

Economics 103 – Statistics for Economists

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Lecture # 6

Basic Probability – Part III

Four Volunteers Please!

The Lie Detector Problem

From accounting records, we know that 10% of employees in the store are stealing merchandise.

The managers want to fire the thieves, but their only tool in distinguishing is a lie detector test that is 80% accurate:

Innocent \Rightarrow Pass test with 80% Probability

Thief \Rightarrow Fail test with 80% Probability

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What is the probability that someone is a thief *given* that she has failed the lie detector test?



Monte Carlo Simulation – Roll a 10-sided Die Twice

Managers will split up and visit employees. Employees roll the die twice **but keep the results secret!**

First Roll – Thief or not?

0 \Rightarrow Thief, 1 – 9 \Rightarrow Innocent

Second Roll – Lie Detector Test

0, 1 \Rightarrow Incorrect Test Result, 2 – 9 Correct Test Result

	0 or 1	2–9
Thief	Pass	Fail
Innocent	Fail	Pass

What percentage of those who failed the test are guilty?

Who Failed Lie Detector Test:

Of Thieves Among Those Who Failed:

Base Rate Fallacy – Failure to Consider Prior Information

Base Rate – Prior Information

Before the test we know that 10% of Employees are stealing.

People tend to focus on the fact that the test is 80% accurate and ignore the fact that only 10% of the employees are thieves.

Thief (Y/N), Lie Detector (P/F)

	0	1	2	3	4	5	6	7	8	9
0	YP	YP	YF	YF	YF	YF	YF	YF	YF	YF
1	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
2	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
3	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
4	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
5	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
6	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
7	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
8	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
9	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP

Table: Each outcome in the table is equally likely. The 26 given in red correspond to failing the test, but only 8 of these (YF) correspond to being a thief.

Base Rate of Thievery is 10%

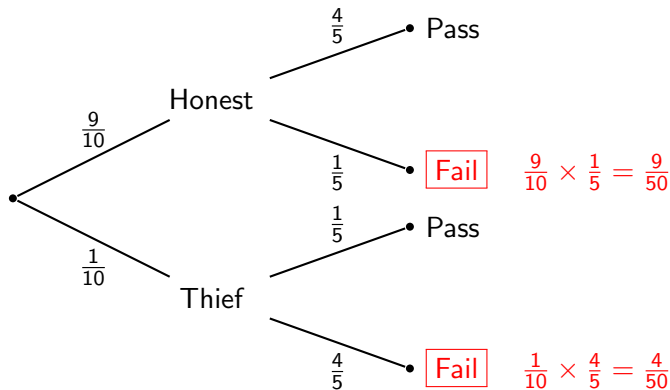


Figure: Although $\frac{9}{50} + \frac{4}{50} = \frac{13}{50}$ fail the test, only $\frac{4/50}{13/50} = \frac{4}{13} \approx 0.31$ are actually thieves!

Deriving Bayes' Rule

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Finally, combining these

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Understanding Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Reversing the Conditioning

Express $P(A|B)$ in terms of $P(B|A)$. *Relative magnitudes* of the two conditional probabilities determined by the ratio $P(A)/P(B)$.

Base Rate

$P(A)$ is called the “base rate” or the “prior probability.”

Denominator

Typically, we calculate $P(B)$ using the law of total probability

In General $P(A|B) \neq P(B|A)$



Question

Most college students are Democrats. Does it follow that most Democrats are college students?

(A = YES, B = NO)

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Answer

There are many more Democrats than college students:

$$P(\text{Dem}) > P(\text{Student})$$

so $P(\text{Student}|\text{Dem})$ is small even though $P(\text{Dem}|\text{Student})$ is large.

Solving the Lie Detector Problem with Bayes' Rule

T = Employee is a Thief, F = Employee Fails Lie Detector Test

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$$P(T|F) = \frac{0.08}{0.26} =$$

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$$P(T|F) = \frac{0.08}{0.26} = \frac{8}{26} = \frac{4}{13} \approx 0.31$$

“Odd” Question # 5

There are two kinds of taxis: green cabs and blue cabs. Of all the cabs on the road, 85% are green cabs. On a misty winter night a taxi sideswiped another car and drove off. A witness says it was a blue cab. The witness is tested under conditions like those on the night of the accident, and 80% of the time she correctly reports the color of the cab that is seen. That is, regardless of whether she is shown a blue or a green cab in misty evening light, she gets the color right 80% of the time.

Given that the witness said she saw a blue cab, what is the probability that a blue cab was the sideswiper?

Solving The Taxi Problem

G = Taxi is Green, $P(G) = 0.85$

B = Taxi is Blue, $P(B) = 0.15$

W_B = Witness says Taxi is Blue, $P(W_B|B) = 0.8$, $P(W_B|G) = 0.2$

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$$\begin{aligned}P(W_B) &= P(W_B|B)P(B) + P(W_B|G)P(G) \\ &= 0.8 \times 0.15 + 0.2 \times 0.85\end{aligned}$$

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$$\begin{aligned}P(W_B) &= P(W_B|B)P(B) + P(W_B|G)P(G) \\&= 0.8 \times 0.15 + 0.2 \times 0.85 \\&= 0.12 + 0.17 = 0.29\end{aligned}$$

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$$P(B|W_B) = 0.12/0.29 = 12/29 \approx 0.41$$

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$$P(B|W_B) = 0.12/0.29 = 12/29 \approx 0.41$$

$$P(G|W_B) = 1 - (12/19) \approx 0.59$$

Random Variables

Random Variables

A random variable is neither random nor a variable.

Random Variable (RV): X

A *fixed* function that assigns a *number* to each basic outcome of a random experiment.

Realization: x

A particular numeric value that an RV could take on. We write $\{X = x\}$ to refer to the *event* that the RV X took on the value x .

Support Set (aka Support)

The set of all possible realizations of a RV.

Random Variables (continued)

Notation

Capital latin letters for RVs, e.g. X , Y , Z , and the corresponding lowercase letters for their realizations, e.g. x , y , z .

Intuition

You can think of an RV as a machine that spits out random numbers: although the machine is deterministic, its inputs, the outcomes of a random experiment, are not.

Example: Coin Flip Random Variable

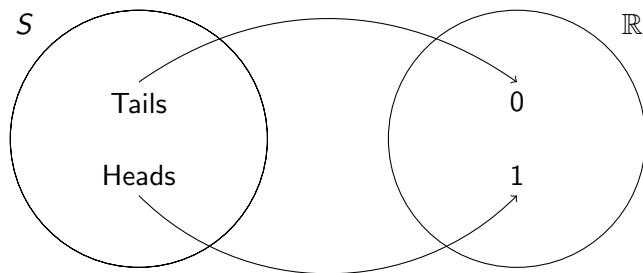


Figure: This random variable assigns numeric values to the random experiment of flipping a fair coin once: Heads is assigned 1 and Tails 0.

Which of these is a realization of the Coin Flip RV?



- (a) Tails
- (b) 2
- (c) 0
- (d) Heads
- (e) $1/2$

What is the support set of the Coin Flip RV?



- (a) {Heads, Tails}
- (b) $1/2$
- (c) 0
- (d) $\{0, 1\}$
- (e) 1

Let X denote the Coin Flip RV



What is $P(X = 1)$?

- (a) 0
- (b) 1
- (c) $1/2$
- (d) Not enough information to determine

Two Kinds of RVs: Discrete and Continuous

Discrete support set is discrete, e.g. $\{0, 1, 2\}$,
 $\{\dots, -2, -1, 0, 1, 2, \dots\}$

Continuous support set is continuous, e.g. $[-1, 1]$, \mathbb{R} .

Start with the discrete case since it's easier, but most of the ideas we learn will carry over to the continuous case.

Discrete Random Variables I

Probability Mass Function (pmf)

A function that gives $P(X = x)$ for any realization x in the support set of a discrete RV X . We use the following notation for the pmf:

$$p(x) = P(X = x)$$

Plug in a realization x , get out a probability $p(x)$.

Probability Mass Function for Coin Flip RV

$$X = \begin{cases} 0, \text{Tails} \\ 1, \text{Heads} \end{cases}$$

$$p(0) = 1/2$$

$$p(1) = 1/2$$

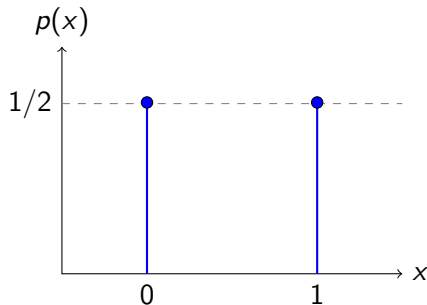


Figure: Plot of pmf for Coin Flip Random Variable

Important Note about Support Sets

Whenever you write down the pmf of a RV, it is **crucial** to also write down its Support Set. Recall that this is the set of *all possible realizations for a RV*. Outside of the support set, all probabilities are zero. In other words, the pmf is **only defined** on the support.

Properties of Probability Mass Functions

If $p(x)$ is the pmf of a random variable X , then

(i) $0 \leq p(x) \leq 1$ for all x

(ii) $\sum_{\text{all } x} p(x) = 1$

where “all x ” is shorthand for “all x in the support of X .”