

Economics 103 – Statistics for Economists

Minsu Chang

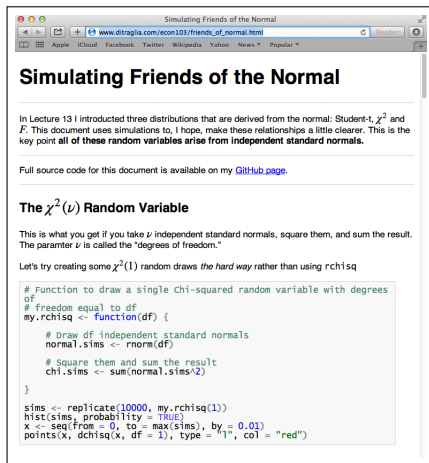
University of Pennsylvania

Lecture # 13

Continuous RVs – Part III



Figure: The Normal Distribution and Friends



Simulating Friends of the Normal

In Lecture 13 I introduced three distributions that are derived from the normal: Student-t, χ^2 and F . This document uses simulations to, I hope, make these relationships a little clearer. This is the key point **all of these random variables arise from independent standard normals**.

Full source code for this document is available on my [GitHub page](#).

The $\chi^2(\nu)$ Random Variable

This is what you get if you take ν independent standard normals, square them, and sum the result. The paramter ν is called the "degrees of freedom."

Let's try creating some $\chi^2(1)$ random draws *the hard way* rather than using `rchisq`

```
# Function to draw a single Chi-squared random variable with degrees
# of
# freedom equal to df
my.rchisq <- function(df) {
  # Draw df independent standard normals
  normal.sims <- rnorm(df)

  # Square them and sum the result
  chi.sims <- sum(normal.sims^2)
}

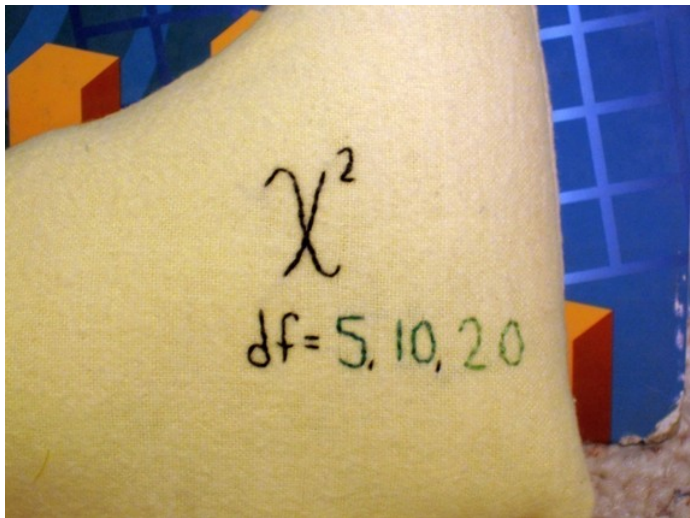
sims <- replicate(10000, my.rchisq(1))
hist(sims, probability = TRUE)
x <- seq(from = 0, to = max(sims), by = 0.01)
points(x, dchisq(x, df = 1), type = "l", col = "red")
```

Functions of Independent RVs are Independent

If X and Y are independent random variables and g and h are functions, then the random variables $g(X)$ and $h(Y)$ are also independent.



Figure: PDF for χ^2 -Distribution



χ^2 Random Variable

Let $X_1, \dots, X_\nu \sim \text{iid } N(0, 1)$. Then,

$$(X_1^2 + \dots + X_\nu^2) \sim \chi^2(\nu)$$

where the parameter ν is the *degrees of freedom*

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Support = $(0, \infty)$

χ^2 PDFs

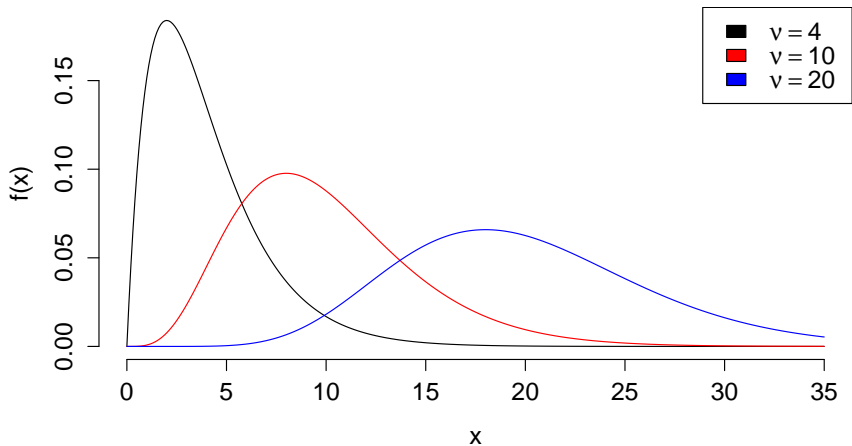
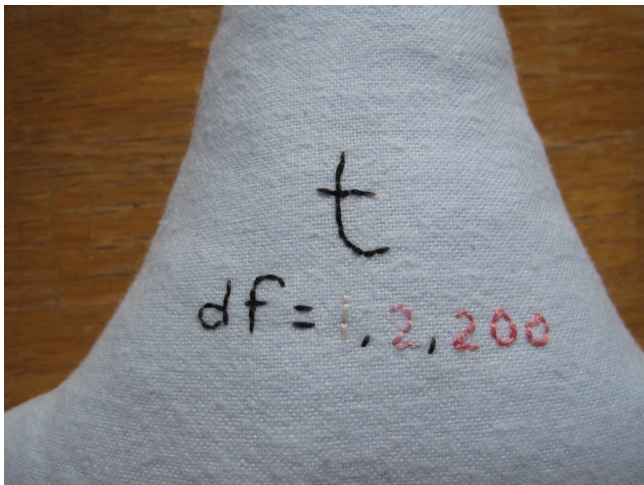




Figure: PDF for Student-t Distribution



Student-t Random Variable

Let $X \sim N(0, 1)$ independent of $Y \sim \chi^2(\nu)$. Then,

$$\frac{X}{\sqrt{Y/\nu}} \sim t(\nu)$$

where the parameter ν is the degrees of freedom.

Student-t Random Variable

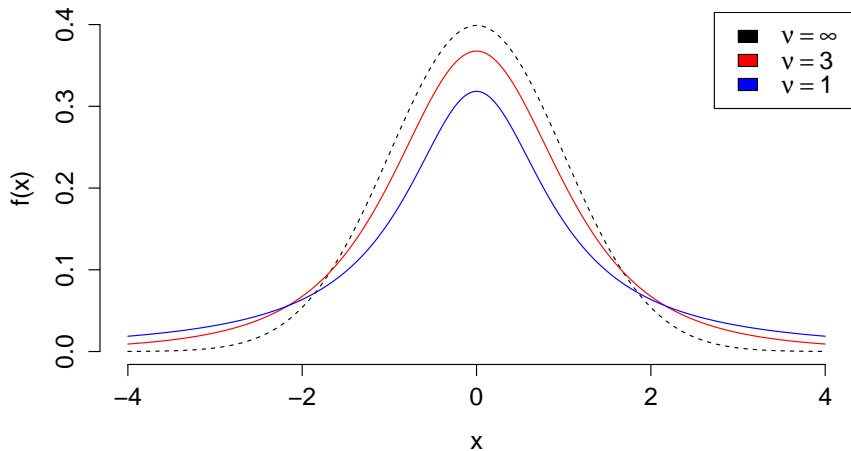
Let $X \sim N(0, 1)$ independent of $Y \sim \chi^2(\nu)$. Then,

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where the parameter ν is the degrees of freedom.

- ▶ Support = $(-\infty, \infty)$
- ▶ As $\nu \rightarrow \infty$, $t \rightarrow$ Standard Normal.
- ▶ Symmetric around zero, but mean and variance may not exist!
- ▶ Degrees of freedom ν control “thickness of tails”

Student-t PDFs



F Random Variable

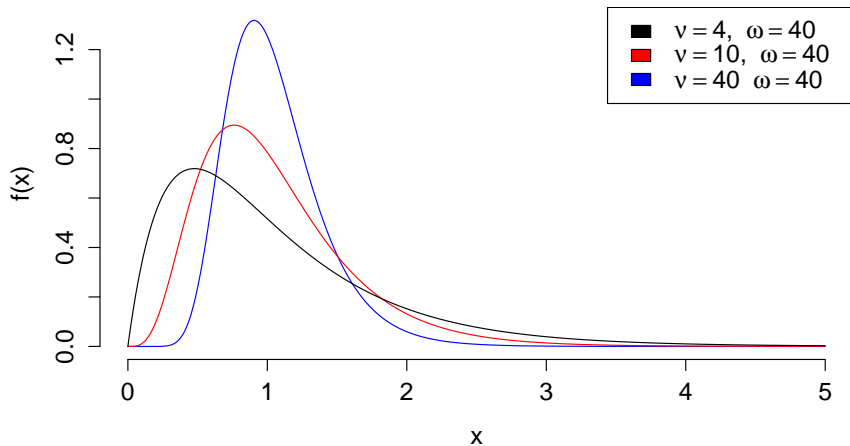
Suppose $X \sim \chi^2(\nu)$ independent of $Y \sim \chi^2(\omega)$. Then,

$$\frac{X/\nu}{Y/\omega} \sim F(\nu, \omega)$$

where ν is the numerator degrees of freedom and ω is the denominator degrees of freedom.

Support = $(0, \infty)$

F PDFs



R Commands – CDFs and Quantile Functions

$F(x) = P(X \leq x)$ is the CDF, $Q(p) = F^{-1}(p)$ the Quantile Function

	$F(x)$	$Q(p)$
$N(\mu, \sigma^2)$	<code>pnorm(x, mean = μ, sd = σ)</code>	<code>qnorm(p, mean = μ, sd = σ)</code>
$\chi^2(\nu)$	<code>pchisq(x, df = ν)</code>	<code>qchisq(p, df = ν)</code>
$t(\nu)$	<code>pt(x, df = ν)</code>	<code>qt(p, df = ν)</code>
$F(\nu, \omega)$	<code>pf(x, df1 = ν, df2 = ω)</code>	<code>qf(p, df1 = ν, df2 = ω)</code>

Mnemonic: “p” is for Probability, “q” is for Quantile.

R Commands – PDFs and Random Draws

	$f(x)$	Make n iid Random Draws
$N(\mu, \sigma^2)$	<code>dnorm(x, mean = μ, sd = σ)</code>	<code>rnorm(n, mean = μ, sd = σ)</code>
$\chi^2(\nu)$	<code>dchisq(x, df = ν)</code>	<code>rchisq(n, df = ν)</code>
$t(\nu)$	<code>dt(x, df = ν)</code>	<code>rt(n, df = ν)</code>
$F(\nu, \omega)$	<code>df(x, df1 = ν, df2 = ω)</code>	<code>rf(n, df1 = ν, df2 = ω)</code>

Mnemonic: “d” is for Density, “r” is for Random.

Example: $X_1, X_2, X_3 \sim \text{iid } N(0, 1)$

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What is the distribution of $Z = X_3/\sqrt{Y_1/2}$?

Ratio of standard normal and square root of independent χ^2 RV divided by its degrees of freedom $\Rightarrow Z \sim t(2)$

Suppose $X_1, X_2, \sim \text{iid } N(\mu, \sigma^2)$



Let $Y = (X_1 - \mu)^2 + (X_2 - \mu)^2$. What is the distribution of Y/σ^2 ?

(a) $F(2, 1)$

(b) $\chi^2(2)$

(c) $t(2)$

(d) $N(\mu, \sigma)$

(e) None of the above

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`pf(5, df1 = 2, df2 = 1)` ≈ 0.7

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What is the median of Y_1 ?

$$\text{qchisq}(0.5, \text{df} = 2) \approx 1.4$$

What is $P(Y_2 \leq 5)$?

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What value of c gives $P(-c \leq Z \leq c) = 0.5$?

Use Symmetry (like normal)

$$c = \text{qt}(0.75, \text{df} = 2) \approx 0.8$$

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Use Symmetry (like normal)

$$c = \text{qt}(0.75, \text{df} = 2) \approx 0.8$$

$$\text{or equivalently } -c = \text{qt}(0.25, \text{df} = 2) \approx -0.8$$