Economics 103 – Statistics for Economists

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Lecture 15

Confidence Intervals – Part I

What We've Done So Far

- Random Sampling: $X_1, \ldots, X_n \sim$ iid
- Use estimator $\widehat{\theta}$ to learn about population parameter θ_0
- Estimator $\widehat{\theta}$ is a random variable:
  - Distribution of $\widehat{\theta}$ is called *sampling distribution*
  - Bias of an estimator
  - Variance of an estimator
  - Mean-squared Error (MSE) of an estimator
  - Consistency of an Estimator

Inference

Confidence Intervals

What values of θ_0 are consistent with the data we observed?

Hypothesis Testing

I think that $\theta_0 = 0$. Do the data we observed suggest that I should change my mind?

Am I Taller Than The Average American Male?

Source: Centers for Disease Control (pg. 16)

My height is 73 inches. Based on a sample of US males aged 20 and over, the Centers for Disease Control (CDC) reported a mean height of about 69 inches in a recent report.

Clearly I'm taller than the average American male!

Do you agree or disagree?

(a) Agree

(b) Disagree

(c) Not Sure

Remember: The Sample Mean is Random!

Just because the sample mean is 69 inches it doesn't follow that the population mean is 69 inches!

What Else Should We Consider?

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  - If the sample was very small there's a higher chance that it won't be representative of the population as a whole

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What Else Should We Consider?

- How big was the sample?
  - If the sample was very small there's a higher chance that it won't be representative of the population as a whole
  - Why? The variance of the sample mean is *decreasing with sample size* so bigger samples are less noisy.

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Just because the sample mean is 69 inches it doesn't follow that the population mean is 69 inches!

What Else Should We Consider?

- How big was the sample?
  - If the sample was very small there's a higher chance that it won't be representative of the population as a whole
  - Why? The variance of the sample mean is *decreasing with sample size* so bigger samples are less noisy.
- How much variability is there in height in the population?

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- How much variability is there in height in the population?
  - If everyone is very similar in height, any sample we take will be representative of the population.

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  - Why? The variance of the sample mean is *decreasing with sample size* so bigger samples are less noisy.
- How much variability is there in height in the population?
  - If everyone is very similar in height, any sample we take will be representative of the population.
  - Remember: the variance of the sample mean is *increasing* with the population standard deviation.

Am I Taller Than The Average American Male?

Source: Centers for Disease Control (pg. 16)

Table: Height in inches for Males aged 20 and over (approximate)

Sample Mean	69 inches
Sample Std. Dev.	6 inches
Sample Size	5647
My Height	73 inches

We'll return to this example later.

For Now – Single Population, Normally Distributed

$$\boxed{X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

Later we'll look at more than one population and talk about what happens if Normality doesn't hold.

Suppose $X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. What is the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$?

(a) $N(\mu, \sigma^2)$

(b) $N(0, 1)$

(c) $N(0, \sigma)$

(d) $N(\mu, 1)$

(e) Not enough information to determine.

Z-score!

Suppose $X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. From above,

$$
\begin{aligned}
E[\bar{X}_n] &= \mu \\
Var(\bar{X}_n) &= \sigma^2/n \\
\Rightarrow SD(\bar{X}_n) &= \sigma/\sqrt{n}
\end{aligned}
$$

$$\text{Thus, } \sqrt{n}(\bar{X}_n - \mu)/\sigma = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X}_n - E[\bar{X}_n]}{SD(\bar{X}_n)} \sim N(0,1)$$

Remember that we call the standard deviation of a sampling distribution the standard error, written SE, so

$$\frac{\bar{X}_n - \mu}{SE(\bar{X}_n)} \sim N(0,1)$$

Suppose $X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. What is the approximate value of the following?

$$P\left(-2 \leq \frac{\bar{X}_n - \mu}{SE(\bar{X}_n)} \leq 2\right)$$

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$$P\left(-2 \leq \frac{\bar{X}_n - \mu}{SE(\bar{X}_n)} \leq 2\right) \approx 0.95$$

What happens if I rearrange?

$$P\left(-2 \leq \frac{\bar{X}_n - \mu}{SE(\bar{X}_n)} \leq 2\right) \quad = \quad 0.95$$

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P\left(\bar{X}_n - 2 \cdot SE \leq \mu \leq \bar{X}_n + 2 \cdot SE\right) &= 0.95
\end{aligned}
$$

Confidence Intervals

Confidence Interval (CI)

A confidence interval is a range (A, B) constructed from the sample data that has a specified probability of containing a population parameter:

$$P(A \leq \theta_0 \leq B) = 1 - \alpha$$

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A confidence interval is a range (A, B) constructed from the sample data that has a specified probability of containing a population parameter:

$$P(A \leq \theta_0 \leq B) = 1 - \alpha$$

Confidence Level

The specified probability, typically denoted $1 - \alpha$, is called the confidence level. For example, if $\alpha = 0.05$ then the confidence level is 0.95 or 95%.

Confidence Interval for Mean of Normal Population

Population Variance Known

Confidence Interval for Mean of Normal Population

The interval $\boxed{\bar{X}_n \pm 2\sigma/\sqrt{n}}$ has approximately 95% probability of containing the population mean μ, provided that:

$$\boxed{X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

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But What Does This Mean?

Which quantities are random?

Suppose $X_1, X_2, \ldots, X_n \sim$ iid $N(\mu, \sigma^2)$. Which quantities are random variables?

(a) μ only

(b) σ and μ

(c) σ only

(d) σ, μ and $\bar{X}_n$

(e) $\bar{X}_n$ only

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Confidence Interval is a Random Variable!

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1. $X_1, \ldots, X_n$ are RVs $\Rightarrow \bar{X}_n$ is a RV (repeated sampling)
2. μ, σ and n are constants
3. Confidence Interval $\bar{X}_n \pm 2\sigma/\sqrt{n}$ is also a RV!

Meaning of Confidence Interval

Meaning of Confidence Interval

If we sampled many times we'd get many different sample means, each leading to a different confidence interval. Approximately 95% of these intervals will contain μ.

Rough Intuition

What values of μ are consistent with the data?

CI for Population Mean: Repeated Sampling

$$\boxed{X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

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Sample 1

CI for Population Mean: Repeated Sampling

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Sample 1

$$\boxed{\bar{x}_1}$$

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$$\downarrow$$

Sample 1

$$\downarrow$$

$$\boxed{\bar{x}_1}$$

$$\downarrow$$

$$\bar{x}_1 \pm 2\sigma/\sqrt{n}$$

CI for Population Mean: Repeated Sampling

$$X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)$$

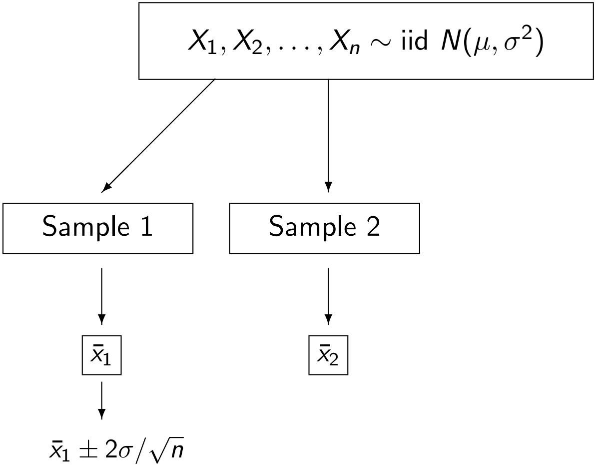
Sample 1

Sample 2

$$\boxed{\bar{x}_1}$$

$$\bar{x}_1 \pm 2\sigma/\sqrt{n}$$

CI for Population Mean: Repeated Sampling



CI for Population Mean: Repeated Sampling

$$
\begin{tikzcd}
 & \boxed{X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)} \arrow[dl] \arrow[d] \\
\boxed{\text{Sample 1}} \arrow[d] & \boxed{\text{Sample 2}} \arrow[d] \\
\boxed{\bar{x}_1} \arrow[d] & \boxed{\bar{x}_2} \arrow[d] \\
\bar{x}_1 \pm 2\sigma/\sqrt{n} & \bar{x}_2 \pm 2\sigma/\sqrt{n}
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\boxed{\text{Sample 1}} \arrow[d] & \boxed{\text{Sample 2}} \arrow[d] & \cdots & \boxed{\text{Sample M}} \\
\boxed{\bar{x}_1} \arrow[d] & \boxed{\bar{x}_2} \arrow[d] & \cdots & \\
\bar{x}_1 \pm 2\sigma/\sqrt{n} & \bar{x}_2 \pm 2\sigma/\sqrt{n} & &
\end{tikzcd}
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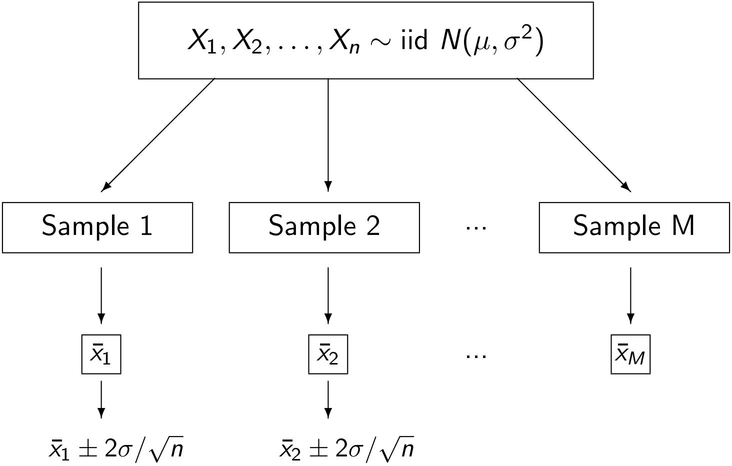
CI for Population Mean: Repeated Sampling



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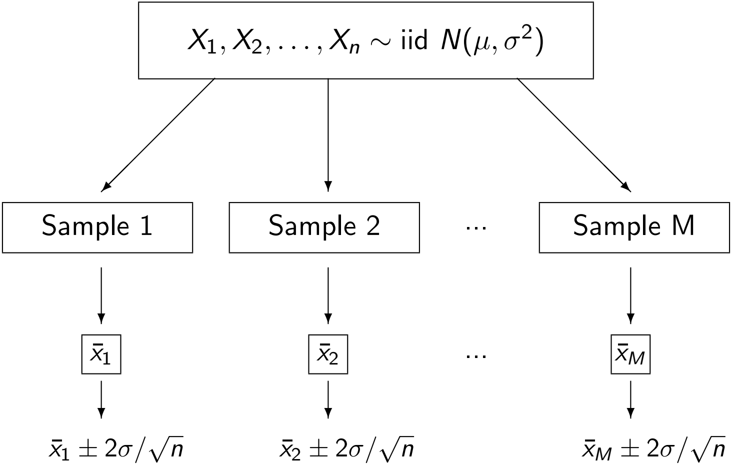

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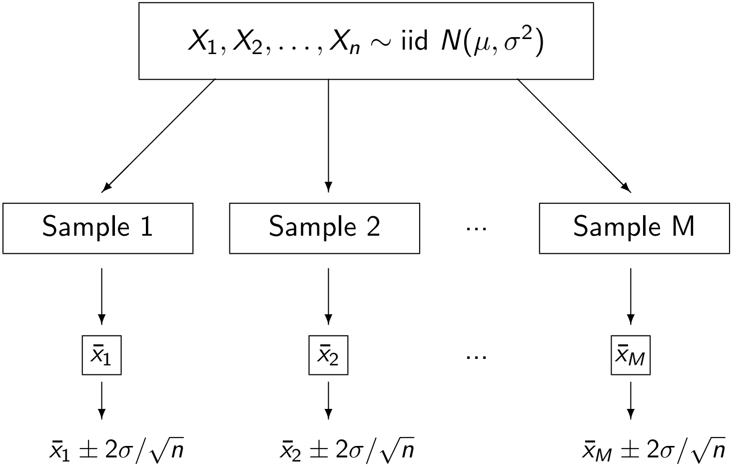

Repeat M times $\rightarrow$ get M different intervals

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\bar{x}_1 \pm 2\sigma/\sqrt{n} & \bar{x}_2 \pm 2\sigma/\sqrt{n} & & \bar{x}_M \pm 2\sigma/\sqrt{n}
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Repeat M times $\rightarrow$ get M different intervals

Large M $\Rightarrow$ Approx. 95% of these Intervals Contain μ

Simulation Example: $X_1, \ldots, X_5 \sim$ iid $N(0,1)$, $M = 20$

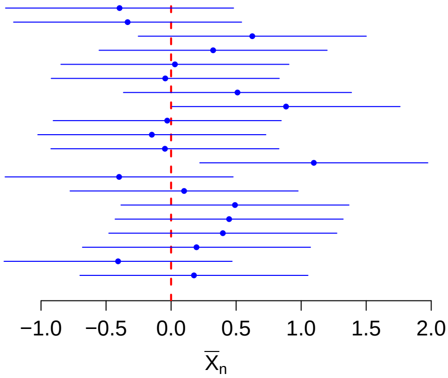

Figure: Twenty confidence intervals of the form $\bar{X}_n \pm 2\sigma/\sqrt{n}$ where $n = 5$, $\sigma^2 = 1$ and the true population mean is 0.

Meaning of Confidence Interval for θ_0

$$\boxed{P(A \leq \theta_0 \leq B) = 1 - \alpha}$$

Each time we sample we'll get a different confidence interval, corresponding to different realizations of the random variables A and B. If we sample many times, approximately $100 \times (1 - \alpha)\%$ of these intervals will contain the population parameter θ_0.

True or False?

Suppose

$$\boxed{X_1, X_2, \ldots, X_n \sim \text{iid } N(\mu, \sigma^2)}$$

Then the population mean μ has approximately a 95% chance of falling in the interval $\bar{X}_n \pm 2\sigma/\sqrt{n}$.

(a) True

(b) False

True or False?

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(a) True

(b) False

FALSE! – μ is a constant!

Confidence Intervals: Some Terminology

Margin of Error

When a CI takes the form $\widehat{\theta} \pm ME$, ME is the Margin of Error.

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Lower and Upper Confidence Limits

The lower endpoint of a CI is the lower confidence limit (LCL), while the upper endpoint is the upper confidence limit (UCL).

Confidence Intervals: Some Terminology

Margin of Error

When a CI takes the form $\widehat{\theta} \pm ME$, ME is the Margin of Error.

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The lower endpoint of a CI is the lower confidence limit (LCL), while the upper endpoint is the upper confidence limit (UCL).

Width of a Confidence Interval

The distance $|\text{UCL} - \text{LCL}|$ is called the width of a CI. This means exactly what it says.

What is the Margin of Error

In the preceding example of a 95% confidence interval for the mean of a normal population when the population variance is known, which of these is the margin of error?

(a) $\sigma/\sqrt{n}$

(b) $\bar{X}_n$

(c) σ

(d) $2\sigma/\sqrt{n}$

(e) $1/\sqrt{n}$

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$2\sigma/\sqrt{n}$, since the CI is $\bar{X}_n \pm 2\sigma/\sqrt{n}$

What is the Width?

In the preceding example of a 95% confidence interval for the mean of a normal population when the population variance is known, which of these is the width of the interval?

(a) $\sigma/\sqrt{n}$

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(c) $3\sigma/\sqrt{n}$

(d) $4\sigma/\sqrt{n}$

(e) $5\sigma/\sqrt{n}$

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$4\sigma/\sqrt{n}$, since the CI is $\bar{X}_n \pm 2\sigma/\sqrt{n}$

Example: Calculate the Margin of Error

$X_1, \ldots, X_{100} \sim$ iid $N(\mu, 1)$ but we don't know μ.
Want to create a 95% confidence interval for μ.

What is the margin of error?

Example: Calculate the Margin of Error

$X_1, \ldots, X_{100} \sim$ iid $N(\mu, 1)$ but we don't know μ.
Want to create a 95% confidence interval for μ.

What is the margin of error?

The confidence interval is $\bar{X}_n \pm 2\sigma/\sqrt{n}$ so

$$ME = 2\sigma/\sqrt{n} = 2 \cdot 1/\sqrt{100} = 2/10 = 0.2$$

Example: Calculate the Lower Confidence Limit

$\boxed{\begin{array}{l} X_1, \ldots, X_{100} \sim N(\mu, 1) \text{ but we don't know } \mu. \\ \text{Want to create a 95\% confidence interval for } \mu. \end{array}}$

We found that $ME = 0.2$. The sample mean $\bar{x} = 4.9$. What is the lower confidence limit?

Example: Calculate the Lower Confidence Limit

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We found that $ME = 0.2$. The sample mean $\bar{x} = 4.9$. What is the lower confidence limit?

$$\text{LCL} = \bar{x} - ME = 4.9 - 0.2 = 4.7$$

Example: Similarly for the Upper Confidence Limit...

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Want to create a 95% confidence interval for μ.

We found that $ME = 0.2$. The sample mean $\bar{x} = 4.9$. What is the upper confidence limit?

$$\text{UCL} = \bar{x} + ME = 4.9 + 0.2 = 5.1$$

Example: 95% CI for Normal Mean, Popn. Var. Known

$$\boxed{X_1, \ldots, X_{100} \sim N(\mu, 1) \text{ but we don't know } \mu.}$$

95% CI for $\mu = [4.7, 5.1]$

Want to be more certain? Use higher confidence level.

What value of c should we use to get a $100 \times (1 - \alpha)\%$ CI for μ?

$$P\left(-c \le \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \le c\right) \;=\; 1 - \alpha$$

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$$P\left(-c \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq c\right) \;=\; 1-\alpha$$

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Take $c = \texttt{qnorm}(1-\alpha/2)$

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$$\bar{X}_n \pm \texttt{qnorm}(1 - \alpha/2) \times \sigma/\sqrt{n}$$

Confidence Interval for a Normal Mean, σ Known

$$\boxed{\bar{X}_n \pm \texttt{qnorm}(1 - \alpha/2) \times \sigma/\sqrt{n}}$$

What Affects the Margin of Error?

$$\boxed{\bar{X}_n \pm \texttt{qnorm}(1-\alpha/2) \times \sigma/\sqrt{n}}$$

Sample Size n

ME decreases with n: bigger sample $\implies$ tighter interval

Population Std. Dev. σ

ME increases with σ: more variable population $\implies$ wider interval

Confidence Level $1-\alpha$

ME increases with $1-\alpha$: higher conf. level $\implies$ wider interval

What Affects the Margin of Error?

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Sample Size n

ME decreases with n: bigger sample $\implies$ tighter interval

Population Std. Dev. σ

ME increases with σ: more variable population $\implies$ wider interval

Confidence Level $1-\alpha$

ME increases with $1-\alpha$: higher conf. level $\implies$ wider interval

Conf. Level	90%	95%	99%
α	0.1	0.05	0.01
$\texttt{qnorm}(1-\alpha/2)$	1.64	1.96	2.56

But What if σ is Unknown?

- What we've done so far assumed that σ was known.
- In real applications this is typically not the case.

Why not try using the sample standard deviation s?

This works, but requires a small change. Instead of basing the interval on quantiles of a normal distribution, we need to use a t distribution. We'll look at this next time.