

# Economics 103 – Statistics for Economists

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Lecture # 6

# Basic Probability – Part III

# The Lie Detector Problem

From accounting records, we know that 10% of employees in the store are stealing merchandise.

The managers want to fire the thieves, but their only tool in distinguishing is a lie detector test that is 80% accurate:

Innocent  $\Rightarrow$  Pass test with 80% Probability

Thief  $\Rightarrow$  Fail test with 80% Probability

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What is the probability that someone is a thief *given* that she has failed the lie detector test?



# Monte Carlo Simulation – Roll a 10-sided Die Twice

Managers will split up and visit employees. Employees roll the die twice **but keep the results secret!**

## First Roll – Thief or not?

0  $\Rightarrow$  Thief, 1 – 9  $\Rightarrow$  Innocent

## Second Roll – Lie Detector Test

0, 1  $\Rightarrow$  Incorrect Test Result, 2 – 9 Correct Test Result

	0 or 1	2–9
Thief	Pass	<b>Fail</b>
Innocent	<b>Fail</b>	Pass

What percentage of those who failed the test are guilty?

# Who Failed Lie Detector Test:

# Of Thieves Among Those Who Failed:

# Base Rate Fallacy – Failure to Consider Prior Information

## Base Rate – Prior Information

Before the test we know that 10% of Employees are stealing.

People tend to focus on the fact that the test is 80% accurate and ignore the fact that only 10% of the employees are thieves.

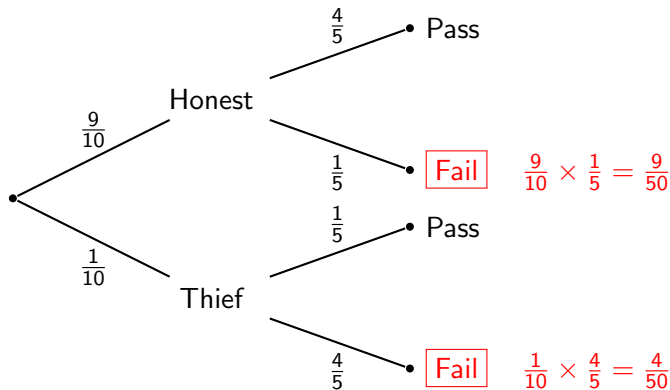
## Thief (Y/N), Lie Detector (P/F)

	0	1	2	3	4	5	6	7	8	9
0	YP	YP	YF	YF	YF	YF	YF	YF	YF	YF
1	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
2	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
3	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
4	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
5	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
6	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
7	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
8	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
9	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP

**Table:** Each outcome in the table is equally likely. The 26 given in red correspond to failing the test, but only 8 of these (YF) correspond to being a thief.



## Base Rate of Thievery is 10%



**Figure:** Although  $\frac{9}{50} + \frac{4}{50} = \frac{13}{50}$  fail the test, only  $\frac{4/50}{13/50} = \frac{4}{13} \approx 0.31$  are actually thieves!

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Finally, combining these

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

# Understanding Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

## Reversing the Conditioning

Express  $P(A|B)$  in terms of  $P(B|A)$ . *Relative magnitudes* of the two conditional probabilities determined by the ratio  $P(A)/P(B)$ .

## Base Rate

$P(A)$  is called the “base rate” or the “prior probability.”

## Denominator

Typically, we calculate  $P(B)$  using the law of total probability

In General  $P(A|B) \neq P(B|A)$



### Question

Most college students are Democrats. Does it follow that most Democrats are college students?

(A = YES, B = NO)

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### Answer

There are many more Democrats than college students:

$$P(\text{Dem}) > P(\text{Student})$$

so  $P(\text{Student}|\text{Dem})$  is small even though  $P(\text{Dem}|\text{Student})$  is large.



## Solving the Lie Detector Problem with Bayes' Rule

$T$  = Employee is a Thief,  $F$  = Employee Fails Lie Detector Test

$$P(T|F) = \frac{P(F|T)P(T)}{P(F)}$$

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$$P(T|F) = \frac{0.08}{0.26} = \frac{8}{26} = \frac{4}{13} \approx 0.31$$

## “Odd” Question # 3

There are two kinds of taxis: green cabs and blue cabs. Of all the cabs on the road, 85% are green cabs. On a misty winter night a taxi sideswiped another car and drove off. A witness says it was a blue cab. The witness is tested under conditions like those on the night of the accident, and 80% of the time she correctly reports the color of the cab that is seen. That is, regardless of whether she is shown a blue or a green cab in misty evening light, she gets the color right 80% of the time.

Given that the witness said she saw a blue cab, what is the probability that a blue cab was the sideswiper?

## Solving The Taxi Problem

$G$  = Taxi is Green,  $P(G) = 0.85$

$B$  = Taxi is Blue,  $P(B) = 0.15$

$W_B$  = Witness says Taxi is Blue,  $P(W_B|B) = 0.8$ ,  $P(W_B|G) = 0.2$



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$$P(B|W_B) = 0.12/0.29 = 12/29 \approx 0.41$$

$$P(G|W_B) = 1 - (12/19) \approx 0.59$$

# Random Variables

# Random Variables

*A random variable is neither random nor a variable.*

## Random Variable (RV): $X$

A *fixed* function that assigns a *number* to each basic outcome of a random experiment.

## Realization: $x$

A particular numeric value that an RV could take on. We write  $\{X = x\}$  to refer to the *event* that the RV  $X$  took on the value  $x$ .

## Support Set (aka Support)

The set of all possible realizations of a RV.



# Random Variables (continued)

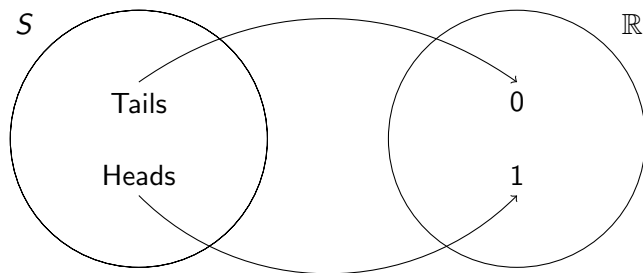
## Notation

Capital latin letters for RVs, e.g.  $X$ ,  $Y$ ,  $Z$ , and the corresponding lowercase letters for their realizations, e.g.  $x$ ,  $y$ ,  $z$ .

## Intuition

You can think of an RV as a machine that spits out random numbers: although the machine is deterministic, its inputs, the outcomes of a random experiment, are not.

## Example: Coin Flip Random Variable



**Figure:** This random variable assigns numeric values to the random experiment of flipping a fair coin once: Heads is assigned 1 and Tails 0.

Which of these is a realization of the Coin Flip RV?



- (a) Tails
- (b) 2
- (c) 0
- (d) Heads
- (e)  $1/2$

# What is the support set of the Coin Flip RV?



- (a) {Heads, Tails}
- (b)  $1/2$
- (c) 0
- (d)  $\{0, 1\}$
- (e) 1

Let  $X$  denote the Coin Flip RV



What is  $P(X = 1)$ ?

- (a) 0
- (b) 1
- (c)  $1/2$
- (d) Not enough information to determine

## Two Kinds of RVs: Discrete and Continuous

**Discrete** support set is discrete, e.g.  $\{0, 1, 2\}$ ,  
 $\{\dots, -2, -1, 0, 1, 2, \dots\}$

**Continuous** support set is continuous, e.g.  $[-1, 1]$ ,  $\mathbb{R}$ .

Start with the discrete case since it's easier, but most of the ideas we learn will carry over to the continuous case.

# Discrete Random Variables I

## Probability Mass Function (pmf)

A function that gives  $P(X = x)$  for any realization  $x$  in the support set of a discrete RV  $X$ . We use the following notation for the pmf:

$$p(x) = P(X = x)$$

Plug in a realization  $x$ , get out a probability  $p(x)$ .



# Probability Mass Function for Coin Flip RV

$$X = \begin{cases} 0, \text{Tails} \\ 1, \text{Heads} \end{cases}$$

$$p(0) = 1/2$$

$$p(1) = 1/2$$

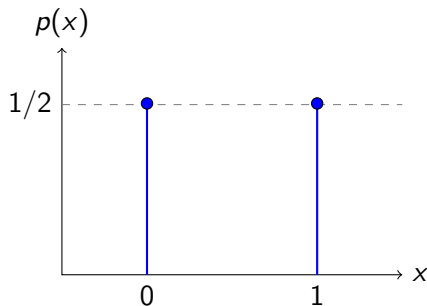


Figure: Plot of pmf for Coin Flip Random Variable

## Important Note about Support Sets

Whenever you write down the pmf of a RV, it is **crucial** to also write down its Support Set. Recall that this is the set of *all possible realizations for a RV*. Outside of the support set, all probabilities are zero. In other words, the pmf is **only defined** on the support.

# Properties of Probability Mass Functions

If  $p(x)$  is the pmf of a random variable  $X$ , then

(i)  $0 \leq p(x) \leq 1$  for all  $x$

(ii)  $\sum_{\text{all } x} p(x) = 1$

where “all  $x$ ” is shorthand for “all  $x$  in the support of  $X$ .”