Economics 103 – Statistics for Economists

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Lecture # 10

Continuous RVs – Part I

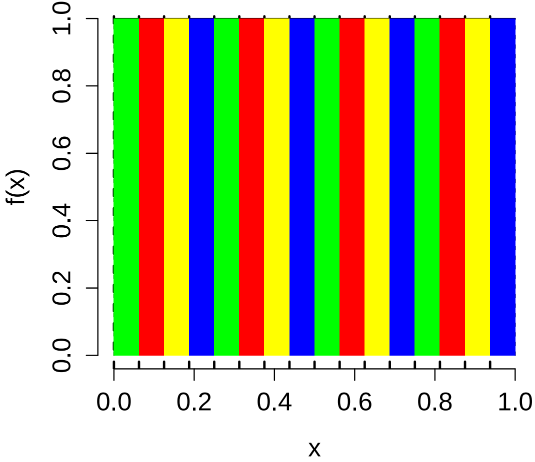
Continuous RVs – What Changes?

1. Probability Density Functions replace Probability Mass Functions (aka Probability Distributions)
2. Integrals Replace Sums

Everything Else is Essentially Unchanged!

What is the probability of “Yellow?”

From Twister to Density – Probability as *Area*



Continuous Random Variables

For continuous RVs, probability is a matter of finding the area of *intervals*. Individual *points* have *zero* probability.

Probability Density Function (PDF)

For a continuous random variable X,

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

where $f(x)$ is the *probability density function* for X.

Extremely Important

For any realization x, $P(X = x) = 0 \neq f(x)$!

Properties of PDFs

1. $\int_{-\infty}^{\infty} f(x)\, dx = 1$
2. $f(x) \geq 0$ for all x
3. $f(x)$ is *not* a probability and can be greater than one!
4. $P(X \leq x_0) = F(x_0) = \int_{-\infty}^{x_0} f(x)\, dx$

Simplest Possible Continuous RV: Uniform$(0, 1)$

You'll look at a generalization, Uniform(a, b) for homework.

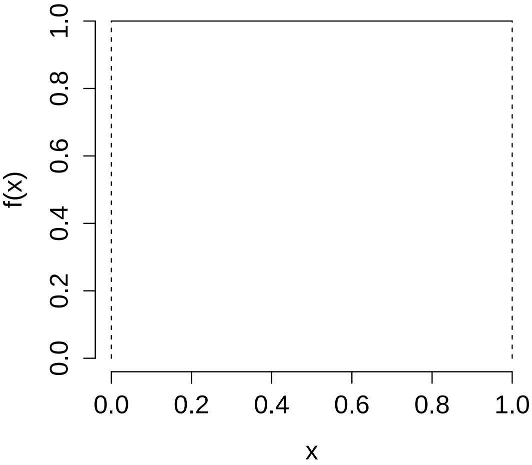
$X \sim$ Uniform$(0, 1)$

A Uniform(0,1) RV is equally likely to take on *any value* in the range $[0, 1]$ and never takes on a value outside this range.

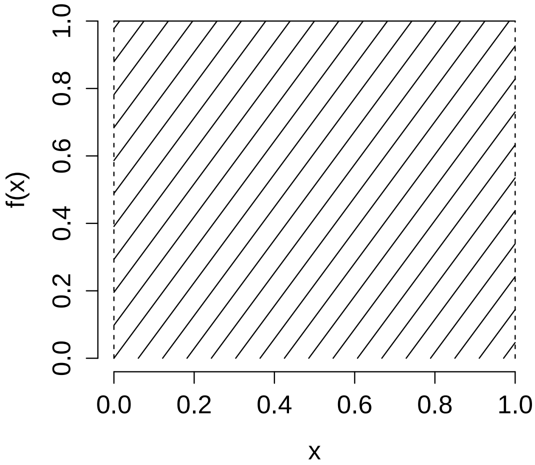
Uniform PDF

$f(x) = 1$ for $0 \leq x \leq 1$, zero elsewhere.

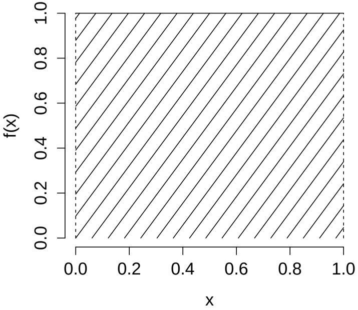
Uniform$(0, 1)$ PDF



What is the area of the shaded region?

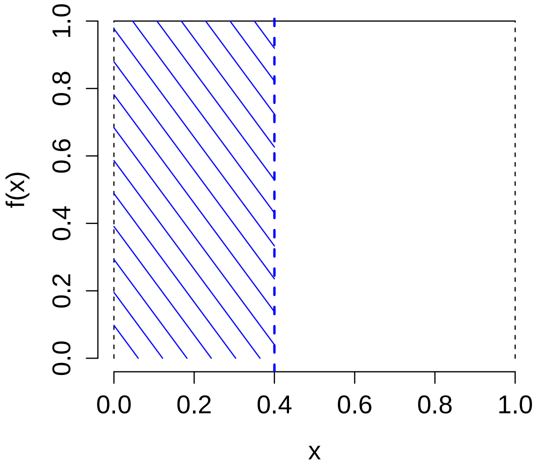


What is the area of the shaded region?

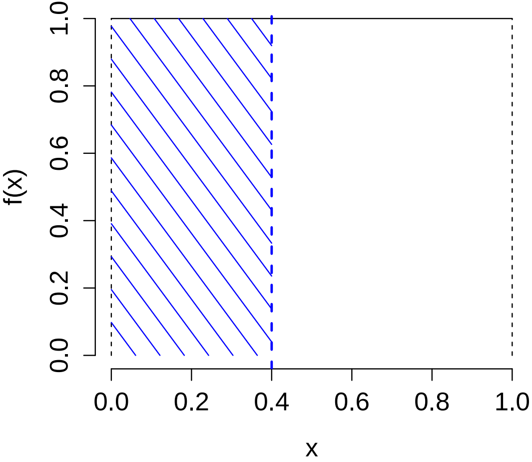


$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{1} 1\, dx = x|_{0}^{1} = 1 - 0 = 1$$

What is the area of the shaded region?



$F(0.4) = P(X \leq 0.4) = 0.4$



Relationship between PDF and CDF

Integrate pdf $\rightarrow$ CDF

$$F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(x)\, dx$$

Differentiate CDF $\rightarrow$ pdf

$$f(x) = \frac{d}{dx} F(x)$$

This is just the First Fundamental Theorem of Calculus.

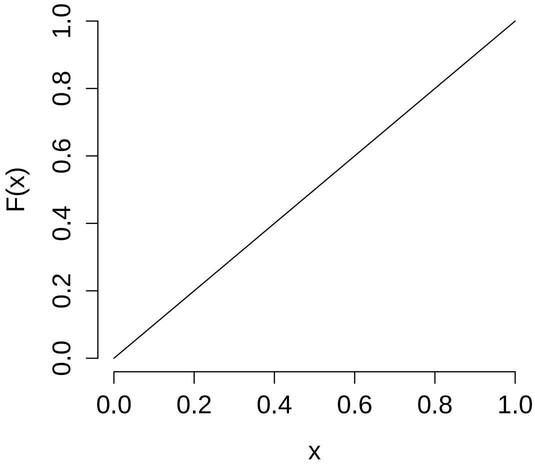
Example: Uniform(0, 1) RV

Integrate the pdf, $f(x) = 1$, to get the CDF

$$F(x_0) = \int_{-\infty}^{x_0} f(x)\, dx = \int_0^{x_0} 1\, dx = x|_0^{x_0} = x_0 - 0 = x_0$$

$$F(x_0) = \begin{cases} 0, x_0 < 0 \\ x_0, 0 \leq x_0 \leq 1 \\ 1, x_0 > 1 \end{cases}$$

Uniform$(0, 1)$ CDF

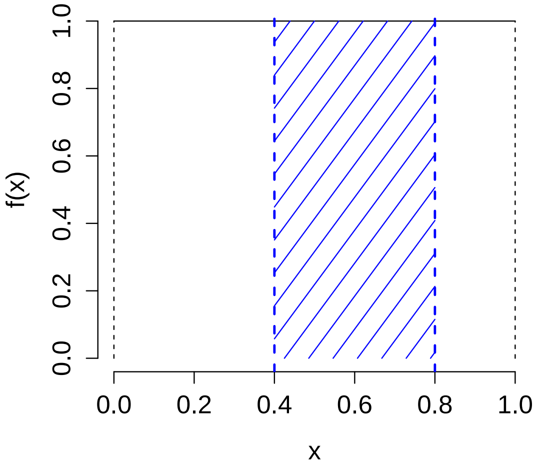


Example: Uniform$(0, 1)$ RV

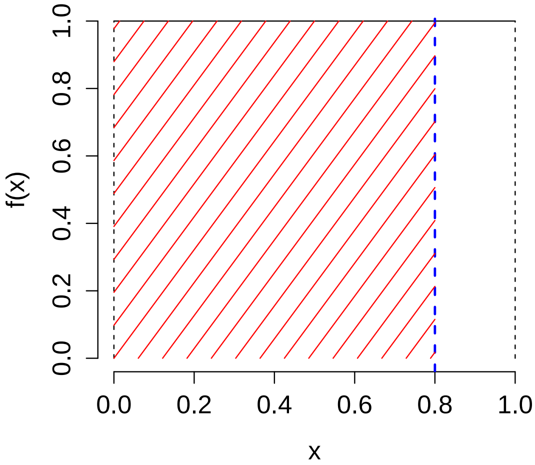
Differentiate the CDF, $F(x_0) = x_0$, to get the pdf

$$\frac{d}{dx} F(x) = 1 = f(x)$$

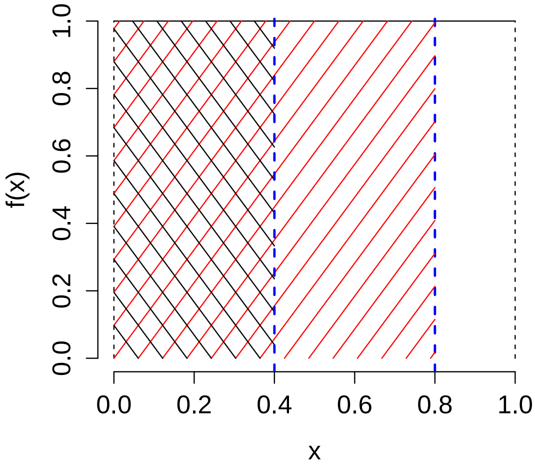
What is $P(0.4 \leq X \leq 0.8)$ if $X \sim \text{Uniform}(0, 1)$?



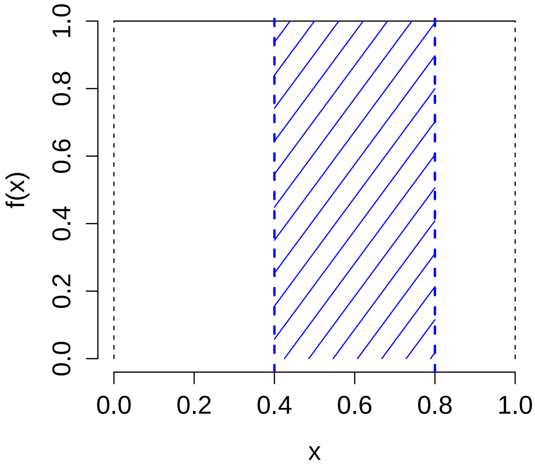
$F(0.8) = P(X \leq 0.8)$



$F(0.8) - F(0.4) = ?$



$F(0.8) - F(0.4) = P(0.4 \leq X \leq 0.8) = 0.4$



Key Idea: Probability of Interval for Continuous RV

$$\boxed{P(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)}$$

This is just the Second Fundamental Theorem of Calculus.

Expected Value for Continuous RVs

$$\boxed{\int_{-\infty}^{\infty} x f(x)\, dx}$$

Remember: Integrals Replace Sums!

Example: Uniform(0,1) Random Variable

$$E[X] \;=\; \int_{-\infty}^{\infty} x f(x)\, dx =$$

Example: Uniform(0,1) Random Variable

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf(x)\, dx = \int_0^1 x \cdot 1\, dx \\
&= \left.\frac{x^2}{2}\right|_0^1 = 1/2 - 0 = 1/2
\end{aligned}
$$

Expected Value of a Function of a Continuous RV

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx}$$

Example: Uniform(0,1) Random Variable

$$E[X^2] \;=\; \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

Example: Uniform(0,1) Random Variable

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2 \cdot 1\, dx \\
&= \left. \frac{x^3}{3} \right|_0^1 = 1/3
\end{aligned}
$$

What about all those rules for expected value?

- The only difference between expectation for continuous versus discrete is how we do the *calculation*.
- Sum for discrete; integral for continuous.
- All *properties* of expected value continue to hold!
- Includes linearity, shortcut for variance, etc.

Variance of Continuous RV

$$\boxed{Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\ dx}$$

where

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\ dx$$

Shortcut formula still holds for continuous RVs!

$$Var(X) = E[X^2] - (E[X])^2$$

Example: Uniform$(0,1)$ RV

$$
\begin{aligned}
Var(X) &= E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2 \\
&= 1/3 - (1/2)^2 \\
&= 1/12 \\
&\approx 0.083
\end{aligned}
$$

Much More Complicated Without the Shortcut Formula!

$$
\begin{aligned}
Var(X) &= E\left[(X - E[X])^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \\
&= \int_0^1 (x - 1/2)^2 \cdot 1\, dx = \int_0^1 (x^2 - x + 1/4)\, dx \\
&= \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right)\Big|_0^1 = 1/3 - 1/2 + 1/4 \\
&= 4/12 - 6/12 + 3/12 = 1/12
\end{aligned}
$$

We Won't Say More About These, But Just So You're Aware of Them...

Joint Density

$$P(a \leq X \leq b \cap c \leq Y \leq d) = \int_c^d \int_a^b f(x,y)\, dxdy$$

Marginal Densities

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

Independence in Terms of Joint and Marginal Densities

$f_{XY}(x,y) = f_X(x) f_Y(y)$

Conditional Density

$f_{Y|X} = f_{XY}(x,y)/f_X(x)$

So where does that leave us?

What We've Accomplished

We've covered all the basic properties of RVs. (Please read RandomVariablesHandout.pdf on Canvas.)

Where are we headed next?

Next up is the most important RV of all: the normal RV.

How should you be studying?

If you *master* the material on RVs (both continuous and discrete) and in particular the normal RV the rest of the semester will seem easy. If you don't, you're in for a rough time...