Potentials and Fields

We seek the general solution to Maxwell's equations

$$\left.\begin{array}{llll}\text{(i)} & \nabla\cdot\boldsymbol{E}=\dfrac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t}\\[2ex] \text{(ii)} & \nabla\cdot\boldsymbol{B}=0, & \text{(iv)} & \nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\dfrac{\partial\boldsymbol{E}}{\partial t}\end{array}\right\}\tag{10.1}$$

From Eq. (10.1ii), we can write

$$\boldsymbol{B}=\nabla\times\boldsymbol{A}\tag{10.2}$$

Putting this into Faraday's law (iii) yields

$$\begin{aligned}\nabla\times\boldsymbol{E}=-\frac{\partial}{\partial t}(\nabla\times\boldsymbol{A})&\Rightarrow\nabla\times\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right)=0\\ &\Rightarrow\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right)=-\nabla V\end{aligned}$$

Scalar and Vector Potentials (cont'd)

In terms of V and $\boldsymbol{A}$, then,

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \tag{10.3}$$

The potential representation (Eqs. (10.2) and (10.3)) automatically fulfills the two homogeneous Maxwell equations, (ii) and (iii). Putting Eq. (10.3) into (i), we find that

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{1}{\epsilon_0}\rho \tag{10.4}$$

It reduces to the Poisson's equation in the static case. Putting Eqs. (10.2) and (10.3) into (iv) yields

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{J} - \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2}$$

Scalar and Vector Potentials (cont'd)

or, using the vector identity $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$,

$$\left(\nabla^2 \boldsymbol{A} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \boldsymbol{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \boldsymbol{J} \qquad (10.5)$$

Eqs. (10.4) and (10.5) contain all the information in Maxwell's equations.

Example

Find the charge and current distributions that would give rise to the potentials

$$V = 0, \quad \boldsymbol{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\boldsymbol{z}}, & \text{for } |x| < ct \\ 0, & \text{for } |x| > ct \end{cases}$$

where k is a constant, and $c = 1/\sqrt{\mu_0 \epsilon_0}$.

Solution

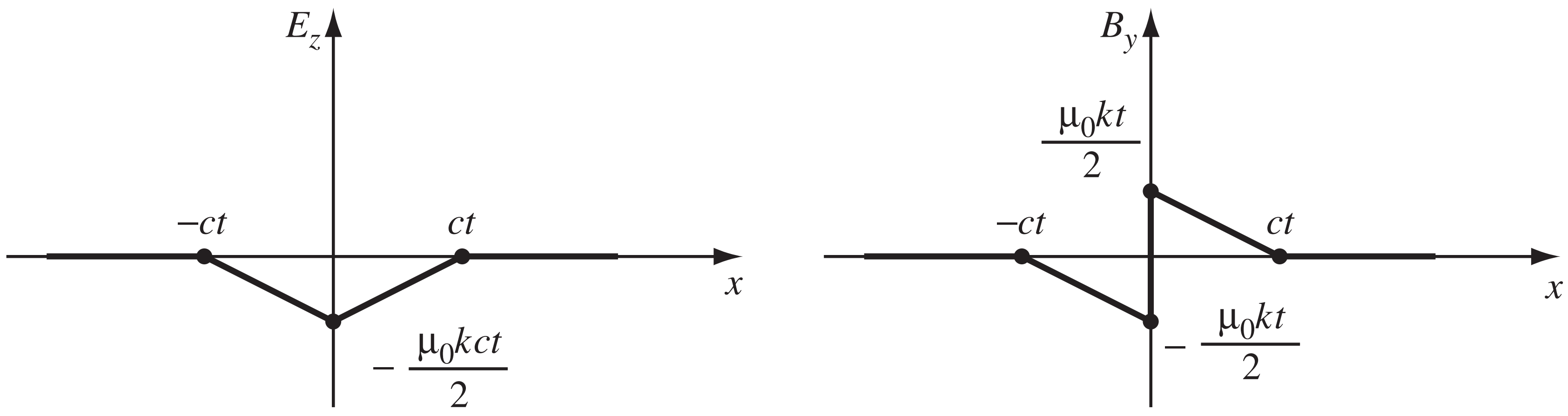
Using Eqs. (10.2) and (10.3):

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\,\hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\,\hat{\boldsymbol{y}} = \pm\frac{\mu_0 k}{2c}(ct - |x|)\,\hat{\boldsymbol{y}}$$

(plus for $x > 0$, minus for $x < 0$).

Example (cont'd)



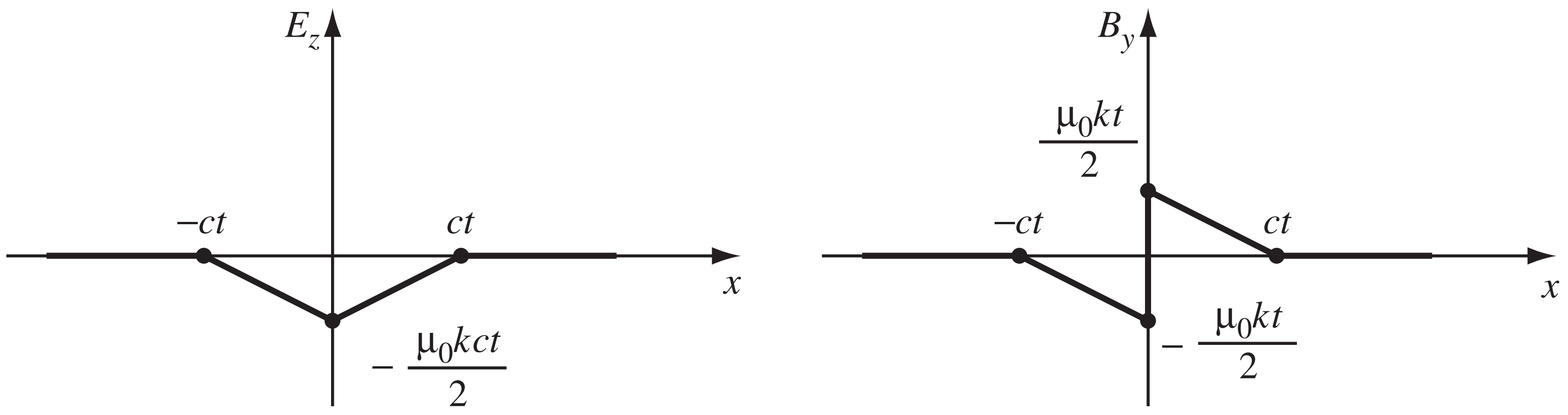
These are for $|x| < ct$; when $|x| > ct$, $\boldsymbol{E} = \boldsymbol{B} = \mathbf{0}$.

Now, we have

$$\nabla \cdot \boldsymbol{E} = 0,\ \nabla \cdot \boldsymbol{B} = 0,\ \nabla \times \boldsymbol{E} = \mp \frac{\mu_0 k}{2}\,\hat{\boldsymbol{y}},\ \nabla \times \boldsymbol{B} = -\frac{\mu_0 k}{2c}\,\hat{\boldsymbol{z}}$$

$$\frac{\partial \boldsymbol{E}}{\partial t} = -\frac{\mu_0 kc}{2}\,\hat{\boldsymbol{z}},\quad \frac{\partial \boldsymbol{B}}{\partial t} = \pm \frac{\mu_0 k}{2}\,\hat{\boldsymbol{y}}$$

Example (cont'd)



Maxwell's equations are all satisfied, with ρ and $\boldsymbol{J}$ both zero.

However, $\boldsymbol{B}$ has a discontinuity at $x = 0$, and this signals the presence of a surface current $\boldsymbol{K}$ in the yz plane.

Now, boundary condition gives

$$kt\,\hat{\boldsymbol{y}} = \boldsymbol{K} \times \hat{\boldsymbol{x}}$$

and hence

$$\boldsymbol{K} = kt\,\hat{\boldsymbol{z}}$$

Gauge Transformations

Gauge Transformations

Eqs. (10.2) and (10.3) do not uniquely define the potentials; we are free to impose extra conditions on V and $\boldsymbol{A}$, as long as nothing happens to $\boldsymbol{E}$ and $\boldsymbol{B}$.

Suppose we have two sets of potentials, $(V, \boldsymbol{A})$ and $(V', \boldsymbol{A}')$, which correspond to the *same* electric and magnetic fields. Write

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\alpha} \quad \text{and} \quad V' = V + \beta$$

Since the two $\boldsymbol{A}$'s give the same $\boldsymbol{B}$, their curls must be equal, and hence

$$\nabla \times \boldsymbol{\alpha} = \boldsymbol{0}$$
$$\Rightarrow \boldsymbol{\alpha} = \nabla\lambda$$

Gauge Transformations (cont'd)

The two potentials also give the same $\boldsymbol{E}$, so

$$\nabla\beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} = \mathbf{0}$$
$$\Rightarrow \nabla\left(\beta + \frac{\partial\lambda}{\partial t}\right) = \mathbf{0}$$

The term in parentheses is therefore independent of position (it could, however, depend on time); call it $k(t)$:

$$\beta = -\frac{\partial\lambda}{\partial t} + k(t)$$

Absorbing $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')\,\mathrm{d}t'$ to the old one. This will not affect the gradient of λ; it just adds $k(t)$ to $\partial\lambda/\partial t$.

Gauge Transformations (cont'd)

It follows that

$$\left.\begin{aligned} \boldsymbol{A}' &= \boldsymbol{A} + \nabla\lambda, \\ V' &= V - \frac{\partial\lambda}{\partial t} \end{aligned}\right\} \tag{10.6}$$

- For any scalar function λ, we can add $\nabla\lambda$ to $\boldsymbol{A}$, provided we simultaneously subtract $\partial\lambda/\partial t$ from V. None of this will affect the physical quantities $\boldsymbol{E}$ and $\boldsymbol{B}$. Such changes in V and $\boldsymbol{A}$ are called **gauge transformations**.

- They can be exploited to simplify Eqs. (10.4) and (10.5). In magnetostatics, it was best to choose $\nabla \cdot \boldsymbol{A} = 0$.

Coulomb Gauge

As in magnetostatics, we pick

$$\nabla \cdot \boldsymbol{A} = 0 \tag{10.7}$$

With this, Eq. (10.4) becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{10.8}$$

This is Poisson's equation. Setting $V = 0$ at infinity,

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}', t)}{\mathscr{r}}\,\mathrm{d}\tau' \tag{10.9}$$

where $\mathscr{r} = |\boldsymbol{\mathscr{r}}|$ and $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$.

Coulomb Gauge (cont'd)

- **Advantage:** Scalar potential is relatively simple to calculate
- **Disadvantage:** $\boldsymbol{A}$ is particularly difficult to calculate. The differential equation for $\boldsymbol{A}$ (10.5) in the Coulomb gauge reads

$$\nabla^2 \boldsymbol{A} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right) \tag{10.10}$$

Lorenz gauge

In the Lorenz gauge we pick

$$\nabla \cdot \boldsymbol{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \tag{10.11}$$

This is designed to eliminate the middle term in Eq. (10.5). With this,

$$\nabla^2 \boldsymbol{A} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} \tag{10.12}$$

Meanwhile, the differential equation for V, Eq. (10.4), becomes

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho \tag{10.13}$$

Lorenz gauge (cont'd)

The virtue of the Lorenz gauge is that it treats V and $\boldsymbol{A}$ on an equal footing: the same differential operator

$$\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \equiv \Box^2 \qquad (10.14)$$

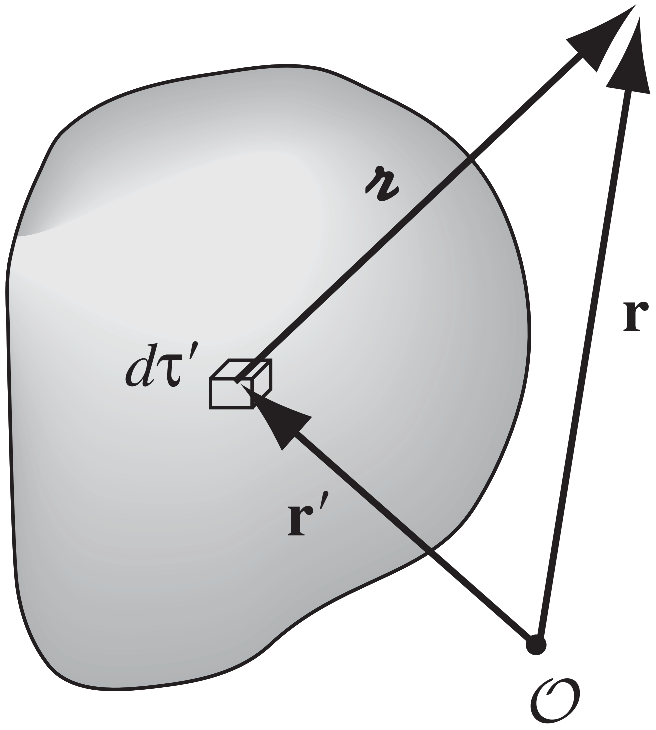
(called the d'Alembertian) occurs in both equations:

$$\begin{aligned} &\text{(i)} \quad \Box^2 V = -\frac{1}{\epsilon_0}\rho \\ &\text{(ii)} \quad \Box^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J} \end{aligned} \qquad (10.15)$$

In the Lorenz gauge V and $\boldsymbol{A}$ satisfy the inhomogeneous wave equation, with a "source" term (in place of zero) on the right.

From now on, Lorenz gauge will be used exclusively, and the whole of electrodynamics reduces to the problem of solving the inhomogeneous wave equation for specified sources.

Retarded Potentials of a Continuous Distributions



For static cases, we have

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \quad \nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

with solutions

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\mathscr{r}}\,\mathrm{d}\tau', \;\; \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}}\,\mathrm{d}\tau' \tag{10.16}$$

where $\mathscr{r}$ is the distance from the source point $\boldsymbol{r}'$ to the field point $\boldsymbol{r}$.

Retarded Potentials (cont'd)

Since electromagnetic "news" travels at the speed of light, in the nonstatic case, it's not the status of the source right now that matters, but rather its condition at some earlier time t_r (called the retarded time) when the "message" left.

Since this message must travel a distance $\mathscr{r}$, the delay is $\mathscr{r}/c$:

$$t_r \equiv t - \frac{\mathscr{r}}{c} \tag{10.17}$$

The generalization of Eq. (10.16) for nonstatic sources is therefore

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}', t_r)}{\mathscr{r}}\, \mathrm{d}\tau', \quad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}', t_r)}{\mathscr{r}}\, \mathrm{d}\tau' \tag{10.18}$$

Here $\rho(\boldsymbol{r}', t_r)$ is the charge density that prevailed at point $\boldsymbol{r}'$ at the retarded time t_r. Because the integrands are evaluated at the retarded time, these are called **retarded potentials**.

Retarded Potentials (cont'd)

Let's check that the retarded scalar potential satisfies Eq. (10.15).

In calculating the Laplacian of $V(\boldsymbol{r}, t)$, note that the integrand (in Eq. (10.18)) depends on $\boldsymbol{r}$ in two places: explicitly, in the denominator $(\mathscr{r} = |\boldsymbol{r} - \boldsymbol{r}'|)$, and implicitly, through $t_r = t - \mathscr{r}/c$, in the numerator. Thus

$$\nabla V = \frac{1}{4\pi\epsilon_0} \int \left[(\nabla\rho)\frac{1}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right] \mathrm{d}\tau' \tag{10.19}$$

and

$$\nabla\rho = \dot{\rho}\nabla t_r = -\frac{1}{c}\dot{\rho}\nabla\mathscr{r} \tag{10.20}$$

(the dot denotes differentiation with respect to time).

Now $\nabla\mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$ and $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, so

$$\nabla V = \frac{1}{4\pi\epsilon_0} \int \left[-\frac{\dot{\rho}}{c}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} - \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right] \mathrm{d}\tau' \tag{10.21}$$

Retarded Potentials (cont'd)

Taking the divergence,

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\cdot(\nabla\dot{\rho}) + \dot{\rho}\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right)\right]\right.$$
$$\left. -\left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot(\nabla\rho) + \rho\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\right\} d\tau'$$

But

$$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\nabla\mathscr{r} = -\frac{1}{c}\ddot{\rho}\hat{\boldsymbol{\mathscr{r}}}$$

and

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$$

whereas

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

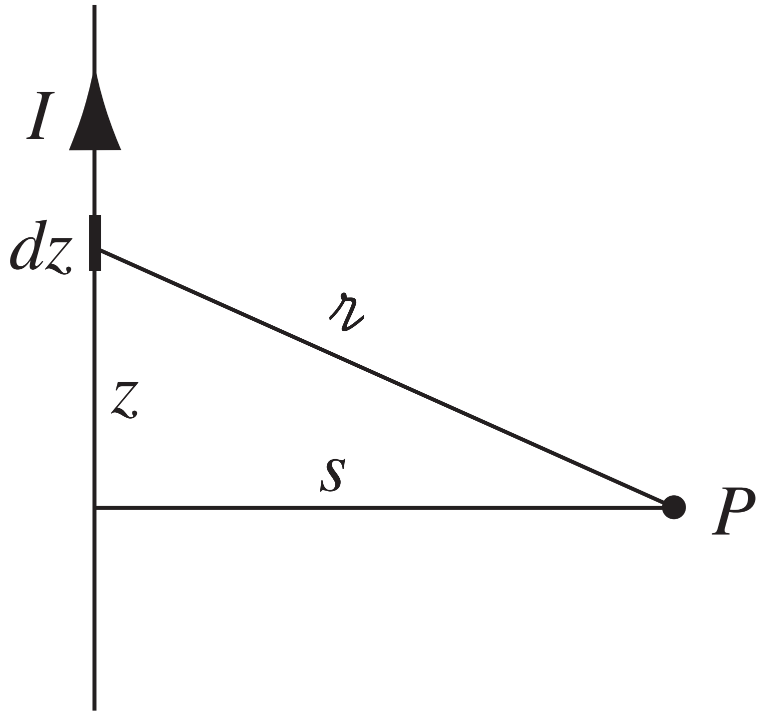
Retarded Potentials (cont'd)

So,

$$\begin{aligned}\nabla^2 V &= \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}})\right]\,\mathrm{d}\tau' \\ &= \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\boldsymbol{r},t)\end{aligned}$$

confirming that the retarded potential Eq. (10.18) satisfies the inhomogeneous wave equation Eq. (10.15)

Example



An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & \text{for } t \le 0 \\ I_0, & \text{for } t > 0 \end{cases}$$

That is, a constant current I_0 is turned on abruptly at $t = 0$. Find the resulting electric and magnetic fields, assuming the wire is electrically neutral.

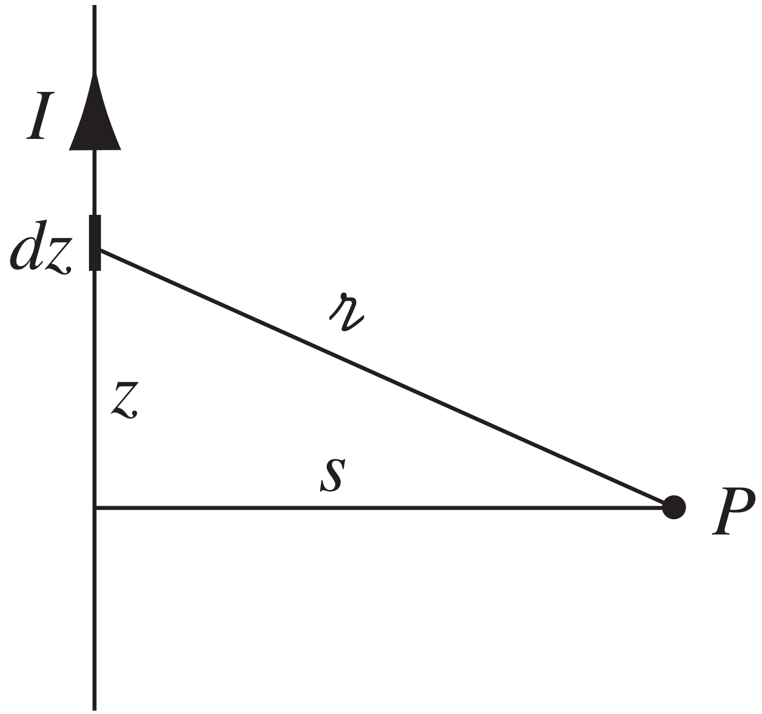
Solution

Wire is electrically neutral, so the scalar potential is zero. Let the wire lie along the z axis.

The retarded vector potential at point P is

$$\boldsymbol{A}(s,t) = \frac{\mu_0}{4\pi}\hat{\boldsymbol{z}} \int_{-\infty}^{\infty} \frac{I(t_r)}{\mathscr{r}}\, \mathrm{d}z$$

Example (cont'd)



For $t < s/c$, the "news" has not yet reached P, and the potential is zero. For $t > s/c$, only the segment

$$|z| \leq \sqrt{(ct)^2 - s^2} \tag{10.22}$$

contributes (outside this range t_r is negative, so $I(t_r) = 0$);

thus

$$\boldsymbol{A}(s,t) = \left(\frac{\mu_0 I_0}{4\pi}\hat{\boldsymbol{z}}\right) 2\int_0^{\sqrt{(ct)^2 - s^2}} \frac{dz}{\sqrt{s^2 + z^2}}$$

Example (cont'd)

$$\boldsymbol{A}(s,t) = \frac{\mu_0 I_0}{2\pi}\hat{\boldsymbol{z}} \ln(\sqrt{s^2+z^2}+z)\Big|_0^{\sqrt{(ct)^2-s^2}}$$

$$= \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)\hat{\boldsymbol{z}}$$

The electric field is

$$\boldsymbol{E}(s,t) = -\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-s^2}}\hat{\boldsymbol{z}}$$

and the magnetic field is

$$\boldsymbol{B}(s,t) = \nabla\times\boldsymbol{A} = -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2-s^2}}\hat{\boldsymbol{\phi}}$$

Jefimenko’s Equations

Jefimenko's Equations

Given the retarded potentials

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}', t_r)}{\mathscr{r}}\,\mathrm{d}\tau', \quad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}', t_r)}{\mathscr{r}}\,\mathrm{d}\tau' \tag{10.23}$$

the fields are

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{10.24}$$

Gradient of V (Eq. (10.21)) is known. The time derivative of $\boldsymbol{A}$ is

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\dot{\boldsymbol{J}}}{\mathscr{r}}\,\mathrm{d}\tau' \tag{10.25}$$

Putting them together (and using $c^2 = 1/\mu_0\epsilon_0$):

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\boldsymbol{r}', t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}(\boldsymbol{r}', t_r)}{c\,\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}} - \frac{\dot{\boldsymbol{J}}(\boldsymbol{r}', t_r)}{c^2\,\mathscr{r}}\right]\mathrm{d}\tau' \tag{10.26}$$

Jefimenko's Equations (cont'd)

As for $\boldsymbol{B}$, the curl of $\boldsymbol{A}$ contains two terms:

$$\nabla \times \boldsymbol{A} = \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathscr{r}} (\nabla \times \boldsymbol{J}) - \boldsymbol{J} \times \nabla \left(\frac{1}{\mathscr{r}} \right) \right] \mathrm{d}\tau'$$

Now

$$(\nabla \times \boldsymbol{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z}$$

and

$$\frac{\partial J_z}{\partial y} = \dot{\boldsymbol{J}}_z \frac{\partial t_r}{\partial y} = -\frac{1}{c} \dot{\boldsymbol{J}}_z \frac{\partial \mathscr{r}}{\partial y}$$

so

$$(\nabla \times \boldsymbol{J})_x = -\frac{1}{c} \left(\dot{\boldsymbol{J}}_z \frac{\partial \mathscr{r}}{\partial y} - \dot{\boldsymbol{J}}_y \frac{\partial \mathscr{r}}{\partial z} \right) = \frac{1}{c} [\dot{\boldsymbol{J}} \times (\nabla \mathscr{r})]_x$$

But $\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$, so

$$\nabla \times \boldsymbol{J} = \frac{1}{c} \dot{\boldsymbol{J}} \times \hat{\boldsymbol{\mathscr{r}}} \qquad (10.27)$$

Jefimenko's Equations (cont'd)

Meanwhile $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$, and hence

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int\left[\frac{\boldsymbol{J}(\boldsymbol{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\boldsymbol{J}}(\boldsymbol{r}', t_r)}{c\,\mathscr{r}}\right]\times\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \qquad (10.28)$$

This is the time-dependent generalization of the Biot-Savart law, to which it reduces in the static case.

Equations (10.26) and (10.28) are the solutions to Maxwell's equations.

The Fields of a Moving Point Charge

The Fields of a Moving Point Charge

The fields of a moving point charge are given by

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u}+\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})] \tag{10.29}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\boldsymbol{E}(\boldsymbol{r},t) \tag{10.30}$$

where vector $\boldsymbol{u}$ is defined as

$$\boldsymbol{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}, \tag{10.31}$$

$\boldsymbol{v}$ is the velocity of the charge at the retarded time, and $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point $\boldsymbol{r}$.

The Fields of a Moving Point Charge (cont'd)

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u}+\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})]$$

- The first term in $\boldsymbol{E}$ (involving $(c^2-v^2)\boldsymbol{u}$) falls off as the inverse *square* of the distance from the particle. If velocity and acceleration are both zero, this term alone survives and reduces to

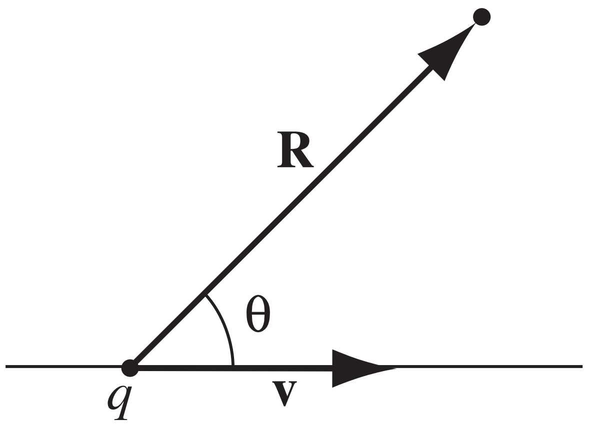
$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

  This term is called the **generalized Coulomb field** (or **velocity field**).

- The second term (involving $\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})$) falls off as the inverse *first* power of $\mathscr{r}$ and is dominant at large distances. It is called the **radiation field** or **acceleration field**.

Fields of a point charge moving with constant velocity

The electric field of a point charge moving with *constant* velocity $\boldsymbol{v}$

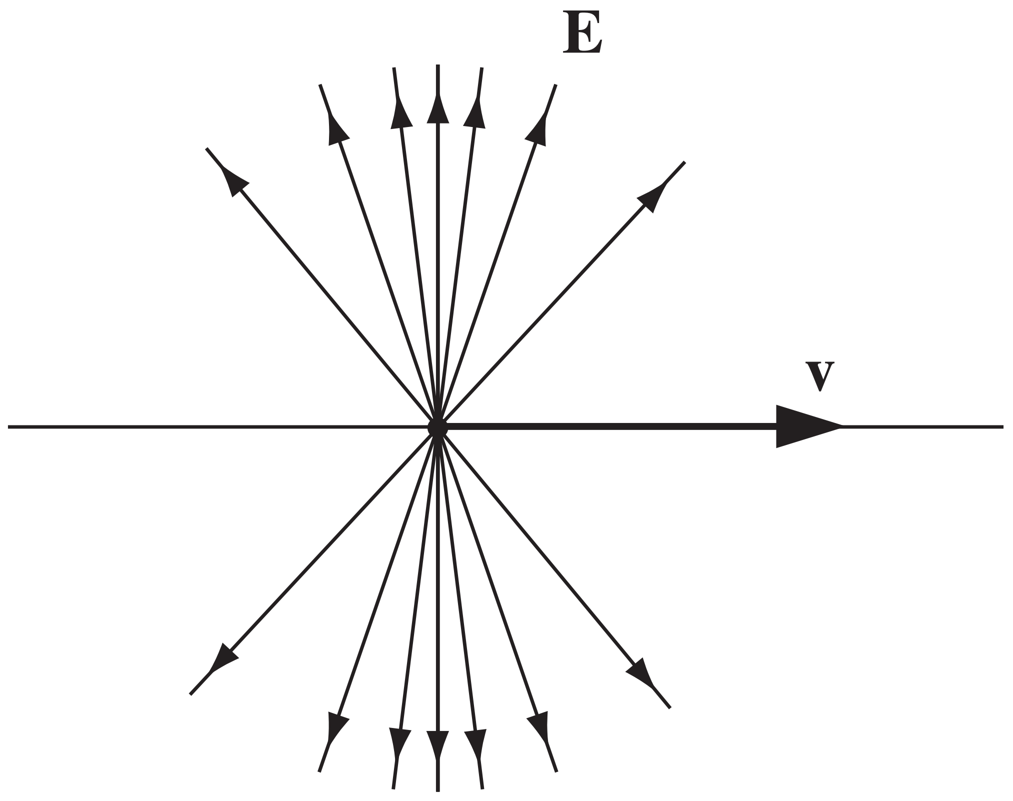


is given by

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta/c^2\right)^{3/2}} \frac{\hat{\boldsymbol{R}}}{R^2} \tag{10.32}$$

where $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{v}t$ is the vector from the *present* location of the charge to field point $\boldsymbol{r}$, and θ is the angle between $\boldsymbol{R}$ and $\boldsymbol{v}$.

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\boldsymbol{R}}}{R^2}$$



- $\boldsymbol{E}$ points along the line from the present position of the particle.

  Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion

- In the forward and backward directions $\boldsymbol{E}$ is reduced by a factor $(1-v^2/c^2)$ relative to the field of a charge at rest.

- In the perpendicular direction, it is enhanced by a factor $1/\sqrt{1-v^2/c^2}$

The magnetic field of a point charge moving with *constant* velocity $\boldsymbol{v}$ is given by

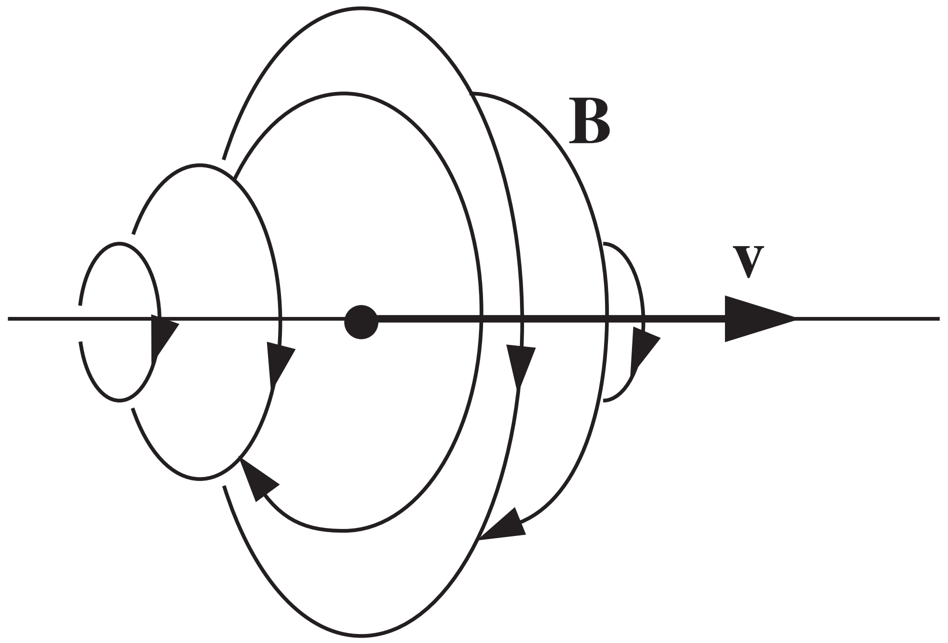
$$\boldsymbol{B} = \frac{1}{c^2}(\boldsymbol{v} \times \boldsymbol{E}) \qquad (10.33)$$

Lines of $\boldsymbol{B}$ circle around the charge.

When $v^2 \ll c^2$, Eqs. (10.32) and (10.33) reduce to

$$\boldsymbol{E}(\boldsymbol{r}, t) \approx \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\boldsymbol{R}}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) \approx \frac{\mu_0}{4\pi}\frac{q}{R^2}(\boldsymbol{v} \times \hat{\boldsymbol{R}}) \qquad (10.34)$$



The first is essentially Coulomb's law, and the latter is the Biot-Savart law for a point charge.