Problem Set (Week 2)

Econ 103

Lecture 4 - 6

1. Suppose you flip a fair coin twice.
   (a) List all the basic outcomes in the sample space.
   (b) Let A be the event that you get at least one head. List all the basic outcomes in A.
   (c) What is the probability of A?
   (d) What is the probability of A^c?

2. Suppose I deal two cards at random from a well-shuffled deck of 52 playing cards. What is the probability that I get a pair of aces?

3. (Adapted from Mosteller, 1965) A jury has three members: the first flips a coin for each decision, and each of the remaining two independently has probability p of reaching the correct decision. Call these two the "serious" jurors and the other the "flippant" juror (pun intended).
   (a) What is the probability that the serious jurors both reach the same decision?
   (b) What is the probability that the serious jurors each reach different decisions?
   (c) What is the probability that the jury reaches the correct decision? Majority rules.

4. This question refers to the prediction market example from lecture. Imagine it is October 2012. Let O be a contract paying \$10 if Obama wins the election, zero otherwise, and R be a contract paying \$10 if Romney wins the election, zero otherwise. Let Price(O) and Price(R) be the respective prices of these contracts. (Assumption: The only possible outcomes are Obama or Romney winning the election.)
   (a) Suppose you *buy* one of each contract. What is your profit?
   (b) Suppose you *sell* one of each contract. What is your profit?
   (c) What must be true about Price(O) and Price(R), to prevent an opportunity for statistical arbitrage?
   (d) How is your answer to part (c) related to the Complement Rule?

Lecture 7 - 9

5. Suppose X is a random variable with support $\{-1, 0, 1\}$ where $p(-1) = q$ and $p(1) = p$.
   (a) What is $p(0)$?
   (b) Calculate the CDF, $F(x_0)$, of X.
   (c) Calculate $E[X]$.
   (d) What relationship must hold between p and q to ensure $E[X] = 0$?

6. Suppose that X is a random variable with support $\{1, 2\}$ and Y is a random variable with support $\{0, 1\}$ where X and Y have the following joint distribution:

$$\begin{aligned} p_{XY}(1,0) &= 0.20, \qquad & p_{XY}(1,1) &= 0.30 \\ p_{XY}(2,0) &= 0.25, \qquad & p_{XY}(2,1) &= 0.25 \end{aligned}$$

   (a) Express the joint distribution in a 2×2 table.
   (b) Using the table, calculate the marginal probability distributions of X and Y.
   (c) Calculate the conditional probability distribution of $Y|X = 1$ and $Y|X = 2$.
   (d) Calculate $E[Y|X]$.
   (e) What is $E[E[Y|X]]$?
   (f) Calculate the covariance between X and Y using the shortcut formula.

7. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:
   (a) $Cov(a + bX, c + dY) = bdCov(X, Y)$
   (b) $Corr(a + bX, c + dY) = Corr(X, Y)$

8. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1 - \omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0, 1]$, so that *negative* weights are not allowed. (This rules out short-selling.)
   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.
   (b) If $\omega \in [0, 1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.
   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.
   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)

(e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio.*