \qquad (9.79)$$

(ii): $[(\tilde{\boldsymbol{B}}_{0_I} + \tilde{\boldsymbol{B}}_{0_R})_z = (\tilde{\boldsymbol{B}}_{0_T})_z]$ adds nothing (0 = 0), since the magnetic fields have no z components.

(iii): $[(\tilde{\boldsymbol{E}}_{0_I} + \tilde{\boldsymbol{E}}_{0_R})_{x,y} = (\tilde{\boldsymbol{E}}_{0_T})_{x,y}]$ becomes

$$\begin{aligned}(\tilde{E}_{0_I}\cos\theta_I + \tilde{E}_{0_R}\cos\theta_R) \\ = \tilde{E}_{0_T}\cos\theta_T\end{aligned} \qquad (9.80)$$



(iv): $[\frac{1}{\mu_1}(\tilde{\boldsymbol{B}}_{0_I} + \tilde{\boldsymbol{B}}_{0_R})_{x,y} = \frac{1}{\mu_2}(\tilde{\boldsymbol{B}}_{0_T})_{x,y}]$ says

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0_T} \qquad (9.81)$$

Given the laws of reflection and refraction, Eqs. (9.79) and (9.81) both reduce to

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T} \qquad (9.82)$$

where

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \qquad (9.83)$$

Fresnel Equations (cont'd)

Eq. (9.80) says

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \alpha \tilde{E}_{0_T} \tag{9.84}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \tag{9.85}$$

Solving Eqs. (9.82) and (9.84) for the reflected and transmitted amplitudes, we obtain

$$\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha + \beta}\right) \tilde{E}_{0_I} \tag{9.86}$$

These are the **Fresnel's equations**, for the case of polarization in the plane of incidence.

The transmitted wave is always in phase with the incident one; the reflected wave is either in phase ("right side up"), if $\alpha > \beta$, or $180°$ out of phase ("upside down"), if $\alpha < \beta$

Fresnel Equations (cont'd)

The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because α is a function of θ_I:

$$\alpha = \frac{\sqrt{1 - \sin^2 \theta_T}}{\cos \theta_I} = \frac{\sqrt{1 - [(n_1/n_2) \sin \theta_I]^2}}{\cos \theta_I} \tag{9.87}$$

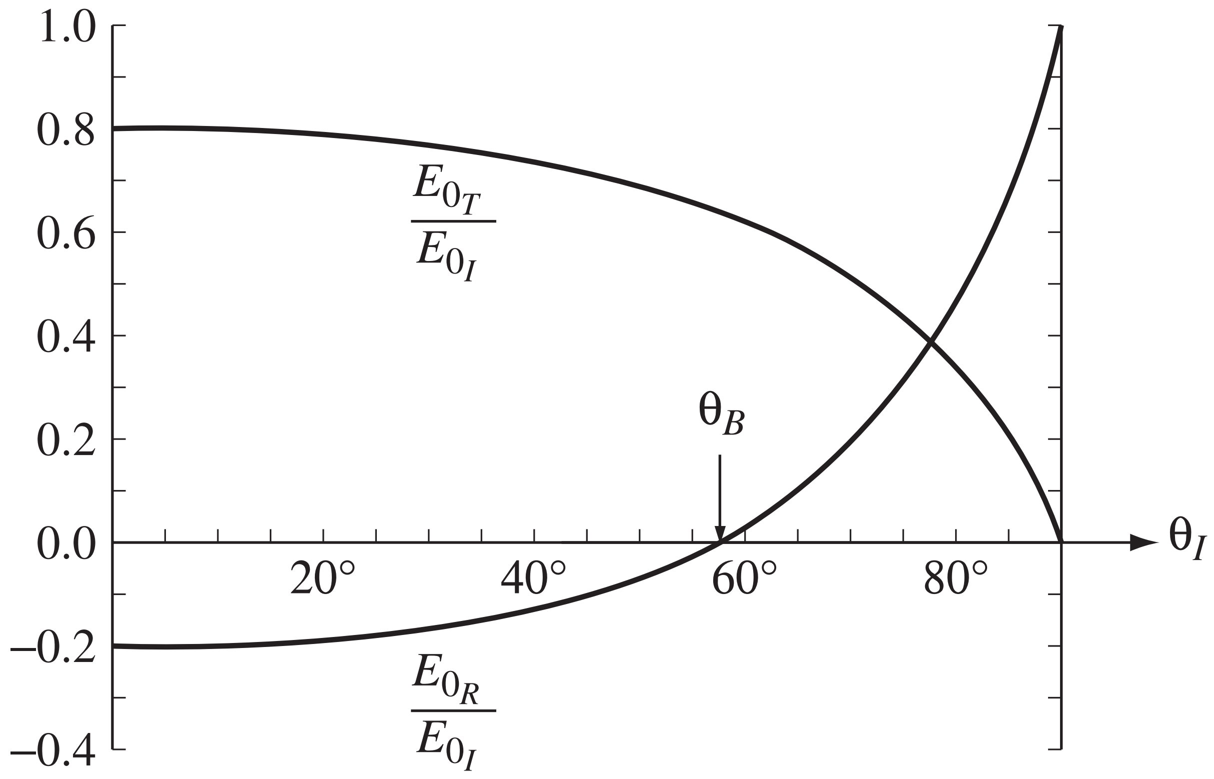
In the case of normal incidence ($\theta_I = 0$), $\alpha = 1$, and we recover Eq. (9.59). At angle θ_B (called **Brewster's angle**), the reflected wave is completely extinguished. According to Eq. (9.86), this occurs when $\alpha = \beta$, or

$$\sin^2 \theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2} \tag{9.88}$$

For the typical case $\mu_1 \cong \mu_2$, so $\beta \cong n_2/n_1$, $\sin^2 \theta_B \cong \beta^2/(1 + \beta^2)$, and hence

$$\tan \theta_B \cong \frac{n_2}{n_1} \tag{9.89}$$

Fresnel Equations (cont'd)



A plot of the transmitted and reflected amplitudes as functions of θ_I, for light incident on glass $(n_2 = 1.5)$ from air $(n_1 = 1)$. A negative number on the graph indicates that the wave is 180° out of phase with the incident beam.

Fresnel Equations (cont'd)

The power per unit area striking the interface is $\boldsymbol{S} \cdot \hat{\boldsymbol{z}}$. Thus the incident intensity is

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0_I}^2 \cos\theta_I \tag{9.90}$$

while the reflected and transmitted intensities are

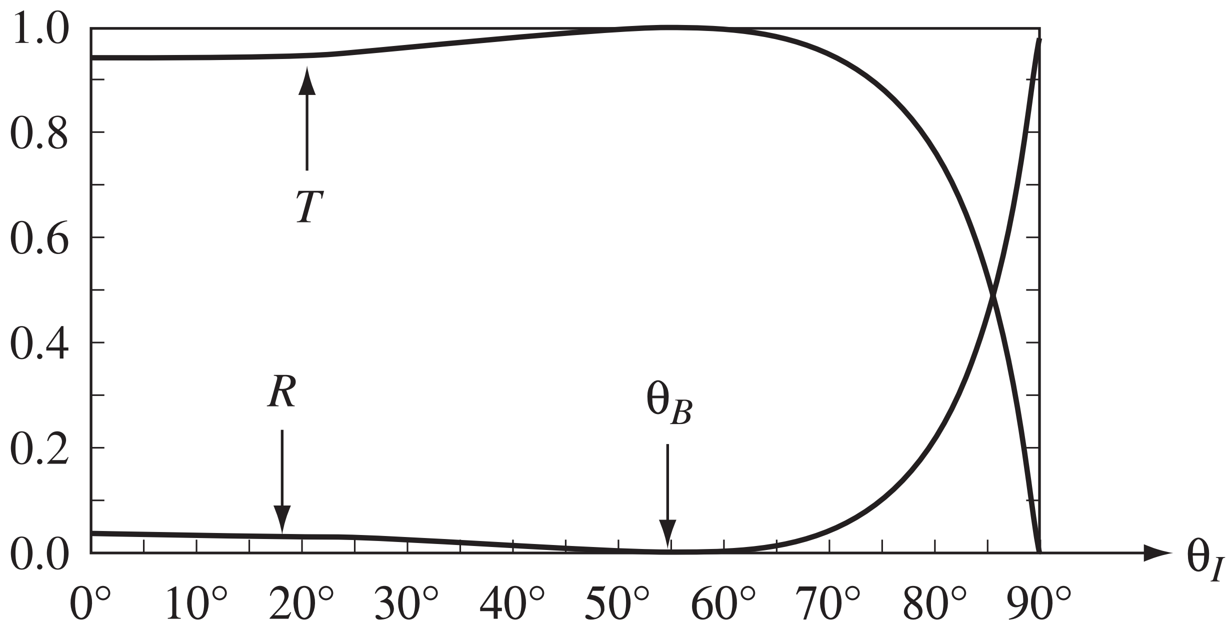
$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0_R}^2 \cos\theta_R, \quad I_T = \frac{1}{2}\epsilon_2 v_2 E_{0_T}^2 \cos\theta_T \tag{9.91}$$

The reflection and transmission coefficients for waves polarized parallel to the plane of incidence are

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \tag{9.92}$$

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2 \tag{9.93}$$

Fresnel Equations (cont'd)



They are plotted as functions of the angle of incidence for the air/glass interface.

Electromagnetic Waves in Conductors

Electromagnetic Waves in Conductors

From Ohm's law, the (free) current density in a conductor is proportional to the electric field:

$$\boldsymbol{J}_f = \sigma \boldsymbol{E} \tag{9.94}$$

With this, Maxwell's equations for linear media assume the form

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon}\rho_f, & \text{(iii)} & \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ \text{(ii)} & \nabla \cdot \boldsymbol{B} = 0, & \text{(iv)} & \nabla \times \boldsymbol{B} = \mu\sigma \boldsymbol{E} + \mu\epsilon \frac{\partial \boldsymbol{E}}{\partial t} \end{array}\right\} \tag{9.95}$$

Now the continuity equation for free charge,

$$\nabla \cdot \boldsymbol{J}_f = -\frac{\partial \rho_f}{\partial t} \tag{9.96}$$

together with Ohm's law and Gauss's law (i), gives

$$\frac{\partial \rho_f}{\partial t} = -\sigma(\nabla \cdot \boldsymbol{E}) = -\frac{\sigma}{\epsilon}\rho_f$$

Electromagnetic Waves in Conductors (cont'd)

for a homogeneous linear medium, from which it follows that

$$\rho_f(t) = e^{-(\sigma/\epsilon)t}\rho_f(0) \tag{9.97}$$

Thus any initial free charge density $\rho_f(0)$ dissipates in a characteristic time $\tau \equiv \epsilon/\sigma$. This reflects the fact that if you put some free charge on a conductor, it will flow out to the edges.

Ignoring this transient behavior, we set $\rho_f = 0$, and obtain

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\boldsymbol{E} = 0, & \text{(iii)} & \nabla\times\boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \text{(ii)} & \nabla\cdot\boldsymbol{B} = 0, & \text{(iv)} & \nabla\times\boldsymbol{B} = \mu\epsilon\dfrac{\partial \boldsymbol{E}}{\partial t} + \mu\sigma\boldsymbol{E} \end{array}\right\} \tag{9.98}$$

These differ from the corresponding equations for nonconducting media Eqs. (9.44) only in the addition of the last term in (iv).

Electromagnetic Waves in Conductors (cont'd)

Applying the curl to (iii) and (iv), we obtain modified wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\nabla^2 \boldsymbol{E} = \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t}, \quad \nabla^2 \boldsymbol{B} = \mu\epsilon \frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} \tag{9.99}$$

These equations still admit plane-wave solutions,

$$\tilde{\boldsymbol{E}}(z,t) = \tilde{\boldsymbol{E}}_0 e^{i(\tilde{k}z - \omega t)}, \quad \tilde{\boldsymbol{B}}(z,t) = \tilde{\boldsymbol{B}}_0 e^{i(\tilde{k}z - \omega t)} \tag{9.100}$$

but this time the "wave number" $\tilde{k}$ is complex:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{9.101}$$

Taking the square root,

$$\tilde{k} = k + i\kappa \tag{9.102}$$

where

$$k \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2}, \quad \kappa \equiv \omega \sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2} \tag{9.103}$$

The imaginary part of $\tilde{k}$ results in an attenuation of the wave (decreasing amplitude with increasing z):

$$\tilde{\boldsymbol{E}}(z,t) = \tilde{\boldsymbol{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\boldsymbol{B}}(z,t) = \tilde{\boldsymbol{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{9.104}$$

The distance it takes to reduce the amplitude by a factor of $1/e$ is called the **skin depth**:

$$d \equiv \frac{1}{\kappa} \tag{9.105}$$

it is a measure of how far the wave penetrates into the conductor.

Electromagnetic Waves in Conductors (cont'd)

The real part of $\tilde{k}$ determines the wavelength, the propagation speed, and the index of refraction:

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega} \tag{9.106}$$

Maxwell's equations (9.98) impose further constraints on $\boldsymbol{E}$ and $\boldsymbol{B}$. (i) and (ii) rule out any z components: the fields are transverse. We may orient our axes so that $\boldsymbol{E}$ is polarized along the x direction:

$$\tilde{\boldsymbol{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\boldsymbol{x}} \tag{9.107}$$

Then (iii) gives

$$\tilde{\boldsymbol{B}}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\boldsymbol{y}} \tag{9.108}$$

(Equation (iv) says the same thing.) Once again, the electric and magnetic fields are mutually perpendicular.

Electromagnetic Waves in Conductors (cont'd)

Now, $\tilde{k}$ can be expressed in terms of its modulus and phase:

$$\tilde{k} = K e^{i\phi} \tag{9.109}$$

where

$$K \equiv |\tilde{k}| = \sqrt{k^2 + \kappa^2} = \omega \sqrt{\epsilon\mu \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}} \tag{9.110}$$

and

$$\phi \equiv \tan^{-1}(\kappa/k) \tag{9.111}$$

According to Eqs. (9.107) and (9.108), the complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ and $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related by

$$B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E} \tag{9.112}$$

Electromagnetic Waves in Conductors (cont'd)

The electric and magnetic fields are no longer in phase:

$$\delta_B - \delta_E = \phi \tag{9.113}$$

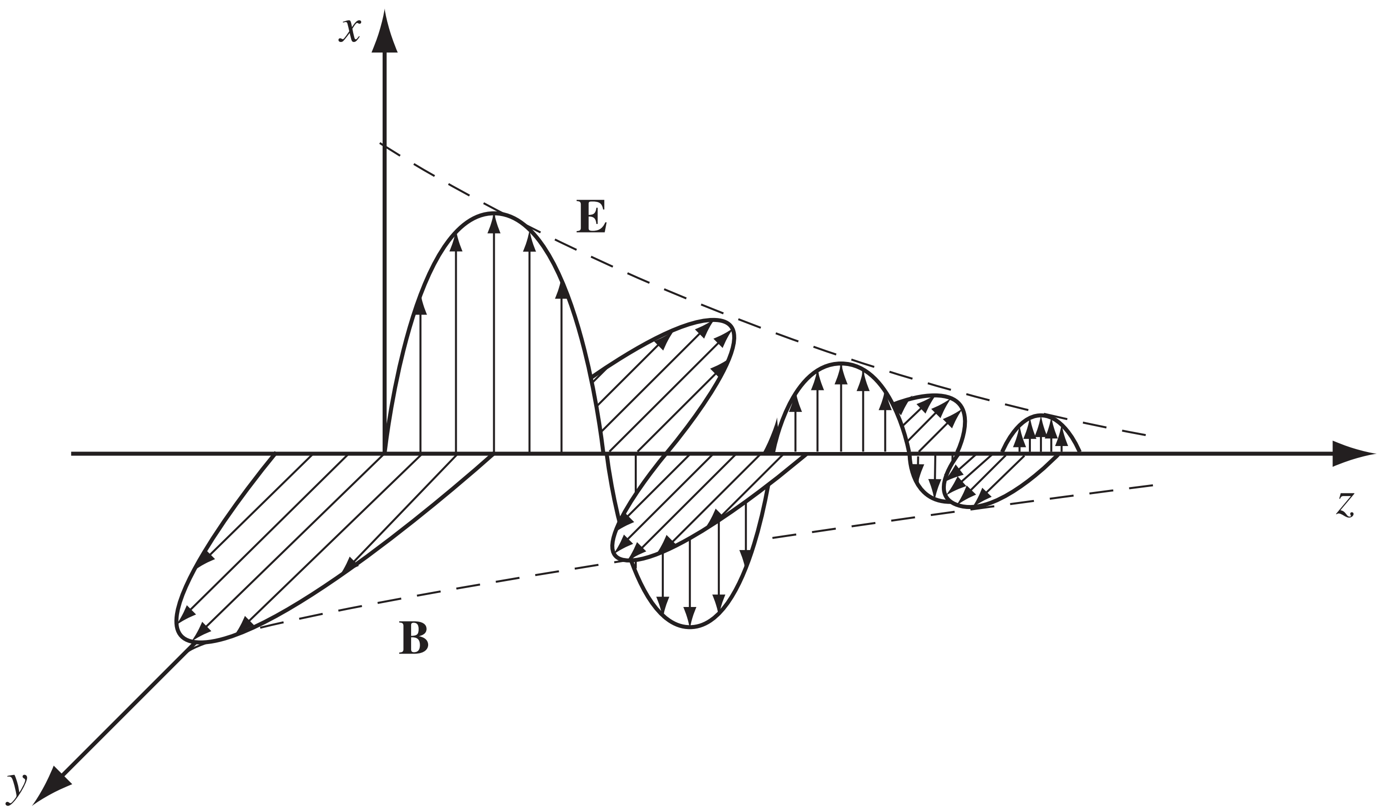
the magnetic field lags behind the electric field. Meanwhile, the (real) amplitudes of $\boldsymbol{E}$ and $\boldsymbol{B}$ are related by

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \tag{9.114}$$

The (real) electric and magnetic fields are, finally,

$$\left.\begin{aligned} \boldsymbol{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{\boldsymbol{x}} \\ \boldsymbol{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\hat{\boldsymbol{y}} \end{aligned}\right\} \tag{9.115}$$

Electromagnetic Waves in Conductors (cont'd)



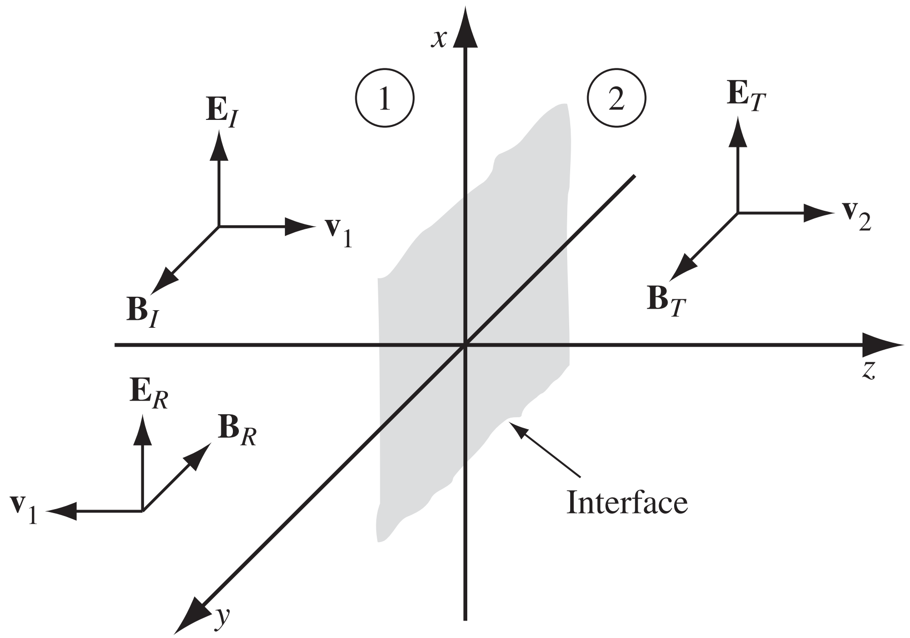
Reflection at a Conducting Surface

The boundary conditions we used to analyze reflection and refraction at an interface between two dielectrics do not hold in the presence of free charges and currents. We have to use the more general relations:

$$\left.\begin{array}{llll} \text{(i)} & \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f\,, & \text{(iii)} & \boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0 \\ \text{(ii)} & \boldsymbol{B}_1^{\perp} - \boldsymbol{B}_2^{\perp} = 0, & \text{(iv)} & \frac{1}{\mu_1}\boldsymbol{B}_1^{\parallel} - \frac{1}{\mu_2}\boldsymbol{B}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}} \end{array}\right\} \tag{9.116}$$

where σ_f is the free surface charge, $\boldsymbol{K}_f$ the free surface current, and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1).

For ohmic conductors ($\boldsymbol{J}_f = \sigma \boldsymbol{E}$) there can be no free surface current.



Suppose xy plane forms the boundary between a nonconducting linear medium (1) and a conductor (2). A monochromatic plane wave, traveling in z direction and polarized in x direction, approaches from left:

$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_I(z,t) &= \tilde{E}_{0_I} e^{i(k_1 z - \omega t)} \hat{\boldsymbol{x}} \\ \tilde{\boldsymbol{B}}_I(z,t) &= \frac{1}{v_1} \tilde{E}_{0_I} e^{i(k_1 z - \omega t)} \hat{\boldsymbol{y}} \end{aligned}\right\} \qquad (9.117)$$

It gives rise to a reflected wave

$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_R(z,t) &= \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{\boldsymbol{x}} \\ \tilde{\boldsymbol{B}}_R(z,t) &= -\frac{1}{v_1} \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{\boldsymbol{y}} \end{aligned}\right\} \qquad (9.118)$$

and a transmitted wave in medium (2)

$$\left.\begin{aligned} \tilde{\boldsymbol{E}}_T(z,t) &= \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)} \hat{\boldsymbol{x}} \\ \tilde{\boldsymbol{B}}_T(z,t) &= \frac{\tilde{k}_2}{\omega} \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)} \hat{\boldsymbol{y}} \end{aligned}\right\} \qquad (9.119)$$

Reflection at a Conducting Surface (cont'd)

The wave is attenuated as it penetrates into the conductor.

At $z = 0$, the combined wave in medium (1) must join the wave in medium (2). Since $E^{\perp} = 0$ on both sides, boundary condition (i) yields $\sigma_f = 0$. Since $B^{\perp} = 0$, (ii) is automatically satisfied.

Meanwhile, (iii) gives

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T} \tag{9.120}$$

and (iv) (with $\boldsymbol{K}_f = 0$) says

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0_T} = 0 \tag{9.121}$$

or

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta}\,\tilde{E}_{0_T} \tag{9.122}$$

Reflection at a Conducting Surface (cont'd)

where

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2 \tag{9.123}$$

It follows that

$$\tilde{E}_{0_R} = \left(\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1 + \tilde{\beta}} \right) \tilde{E}_{0_I}. \tag{9.124}$$

For a conductor with a large conductivity, $\tilde{k}_2$ (Eq. (9.103)) and hence $\tilde{\beta}$ are large as well. Thus,

$$\tilde{E}_{0_R} \approx -\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} \approx 0 \tag{9.125}$$

In this case the wave is totally reflected, with a 180° phase shift.