1 PRELIMINARY DEFINITIONS, EXAMPLES, AND LEMMATA

Definition 1. *A **branch-and-bound algebra** is a semiring $\mathcal{S} = (S, +, \times, 0, 1)$ equipped with orders $(\leq, \sqsubseteq)$ such that:*

(1) $(S, \leq)$ is a total order,
(2) $(S, \sqsubseteq)$ is a join–semilattice with join $\sqcup$ where $\sqsubseteq$ respects $+, \times$,
*(3) $\leq, \sqsubseteq$ are compatible in the sense that for all $a \sqsubseteq b$, we also have $a \leq b$. We will henceforth call this **compatibility**.*

Example 1. The nonnegative real numbers $\mathbb{R}^{\geq 0}$ forms a branch-and-bound algebra with the usual semiring structure of $\mathbb{R}^{\geq 0}$ and the usual order serving as both $\leq$ and $\sqsubseteq$, with join being the max function. The standard extension $\mathbb{R}^{\geq 0} \cup \{\infty\}$ is also a branch-and-bound algebra with the usual extended operations.

Example 2. The Boolean semiring $\mathbb{B} = \{\top, \bot\}$ with $+ = \vee, \times = \wedge, 0 = \bot, 1 = \top$ forms a branch-and-bound algebra with the order $\bot \leq \top$ with join $\wedge$.

Example 3. The expected utility semiring $\mathbb{R}^{\geq 0} \times \mathbb{R}$ with the usual semiring operations forms a branch-and-bound algebra with:

(1) $(p, u) \leq (q, v)$ iff $u \leq v$ or $u = v$ and $p \leq q$
(2) $(p, u) \sqsubseteq (q, v)$ iff $p \leq q$ and $u \leq v$, with join being a coordinatewise max.

It is straightforward to see that these are compatible.

Example 4. For any branch-and-bound algebra $\mathcal{B} = (\mathcal{B}, 0, 1, +, \times, \leq, \sqsubseteq)$ consider the collection of finite sets with elements in $\mathcal{B}$, $\mathcal{P}_{<\omega}(\mathcal{B})$. This forms a semiring with additive and multiplicative identities $\{0\}, \{1\}$ with:

(1) $A + B = \cup_{A,B}\{a + b\}$,
(2) $A \times B = \cup_{A,B}\{a \times b\}$.

Moreover it becomes a branch-and-bound algebra with:

(1) $A \leq B$ iff $\max A \leq \max B$, where max is the greatest in the set with respect to $\leq$,
(2) $A \sqsubseteq B$ iff for all $a \in A$ there exists $b \in B$ with $a \sqsubseteq b$, with join

$$A \sqcup B = \cup_{A,B} \begin{cases} a & a \sqcup b = a \\ b & a \sqcup b = b \\ \{a, b\} & else. \end{cases} \tag{1}$$

The intuition here is that $\leq$ is a total order that allows for a selection between "fully evaluated" values and $\sqsubseteq$ is a partial order that allows for comparisons between "partially evaluated" values. The compatibility condition is effectively saying that "comparable partially evaluated values will stay comparable once fully evaluated".

Definition 2. *Let X be a set and $Y(X)$ a set disjoint but possibly dependent on X. $\mathcal{B}$ a branch-and-bound algebra. Let $f : X \times Y \to \mathcal{B}$ be a function. Then the **max-sum problem (MSP)** associated to f is*

$$\max_{x \in X} \sum_{y \in Y(X)} f(x, y). \tag{2}$$

where max *is taken with respect to the total order $\leq$ of the branch and bound algebra. Relatedly, the **join-sum problem (JSP)** associated to f is*

$$\bigsqcup_{x \in X} \sum_{y \in Y(X)} f(x, y). \tag{3}$$

Example 5. The marginal MAP problem is the MSP problem with $X = inst(M)$ instantiations of MAP variables, $Y = inst(V)$ instantiations of the marginal variables, and

$$f(m, v) = \Pr(M = m, V = v \mid E = e).$$

Example 6. The maximum expected utility problem is the MSP problem with $X = \pi$ policies, $Y = E$ the event in which the policy $x \in X$ was taken, and

$$f(\pi, E) = \sum_{\omega \in E} \Pr(\omega) U(\omega)$$

where $U(\omega)$ is additive coordinatewise if ω if the probability distribution is a joint distribution.

Example 7. The weak weighted SAT problem asks for a boolean formula φ over variables V the maximum numbers of true variables in a satisfying assignment. This the MSP problem with $X = inst(V)$, $Y = V$, and

$$f(x, v) = \begin{cases} 1 & x(v) = \top, \\ 0 & x(v) = \bot \end{cases}$$

if $\varphi(x)$ is SAT and $f(x, \_) = 0$ if $\varphi(x)$ is UNSAT.

Lemma 1 (JSP is weak MSP[1]). *Let X, Y be disjoint and $\mathcal{B}$ a branch-and-bound algebra. Let $f : X \times Y \to \mathcal{B}$ be a function. Let MSP, JSP be the max-sum problem and the join-sum problem with respect to f as in 2. Then we have*

$$MSP \leq JSP. \tag{4}$$

Proof. It suffices to show for all $a, b \in \mathcal{B}$, $\max\{a, b\} \leq a \sqcup b$. Note that $\max\{a, b\} = a$ or $= b$; by definition of join we have $a, b \sqsubseteq a \sqcup b$; by compatibility we are done. □

Definition 3. *Let φ be a Boolean formula over variables V and let $X \subseteq V$ a subsest of variables. Let $Y(\varphi|_X)$ be a set depending on φ conditioned on instantiations of X. Let $\mathcal{B}$ a branch-and-bound algebra. Let $f : X \times Y \to \mathcal{B}$ a function. Then the* ***Boolean max-sum problem*** *is*

$$\max_{x \in X} \sum_{y \in Y(\varphi|_X)} f(x, y). \tag{5}$$

where max *is taken with respect to the total order $\leq$ of the branch and bound algebra. Relatedly, the* ***join-sum problem (JSP)*** *associated to f is*

$$\bigsqcup_{x \in X} \sum_{y \in Y(\varphi|_X)} f(x, y). \tag{6}$$

It is easy to see that if $Y(X)$ in Definition 2 is a function mapping X to a set, then $Y(\varphi|_X)$ in Definition 3 is identical; it maps X to a function mapping $\varphi|_X$ to a set. This lends itself to the fact that Lemma 1 naturally extends to Definition 3.

2 SOLVING BOOLEAN MAX-SUM PROBLEMS EXACTLY THROUGH BRANCH-AND-BOUND SEARCH

[1]One may think that this is reminiscent of weak duality. Unfortunately weak duality is completely irrelevant to this Lemma and is only used for namesake purposes.