

Problem Set (Week 3)

Econ 103

Lecture 10

1. Suppose that X is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Graph the PDF of X .

Solution: It's an isosceles triangle with base from (0,0) to (2,0) and height 1.

- (b) Show that $\int_{-\infty}^{\infty} f(x) dx = 1$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_0^1 x dx + \int_1^2 (2 - x) dx = \left. \frac{x^2}{2} \right|_0^1 + \left. \left(2x - \frac{x^2}{2} \right) \right|_1^2 \\ &= 1/2 + (4 - 2) - (2 - 1/2) = 1 \end{aligned}$$

- (c) What is $P(0.5 < X < 1.5)$?

Solution:

$$\begin{aligned}P(0.5 < X < 1.5) &= \int_{0.5}^{1.5} f(x) dx = \int_{0.5}^1 x dx + \int_1^{1.5} (2 - x) dx \\&= \left. \frac{x^2}{2} \right|_{0.5}^1 + \left(2x - \frac{x^2}{2} \right) \Big|_1^{1.5} \\&= (1/2 - 1/8) + (3 - 9/8) - (2 - 1/2) \\&= 3/8 + 15/8 - 2 + 1/2 = 18/8 - 16/8 + 4/8 \\&= 6/8 = 3/4 = 0.75\end{aligned}$$

2. A random variable is said to follow a $\text{Uniform}(a, b)$ distribution if it is equally likely to take on any value in the range $[a, b]$ and never takes a value outside this range. Suppose that X is such a random variable, i.e. $X \sim \text{Uniform}(a, b)$.

- (a) What is the support of X ?

Solution: $[a, b]$

- (b) Explain why the PDF of X is $f(x) = 1/(b - a)$ for $a \leq x \leq b$, zero elsewhere.

Solution: This simply generalizes the $\text{Uniform}(0, 1)$ random variable from class. To capture the idea that X is equally likely to take on any value in the range $[a, b]$, the PDF must be constant. To ensure that it integrates to 1, the denominator must be $b - a$.

- (c) Using the PDF from part (b), calculate the CDF of X .

Solution:

$$F(x_0) = \int_{-\infty}^{x_0} f(x) dx = \int_a^{x_0} \frac{dx}{b - a} = \frac{x}{b - a} \Big|_a^{x_0} = \frac{x_0 - a}{b - a}$$

- (d) Verify that $f(x) = F'(x)$ for the present example.

Solution:

$$F'(x) = \frac{d}{dx} \left(\frac{x - a}{b - a} \right) = \frac{1}{b - a} = f(x)$$

(e) Calculate $E[X]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{x^2}{2(b-a)} \Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

(f) Calculate $E[X^2]$. *Hint:* recall that $b^3 - a^3$ can be factorized as $(b-a)(b^2 + a^2 + ab)$.

Solution:

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x) dx = \int_a^b \frac{x^2}{b-a} = \frac{x^3}{3(b-a)} \Big|_a^b = \frac{b^3 - a^3}{3(b-a)} \\ &= \frac{(b-a)(b^2 + a^2 + ab)}{3(b-a)} = \frac{b^2 + a^2 + ab}{3} \end{aligned}$$

(g) Using the shortcut formula and parts (e) and (f), show that $Var(X) = (b-a)^2/12$.

Solution:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 = \frac{b^2 + a^2 + ab}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^2 + a^2 + ab}{3} - \frac{a^2 + 2ab + b^2}{4} \\ &= \frac{4b^2 + 4a^2 + 4ab - 3a^2 - 6ab - 3b^2}{12} \\ &= \frac{b^2 + a^2 - 2ab}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

Lecture 11

3. Suppose that $X \sim N(0, 16)$ independent of $Y \sim N(2, 4)$. Recall that our convention is to express the normal distribution in terms of its mean and variance, i.e. $N(\mu, \sigma^2)$.

Hence, X has a mean of zero and variance of 16, while Y has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.

- (a) Calculate $P(-8 \leq X \leq 8)$.

Solution:

$$P(-8 \leq X \leq 8) = P(-8/4 \leq X/4 \leq 8/4) = P(-2 \leq Z \leq 2) \approx 0.95$$

where Z is a standard normal random variable.

- (b) Calculate $P(0 \leq Y \leq 4)$.

Solution:

$$P(0 \leq Y \leq 4) = P\left(\frac{0-2}{2} \leq \frac{Y-2}{2} \leq \frac{4-2}{2}\right) = P(-1 \leq Z \leq 1) \approx 0.68$$

where Z is a standard normal random variable.

- (c) Calculate $P(-1 \leq Y \leq 6)$.

Solution:

$$\begin{aligned} P(-1 \leq Y \leq 6) &= P\left(\frac{-1-2}{2} \leq \frac{Y-2}{2} \leq \frac{6-2}{2}\right) \\ &= P(-1.5 \leq Z \leq 2) \\ &= \Phi(2) - \Phi(-1.5) \\ &= \text{pnorm}(2) - \text{pnorm}(-1.5) \\ &\approx 0.91 \end{aligned}$$

where Z is a standard normal random variable.

- (d) Calculate $P(X \geq 10)$.

Solution:

$$\begin{aligned} P(X \geq 10) &= 1 - P(X \leq 10) = 1 - P(X/4 \leq 10/4) = 1 - P(Z \leq 2.5) \\ &= 1 - \Phi(2.5) = 1 - \text{pnorm}(2.5) \\ &\approx 0.006 \end{aligned}$$