# The Answer of Assignment 2

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### Problem 1 Solution

**(1)** According to the theory of isomorphism of linear spaces, any two linear spaces of the same dimension are isomorphic. Therefore, $\sum_{i,j=0}^{1} c_{ij} |i,j\rangle$ is isomorphic to $\{(x_1, x_2, x_3, x_4)\}, x \in \mathbb{C}$. That is, $v_1 = (c_{00}, c_{01}, c_{10}, c_{11})^T$, $v_2 = (d_{00}, d_{01}, d_{10}, d_{11})^T$.

**(2)** $\langle \psi_1 | \psi_2 \rangle = v_1^\dagger v_2$ 成立, 这是因为 $\langle \psi_1 | \psi_2 \rangle = (\sum_{i,j=0}^{1} c_{i,j}^* \langle ij|)(\sum i,,j=0^1 c_{i,j} |ij\rangle) = v_1^\dagger v_2$

**(3)** $O |pq\rangle = O_{ij,kl} |ij\rangle \langle kl| |pq\rangle = O_{ij,kl} |ij\rangle \delta_{kp}\delta_{lq} = O_{ij,pq} |ij\rangle$ if we define $|00\rangle$ as $e_1$, $|01\rangle$ as $e_2$, $|10\rangle$ as $e_3$, $|11\rangle$ as $e_4$, then we can get the matrix representation of $O$: $Oe_i = \sum_{j=1}^{4} O_{ji}e_j$ Therefore, the matrix representation of $O$ is:

$$M = \begin{pmatrix} O_{11} & O_{12} & O_{13} & O_{14} \\ O_{21} & O_{22} & O_{23} & O_{24} \\ O_{31} & O_{32} & O_{33} & O_{34} \\ O_{41} & O_{42} & O_{43} & O_{44} \end{pmatrix}$$

**(4)** $O |\psi_1\rangle$ is the same as $Mv_1$, because:

$$O |\psi_1\rangle = O_{ij,kl} |ij\rangle \langle kl| \cdot c_{mn} |mn\rangle = O_{ij,kl} |ij\rangle c_{mn}\delta_{km}\delta_{ln} = O_{ij,mn}c_{mn} |ij\rangle$$

this is equivalent to $Mv_1$ if we define $|00\rangle$ as $e_1$, $|01\rangle$ as $e_2$, $|10\rangle$ as $e_3$, $|11\rangle$ as $e_4$.:

### Problem 2 Solution

**(1)** Starting from $[b_i^\dagger, b_i^\dagger] = 0$, show that $[b_i, b_j] = 0$. According to the definition of commutation relation, we have:

$$[b_i^\dagger, b_j^\dagger] = b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger = 0$$

Taking the Hermitian conjugate of both sides, we get:

$$(b_i^\dagger b_j^\dagger - b_j^\dagger b_i^\dagger)^\dagger = 0^\dagger$$

$$b_j b_i - b_i b_j = 0$$

so we have: $[b_i, b_j] = 0$

**(2)**

$$\langle n_1' n_2' \cdots n_k' | b_i^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle = \langle \overline{n_1 n_2 \cdots n_k} | b_i | n_1' n_2' \cdots n_k' \rangle^* = \delta_{n_1,n_1'} \delta_{n_2,n_2'} \cdots \delta_{n_i,n_i'-1} \cdots \delta_{n_k,n_k'} \sqrt{n_i'}$$

and

$$b_i^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle = \sqrt{n_i + 1} \, | \overline{n_1 n_2 \cdots (n_i + 1) \cdots n_k} \rangle$$

we can get the relation between $b_i$ and $b_i^\dagger$ as follows:

$$b_i | \overline{n_1 n_2 \cdots n_k} \rangle = \sqrt{n_i} \, | \overline{n_1 n_2 \cdots (n_i - 1) \cdots n_k} \rangle$$

**(3)**

We have:

$$
\begin{aligned}
[b_i, b_j^\dagger] | \overline{n_1 n_2 \cdots n_k} \rangle &= b_i b_j^\dagger | \overline{n_1 n_2 \cdots n_k} \rangle - b_j^\dagger b_i | \overline{n_1 n_2 \cdots n_k} \rangle \\
&= b_i \sqrt{n_j + 1} \, | \overline{n_1 n_2 \cdots (n_j + 1) \cdots n_k} \rangle \\
&\quad - b_j^\dagger \sqrt{n_i} \, | \overline{n_1 n_2 \cdots (n_i - 1) \cdots n_k} \rangle \\
&= \sqrt{n_i(n_j + 1)} \, | \overline{n_1 n_2 \cdots (n_i - 1) \cdots (n_j + 1) \cdots n_k} \rangle \cdot 0
\end{aligned}
$$

**(4)** if $i = j$, we have:

$$[b_i, b_i^\dagger] | \overline{n_1 n_2 \cdots n_k} \rangle = (n_i + 1 - n_i) | \overline{n_1 n_2 \cdots n_k} \rangle = | \overline{n_1 n_2 \cdots n_k} \rangle$$

**(5)**

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

When $i = j$, this becomes

$$\{f_i, f_i\} = f_i f_i + f_i f_i = 2 f_i^2$$

Since $f_i$ is a fermionic annihilation operator, it satisfies $f_i^2 = 0$. Therefore,

$$\{f_i, f_i\} = 2 \times 0 = 0$$

So, when $i = j$, the anticommutation relation $\{f_i, f_j\} = 0$ holds.

**(6)**

Suppose $i \neq j$. Calculate $\{f_i, f_j\} | n_1 n_2 \cdots \rangle$.

Recall the definition:

$$\{f_i, f_j\} = f_i f_j + f_j f_i$$

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We need to consider the action of $f_i$ and $f_j$ on the occupation number basis $|n_1 n_2 \cdots\rangle$, where $n_k = 0$ or $1$ for fermions.

Consider the following cases:

**Case 1:** $n_i = 0$ **or** $n_j = 0$

- If $n_i = 0$, then $f_i |n_1 n_2 \cdots\rangle = 0$. - If $n_j = 0$, then $f_j |n_1 n_2 \cdots\rangle = 0$.

Therefore, in either case, both $f_i f_j |n_1 n_2 \cdots\rangle = 0$ and $f_j f_i |n_1 n_2 \cdots\rangle = 0$, so

$$\{f_i, f_j\} |n_1 n_2 \cdots\rangle = 0$$

**Case 2:** $n_i = 1$ **and** $n_j = 1$

- $f_j |n_1 n_2 \cdots n_j = 1 \cdots n_i = 1 \cdots\rangle = (-1)^{s_1} |n_1 \cdots n_j = 0 \cdots n_i = 1 \cdots\rangle$ - Then $f_i$ acts on this state: $f_i |n_1 \cdots n_j = 0 \cdots n_i = 1 \cdots\rangle = (-1)^{s_2} |n_1 \cdots n_j = 0 \cdots n_i = 0 \cdots\rangle$

Similarly, $f_i f_j |n_1 n_2 \cdots\rangle$ and $f_j f_i |n_1 n_2 \cdots\rangle$ will differ by a sign, but since $i \neq j$, the sum $f_i f_j + f_j f_i$ will always cancel out due to the anticommutation property of fermionic operators.

Therefore,

$$\{f_i, f_j\} |n_1 n_2 \cdots\rangle = 0$$

**(7) case 1:** $i = j$

$$f_i, f_j^\dagger = f_i f_i^\dagger + f_i^\dagger f_i$$

when $n_i = 0$, we have:

$$\begin{aligned}
&f_i f_i^\dagger |n_1 n_2 \cdots n_i = 0 \cdots\rangle \\
&= f_i (-1)^s |n_1 n_2 \cdots n_i = 1 \cdots\rangle \\
&= (-1)^s (-1)^s |n_1 n_2 \cdots n_i = 0 \cdots\rangle \\
&= |n_1 n_2 \cdots n_i = 0 \cdots\rangle
\end{aligned}$$

when $n_i = 1$, we have:

$$\begin{aligned}
f_i^\dagger f_i |n_1 n_2 \cdots n_i = 1 \cdots\rangle &= f_i^\dagger (-1)^s |n_1 n_2 \cdots n_i = 0 \cdots\rangle \\
&= (-1)^s (-1)^s |n_1 n_2 \cdots n_i = 1 \cdots\rangle = |n_1 n_2 \cdots n_i = 1 \cdots\rangle
\end{aligned}$$

**case 2:** $i \neq j$

$$\{f_i, f_j^\dagger\} = f_i f_j^\dagger + f_j^\dagger f_i$$

when $n_i = 0$ or $n_j = 0$, we have: $f_i f_j^\dagger \psi = 0 f_j^\dagger f_i \psi = 0$ when $n_i = 1$ and $n_j = 1$, we have the same conclusion as above.

when $n_i = 1$ and $n_j = 0$, we have:

$$\{f_i f_j^\dagger\} |n_1 n_2 \cdots n_i = 1 \cdots n_j = 0 \cdots\rangle = f_i (-1)^{\sum_1^{j-1} n_x} |n_1 n_2 \cdots n_i = 1 \cdots n_j = 1 \cdots\rangle$$

$$+ f_j^\dagger (-1)^{\sum_1^{i-1} n_x} |n_1 n_2 \cdots n_i = 0 \cdots n_j = 0 \cdots\rangle = ((-1)^{s1+s2} + (-1)^{s1+s2+1}) |n_1, \cdots n_i = 0 \cdots n_j = 1 \cdots\rangle$$

# Problem 3 Solution

**(1)** We define $b_I^\dagger$ in

$$b_i^\dagger \left|n_1, n_2, \cdots, n_k\right\rangle = \left|n_1, n_2, \cdots, n_i + 1, \cdots, n_L\right\rangle = \left|n_1, n_2, \cdots, 1, n_{i+1}, \cdots, n_L\right\rangle \text{ if } n_i = 0$$

$$b_i^\dagger \left|n_1, n_2, \cdots, n_k\right\rangle = 0 \text{ if } n_i = 1$$

$$b_i \left|n_1, n_2, \cdots, n_k\right\rangle = \left|n_1, n_2, \cdots, n_i - 1, \cdots, n_L\right\rangle = \left|n_1, n_2, \cdots, 0, n_{i+1}, \cdots, n_L\right\rangle \text{ if } n_i = 1$$

$$b_i \left|n_1, n_2, \cdots, n_k\right\rangle = 0 \text{ if } n_i = 0$$

$[b_i, b_j] = 0$ for any i,j because when i=j $(b_i b_j - b_j b_i)\left|n_1, n_2, \cdots, n_L\right\rangle = 0$ if $n_i = 0$ $[b_i, b_j] = 0$ if $n_i = 1$ because $b_i b_j \left|n_1, n_2, \cdots, n_L\right\rangle = \left|n_1, n_2, \cdots, n_L\right\rangle$ and $b_j b_i \left|n_1, n_2, \cdots, n_L\right\rangle = \left|n_1, n_2, \cdots, n_L\right\rangle$ when $i \neq j$, $(b_i b_j - b_j b_i)\left|n_1, n_2, \cdots, n_L\right\rangle = 0$ because $b_i b_j \left|n_1, n_2, \cdots, n_L\right\rangle$ and $b_j b_i \left|n_1, n_2, \cdots, n_L\right\rangle$ are the same state. $[b_i, b_j^\dagger] = \delta_{ij}$

**case 1: i=j**

$$[b_i, b_i^\dagger] = b_i b_i^\dagger - b_i^\dagger b_i$$

if $n_i = 0$, we have:

$$b_i b_i^\dagger \left|n_1, n_2, \cdots, n_k\right\rangle = b_i \left|n_1, n_2, \cdots, 1, n_{i+1}, \cdots, n_L\right\rangle$$

$$= \left|n_1, n_2, \cdots, 0, n_{i+1}, \cdots, n_L\right\rangle = \left|n_1, n_2, \cdots, n_k\right\rangle$$

$b_i^\dagger b_i \left|n_1, n_2, \cdots, n_k\right\rangle = b_i^\dagger 0 = 0$

case 2: i≠j

$[b_i, b_j^\dagger] = b_i b_j^\dagger - b_j^\dagger b_i = 0$ because when $n_i = 0$ or $n_j = 0$, $b_i b_j^\dagger \left|n_1, n_2, \cdots, n_L\right\rangle = 0$ and $b_j^\dagger b_i \left|n_1, n_2, \cdots, n_L\right\rangle = 0$

when $n_i = 1$ or $n_j = 1$, $b_i b_j^\dagger \left|n_1, n_2, \cdots, n_L\right\rangle$ and $b_j^\dagger b_i \left|n_1, n_2, \cdots, n_L\right\rangle$ are the same state. only when $n_i = 1$ and $n_j = 0$, we have:

$$b_i b_j^\dagger \left|n_1, n_2, \cdots, 1, \cdots, 0, \cdots, n_L\right\rangle = b_i \left|n_1, n_2, \cdots, 1, \cdots, 1, \cdots, n_L\right\rangle$$

$$= \left|n_1, n_2, \cdots, 0, \cdots, 1, \cdots, n_L\right\rangle = b_j^\dagger b_i$$

**(2)** $\sigma_i^\pm = (\sigma_i^x \pm \sigma_i^y \cdot i)/2$ if we call

**case 1** $[\sigma_i^-, \sigma_j^-] = 0$

when $i = j$, we have:

$$[\sigma_i^-, \sigma_i^-] = \sigma_i^- \sigma_i^- - \sigma_i^- \sigma_i^- = 0$$

**(3)**

when $i \neq j$, we have: **(4)** how should the Pauli operators $\sigma_i^+$ and $\sigma_i^z$ can be written in terms of the $f$ operators? we have the definition that:

$$f_i^\dagger = \left(\prod_{j<i} \sigma_j^z\right) \sigma_i^+$$

so from $\langle \overline{n_{1'}n_{2'} \cdot n_{i'}} | f_i^\dagger | \overline{n_1 n_2 n_3 \cdot n_i} \rangle = \langle \overline{n_1 n_2 \cdot n_i} | f_i | \overline{n_{1'} n_{2'} \cdot n_{i'}} \rangle^*$ we can get the relation between $f_i$ and $\sigma_i^+$, $\sigma_i^z$ as follows:

$$f_i = \left( \prod_{j<i} \sigma_j^z \right) \sigma_i^-$$

then try to represent $\sigma_i^z$ and $\sigma_i^+$ in terms of $f_i$: we use the matrix representation of the Pauli operators:

$$\sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\sigma_i^z$ can be expressed as: $2\sigma_i^+ \sigma_i^- - I = 2 f_i^\dagger f_i - I$

like wise:

$$\sigma_i^+ = f_i^\dagger \left( \prod_{j<i} \sigma_j^z \right)^{-1}$$

$$= f_i^\dagger \left( \prod_{j<i} (2 f_j^\dagger f_j - I) \right)$$