Problem Set (Week 2)

Econ 103

Lecture 4 - 6

1. Suppose you flip a fair coin twice.

   (a) List all the basic outcomes in the sample space.

   > **Solution:** $S = \{HH, HT, TT, TH\}$

   (b) Let A be the event that you get at least one head. List all the basic outcomes in A.

   > **Solution:** $A = \{HH, HT, TH\}$

   (c) What is the probability of A?

   > **Solution:** $P(A) = 3/4 = 0.75$

   (d) What is the probability of A^c?

   > **Solution:** $P(A^c) = 1/4$

2. Suppose I deal two cards at random from a well-shuffled deck of 52 playing cards. What is the probability that I get a pair of aces?

> **Solution:** You can either solve this assuming that order doesn't matter:
>
> $$\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{4!/(2! \times 2!)}{52!/(50! \times 2!)} = \frac{6}{(52 \times 51)/2} = 6/1326 = 1/221$$
>
> or that it does:
>
> $$\frac{P_2^4}{P_2^{52}} = \frac{4!/2!}{52!/50!} = \frac{(4 \times 3)}{(52 \times 51)} = 12/2652 = 1/221$$

In either case, the answer is the same: $1/221 \approx 0.005$

3. (Adapted from Mosteller, 1965) A jury has three members: the first flips a coin for each decision, and each of the remaining two independently has probability p of reaching the correct decision. Call these two the "serious" jurors and the other the "flippant" juror (pun intended).

   (a) What is the probability that the serious jurors both reach the same decision?

   Solution: There are two ways for them to agree: they can either make the right decision, p^2, or the wrong decision, $(1-p)^2$. These are mutually exclusive, so we sum the probabilities for a total of $p^2 + (1-p)^2$

   (b) What is the probability that the serious jurors each reach different decisions?

   Solution: There are two ways for them to disagree: either the first makes the wrong decision, $p(1-p)$, or the second makes the wrong decision, $(1-p)p$. These are mutually exclusive, so we sum the probabilities for a total of $2p(1-p)$.

   (c) What is the probability that the jury reaches the correct decision? Majority rules.

   Solution: With probability p^2 the serious jurors agree and make the correct decision so the flippant juror is irrelevant. With probability $2p(1-p)$ they disagree. In half of these cases the flippant juror makes the correct decision. Thus, the overall probability is $p^2 + p(1-p) = p$.

4. This question refers to the prediction market example from lecture. Imagine it is October 2012. Let O be a contract paying \$10 if Obama wins the election, zero otherwise, and R be a contract paying \$10 if Romney wins the election, zero otherwise. Let $\text{Price}(O)$ and $\text{Price}(R)$ be the respective prices of these contracts. (Assumption: The only possible outcomes are Obama or Romney winning the election.)

   (a) Suppose you *buy* one of each contract. What is your profit?

   Solution: Regardless of whether Romney or Obama wins, you get \$10. Thus, your profit is

   $$10 - \text{Price}(O) - \text{Price}(R)$$

   (b) Suppose you *sell* one of each contract. What is your profit?

Solution: Regardless of whether Romney or Obama wins, you have to pay out \$10. Thus, your profit is

$$\text{Price}(O) + \text{Price}(R) - 10$$

(c) What must be true about Price(O) and Price(R), to prevent an opportunity for statistical arbitrage?

Solution: From (a) we see that you can earn a guaranteed, risk-free profit from *buying* one of each contract whenever $10 > \text{Price}(O) + \text{Price}(R)$. From (b) we see that you can earn a guaranteeed, risk-free profit by *selling* one of each contract whenever $\text{Price}(O) + \text{Price}(R) > 10$. Therefore, the only way to prevent statistical arbitrage is to have $\text{Price}(O) + \text{Price}(R) = 10$.

(d) How is your answer to part (c) related to the Complement Rule?

Solution: In class we discussed how the market price of a prediction contract can be viewed as a subjective probability assessment. To find the implied probability we divide the price of the contract by the amount that is pays out, in this case \$10. Hence, dividing through by \$10, we see that the condition from part (b) when stated in probability terms is

$$P(O) = 1 - P(R)$$

This is precisely the Complement Rule because $R = O^c$.

Lecture 7 - 9

5. Suppose X is a random variable with support $\{-1, 0, 1\}$ where $p(-1) = q$ and $p(1) = p$.

(a) What is $p(0)$?

Solution: By the complement rule $p(0) = 1 - p - q$.

(b) Calculate the CDF, $F(x_0)$, of X.

Solution:

$$F(x_0) = \begin{cases} 0, \ x_0 < -1 \\ q, \ -1 \leq x_0 < 0 \\ 1-p, \ 0 \leq x_0 < 1 \\ 1, \ x_0 \geq 1 \end{cases}$$

(c) Calculate $E[X]$.

Solution: $E[X] = -1 \cdot q + 0 \cdot (1-p-q) + p \cdot 1 = p - q$

(d) What relationship must hold between p and q to ensure $E[X] = 0$?

Solution: $p = q$

6. Suppose that X is a random variable with support $\{1, 2\}$ and Y is a random variable with support $\{0, 1\}$ where X and Y have the following joint distribution:

$$p_{XY}(1,0) = 0.20, \qquad p_{XY}(1,1) = 0.30$$
$$p_{XY}(2,0) = 0.25, \qquad p_{XY}(2,1) = 0.25$$

(a) Express the joint distribution in a 2×2 table.

Solution:

	X	
	1	2
0	0.20	0.25
1	0.30	0.25

(b) Using the table, calculate the marginal probability distributions of X and Y.

Solution:

$$\begin{aligned} p_X(1) &= p_{XY}(1,0) + p_{XY}(1,1) = 0.20 + 0.30 = 0.50 \\ p_X(2) &= p_{XY}(2,0) + p_{XY}(2,1) = 0.25 + 0.25 = 0.50 \\ p_Y(0) &= p_{XY}(1,0) + p_{XY}(2,0) = 0.20 + 0.25 = 0.45 \\ p_Y(1) &= p_{XY}(1,1) + p_{XY}(2,1) = 0.30 + 0.25 = 0.55 \end{aligned}$$

(c) Calculate the conditional probability distribution of $Y|X = 1$ and $Y|X = 2$.

Solution: The distribution of $Y|X = 1$ is

$$P(Y = 0|X = 1) = \frac{p_{XY}(1, 0)}{p_X(1)} = \frac{0.2}{0.5} = 0.4$$

$$P(Y = 1|X = 1) = \frac{p_{XY}(1, 1)}{p_X(1)} = \frac{0.3}{0.5} = 0.6$$

while the distribution of $Y|X = 2$ is

$$P(Y = 0|X = 2) = \frac{p_{XY}(2, 0)}{p_X(2)} = \frac{0.25}{0.5} = 0.5$$

$$P(Y = 1|X = 2) = \frac{p_{XY}(2, 1)}{p_X(2)} = \frac{0.25}{0.5} = 0.5$$

(d) Calculate $E[Y|X]$.

Solution:

$$\begin{aligned} E[Y|X = 1] &= 0 \times 0.4 + 1 \times 0.6 = 0.6 \\ E[Y|X = 2] &= 0 \times 0.5 + 1 \times 0.5 = 0.5 \end{aligned}$$

Hence,

$$E[Y|X] = \begin{cases} 0.6 & \text{with probability } 0.5 \\ 0.5 & \text{with probability } 0.5 \end{cases}$$

since $p_X(1) = 0.5$ and $p_X(2) = 0.5$.

(e) What is $E[E[Y|X]]$?

Solution: $E[E[Y|X]] = 0.5 \times 0.6 + 0.5 \times 0.5 = 0.3 + 0.25 = 0.55$. Note that this equals the expectation of Y calculated from its marginal distribution, since $E[Y] = 0 \times 0.45 + 1 \times 0.55$. This illustrates the so-called "Law of Iterated Expectations."

(f) Calculate the covariance between X and Y using the shortcut formula.

Solution: First, from the marginal distributions, $E[X] = 1 \cdot 0.5 + 2 \cdot 0.5 = 1.5$ and $E[Y] = 0 \cdot 0.45 + 1 \cdot 0.55 = 0.55$. Hence $E[X]E[Y] = 1.5 \cdot 0.55 = 0.825$.

Second,

$$\begin{aligned} E[XY] &= (0 \cdot 1) \cdot 0.2 + (0 \cdot 2) \cdot 0.25 + (1 \cdot 1) \cdot 0.3 + (1 \cdot 2)0.25 \\ &= 0.3 + 0.5 = 0.8 \end{aligned}$$

Finally $Cov(X, Y) = E[XY] - E[X]E[Y] = 0.8 - 0.825 = -0.025$

7. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:

   (a) $Cov(a + bX, c + dY) = bdCov(X, Y)$

Solution: Let $\mu_X = E[X]$ and $\mu_Y = E[Y]$. By the linearity of expectation,

$$\begin{aligned} E[a + bX] &= a + b\mu_X \\ E[c + dY] &= c + d\mu_Y \end{aligned}$$

Thus, we have

$$\begin{aligned} (a + bx) - E[a + bX] &= b(x - \mu_X) \\ (c + dy) - E[c + dY] &= d(y - \mu_Y) \end{aligned}$$

Substituting these into the formula for the covariance between two discrete random variables,

$$\begin{aligned} Cov(a + bX, c + dY) &= \sum_x \sum_y [b(x - \mu_X)] [d(y - \mu_Y)] \, p(x, y) \\ &= bd \sum_x \sum_y (x - \mu_X)(y - \mu_Y) p(x, y) \\ &= bdCov(X, Y) \end{aligned}$$

   (b) $Corr(a + bX, c + dY) = Corr(X, Y)$

Solution:

$$\begin{aligned} Corr(a+bX, c+dY) &= \frac{Cov(a+bX, c+dY)}{\sqrt{Var(a+bX)Var(c+dY)}} \\ &= \frac{bdCov(X,Y)}{\sqrt{b^2Var(X)d^2Var(Y)}} \\ &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= Corr(X,Y) \end{aligned}$$

8. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1-\omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0,1]$, so that *negative* weights are not allowed. (This rules out short-selling.)

   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.

   Solution:

   $$\begin{aligned} E[\Pi(\omega)] &= E[\omega X_1 + (1-\omega)X_2] = \omega E[X_1] + (1-\omega)E[X_2] \\ &= \omega\mu_1 + (1-\omega)\mu_2 \end{aligned}$$

   (b) If $\omega \in [0,1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.

   Solution: No. If short-selling is disallowed, the portfolio expected return must be between μ_1 and μ_2.

   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.

   Solution: $Cov(X,Y) = \rho\sigma_1\sigma_2$

   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)

Solution:

$$\begin{aligned} Var[\Pi(\omega)] &= Var[\omega X_1 + (1-\omega)X_2] \\ &= \omega^2 Var(X_1) + (1-\omega)^2 Var(X_2) + 2\omega(1-\omega)Cov(X_1, X_2) \\ &= \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho\sigma_1\sigma_2 \end{aligned}$$

(e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio*.

Solution: The First Order Condition is:

$$2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2 + (2-4\omega)\rho\sigma_1\sigma_2 = 0$$

Dividing both sides by two and rearranging:

$$\begin{aligned} \omega\sigma_1^2 - (1-\omega)\sigma_2^2 + (1-2\omega)\rho\sigma_1\sigma_2 &= 0 \\ \omega\sigma_1^2 - \sigma_2^2 + \omega\sigma_2^2 + \rho\sigma_1\sigma_2 - 2\omega\rho\sigma_1\sigma_2 &= 0 \\ \omega(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) &= \sigma_2^2 - \rho\sigma_1\sigma_2 \end{aligned}$$

So we have

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$