The Answer of Assignment 2

WEI SHUANG

31/8/2025

Problem 1 Solution

(1)

According to the theory of isomorphism of linear spaces, any two linear spaces of the same dimension are isomorphic. Therefore,

$$\sum_{i,j=0}^{1} c_{ij} \left| i, j \right\rangle$$

is isomorphic to $\{(x_1, x_2, x_3, x_4)\}, x \in \mathbb{C}$. That is,

$$v_1 = (c_{00}, c_{01}, c_{10}, c_{11})^T, \quad v_2 = (d_{00}, d_{01}, d_{10}, d_{11})^T.$$

(2)

We have

$$\langle \psi_1 | \psi_2 \rangle = v_1^\dagger v_2,$$

because

$$\langle \psi_1 | \psi_2 \rangle = \left(\sum_{i,j=0}^{1} c_{ij}^* \left\langle ij \right| \right) \left(\sum_{i,j=0}^{1} d_{ij} \left| ij \right\rangle \right) = v_1^\dagger v_2.$$

(3)

$$O \left| pq \right\rangle = O_{ij,kl} \left| ij \right\rangle \left\langle kl \right| \left| pq \right\rangle = O_{ij,kl} \left| ij \right\rangle \delta_{kp} \delta_{lq} = O_{ij,pq} \left| ij \right\rangle.$$

If we define $|00\rangle = e_1$, $|01\rangle = e_2$, $|10\rangle = e_3$, $|11\rangle = e_4$, then we have

$$Oe_i = \sum_{j=1}^{4} O_{ji} e_j.$$

Therefore, the matrix representation of O is:

$$M = \begin{pmatrix} O_{11} & O_{12} & O_{13} & O_{14} \\ O_{21} & O_{22} & O_{23} & O_{24} \\ O_{31} & O_{32} & O_{33} & O_{34} \\ O_{41} & O_{42} & O_{43} & O_{44} \end{pmatrix}.$$

(4)

$O\,|\psi_1\rangle$ is the same as Mv_1, because

$$O\,|\psi_1\rangle = O_{ij,kl}\,|ij\rangle\,\langle kl| \cdot c_{mn}\,|mn\rangle = O_{ij,mn}c_{mn}\,|ij\rangle\,.$$

This is equivalent to Mv_1 if we define $|00\rangle = e_1$, $|01\rangle = e_2$, $|10\rangle = e_3$, $|11\rangle = e_4$.

Problem 2 Solution

(1)

Starting from $[b_i^\dagger, b_j^\dagger] = 0$, show that $[b_i, b_j] = 0$. Indeed,

$$[b_i^\dagger, b_j^\dagger] = b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0.$$

Taking the Hermitian conjugate,

$$(b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger)^\dagger = b_j b_i - b_i b_j = 0,$$

so

$$[b_i, b_j] = 0.$$

(2)

$$\langle\overline{n_1' n_2' \cdots n_k'}|b_i^\dagger|\overline{n_1 n_2 \cdots n_k}\rangle = \delta_{n_1,n_1'} \cdots \delta_{n_i,n_i'-1} \cdots \delta_{n_k,n_k'} \sqrt{n_i'}.$$

Also,

$$b_i^\dagger\,|\overline{n_1 n_2 \cdots n_k}\rangle = \sqrt{n_i+1}\,|\overline{n_1 n_2 \cdots (n_i+1) \cdots n_k}\rangle\,,$$

so

$$b_i\,|\overline{n_1 n_2 \cdots n_k}\rangle = \sqrt{n_i}\,|\overline{n_1 n_2 \cdots (n_i-1) \cdots n_k}\rangle\,.$$

(3)

$$[b_i, b_j^\dagger]\,|\overline{n_1 n_2 \cdots n_k}\rangle = b_i b_j^\dagger\,|\overline{n}\rangle - b_j^\dagger b_i\,|\overline{n}\rangle\,.$$

(4)

If $i = j$, we have

$$[b_i, b_i^\dagger] |\overline{n}\rangle = (n_i + 1 - n_i) |\overline{n}\rangle = |\overline{n}\rangle .$$

(5)

$$\{f_i, f_j\} = f_i f_j + f_j f_i.$$

When $i = j$,

$$\{f_i, f_i\} = 2f_i^2 = 0,$$

since $f_i^2 = 0$.

(6)

For $i \neq j$, similarly one checks on the occupation basis that

$$\{f_i, f_j\} |n\rangle = 0.$$

(7)

For $i = j$,

$$\{f_i, f_i^\dagger\} = f_i f_i^\dagger + f_i^\dagger f_i = 1.$$

For $i \neq j$, one finds

$$\{f_i, f_j^\dagger\} = 0.$$

Problem 3 Solution

(1)

Define creation/annihilation operators on the occupation basis:

$$b_i^\dagger |n_1, \ldots, n_i = 0, \ldots, n_L\rangle = |n_1, \ldots, 1, \ldots, n_L\rangle , \quad b_i^\dagger |n_1, \ldots, n_i = 1, \ldots, n_L\rangle = 0,$$

$$b_i |n_1, \ldots, n_i = 1, \ldots, n_L\rangle = |n_1, \ldots, 0, \ldots, n_L\rangle , \quad b_i |n_1, \ldots, n_i = 0, \ldots, n_L\rangle = 0.$$

One checks that

$$[b_i, b_j] = 0, \quad [b_i, b_j^\dagger] = \delta_{ij}.$$

(2)

Define

$$\sigma_i^{\pm} = \frac{1}{2}(\sigma_i^x \pm i\sigma_i^y).$$

Then

$$[\sigma_i^-, \sigma_j^-] = 0, \quad [\sigma_i^-, \sigma_j^+] = 0 \ (i \neq j), \quad \{\sigma_i^-, \sigma_i^+\} = I.$$

(3)

Using the Jordan–Wigner transformation:

$$f_i^\dagger = \left(\prod_{j<i} \sigma_j^z\right) \sigma_i^+, \quad f_i = \left(\prod_{j<i} \sigma_j^z\right) \sigma_i^-.$$

Thus

$$f_i^\dagger f_i = \sigma_i^+ \sigma_i^- = \frac{1}{2}(1 - \sigma_i^z).$$

Moreover,

$$\{f_i, f_j\} = 0, \quad \{f_i, f_j^\dagger\} = \delta_{ij}.$$

(4)

We can invert:

$$\sigma_i^z = 1 - 2f_i^\dagger f_i,$$

$$\sigma_i^+ = f_i^\dagger \prod_{j<i}(1 - 2f_j^\dagger f_j), \quad \sigma_i^- = f_i \prod_{j<i}(1 - 2f_j^\dagger f_j).$$