

LIMIT THEOREMS FOR HIGHER RANK ACTIONS ON HEISENBERG NILMANIFOLDS

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ABSTRACT. The main result of this paper is a construction of finitely additive measures for higher rank abelian actions on Heisenberg nilmanifolds. Under a full measure set of Diophantine condition for the generators of the action, we construct *Bufetov functional* on rectangles on $2g + 1$ dimensional Heisenberg manifold. We prove the deviation of Birkhoff integral of higher rank actions described by asymptotics of Bufetov functionals for a sufficiently smooth function. As a corollary, a distribution of normalized Birkhoff integrals to have a variance 1 converges to a non-degenerate compactly supported measure on the real line.

CONTENTS

1. Introduction	2
1.1. Introduction	2
1.2. Definitions and statement of results	3
Main results	5
2. Analysis on Heisenberg manifolds	7
2.1. Sobolev space	7
2.2. Invariant currents	8
2.3. Diophantine condition	9
3. Constructions of the functionals	11
3.1. Remainder estimates	11
3.2. Constructions of the functionals	15
3.3. Proof of Theorem 1.3	17
3.4. Proof of Theorem 1.6	19
4. Limit distributions	20
4.1. Limiting distributions	21
4.2. Proof of Theorem 1.7	23
5. L^2 -lower bounds	24
5.1. Structure of return map	24
5.2. Higher cohomology for \mathbb{Z}^d -action of skew-shifts	25
5.3. Changes of coordinates	26
5.4. L^2 -lower bound of functional	27
6. Analyticity of functionals	30
6.1. Analyticity	30
6.2. Extensions of domain	33

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1	7. Measure estimation for bounded-type	34
2	Acknowledgement	36
3	References	36

4

1. INTRODUCTION

5 **1.1. Introduction.** The asymptotic behavior and limiting distribution of ergodic
6 averages of translation flow were studied by Alexander Bufetov in the series of his
7 works [Buf14a, Buf10, Buf14b]. He constructed finitely-additive Hölder measures
8 and cocycles over translation flows that are known as *Bufetov functionals*. Such
9 functionals are constructed to derive the deviation of ergodic integrals of translation
10 flows by providing a new proof of the celebrated work of G. Forni [For02]. Bufetov
11 noticed a duality between his finitely-additive Hölder measures and Forni's invariant
12 distributions, which played a crucial role in his work of cohomological equations.
13 Following these observations, the duality between such functionals and the space
14 of invariant distributions was constructed analogously for other parabolic flows
15 and higher rank actions (e.g. [Buf13, BS13, BF14, FK20b, For20a]). Bufetov also
16 proved the probabilistic results for normalized ergodic integrals as a part of his
17 main theorem. Furthermore, it turned out that his construction of functional is also
18 closely related to limit shapes of ergodic sums for interval exchange transformation
19 [MMY10, MUY20].

20 In this paper, our main results are on effective equidistributions for higher rank
21 abelian actions and limit theorems on Heisenberg nilmanifolds. We firstly intro-
22 duce the construction of the Bufetov functional for higher rank abelian actions.
23 The main argument is based on the renormalizations and induction argument from
24 the work of Cosentino and Flaminio [CF15]. However, in our case, we extended
25 their constructions to the rectangular domain and imposed more flexibilities on
26 each directions of shape. We assume that the functional exists for a full measure
27 set of frame α under a necessary Diophantine condition obtained from recurrent
28 conditions (see §2.3.2). Likewise, we also constructed the duality between function-
29 als and invariant currents in [CF15], enabling the deviation formula. As a corollary,
30 we prove an existence to the limit distributions of (normalized) ergodic integrals for
31 higher rank actions. More specifically, as a random variable, normalized ergodic in-
32 tegrals converge in distribution to a non-degenerate, compactly supported measure
33 on the real line.

34 As a minor application, it is not surprising that the related result for exponential
35 sums appeared from connections between the Heisenberg group and theta series in
36 several contexts. For Heisenberg flows, as a first return map on a transverse torus
37 is obtained as a skew-translation, its limit theorem of theta sums was studied by J.
38 Griffin J. Marklof and Cellarosi-Marklof [GM14, CM16]. It is recently generalized by
39 Forni and Kanigowski [FK20b]. Likewise, we obtain limit theorem for theta series on
40 Siegel half spaces by generalizing results [GG04, Mar99] (see [Tol78, MM07, MNN07]
41 for a general introduction).

42 Furthermore, L^2 -lower bound of Bufetov functional on a transverse torus and
43 its analyticity on the higher dimensional rectangular domain follows. This part
44 contributes that our functional does not vanish and extends to the complex domain
45 in the weighted space of fast-decaying coefficient of functionals. From these results,

polynomial type of lower bounds for the analytic function is obtained on certain sub-level sets. Such polynomial estimates on the measure are derived by real analyticity of a functional along the leaves of a foliation transverse to the actions based on results of [Bru99]. This tool was already devised by Forni and Kanigowski [FK20b] to study the bound of correlations for time-changes of nilflows. Unfortunately, it turned out that we can not obtain similar results because the time-changes of higher rank abelian actions on Heisenberg nilmanifolds are all trivial. It follows from the triviality of the first cohomology group (see [CF15, Theorem 3.16]) thus, they are conjugated to the linear action and never mixing.

As a rank 1-action, mixing properties for time changes of Heisenberg nilflows were firstly studied in [AFU11]. Then, it was extended in the dense set of non-trivial time changes for any uniquely ergodic nilflows on general nilmanifolds [Rav18, AFRU21]. As a special case, on time-changes of Heisenberg flows, multiple mixing [FK20a] was proved by Ratner property. In \mathbb{Z}^k -actions, mixing of shapes for automorphisms on nilmanifold is proved in [GS14, GS15].

Comprehensive studies of spatial and temporal limit theorems for horocycle flows have been carried out in the work of D. Dolgopyat and O. Sarig [DS17]. Recent work of Ravotti [Rav21] proved spatial and temporal limit theorems for horocycle flows in alternative ways by using Ratner's argument. His work did not rely on the methods from invariant distributions of solving the cohomological equations. However, it is still unknown if temporal (or weaker) limit theorems for Heisenberg flows can be proved, and it will be a possible subject of further works.

1.2. Definitions and statement of results. We review definitions about Heisenberg manifold and its moduli space.

1.2.1. Heisenberg manifold. Let $g \geq 1$ and H^g be the standard $2g + 1$ dimensional Heisenberg group and set $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$ a discrete and co-compact subgroup of H^g . We shall call it standard lattice of H^g and the quotient $M := H^g/\Gamma$ will be called *Heisenberg manifold*. Lie algebra $\mathfrak{h}^g = Lie(H^g)$ is equipped with a basis $(X_1, \dots, X_g, Y_1, \dots, Y_g, Z)$ satisfying canonical commutation relations

$$(1) \quad [X_i, Y_j] = \delta_{ij}Z, \quad \text{where } \delta_{ij} = 1 \text{ if } i = j, \text{ and } \delta_{ij} = 0 \text{ otherwise.}$$

For $1 \leq d \leq g$, let P^d be a subgroup of H^g where its Lie algebra $\mathfrak{p} := \mathfrak{p}_d = Lie(P^d)$ is generated by $\{X_1, \dots, X_d\}$. For any $\alpha \in Sp_{2g}(\mathbb{R})$, set $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$ for $1 \leq i \leq d$. We define a parametrization of the subgroup $P^{d,\alpha} = \alpha^{-1}(P^d)$ according to

$$P_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \dots + x_d X_d^\alpha), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

By central extension of \mathbb{R}^{2g} by \mathbb{R} , we have an exact sequence

$$0 \rightarrow Z(H^g) \rightarrow H^g \rightarrow \mathbb{R}^{2g} \rightarrow 0.$$

The natural projection $pr : M \rightarrow H^g/(\Gamma Z(H^g))$ maps M onto a $2g$ -dimensional torus $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

1.2.2. Moduli space. The group of automorphisms of H^g that are trivial on the center is denoted by $Aut_0(H^g) = Sp_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$. Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to $Sp_{2g}(\mathbb{R})$. We regard $Sp_{2g}(\mathbb{R})$ as the *deformation space* of the standard Heisenberg manifold M and call the quotient $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ the *moduli space* of (standard) Heisenberg manifold.

1 *Siegel modular variety* is a double coset space $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R}) / Sp_{2g}(\mathbb{Z})$ where
 2 K_g is a maximal compact subgroup $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$ of $Sp_{2g}(\mathbb{R})$. For $\alpha \in$
 3 $Sp_{2g}(\mathbb{R})$, we denote $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$ by its projection on the moduli space \mathfrak{M}_g
 4 and write $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$ the projection of α to the Siegel modular variety
 5 Σ_g .

6 Double coset $K_g \backslash Sp_{2g}(\mathbb{R}) / 1_{2g}$ is identified to the Siegel upper half space $\mathfrak{H}_g :=$
 7 $\{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$. *Siegel upper half space* of genus g is complex manifold
 8 of symmetric complex $g \times g$ matrices $Z = X + iY$ with positive-definite symmetric
 9 imaginary part $\Im(Z) = Y$ and arbitrary (symmetric) real part X . Denote by
 10 $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$.

11 1.2.3. *Representation.* We write the Hilbert sum decomposition of

$$(2) \quad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n$$

into closed H^g -invariant subspaces. Set $f = \sum_{n \in \mathbb{Z}} f_n \in L^2(M)$ and $f_n \in H_n$ where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi i n K t)f\}$$

12 for some fixed $K > 0$. The center $Z(H^g)$ has spectrum $2\pi\mathbb{Z} \setminus \{0\}$ and the space
 13 $L^2(M)$ splits as Hilbert sum of H^g -module H_n , which is equivalent to irreducible
 14 representation π .

By Stone-Von Neumann theorem, the unitary irreducible representation π of the
 Heisenberg group of non-zero central parameter $K > 0$ is unitarily equivalent to the
 Schrödinger representation π . By differentiating the Schrödinger representation, we
 obtain a representation of the Lie algebra \mathfrak{h}^g on Schwartz space $\mathcal{S}(\mathbb{R}^g) \subset L^2(\mathbb{R}^g)$
 (as a C^∞ -vector). This is called *infinitesimal derived representation* $d\pi_*$ with pa-
 rameter $n \in \mathbb{Z}$, and for each $k = 1, 2, \dots, g$,

$$d\pi_*(X_k) = \frac{\partial}{\partial x_k}, \quad d\pi_*(Y_k) = 2\pi i n K x_k, \quad d\pi_*(Z) = 2\pi i n K$$

15 acts on $L^2(\mathbb{R}^g) \simeq L^2(H_n)$.

16 Given a basis (V_i) of the Lie algebra, we set a Laplacian $\Delta = -\sum_i V_i^2$ and define
 17 L^2 -Sobolev norm $\|f\|_s^2 = \langle f, (1 + \Delta)^s f \rangle$ where $\langle \cdot, \cdot \rangle$ is an ordinary inner product.
 18 For $s > 0$, the *Sobolev space* $W^s(M)$ is defined by a completion of $C^\infty(M)$ equipped
 19 with the norm $\|\cdot\|_s$. The Sobolev space $W^s(M) = \bigoplus_{n \in \mathbb{Z}} W^s(H_n)$ decomposes to
 20 closed H^g -invariant subspaces $W^s(H_n) = W^s(M) \cap H_n$.

21 1.2.4. *Renormalization flow.* Denote diagonal matrix $\delta_i = \text{diag}(d_1, \dots, d_g)$ with
 22 $d_i = 1$, $d_k = 0$ if $k \neq i$. Then, for each $1 \leq i \leq g$, we denote $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in$
 23 $\mathfrak{sp}_{2g} = \text{Lie}(Sp_{2g}(\mathbb{R}))$.

24 Any such $\hat{\delta}_i$ generate a one-parameter subgroup of automorphism (renormaliza-
 25 tion flow) $r_i^t := e^{t\hat{\delta}_i}$. We denote (rank d) renormalization actions $r_t := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$
 26 for $t = (t_1, \dots, t_d)$ and $1 \leq i_1, \dots, i_d \leq g$. We also write the corresponding auto-
 27 morphism

$$(3) \quad \exp(t\hat{\delta}(d)) : (x, y, z) \mapsto (e^{t\hat{\delta}}x, e^{-t\hat{\delta}}y, z)$$

28 of the Heisenberg group (see §2.2.3 for its use).

1 **Main results.** One of the main objects in this paper is to construct finitely-
 2 additive measures defined on the space of rectangles on Heisenberg manifold M .
 3 We state our results beginning with an overview of Bufetov functional.

4 **Definition 1.1.** For $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$, denote the *standard rectangle* for action \mathbf{P} ,

$$(4) \quad \Gamma_{\mathbf{T}}^X(m) = \{\mathbf{P}_{\mathbf{t}}^{d,\alpha}(m) \mid \mathbf{t} \in U(\mathbf{T}) = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]\}.$$

5 Let $\mathbf{Q}_y^{d,Y} := \exp(y_1 Y_1 + \cdots + y_d Y_d)$, $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ be an action generated
 6 by elements Y_i of standard basis. Let $\phi_z^Z := \exp(zZ)$, $z \in \mathbb{R}$ be a flow generated
 7 by the central element Z .

8 **Definition 1.2.** Let \mathfrak{R} be the collection of the *generalized rectangles* in M . For
 9 any $1 \leq j \leq d$ and $\mathbf{t} = (t_1, \dots, t_d)$, we set

$$(5) \quad \mathfrak{R} := \bigcup_{1 \leq i \leq d} \bigcup_{(y,z) \in \mathbb{R}^j \times \mathbb{R}} \bigcup_{(m,\mathbf{T}) \in M \times \mathbb{R}_+^d} \{(\phi_{t_{iz}}^Z) \circ \mathbf{Q}_y^{j,Y} \circ \mathbf{P}_{\mathbf{t}}^{d,\alpha}(m) \mid \mathbf{t} \in U(\mathbf{T})\}.$$

10 **Theorem 1.3.** For any irreducible representation H , there exists a measure $\hat{\beta}_H(\Gamma) \in$
 11 \mathbb{C} for every standard d -rectangle Γ such that the following holds:

(1) (Additive property) For any decomposition of disjoint rectangles $\Gamma = \bigcup_{i=1}^n \Gamma_i$
 or whose intersections have zero measure,

$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^n \hat{\beta}_H(\alpha, \Gamma_i).$$

(2) (Scaling property) For $\mathbf{t} \in \mathbb{R}^d$,

$$\hat{\beta}_H(r_{\mathbf{t}}(\alpha), \Gamma) = e^{-(t_1 + \cdots + t_d)/2} \hat{\beta}_H(\alpha, \Gamma).$$

(3) (Invariance property) For any action $\mathbf{Q}_{\tau}^{j,Y}$ generated by Y_i 's for $\tau \in \mathbb{R}_+^j$
 and $1 \leq j \leq d$,

$$\hat{\beta}_H(\alpha, (\mathbf{Q}_{\tau}^{j,Y})_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

(4) (Bounded property) For any rectangle $\Gamma \in \mathfrak{R}$, there exists a constant $C(\Gamma) >$
 0 such that for $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$,

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma) (\int_{\Gamma} |\hat{X}|)^{d/2}.$$

12 **Corollary 1.4.** The functional $\hat{\beta}_H$ defined on standard rectangle $\Gamma_{\mathbf{T}}^X$ extends to the
 13 class \mathfrak{R} .

For arbitrary rectangle $U_{\mathbf{T}} = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]$, pick $\mathbf{T}'^{(i)} \in [0, \mathbf{T}^{(i)}]$ for
 each i to decompose $U_{\mathbf{T}}$ into 2^d sub-rectangles. We write $\mathbf{P}(\mathbf{T}')$ collection of 2^d
 vertices $v = (v^{(1)}, v^{(2)}, \dots, v^{(d)})$ where $v^{(i)} \in \{0, \mathbf{T}'^{(i)}\}$. Let $U_{\mathbf{T},v}$ be a rectangle
 whose sides $\mathbf{I}_v = (I^{(1)}, I^{(2)}, \dots, I^{(d)}) \in \mathbb{R}_+^d$ where

$$I^{(l)} = \begin{cases} T^{(l)} - T'^{(l)} & \text{if } v^{(l)} = T'^{(l)} \\ T'^{(l)} & \text{if } v^{(l)} = 0. \end{cases}$$

14 Then, we have $U_{\mathbf{T}} = \bigcup_{v \in \mathbf{P}(\mathbf{T}')} U_{\mathbf{T},v}$.

15 **Corollary 1.5.** Let us denote

$$(6) \quad \beta_H(\alpha, m, \mathbf{T}) := \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(m)).$$

16 The function β_H satisfies the following properties:

(1) (Cocycle property) For all $(m, \mathbf{T}_1, \mathbf{T}_2) \in M \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\beta_H(\alpha, m, \mathbf{T}_1 + \mathbf{T}_2) = \sum_{v \in \mathbf{P}(\mathbf{T}_1)} \beta_H(\alpha, \mathbf{P}_v^{d, \alpha}(m), \mathbf{I}_v).$$

(2) (Scaling property) For all $m \in M$ and $\mathbf{t} = (t_1, \dots, t_d)$,

$$\beta_H(r_{\mathbf{t}}(\alpha), m, \mathbf{T}) = e^{-(t_1 + \dots + t_d)/2} \beta_H(\alpha, m, \mathbf{T}).$$

(3) (Bounded property) Let us denote the largest length of side $T_{\max} = \max_i \mathbf{T}^{(i)}$. Then there exists a constant $C_H > 0$ such that

$$\beta_H(\alpha, m, \mathbf{T}) \leq C_H T_{\max}^{d/2}.$$

(4) (Orthogonality) For all $\mathbf{T} \in \mathbb{R}^d$, $\beta_H(\alpha, \cdot, \mathbf{T})$ belongs to the irreducible component, i.e.,

$$\beta_H(\alpha, \cdot, \mathbf{T}) \in H \subset L^2(M).$$

By representation theory introduced (2), for any $f = \sum_H f_H \in W^s(M)$, define a Buzetov cocycle associated to f as a sum

$$(7) \quad \beta^f(\alpha, m, \mathbf{T}) = \sum_H D_\alpha^H(f) \beta_H(\alpha, m, \mathbf{T}).$$

Notation. Given a Jordan region U and a point $m \in M$, set $\mathcal{P}_U^{d, \alpha} m$ the Birkhoff integrals (currents) associated to the action $\mathbf{P}_x^{d, \alpha}$ given by

$$\left\langle \mathcal{P}_U^{d, \alpha} m, \omega_f \right\rangle := \int_U f(\mathbf{P}_x^{d, \alpha} m) dx_1 \cdots dx_d$$

for any degree \mathbf{p} -form $\omega_f = f \hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$ with smooth function with zero averages $f \in C_0^\infty(M)$.

Theorem 1.6. For all $s > d(d+11)/4 + g + 1$, there exists a constant $C_s > 0$ such that for almost all frequency α and for all $f \in W^s(M)$ and for all $(m, \mathbf{T}) \in M \times \mathbb{R}^d$, we have

$$(8) \quad \left| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha} m, \omega_f \right\rangle - \beta^f(\alpha, m, \mathbf{T}) \right| \leq C_s \|\omega_f\|_s.$$

for $U(\mathbf{T}) = [0, T^{(1)}] \times \cdots \times [0, T^{(d)}]$ and $\omega_f = f \omega^{d, \alpha} \in \Lambda^d \mathbf{p} \otimes W^s(M)$.

Let the family of random variable

$$E_{T_n}(f) := \frac{1}{\text{vol}(U(\mathbf{T}_n))^{1/2}} \left\langle \mathcal{P}_{U(\mathbf{T}_n)}^{d, \alpha}(m), \omega_f \right\rangle$$

where $U(\mathbf{T}_n)$ is a sequence of rectangles. The point $m \in M$ is distributed accordingly to the probability measure vol on M . Our goal is to understand the asymptotic behavior of the probability distributions of $E_{T_n}(f)$.

Theorem 1.7 (Theorem 4.6). For every closed form $\omega_f \in \Lambda^d \mathbf{p} \otimes W^s(M)$ with $s > s_{d, g} = d(d+11)/4 + g + 1/2$, which is not a coboundary, the limit distribution of the family of random variables $E_{T_n}(f)$ exists, and for almost all frequency α , it has compact support on the real line.

We finish the section by giving some remarks on the new adaptation of transfer operator techniques from hyperbolic dynamics. The method stems from the analysis of the transfer operator, firstly treated by P. Gieulletti and Liverani [GL19]. They set up a non-linear flows on the torus and proved asymptotics of ergodic averages

1 with expansions of invariant distributions and eigenvalues of transfer operators
 2 called *Ruelle resonances* (see also [AB18, FGL19, For20b, B19] for applications in
 3 parabolic flows).

4 A recent work of L. Simonelli and O. Butterley [BS20] rephrased the work of
 5 Flaminio and Forni [FF06] by reproving the deviation of ergodic averages for Heisen-
 6 berg nilflows. Although their work is restricted to the periodic type of α in the
 7 moduli space \mathfrak{M} (the flow is only renormalized by partially hyperbolic diffeomor-
 8 phism, or α is periodic) their methods showed indirect similarities with [FK20b]
 9 and our current work.¹ However, it is still not known if it is possible to extend
 10 their approach to full measure set of α , requiring construction of renormalization
 11 cocycle, so called *Transfer cocycle*. It is also not well-studied how to replace the
 12 use of Anisotropic norm, not relying on the result of Faure-Tsujii [FT15].²

13 **Open problem.** Our work leaves open questions.

14 *Problem 1.* Can we prove measure estimation of β^f in §7 for the full measure
 15 set of α ?

16 *Problem 2.* Can we construct Bufetov functionals on higher step nilmanifolds?

17 *Problem 3.* Can we prove the deviation of ergodic averages for higher rank
 18 abelian actions by adopting Gieulletti-Liverani approach?

19 Since moduli space is trivial and there does not exist renormalization flows, we
 20 could not obtain the results in the same method. However, we still hope other meth-
 21 ods in handling non-renormalizable flows are possibly applied (cf. [FF14, FFT16,
 22 Kim21]).

23 *Outline of the paper.* In section 2, we give basic definitions on Higher rank
 24 actions, moduli spaces and Sobolev spaces. In section 3, we state main theorem
 25 and prove constructions of Bufetov functionals with main properties. In section 4,
 26 we prove asymptotic formula of Birkhoff integrals and its limit theorems. In section
 27 5, we prove L^2 -lower bound of the Bufetov functionals. In Section 6, analyticity of
 28 functional and extensions of domain are provided. In section 7, there exist measure
 29 estimates of functionals on the sets where values of functionals are small. This
 30 result only holds when frame α is of bounded type.

31 2. ANALYSIS ON HEISENBERG MANIFOLDS

32 In this section, we review definitions of Sobolev space, currents, representation
 33 and renormalization flows on moduli space.

34 2.1. Sobolev space.

2.1.1. *Sobolev bundles.* Given a basis (V_i) of the Lie algebra \mathfrak{h}^g , we write a new basis
 $((\alpha^{-1})_* V_i)$ for $\alpha \in \text{Aut}_0(\mathbb{H}^g)$. Similarly, denote by Laplacian $\Delta_\alpha = -\sum_i (\alpha^{-1})_* V_i^2$
 with respect to the new basis. For any $s \in \mathbb{R}$ and any C^∞ -function $f \in L^2(M)$,
Sobolev norm is defined by

$$\|f\|_{\alpha, s} = \langle f, (1 + \Delta_\alpha)^s f \rangle^{1/2}.$$

¹For instance, it is plausible to view their construction of functionals obtained by spectral
 projection as a functional $\hat{\beta}_H$ on each sub-representation H in our setting (see [BS20, §4]).

²Furthermore, it is unknown how to apply recurrent conditions instead of relying on the meth-
 ods in Sobolev constant techniques.

- 1 Let $W_\alpha^s(M)$ be a completion of $C^\infty(M)$ with the norm above. The dual space of
 2 $W_\alpha^s(M)$ is denoted by $W_\alpha^{-s}(M)$ and it is isomorphic to $W_\alpha^s(M)$. Extending it to
 3 the exterior algebra, define the Sobolev spaces $\Lambda^d \mathfrak{p} \otimes W_\alpha^s(M)$ of cochains of degree
 4 d and we use the same notations for the norm.

2.1.2. *Sobolev bundle.* The group $Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \rightarrow Sp_{2g}(\mathbb{R})$$

where

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma = (\alpha\gamma, \gamma^*\varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \rightarrow \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

- 5 and the fiber over $[\alpha] \in \mathfrak{M}_g$ is locally identified with the space $W_\alpha^s(M)$.

By invariance of $Sp_{2g}(\mathbb{Z})$ action, the class of (α, φ) is denoted by $[\alpha, \varphi]$ and $Sp_{2g}(\mathbb{Z})$ -invariant Sobolev norm is written by

$$\|f\|_{\alpha, s} := \|[\alpha, f]\|_s.$$

We denote the bundle of \mathfrak{p} -forms of degree j of Sobolev order s by $A^j(\mathfrak{p}, \mathfrak{M}^s)$. The space of continuous linear functional on $A^j(\mathfrak{p}, \mathfrak{M}^s)$ will be called the *space of currents* of dimension j and denoted by $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$. There is a flat bundle of (currents) distribution $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$ whose fiber over $[\alpha]$ is locally identified with the space $W_\alpha^{-s}(M)$. Similarly, we write the norm for form ω and currents \mathcal{D} by

$$\|\omega\|_{\alpha, s} := \|[\alpha, \omega]\|_s, \quad \|\mathcal{D}\|_{\alpha, s} := \|[\alpha, \mathcal{D}]\|_s.$$

- 6 2.1.3. *Best Sobolev constant.* The *Sobolev embedding theorem* implies that for any
 7 $\alpha \in Sp_{2g}(\mathbb{R})$ and $s > g + 1/2$, there exists a constant $B_s(\alpha)$ such that for any
 8 $f \in W_\alpha^s(M)$,

$$(9) \quad B_s(\alpha) := \sup_{f \in W_\alpha^s(M) \setminus \{0\}} \frac{\|f\|_\infty}{\|f\|_{\alpha, s}}.$$

- 9 By Lemma 4.4 in [CF15], the *best Sobolev constant* B_s is a $Sp_{2g}(\mathbb{Z})$ -modular func-
 10 tion on \mathfrak{H}_g . Thus, we shall write $B_s([\alpha])$ or $B_s([\alpha])$ for $B_s(\alpha)$. By the Sobolev
 11 embedding theorem, we have a bound for the Birkhoff integral current.

Lemma 2.1. [CF15, Lemma 5.5] *For any Jordan region $U \subset \mathbb{R}^d$ with Lebesgue measure $|U|$, for any $s > g + 1/2$ and all $m \in M$,*

$$\left\| [\alpha, \mathcal{D}_U^{d, \alpha} m] \right\|_{-s} \leq B_s([\alpha])|U|.$$

12 2.2. Invariant currents.

2.2.1. *Identification.* We denote $I_d(\mathfrak{p}, W^{-s}(M))$ the space of \mathfrak{P} -invariant currents of Sobolev order s . By formal identities $\langle D, X_i^\alpha(f) \rangle = 0$ for all test function f and $i \in [1, d]$. The *boundary operators*

$$\partial : A_j(\mathfrak{p}, W^{-s}(M)) \rightarrow A_{j-1}(\mathfrak{p}, W^{-s}(M))$$

- 13 are adjoint of the differentials d such that $\langle \partial D, \omega \rangle = \langle D, d\omega \rangle$. A current D is called
 14 *closed* if $\partial D = 0$. For $s > 0$, we denote $D \in Z_d(\mathfrak{p}, W^{-s}(M))$ the space of closed
 15 currents of dimension d and Sobolev order s .

- 1 **Proposition 2.2.** [CF15, Proposition 3.13] For any $s > d/2$, with $d = \dim \mathbf{P}$, we
 2 have $I_d(\mathbf{p}, \mathcal{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$ for all $\epsilon > 0$. Additionally,
 3 • $I_d(\mathbf{p}, \mathcal{S}(\mathbb{R}^g))$ is one dimensional space if $\dim \mathbf{P} = g$,
 4 • $I_d(\mathbf{p}, \mathcal{S}(\mathbb{R}^g))$ is an infinite-dimensional space if $\dim \mathbf{P} < g$,
 5 • $I_d(\mathbf{p}, \mathcal{S}(\mathbb{R}^g)) = Z_d(\mathbf{p}, W^{-s}(M))$ for any $1 \leq d \leq g$.

2.2.2. *Basic currents.* The current B_α is called *basic* if for all $j \in \{i_1, \dots, i_d\}$,

$$\iota_{X_j} B_\alpha = L_{X_j} B_\alpha = 0.$$

- 6 For an irreducible representation H , there exists a *basic current* B_α^H associated to
 7 invariant distribution D_α^H . It is defined by $B_\alpha^H = D_\alpha^H \eta_X$ and this formula implies
 8 that for every d -form ξ ,

$$(10) \quad B_\alpha^H(\xi) = D_\alpha^H \left(\frac{\eta_X \wedge \xi}{\omega_{vol}} \right)$$

- 9 where $\eta_X := \iota_{X_{i_1}} \dots \iota_{X_{i_d}} \omega_{vol}$ and ω_{vol} is an invariant volume form.
 10 We recall that the space of Sobolev currents $A_j(\mathbf{p}, W^{-s}(M))$ of dimension j with
 11 order s is identified with $\Lambda^j \mathbf{p} \otimes W^{-s}(M)$. It follows that for all $s > d/2$, by Sobolev
 12 embedding theorem, $B_\alpha^H \in A_j(\mathbf{p}, W_\alpha^{-s}(M))$ if and only if $D_\alpha^H \in W_\alpha^{-s}(M)$ for all
 13 $s > d/2$.

Remark. The formula (10) yields a duality between the space of basic (closed) currents and invariant distributions (see also [For02, §6.1] and [BF14, §2.1]). For any d -dimensional \mathbf{p} -form $\omega_f = f dX_1 \wedge \dots \wedge dX_d$ and $f \in C^\infty(M)$, we will identify currents \mathcal{D} with distribution D by writing

$$\langle D, f \rangle = \langle \mathcal{D}, \omega_f \rangle.$$

2.2.3. *Renormalization.* Let $s > d/2$. Recall the definition of renormalization flow (3) in §1.2.4. For $\omega \in \Lambda^d \mathbf{p} \otimes W^s(\mathbb{R}^g)$, $\mathcal{D} \in Z_d(\mathbf{p}, W^{-s}(M))$, and $t \in \mathbb{R}$

$$r_i^t[\alpha, \omega] = [r_i^t \alpha, \omega], \quad r_i^t[\alpha, \mathcal{D}] = [r_i^t \alpha, \mathcal{D}].$$

- 14 By Proposition 5.2 in [CF15], the sub-bundle $Z_d(\mathbf{p}, W^{-s}(M))$ is invariant under
 15 the renormalization flows r_i^t . Furthermore, we have

$$\|r_1^{t_1} \dots r_d^{t_d} [\alpha, \mathcal{D}]\|_{-s} = e^{-(t_1 + \dots + t_d)/2} \|[\alpha, \mathcal{D}]\|_{-s}.$$

- 16 Let $U_{\mathbf{t}} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be unitary operator for $\mathbf{t} = (t_1, \dots, t_d)$,

$$(11) \quad U_{\mathbf{t}} f(x) = e^{-(t_1 + \dots + t_d)/2} f(e^{t_1} x_1, \dots, e^{t_d} x_d).$$

Then, for any invariant distribution D_α^H ,

$$D_{r_{\mathbf{t}}(\alpha)}^H = e^{(t_1 + \dots + t_d)/2} D_\alpha^H.$$

- 17 By reparametrization of \mathbb{R}^d -action $r_{\mathbf{t}} = r_1^{t_1} \dots r_d^{t_d}$,

$$(12) \quad \mathbf{p}_x^{d, (r_1^{t_1} \dots r_d^{t_d} \alpha)} = \mathbf{p}_{(e^{-t_1} x_1, \dots, e^{-t_d} x_d)}^{d, \alpha}.$$

- 18 Then the Birkhoff integral current also satisfies the following identity

$$(13) \quad \mathcal{P}_U^{d, (r_1^{t_1} \dots r_d^{t_d} \alpha)} m = e^{(t_1 + \dots + t_d)/2} \mathcal{P}_{(e^{-t_1}, \dots, e^{-t_d})U}^{d, \alpha} m.$$

- 19 **2.3. Diophantine condition.**

2.3.1. *Height function.*

Definition 2.3 (Height function). The *height* of a point $Z \in \mathfrak{H}_g$ in Siegel upper half space is the positive number

$$hgt(Z) := \det \Im(Z).$$

The *maximal height function* $Hgt : \Sigma_g \rightarrow \mathbb{R}^+$ is the maximal height of a $Sp_{2g}(\mathbb{Z})$ orbit of Z . That is, for the class of $[Z] \in \Sigma_g$,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

By Proposition 4.8 of [CF15], there exists a universal constant $C(s) > 0$ such that the best Sobolev constant satisfies the estimate

$$(14) \quad B_s([\alpha]) \leq C(s) \cdot (Hgt([\alpha]))^{1/4}.$$

We rephrase Lemma 4.9 in [CF15] regarding the bound of renormalized height.

Lemma 2.4. For any $[\alpha] \in \mathfrak{M}_g$ and any $\mathbf{t} \in \mathbb{R}_+^d$,

$$(15) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))\alpha]) \leq (\det(e^{\mathbf{t}\hat{\delta}}))^2 Hgt([\alpha]).$$

2.3.2. *Diophantine condition.*

Definition 2.5. An automorphism $\alpha \in Sp_{2g}(\mathbb{R})$ or a point $[\alpha] \in \mathfrak{M}_g$ is $\hat{\delta}(d)$ -*Diophantine* of type σ if there exists a $\sigma > 0$ and a constant $C > 0$ such that

$$(16) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))\alpha]) \leq CHgt([\exp(-\mathbf{t}\hat{\delta}(d))])^{(1-\sigma)} Hgt([\alpha]), \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

This states that $\alpha \in Sp_{2g}(\mathbb{R})$ satisfies a $\hat{\delta}(d)$ -Diophantine if the height of the projection of $\exp(-\mathbf{t}\hat{\delta}(d))\alpha$ in the Siegel modular variety Σ_g is bounded by $e^{2(t_1+\dots+t_d)(1-\sigma)}$. Furthermore,

- $[\alpha] \in \mathfrak{M}_g$ satisfies a $\hat{\delta}(d)$ -*Roth condition* if for any $\epsilon > 0$ there exists a constant $C > 0$ such that

$$(17) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))\alpha]) \leq CHgt([\exp(-\mathbf{t}\hat{\delta}(d))])^\epsilon Hgt([\alpha]), \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

That is, $\hat{\delta}(d)$ -Diophantine of type $0 < \sigma < 1$.

- $[\alpha]$ is of *bounded type* if there exists a constant $C > 0$ such that

$$(18) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))]) \leq C, \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

Definition 2.6. Let $X = G/\Lambda$ be a homogeneous space equipped with the probability Haar measure μ . A function $\phi : X \rightarrow \mathbb{R}$ is said k -DL (distance like) for some exponent $k > 0$ if it is uniformly continuous and if there exist constants $C_1, C_2 > 0$ such that

$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

By the work of Kleinblock and Margulis, a multi-parameter generalization of Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

Theorem 2.7. [KM99, Theorem 1.9] Let G be a connected semisimple Lie group without compact factors, μ its normalized Haar measure, $\Lambda \subset G$ an irreducible lattice, \mathfrak{a} a Cartan subalgebra of the Lie algebra of G . Let \mathfrak{d}_+ be a non-empty open cone in a d -dimensional subalgebra \mathfrak{d} of \mathfrak{a} ($1 \leq d \leq \text{rank}_{\mathbb{R}}(G)$). If $\phi : G/\Lambda \rightarrow \mathbb{R}$ is a k -DL function for some $k > 0$, then for μ -almost all $x \in G/\Lambda$ one has

$$\limsup_{\mathbf{z} \in \mathfrak{d}_+, \mathbf{z} \rightarrow \infty} \frac{\phi(\exp(\mathbf{z})x)}{\log \|\mathbf{z}\|} = \frac{d}{k}$$

By Lemma 4.7 of [CF15], the logarithm of Height function is DL-function with exponent $k = \frac{g+1}{2}$ on the Siegel variety Σ_g (and induces on $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$). Hence, we obtain the following proposition.

Proposition 2.8. *Under the assumption $X = \mathfrak{M}_g$ of Theorem 2.7, for $s > g+1/2$, there exists a full measure set $\Omega_g(\hat{\delta})$ and for all $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$*

$$(19) \quad \limsup_{t \in \mathbb{R}_+^d, t \rightarrow \infty} \frac{\log Hgt([\exp(-t\hat{\delta}(d))\alpha])}{\log \|t\|} \leq \frac{2d}{g+1}.$$

Any such $[\alpha]$ satisfies a $\hat{\delta}(d)$ -Roth condition (17).

For any $L > 0$ and $1 \leq d \leq g$, let $DC(L)$ denote the set of $[\alpha] \in \mathfrak{M}_g$ such that

$$(20) \quad \int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots + t_d)/2} \text{Hgt}([r_{-t}\alpha])^{1/4} dt_1 \cdots dt_d \leq L.$$

Let DC denote the union of the sets $DC(L)$ over all $L > 0$. It follows immediately that the set $DC \subset \mathfrak{M}_g$ has full Haar volume.

Remark. In the work of Cosentino-Flaminio [CF15], Diophantine condition is restricted to one-parameter subgroup $\exp(-t\hat{\delta})$ for non-negative (g -dimensional) diagonal directions. This is based on the easy part of theorem in Kleinblock and Margulis [KM99, Theorem 1.7]. Our condition is based on d -dimensional actions for renormalizations. This is key ingredient to construct functionals on rectangles. We keep the notation $\exp(-t\hat{\delta}(d))$ for consistency and to show dependence on dimensions for renormalization actions.

3. CONSTRUCTIONS OF THE FUNCTIONALS

In this section, we construct Bufetov functionals for higher rank actions on the standard rectangular domain. Firstly, we prove the bound of Birkhoff integrals on rectangles. It consists of similar proofs for Theorem 5.10 in [CF15], but this type of estimate is used to control remainder estimate (Lemma 3.4) to define functionals under Diophantine condition (Lemma 3.5). Furthermore, we will verify the properties of functionals $\hat{\beta}_H$ and cocycles β_H stated in Theorem 1.3.

3.1. Remainder estimates. For any exponent $s > d/2$, Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^\perp$. Denote by \mathcal{I}^{-s} and \mathcal{R}^{-s} the corresponding orthogonal projection operator and by \mathcal{I}_α^{-s} and \mathcal{R}_α^{-s} the restrictions to the fiber over $[\alpha] \in \mathfrak{M}$ for $\alpha \in Sp_{2g}(\mathbb{R})$. In particular, for the current (Birkhoff integrals) $\mathcal{D} = \mathcal{D}_U^{d,\alpha} m$, we call $\mathcal{I}_\alpha^{-s}(\mathcal{D}) = \mathcal{I}^{-s}[\alpha, \mathcal{D}]$ boundary term and $\mathcal{R}_\alpha^{-s}(\mathcal{D}) = \mathcal{R}^{-s}[\alpha, \mathcal{D}]$ remainder term respectively. Consider the orthogonal projection

$$(21) \quad \mathcal{D} = \mathcal{I}_{r_{-t}[\alpha]}^{-s}(\mathcal{D}) + \mathcal{R}_{r_{-t}[\alpha]}^{-s}(\mathcal{D}).$$

We firstly recall the following estimate of boundary term.

Lemma 3.1. [CF15, Lemma 5.7] *Let $s > d/2 + 2$. There exists a constant $C = C(s) > 0$ such that for all $t_i \geq 0$ for $1 \leq i \leq d$, we have*

$$\begin{aligned} \|\mathcal{I}^{-s}[\alpha, \mathcal{D}]\|_{-s} &\leq e^{-(t_1+\dots+t_d)/2} \|\mathcal{I}^{-s}[r_1^{-t_1} \dots r_d^{-t_d} \alpha, \mathcal{D}]\|_{-s} \\ &\quad + C_1 |t_1 + \dots + t_d| \int_0^1 e^{-u(t_1+\dots+t_d)/2} \|\mathcal{R}^{-s}[r_1^{-ut_1} \dots r_d^{-ut_d} \alpha, \mathcal{D}]\|_{-(s-2)} du. \end{aligned}$$

1 By Stokes' theorem, we have the following remainder estimate.

Lemma 3.2. [CF15, Lemma 5.6] *For any non-negative $s' < s - (d+1)/2$ and Jordan region $U \subset \mathbb{R}^d$, there exists $C = C(d, g, s, s') > 0$ such that*

$$\|\mathcal{R}^{-s}[\alpha, (\mathcal{P}_U^{d,\alpha} m)]\|_{-s} \leq C \|\alpha, \partial(\mathcal{P}_U^{d,\alpha} m)\|_{-s'}.$$

2 Quantitative bound of Birkhoff integrals on the square domain was obtained
3 in [CF15], but we need to extend the result to the rectangular shapes for analyticity
4 of functionals in the section §6.

5 From now on, assume that $s_{d,g} := d(d+11)/4 + g + 1/2$.

Theorem 3.3. *For $s > s_{d,g}$, there exists a constant $C(s, d) > 0$ such that the following holds. For any $t_i > 0$, $m \in M$ and $U_d(t) = [0, e^{t_1}] \times \dots \times [0, e^{t_d}]$, we have*

(22)

$$\begin{aligned} \|\alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)\|_{-s} &\leq C \sum_{k=0}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \int_0^{t_{i_k}} \dots \int_0^{t_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^d t_l - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \\ &\quad \times Hgt\left(\left[\prod_{1 \leq j \leq d} r_j^{-t_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}} \alpha\right]\right)^{1/4} du_{i_1} \dots du_{i_k}. \end{aligned}$$

6 *Proof.* We proceed by induction. For $d = 1$, it follows from the Theorem 5.8
7 in [CF15]. We assume that the result holds for $d - 1$. Decompose the current as a
8 sum of boundary and remainder term as in (21).

Step 1. We estimate the boundary term. By Lemma 3.1, renormalize terms with $r^u = r_1^u \dots r_d^u$. Then, we have

$$\begin{aligned} \|\mathcal{I}^{-s}[\alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)]\|_{-s} &\leq e^{-(t_1+\dots+t_d)/2} \|\mathcal{I}^{-s}[r_1^{-t_1} \dots r_d^{-t_d} \alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)]\|_{-s} \\ (23) \quad &+ C_1(s) \int_0^{t_1+\dots+t_d} e^{-ud/2} \|\mathcal{R}^{-s}[r^{-u} \alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)]\|_{-(s-2)} du \\ &:= (I) + (II). \end{aligned}$$

By renormalization (13) and Lemma 2.1 for unit volume,

$$\begin{aligned} \|\mathcal{I}^{-s}[r_1^{-t_1} \dots r_d^{-t_d} \alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)]\|_{-s} &= e^{t_1+\dots+t_d} \left\| \mathcal{I}^{-s}[r_1^{-t_1} \dots r_d^{-t_d} \alpha, (\mathcal{P}_{U_d(0)}^{d, r_1^{-t_1} \dots r_d^{-t_d} \alpha} m)] \right\|_{-s} \\ &\leq C_2 e^{t_1+\dots+t_d} Hgt([r_1^{-t_1} \dots r_d^{-t_d} \alpha])^{1/4}. \end{aligned}$$

Hence

$$I \leq C_2 e^{(t_1+\dots+t_d)/2} Hgt([r_1^{-t_1} \dots r_d^{-t_d} \alpha])^{1/4},$$

9 where the sum corresponds to the first term ($k = 0$) in the statement.

Step 2. To estimate (II),

$$(24) \quad \left\| \mathcal{R}^{-s}[r^{-u}\alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)] \right\|_{-(s-2)} = \left\| e^{ud} \mathcal{R}^{-s}[r^{-u}\alpha, (\mathcal{P}_{U_d(t-u)}^{d,r^{-u}\alpha} m)] \right\|_{-(s-2)} \\ \leq C_3(s, s') e^{ud} \left\| [r^{-u}\alpha, \partial(\mathcal{P}_{U_d(t-u)}^{d,r^{-u}\alpha} m)] \right\|_{-s'}.$$

The boundary $\partial(\mathcal{P}_{U_d}^{d,r^{-t}\alpha})$ is the sum of $2d$ currents of dimension $d-1$. These currents are Birkhoff sums of d face subgroups obtained from $\mathcal{P}_{U_d}^{d,r^{-t}\alpha}$ by omitting one of the base vector fields X_i . It is reduced to $(d-1)$ dimensional shape obtained from $U_d(t-u) := [0, e^{t_1-u}] \times \cdots \times [0, e^{t_d-u}]$. For each $1 \leq j \leq d$, there are Birkhoff sums along $d-1$ dimensional cubes. By induction hypothesis, we add all the $d-1$ dimensional cubes by adding all the terms along j :

$$(25) \quad \left\| [r^{-u}\alpha, (\mathcal{P}_{U_{d-1}(t-u)}^{d-1,r^{-u}\alpha} m)] \right\|_{-s'} \leq C_4(s', d-1) \sum_{j=1}^d \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq d \\ i_l \neq j, \forall l}} \int_0^{t_{i_k}-u} \cdots \int_0^{t_{i_1}-u} \\ \exp\left(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^d (t_l - u) - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \text{Hgt}([\prod_{\substack{1 \leq l \leq d \\ l \neq j}} r_l^{-(t_l-u)} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r^{-u}\alpha)])^{1/4} du_{i_1} \cdots du_{i_k}.$$

Combining (23) and (24), we obtain the estimate for (II).

$$(II) \leq C_5(s', d-1) \sum_{j=1}^d \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq d \\ i_l \neq j}} \int_0^{t_1+\cdots+t_d} \int_0^{t_{i_k}-u} \cdots \int_0^{t_{i_1}-u} du_{i_1} \cdots du_{i_k} du \\ \times \exp\left(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^d t_l - \frac{1}{2} u - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \text{Hgt}([\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_j^{-u+t_j}\alpha)])^{1/4}.$$

Applying the change of variable $u_j = t_j - u$, we obtain

$$(II) \leq C_6(s', d-1) \sum_{j=1}^d \sum_{k=1}^{d-1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq d \\ i_l \neq j}} \\ \times \left(\int_{-(t_1+\cdots+t_d)+t_j}^{t_j} \int_0^{t_{i_k}-u} \cdots \int_0^{t_{i_1}-u} du_{i_1} \cdots du_{i_k} du_j \right. \\ \left. \times \exp\left(\frac{1}{2} (t_1 + \cdots + t_d) - \frac{1}{2} u_j - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \text{Hgt}([\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_j^{-u_j}\alpha)])^{1/4} \right).$$

Simplifying multi-summation above, (with $-(t_1 + \cdots + t_d) + t_j \leq 0$)

$$(II) \leq C_7(s', d) \sum_{k=1}^d \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} du_{i_1} \cdots du_{i_k} \\ \times \exp\left(\frac{1}{2} (t_1 + \cdots + t_d) - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \text{Hgt}([\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} \alpha])^{1/4}.$$

1 *Step 3. (Remainder estimate).* The remainder term is obtained from Lemma 3.2
 2 (Stokes' theorem). Following step 2, estimate of remainder reduces to that of $d-1$
 3 form. Combining with the step 1, we have the following
 (26)

$$\left\| \mathcal{R}^{-s}[\alpha, \partial(\mathcal{P}_{U_d}^{d,\alpha} m)] \right\|_{-s} \leq C(s) \sum_{i=1}^{d-1} \left\| \mathcal{I}^{-s}[\alpha, (\mathcal{P}_{U_i}^{i,\alpha} m)] \right\|_{-s} + \left\| \mathcal{R}^{-s}[\alpha, (\mathcal{P}_{U_1}^{1,\alpha} m)] \right\|_{-s}$$

where U_i is i -dimensional rectangle. Sum of the boundary terms are absorbed in the bound of (I) + (II). For 1-dimensional remainder with interval Γ_T , the boundary is a 0-dimensional current. Then,

$$\langle \partial(\mathcal{P}_{[0,T]}^{1,\alpha} m), f \rangle = f(\mathbf{P}_T^{1,\alpha} m) - f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (9) and (14),

$$\left\| \mathcal{R}^{-s}[\alpha, \partial(\mathcal{P}_{[0,T]}^{1,\alpha} m)] \right\|_{-s} \leq 2B_s([\alpha]) \leq C(s) Hgt([\alpha])^{1/4}.$$

Then, by inequality (15)

$$\begin{aligned} C(s) Hgt([\alpha])^{1/4} &= C(s) Hgt([r_{\mathbf{t}} r_{-\mathbf{t}} \alpha])^{1/4} \\ &\leq C(s) e^{(t_1 + \dots + t_d)/2} Hgt([r_1^{-t_1} \dots r_d^{-t_d} \alpha])^{1/4}. \end{aligned}$$

4 This implies that remainder term produces one more term like the bound of (I).
 5 Therefore, the Theorem follows from combining all the terms (I), (II), and remain-
 6 der. \square

7 **Remark.** It is also possible to renormalize a rectangle $U_d(t)$ by several squares
 8 and sum up all of these sub-dimeinsonal ($\dim < d$) squares. However, it may also
 9 involve computational difficulty and such approach of summing up squares may
 10 provide a rough upper bound. To obtain necessary bound for the Lemma 3.4, we
 11 rather simply generalized the methods of Theorem 5.10 in [CF15].

12 Let us set

$$(27) \quad \mathcal{K}_{\alpha, \mathbf{t}, s}(\Gamma) := \left\| \mathcal{R}^{-s}[r_{-\mathbf{t}}(\alpha), (\mathcal{P}_{U_\Gamma}^{d, r_{-\mathbf{t}}(\alpha)} m)] \right\|_{-(s+1)}$$

13 Now we prove the remainder estimate that will be used in (34).

Lemma 3.4. *Let $s > s_{d,g}$. There exists a constant $C(s, \Gamma) > 0$ such that for any rectangle $U_\Gamma = [0, e^{\Gamma_1}] \times \dots \times [0, e^{\Gamma_d}]$,*

$$\mathcal{K}_{\alpha, \mathbf{t}, s}(\Gamma) \leq C(s, \Gamma) Hgt([r_{-\mathbf{t}}(\alpha)])^{1/4}.$$

Proof. By Lemma 3.2, we begin to find the bound of $d-1$ (renormalized) currents. By Theorem 3.3, we obtain the remainder estimate

$$\begin{aligned} \mathcal{K}_{\alpha, \mathbf{t}}(\Gamma) &\leq C \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \int_0^{\Gamma_{i_k}} \dots \int_0^{\Gamma_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^{d-1} \Gamma_l - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \\ &\quad \times Hgt\left([\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}}(r_{-\mathbf{t}}(\alpha))]\right)^{1/4} du_{i_1} \dots du_{i_k}. \end{aligned}$$

It follows from (15) that for $0 \leq k \leq d-1$,

$$Hgt\left(\left[\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}}(r_{-\mathbf{t}}(\alpha))\right]\right)^{1/4} \leq e^{\frac{1}{2}(\sum_{i=1}^k u_{i_l} - \sum_{l=1}^d \Gamma_l)} Hgt\left(\left[r_{-\mathbf{t}}(\alpha)\right]\right)^{1/4}.$$

Then, we obtain

(28)

$$\begin{aligned} \left\| \mathcal{R}^{-s}[r_{-\mathbf{t}}(\alpha), (\mathcal{P}_{U_\Gamma}^{d, r_{-\mathbf{t}} \alpha} m)] \right\|_{-s} &\leq C(s) \left(\sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \prod_{l=1}^k \Gamma_{i_l} \right) Hgt\left(\left[r_{-\mathbf{t}}(\alpha)\right]\right)^{1/4} \\ &\leq C(s, \Gamma) Hgt\left(\left[r_{-\mathbf{t}}(\alpha)\right]\right)^{1/4}. \end{aligned}$$

Therefore, we obtain the conclusion. \square

3.2. Constructions of the functionals. For fixed $\alpha \in \text{Aut}_0(\mathcal{H}^g)$, let $\Pi_H^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \rightarrow A_d(\mathfrak{p}, W_\alpha^{-s}(H))$ denote the orthogonal projection on a single irreducible unitary representation H . We further decompose projection operators with basic current $B_\alpha^{-s, H}$ and its remainder $R_\alpha^{-s, H}$ given by

$$\Pi_H^{-s} = \mathcal{B}_\alpha^{-s}(\Gamma) B_\alpha^{-s, H} + R_\alpha^{-s, H}$$

where $\mathcal{B}_{H, \alpha}^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \rightarrow \mathbb{C}$ denote the orthogonal component map in the direction of basic current, supported on a single irreducible unitary representation.

The *Bufovetov functionals* on a standard rectangle Γ are defined for all $[\alpha] \in DC$ as follows.

Lemma 3.5. *Let $[\alpha] \in DC(L)$. For $s > s_{d, g}$, the limit*

$$\hat{\beta}_H(\alpha, \Gamma) = \lim_{t_1, \dots, t_d \rightarrow \infty} e^{-(t_1 + \dots + t_d)/2} \mathcal{B}_{H, r_{-\mathbf{t}}(\alpha)}^{-s}(\Gamma)$$

exists and it defines a finitely-additive measure on the set of standard rectangles.

Moreover, there exists a constant $C(s, \Gamma) > 0$ such that the following estimate holds:

$$(29) \quad |\Pi_{H, \alpha}^{-s}(\Gamma) - \hat{\beta}_H(\alpha, \Gamma) B_\alpha^H|_{\alpha, -s} \leq C(s, \Gamma)(1 + L).$$

Proof. For simplicity, we omit dependence of H . For every $\mathbf{t} \in \mathbb{R}^d$, we have the following orthogonal splitting:

$$\Pi_{H, \alpha}^{-s}(\Gamma) = \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(\Gamma) B_{\alpha, \mathbf{t}} + R_{\alpha, \mathbf{t}},$$

where

$$\mathcal{B}_{\alpha, \mathbf{t}}^{-s} := \mathcal{B}_{H, r_{-\mathbf{t}}(\alpha)}^{-s}, \quad B_{\alpha, \mathbf{t}} := B_{r_{-\mathbf{t}}(\alpha)}^{-s, H}, \quad R_{\alpha, \mathbf{t}} := R_{r_{-\mathbf{t}}(\alpha)}^{-s, H}.$$

For any $\mathbf{h} \in \mathbb{R}^d$, we have

$$\mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(\Gamma) B_{\alpha, \mathbf{t}+\mathbf{h}} + R_{\alpha, \mathbf{t}+\mathbf{h}} = \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(\Gamma) B_{\alpha, \mathbf{t}} + R_{\alpha, \mathbf{t}}.$$

By reparametrization (12), we have $B_{\mathbf{t}+\mathbf{h}} = e^{-(h_1 + \dots + h_d)/2} B_{\mathbf{t}}$ and

$$(30) \quad \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{(h_1 + \dots + h_d)/2} \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(\Gamma) + \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}})$$

and it follows that

$$\mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{h_1/2} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}).$$

By differentiating at $h_1 = 0$,

$$(31) \quad \frac{d}{dt_1} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \left[\frac{d}{dh_1} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) \right]_{h_1=0}.$$

Therefore, we solve the following first order ODE

$$\frac{d}{dt_1} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \mathcal{K}_{\alpha, \mathbf{t}}^{(1)}(\Gamma)$$

where

$$\mathcal{K}_{\alpha, \mathbf{t}, s}^{(1)}(\Gamma) := \left[\frac{d}{dh_1} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) \right]_{h_1=0}.$$

Then, the solution of the differential equation is

$$\begin{aligned} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) &= e^{t_1/2} \left(\mathcal{B}_{\alpha, 0, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{-\tau_1/2} \mathcal{K}_{\alpha, \tau, s}^{(1)}(\Gamma) d\tau_1 \right) \\ &= e^{t_1/2} \mathcal{B}_{\alpha, 0, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha, \tau, s}^{(1)}(\Gamma) d\tau_1. \end{aligned}$$

Note by reparametrization

$$e^{t_1/2} \mathcal{B}_{\alpha, 0, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) = e^{h_2/2} \mathcal{B}_{\alpha, t_1, t_2, t_3+h_3, \dots, t_d+h_d}^{-s}(\Gamma)$$

1 and it is possible to differentiate the previous equation with respect to h_2 again.

2 Then

$$(32) \quad \frac{d}{dt_2} \mathcal{B}_{\alpha, t_1, t_2, \dots, t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha, t_1, t_2, t_3+h_3, \dots, t_d+h_d}^{-s} + \int_0^{t_1} e^{-\tau_1/2} \mathcal{K}_{\alpha, \tau, s}^{(2)}(\Gamma) d\tau_1$$

3 where $\mathcal{K}_{\alpha, \tau, s}^{(2)}(\Gamma) = \frac{d}{dh_2} \mathcal{K}_{\alpha, \tau, s}^{(1)}(\Gamma)$.

Then, the solution of equation (32) is

$$\begin{aligned} &\mathcal{B}_{\alpha, t_1, t_2, \dots, t_d+h_d}^{-s}(\Gamma) \\ &= e^{t_2/2} \left(\mathcal{B}_{\alpha, t_1, 0, t_3+h_3, \dots, t_d+h_d}^{-s} + \int_0^{t_2} e^{-\tau_2/2} \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha, \tau, s}^{(2)}(\Gamma) d\tau_1 d\tau_2 \right) \\ &= e^{h_3/2} \mathcal{B}_{\alpha, t_1, t_2, t_3, \dots, t_d+h_d}^{-s} + \int_0^{t_2} e^{(t_2-\tau_2)/2} \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha, \tau, s}^{(2)}(\Gamma) d\tau_1 d\tau_2. \end{aligned}$$

4 Inductively, we solve the first order ODE repeatedly and obtain the following

5 solution

$$(33) \quad \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(\Gamma) = e^{(t_1+\dots+t_d)/2} \left(\mathcal{B}_{\alpha, 0}^{-s} + \int_0^{t_d} \dots \int_0^{t_1} e^{-(\tau_1+\dots+\tau_d)/2} \mathcal{K}_{\alpha, \tau, s}^{(d)}(\Gamma) d\tau_1 \dots d\tau_d \right)$$

where

$$\mathcal{K}_{\alpha, \mathbf{t}, s}^{(d)}(\Gamma) = \left[\frac{d}{dh_d} \dots \frac{d}{dh_1} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) \right]_{h_d \dots h_1=0}.$$

Let $\langle \cdot, \cdot \rangle_{\alpha, \mathbf{t}}$ denote the inner product in the space of Hilbert current $A_d(\mathfrak{p}, W_{r-\mathbf{t}(\alpha)}^{-s}(H))$.

By the intertwining formula (11),

$$\begin{aligned} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) &= \langle R_{\alpha, \mathbf{t}}, \frac{B_{\alpha, \mathbf{t}+\mathbf{h}}}{|B_{\alpha, \mathbf{t}+\mathbf{h}}|_{\mathbf{t}+\mathbf{h}}^2} \rangle_{\alpha, \mathbf{t}+\mathbf{h}} \\ &= \langle R_{\alpha, \mathbf{t}} \circ U_{-\mathbf{h}}, \frac{B_{\alpha, \mathbf{t}+\mathbf{h}} \circ U_{-\mathbf{h}}}{|B_{\alpha, \mathbf{t}+\mathbf{h}}|_{\mathbf{t}+\mathbf{h}}^2} \rangle_{\alpha, \mathbf{t}} \\ &= \langle R_{\alpha, \mathbf{t}} \circ U_{-\mathbf{h}}, \frac{B_{\alpha, \mathbf{t}}}{|B_{\alpha, \mathbf{t}}|_{\mathbf{t}}^2} \rangle_{\alpha, \mathbf{t}} = \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(R_{\alpha, \mathbf{t}} \circ U_{-\mathbf{h}}). \end{aligned}$$

In the sense of distributions,

$$\begin{aligned} \frac{d}{dh_d} \cdots \frac{d}{dh_1} (R_{\alpha,t} \circ U_{-\mathbf{h}}) &= -R_{\alpha,t} \circ \left(\frac{d}{2} + \sum_{i=1}^d X_i(t) \right) \circ U_{-\mathbf{h}} \\ &= \left[\left(\sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha,t} \right] \circ U_{-\mathbf{h}}. \end{aligned}$$

Then we compute

$$\left[\frac{d}{dh_d} \cdots \frac{d}{dh_1} (\mathcal{B}_{\alpha,t+\mathbf{h}}^{-s}(R_{\alpha,t})) \right]_{\mathbf{h}=0} = -\mathcal{B}_{\alpha,t}^{-s} \left(\left(\sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha,t} \right).$$

Recall by (27) that $\mathcal{K}_{\alpha,t,s}(\Gamma) = |R_{\alpha,t}|_{r_{-t}(\alpha), -(s+1)}$. By Proposition 2.8 and Lemma 3.4, for any $\epsilon > 0$

$$\begin{aligned} (34) \quad |\mathcal{B}_{\alpha,t}^{-s} \left(\left(\sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha,t} \right)| &\leq \mathcal{K}_{\alpha,t,s}(\Gamma) \\ &\leq C(s, \Gamma) \text{Hgt}([r_{-t}(\alpha)])^{1/4} \leq C(s, \Gamma) (t_1 \cdots t_d)^{\frac{d}{2(g+1)} + \epsilon}. \end{aligned}$$

Therefore, the solution of equation (33) exists under Diophantine condition (20) and the following holds:

$$\lim_{t_1, \dots, t_d \rightarrow \infty} e^{-(t_1 + \cdots + t_d)/2} \mathcal{B}_{\alpha,t}^{-s}(\Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

Moreover, the complex number

$$\hat{\beta}_H(\alpha, \Gamma) = \mathcal{B}_{\alpha,0}^{-s} + \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots + \tau_d)/2} \mathcal{K}_{\alpha,\tau,s}(\Gamma) d\tau_1 \cdots d\tau_d$$

depends continuously on $\alpha \in DC(L)$. Since we have

$$\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B_\alpha^H = R_0 - \left(\int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots + \tau_d)/2} \mathcal{K}_{\alpha,\tau,s}(\Gamma) d\tau_1 \cdots d\tau_d \right) B_\alpha^H,$$

by Diophantine condition again,

$$|\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}_H(\alpha, \Gamma) B_\alpha^H|_{\alpha, -s} \leq C(s, \Gamma)(1 + L).$$

1

□

2 **3.3. Proof of Theorem 1.3.** The proof of Theorem 1.3 follows immediately from
 3 the refinement to the constructions of Bufetov functionals (see also [BF14, §2.5] for
 4 horocycle flows).

5 *Notation.* The action of flow $\{r_t\}_{t \in \mathbb{R}}$ on a current \mathcal{C} is defined by pull-back as
 6 follows:

$$(r_t^* \mathcal{C})(\omega) = \mathcal{C}(r_{-t}^* \omega), \quad \text{for any smooth form } \omega.$$

Lemma 3.6 (Invariance). *Let $1 \leq d \leq g$. The functional $\hat{\beta}_H$ defined on d -standard rectangle $\Gamma = \Gamma_T^X$ is invariant under the action of $(Q_y^{j,Y})$ for any $y \in \mathbb{R}_+^j$ and $1 \leq j \leq d$. That is,*

$$\hat{\beta}_H(\alpha, (Q_y^{j,Y})_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

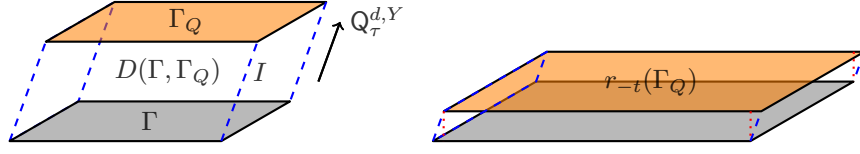


FIGURE 1. Illustration of the standard d -rectangles Γ , Γ_Q , $d+1$ dimensional current $D(\Gamma, \Gamma_Q)$ and supports of $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_Q)$.

1 *Proof.* We will prove the functional $\hat{\beta}_H$ exists with respect to rectangle $(Q_y^{j,Y})_*\Gamma$
 2 and prove its invariance under the action $Q_y^{j,Y}$. It suffices to verify the invariance
 3 property under the rank 1 action $Q_\tau^{1,Y}$ for $\tau \in \mathbb{R}$ since we can apply the statement
 4 for d -rank actions repeatedly.

5 Given a standard d -dimensional rectangle Γ , set $\Gamma_Q := (Q_\tau^{1,Y})_*\Gamma$. Let $D(\Gamma, \Gamma_Q)$
 6 be the $(d+1)$ dimensional space spanned by trajectories of the action of $Q_\tau^{1,Y}$
 7 projecting Γ onto Γ_Q . Then $D(\Gamma, \Gamma_Q)$ is a union of all orbits I of action $Q_\tau^{1,Y}$ such
 8 that the boundary of I , d -dimensional faces, is contained in $\Gamma \cup \Gamma_Q$, and interior of I
 9 is disjoint from $\Gamma \cup \Gamma_Q$. $D(\Gamma, \Gamma_Q)$ is defined by integration and it is a $(d+1)$ -current.

For convenience, let us denote renormalization flow by $r_t := r_i^t$ for some i -th
 coordinates. Then, $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_Q)$ are respectively the support of the currents
 $r_t^*\Gamma$ and $r_t^*\Gamma_Q$. Thus, we have the following identity

$$r_t^*D(\Gamma, \Gamma_Q) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q)).$$

10 Since the current $\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)$ is composed of orbits for the action $Q_\tau^{1,Y}$,
 11 it follows that

$$(35) \quad \partial[r_t^*D(\Gamma, \Gamma_Q)] - (r_t^*\Gamma - r_t^*\Gamma_Q) = r_t^*[\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)] \rightarrow 0.$$

Now, we turn to prove the volume of $D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q))$ is uniformly bounded
 for all $t > 0$. For any $p \in \Gamma$, set $\tau(p)$ be length of the arc lying in $D_Q := D(\Gamma, \Gamma_Q)$,
 and set $\tau_\Gamma := \sup\{\tau(p) \mid p \in \Gamma\} < \infty$. We write

$$vol_{d+1}(D_Q) = \int_\Gamma \tau dvol_d.$$

12 Since $vol_d(r_{-t}(\Gamma)) \leq e^t vol_d(\Gamma)$,

$$(36) \quad vol_{d+1}(r_{-t}(D_Q)) = \int_{r_{-t}(\Gamma)} \tau dvol_d \leq \tau_\Gamma e^{-t} vol_d(r_{-t}(\Gamma)) \leq \tau_\Gamma vol_d(\Gamma) < \infty.$$

13 Note that d -dimensional current $(Q_\tau^{1,Y})_*\Gamma - \Gamma$ is equal to the boundary of the
 14 $(d+1)$ dimensional current D_Q . By arguments in remainder estimate (or Sobolev
 15 embedding theorem),

$$(37) \quad |(Q_\tau^{1,Y})_*\Gamma - \Gamma|_{r^{-t}\alpha, -s} \leq C_s \tau B_s([r^{-t}\alpha]) \leq C_s \tau \text{Hgt}([r^{-t}\alpha])^{1/4}$$

16 is finite for all $t > 0$.

Then, by (35), (36) and existence of Bufetov functional $\hat{\beta}_H(\alpha, \Gamma)$, the last in-
 equality holds:

$$|\mathcal{B}_{\alpha, t}^{-s}((Q_\tau^{1,Y})_*\Gamma) - \mathcal{B}_{\alpha, t}^{-s}(\Gamma)|_{\alpha, -s} < \infty.$$

17 Therefore, by the definition of Bufetov functional in the Lemma 3.5, $\hat{\beta}_H(\alpha, (Q_\tau^{1,Y})_*\Gamma)$
 18 exists and $\hat{\beta}_H(\alpha, \Gamma)$ is invariant under the action of $Q_\tau^{1,Y}$. \square

1 *Proof of Theorem 1.3. Additive property.* It follows from the linearity of projections
 2 and limit.

3 *Scaling property.* It is immediate from the definition.

Bounded property. By the scaling property, for $r_t = r_1^t \cdots r_d^t$ with $t > 0$,

$$\hat{\beta}_H(\alpha, \Gamma) = e^{dt/2} \hat{\beta}_H(r_t(\alpha), \Gamma).$$

Choose $t = \log(\int_\Gamma |\hat{X}|)$ and $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$, then Bufetov functional on the rectangle Γ is bounded:

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma) \left(\int_\Gamma |\hat{X}| \right)^{d/2}.$$

4 *Invariance property.* Now it follows directly from the Lemma 3.6. \square

5 *Notation.* We write for $\mathbf{T} = (T^{(i)}) \in \mathbb{R}^d$ and $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$,

$$(38) \quad \text{vol}(U(\mathbf{T})) := \prod_{i=1}^d T^{(i)}, \quad \text{vol}(U(\mathbf{t})) = \prod_{i=1}^d t_i.$$

6 Now we extend the properties of functional $\hat{\beta}_H$ to cocycle β_H .

7 *Proof of Corollary 1.5.* Cocycle property of β_H follows from the additive property
 8 of $\hat{\beta}_H$. Scaling and bounded properties are immediate. Denote $\mathbf{tT} := (t_1 T^{(1)}, \dots, t_d T^{(d)}) \in$
 9 \mathbb{R}_+^d , and we obtain

$$(39) \quad \beta_H(\alpha, m, \mathbf{tT}) = \text{vol}(U(\mathbf{T}))^{1/2} \beta_H(r_{\log \mathbf{T}}(\alpha), m, \mathbf{t}).$$

By Lemma 3.5 and scaling property, we have

$$\begin{aligned} D_\alpha^H(f) \beta_H(\alpha, m, \mathbf{t}) &= \lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \beta_H(r_{-\log \mathbf{t}}(\alpha), m, \mathbf{tT}) \\ &= \lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \left\langle \mathcal{D}_{U(\mathbf{tT})}^{d, r_{-\log \mathbf{t}}(\alpha)} m, \omega_{f_H} \right\rangle. \end{aligned}$$

10 It follows that $\beta_H(\alpha, \cdot, \mathbf{t}) \in H$ as a point-wise limit of Birkhoff integrals. This
 11 implies *Orthogonal property*. \square

12 For $\gamma \in Sp_{2g}(\mathbb{Z})$, we have

$$\beta_H(\gamma\alpha, \gamma(m), \mathbf{T}) = \beta_H(\alpha, m, \mathbf{T}).$$

13 It means that the function $\beta_H(\cdot, m, \mathbf{T})$ is well-defined on the moduli space \mathfrak{M}_g .

3.4. Proof of Theorem 1.6. We define the excursion function

$$\begin{aligned} E_{\mathfrak{M}}(\alpha, \mathbf{T}) &:= \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1 + \cdots + t_d)/2} \text{Hgt}([r_{\mathbf{t} - \log \mathbf{T}}(\alpha)])^{1/4} dt_1 \cdots dt_d \\ &= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1 + \cdots + t_d)/2} \text{Hgt}([r_{\mathbf{t}}(\alpha)])^{1/4} dt_1 \cdots dt_d. \end{aligned}$$

Lemma 3.7 (Bound of cocycle). *For any Diophantine $[\alpha] \in DC(L)$ and for any $f \in W^s(M)$ for $s > s_{d,g} + 1/2$, the Bufetov cocycle β^f is defined by a uniformly convergent series:*

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_s \left(L + \text{vol}(U(\mathbf{T}))^{1/2} (1 + \text{vol}(U(\mathbf{t})) + E_{\mathfrak{M}}(\alpha, \mathbf{T})) \right) \|f\|_{\alpha, s}.$$

Proof. Recall that by (7),

$$\beta^f(\alpha, m, \mathbf{T}) = \sum_H D_\alpha^H(f) \beta_H(\alpha, m, \mathbf{T}).$$

- 1 It follows from Lemma 3.5 that there exists a constant $C > 0$ such that whenever
 2 $[\alpha] \in DC(L)$, then

$$(40) \quad |\beta_H(\alpha, m, \mathbf{t})| \leq C(1 + L + \text{vol}(U(\mathbf{t}))), \quad (m, \mathbf{t}) \in M \times \mathbb{R}_+^d.$$

By Diophantine condition (20), whenever $[\alpha] \in DC(L)$ then $[r_{\log \mathbf{T}}(\alpha)] \in DC(L_{\mathbf{T}})$ with

$$L_{\mathbf{T}} \leq L \text{vol}(U(\mathbf{T}))^{-1/2} + E_{\mathfrak{M}}(\alpha, \mathbf{T}).$$

Thus by (40), we obtain for all $(m, \mathbf{t}) \in M \times \mathbb{R}^d$,

$$|\beta_H(r_{\log \mathbf{T}}(\alpha), m, \mathbf{t})| \leq C(1 + L_{\mathbf{T}} + \text{vol}(U(\mathbf{t}))).$$

By the scaling property (39), it follows that for all $s > s_{d,g}$ and $q > 1/2$

$$\begin{aligned} |\beta^f(\alpha, m, \mathbf{tT})| &\leq C_s \text{vol}(U(\mathbf{T}))^{1/2} (1 + L + \text{vol}(U(\mathbf{t}))) \sum_{n \in \mathbb{Z}} \|f_n\|_{\alpha, s} \\ &\leq C_s \text{vol}(U(\mathbf{T}))^{1/2} (1 + L_{\mathbf{T}} + \text{vol}(U(\mathbf{t}))) \left(\sum_{n \in \mathbb{Z}} (1 + n^2)^q \right)^{-1/2} \\ &\quad \times \left(\sum_{n \in \mathbb{Z}} \left\| (1 - Z^2)^{q/2} f_n \right\|_{\alpha, s}^2 \right)^{1/2}. \end{aligned}$$

Therefore, for all $s' = s_{d,g} + q$, there exists a constant $C_{s'} > 0$ such that

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_{s'} \text{vol}(U(\mathbf{T}))^{1/2} (1 + L_{\mathbf{T}} + \text{vol}(U(\mathbf{t}))) \|f\|_{\alpha, s'}.$$

3

□

- 4 By Lemma 3.5, 3.7 and identification of the norm for the form ω_f , asymptotic
 5 formula on each irreducible component provides the following corollary.

- 6 **Corollary 3.8.** *For all $s > s_{d,g} + 1/2$, there exists a constant $C_s > 0$ such that for*
 7 *all $[\alpha] \in DC(L)$, for all $f \in W_\alpha^s(M)$ and for all $(m, \mathbf{T}) \in M \times \mathbb{R}^d$, we have*

$$(41) \quad \left| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha} m, \omega_f \right\rangle - \beta^f(\alpha, m, \mathbf{T}) \right| \leq C_s (1 + L) \|\omega_f\|_{\alpha, s}$$

- 8 *for $U(\mathbf{T}) = [0, T^{(1)}] \times \cdots \times [0, T^{(d)}]$ and $\omega_f = f \omega^{d, \alpha} \in \Lambda^d \mathfrak{p} \otimes W_\alpha^s(M)$.*

- 9 *Proof of Theorem 1.6.* The theorem follows from Corollary 3.8 for $\alpha \in \bigcup_{L>0} DC(L)$.
 10 □

11

4. LIMIT DISTRIBUTIONS

- 12 In this section, we prove Theorem 1.7, limit distribution of Birkhoff sums of
 13 higher rank actions on rectangles.

1 **4.1. Limiting distributions.**

2 **Lemma 4.1.** *There exists a continuous modular function $\theta_H : \text{Aut}_0(\mathbf{H}^g) \rightarrow H \subset$*
 3 *$L^2(M)$ such that for any $\omega_f = f\omega^{d,\alpha} \in \Lambda^d \mathfrak{p} \otimes W_\alpha^s(H)$ with $s > d/2$,*

$$(42) \quad \lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\cdot), \omega_f \right\rangle - \theta_H(r_{\log \mathbf{T}}(\alpha)) D_\alpha^H(f) \right\|_{L^2(M)} = 0.$$

4 *The family $\{\theta_H(\alpha) \mid \alpha \in \text{Aut}_0(\mathbf{H}^g)\}$ has a constant norm in $L^2(M)$.*

Proof. By the Fourier transform, the space of smooth vectors and Sobolev space $W^s(H)$ is represented as the Schwartz space $\mathcal{S}^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \left| \left(1 + \sum_{i=1}^d \left(\frac{\partial^2}{\partial u_i^2} + u_i^2 \right) \right)^{s/2} \hat{f}(u) \right|^2 du < \infty.$$

Let $\mathbf{t}, \mathbf{u} \in \mathbb{R}^d$. Then we claim for any $f \in \mathcal{S}^s(\mathbb{R}^d)$, there exists a function $\theta(\alpha)(\cdot) \in L^2(\mathbb{R}^d)$ such that

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \int_{U(\mathbf{T})} f(\mathbf{u} + \mathbf{t}) d\mathbf{t} - \theta_H(r_{\log \mathbf{T}}(\alpha))(\mathbf{u}) \text{Leb}(f) \right\|_{L^2(\mathbb{R}^d, d\mathbf{u})} = 0.$$

This is equivalent to the statement (42). By the standard Fourier transform on \mathbb{R}^d , equivalently

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \int_{U(\mathbf{T})} e^{i\mathbf{t} \cdot \hat{\mathbf{u}}} \hat{f}(\hat{\mathbf{u}}) d\mathbf{t} - \hat{\theta}_H(r_{\log \mathbf{T}}(\alpha))(\hat{\mathbf{u}}) \hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{\mathbf{u}})} = 0.$$

For $\chi \in L^2(\mathbb{R}^d, d\hat{\mathbf{u}})$ and $\hat{\mathbf{u}} = (\hat{u}_j)_{1 \leq j \leq d}$, we denote

$$\chi_j(\hat{\mathbf{u}}) = \frac{e^{i\hat{u}_j} - 1}{i\hat{u}_j}, \quad \chi(\hat{\mathbf{u}}) = \prod_{j=1}^d \chi_j(\hat{\mathbf{u}}).$$

Let $\hat{\theta}(\alpha)(\hat{\mathbf{u}}) := \chi(\hat{\mathbf{u}})$ for all $\hat{\mathbf{u}} \in \mathbb{R}^d$. Now we will compute $\theta(r_{\log \mathbf{T}}(\alpha))$. By intertwining formula (11) for $\mathbf{T} \in \mathbb{R}^d$,

$$U_{\mathbf{T}}(f)(\hat{\mathbf{u}}) = \prod_{i=1}^d (T^{(i)})^{1/2} f(\mathbf{T}\hat{\mathbf{u}}), \text{ for } \mathbf{T}\hat{\mathbf{u}} = (T^{(1)}\hat{u}_1, \dots, T^{(d)}\hat{u}_d).$$

Then, for all $\alpha \in \text{Aut}_0(\mathbf{H}^g)$,

$$\hat{\theta}(r_{\log \mathbf{T}}(\alpha))(\hat{\mathbf{u}}) = U_{\mathbf{T}}(\chi)(\hat{\mathbf{u}}) = \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{\mathbf{u}}).$$

The function $\theta(\alpha)$ is defined by inverse Fourier transform of $\hat{\theta}(\alpha)$ and

$$\|\theta_H(\alpha)\|_H = \|\theta(\alpha)\|_{L^2(\mathbb{R}^d)} = \|\hat{\theta}(\alpha)\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{\mathbf{u}})\|_{L^2(\mathbb{R}^d, d\hat{\mathbf{u}})} = C > 0.$$

By integration,

$$(43) \quad \int_0^{T^{(d)}} \dots \int_0^{T^{(1)}} e^{i\mathbf{t} \cdot \hat{\mathbf{u}}} \hat{f}(\hat{\mathbf{u}}) d\mathbf{t} = \text{vol}(U(\mathbf{T})) \chi(\mathbf{T}\hat{\mathbf{u}}) \hat{f}(\hat{\mathbf{u}}) \\ = \text{vol}(U(\mathbf{T})) \chi(\mathbf{T}\hat{\mathbf{u}}) (\hat{f}(\hat{\mathbf{u}}) - \hat{f}(0)) + \text{vol}(U(\mathbf{T}))^{1/2} \hat{\theta}(r_{\log \mathbf{T}}(\alpha))(\hat{\mathbf{u}}) \hat{f}(0).$$

Then the claim reduces to the following:

$$\limsup_{|U(\mathbf{T})| \rightarrow \infty} \left\| \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{\mathbf{u}})(\hat{f}(\hat{\mathbf{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If $f \in \mathcal{S}^s(\mathbb{R}^g)$ with $s > d/2$, function $\hat{f} \in C^0(\mathbb{R}^d)$ and bounded. Thus, by Dominated convergence theorem and change of variables,

$$\left\| \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{\mathbf{u}})(\hat{f}(\hat{\mathbf{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{\mathbf{u}})} = \left\| \chi(\nu)(\hat{f}(\frac{\nu}{\mathbf{T}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\nu)} \rightarrow 0.$$

1

□

Corollary 4.2. *For any $s > d/2$, $\alpha \in \text{Aut}_0(\mathbb{H}^g)$ and $f \in W_\alpha^s(H)$, there exists a constant $C > 0$ such that*

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} m, \omega_f \right\rangle \right\|_{L^2(M)} = C |D_\alpha^H(f)|.$$

2

From Corollary 4.2, we derive the following limit result for the L^2 -norm of Bufe-

3

tov functionals.

Corollary 4.3. *For every irreducible component H and $[\alpha] \in DC$, there exists $C > 0$ such that*

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \|\beta_H(\alpha, \cdot, \mathbf{T})\|_{L^2(M)} = C.$$

Proof. By the normalization of invariant distribution in Sobolev space, for any $\alpha \in \text{Aut}_0(\mathbb{H}^g)$, there exists a function $f_\alpha^H \in W_\alpha^s(H)$ such that $D_\alpha(f_\alpha^H) = \|f_\alpha^H\|_s = 1$. For all $[\alpha] \in DC(L)$, by asymptotic formula (41) for $f = f_H$,

$$\left| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} m, \omega_f \right\rangle - \beta^f(\alpha, m, \mathbf{T}) \right| \leq C_s(1 + L).$$

4

Therefore, L^2 -estimate follows from Corollary 4.2.

□

5

A relation between the Bufetov functional and the modular function θ_H is established below.

6

Corollary 4.4. *For every irreducible component $H \subset L^2(M)$, the following holds. For any $L > 0$ and any $r_{\mathbf{t}}$ -invariant probability measure μ supported on $DC(L) \subset \mathfrak{M}_g$,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H(\alpha)(\cdot), \quad \text{for } \mu\text{-almost all } [\alpha] \in \mathfrak{M}_g.$$

7

Proof. By Theorem 1.6 and Lemma 4.1, there exists a constant $C > 0$ such that for all $[\alpha] \in \text{supp}(\mu) \subset DC(L)$ and $\mathbf{T} \in \mathbb{R}_+^d$, we have

8

$$(44) \quad \lim_{|U(\mathbf{T})| \rightarrow \infty} \|\beta_H(r_{\log \mathbf{T}}(\alpha), \cdot, 1) - \theta_H(r_{\log \mathbf{T}}(\alpha))\|_{L^2(M)} \leq \frac{C_\mu}{\text{vol}(U(\mathbf{T}))^{1/2}}.$$

9

By Luzin's theorem, for any $\delta > 0$ there exists a compact subset $E(\delta) \subset \mathfrak{M}$ such that we have the measure bound $\mu(\mathfrak{M} \setminus E(\delta)) < \delta$ and the function $\beta_H(\alpha, \cdot, 1) \in L^2(M)$ depends continuously on $[\alpha] \in E(\delta)$. By Poincaré recurrence, there is a full measure set $F \subset \mathfrak{M}$ for \mathbb{R}^d -action. Denote by a full measure set $E'(\delta) = E(\delta) \cap F \subset E(\delta)$.

10

11

12

For every $\alpha_0 \in E'(\delta)$, there is diverging sequence (t_n) such that $\{r_{t_n}(\alpha_0)\} \subset E(\delta)$ and $\lim_{n \rightarrow \infty} r_{t_n}(\alpha_0) = \alpha_0$. By continuity of θ_H and β_H at α_0 , we have

$$(45) \quad \|\beta_H(\alpha_0, \cdot, 1) - \theta_H(\alpha_0)\|_{L^2(M)} = \lim_{n \rightarrow \infty} \|\beta_H(r_{t_n}(\alpha_0), \cdot, 1) - \theta_H(r_{t_n}(\alpha_0))\|_{L^2(M)} = 0.$$

Thus, we have $\beta_H(\alpha, \cdot, 1) = \theta_H(\alpha) \in L^2(M)$ for all $\alpha \in E'(\delta)$. It follows that the set where the equality (45) fails has a measure less than any $\delta > 0$, thus the identity holds for μ -almost all $\alpha \in \text{Aut}_0(\mathbf{H}^g)$. \square

For all $\alpha \in \text{Aut}_0(\mathbf{H}^g)$, smooth function $f \in W^s(M)$ for $s > s_{d,g} + 1/2$ decompose as an infinite sum, and the functional θ^f is defined by a convergent series

$$(46) \quad \theta^f(\alpha) := \sum_H D_\alpha^H(f) \theta_H(\alpha).$$

Hence, the modular function $\theta^f : \text{Aut}_0(\mathbf{H}^g) \rightarrow L^2(M)$ is continuous.

The following result is an extension of Lemma 4.1 to an asymptotic formula.

Lemma 4.5. *For all $\alpha \in \text{Aut}_0(\mathbf{H}^g)$, $f \in W^s(M)$ and $s > s_{d,g} + 1/2$,*

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}_n))^{1/2}} \left\langle \mathcal{P}_{U(\mathbf{T}_n)}^{d,\alpha} m, \omega_f \right\rangle - \theta^f(r_{\log \mathbf{T}_n}(\alpha)) \right\|_{L^2(M)} = 0.$$

4.2. Proof of Theorem 1.7. We summarize our results on limit distributions for higher rank actions.

Theorem 4.6. *Let (\mathbf{T}_n) be any sequence such that*

$$\lim_{n \rightarrow \infty} r_{\log \mathbf{T}_n}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_{d,g} + 1/2$, which is not a coboundary, the limit distribution of the family of random variables

$$E_{T_n}(f) := \frac{1}{\text{vol}(U(\mathbf{T}_n))^{1/2}} \left\langle \mathcal{P}_{U(\mathbf{T}_n)}^{d,\alpha}(\cdot), \omega_f \right\rangle$$

exists and is equal to the distribution of the function $\theta^f(\alpha_\infty) = \beta(\alpha_\infty, \cdot, 1) \in L^2(M)$. If $\alpha_\infty \in DC$, then $\theta^f(\alpha_\infty)$ is a bounded function on M , and the limit distribution has compact support.

Proof of Theorem 1.7. Since $\alpha_\infty \in \mathfrak{M}_g$, the existence of limit follows from the Lemma 4.5. \square

A relation with Birkhoff integrals and theta sum was introduced in [CF15, §5.3], and as an application, we derive the limit theorem of theta sums.

Corollary 4.7. *Let $\mathcal{Q}[x] = x^\top \mathcal{Q} x$ be the quadratic forms defined by $g \times g$ real matrix \mathcal{Q} , where $\alpha = \begin{pmatrix} I & 0 \\ \mathcal{Q} & I \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{R})$ and $\ell(x) = \ell^\top x$ is the linear form defined by $\ell \in \mathbb{R}^g$. Then the theta sum*

$$\Theta(\mathcal{Q}, \ell; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0, N]} \exp(2\pi i(\mathcal{Q}[n] + \ell(n)))$$

has a limit distribution and it has compact support.

5. L^2 -LOWER BOUNDS

In this section we prove L^2 - lower bounds of ergodic integrals on transverse torus.

5.1. Structure of return map. Given the standard frame (X_i, Y_i, Z) , $(g+1)$ -dimensional torus is denoted by

$$\mathbb{T}_\Gamma^{g+1} := \{\Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

It is convenient to work with the polarized Heisenberg group. Set $\mathbf{H}_{pol}^g \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ equipped with the group law $(x, y, z) \cdot (x', y', z') = (x+x', y+y', z+z'+yx')$.

Definition 5.1. Reduced standard Heisenberg group \mathbf{H}_{red}^g is defined by quotient $\mathbf{H}_{pol}^g / (\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R} / \frac{1}{2}\mathbb{Z}$. Reduced standard lattice Γ_{red}^g is $\mathbb{Z}^g \times \mathbb{Z}^g \times \{0\}$ and the quotient $\mathbf{H}_{red}^g / \Gamma_{red}^g$ is isomorphic to the standard Heisenberg manifold \mathbf{H}^g / Γ .

Now, we consider a return map of $\mathbf{P}^{d,\alpha}$ on \mathbb{T}_Γ^{g+1} . For $x = (x_1, \dots, x_g) \in \mathbb{R}^g$, we write in the coordinate of \mathbf{H}_{red}^g

$$\exp(x_1 X_1^\alpha + \dots + x_g X_g^\alpha) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta, \text{ and } w \in \mathbb{R}^d.$$

Then by group law,

$$\exp(x_1 X_1^\alpha + \dots + x_g X_g^\alpha) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x)$$

and given $(n, m, 0) \in \Gamma_{red}^g$,

$$(47) \quad \exp(x_1 X_1^\alpha + \dots + x_g X_g^\alpha) \cdot (0, y, z) \cdot (n, m, 0) = \exp(x'_1 X_1^\alpha + \dots + x'_g X_g^\alpha) \cdot (0, y', z')$$

if and only if

$$x'_\alpha = x_\alpha + n, y' = y + (x_\beta - x'_\beta) + m \text{ and } z' = z + (w - w') \cdot x + n^\top (y + x_\beta).$$

Assume $\langle X_i^\alpha, X_j \rangle \neq 0$ for all i, j , and we write the first return time for $\mathbf{P}^{d,\alpha}$ action

$$t_{Ret} = (t_{Ret,1}, \dots, t_{Ret,d}) \in \mathbb{R}^d$$

on transverse torus \mathbb{T}_Γ^{g+1} . We denote the domain for return time $U(t_{Ret}) = [0, t_{Ret,1}] \times \dots \times [0, t_{Ret,d}]$. Return map of action $\mathbf{P}^{d,\alpha}$ on \mathbb{T}_Γ^{g+1} has a form of skew-shift

$$(48) \quad A_{\rho,\tau}(y, z) = (y + \rho, z + v \cdot y + \tau) \text{ on } \mathbb{R}^g / \mathbb{Z}^g \times \mathbb{R} / K^{-1}\mathbb{Z}.$$

for some non-zero vectors $\rho, v \in \mathbb{R}^g$ and $\tau \in \mathbb{R}$.

Furthermore, $A_{\rho,\tau} = A_{d,\rho,\tau} \circ \dots \circ A_{1,\rho,\tau}$ decomposes with commuting linear skew-shifts

$$(49) \quad A_{i,\rho,\tau}(y, z) = (y + \rho_i, z + v_i \cdot y + \tau_i) \text{ on } \mathbb{R}^g / \mathbb{Z}^g \times \mathbb{R} / K^{-1}\mathbb{Z}$$

for some $\rho_i, v_i \in \mathbb{R}^g$ and $\tau_i \in \mathbb{R}$. For each $j \neq k$, it is easily verified that

$$A_{j,\rho,\tau} \circ A_{k,\rho,\tau} = A_{k,\rho,\tau} \circ A_{j,\rho,\tau}.$$

Given pair $(\mathbf{m}, n) \in \mathbb{Z}_{K|n|}^g \times \mathbb{Z}$, let $H_{(\mathbf{m},n)}$ denote the corresponding factor and $C^\infty(H_{(\mathbf{m},n)})$ be a subspace of smooth functions on $H_{(\mathbf{m},n)}$. Denote $\{e_{\mathbf{m},n} \mid (\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z}\}$ the basis of characters on \mathbb{T}_Γ^{g+1} and for all $(y, z) \in \mathbb{T}^g \times \mathbb{T}$,

$$e_{\mathbf{m},n}(y, z) := \exp[2\pi i(\mathbf{m} \cdot y + nKz)].$$

For each $A_{i,\rho,\sigma}$, we set $\mathbf{v}_i = (v_{i1}, \dots, v_{ig}) \in \mathbb{Z}_{K|n|}^g$. Then the orbit can be identified with the following dual orbit

$$\begin{aligned}\mathcal{O}_{A_i}(\mathbf{m}, n) &= \{(\mathbf{m} + (nj_i)\mathbf{v}_i, n), j_i \in \mathbb{Z}\} \\ &= \{(m_1 + (nv_{i1})j_i, \dots, m_g + (nv_{ig})j_i, n), j_i \in \mathbb{Z}\}.\end{aligned}$$

If $n = 0$, the orbit $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$ of $(\mathbf{m}, 0)$ is reduced to a single element. If $n \neq 0$, then the dual orbit $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$ of (\mathbf{m}, n) for higher rank actions is described as follows:

$$\mathcal{O}_A(\mathbf{m}, n) = \{(m_k + n \sum_{i=1}^d (v_{ik}j_i), n)_{1 \leq k \leq d} : j = (j_1, \dots, j_d) \in \mathbb{Z}^d\}.$$

It follows that every A -orbit for rank \mathbb{R}^d -action (or A_i -orbit) can be labeled uniquely by a pair $(\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \setminus \{0\}$ with $\mathbf{m} = (m_1, \dots, m_g)$. Thus, the subspace of functions with non-zero central characters splits as a direct sum of components $H_{(\mathbf{m}, n)}$

$$L^2(\mathbb{T}_\Gamma^{g+1}) = \bigoplus_{\omega \in \mathcal{O}_A} H_\omega, \quad \text{where } H_\omega = \bigoplus_{(\mathbf{m}, n) \in \omega} \mathbb{C}e_{(\mathbf{m}, n)}.$$

- 1 **5.2. Higher cohomology for \mathbb{Z}^d -action of skew-shifts.** In this subsection, we
 2 will find a relations with the return map for \mathbb{Z}^d action of $\mathbb{P}^{d,\alpha}$ on the torus \mathbb{T}_Γ^{g+1}
 3 and obstructions to cohomological equation $\omega = d\Omega$.
 4 We will restrict our interest to the following cocycle equation

$$(50) \quad \varphi(x, t) = \mathfrak{D}\Phi(x, t), \quad x \in \mathbb{T}^g, t \in \mathbb{Z}^d$$

where d -cocycle $\varphi : \mathbb{T}_\Gamma^{g+1} \times \mathbb{Z}^d \rightarrow \mathbb{R}$, $(d-1)$ cochain $\Phi : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathbb{R}^d$, $\Phi = (\Phi_1, \dots, \Phi_d)$, and \mathfrak{D} is coboundary operator

$$\mathfrak{D}\Phi = \sum_{i=1}^d (-1)^{i+1} \Delta_i \Phi_i$$

- 5 where $\Delta_i \Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$.
 6 In the work of Katok [KK95, §2], he proved the existence of solutions for cocycle
 7 equations by studying the dual equations $\hat{\varphi} = \mathfrak{D}\hat{\Phi}$ in the space of Fourier coeffi-
 8 cients (dual orbit). We apply this result to our A -orbit and verify the invariant
 9 distributions (or currents) explicitly.

10 **Proposition 5.2.** [KK95, Proposition 2.2] *A cocycle $\hat{\varphi}$ satisfies cocycle equation*
 11 *(50) if and only if $\sum_{j \in \mathbb{Z}^d} \hat{\varphi}(\mathbf{m}, n) \circ A^j = 0$.*

12 For fixed $(\mathbf{m}, n) \in \mathbb{Z}^g \times \mathbb{Z}$, we denote an obstruction of cohomological equation
 13 restricted to the A -orbit of (\mathbf{m}, n) by $\mathcal{D}_{\mathbf{m}, n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \hat{\varphi}(\mathbf{m}, n) \circ A^j$. Since A^j is
 14 composition of commuting toral automorphisms, we obtain the following generalized
 15 formula (see [AFU11, §5] and [Ka03, §11] for rank 1 map on \mathbb{T}^2).

16 **Lemma 5.3.** *There exists a distributional obstruction to the existence of a smooth*
 17 *solution $\varphi \in C^\infty(H_{(\mathbf{m}, n)})$ of the cohomological equation (50).*

A generator of the space of invariant distribution $\mathcal{D}_{\mathbf{m}, n}$ has form of

$$\mathcal{D}_{\mathbf{m}, n}(e_{a,b}) := \begin{cases} e^{-2\pi i \sum_{i=1}^d [(m_i \rho_i + n K \tau_i) j_i + n K \tau_i (\frac{j_i}{2})]} & \text{if } (a, b) = (m_k + K \sum_{i=1}^d (v_{ik} j_i), n)_{1 \leq k \leq g} \\ 0 & \text{otherwise.} \end{cases}$$

1 *Proof.* From previous observation, there exists an obstruction

$$(51) \quad \mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}_\Gamma^{g+1}} \varphi(x, y) \overline{e_{\mathbf{m},n} \circ A_{\rho,\tau}^j} dx dy.$$

By direct computation, for fixed $j = (j_1, \dots, j_d)$,

$$e_{\mathbf{m},n} \circ A_{\rho,\tau}^j(y, z) = \prod_{i=1}^d \left(e^{2\pi i [(\mathbf{m} \cdot \rho_i + nK\tau_i)j_i + nK\tau_i \binom{j_i}{2}]} \right) \left(e^{2\pi i (\mathbf{m} \cdot y + K(z + n \sum_{k=1}^d (v_{ik}j_i)y_k))} \right).$$

2 Then, we choose $\hat{\varphi} = e_{a,b}$ for $(a, b) = (m_k + K \sum_{i=1}^d (v_{ik}j_i), n)_{1 \leq k \leq g}$ in the non-
3 trivial orbit ($n \neq 0$),

$$(52) \quad \mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi i \sum_{i=1}^d [(\mathbf{m} \cdot \rho_i + nK\tau_i)j_i + nK\tau_i \binom{j_i}{2}]}.$$

4

□

5.3. Changes of coordinates. For any frame $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$, denote transverse cylinder for any $m \in M$,

$$\mathcal{C}_{\alpha,m} := \{m \exp(\sum_{i=1}^g y'_i Y_i^\alpha + z'Z) \mid (y', z') \in U(t_{Ret}^{-1}) \times \mathbb{T}\}.$$

Let $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathcal{C}_{\alpha,m}$ denote the maps: for any $\xi \in \mathbb{T}_\Gamma^{g+1}$, let $\xi' \in \mathcal{C}_{\alpha,m}$ denote first intersection of the orbit $\{\mathbf{P}_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}_+^d\}$ with transverse cylinder $\mathcal{C}_{\alpha,m}$. Then, there exists a first return time to the cylinder $t(\xi) = (t_1(\xi), \dots, t_d(\xi)) \in \mathbb{R}_+^d$ such that

$$\xi' = \Phi_{\alpha,m}(\xi) = \mathbf{P}_{t(\xi)}^{d,\alpha}(\xi), \quad \forall \xi \in \mathbb{T}_\Gamma^{g+1}.$$

Let (y, z) and (y', z') denote the coordinates on \mathbb{T}_Γ^{g+1} and $\mathcal{C}_{\alpha,m}$ given by the exponential map respectively,

$$(y, z) \rightarrow \xi_{y,z} := \Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ), \quad (y', z') \rightarrow m \exp(\sum_{i=1}^g y'_i Y_i^\alpha + z'Z).$$

5 Recall that if $\alpha \in Sp_{2g}(\mathbb{R})$, then for $1 \leq i, j \leq g$ there exist matrices $A =$
6 $(a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$ such that

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2g}(\mathbb{R}),$$

satisfying $A^t D - C^t B = I_{2g}$, $C^t A = A^t C$, $D^t B = B^t D$, and $\det(A) \neq 0$. Set

$$X_i^\alpha = \sum_{j=1}^g (a_{ij} X_j + b_{ij} Y_j) + w_i Z \quad \text{and} \quad Y_i^\alpha = \sum_{j=1}^g (c_{ij} X_j + d_{ij} Y_j) + v_i Z.$$

7 Let $x = \Gamma \exp(\sum_{i=1}^d y_{x,i} Y_i + z_x Z) \exp(\sum_{i=1}^d t_{x,i} X_i)$, for some $(y_x, z_x) \in \mathbb{T}^d \times$
8 $\mathbb{R}/K\mathbb{Z}$ and $t_x = (t_{x,i}) \in [0, 1]^d$. Then, the map $\Phi_{\alpha,x} : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathcal{C}_{\alpha,m}$ is defined by
9 $\Phi_{\alpha,x}(y, z) = (y', z')$ where

$$(53) \quad \begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix},$$

1 and $z' = z + P(\alpha, x, y)$ for some degree 4 polynomial P .

Therefore, the map $\Phi_{\alpha, x}$ is invertible with

$$\Phi_{\alpha, x}^*(dy'_1 \wedge \cdots \wedge dy'_g \wedge dz') = \frac{1}{\det(A)} dy_1 \wedge \cdots \wedge dy_g \wedge dz.$$

2 Since $A^t D - C^t B = I_{2g}$, by direct computation we obtain return time, we have

$$(54) \quad \begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}.$$

Then,

$$\|t(\xi)\| \leq \max_i |t_i(\xi)|^g \leq \max_i \left| \sum_{j=1}^g d_{ij} t_{x,i} + c_{ij} (y_i - y_{x,i}) \right|^g \leq \max_i \|Y_i^\alpha\|^g.$$

3 5.4. **L^2 -lower bound of functional.** We will prove bounds for the square mean
4 of integrals along leaves of foliations of the torus \mathbb{T}_Γ^{g+1} .

Lemma 5.4. *There exists a constant $C > 0$ such that for all $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$, and for every irreducible component $H := H_n$ of central parameter $n \neq 0$, there exists a function f_H such that*

$$|f_H|_{L^\infty(H)} \leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)|,$$

$$|f_H|_{\alpha, s} \leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)| \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|^s (1+n^2)^{s/2}\right)$$

5 where $\|Y\| := \max_{1 \leq i \leq g} \|Y_i^\alpha\|$ and $\Sigma(t_{Ret}) = \sum_{i=1}^g t_{Ret, i}$.

6

7 On rectangular domain $U(\mathbf{T})$, for all $m \in \mathbb{T}_\Gamma^{g+1}$ and $T^{(i)} \in \mathbb{Z}_{t_{Ret, i}}$

$$(55) \quad \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha}(\mathbf{Q}_y^{g, Y} m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = |\mathcal{D}_\alpha^H(f_H)| \left(\frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2}.$$

In addition, whenever $H \perp H' \subset L^2(M)$ the functions

$$\left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha}(\mathbf{Q}_y^{g, Y} m), \omega_H \right\rangle \quad \text{and} \quad \left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha}(\mathbf{Q}_y^{g, Y} m), \omega_{H'} \right\rangle$$

8 are orthogonal in $L^2(\mathbb{T}^g, dy)$.

9 *Proof.* The operator $I_\alpha : L^2(M) \rightarrow L^2(\mathbb{T}_\Gamma^{g+1})$ is defined by

$$(56) \quad f \rightarrow I_\alpha(f) := \int_{U(t_{Ret})} f \circ \mathbf{P}_s^{d, \alpha}(\cdot) ds.$$

Then operator I_α is surjective linear map of $L^2(M)$ onto $L^2(\mathbb{T}_\Gamma^{g+1})$ with a right inverse R_α^χ defined as follows. Let $\chi \in C_0^\infty(0, 1)^g$ be any function of jointly integrable with integral 1. For any $F \in L^2(\mathbb{T}_\Gamma^{g+1})$, let $R_\alpha^\chi(F) \in L^2(M)$ be a function defined by

$$R_\alpha^\chi(F)(\mathbf{P}_v^{d, \alpha}(x)) = \frac{1}{\text{vol}(U(t_{Ret}))} \chi\left(\frac{v}{t_{Ret}}\right) F(x), \quad (x, v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows that there exists a constant $C_\chi > 0$ such that

$$(57) \quad \begin{aligned} |R_\alpha^\chi(F)|_{\alpha,s} &\leq C_\chi \text{vol}(U(t_{Ret}))^{-1} \left(1 + \sum_{i=1}^g t_{Ret,i}^{-1} \|Y_i^\alpha\|\right)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})} \\ &\leq C_\chi \text{vol}(U(t_{Ret}))^{-1} \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|\right)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}. \end{aligned}$$

As explained in §5.1, the space $L^2(\mathbb{T}_\Gamma^{g+1})$ decompose as a direct sum of irreducible subspaces invariant under the action of each $A_{j,\rho,\sigma}$. It follows that the subspace of functions with non-zero central character can be split as direct sum of components $H_{(\mathbf{m},n)}$ with $(\mathbf{m},n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \setminus \{0\}$ with $\mathbf{m} = (m_1, \dots, m_g)$. For a function $F \in H_{(\mathbf{m},n)}$, it is characterized by Fourier expansion

$$F = \sum_{j \in \mathbb{Z}^d} F_j e^{A^j(\mathbf{m},n)} = \sum_{j \in \mathbb{Z}^d} F_j e^{(m_k + K \sum_{i=1}^d (v_{ik} j_i), n)}.$$

Then, we can choose $f_H := R_\alpha^\chi(e_{\mathbf{m},n}) \in C^\infty(H)$ such that

$$(58) \quad |D_\alpha(f_\alpha^H)| = |D_{\mathbf{m},n}(e_{\mathbf{m},n})| = 1,$$

$$(59) \quad \int_{U(t_{Ret})} f_H \circ \mathcal{P}_t^{d,\alpha}(y,z) dt = e_{\mathbf{m},n}(y,z), \text{ for } (y,z) \in \mathbb{T}_\Gamma^{g+1}.$$

Therefore, it follows from (57) that

$$\begin{aligned} |f_H|_{L^\infty(H)} &\leq C_\chi \text{vol}(U(t_{Ret}))^{-1}, \\ |f_H|_{\alpha,s} &\leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)| \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|\right)^s (1+n^2)^{s/2}. \end{aligned}$$

Moreover, since $\{e_{\mathbf{m},n} \circ A_{\rho,\tau}^j\}_{j \in \mathbb{Z}^d} \subset L^2(\mathbb{T}_\Gamma^g, dy)$ is orthonormal, we verify

$$\begin{aligned} \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} x), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} &= \left\| \sum_{j_d=0}^{[\frac{T^{(d)}}{t_{Ret,d}}]-1} \cdots \sum_{j_1=0}^{[\frac{T^{(1)}}{t_{Ret,1}}]-1} e_{\mathbf{m},n} \circ A_{\rho,\tau}^j \right\|_{L^2(\mathbb{T}^g, dy)} \\ &= \left(\frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2}. \end{aligned}$$

1

□

2 Recall that

$$(60) \quad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n := \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i=1}^{\mu(n)} H_{i,n}$$

3 where $H_n = \bigoplus_{i=1}^{\mu(n)} H_{i,n}$ is irreducible representation with central parameter n and
 4 $\mu(n)$ is countable by Howe-Richardson multiplicity formula.

For any infinite dimensional vector $\mathbf{c} := (c_{i,n}) \in \ell^2$, let $\beta_{\mathbf{c}}$ denote Bufetov functional

$$\beta_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} \beta^{i,n}.$$

By orthogonal property and from Corollary 4.3, the function $\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T}) \in L^2(M)$ for all $(\alpha, \mathbf{T}) \in \text{Aut}_0(H^g) \times \mathbb{R}_+^d$. Furthermore,

$$\|\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T})\|_{L^2(M)}^2 = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} |c_{i,n}|^2 \|\beta^{i,n}(\alpha, \cdot, \mathbf{T})\|_{L^2(H_{i,n})}^2 \leq C^2 |\mathbf{c}|_{\ell^2}^2 \text{vol}(U(\mathbf{T})).$$

1 For any $\mathbf{c} := (c_{i,n})$, let $|\mathbf{c}|_s$ denote the norm defined as

$$(61) \quad |\mathbf{c}|_s^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{i=1}^{\mu(n)} (1 + K^2 n^2)^s |c_{i,n}|^2.$$

Lemma 5.5. *For any $s > s_{d,g} + 1/2$, there exists a constant $C_s > 0$ such that for all $\alpha \in DC(L)$, for all $\mathbf{c} \in \ell^2$, for all $z \in \mathbb{T}$ and all $T > 0$,*

$$\left| \|\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T})\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \leq C_s (\text{vol}(U(t_{Ret})) + \text{vol}(U(t_{Ret}))^{-1}) (1 + L) \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\| \right)^s |\mathbf{c}|_s.$$

Proof. By Lemma 5.3, there exists a function $f_{i,n} \in C^\infty(H_{i,n})$ with $|\mathcal{D}^{i,n}(f_{i,n})| = 1$. Let $f_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n} \in C^\infty(M)$, summing up all the functions on irreducibles. Then by the estimates in the Lemma 5.4 and (61),

$$(62) \quad \|f_{\mathbf{c}}\|_{L^\infty(M)} \leq C |\mathbf{c}|_{\ell^1}.$$

$$(63) \quad \|f_{\mathbf{c}}\|_{\alpha,s} \leq C \text{vol}(U(t_{Ret}))^{-1} \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\| \right)^s |\mathbf{c}|_s.$$

By orthogonality,

$$\left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} \circ \mathbf{Q}_y^{g,Y}, \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = \left(\frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0.$$

By the estimate in Lemma 5.4 for each $f_{i,n}$, for every $z \in \mathbb{T}$ and all $\mathbf{T} \in \mathbb{R}_+^d$, we have

$$\left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq 2 \|f_{\mathbf{c}}\|_{L^\infty(M)} \|Y\|.$$

Let $U(\mathbf{T}_{Ret}) = [0, \mathbf{T}_{Ret,1}] \times \cdots \times [0, \mathbf{T}_{Ret,g}]$ where $\mathbf{T}_{Ret,i} := t_{Ret,i}([T^{(i)}/t_{Ret,i}] + 1)$. Then,

$$\left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathcal{P}_{U(\mathbf{T}_{Ret})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq \text{vol}(U(t_{Ret})) \|f_{\mathbf{c}}\|_{L^\infty(M)}.$$

Therefore, there exists a constant $C' > 0$ such that

$$\left| \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \leq C' \text{vol}(U(t_{Ret})) |\mathbf{c}|_{\ell^1}.$$

For all $s > s_{d,g} + 1/2$, by asymptotic property of Theorem 1.6, for some constant $C_s > 0$,

$$\left| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} m, \omega \right\rangle - \beta_H(\alpha, m, \mathbf{T}) \mathcal{D}_\alpha^H(f_H) \right| \leq C_s (1 + L) \|f\|_{\alpha,s}.$$

Applying $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$ and combining bounds on the function $f_{\mathbf{c}}$ with (62),

$$\begin{aligned} & \left| \|\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T})\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \\ & \leq C' \text{vol}(U(t_{Ret})) |\mathbf{c}|_{\ell^1} + C_s \text{vol}(U(t_{Ret}))^{-1} (1+L) |f_{\mathbf{c}}|_{\alpha, s} \\ & \leq C'_s (\text{vol}(U(t_{Ret})) + \text{vol}(U(t_{Ret}))^{-1}) (1+L) \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|\right)^s |\mathbf{c}|_s. \end{aligned}$$

Therefore, we derive the estimates in the statement. \square

6. ANALYTICITY OF FUNCTIONALS

In this section we will prove that for all $\alpha \in DC$, the Bufetov functionals on any square are real analytic.

6.1. Analyticity.

Definition 6.1. For every $t \in \mathbb{R}$, $1 \leq i \leq d$, and $m \in M$, the *stretched (in direction of Z) rectangle* is denoted by

$$(64) \quad [\Gamma_{\mathbf{T}}]_{i,t}^Z(m) := \{(\phi_{ts_i}^Z) \circ \mathbf{P}_{\mathbf{s}}^{d,\alpha}(m) \mid \mathbf{s} \in U(\mathbf{T})\}.$$

For $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}^d$, let us denote by the standard rectangle $\Gamma_{\mathbf{T}}(\mathbf{s}) := (\gamma_1(s_1), \dots, \gamma_d(s_d))$ for $\gamma_i(s_i) = \exp(s_i X_i)$. Similarly, we also write the stretched rectangle

$$(65) \quad [\Gamma_{\mathbf{T}}]_{i,t}^Z(\mathbf{s}) := (\gamma_1(s_1), \dots, \gamma_{i,t}^Z(s_i), \dots, \gamma_d(s_d))$$

where $\gamma_{i,t}^Z(s_i) := \phi_{ts_i}^Z(\gamma_i(s_i))$ is a stretched curve.

Definition 6.2. The *restricted rectangle* $\Gamma_{T,i,s}$ of the standard rectangle $\Gamma_{\mathbf{T}}$ is defined as a restriction

$$\Gamma_{T,i,s}(\mathbf{s}) := \Gamma_{\mathbf{T}}|_{U_{T,i,s}}(\mathbf{s}),$$

where $U_{T,i,s} = [0, T^{(1)}] \times \dots \times \underbrace{[0, s]}_{i\text{-th}} \times \dots \times [0, T^{(d)}]$ for some $0 < s \leq T^{(i)}$.

Recall the orthogonal property on a irreducible component H (central parameter $n \in \mathbb{Z} \setminus \{0\}$). For any $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$ and $t \in \mathbb{R}$,

$$(66) \quad \beta_H(\alpha, \phi_t^Z(m), \mathbf{T}) = e^{2\pi i K n t} \beta_H(\alpha, m, \mathbf{T}).$$

We obtain the following lemma for stretched rectangle by applying orthogonal property.

Lemma 6.3. For fixed elements (X_i, Y_i, Z) satisfying commutation relation (1), the following formula for rank 1 action holds:

$$\hat{\beta}_H(\alpha, [\Gamma_{\mathbf{T}}]_{i,t}^Z) = e^{2\pi i t n K T^{(i)}} \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}) - 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \hat{\beta}_H(\alpha, \Gamma_{T,i,s}) ds_i.$$

Proof. Let $\alpha = (X_i, Y_i, Z)$ and ω be d -form supported on a single irreducible representation H . We obtain following the formula for stretches of curve $\gamma_{i,t}^Z$ (see [FK20b, §4 and Lemma 9.1]),

$$\frac{d\gamma_{i,t}^Z}{ds_i} = D\phi_{ts_i}^Z\left(\frac{d\gamma_i}{ds_i}\right) + tZ \circ \gamma_{i,t}^Z.$$

It follows that pairing is given by

$$\begin{aligned}\langle [\Gamma_{\mathbf{T}}]_{i,t}^Z, \omega \rangle &= \int_{U(\mathbf{T})} \omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_{i,t}^Z}{ds_i}(s_i), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) d\mathbf{s} \\ &= \int_{U(\mathbf{T})} e^{2\pi i n K t s_i} \left[\omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] + \iota_Z \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z(\mathbf{s}) d\mathbf{s}\end{aligned}$$

Denote $d-1$ dimensional triangle $U_{d-1}(\mathbf{T})$ with $U(\mathbf{T}) = U_{d-1}(\mathbf{T}) \times [0, T^{(i)}]$. Integration by parts for a fixed i -th integral gives

$$\begin{aligned}& \int_{U(\mathbf{T})} e^{2\pi i n K t s_i} \left[\omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] d\mathbf{s} \\ &= e^{2\pi i n K t T^{(i)}} \int_{U(\mathbf{T})} \left[\omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] d\mathbf{s} \\ &\quad - 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \int_{U_{d-1}(\mathbf{T})} \left(\int_0^{s_i} \left[\omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_i}{ds_i}(r), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] dr \right) ds_i.\end{aligned}$$

Then, we have the following formula

$$\begin{aligned}\langle [\Gamma_{\mathbf{T}}]_{i,t}^Z, \omega \rangle &= e^{2\pi i n K t T^{(i)}} \langle [\Gamma_{\mathbf{T}}], \omega \rangle - 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \langle \Gamma_{T,i,s}, \omega \rangle ds_i \\ &\quad + \int_{U(\mathbf{T})} (\iota_Z \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z)(\mathbf{s}) d\mathbf{s}.\end{aligned}$$

Since the action of $\mathbf{P}_{\mathbf{t}}^{d,X}$ for $\mathbf{t} \in \mathbb{R}^d$ is identity on the center Z ,

$$\lim_{t_1, \dots, t_d \rightarrow \infty} e^{-(t_1 + \dots + t_d)/2} \int_{U(\mathbf{T})} (\iota_Z (\mathbf{P}_{\mathbf{t}}^{d,X})^* \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z)(\mathbf{s}) d\mathbf{s} = 0.$$

- 1 Thus, in particular concerning $d = 1$, it follows by the definition of Bufetov func-
2 tional (Lemma 3.5), the statement holds. \square

Here we define a restricted vector $\mathbf{T}_{i,s}$ of $\mathbf{T} = (T^{(1)}, \dots, T^{(d)}) \in \mathbb{R}^d$. For fixed $i \in [1, d]$, pick $s_i \in [0, T^{(i)}]$ such that $\mathbf{T}_{i,s} \in \mathbb{R}^d$ is a vector with its coordinates

$$T_{i,s}^{(j)} = \begin{cases} T^{(j)} & \text{if } j \neq i \\ s_i & \text{if } j = i. \end{cases}$$

- 3 Similarly, $\mathbf{T}_{i_1, \dots, i_k, s}$ is a vector whose i_1, \dots, i_k -th coordinates are replaced by
4 s_{i_1}, \dots, s_{i_k} .

Lemma 6.4. *For $m \in M$ and $y_i \in \mathbb{R}$, the following property holds:*

$$\begin{aligned}\beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) &= \\ e^{-2\pi i y_i n K T^{(i)}} \beta_H(\alpha, m, \mathbf{T}) &+ 2\pi i n K y_i \int_0^{T^{(i)}} e^{-2\pi i y_i n K s_i} \beta_H(\alpha, m, \mathbf{T}_{i,s}) ds_i.\end{aligned}$$

Proof. By definition (4), (65) and commutation relation (1), it follows that

$$\phi_{y_i}^{Y_i}(\Gamma_{\mathbf{T}}^X(m)) = [\Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))]_{i,t}^Z.$$

By the invariance property of Bufetov functional and Lemma 6.3,

$$\begin{aligned}\beta_H(\alpha, m, \mathbf{T}) &= \hat{\beta}_H(\alpha, \phi_{y_i}^{Y_i}(\Gamma_{\mathbf{T}}^X(m))) \\ &= e^{2\pi\iota y_i n K T^{(i)}} \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))) - 2\pi\iota n K y_i \int_0^{T^{(i)}} e^{2\pi\iota n K y_i s_i} \hat{\beta}_H(\alpha, \Gamma_{T, i, s}^X(\phi_{s_i}^{Y_i}(m))) ds_i \\ &= e^{2\pi\iota y_i n K T^{(i)}} \beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) - 2\pi\iota n K y_i \int_0^{T^{(i)}} e^{2\pi\iota n K y_i s_i} \beta_H(\alpha, \phi_{s_i}^{Y_i}(m), \mathbf{T}_{i, s}) ds_i.\end{aligned}$$

1 Then statement follows immediately. \square

2 We extend previous Lemma 6.4 to higher rank actions by induction argument.

Lemma 6.5 (Rank- d action). *For $m \in M$ and $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, the following identity for the cocycle β_H holds:*

$$\begin{aligned}(67) \quad \beta_H(\alpha, \mathbf{Q}_y^{d, Y}(m), \mathbf{T}) &= e^{-2\pi\iota \sum_{j=1}^d y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \\ &+ \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\ &\times \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1}.\end{aligned}$$

Proof. We verified that the statement works for $d = 1$ in Lemma 6.4. Assume that (67) holds for rank $d - 1$ action $\mathbf{Q}_{y'}^{d-1, Y}$ by induction hypothesis. For convenience, we write

$$\mathbf{Q}_y^{d, Y}(m) = \phi_{y_d}^{Y_d} \circ \mathbf{Q}_{y'}^{d-1, Y}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 6.4,

$$\begin{aligned}(68) \quad \beta_H(\alpha, \mathbf{Q}_y^{d, Y}(m), \mathbf{T}) &= e^{-2\pi\iota y_d n K T^{(d)}} \beta_H(\alpha, \mathbf{Q}_{y'}^{d-1, Y}(m), \mathbf{T}) \\ &+ 2\pi\iota n K y_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d n K s_d} \beta_H(\alpha, \mathbf{Q}_{y'}^{d-1, Y}(m), \mathbf{T}_{d, s}) ds_d \\ &:= I + II.\end{aligned}$$

Firstly, by induction hypothesis,

$$\begin{aligned}I &= e^{-2\pi\iota y_d n K T^{(d)}} \left(e^{-2\pi\iota \sum_{j=1}^{d-1} y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \right. \\ &+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)} + y_d T^{(d)})} \\ &\times \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1} \Big) \\ &= e^{-2\pi\iota \sum_{j=1}^d y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) + III.\end{aligned}$$

3 Then term III contains 0 to $d - 1$ th iterated integrals on the restricted rectangles
4 (from 0 to $d - 1$ -th) containing $e^{-2\pi\iota n K y_d T^{(d)}}$ outside of iterated integrals.

5 For the second part, we apply induction hypothesis again for restricted rectangle
6 $\mathbf{T}_{d, s}$. Then,

$$\begin{aligned}
II &= 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d} [e^{-2\pi\iota \sum_{j=1}^{d-1} y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T}_{d,s})] ds_d \\
&+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} (2\pi\iota nKy_d) \prod_{j=1}^k (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\
&\times \int_0^{T^{(d)}} \left(\int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} ds_{i_k} \dots ds_{i_1} \right) ds_d \\
&\times e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \dots + y_{i_k}s_{i_k} + y_d s_d)} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, d, s}).
\end{aligned}$$

1 The term II consist of 1 to d -th iterated integrals on the (from 1 to d -th) restricted
2 rectangles containing $e^{-2\pi\iota nKy_d s_d}$ inside of iterated integrals. Thus, by rearranging
3 terms II and III , we obtain all the terms in the expression (67). \square

4 **6.2. Extensions of domain.** In this subsection, we extend the domain of Bufetov
5 functional defined on standard rectangle $\Gamma_{\mathbf{T}}^X$ to the class \mathfrak{R} (see Definition 1.1). We
6 also extend the functional to holomorphic function on a complex domain.

7 *Proof of Corollary 1.4.* Firstly, we can extend our functional to class $(Q_y^{d,Y})_* \Gamma_{\mathbf{T}}^X$ for
8 any $y \in \mathbb{R}^d$ by invariance property (Lemma 3.6). Similarly, by Lemma 6.3, Bufetov
9 functional defined on the standard rectangles extends to the class of generalized
10 rectangle $(\phi_{t_{iz}}^Z) \circ P_t^{d,\alpha}(m)$. Since the flow generated by Z commutes with other
11 actions P and Q , for any standard rectangle $\Gamma = \Gamma(m)$ with a fixed point $m \in M$,
12 we have $(\phi_z^Z)_* \Gamma(m) = \Gamma(\phi_z^Z(m))$ for any $z \in \mathbb{R}$. Therefore, by combining with the
13 invariance under the action Q from Lemma 3.6, the domain of Bufetov functional
14 extends to the class \mathfrak{R} . \square

For any $R > 0$, the *analytic norm* defined for all $\mathbf{c} \in \ell^2$ as

$$\|\mathbf{c}\|_{\omega, R} = \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}|.$$

15 Let Ω_R denote the subspace of $\mathbf{c} \in \ell^2$ such that $\|\mathbf{c}\|_{\omega, R}$ is finite.

Lemma 6.6. For $\mathbf{c} \in \Omega_R$ and $\mathbf{T} \in \mathbb{R}_+^d$, the function

$$\beta_{\mathbf{c}}(\alpha, Q_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}), \quad (y, z) \in \mathbb{R}^d \times \mathbb{T}$$

16 extends to a holomorphic function in the domain

$$(69) \quad D_{R, \mathbf{T}} := \{(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \mid \sum_{i=1}^d |Im(y_i)| T^{(i)} + |Im(z)| < \frac{R}{2\pi K}\}.$$

The following bound holds: for any $R' < R$ there exists a constant $C > 0$ such that, for all $(y, z) \in D_{R', \mathbf{T}}$ we have

$$\begin{aligned}
&|\beta_{\mathbf{c}}(\alpha, Q_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})| \\
&\leq C_{R, R'} \|\mathbf{c}\|_{\omega, R} (L + vol(U(\mathbf{T}))^{1/2} (1 + E_M(a, \mathbf{T})) (1 + K \sum_{i=1}^d |Im(y_i)| T^{(i)}).
\end{aligned}$$

Proof. By Lemma 6.5 and (66),

$$\begin{aligned} \beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}) &= e^{(z-2\pi i \sum_{j=1}^d y_j n K T^{(j)})} \beta_H(\alpha, m, \mathbf{T}) \\ &+ \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (2\pi i n K y_{i_j}) e^{-2\pi i n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\ &\times e^{2\pi i n K z} \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi i n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1}. \end{aligned}$$

As a consequence, by Lemma 3.7 for each variable $(y_i, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$, Then for the rank d -action, by induction, for $(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ we have

$$\begin{aligned} &|\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(x), \mathbf{T})| \\ &\leq (L + \text{vol}(U(\mathbf{T}))^{1/2} (1 + E_M(a, \mathbf{T}))) \left(C_1 \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}| e^{2\pi |Im(z - \sum_{i=1}^d T^{(i)} y_i)| n K} \right. \\ &+ \sum_{k=1}^d C_k \left(\sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (|Im(y_{i_j})| T^{(i_j)}) \right. \\ &\times \left. \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} n |c_{i,n}| e^{2\pi (|Im(z)| + \sum_{j=1}^k T^{(i_j)} |Im(y_{i_j})|) n K} \right) \Bigg). \end{aligned}$$

- 1 Therefore, the functional $\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})$ is bounded by a series of holo-
 2 morphic functions on $\mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ and it converges uniformly on compact subsets of
 3 domain $D_{R,T}$. Thus it is holomorphic on the set. \square

4 7. MEASURE ESTIMATION FOR BOUNDED-TYPE

5 In this section, we prove a measure estimation of Bufetov functional with bounded-
 6 type α . This result is a generalization of §11 in [FK20b].

7 Let \mathcal{O}_r denote the space of holomorphic functions on the ball $B_{\mathbb{C}}(0, r) \subset \mathbb{C}^n$.

Theorem 7.1. [Bru99, Theorem 1.9] For any $f \in \mathcal{O}_r$, there is a constant $d := d_f(r) > 0$ such that for any convex set $D \subset B_{\mathbb{R}}(0, 1) := B_{\mathbb{C}}(0, 1) \cap \mathbb{R}^n$, for any measurable subset $U \subset D$

$$\sup_D |f| \leq \left(\frac{4n \text{Leb}(D)}{\text{Leb}(U)} \right)^d \sup_U |f|.$$

- 8 We say that a holomorphic function f defined in a disk is p -valent if it assumes
 9 no value more than p -times there. We also say that f is θ -valent if it is a constant.

Definition 7.2. [Bru99, Def 1.6] Let \mathcal{L}_t denote the set of one-dimensional complex affine spaces $L \subset \mathbb{C}^n$ such that $L \cap B_{\mathbb{C}}(0, t) \neq \emptyset$. For $f \in \mathcal{O}_r$, the number

$$\nu_f(t) := \sup_{L \in \mathcal{L}_t} \{\text{valency of } f \mid L \cap B_{\mathbb{C}}(0, t) \neq \emptyset\}$$

- 10 is called the *valency* of f in $B_{\mathbb{C}}(0, t)$.

- 11 By Proposition 1.7 of [Bru99], for any $f \in \mathcal{O}_r$ with finite valency $\nu_f(t)$ for any
 12 $t \in [1, r)$, there is a constant $c := c(r) > 0$ such that

$$(70) \quad d_f(r) \leq c \nu_f\left(\frac{1+r}{2}\right).$$

Lemma 7.3. *[FK20b, Lemma 10.3] Let $R > r > 1$. For any normal family $\mathcal{F} \subset \mathcal{O}_R$, assume that no functions in $\overline{\mathcal{F}} = \emptyset$ are constant along a one-dimensional complex line. Then we have*

$$\sup_{f \in \mathcal{F}} \nu_f(r) < \infty.$$

Lemma 7.4. *Let $L > 0$ and $\mathcal{B} \subset DC(L)$ be a bounded subset. Given $R > 0$, for all $\mathbf{c} \in \Omega_R$ and all $\mathbf{T}^{(i)} > 0$, denote $\mathcal{F}(\mathbf{c}, \mathbf{T})$ by the family of real analytic functions of the variable $y \in [0, 1]^d$ and*

$$\mathcal{F}(\mathbf{c}, \mathbf{T}) := \{\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}) \mid (\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}\}.$$

Then there exist $\mathbf{T}_{\mathcal{B}} := (\mathbf{T}_{\mathcal{B}}^{(i)})$ and $\rho_{\mathcal{B}} > 0$, such that for every (R, \mathbf{T}) with $R/\mathbf{T}^{(i)} \geq \rho_{\mathcal{B}}$, $\mathbf{T}^{(i)} \geq \mathbf{T}_{\mathcal{B}}^{(i)}$ and for all $\mathbf{c} \in \Omega_R \setminus \{0\}$, we have

$$(71) \quad \sup_{f \in \mathcal{F}(\mathbf{c}, \mathbf{T})} \nu_f < \infty.$$

Proof. Since $\mathcal{B} \subset \mathfrak{M}$ is bounded, for each time $t_i \in \mathbb{R}$ and $1 \leq i \leq g$,

$$0 < t_{i, \mathcal{B}}^{\min} = \min_i \inf_{\alpha \in \mathcal{B}} t_{Ret, i, \alpha} \leq \max_i \sup_{\alpha \in \mathcal{B}} t_{Ret, i, \alpha} = t_{\mathcal{B}}^{\max} < \infty.$$

For any $\alpha \in \mathcal{B}$ and $x \in M$, the map $\Phi_{\alpha, x} : [0, 1]^d \times \mathbb{T} \rightarrow \prod_{i=1}^d [0, t_{\alpha, i}) \times \mathbb{T}$ in (53) extends to a complex analytic diffeomorphism $\hat{\Phi}_{\alpha, x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$. By Lemma 6.6, it follows that for fixed $z \in \mathbb{T}$, real analytic function $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T})$ extends to a holomorphic function defined on a region

$$H_{\alpha, m, R, t} := \{y \in \mathbb{C}^d \mid \sum_{i=1}^d |Im(y_i)| \leq h_{\alpha, m, R, t}\}.$$

By boundedness of the set $\mathcal{B} \subset \mathfrak{M}$, it follows that

$$\inf_{(\alpha, x) \in \mathcal{B} \times M} h_{\alpha, m, R, t} := h_{R, T} > 0.$$

We remark that the function $h_{\alpha, m, R, t}$ and its lower bound $h_{R, T}$ can be obtained from the formula (53) for the polynomial $\Phi_{\alpha, x}$ and the definition of the domain $D_{R, T}$ in formula (69).

For every $r > 1$, there exists $\rho_{\mathcal{B}} > 1$ such that for every R and \mathbf{T} with $R/\mathbf{T}^{(i)} > \rho_{\mathcal{B}}$,

$$(72) \quad \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}) \in \mathcal{O}_r$$

as a function of $y \in \mathbb{T}^d$.

By Lemma 6.6, the family $\mathcal{F}(\mathbf{c}, \mathbf{T})$ is uniformly bounded and normal. By Lemma 5.5 for the non-zero L^2 -lower bound of functionals, for sufficiently large pair \mathbf{T} , no sequence from $\mathcal{F}(\mathbf{c}, \mathbf{T})$ can converge to a constant. Therefore, by Lemma 7.3 for the family $\mathcal{F} = \mathcal{F}(\mathbf{c}, \mathbf{T})$, the main statement follows. \square

We derive measure estimates of Bufetov functionals on the rectangular domain.

Lemma 7.5. *Let $\alpha \in DC$ such that the forward orbit of \mathbb{R}^d -action $\{r_t[\alpha]\}_{t \in \mathbb{R}_+^d}$ is contained in a compact set of \mathfrak{M}_g . There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}_+^d$ such that, for every $\mathbf{c} \in \Omega_R \setminus \{0\}$, $T \geq T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{m \in M \mid |\beta_{\mathbf{c}}(\alpha, m, \mathbf{T})| \leq \epsilon vol(U(\mathbf{T}))^{1/2}\}) \leq C\epsilon^\delta.$$

Proof. Since $\alpha \in DC$ and the orbit $\{r_t[\alpha]\}_{t \in \mathbb{R}_+}$ is contained in a compact set, there exists $L > 0$ such that $r_t[\alpha] \in DC(L)$ for all $t \in \mathbb{R}_+^d$. Then, we choose $\mathbf{T}_0 \in \mathbb{R}^d$ and $R > 0$ from the conclusion of Lemma 7.4. By the scaling property,

$$\beta_{\mathbf{c}}(\alpha, m, \mathbf{T}) = \left(\frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(\mathbf{T}_0))} \right)^{1/2} \beta_{\mathbf{c}}(g_{\log(\mathbf{T}/\mathbf{T}_0)}[\alpha], m, \mathbf{T}_0).$$

By Fubini's theorem, it suffices to estimate

$$\text{Leb}(\{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\}).$$

Let $\delta^{-1} := c(r) \sup_{f \in \mathcal{F}(\mathbf{c}, \mathbf{T}_0)} \nu_f(\frac{1+r}{2}) < \infty$ as in (70) and (71). By Lemma 5.5, we have

$$\inf_{(\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}} \sup_{y \in [0, 1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| > 0$$

so that the functional is not trivial. By Theorem 7.1 for the unit ball $D = B_{\mathbb{R}}(0, 1)$ and setting

$$U = \{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\},$$

by the bound in (70) for $d_f(r)$, there exists a constant $C > 0$ such that for all $\epsilon > 0$ and $(\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}$,

$$\text{Leb}(\{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\}) \leq C\epsilon^\delta.$$

1 Then the statement follows from the Fubini theorem. \square

Corollary 7.6. *Let α be as in the previous Lemma 7.5. There exist $R, C, \delta > 0$ and $\mathbf{T}_0 \in \mathbb{R}_+^d$ such that, for every $\mathbf{c} \in \Omega_R \setminus \{0\}$, $\mathbf{T} \geq \mathbf{T}_0$ and for every $\epsilon > 0$, we have*

$$\text{vol}(\{m \in M \mid |\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha} m, \omega_{\mathbf{c}} \rangle| \leq \epsilon \text{vol}(U(\mathbf{T}))^{1/2}\}) \leq C\epsilon^\delta.$$

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