Definition 1. *A **branch-and-bound algebra** is a semiring $\mathcal{S} = (S, +, \times, 0, 1)$ equipped with orders $(\leq, \sqsubseteq)$ such that:*

(1) $(S, \leq)$ is a total order,
(2) $(S, \sqsubseteq)$ is a join–semilattice with join $\sqcup$ where $\sqsubseteq$ respects $+, \times$,
(3) $\leq, \sqsubseteq$ are compatible in the sense that for all $a \sqsubseteq b$, we also have $a \leq b$.

Example 1. The nonnegative real numbers $\mathbb{R}^{\geq 0}$ forms a branch-and-bound algebra with the usual semiring structure of $\mathbb{R}^{\geq 0}$ and the usual order serving as both $\leq$ and $\sqsubseteq$, with join being the max function.

Example 2. The Boolean semiring $\mathbb{B} = \{\top, \bot\}$ with $+ = \vee, \times = \wedge, 0 = \bot, 1 = \top$ forms a branch-and-bound algebra with the order $\bot \leq \top$ with join $\wedge$.

Example 3. The expected utility semiring $\mathbb{R}^{\geq 0} \times \mathbb{R}$ with the usual semiring operations forms a branch-and-bound algebra with:

(1) $(p, u) \leq (q, v)$ iff $u \leq v$ or $u = v$ and $p \leq q$
(2) $(p, u) \sqsubseteq (q, v)$ iff $p \leq q$ and $u \leq v$, with join being a coordinatewise max.

It is straightforward to see that these are compatible.

Example 4. For any branch-and-bound algebra $\mathcal{B} = (\mathcal{B}, 0, 1, +, \times, \leq, \sqsubseteq)$ consider the collection of finite sets with elements in $\mathcal{B}$, $\mathcal{P}_{<\omega}(\mathcal{B})$. This forms a semiring with additive and multiplicative identities $\{0\}, \{1\}$ with:

(1) $A + B = \cup_{A,B}\{a + b\}$,
(2) $A \times B = \cup_{A,B}\{a \times b\}$.

Moreover it becomes a branch-and-bound algebra with:

(1) $A \leq B$ iff $\max A \leq \max B$, where max is the greatest in the set with respect to $\leq$,
(2) $A \sqsubseteq B$ iff for all $a \in A$ there exists $b \in B$ with $a \sqsubseteq b$, with join

$$A \sqcup B = \cup_{A,B} \begin{cases} a & a \sqcup b = a \\ b & a \sqcup b = b \\ \{a, b\} & else. \end{cases} \tag{1}$$

The intuition here is that $\leq$ is a total order that allows for a selection between "fully evaluated" values and $\sqsubseteq$ is a partial order that allows for comparisons between "partially evaluated" values. The compatibility condition is effectively saying that "comparable partially evaluated values will stay comparable once fully evaluated".

Definition 2. *Let X, D be disjoint sets of variables and let φ a formula with variables $X \cup D$. Let $\pi(X), \pi(D)$ denote assignments to X, D respectively. Fix $\mathcal{B}$ a branch-and-bound algebra; let $f : \mathcal{P}X \times \mathcal{P}D \to \mathcal{B}$ a function; the associated **max-sum** problem is*

$$\max_{x \in \pi(X)} \sum_{y \in} \tag{2}$$