LIMIT THEOREMS FOR HIGHER RANK ACTIONS ON HEISENBERG NILMANIFOLDS

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ABSTRACT. The main result of this paper is a construction of finitely additive measures for higher rank abelian actions on Heisenberg nilmanifolds. Under a full measure set of Diophantine condition for the generators of the action, we construct *Bufetov functional* on rectangles on $2g+1$ dimensional Heisenberg manifold. We prove the deviation of Birkhoff integral of higher rank actions described by asymptotics of Bufetov functionals for a sufficiently smooth function. As a corollary, a distribution of normalized Birkhoff integrals to have a variance 1 converges to a non-degenerate compactly supported measure on the real line.

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1. Introduction

1.1. **Introduction.** The asymptotic behavior and limiting distribution of ergodic averages of translation flow were studied by Alexander Bufetov in the series of his works [Buf14a, Buf10, Buf14b]. He constructed finitely-additive Hölder measures and cocycles over translation flows that are known as *Bufetov functionals*. The construction of such functionals is used to derive the deviation of ergodic integrals and results on probabilistic behavior of ergodic averages of translation flows. He noticed that there was a duality between the Bufetov functionals and invariant distributions which played a key role in the work of G. Forni [For02]. Following these observations, the duality between such functionals and the space of invariant distributions, was constructed analogously for parabolic flows or other settings (e.g [Buf13, BS13, BF14, FK20b, For20a]). Furthermore, it turned out that his construction of functional is also closely related to limit shapes of ergodic sums for interval exchange transformation [MMY10, MUY20].

In this paper, our main results are on limit distributions of higher rank abelian actions. We firstly introduce the construction of Bufetov functional for higher rank abelian actions. Our main argument is based on the renormalization for higher rank actions by induction argument. It is a key idea used in the work of Cosentino and Flaminio [CF15], but we extend their constructions to rectangular shape and derive the deviation of ergodic integrals for higher rank actions. Likewise, this explains the duality between Bufetov functionals and approach for basic currents. The technical part is handling the estimate of deviation ergodic integrals on (stretched) rectangles, and this enables to derive our main theorems.

As a corollary, we prove there exists a limit distributions of (normalized) ergodic integrals of abelian actions. More specifically, as a random variable, normalized ergodic integrals converge in distribution to a non-degenerate, compactly supported measure on the real line. This generalizes the limit theorem for theta series on Siegel half spaces, introduced by Götze and Gordin [GG04] and Marklof [Mar99] (see [Tol78, MM07, MNN07] for general introduction and nilflow case [GM14, CM16]). In the last section, the analyticity of Bufetov functional on higher dimensional rectangular domain is proved.

From these results, polynomial type of lower bounds of sub-level sets of analytic function is obtained. Such polynomial estimates on the measure are derived by real analyticity of a functional along the leaves of a foliation transverse to the actions based on results of [Bru99]. This tool was devised to study the bound of correlations for time-changes of nilflows in [FK20b]. However, in the higher rank setting, time-changes of higher rank abelian actions on Heisenberg nilmanifold are all trivial (by triviality of first cohomology group, see [CF15, Theorem 3.16]). Thus, they are conjugated to the linear action and never mixing.

As a rank 1-action, mixing properties for time changes of Heisenberg nilflows were firstly studied in [AFU11]. Then, it was extended in the dense set of non-trivial time changes for any uniquely ergodic nilflows on general nilmanifolds [Rav18, AFRU21].

As a special special case, on time-changes of Heisenberg flows, multiple mixing [FK20a] was proved by Ratner property. In $\mathbb{Z}^k$-actions, mixing of shapes for automorphisms on nilmanifold is proved in [GS14, GS15].

Comprehensive studies of spatial and temporal limit theorems for horocycle flows have been carried out in the work of D. Dolgopyat and O. Sarig [DS17]. Recent work of Ravotti [Rav21] proved spatial and temporal limit theorem for horocycle flows in an alternative ways by using Ratner's argument, not relying on invariant distributions from the solving cohomological equation. Temporal limit theorems for Heisenberg flows (and its time-changes) will be the subject of further works.

1.2. **Definition and statement of results.** We review definitions about Heisenberg manifold and its moduli space.

1.2.1. *Heisenberg manifold.* Let H^g be the standard $2g+1$ dimensional Heisenberg group and set $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$ a discrete and co-compact subgroup of H^g. We shall call it standard lattice of H^g and the quotient $M := \mathsf{H}^g/\Gamma$ will be called *Heisenberg manifold*. Lie algebra $\mathfrak{h}^g = Lie(\mathsf{H}^g)$ is equipped with a basis $(X_1, \cdots, X_g, Y_1, \cdots Y_g, Z)$ satisfying canonical commutation relations

$$[X_i, Y_j] = \delta_{ij} Z, \quad \text{where } \delta_{ij} = 1 \text{ if } i = j, \text{ and } \delta_{ij} = 0 \text{ otherwise.} \tag{1}$$

For $1 \le d \le g$, let P^d be a subgroup of H^g where its Lie algebra $\mathfrak{p} := \mathfrak{p}_d = Lie(\mathsf{P}^d)$ is generated by $\{X_1, \cdots, X_d\}$. For any $\alpha \in Sp_{2g}(\mathbb{R})$, set $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$ for $1 \le i \le d$. We define parametrization of the subgroup $\mathsf{P}^{d,\alpha}$ by

$$\mathsf{P}_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \cdots + x_d X_d^\alpha), \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

By central extension of $\mathbb{R}^{2g}$ by $\mathbb{R}$, we have an exact sequence

$$0 \to Z(H^g) \to H^g \to \mathbb{R}^{2g} \to 0.$$

The natural projection map $pr : M \to H^g/(\Gamma Z(H^g))$ maps M onto a $2g$-dimensional torus $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

1.2.2. *Moduli space.* The group of automorphisms of H^g that are trivial on the center is $Aut_0(\mathsf{H}^g) = Sp_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$. Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to $Sp_{2g}(\mathbb{R})$. We call *moduli space* of the standard Heisenbeg manifold the quotient $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$. We regard $Sp_{2g}(\mathbb{R})$ as the deformation space of the standard Heisenberg manifold M and $\mathfrak{M}_g$ as the moduli space of M.

Siegel modular variety is a double coset space $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ where K_g is a maximal compact subgroup $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$ of $Sp_{2g}(\mathbb{R})$. For $\alpha \in Sp_{2g}(\mathbb{R})$, we denote $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$ by its projection on the moduli space $\mathfrak{M}_g$ and write $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$ the projection of α to the Siegel modular variety Σ_g.

Double coset $K_g \backslash Sp_{2g}(\mathbb{R})/1_{2g}$ is identifed to the Siegel upper half space $\mathfrak{H}_g := \{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$. *Siegel upper half space* of genus g is complex manifold of symmetric complex $g \times g$ matrices $Z = X + iY$ with positive-definite symmetric imaginary part $\Im(Z) = Y$ and arbitrary (symmetric) real part X. Denote by $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$.

1.2.3. *Representation.* We write the Hilbert sum decomposition of

$$L^2(M) = \bigoplus_{n\in\mathbb{Z}} H_n \tag{2}$$

into closed H^g-invariant subspaces. Set $f = \sum_{n\in\mathbb{Z}} f_n \in L^2(M)$ and $f_n \in H_n$ where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi\imath nKt)f\}$$

for some fixed $K > 0$. The center $Z(H^g)$ has spectrum $2\pi\mathbb{Z}\backslash\{0\}$ and the space $L^2(M)$ splits as Hilbert sum of H^g-module H_n, which is equivalent to irreducible representation π.

By Stone-Von Neumann theorem, the unitary irreducible representation π of the Heisenberg group of non-zero central parameter $K > 0$ is unitarily equivalent to the Schrödinger representation π. By differentiating the Schrödinger representation, we obtain a representation of the Lie algebra $\mathfrak{h}^g$ on Schwartz space $\mathscr{S}(\mathbb{R}^g) \subset L^2(\mathbb{R}^g)$ (as a C^∞-vector). This is called *infinitesimal derived representation* $d\pi_*$ with parameter $n \in \mathbb{Z}$, and for each $k = 1, 2, \cdots, g$

$$d\pi_*(X_k) = \frac{\partial}{\partial x_k}, \quad d\pi_*(Y_k) = 2\pi\imath nKx_k, \quad d\pi_*(Z) = 2\pi\imath nK$$

acts on $L^2(\mathbb{R}^g) \simeq L^2(H_n)$.

Given a basis (V_i) of the Lie algebra, we set a Laplacian $\Delta = -\sum_i V_i^2$ and define L^2-Sobolev norm $\|f\|_s^2 = \langle f, (1+\Delta)^s f\rangle$ where $\langle\cdot,\cdot\rangle$ is an ordinary inner product. For $s > 0$, the *Sobolev space* $W^s(M)$ is defined as a completion of $C^\infty(M)$ equipped with the norm $\|\cdot\|_s$. The Sobolev space $W^s(M) = \bigoplus_{n\in\mathbb{Z}} W^s(H_n)$ decomposes to closed H^g-invariant subspaces $W^s(H_n) = W^s(M) \cap H_n$.

1.2.4. *Renormalization flow.* Denote diagonal matrix $\delta_i = diag(d_1, \cdots, d_g)$ with $d_i = 1$, $d_k = 0$ if $k \neq i$. Then, for each $1 \leq i \leq g$, we denote $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in \mathfrak{sp}_{2g}$.

Any such $\hat{\delta}_i$ generate a one-parameter subgroup of automorphism $r_i^t = e^{t\hat{\delta}_i}$. We denote (rank d) renormalization flow $r_{\mathbf{t}} := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$ for $\mathbf{t} = (t_1, \cdots, t_d)$.

Main results. One of the main objects in this paper is to construct finitely-additive measures defined on the space of rectangles on Heisenberg manifold M. We state our results beginning with an overview of Bufetov functional.

Definition 1.1. For $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$, denote the *standard rectangle* for action P,

$$\Gamma_{\mathbf{T}}^X(m) = \{\mathsf{P}_{\boldsymbol{t}}^{d,\alpha}(m) \mid \boldsymbol{t} \in U(\mathbf{T}) = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]\}. \tag{3}$$

Let $\mathsf{Q}_y^{d,Y} := \exp(y_1Y_1 + \cdots + y_dY_d)$, $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$ be an action generated by elements Y_i of standard basis. Let $\phi_z^Z := \exp(zZ)$, $z \in \mathbb{R}$ be a flow generated by the central element Z.

Definition 1.2. Let $\mathfrak{R}$ be the collection of the *generalized rectangles* in M. For each $1 \leq d \leq g$ and $\boldsymbol{t} = (t_1, \cdots, t_d)$, we set

$$\mathfrak{R} := \bigcup_{1\leq i\leq d} \bigcup_{(y,z)\in\mathbb{R}^d\times\mathbb{R}} \bigcup_{(m,\mathbf{T})\in M\times\mathbb{R}_+^d} \{(\phi_{t_i z}^Z) \circ \mathsf{Q}_y^{d,Y} \circ \mathsf{P}_{\boldsymbol{t}}^{d,\alpha}(m) \mid \boldsymbol{t} \in U(\mathbf{T})\}. \tag{4}$$

Theorem 1.3. *For any irreducible representation H, there exists a measure $\hat{\beta}_H(\Gamma) \in \mathbb{C}$ for every rectangle $\Gamma \in \mathfrak{R}$ such that the following holds:*

(1) (Additive property) *For any decomposition of disjoint rectangles* $\Gamma = \bigcup_{i=1}^n \Gamma_i$ *or whose intersections have zero measure,*

$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^n \hat{\beta}_H(\alpha, \Gamma_i).$$

(2) (Scaling property) *For* $\boldsymbol{t} \in \mathbb{R}^d$*,*

$$\hat{\beta}_H(r_{\boldsymbol{t}}[\alpha], \Gamma) = e^{-(t_1+\cdots t_d)/2}\hat{\beta}_H(\alpha, \Gamma).$$

(3) (Invariance property) *For any action* $\mathsf{Q}_\tau^{d,Y}$ *generated by* Y_i*'s and* $\tau \in \mathbb{R}_+^d$*,*

$$\hat{\beta}_H(\alpha, (\mathsf{Q}_\tau^{d,Y})_*\Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

(4) (Bounded property) *For any rectangle* $\Gamma \in \mathfrak{R}$*, there exists a constant* $C(\Gamma) > 0$ *such that for* $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$*,*

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

For arbitrary rectangle $U_{\mathbf{T}} = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]$, pick $\mathbf{T}'^{(i)} \in [0, \mathbf{T}^{(i)}]$ for each i to decompose $U_{\mathbf{T}}$ into 2^d sub-rectangles. We write $\mathbf{P}(\mathbf{T}')$ collection of 2^d vertices $v = (v^{(1)}, v^{(2)}, \cdots, v^{(d)})$ where $v^{(i)} \in \{0, \mathbf{T}'^{(i)}\}$. Let $U_{\mathbf{T},v}$ be a rectangle whose sides $\boldsymbol{I}_v = (I^{(1)}, I^{(2)}, \cdots, I^{(d)}) \in \mathbb{R}_+^d$ where

$$I^{(l)} = \begin{cases} T^{(l)} - T'^{(l)} & \text{if} \quad v^{(l)} = T'^{(l)} \\ T'^{(l)} & \text{if} \quad v^{(l)} = 0. \end{cases}$$

Then, we have $U_{\mathbf{T}} = \bigcup_{v \in \mathbf{P}(\mathbf{T}')} U_{\mathbf{T},v}$.

Theorem 1.4. *Let us denote* $\beta_H(\alpha, m, \boldsymbol{T}) := \hat{\beta}_H(\alpha, \Gamma_{\boldsymbol{T}}^X(m))$*. The function* β_H *satisfies the following properties:*

(1) (Cocycle property) *For all* $(m, \boldsymbol{T}_1, \boldsymbol{T}_2) \in M \times \mathbb{R}^d \times \mathbb{R}^d$*,*

$$\beta_H(\alpha, m, \boldsymbol{T}_1 + \boldsymbol{T}_2) = \sum_{v \in \mathbf{P}(\boldsymbol{T}_1)} \beta_H(\alpha, \mathsf{P}_v^{d,\alpha}(m), \boldsymbol{I}_v).$$

(2) (Scaling property) *For all* $m \in M$ *and* $\boldsymbol{t} = (t_1, \cdots, t_d)$*,*

$$\beta_H(r_{\boldsymbol{t}}\alpha, m, \boldsymbol{T}) = e^{(t_1+\cdots t_d)/2}\beta_H(\alpha, m, \boldsymbol{T}).$$

(3) (Bounded property) *Let us denote the largest length of side* $T_{max} = \max_i \boldsymbol{T}^{(i)}$*. Then there exists a constant* $C_H > 0$ *such that*

$$\beta_H(\alpha, m, \boldsymbol{T}) \leq C_H T_{max}^{d/2}.$$

(4) (Orthogonality) *For all* $\boldsymbol{T} \in \mathbb{R}^d$*, bounded function* $\beta_H(\alpha, \cdot, \boldsymbol{T})$ *belongs to the irreducible component, i.e,*

$$\beta_H(\alpha, \cdot, \boldsymbol{T}) \in H \subset L^2(M).$$

By representation theory introduced (2), for any $f = \sum_H f_H \in W^s(M)$, we define a *Bufetov cocycle* associated to f as a sum

$$\beta^f(\alpha, m, \boldsymbol{T}) = \sum_H D_\alpha^H(f)\beta_H(\alpha, m, \boldsymbol{T}). \tag{5}$$

Notation. Given a Jordan region U and a point $m \in M$, set $\mathscr{P}_U^{d,\alpha} m$ the Birkhoff integrals (currents) associated to the action $\mathsf{P}_x^{d,\alpha}$ given by

$$\left\langle \mathscr{P}_U^{d,\alpha} m, \omega_f \right\rangle := \int_U f(\mathsf{P}_x^{d,\alpha} m) dx_1 \cdots dx_d$$

for any degree $\mathfrak{p}$-form $\omega_f = f \hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$ with smooth function with zero averages $f \in C_0^\infty(M)$.

Theorem 1.5. *For all $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for almost all frequency α and for all $f \in W^s(M)$ and for all $(m, \boldsymbol{T}) \in M \times \mathbb{R}^d$, we have*

$$|\left\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha} m, \omega_f \right\rangle - \beta^f(\alpha, m, \boldsymbol{T})| \leq C_s \left\| \omega_f \right\|_{\alpha,s}. \tag{6}$$

for $U(\boldsymbol{T}) = [0, T^{(1)}] \times \cdots [0, T^{(d)}]$ and $\omega = f\omega^{d,\alpha} \in \Lambda^d \mathfrak{p} \otimes W^s(M)$.

Let the family of random variable

$$E_{T_n}(f) := \frac{1}{vol(U(\boldsymbol{T_n}))^{1/2}} \left\langle \mathscr{P}_{U(\boldsymbol{T_n})}^{d,\alpha}(m), \omega_f \right\rangle$$

where $U(\boldsymbol{T_n})$ is a sequence of rectangles. The point $m \in M$ is distributed accordingly to the probability measure *vol* on M. Our goal is to understand the asymptotic behavior of the probability distributions of $E_{T_n}(f)$.

Theorem 1.6. *For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_{d,g} = d(d + 11)/4 + g + 1/2$, which is not a coboundary, the limit distribution of the family of random variables $E_{T_n}(f)$ exists, and for almost all frequency α, it has compact support on the real line.*

We finish the section by giving some remarks on the new adaptation of transfer operator techniques from hyperbolic dynamics. The method stems from the analysis of the transfer operator, firstly treated by P. Gieuletti and Liverani [GL19]. They set up a non-linear flows on the torus and proved asymptotics of ergodic averages with expansions of invariant distributions and eigenvalues of transfer operators called *Ruelle resonances* (see also [AB18, FGL19, For20b, B19]).

A recent work of L. Simonelli and O. Butterley [BS20] rephrased the work of Flaminio and Forni [FF06] by reproving the deviation of ergodic averages on Heisenberg nilflows with expansions of such resonances of transfer operators defined on an anisotropic space. Although their work is restricted to the periodic type where the flow is only renormalized by partially hyperbolic diffeomorphism (or α in the moduli space $\mathfrak{M}$ is periodic), their methods showed indirect similarities with [FK20b] and our current work (see also [BS20, §4]). However, it is still not known if it is possible to extend their approach for recurrent orbits requiring construction of renormalization cocycle. Furthermore, it is not well-studied how to replace the use of Anisotropic norm, not relying on result of Faure-Tsujii [FT15] and improve their argument by adopting necessary Diophantine conditions.

Open problem. Our work leaves open questions.

Problem 1. Can we prove measure estimation of β^f in §7 for the full measure set of α?

Problem 2. Can we construct Bufetov functionals on higher step nilmanifolds?

Problem 3. Can we prove the deviation of ergodic averages for higher rank abelian actions by adopting Gieuletti-Liverani approach?

Since moduli space is trivial and there does not exist renormalization flows, we could not obtain the results in the same method. However, we still hope other methods in handling non-renormalizable flows are possibly applied (cf. [FF14, FFT16, Kim21]).

Outline of the paper. In section 2, we give basic definitions on Higher rank actions, moduli spaces and Sobolev spaces. In section 3, we state main theorem and prove constructions of Bufetov functionals with main properties. In section 4, we prove asymptotic formula of Birkhoff integrals and its limit theorems. In section 5, we prove L^2-lower bound of the Bufetov functionals. In Section 6, analyticity of functional and extensions of domain are provided. In section 7, there exist measure estimates of functionals on the sets where values of functionals are small. This result only holds when frame α is of bounded type.

2. Analysis on Heisenberg manifolds

In this section, we review definitions of Sobolev space, currents, representation and renormalization flows on moduli space.

2.1. Sobolev space.

2.1.1. *Sobolev bundles.* Given a basis (V_i) of the Lie algebra $\mathfrak{h}^g$, we write a new basis $((\alpha^{-1})_*V_i)$ for $\alpha \in Aut_0(\mathsf{H}^g)$. Similarly, denote by Laplacian $\Delta_\alpha = -\sum_i (\alpha^{-1})_* V_i^2$. For any $s \in \mathbb{R}$ and any C^∞ function $f \in L^2(M)$, Sobolev norm is defined by

$$\|f\|_{\alpha,s} = \langle f, (1+\Delta_\alpha)^s f\rangle^{1/2}.$$

Let $W^s_\alpha(M)$ be a completion of $C^\infty(M)$ with the norm above. The dual space of $W^s_\alpha(M)$ is denoted by $W^{-s}_\alpha(M)$ and it is isomorphic to $W^s_\alpha(M)$. Extending it to the exterior algebra, define the Sobolev spaces $\Lambda^d\mathfrak{p} \otimes W^s_\alpha(M)$ of cochains of degree d and we use the same notations for the norm.

The group $Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \to Sp_{2g}(\mathbb{R})$$

where

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma = (\alpha\gamma, \gamma^*\varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \to \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

the fiber over $[\alpha] \in \mathfrak{M}_g$ is locally identified with the space $W^s_\alpha(M)$.

By invariance of $Sp_{2g}(\mathbb{Z})$ action, we denote the class (α, φ) by $[\alpha, \varphi]$ and write $Sp_{2g}(\mathbb{Z})$-invariant Sobolev norm

$$\|(\alpha, f)\|_s := \|f\|_{\alpha,s}.$$

We denote the bundle of $\mathfrak{p}$-forms of degree j of Sobolev order s by $A^j(\mathfrak{p}, \mathfrak{M}^s)$. The space of continuous linear functional on $A^j(\mathfrak{p}, \mathfrak{M}^s)$ will be called the *space of currents* of dimension j and denoted by $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$. Similarly, there is a flat bundle of (currents) distribution $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$ whose fiber over $[\alpha]$ is locally identified with the space $W^{-s}_\alpha(M)$ normed by $\|\cdot\|_{\alpha,-s}$.

2.1.2. *Best Sobolev constant.* The *Sobolev embedding theorem* implies that for any $\alpha \in Sp_{2g}(\mathbb{R})$ and $s > g + 1/2$, there exists a constant $B_s(\alpha)$ such that for any $f \in W^s_\alpha(M)$,

$$B_s(\alpha) := \sup_{f \in W^s_\alpha(M) \setminus \{0\}} \frac{\|f\|_\infty}{\|f\|_{s,\alpha}}. \tag{7}$$

The *best Sobolev constant* B_s is a $Sp_{2g}(\mathbb{Z})$-modular function on $\mathfrak{H}_g$ [CF15, Lemma 4.4]. From the Sobolev embedding theorem, we have the following bound for the Birkhoff integral current.

Lemma 2.1. *[CF15, Lemma 5.5] For any Jordan region $U \subset \mathbb{R}^d$ with Lebesgue measure $|U|$, for any $s > g + 1/2$ and all $m \in M$,*

$$\left\| [\alpha, (\mathscr{P}^{d,\alpha}_U m)] \right\|_{-s} \leq B_s(\alpha)|U|.$$

2.2. **Invariant currents.**

2.2.1. *Identification.* We denote $I_d(\mathfrak{p}, W^{-s}(M))$ the space of P-invariant currents of Sobolev order s. By formal identities $\langle D, X^\alpha_i(f)\rangle = 0$ for all test function f and $i \in [1, d]$.

The boundary operators

$$\partial : A_j(\mathfrak{p}, W^{-s}(M)) \to A_{j-1}(\mathfrak{p}, W^{-s}(M))$$

are adjoint of the differentials d such that $\langle \partial D, \omega\rangle = \langle D, d\omega\rangle$. A current D is called *closed* if $\partial D = 0$. For $s > 0$, we denote $D \in Z_d(\mathfrak{p}, W^{-s}(M))$ the space of closed currents of dimension d and Sobolev order s.

Proposition 2.2. *[CF15, Proposition 3.13] For any $s > d/2$, with $d = \dim \mathsf{P}$, we have $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$ for all $\epsilon > 0$. Additionally,*

- *$I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is one dimensional space if $\dim \mathsf{P} = g$,*
- *$I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is an infinite-dimensional space if $\dim \mathsf{P} < g$,*
- *$I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) = Z_d(\mathfrak{p}, W^{-s}(M))$ for any $1 \leq d \leq g$.*

2.2.2. *Basic currents.* The current B_α is called *basic* if for all $j \in \{i_1, \cdots, i_d\}$,

$$\iota_{X_j} B_\alpha = L_{X_j} B_\alpha = 0.$$

For an irreducible representation H, there exists *basic current* B^H_α associated to invariant distribution D^H_α. It is defined by $B^H_\alpha = D^H_\alpha \eta_X$ and this formula implies that for every d-form ξ,

$$B^H_\alpha(\xi) = D^H_\alpha \left(\frac{\eta_X \wedge \xi}{\omega} \right) \tag{8}$$

where $\eta_X := \iota_{X_{i_1}} \cdots \iota_{X_{i_d}} \omega$ and ω is an invariant volume form.

We recall that the space of Sobolev currents $A_j(\mathfrak{p}, W^{-s}(M))$ of dimension j with order s is identified with $\Lambda^j \mathfrak{p} \otimes W^{-s}(M)$. It follows that for all $s > d/2$, by Sobolev embedding theorem, $B^H_\alpha \in A_j(\mathfrak{p}, W^{-s}(M))$ if and only if $D^H_\alpha \in W^{-s}_\alpha(M)$ for all $s > d/2$.

Remark. *The formula (8) yields a duality between the space of basic (closed) currents and invariant distributions (see also [For02, §6.1] and [BF14, §2.1]). For a d-dimensional $\mathfrak{p}$-form $\omega^{d,\alpha}_f = f dX^\alpha_1 \wedge \cdots \wedge dX^\alpha_d$ and $f \in C^\infty(M)$, we will identify its currents $\mathcal{D}$ with distribution D by giving*

$$\langle D, f\rangle = \langle \mathcal{D}_\alpha, \omega^{d,\alpha}_f\rangle.$$

2.2.3. *Renormalization.* By definition of renormalization flow in §1.2.4, for $\omega \in \Lambda^d\mathfrak{p} \otimes W^s(\mathbb{R}^g)$ and $\mathcal{D} \in Z_d(\mathfrak{p}, W^{-s}(M))$, we have

$$r_i^t[\alpha,\omega] = [r_i^t\alpha,\omega], \quad r_i^t[\alpha,\mathcal{D}] = [r_i^t\alpha,\mathcal{D}].$$

Let $U_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be unitary operator for $\mathbf{t} = (t_1, \cdots, t_d)$,

$$U_t f(x) = e^{-(t_1+\cdots+t_d)/2} f(e^{t_1}x_1, \cdots e^{t_d}x_d). \tag{9}$$

Then, for any invariant currents D_α^H,

$$D^H_{r_t(\alpha)} = e^{(t_1+\cdots+t_d)/2} D^H_\alpha.$$

Then, the $\mathbb{R}^d$-action is defined by reparametrization

$$\mathsf{P}_x^{d,(r_1^{t_1}\cdots r_d^{t_d}\alpha)} = \mathsf{P}^{d,\alpha}_{(e^{-t_1}x_1,\dots,e^{-t_d}x_d)} \tag{10}$$

and the Birkhoff integral current satisfies the following identity

$$\mathscr{P}_U^{d,(r_1^{t_1}\cdots r_d^{t_d}\alpha)} m = e^{(t_1+\cdots+t_d)/2} \mathscr{P}^{d,\alpha}_{(e^{-t_1},\dots,e^{-t_d})U} m. \tag{11}$$

2.3. **Diophantine condition.**

2.3.1. *Height function.*

Definition 2.3 (Height function)**.** The *height* of a point $Z \in \mathfrak{H}_g$ in Siegel upper half space is the positive number

$$hgt(Z) := \det \Im(Z).$$

The *maximal height function* $Hgt : \Sigma_g \to \mathbb{R}^+$ is the maximal height of a $Sp_{2g}(\mathbb{Z})$ orbit of Z. That is, for the class of $[Z] \in \Sigma_g$,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

By Proposition 4.8 of [CF15], there exists a universal constant $C(s) > 0$ such that the best Sobolev constant satisfies the estimate

$$B_s([[\alpha]]) \le C(s) \cdot (Hgt[[\alpha]])^{1/4}. \tag{12}$$

We rephrase Lemma 4.9 in [CF15] regarding the bound of renormalized height.

Lemma 2.4. *For any $[\alpha] \in \mathfrak{M}_g$ and any $t \ge 0$,*

$$Hgt([[\exp(t\hat{\delta}(d))\alpha]]) \le (det(e^{t\delta}))^2 Hgt([[\alpha]]). \tag{13}$$

2.3.2. *Diophantine condition.* We recall Diophantine condition introduced in [CF15, Definition 4.10] by Flaminio and Cosentino.

Let $\exp \boldsymbol{t}\hat{\delta}(d)$ be the subgroup of $Sp_{2g}(\mathbb{R})$ defined by $\exp(\boldsymbol{t}\hat{\delta}(d))X_i = e^{t_i}X_i$, for $i = 1, \cdots, d$, and $\exp(\boldsymbol{t}\hat{\delta}(d))X_i = X_i$ for $i = d+1, \cdots g$. We also denote $r_{\boldsymbol{t}} = \exp \boldsymbol{t}\hat{\delta}(d)$.

Definition 2.5. An automorphism $\alpha \in Sp_{2g}(\mathbb{R})$ or a point $[\alpha] \in \mathfrak{M}_g$ is $\hat{\delta}(d)$-*Diophantine* of type σ if there exists a $\sigma > 0$ and a constant $C > 0$ such that

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \le CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{(1-\sigma)} Hgt([[\alpha]]), \ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \tag{14}$$

This states that $\alpha \in Sp_{2g}(\mathbb{R})$ satisfies a $\hat{\delta}(d)$-Diophantine if the height of the projection of $\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha$ in the Siegel modular variety Σ_g is bounded by $e^{2(t_1+\cdots t_d)(1-\sigma)}$. Furthermore,

- $[\alpha] \in \mathfrak{M}_g$ satisfies a *$\hat{\delta}$-Roth condition* if for any $\epsilon > 0$ there exists a constant $C > 0$ such that

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \leq CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{\epsilon}Hgt([[\alpha]]), \ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \tag{15}$$

That is, $\hat{\delta}(d)$-Diophantine of type $0 < \sigma < 1$.

- $[\alpha]$ is of *bounded type* if there exists a constant $C > 0$ such that

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]]) \leq C, \ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \tag{16}$$

Definition 2.6. Let $X = G/\Lambda$ be a homogeneous space equipped with the probability Haar measure μ. A function $\phi : X \to \mathbb{R}$ is said k-DL (distance like) for some exponent $k > 0$ if it is uniformly continuous and if there exist constants $C_1, C_2 > 0$ such that

$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

By the work of Margulis and Kleinblock, a generalization of Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

Theorem 2.7. *[KM99, Theorem 1.9] Let G be a connected semisimple Lie group without compact factors, μ its normalized Haar measure, $\Lambda \subset G$ an irreducible lattice, $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G. Let $\mathfrak{d}_+$ be a non-empty open cone in a d-dimensional subalgebra $\mathfrak{d}$ of $\mathfrak{a}$. If $\phi : G/\Lambda \to \mathbb{R}$ is a k-DL function for some $k > 0$, then for μ-almost all $x \in G/\Lambda$ one has*

$$\limsup_{\mathbf{z} \in \mathfrak{d}_+, \mathbf{z} \to \infty} \frac{\phi(\exp(\mathbf{z})x)}{\log \|\mathbf{z}\|} = \frac{d}{k}$$

By Lemma 4.7 of [CF15], the logarithm of Height function is DL-function with exponent $k = \frac{g+1}{2}$ on the Siegel variety Σ_g (and induces on $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$). Hence, we obtain the following proposition.

Proposition 2.8. *Under the assumption $X = \mathfrak{M}_g$ of Theorem 2.7, for $s > g+1/2$, there exists a full measure set $\Omega_g(\hat{\delta})$ and for all $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$*

$$\limsup_{\boldsymbol{t} \to \infty} \frac{\log Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]])}{\log \|\boldsymbol{t}\|} \leq \frac{2d}{g+1}. \tag{17}$$

Any such $[\alpha]$ satisfies a $\hat{\delta}$-Roth condition (15).

For any $L > 0$ and $1 \leq d \leq g$, let $\mathrm{DC}(L)$ denote the set of $[\alpha] \in \mathfrak{M}_g$ such that

$$\int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots t_d)/2} \mathrm{Hgt}([[r_{\text{-}\mathbf{t}}\alpha]])^{1/4} dt_1 \cdots dt_d \leq L. \tag{18}$$

Let DC denote the union of the sets $DC(L)$ over all $L > 0$. It follows immediately that the set $DC \subset \mathfrak{M}_g$ has full Haar volume.

Remark. *We simply rewrite Cosentino-Flaminio's notation to faciliate decomposition of actions $r_{\boldsymbol{t}} = \exp \boldsymbol{t}\hat{\delta}(d)$ and to avoid abusing notation r_t in the following next sections. Therefore, our choice of t_i in $\boldsymbol{t} = (t_i) \in \mathbb{R}^d$ is restrictive since $\boldsymbol{t}$ lies in an open cone in a d-dimensional subspace. I.e, there always exists non-zero constant c_j such that $t_i = c_j t_j$ for each $j \neq i$ whenever $\boldsymbol{t}$ tends to infinity.*

3. Constructions of the functionals

3.1. **Remainder estimates.** For any exponent $s > d/2$, Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^{\perp}$. Denote by $\mathcal{I}^{-s}$ and $\mathcal{R}^{-s}$ the corresponding orthogonal projection operator and by $\mathcal{I}_\alpha^{-s}$ and $\mathcal{R}_\alpha^{-s}$ the restrictions to the fiber over $[\alpha] \in \mathfrak{M}$ for $\alpha \in Sp_{2g}(\mathbb{R})$. In particular, for the current (Birkhoff integrals) $\mathscr{D} = \mathscr{P}_U^{d,\alpha} m$, we call $\mathcal{I}_\alpha^{-s}(\mathscr{D}) = \mathcal{I}^{-s}[\alpha, \mathscr{D}]$ boundary term and $\mathcal{R}_\alpha^{-s}(\mathscr{D}) = \mathcal{R}^{-s}[\alpha, \mathscr{D}]$ remainder term respectively. Consider the orthogonal projection

$$\mathscr{D} = \mathcal{I}^{-s}_{r_{-\boldsymbol{t}}[\alpha]}(\mathscr{D}) + \mathcal{R}^{-s}_{r_{-\boldsymbol{t}}[\alpha]}(\mathscr{D}). \tag{19}$$

We firstly recall the following estimate of boundary terms.

Lemma 3.1. *[CF15, Lemma 5.7] Let $s > d/2 + 2$. There exists a constant $C = C(s) > 0$ such that for all $t_i \geq 0$ for $1 \leq i \leq d$, we have*

$$\begin{aligned}\left\|\mathcal{I}^{-s}[\alpha, \mathscr{D}]\right\|_{-s} &\leq e^{-(t_1+\cdots+t_d)/2} \left\|\mathcal{I}^{-s}[r_1^{-t_1}\cdots r_d^{-t_d}\alpha, \mathscr{D}]\right\|_{-s} \\ &\quad + C_1|t_1+\cdots+t_d| \int_0^1 e^{-u(t_1+\cdots+t_d)/2} \left\|\mathcal{R}^{-s}[r_1^{-ut_1}\cdots r_d^{-ut_d}\alpha, \mathscr{D}\right\|_{-(s-2)} du.\end{aligned}$$

By Stokes' theorem, we have the following remainder estimate.

Lemma 3.2. *[CF15, Lemma 5.6] For any non-negative $s' < s - (d+1)/2$ and Jordan region $U \subset \mathbb{R}^d$, there exists $C = C(d, g, s, s') > 0$ such that*

$$\left\|\mathcal{R}^{-s}[\alpha, (\mathscr{P}_U^{d,\alpha} m)]\right\|_{-s} \leq C \left\|[\alpha, \partial(\mathscr{P}_U^{d,\alpha} m)]\right\|_{-s'}.$$

Quantitative bound of Birkhoff integrals on the square domain was obtained in [CF15], but we need to extend the result on rectangular shapes for analyticity in the section §6.

Theorem 3.3. *For $s > s_d$, there exists a constant $C(s,d) > 0$ such that the following holds. For any $t_i > 0$, $m \in M$ and $U_d(t) = [0, e^{t_1}] \times \cdots \times [0, e^{t_d}]$, we have*

$$\begin{aligned}\left\|[\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)} m)]\right\|_{-s} &\leq C \sum_{k=0}^{d} \sum_{1\leq i_1<\cdots<i_k\leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} \exp\Big(\frac{1}{2}\sum_{l=1}^{d} t_l - \frac{1}{2}\sum_{l=1}^{k} u_{i_l}\Big) \\ &\qquad \times Hgt\Big(\Big[\Big[\prod_{1\leq j\leq d} r_j^{-t_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha\Big]\Big]\Big)^{1/4} du_{i_1}\cdots du_{i_k}.\end{aligned} \tag{20}$$

Proof. We proceed by induction. For $d = 1$, it follows from the Theorem 5.8 in [CF15]. We assume that the result holds for $d-1$. Decompose the current as a sum of boundary and remainder term as in (19).

Step 1. We estimate the boundary term. By Lemma 3.1, renormalize terms with $r^u = r_1^u \cdots r_d^u$. Then, we have

$$
\begin{aligned}
&\left\|\mathcal{I}^{-s}[\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)}m)]\right\|_{-s} \leq e^{-(t_1+\cdots+t_d)/2}\left\|\mathcal{I}^{-s}[r_1^{-t_1}\cdots r_d^{-t_d}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)}m)]\right\|_{-s}\\
&+ C_1(s)\int_0^{t_1+\cdots+t_d} e^{-ud/2}\left\|\mathcal{R}^{-s}[r^{-u}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)}m)]\right\|_{-(s-2)} du\\
&:= (I) + (II).
\end{aligned}
\tag{21}
$$

By renormalization (11) and Lemma 2.1 for unit volume,

$$
\begin{aligned}
\left\|\mathcal{I}^{-s}[r_1^{-t_1}\cdots r_d^{-t_d}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)}m)]\right\|_{-s} &= e^{t_1+\cdots+t_d}\left\|\mathcal{I}^{-s}[r_1^{-t_1}\cdots r_d^{-t_d}\alpha, (\mathscr{P}^{d,r_1^{-t_1}\cdots r_d^{-t_d}\alpha}_{U_d(0)}m)]\right\|_{-s}\\
&\leq C_2 e^{t_1+\cdots+t_d} Hgt([[r_1^{-t_1}\cdots r_d^{-t_d}\alpha]])^{1/4}.
\end{aligned}
$$

Hence

$$
I \leq C_2 e^{(t_1+\cdots t_d)/2} Hgt([[r_1^{-t_1}\cdots r_d^{-t_d}\alpha]])^{1/4},
$$

13 where the sum corresponds to the first term ($k = 0$) in the statement.

Step 2. To estimate (II),

$$
\begin{aligned}
\left\|\mathcal{R}^{-s}[r^{-u}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)}m)]\right\|_{-(s-2)} &= \left\|e^{ud}\mathcal{R}^{-s}[r^{-u}\alpha, (\mathscr{P}^{d,r^{-u}\alpha}_{U_d(t-u)}m)]\right\|_{-(s-2)}\\
&\leq C_3(s,s')e^{ud}\left\|[r^{-u}\alpha, \partial(\mathscr{P}^{d,r^{-u}\alpha}_{U_d(t-u)}m)]\right\|_{-s'}.
\end{aligned}
\tag{22}
$$

The boundary $\partial(\mathsf{P}^{d,r^{-t}\alpha}_{U_d})$ is the sum of $2d$ currents of dimension $d-1$. These currents are Birkhoff sums of d face subgroups obtained from $\mathsf{P}^{d,r^{-t}\alpha}_{U_d}$ by omitting one of the base vector fields X_i. It is reduced to $(d-1)$ dimensional shape obtained from $U_d(t-u) := [0, e^{t_1-u}] \times \cdots [0, e^{t_d-u}]$. For each $1 \leq j \leq d$, there are Birkhoff sums along $d-1$ dimensional cubes. By induction hypothesis, we add all the $d-1$ dimensional cubes by adding all the terms along j:

$$
\begin{aligned}
&\left\|[r^{-u}\alpha, (\mathscr{P}^{d-1,r^{-u}\alpha}_{U_{d-1}(t-u)}m)]\right\|_{-s'} \leq C_4(s', d-1)\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j,\forall l}}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u}\\
&\exp\Big(\frac{1}{2}\sum_{\substack{l=1\\ l\neq j}}^{d}(t_l-u)-\frac{1}{2}\sum_{l=1}^{k}u_{i_l}\Big)Hgt\Big([[\prod_{\substack{1\leq l\leq d\\ l\neq j}} r_l^{-(t_l-u)}\prod_{l=1}^{k} r_{i_l}^{u_{i_l}}(r^{-u}\alpha)]]\Big)^{1/4} du_{i_1}\cdots du_{i_k}.
\end{aligned}
\tag{23}
$$

14 Combining (21) and (22), we obtain the estimate for (II).

$$
\begin{aligned}
(II) \leq C_5(s', d-1)&\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j}}\int_0^{t_1+\cdots t_d}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u} du_{i_1}\cdots du_{i_k}du\\
&\times\exp\Big(\frac{1}{2}\sum_{\substack{l=1\\ l\neq j}}^{d} t_l-\frac{1}{2}u-\frac{1}{2}\sum_{l=1}^{k}u_{i_l}\Big)Hgt\Big([[\prod_{1\leq l\leq d} r_l^{-t_l}\prod_{l=1}^{k} r_{i_l}^{u_{i_l}}(r_j^{-u+t_j}\alpha)]]\Big)^{1/4}.
\end{aligned}
$$

Applying the change of variable $u_j = t_j - u$, we obtain

$$(II) \leq C_6(s', d-1) \sum_{j=1}^{d} \sum_{k=1}^{d-1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq d \\ i_l \neq j}}$$
$$\times \Big(\int_{-(t_1+\cdots t_d)+t_j}^{t_j} \int_0^{t_{i_k}-u} \cdots \int_0^{t_{i_1}-u} du_{i_1} \cdots du_{i_k} du_j$$
$$\times \exp(\frac{1}{2}(t_1 + \cdots t_d) - \frac{1}{2}u_j - \frac{1}{2}\sum_{l=1}^{k} u_{i_l}) Hgt([[\prod_{1\leq l \leq d} r_l^{-t_l} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} (r_j^{-u_j}\alpha)]])^{1/4}\Big).$$

Simplifying multi-summation above, (with $-(t_1 + \cdots t_d) + t_j \leq 0$)

$$(II) \leq C_7(s', d) \sum_{k=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} du_{i_1} \cdots du_{i_k}$$
$$\times \exp(\frac{1}{2}(t_1 + \cdots t_d) - \frac{1}{2}\sum_{l=1}^{k} u_{i_l}) Hgt([[\prod_{1\leq l \leq d} r_l^{-t_l} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha]])^{1/4}.$$

Step 3. (Remainder estimate). The remainder term is obtained from Lemma 3.2 (Stokes' theorem). Following step 2, estimate of remainder reduces to that of $d-1$ form. Combining with the step 1, we have the following

(24)
$$\left\| \mathcal{R}^{-s}[\alpha, \partial(\mathscr{P}^{d,\alpha}_{U_d} m)] \right\|_{-s} \leq C(s) \sum_{i=1}^{d-1} \left\| \mathcal{I}^{-s}[\alpha, (\mathscr{P}^{i,\alpha}_{U_i} m)] \right\|_{-s} + \left\| \mathcal{R}^{-s}[\alpha, (\mathscr{P}^{1,\alpha}_{U_1} m)] \right\|_{-s}$$

where U_i is i-dimensional rectangle. Sum of the boundary terms are absorbed in the bound of $(I) + (II)$. For 1-dimensional remainder with interval Γ_T, the boundary is a 0-dimensional current. Then,

$$\langle \partial(\mathscr{P}^{1,\alpha}_{[0,T]} m), f \rangle = f(\mathsf{P}^{1,\alpha}_T m) - f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (7) and (12),

$$\left\| \mathcal{R}^{-s}[\alpha, \partial(\mathscr{P}^{1,\alpha}_{[0,T]} m)] \right\|_{-s} \leq 2B_s([[\alpha]]) \leq C(s) Hgt([[\alpha]])^{1/4}.$$

Then, by inequality (13)

$$\begin{aligned} C(s) Hgt([[\alpha]])^{1/4} &= C(s) Hgt([[r_{\boldsymbol{t}} r_{-\boldsymbol{t}} \alpha]])^{1/4} \\ &\leq C(s) e^{(t_1+\cdots t_d)/2} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d} \alpha]])^{1/4}. \end{aligned}$$

This implies that remainder term produces one more term like the bound of (I). Therefore, the Theorem follows from combining all the terms $(I), (II)$, and remainder. □

Remark. *It is also possible to renormalize a rectangle $U_d(t)$ by several squares and sum up all of these sub-dimeinsonal (dim $< d$) squares. However, it may also involve computational difficulty and such approach of summing up squares may provide a rough upper bound. To obtain necessary bound for the Lemma 3.4, we rather simply generalized the methods of Theorem 5.10 in [CF15].*

Now we prove the remainder estimate that will be used in (31).

Lemma 3.4. *Let $s > s_d$. There exists a constant $C(s,\Gamma) > 0$ such that for any rectangle $U_\Gamma = [0, e^{\Gamma_1}] \times \cdots \times [0, e^{\Gamma_d}]$,*

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C(s,\Gamma) Hgt([r_{-\boldsymbol{t}}\alpha])^{1/4}.$$

Proof. Recall that $\mathcal{K}_{\alpha,t}(\Gamma) = \left\| \mathcal{R}^{-s}[r_{-\boldsymbol{t}}[\alpha], (\mathscr{P}^{d,r_{-\boldsymbol{t}}\alpha}_{U_\Gamma} m)] \right\|_{-(s+1)}$ and by Lemma 3.2, it is equivalent to find the bound of $d-1$ currents. By Theorem 3.3, we obtain the remainder estimate.

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \int_0^{\Gamma_{i_k}} \cdots \int_0^{\Gamma_{i_1}} \exp(\frac{1}{2}\sum_{l=1}^{d-1} \Gamma_l - \frac{1}{2}\sum_{l=1}^{k} u_{i_l})$$
$$\times Hgt([[\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} (r_{-\boldsymbol{t}}\alpha)]])^{1/4} du_{i_1} \cdots du_{i_k}.$$

It follows from (13) that for $0 \leq k \leq d-1$,

$$Hgt([[\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} (r_{-\boldsymbol{t}}\alpha)]])^{1/4} \leq e^{\frac{1}{2}(\sum_{l=1}^{k} u_{i_l} - \sum_{l=1}^{d} \Gamma_l)} Hgt([[r_{-\boldsymbol{t}}\alpha]])^{1/4}.$$

Then, we obtain

$$\left\| \mathcal{R}^{-s}[r_{-\boldsymbol{t}}\alpha, (\mathscr{P}^{d,r_{-\boldsymbol{t}}\alpha}_{U_\Gamma} m)] \right\|_{-s} \leq C(s) \Big(\sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \prod_{l=1}^{k} \Gamma_{i_l} \Big) Hgt([[r_{-\boldsymbol{t}}\alpha]])^{1/4}. \tag{25}$$

Setting $C(s,\Gamma) = C(s)\Big(\sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \prod_{l=1}^{k} \Gamma_{i_l} \Big)$, we obtain the conclusion. □

3.2. **Constructions of the functionals.** For fixed α, let $\Pi_H^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \to A_d(\mathfrak{p}, W_\alpha^{-s}(H))$ denote the orthogonal projection on a single irreducible unitary representation. We further decompose projection operators with

$$\Pi_H^{-s} = \mathscr{B}_\alpha^{-s}(\Gamma) B_\alpha^{-s,H} + R_\alpha^{-s,H}$$

where $\mathscr{B}_{H,\alpha}^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \to \mathbb{C}$ denote the orthogonal component map of P-invariant currents (closed), supported on a single irreducible unitary representation.

The *Bufetov functionals* on a rectangle $\Gamma \in \mathfrak{R}$ are defined for all $\alpha \in DC$ as follows.

Lemma 3.5. *Let $\alpha \in DC(L)$. For $s > s_d = d(d+11)/4 + g + 1/2$, the limit*

$$\hat{\beta}_H(\alpha,\Gamma) = \lim_{t_1,\ldots,t_d \to \infty} e^{-(t_1+\cdots t_d)/2} \mathscr{B}^{-s}_{H,r_{-t}[\alpha]}(\Gamma)$$

exists and define a finitely-additive finite measure on the standard rectangle (3) $\Gamma := \Gamma_{\boldsymbol{T}}^X(m)$ for $m \in M$. Then there exists a constant $C(s,\Gamma) > 0$ such that the following estimate holds:

$$|\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha,\Gamma) B_\alpha^H|_{\alpha,-s} \leq C(s,\Gamma)(1+L). \tag{26}$$

Proof. For simplicity, we omit dependence of H. For every $\mathbf{t} \in \mathbb{R}^d$, we have the following orthogonal splitting:

$$\Pi_{H,\alpha}^{-s}(\Gamma) = \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}},$$

where
$$\mathscr{B}^{-s}_{\alpha,\mathbf{t}} := \mathscr{B}^{-s}_{H,r_{\text{-}\mathbf{t}}[\alpha]},\ B_{\alpha,\mathbf{t}} := B^{-s,H}_{r_{\text{-}\mathbf{t}}[\alpha]},\ R_{\alpha,\mathbf{t}} := R^{-s,H}_{r_{\text{-}\mathbf{t}}[\alpha]}.$$
For any $\mathbf{h} \in \mathbb{R}^d$, we have
$$\mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(\Gamma)B_{\alpha,\mathbf{t+h}} + R_{\alpha,\mathbf{t+h}} = \mathscr{B}^{-s}_{\alpha,\mathbf{t}}(\Gamma)B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}}.$$

By reparametrization (10), we have $B_{\mathbf{t+h}} = e^{-(h_1+\cdots h_d)/2}B_{\mathbf{t}}$ and
$$\mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(\Gamma) = e^{(h_1+\cdots h_d)/2}\mathscr{B}^{-s}_{\alpha,\mathbf{t}}(\Gamma) + \mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(R_{\alpha,\mathbf{t}}) \tag{27}$$
and it follows that
$$\mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(\Gamma) = e^{h_1/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) + \mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(R_{\alpha,\mathbf{t}}).$$

By differentiating at $h_1 = 0$,
$$\frac{d}{dt_1}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) + [\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(R_{\alpha,\mathbf{t}})]_{h_1=0}. \tag{28}$$

Therefore, we solve the following first order ODE
$$\frac{d}{dt_1}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) + \mathcal{K}^{(1)}_{\alpha,\boldsymbol{t}}(\Gamma)$$
where
$$\mathcal{K}^{(1)}_{\alpha,\boldsymbol{t}}(\Gamma) = [\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(R_{\alpha,\mathbf{t}})]_{h_1=0}.$$

Then, the solution of the differential equation is
$$\begin{aligned}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots t_d+h_d}(\Gamma) &= e^{t_1/2}[\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) + \int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)d\tau_1] \\ &= e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) + \int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)d\tau_1.\end{aligned}$$

Note by reparametrization
$$e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) = e^{h_2/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3\cdots t_d+h_d}(\Gamma)$$
and it is possible to differentiate the previous equation with respect to h_2 again. Then
$$\frac{d}{dt_2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1. \tag{29}$$
where $\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma) = \frac{d}{dh_2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)$.

Then, the solution of (29) is
$$\begin{aligned}&\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma) \\ &= e^{t_2/2}[\mathscr{B}^{-s}_{\alpha,t_1,0,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_2} e^{-\tau_2/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2] \\ &= e^{h_3/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3,\cdots t_d+h_d} + \int_0^{t_2} e^{(t_2-\tau_2)/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2.\end{aligned}$$

Inductively, we solve first order ODE repeatedly and obtain the following solution
$$\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}(\Gamma) = e^{(t_1+\cdots t_d)/2}\Big(\mathscr{B}^{-s}_{\alpha,0} + \int_0^{t_d}\cdots\int_0^{t_1} e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma)d\tau_1\cdots d\tau_d\Big) \tag{30}$$

where

$$\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma) = [\frac{d}{dh_d} \cdots \frac{d}{dh_1} \mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}})]_{h_d \cdots h_1 = 0}.$$

Let $\langle \cdot, \cdot \rangle_{\alpha,\boldsymbol{t}}$ denote the inner product in Hilbert space $\Omega^{-s}_{r_{\text{-}\mathbf{t}}[\alpha]}$. By the intertwining formula (9),

$$\begin{aligned}
\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}+\boldsymbol{h}}(R_{\alpha,\boldsymbol{t}}) &= \langle R_{\alpha,\boldsymbol{t}}, \frac{B_{\alpha,\boldsymbol{t}+\boldsymbol{h}}}{|B_{\alpha,\boldsymbol{t}+\boldsymbol{h}}|^2_{\boldsymbol{t}+\boldsymbol{h}}} \rangle_{\alpha,\boldsymbol{t}+\boldsymbol{h}} \\
&= \langle R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}, \frac{B_{\alpha,\boldsymbol{t}+\boldsymbol{h}} \circ U_{-\boldsymbol{h}}}{|B_{\alpha,\boldsymbol{t}+\boldsymbol{h}}|^2_{\boldsymbol{t}+\boldsymbol{h}}} \rangle_{\alpha,\boldsymbol{t}} \\
&= \langle R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}, \frac{B_{\alpha,\boldsymbol{t}}}{|B_{\alpha,\boldsymbol{t}}|^2_t} \rangle_{\alpha,\boldsymbol{t}} = \mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}(R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}).
\end{aligned}$$

In the sense of distributions,

$$\begin{aligned}
\frac{d}{dh_d} \cdots \frac{d}{dh_1}(R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}) &= -R_{\alpha,t} \circ (\frac{d}{2} + \sum_{i=1}^{d} X_i(t)) \circ U_{-\boldsymbol{h}} \\
&= [(\sum_{i=1}^{d} X_i(t) - \frac{d}{2}) R_{\alpha,\boldsymbol{t}}] \circ U_{-\boldsymbol{h}}
\end{aligned}$$

and we compute derivative term of (28) in representation,

$$[\frac{d}{dh_d} \cdots \frac{d}{dh_1}(\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}+\boldsymbol{h}}(R_{\alpha,t}))]_{\boldsymbol{h}=0} = -\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2}) R_{\alpha,\boldsymbol{t}})).$$

Set $\mathcal{K}_{\alpha,t}(\Gamma) = |R_{\alpha,\boldsymbol{t}}|_{r_{-\boldsymbol{t}}\alpha,-(s+1)}$ with a bounded non-negative function, then by Lemma 3.4, we claim that

$$|\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2}) R_{\alpha,\boldsymbol{t}}))| \leq \mathcal{K}_{\alpha,\tau}(\Gamma) \leq C(s,\Gamma) Hgt([r_{-\boldsymbol{t}}\alpha])^{1/4}. \tag{31}$$

Therefore, the solution of equation (30) exists under Diophantine condition (18) and the following holds:

$$\lim_{t_d \to \infty} \cdots \lim_{t_1 \to \infty} e^{-(t_1 + \cdots t_d)/2} \mathscr{B}^{-s}_{\alpha,t}(\Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

Moreover, the complex number

$$\hat{\beta}_H(\alpha,\Gamma) = \mathscr{B}^{-s}_{\alpha,0} + \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots \tau_d)/2} \mathcal{K}_{\alpha,\tau}(\Gamma) d\tau_1 \cdots d\tau_d$$

depends continuously on $\alpha \in DC(L)$. Since we have

$$\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}(\alpha,\Gamma) B^H_\alpha = R_0 - \left(\int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots \tau_d)/2} \mathcal{K}_{\alpha,\tau}(\Gamma) d\tau_1 \cdots d\tau_d \right) B^H_\alpha,$$

by Diophantine condition again,

$$|\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}_H(\alpha,\Gamma) B^H_\alpha|_{\alpha,-s} \leq C(s,\Gamma)(1+L).$$

$\square$

3.3. **Proof of Theorem 1.3.** The proof of Theorem 1.3 follows immediately from the refinement to the constructions of Bufetov functionals (see also [BF14, §2.5] for horocycle flows).

Notation. The action of flow $\{r_t\}_{t\in\mathbb{R}}$ on a current C is defined by pull-back as follows:

$$(r_t^*C)(\omega) = C(r_{-t}^*\omega), \quad \text{for any smooth form } \omega.$$

Lemma 3.6 (Invariance)**.** *Let $1 \leq d \leq g$. The functional $\hat{\beta}_H$ defined on standard rectangle $\Gamma_{\boldsymbol{T}}^X$ is invariant under the action of $(\mathsf{Q}_y^{d,Y})$ for any $y \in \mathbb{R}^d$. That is,*

$$\hat{\beta}_H(\alpha, (\mathsf{Q}_\tau^{d,Y})_*\Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

Proof. First, we prove the functional $\hat{\beta}_H$ exists and it is invariant under the action $\mathsf{Q}_y^{d,Y}$. It suffices to verify the invariance property under the rank 1 action $\mathsf{Q}_\tau^{1,Y}$ for $\tau \in \mathbb{R}$.

Given a standard d-dimensional rectangle Γ, set $\Gamma_{\mathsf{Q}} := (\mathsf{Q}_\tau^{1,Y})_*\Gamma$. Let $D(\Gamma, \Gamma_{\mathsf{Q}})$ be the $(d+1)$ dimensional space spanned by trajectories of the action of $\mathsf{Q}_\tau^{1,Y}$ projecting Γ onto Γ_{Q}. Then $D(\Gamma, \Gamma_{\mathsf{Q}})$ is a union of all orbits I of action $\mathsf{Q}_\tau^{1,Y}$ such that the boundary of I, d-dimensional faces, is contained in $\Gamma\cup\Gamma_{\mathsf{Q}}$, and interior of I is disjoint from $\Gamma\cup\Gamma_{\mathsf{Q}}$. $D(\Gamma, \Gamma_{\mathsf{Q}})$ is defined by integration and it is a $(d+1)$-current.

By definition, denote $r_t := r_i^t$. Then, $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_{\mathsf{Q}})$ are respectively the support of the currents $r_t^*\Gamma$ and $r_t^*\Gamma_{\mathsf{Q}}$. Thus, we have the following identity

$$r_t^*D(\Gamma, \Gamma_{\mathsf{Q}}) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}})).$$

Since the current $\partial D(\Gamma, \Gamma_{\mathsf{Q}}) - (\Gamma - \Gamma_{\mathsf{Q}})$ is composed of orbits for the action $\mathsf{Q}_\tau^{1,Y}$, it follows that

$$\partial[r_t^*D(\Gamma, \Gamma_{\mathsf{Q}})] - (r_t^*\Gamma - r_t^*\Gamma_{\mathsf{Q}}) = r_t^*[\partial D(\Gamma, \Gamma_{\mathsf{Q}}) - (\Gamma - \Gamma_{\mathsf{Q}})] \to 0. \tag{32}$$

Now, we turn to prove the volume of $D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}}))$ is uniformly bounded for all $t > 0$. For any $p \in \Gamma$, set $\tau(p)$ be length of the arc lying in $D_Q := D(\Gamma, \Gamma_{\mathsf{Q}})$, and set $\tau_\Gamma := \sup\{\tau(p) \mid p \in \Gamma\} < \infty$. We write

$$vol_{d+1}(D_Q) = \int_\Gamma \tau dvol_d.$$

Since $vol_d(r_{-t}\Gamma) \leq e^t vol_d(\Gamma)$,

$$vol_{d+1}(r_{-t}D_Q) = \int_{r_{-t}\Gamma} \tau dvol_d \leq \tau_\Gamma e^{-t} vol_d(r_{-t}\Gamma) \leq \tau_\Gamma vol_d(\Gamma) < \infty. \tag{33}$$

Note that d-dimensional current $(\mathsf{Q}_\tau^{1,Y})_*\Gamma - \Gamma$ is equal to the boundary of the $(d+1)$ dimensional current D_Q. By arguments in remainder estimate (or Sobolev embedding theorem),

$$|(\mathsf{Q}_\tau^{1,Y})_*\Gamma - \Gamma|_{r^{-t}\alpha,-s} \leq C_s \tau B_s([[r^{-t}\alpha]]) \leq C_s \tau \mathrm{Hgt}[[r^{-t}\alpha]]^{1/4} \tag{34}$$

is finite for all $t > 0$.

Then, by (32), (33) and existence of Bufetov functional $\hat{\beta}_H(\alpha, \Gamma)$, the last inequality holds:

$$|\mathscr{B}_{\alpha,t}^{-s}((\mathsf{Q}_\tau^{1,Y})_*\Gamma) - \mathscr{B}_{\alpha,t}^{-s}(\Gamma)|_{\alpha,-s} < \infty.$$

Therefore, by the definition of Bufetov functional in the Lemma 3.5, $\hat{\beta}_H(\alpha, (\mathsf{Q}_\tau^{1,Y})_*\Gamma)$ exists and $\hat{\beta}_H(\alpha, \Gamma)$ is invariant under the action of $\mathsf{Q}_\tau^{1,Y}$. $\square$

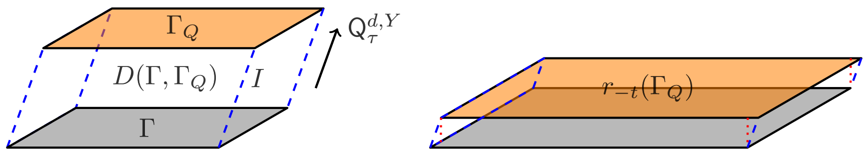

FIGURE 1. Illustration of the standard d-rectangles Γ, Γ_Q, $d+1$ dimensional current $D(\Gamma,\Gamma_{\mathsf{Q}})$ and supports of $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_{\mathsf{Q}})$.

Proof of Theorem 1.3. Additive property. It follows from the linearity of projections and limit. *Scaling property* is immediate from the definition.

Bounded property. By the scaling property,

$$\hat{\beta}_H(\alpha,\Gamma) = e^{dt/2}\hat{\beta}_H(r^t[\alpha],\Gamma).$$

Choose $t = \log(\int_\Gamma |\hat{X}|)$ and $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$, then uniform bound of Bufetov functional on bounded size of rectangles,

$$|\hat{\beta}_H(\alpha,\Gamma)| \leq C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

Invariance property follows directly from the Lemma 3.6. □

3.4. **Proof of Theorem 1.4 and 1.5.** We define the excursion function for $\boldsymbol{T} = (T^{(i)}) \in \mathbb{R}^d$,

$$\begin{aligned} E_{\mathfrak{M}}(\alpha,\mathbf{T}) &:= \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1+\cdots t_d)/2}\mathrm{Hgt}([[r_{\mathbf{t}-\log\mathbf{T}}\alpha]])^{1/4} dt_1\cdots dt_d \\ &= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1+\cdots t_d)/2}\mathrm{Hgt}([[r_{\mathbf{t}}\alpha]])^{1/4} dt_1\cdots dt_d. \end{aligned}$$

Notation. We write

$$vol(U(\boldsymbol{T})) := \prod_{i=1}^d T^{(i)}, \quad vol(U(\boldsymbol{t})) = \prod_{i=1}^d t_i. \tag{35}$$

Lemma 3.7 (Bound of functionals)**.** *For any Diophantine $[\alpha] \in DC(L)$ and for any $f \in W^s(M)$ for $s > s_d + 1/2$, the Bufetov functional β^f is defined by a uniformly convergent series:*

$$|\beta^f(\alpha,m,\boldsymbol{tT})| \leq C_s \left(L + vol(U(\boldsymbol{T}))^{1/2}(1 + vol(U(\boldsymbol{t})) + E_{\mathfrak{M}}(\alpha,\boldsymbol{T}))\right)\|\omega\|_{\alpha,s}$$

for $\omega = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p} \otimes W^s(M)$ and $\boldsymbol{tT} = (t_1T^{(1)},\cdots,t_dT^{(d)}) \in \mathbb{R}^d$.

Proof. It follows from Lemma 3.5 that there exists a constant $C > 0$ such that whenever $\alpha \in DC(L)$, then

$$|\beta_H(\alpha,m,\mathbf{t})| \leq C(1 + L + vol(U(\boldsymbol{t}))), \quad (m,\mathbf{t}) \in M \times \mathbb{R}^d. \tag{36}$$

By the scaling property of Bufetov functionals,

$$\beta_H(\alpha,m,\boldsymbol{tT}) = vol(U(\boldsymbol{T}))^{1/2}\beta_H(r_{\log\mathbf{T}}[\alpha],m,\mathbf{t}). \tag{37}$$

By Diophantine condition (18), whenever $\alpha \in DC(L)$ then $r_{\log\mathbf{T}}[\alpha] \in DC(L_{\mathbf{T}})$ with

$$L_{\mathbf{T}} \leq Lvol(U(\boldsymbol{T}))^{-1/2} + E_{\mathfrak{M}}(\alpha,\mathbf{T}).$$

Thus by (36), we obtain for all $(m, \mathbf{t}) \in M \times \mathbb{R}^d$,

$$|\beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t})| \leq C(1 + L_{\mathbf{T}} + vol(U(\boldsymbol{t}))).$$

From (37), it follows that for all $s > s_d$ and $q > 1/2$

$$\begin{aligned}|\beta^f(\alpha, m, \boldsymbol{tT})| &\leq C_s vol(U(\boldsymbol{T}))^{1/2}(1 + L + vol(U(\boldsymbol{t}))) \sum_{n\in\mathbb{Z}} \|\omega_n\|_{\alpha,s} \\ &\leq C_s vol(U(\boldsymbol{T}))^{1/2}(1 + L_{\mathbf{T}} + vol(U(\boldsymbol{t})))(\sum_{n\in\mathbb{Z}}(1+n^2)^q)^{-1/2} \\ &\times (\sum_{n\in\mathbb{Z}} \left\|(1 - Z^2)^{q/2}\omega_n\right\|^2_{\alpha,s})^{1/2}.\end{aligned}$$

Therefore, for all $s' = s_d + q$, there exists a constant $C_{s'} > 0$ such that

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_{s'} vol(U(\boldsymbol{T}))^{1/2}(1 + L_{\mathbf{T}} + vol(U(\boldsymbol{t}))) \|\omega\|_{\alpha,s'}.$$

$\square$

By Lemma 3.5 and 3.7, asymptotic formula on each irreducible component provides the following.

Corollary 3.8. *For all $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for all $\alpha \in DC(L)$ and for all $f \in W^s(M)$ and for all $(m, \boldsymbol{T}) \in M \times \mathbb{R}^d$, we have*

$$|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_f \right\rangle - \beta^f(\alpha, m, \boldsymbol{T})| \leq C_s(1 + L) \|\omega_f\|_{\alpha,s}. \tag{38}$$

for $U(\boldsymbol{T}) = [0, T^{(1)}] \times \cdots [0, T^{(d)}]$ and $\omega = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p} \otimes W^s(M)$.

Proof of Theorem 1.4. The Lemma 3.7 implies that we can obtain bounded property for the cocycle $\beta(\alpha, m, \mathbf{T})$. Also, all the properties of Bufetov functionals β_H associated to a single irreducible component (Theorem 1.3) are extended to β^f for any $f \in W^s(M), \forall s > s_d + 1/2$ in the similar way (it follows from the convergence of series on irreducibles). $\square$

Proof of Theorem 1.5. The theorem follows from Corollary 3.8 for $\alpha \in \bigcup_{L>0} DC(L)$. $\square$

4. Limit distributions

In this section, we prove Theorem 1.6, limit distribution of Birkhoff sums of higher rank actions on rectangles.

4.1. **Limiting distributions.**

Lemma 4.1. *There exists a continuous modular function $\theta_H : Aut_0(\mathsf{H}^g) \to H \subset L^2(M)$ such that*

$$\lim_{|U(\boldsymbol{T})|\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\cdot), \omega_f \right\rangle - \theta_H(r_{\log \boldsymbol{T}}[\alpha]) D^H_\alpha(f) \right\|_{L^2(M)} = 0. \tag{39}$$

The family $\{\theta_H(\alpha) \mid \alpha \in Aut_0(\mathsf{H}^g)\}$ has a constant norm in $L^2(M)$.

Proof. By the Fourier transform, the space of smooth vectors and Sobolev space $W^s(H)$ is represented as the Schwartz space $\mathscr{S}^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |(1 + \sum_i \frac{\partial^2}{\partial u_i^2} + \sum_i u_i^2)^{s/2} \hat{f}(u)|^2 du < \infty.$$

Let $\boldsymbol{t}, \boldsymbol{u} \in \mathbb{R}^d$. Then we claim for any $f \in \mathscr{S}^s(\mathbb{R}^d)$, there exists a function $\theta([\alpha])(\cdot) \in L^2(\mathbb{R}^d)$ such that

$$\lim_{|U(\boldsymbol{T})|\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} f(\boldsymbol{u}+\boldsymbol{t})d\boldsymbol{t} - \theta_H(r_{\log \boldsymbol{T}}[\alpha])(u) Leb(f) \right\|_{L^2(\mathbb{R}^d, d\boldsymbol{u})} = 0.$$

This is equivalent to the statement (39). By the standard Fourier transform on $\mathbb{R}^d$, equivalently

$$\lim_{|U(\boldsymbol{T})|\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} e^{i\boldsymbol{t}\cdot\hat{\boldsymbol{u}}} \hat{f}(\hat{\boldsymbol{u}})dt - \hat{\theta}_H(r_{\log \boldsymbol{T}}[\alpha])(\hat{u})\hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = 0.$$

For $\chi \in L^2(\mathbb{R}^d, d\hat{u})$ and $\hat{\boldsymbol{u}} = (\hat{u}_j)_{1\le j\le d}$, we denote

$$\chi_j(\hat{u}) = \frac{e^{i\hat{u}_j} - 1}{i\hat{u}_j},\ \chi(\hat{u}) = \prod_{j=1}^d \chi_j(\hat{u}).$$

Let $\hat{\theta}([\alpha])(\hat{\boldsymbol{u}}) := \chi(\hat{\boldsymbol{u}})$ for all $\hat{\boldsymbol{u}} \in \mathbb{R}^d$. Now we will compute $\theta(r_{\log \boldsymbol{T}}[\alpha])$. By intertwining formula (9) for $\mathbf{T} \in \mathbb{R}^d$,

$$U_{\mathbf{T}}(f)(\hat{\boldsymbol{u}}) = \prod_{i=1}^d (T^{(i)})^{1/2} f(\boldsymbol{T}\hat{\boldsymbol{u}}),\ \text{for } \boldsymbol{T}\hat{\boldsymbol{u}} = (T^{(1)}\hat{u}_1, \cdots, T^{(d)}\hat{u}_d).$$

Then, for all $\alpha \in Aut_0(\mathsf{H}^g)$,

$$\hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{\boldsymbol{u}}) = U_{\mathbf{T}}(\chi)(\hat{\boldsymbol{u}}) = vol(U(\boldsymbol{T}))^{1/2}\chi(\boldsymbol{T}\hat{\boldsymbol{u}}).$$

The function $\theta[\alpha]$ is defined by inverse Fourier transform of $\hat{\theta}[\alpha]$

$$\|\theta_H([\alpha])\|_H = \|\theta([\alpha])\|_{L^2(\mathbb{R}^d)} = \left\|\hat{\theta}([\alpha])\right\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{\boldsymbol{u}})\|_{L^2(\mathbb{R}^d, d\hat{u})} = C > 0.$$

By integration,

$$\begin{aligned} \int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{i\boldsymbol{t}\cdot\hat{\boldsymbol{u}}} \hat{f}(\hat{\boldsymbol{u}})dt &= vol(U(\boldsymbol{T}))\chi(\boldsymbol{T}\hat{\boldsymbol{u}})\hat{f}(\hat{\boldsymbol{u}}) \\ &= vol(U(\boldsymbol{T}))\chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{\boldsymbol{u}}) - \hat{f}(0)) + vol(U(\boldsymbol{T}))^{1/2}\hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{\boldsymbol{u}})\hat{f}(0). \end{aligned} \tag{40}$$

Then the claim reduces to the following:

$$\limsup_{|U(\boldsymbol{T})|\to\infty} \left\| vol(U(\boldsymbol{T}))^{1/2}\chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{\boldsymbol{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If $f \in \mathscr{S}^s(\mathbb{R}^g)$ with $s > d/2$, function $\hat{f} \in C^0(\mathbb{R}^d)$ and bounded. Thus, by Dominated convergence theorem and change of variables,

$$\left\| vol(U(\boldsymbol{T}))^{1/2}\chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{\boldsymbol{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = \left\| \chi(\nu)(\hat{f}(\frac{\boldsymbol{\nu}}{\boldsymbol{T}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\nu)} \to 0.$$

19 $\square$

Corollary 4.2. *For any $s > d/2$, $\alpha \in Aut_0(\mathsf{H}^g)$ and $f \in W^s(H)$, there exists a constant $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_f \right\rangle \right\|_{L^2(M)} = C |D^H_\alpha(f)|.$$

From Corollary 4.2, we derive the following limit result for the L^2-norm of Bufetov functionals.

Corollary 4.3. *For every irreducible component H and $\alpha \in DC$, there exists $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \|\beta_H(\alpha, \cdot, \boldsymbol{T})\|_{L^2(M)} = C.$$

Proof. By the normalization of invariant distribution in Sobolev space, for any $\alpha \in Aut_0(\mathsf{H}^g)$, there exists a function $f^H_\alpha \in W^s_\alpha(H)$ such that $D_\alpha(f^H_\alpha) = \left\| f^H_\alpha \right\|_s = 1$. For all $\alpha \in DC(L)$, by asymptotic formula (38)

$$\left| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega \right\rangle - \beta^f(\alpha, m, \boldsymbol{T}) \right| \le C_s(1+L).$$

Therefore, L^2-estimate follows from Corollary 4.2. □

A relation between the Bufetov functional and the modular function θ_H is established below.

Corollary 4.4. *For every irreducible component $H \subset L^2(M)$, the following holds. For any $L > 0$ and any $r_{\mathbf{t}}$-invariant probability measure μ supported on $DC(L) \subset \mathfrak{M}_g$,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H([\alpha])(\cdot), \quad \text{for } \mu\text{-almost all } [\alpha] \in \mathfrak{M}_g.$$

Proof. By Theorem 1.5 and Lemma 4.1, there exists a constant $C > 0$ such that for all $\alpha \in supp(\mu) \subset DC(L)$ and $\mathbf{T} \in \mathbb{R}^d_+$, we have

$$\lim_{|U(\boldsymbol{T})| \to \infty} \|\beta_H(r_{\log \boldsymbol{T}}[\alpha], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T}}[\alpha])\|_{L^2(M)} \le \frac{C_\mu}{vol(U(\boldsymbol{T}))^{1/2}}. \tag{41}$$

By Luzin's theorem, for any $\delta > 0$ there exists a compact subset $E(\delta) \subset \mathfrak{M}$ such that we have the measure bound $\mu(\mathfrak{M} \backslash E(\delta)) < \delta$ and the function $\beta_H(\alpha, \cdot, 1) \in L^2(M)$ depends continuously on $[\alpha] \in E(\delta)$. By Poincaré recurrence, there is a full measure subset $E'(\delta) \subset E(\delta)$ of $\mathbb{R}^d$-action.

For every $\alpha_0 \in E'(\delta)$, there is diverging sequence (t_n) such that $\{r_{t_n}(\alpha_0)\} \subset E(\delta)$ and $\lim_{n\to\infty} r_{t_n}(\alpha_0) = (\alpha_0)$. By continuity of θ_H and β_H at $[\alpha_0]$, we have

$$\begin{aligned} &\|\beta_H([\alpha_0], \cdot, 1) - \theta_H([\alpha_0])\|_{L^2(M)} \\ &\qquad = \lim_{n\to\infty} \|\beta_H(r_{\log \boldsymbol{T_n}}[\alpha_0], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T_n}}[\alpha_0])\|_{L^2(M)} = 0. \end{aligned} \tag{42}$$

Thus, we have $\beta_H([\alpha], \cdot, 1) = \theta_H([\alpha]) \in L^2(M)$ for all $[\alpha] \in E'(\delta)$. It follows that the set where the equality (42) fails has a measure less than any $\delta > 0$, thus the identity holds for μ-almost all $[\alpha]$. □

For all $\alpha \in Aut_0(\mathsf{H}^g)$, smooth function $f \in W^s(M)$ for $s > s_d + 1/2$ decompose as an infinite sum, and the functional θ^f is defined by a convergent series

$$\theta^f(\alpha) := \sum_H D^H_\alpha(f) \theta_H(\alpha). \tag{43}$$

Hence, the function $\theta^f : Aut_0(\mathsf{H}^g) \to L^2(M)$ is continuous.

The following result is an extension of Lemma 4.1 to an asymptotic formula.

Lemma 4.5. *For all $\alpha \in Aut_0(\mathsf{H}^g)$, $f \in W^s(M)$ and $s > s_d + 1/2$,*

$$\lim_{n\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T_n}))^{1/2}} \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T_n})} m, \omega_f \right\rangle - \theta^f(r_{\log \boldsymbol{T_n}}\alpha) \right\|_{L^2(M)} = 0.$$

4.2. **Proof of Theorem 1.6.** By Theorem 4.5, we summarize our results on limit distributions for higher rank actions.

Theorem 4.6. *Let $(\boldsymbol{T_n})$ be any sequence such that*

$$\lim_{n\to\infty} r_{\log \boldsymbol{T_n}}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_d + 1/2$, which is not a coboundary, the limit distribution of the family of random variables

$$E_{T_n}(f) := \frac{1}{vol(U(\boldsymbol{T_n}))^{1/2}} \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T_n})}(\cdot), \omega_f \right\rangle$$

exists and is equal to the distribution of the function $\theta^f(\alpha_\infty) = \beta(\alpha, \cdot, 1) \in L^2(M)$. If $\alpha_\infty \in DC$, then $\theta^f(\alpha_\infty)$ is bounded function on M, and the limit distribution has compact support.

Proof of Theorem 1.6. Since $\alpha_\infty \in \mathfrak{M}_g$, the existence of limit follows from the Lemma 4.5. $\square$

A relation with Birkhoff integrals and theta sum was introduced in [CF15, §5.3], and as an applications, we derive limit theorem of theta sums.

Corollary 4.7. *Let $\mathscr{Q}[x] = x^\top \mathscr{Q} x$ be the quadratic forms defined by $g \times g$ real matrix $\mathscr{Q}$, $\alpha = \begin{pmatrix} I & 0 \\ \mathscr{Q} & I \end{pmatrix} \in Sp_{2g}(\mathbb{R})$, $\ell(x) = \ell^\top x$ be the linear form defined by $\ell \in \mathbb{R}^g$. Then, Theta sum*

$$\Theta(\mathscr{Q}, \ell; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0,N]} \exp(2\pi\iota(\mathscr{Q}[n] + \ell(n)))$$

has a limit distribution and it has compact support.

5. L^2-lower bounds

In this section we prove L^2- lower bounds of ergodic integrals on transverse torus.

5.1. **Structure of return map.** Let $\mathbb{T}^{g+1}_\Gamma$ denote $(g+1)$-dimensional torus with standard frame (X_i, Y_i, Z) with

$$\mathbb{T}^{g+1}_\Gamma := \{\Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

It is convenient to work with the polarized Heisenberg group. Set $H^g_{pol} \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ equipped with the group law $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx')$.

Definition 5.1. Reduced standard Heisenberg group H^g_{red} is defined by quotient $H^g_{pol}/(\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}/\frac{1}{2}\mathbb{Z}$. Reduced standard latice Γ^g_{red} is $\mathbb{Z}^g \times \mathbb{Z}^g \times \{0\}$ and the quotient H^g_{red}/Γ^g_{red} is isomorhic to standard Heisenberg manifold H^g/Γ.

Now, we consider a return map of $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}^{g+1}_{\Gamma}$. For $x = (x_1, \cdots, x_g) \in \mathbb{R}^g$,

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta \in \mathbb{R}^d.$$

In H^g_{red},

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x).$$

Then, given $(n, m, 0) \in \Gamma^g_{red}$,

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) \cdot (n, m, 0) = \exp(x'_1 X_1^\alpha + \cdots + x'_g X_g^\alpha) \cdot (0, y', z') \tag{44}$$

if and only if

$$x'_\alpha = x_\alpha + n, y' = y + (x_\beta - x'_\beta) + m \text{ and } z' = z + (w - w') \cdot x + n(y + x_\beta).$$

Assume $\langle X_i^\alpha, X_j \rangle \neq 0$ for all i, j, and we write first return time for $\mathsf{P}^{d,\alpha}$ action

$$t_{Ret} = (t_{Ret,1}, \cdots, t_{Ret,d}) \in \mathbb{R}^d$$

on transverse torus $\mathbb{T}^{g+1}_{\Gamma}$. We denote the domain for return time $U(t_{Ret}) = [0, t_{Ret,1}] \times \cdots \times [0, t_{Ret,d}]$. Return map of action $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}^{g+1}_{\Gamma}$ has a form of skew-shift

$$A_{\rho,\tau}(y, z) = (y + \rho, z + v \cdot y + \tau) \text{ on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z}. \tag{45}$$

for some non-zero $\rho, v \in \mathbb{R}^g$ and $\tau \in \mathbb{R}$. From computation of each rank 1 action, for each $1 \leq i \leq d$, it is a composition of commuting linear skew-shift

$$A_{i,\rho,\tau}(y, z) = (y + \rho_i, z + v_i \cdot y + \tau_i) \text{ on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z} \tag{46}$$

for some $\rho_i, v_i \in \mathbb{R}^g$ and $v_i \in \mathbb{R}^g$. For each $j \neq k$,

$$A_{j,\rho,\tau} \circ A_{k,\rho,\tau} = A_{k,\rho,\tau} \circ A_{j,\rho,\tau}.$$

Given pair $(\mathbf{m}, n) \in \mathbb{Z}^g_{K|n|} \times \mathbb{Z}$, let $H_{(\mathbf{m},n)}$ denote the corresponding factor and $C^\infty(H_{(\mathbf{m},n)})$ be a subspace of smooth functions on $H_{(\mathbf{m},n)}$. Denote $\{e_{\mathbf{m},n} \mid (\mathbf{m}, n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z}\}$ the basis of characters on $\mathbb{T}^{g+1}_{\Gamma}$ and for all $(y, z) \in \mathbb{T}^g \times \mathbb{T}$,

$$e_{\mathbf{m},n}(y, z) := \exp[2\pi\iota(\mathbf{m} \cdot y + nKz)].$$

For each $A_{i,\rho,\sigma}$, we set $v_i = (v_{i1}, \cdots, v_{ig}) \in \mathbb{Z}^g_{K|n|}$. Then the orbit can be identified with the following dual orbit

$$\begin{aligned} \mathcal{O}_{A_i}(\boldsymbol{m}, n) &= \{(\boldsymbol{m} + (nj_i)v_i, n),\ j_i \in \mathbb{Z}\} \\ &= \{(m_1 + (nv_{i1})j_i, \cdots, m_d + (nv_{id})j_i, n),\ j_i \in \mathbb{Z}\}. \end{aligned}$$

If $n = 0$, the orbit $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$ of $(\mathbf{m}, 0)$ is reduced to a single element. If $n \neq 0$, then the dual orbit $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$ of $(\mathbf{m}, n)$ for higher rank actions is described as follows:

$$\mathcal{O}_A(\boldsymbol{m}, n) = \{(m_k + n\sum_{i=1}^{d}(v_{ik}j_i), n)_{1 \leq k \leq d} : j = (j_1, \cdots j_d) \in \mathbb{Z}^d\}.$$

It follows that every A-orbit for rank $\mathbb{R}^d$-action (or A^j-orbit) can be labeled uniquely by a pair $(\mathbf{m}, n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z} \backslash \{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. Thus, the subspace of functions with non-zero central characters splits as a direct sum of components $H_{(\mathbf{m},n)}$

$$L^2(\mathbb{T}^{g+1}_{\Gamma}) = \bigoplus_{\omega \in \mathcal{O}_A} H_\omega, \quad \text{where } H_\omega = \bigoplus_{(\mathbf{m},n) \in \omega} \mathbb{C}e_{(\mathbf{m},n)}.$$

5.2. **Higher cohomology for $\mathbb{Z}^d$-action of skew-shifts.** We consider a $\mathbb{Z}^d$ action of return map $\mathsf{P}^{d,\alpha}$ on torus $\mathbb{T}^{g+1}_{\mathsf{\Gamma}}$. By identification of cochain complex on torus, it is equivalent to consider the following cohomological equation for degree d form ω,

$$\omega = d\Omega \iff \varphi(x,t) = \mathfrak{D}\Phi(x,t), \quad x \in \mathbb{T}^g,\ t \in \mathbb{Z}^d. \tag{47}$$

We restrict our interest of d-cocycle $\varphi : \mathbb{T}^{g+1}_{\mathsf{\Gamma}} \times \mathbb{Z}^d \to \mathbb{R}$ with $\Phi : \mathbb{T}^{g+1}_{\mathsf{\Gamma}} \to \mathbb{R}^d$, $\Phi = (\Phi_1, \cdots \Phi_d)$ and $\mathfrak{D}$ is coboundary operator $\mathfrak{D}\Phi = \sum_{i=1}^d (-1)^{i+1}\Delta_i \Phi_i$ where $\Delta_i\Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$. The following proposition is a generalization of the argument [KK95, Proposition 2.2]. Let us denote

$$A^j = A^{j_1}_{1,\rho,\tau} \circ \cdots \circ A^{j_d}_{d,\rho,\tau} \text{ for } j = (j_1, \cdots, j_d) \in \mathbb{Z}^d.$$

Now we consider the problem in higher cohomology with Fourier coefficients.

Proposition 5.2. *A cocycle $\hat{\varphi}$ satisfies cohomological equation (47) if and only if $\sum_{j\in\mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j = 0$.*

Proof. We consider a dual equation

$$\hat{\varphi} = \mathfrak{D}\hat{\Phi}. \tag{48}$$

Let us use the following notation:

$$(\delta_i\hat{\varphi})(m_1, \cdots, m_d) = \delta(m_i)\hat{\varphi}(m_1, \cdots, m_d), \text{ and } \quad \delta(0) = 1, \text{ otherwise } 0.$$

$$(\Sigma_i\hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=-\infty}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

$$(\Sigma_i^+\hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=m_i}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

$$(\Sigma_i^-\hat{\varphi})(m_1, \cdots, m_d) = -\sum_{j=-\infty}^{m_i-1} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

It is clear that $\Sigma_i^- - \Sigma_i^+ = \Sigma_i$ and $\Sigma_i^+\hat{\varphi} = \Sigma_i^-\hat{\varphi}$ if and only if $\Sigma_i\hat{\varphi} = 0$. Note that

$$\Sigma_i^+\Delta_i = \Sigma_i^-\Delta_i = id, \quad \Delta_i\Sigma_i^+ = \Delta_i\Sigma_i^- = id. \tag{49}$$

By direct calculation of Fourier coefficient, $\Sigma_i(\hat{\varphi} - \delta_i\Sigma_i\hat{\varphi}) = 0$. Let $\hat{\Phi}_i(\hat{\varphi}) = \Sigma_i^-(\hat{\varphi} - \delta_i\Sigma_i\hat{\varphi})$, then $\hat{\Phi}_i(\hat{\varphi})$ vanishes at ∞. By (49),

$$\hat{\varphi} - \delta_i\Sigma_i\hat{\varphi} = \Delta_i\hat{\Phi}_i(\hat{\varphi}),$$

and we can proceed this by induction

$$\begin{aligned}
\hat{\varphi} - \Sigma_{1,\cdots,d}\hat{\varphi} &= \sum_{i=1}^d (\delta_1 \cdots \delta_{i-1}\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi} - \delta_1 \cdots \delta_i\Sigma_1 \cdots \Sigma_i\hat{\varphi}) \\
&= \sum_{i=1}^d (\delta_1 \cdots \delta_{i-1}\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi} - \delta_i\Sigma_i(\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi})) \\
&= \sum_{i=1}^d (-1)^{i+1}\Delta_i\hat{\Phi}_i(\hat{\varphi})
\end{aligned}$$

where

$$\hat{\Phi}_i(\hat{\varphi}) = (-1)^{i+1}\Sigma_i^-\delta_1 \cdots \delta_{i-1}(\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi} - \delta_i\Sigma_i(\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi})).$$

Thus, $\hat{\Phi}_i$ is a solution of (48) if and only if $\Sigma_1 \cdots \Sigma_d \hat{\varphi} = 0$. $\square$

For fixed $(\mathbf{m}, n) \in \mathbb{Z}^g \times \mathbb{Z}$, we denote an obstruction of cohomological equation restricted to the orbit of $(\mathbf{m}, n)$ by $\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j\in\mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j$.

Lemma 5.3. *There exists a distributional obstruction to the existence of a smooth solution $\varphi \in C^\infty(H_{(\mathbf{m},n)})$ of the cohomological equation (47).*

A generator of the space of invariant distribution $\mathcal{D}_{m,n}$ has form of

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) := \begin{cases} e^{-2\pi\iota \sum_{i=1}^d [(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]} & \text{if } (a,b) = (m_k + K\sum_{i=1}^d (v_{ik}j_i), n)_{1\le k\le g} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From previous observation, there exists an obstruction

$$\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j\in\mathbb{Z}^d} \int_{\mathbb{T}^{g+1}_\Gamma} \varphi(x,y)\overline{e_{\mathbf{m},n} \circ A^j_{\rho,\tau}}dxdy. \tag{50}$$

By direct computation, for fixed $j = (j_1, \cdots j_d)$,

$$e_{\mathbf{m},n} \circ A^j_{\rho,\tau}(y,z) = \prod_{i=1}^d \left(e^{2\pi\iota[(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}\right)\left(e^{2\pi\iota(\mathbf{m}\cdot y + K(z + n\sum_{k=1}^d (v_{ik}j_i)y_k))}\right).$$

Then, we choose $\varphi = e_{a,b}$ for $(a,b) = (m_k + K\sum_{i=1}^d (v_{ik}j_i), n)_{1\le k\le g}$ in the non-trivial orbit $(n \neq 0)$,

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi\iota \sum_{i=1}^d [(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}.$$

$\square$

5.3. **Changes of coordinates.** For any frame $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$, denote transverse cylinder for any $m \in M$,

$$\mathscr{C}_{\alpha,m} := \{m \exp(\sum_{i=1}^g y_i' Y_i^\alpha + z'Z) \mid (y', z') \in U(t_{Ret}^{-1}) \times \mathbb{T}\}.$$

Let $\Phi_{\alpha,m} : \mathbb{T}^{g+1}_\Gamma \to \mathscr{C}_{\alpha,m}$ denote the maps: for any $\xi \in \mathbb{T}^{g+1}_\Gamma$, let $\xi' \in \mathscr{C}_{\alpha,m}$ denote first intersection of the orbit $\{\mathsf{P}_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}^d_+\}$ with transverse cylinder $\mathscr{C}_{\alpha,m}$. Then, there exists a first return time to the cylinder $t(\xi) = (t_1(\xi), \cdots, t_d(\xi)) \in \mathbb{R}^d_+$ such that

$$\xi' = \Phi_{\alpha,m}(\xi) = \mathsf{P}^{d,\alpha}_{t(\xi)}(\xi), \quad \forall \xi \in \mathbb{T}^{g+1}_\Gamma.$$

Let (y,z) and (y',z') denote the coordinates on $\mathbb{T}^{g+1}_\Gamma$ and $\mathscr{C}_{\alpha,m}$ given by the exponential map respectively,

$$(y,z) \to \xi_{y,z} := \Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ), \quad (y',z') \to m\exp(\sum_{i=1}^g y_i' Y_i^\alpha + z'Z).$$

Recall that if $\alpha \in Sp_{2g}(\mathbb{R})$, then for $1 \le i,j \le g$ there exist matrices $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$ such that

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2g}(\mathbb{R}),$$

satisfying $A^tD - C^tB = I_{2g}$, $C^tA = A^tC$, $D^tB = B^tD$, and $\det(A) \neq 0$. Set

$$X_i^\alpha = \sum_{j=1}^{g}(a_{ij}X_j + b_{ij}Y_j) + w_iZ \quad \text{ and } Y_i^\alpha = \sum_{j=1}^{g}(c_{ij}X_j + d_{ij}Y_j) + v_iZ.$$

Let $x = \Gamma \exp(\sum_{i=1}^d y_{x,i}Y_i + z_xZ)\exp(\sum_{i=1}^d t_{x,i}X_i)$, for some $(y_x, z_x) \in \mathbb{T}^d \times \mathbb{R}/K\mathbb{Z}$ and $t_x = (t_{x,i}) \in [0,1)^d$. Then, the map $\Phi_{\alpha,x} : \mathbb{T}_\Gamma^{g+1} \to \mathscr{C}_{\alpha,m}$ is defined by $\Phi_{\alpha,x}(y,z) = (y', z')$ where

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_g' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix}, \tag{51}$$

and $z' = z + P(\alpha, x, y)$ for some degree 4 polynomial P.

Therefore, the map $\Phi_{\alpha,x}$ is invertible with

$$\Phi_{\alpha,x}^*(dy_1' \wedge \cdots dy_g' \wedge dz') = \frac{1}{det(A)} dy_1 \wedge \cdots dy_g \wedge dz.$$

Since $A^tD - C^tB = I_{2g}$, by direct computation we obtain return time, we have

$$\begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}. \tag{52}$$

Then,

$$\|t(\xi)\| \leq \max_i |t_i(\xi)|^g \leq \max_i |\sum_{j=1}^{g} d_{ij}t_{x,i} + c_{ij}(y_i - y_{x,i})|^g \leq \max_i \|Y_i^\alpha\|^g.$$

5.4. **L^2-lower bound of functional.** We will prove bounds for the square mean of integrals along leaves of foliations of the torus $\mathbb{T}_\Gamma^{g+1}$.

Lemma 5.4. *There exists a constant $C > 0$ such that for all $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$, and for every irreducible component $H := H_n$ of central parameter $n \neq 0$, there exists a function f_H such that*

$$|f_H|_{L^\infty(H)} \leq Cvol(U(t_{Ret}))^{-1}|\mathcal{D}_\alpha^H(f_H)|,$$

$$|f_H|_{\alpha,s} \leq Cvol(U(t_{Ret}))^{-1}|\mathcal{D}_\alpha^H(f_H)|(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s(1+n^2)^{s/2}$$

where $\|Y\| := \max_{1\leq i \leq g}\|Y_i^\alpha\|$ and $\Sigma(t_{Ret}) = \sum_{i=1}^g t_{Ret,i}$.

On rectangular domain $U(\boldsymbol{T})$, for all $m \in \mathbb{T}_\Gamma^{g+1}$ and $T^{(i)} \in \mathbb{Z}_{t_{Ret,i}}$

$$\left\| \left\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y}m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = |\mathcal{D}_\alpha^H(f_H)| \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2}. \tag{53}$$

In addition, whenever $H \perp H' \subset L^2(M)$ the functions

$$\left\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y}m), \omega_H \right\rangle \quad \text{and} \quad \left\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y^\alpha}m), \omega_{H'} \right\rangle$$

are orthogonal in $L^2(\mathbb{T}^g, dy)$.

Proof. As explained in §5.1, the space $L^2(\mathbb{T}_\Gamma^{d+1})$ decompose as a direct sum of irreducible subspaces invariant under the action of each $A_{j,\rho,\sigma}$. It follows that the subspace of functions with non-zero central character can be split as direct sum of components $H_{(\mathbf{m},n)}$ with $(\mathbf{m},n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z}\backslash\{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. For $F \in H_{(\mathbf{m},n)}$, the function is characterized by Fourier expansion

$$F = \sum_{j \in \mathbb{Z}^d} F_j e_{A^j(\boldsymbol{m},n)} = \sum_{j \in \mathbb{Z}^d} F_j e_{(m_k + K\sum_{i=1}^d (v_{ik} j_i), n)}.$$

Then, by Lemma 5.3,

$$\mathcal{D}_{(\mathbf{m},n)}(e_{A^j(\boldsymbol{m},n)}) = e^{-2\pi\iota\sum_{i=1}^d [(m\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}. \tag{54}$$

For any irreducible representation H with central parameter $n \neq 0$, there exists $\mathbf{m} \in \mathbb{Z}^d_{|n|}$ such that the operator I_α maps the space H onto $H_{(\mathbf{m},n)}$. The operator $I_\alpha : L^2(M) \to L^2(\mathbb{T}_\Gamma^{g+1})$ is defined by

$$f \to I_\alpha(f) := \int_{U(t_{Ret})} f \circ \mathsf{P}_x^{d,\alpha}(\cdot)dx. \tag{55}$$

Then, operator I_α is surjective linear map of $L^2(M)$ onto $L^2(\mathbb{T}_\Gamma^{g+1})$ with right inverse defined as follows:

Let $\chi \in C_0^\infty(0,1)^g$ be any function of jointly integrable with integral 1. For any $F \in L^2(\mathbb{T}_\Gamma^{g+1})$, let $R_\alpha^\chi(F) \in L^2(M)$ be a function defined by

$$R_\alpha^\chi(F)(\mathsf{P}_v^{d,\alpha}(m)) = \frac{1}{vol(U(t_{Ret}))}\chi(\frac{v}{t_{Ret}})F(m),\ (m,v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows that there exists a constant $C_\chi > 0$ such that

$$\begin{aligned} |R_\alpha^\chi(F)|_{\alpha,s} &\le C_\chi vol(U(t_{Ret}))^{-1}(1 + \sum_{i=1}^g t_{Ret,i}^{-1}\left\|Y_i^\alpha\right\|)^s \left\|F\right\|_{W^s(\mathbb{T}_\Gamma^{g+1})} \\ &\le C_\chi vol(U(t_{Ret}))^{-1}(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\left\|Y\right\|)^s \left\|F\right\|_{W^s(\mathbb{T}_\Gamma^{g+1})}. \end{aligned} \tag{56}$$

Choose $f_H := R_\alpha^\chi(e_{m,n}) \in C^\infty(H)$ such that $I_\alpha(f_H) = e_{m,n}$ and

$$\int_{U(t_{Ret})} f_H \circ \mathsf{P}_t^{d,\alpha}(y,z)dt = e_{m,n}(y,z),\ \text{for } (y,z) \in \mathbb{T}_\Gamma^{g+1}. \tag{57}$$

For P-action, the space of invariant currents $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-s}(\mathbb{R}^g)$ for $s > d/2 + \epsilon$ for all $\epsilon > 0$. That is, by normalization of invariant distributions in the Sobolev space, for any irreducible components H, there exists a non-unique function f_α^H such that

$$\mathcal{D}_\alpha(f_\alpha^H) = \left\|f_\alpha^H\right\|_s = 1. \tag{58}$$

By (54), we have $|\mathcal{D}_H(f_H)| = |\mathcal{D}_{(m,n)}(e_{m,n})| = 1$. Therefore, it follows from (56)

$$|f_H|_{L^\infty(H)} \le C_\chi vol(U(t_{Ret}))^{-1},$$

$$|f_H|_{\alpha,s} \le C vol(U(t_{Ret}))^{-1}|\mathcal{D}_\alpha^H(f_H)|\Big(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\left\|Y\right\|\Big)^s(1+n^2)^{s/2}.$$

Moreover, since $\{e_{m,n} \circ A^j_{\rho,\tau}\}_{j\in\mathbb{Z}^d} \subset L^2(\mathbb{T}^g_\Gamma, dy)$ is orthonormal, we verify

$$\left\|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\mathsf{Q}^{g,Y}_y m), \omega_H \right\rangle\right\|_{L^2(\mathbb{T}^g,dy)} = \left\| \sum_{j_d=0}^{[\frac{T^{(d)}}{t_{Ret,d}}]-1} \cdots \sum_{j_1=0}^{[\frac{T^{(1)}}{t_{Ret,1}}]-1} e_{m,n} \circ A^j_{\rho,\tau} \right\|_{L^2(\mathbb{T}^g,dy)}$$
$$= \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2}.$$

□

Recall that

$$L^2(M) = \bigoplus_{n\in\mathbb{Z}} H_n := \bigoplus_{n\in\mathbb{Z}} \bigoplus_{i=1}^{\mu(n)} H_{i,n} \tag{59}$$

where $H_n = \bigoplus_{i=1}^{\mu(n)} H_{i,n}$ is irreducible representation with central parameter n and $\mu(n)$ is countable by Howe-Richardson multiplicity formula.

For any infinite dimensional vector $\mathbf{c} := (c_{i,n}) \in \ell^2$, let $\beta_{\mathbf{c}}$ denote Bufetov functional

$$\beta_{\mathbf{c}} = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n}\beta^{i,n}.$$

For any $\mathbf{c} := (c_{i,n})$, let $|\mathbf{c}|_s$ denote the norm defined as

$$|\mathbf{c}|_s^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_{i=1}^{\mu(n)} (1 + K^2n^2)^s |c_{i,n}|^2. \tag{60}$$

By Corollary 4.3,

$$\|\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T})\|^2_{L^2(M)} \le C^2 |\mathbf{c}|^2_{\ell^2} vol(U(\boldsymbol{T})). \tag{61}$$

Lemma 5.5. *For any $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for all $\alpha \in DC(L)$, for all $\mathbf{c} \in \ell^2$, for all $z \in \mathbb{T}$ and all $T > 0$,*

$$\left| \|\beta_{\boldsymbol{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})\|_{L^2(\mathbb{T}^g,dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\boldsymbol{c}|_0 \right|$$
$$\le C_s(vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1+L)(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s |\boldsymbol{c}|_s.$$

Proof. By the proof in the Lemma 5.4 and (59), there exists a function $f_{i,n} \in C^\infty(H_{i,n})$ with $\mathcal{D}^{i,n}(f_{i,n}) = 1$. Let $f_{\mathbf{c}} = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n} \in C^\infty(M)$, summing up all the functions on irreducibles. Then by the estimates in the Lemma 5.4 and (60),

$$|f_{\mathbf{c}}|_{L^\infty(M)} \le C|\mathbf{c}|_{\ell^1}. \tag{62}$$

$$|f_{\mathbf{c}}|_{\alpha,s} \le Cvol(U(t_{Ret}))^{-1}(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s |\mathbf{c}|_s. \tag{63}$$

By orthogonality,

$$\left\|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} \circ \mathsf{Q}^{g,Y}_y, \omega_{\mathbf{c}} \right\rangle\right\|_{L^2(\mathbb{T}^g,dy)} = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\mathbf{c}|_0.$$

From estimate on each $f_{i,n}$ in Lemma 5.4, for every $z \in \mathbb{T}$ and all $T > 0$, we have

$$\left\| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g,dy)} \leq 2|f_{\mathbf{c}}|_{L^\infty(M)} \left\| Y \right\| .$$

Let $T_{\alpha,i} = t_{Ret,i}([T/t_{Ret,i}] + 1)$ and $U(t_\alpha) = [0, T_{\alpha,1}] \times \cdots \times [0, T_{\alpha,g}]$. Then,

$$\left\| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathscr{P}^{d,\alpha}_{U(t_\alpha)}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g,dy)} \leq vol(U(t_{Ret}))|f_{\mathbf{c}}|_{L^\infty(M)}.$$

Therefore, there exists a constant $C' > 0$ such that

$$\begin{aligned} &\left| \left\| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g,dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \\ &\qquad\qquad\qquad\qquad \leq C' vol(U(t_{Ret}))|\mathbf{c}|_{\ell^1} . \end{aligned}$$

For all $s > s_d + 1/2$, by asymptotic property of Theorem 1.5, for some constant $C_s > 0$,

$$\left| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega \right\rangle - \beta_H(\alpha, m, \boldsymbol{T}) \mathcal{D}^H_\alpha(f_H) \right| \leq C_s(1 + L) \left\| f \right\|_{\alpha,s} .$$

Applying $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$ and combining bounds on the function $f_{\mathbf{c}}$ with (62),

$$\begin{aligned} &\left| \| \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}) \|_{L^2(\mathbb{T}^g,dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \\ &\leq C' vol(U(t_{Ret}))|\mathbf{c}|_{\ell^1} + C_s vol(U(t_{Ret}))^{-1}(1 + L)|f_{\mathbf{c}}|_{\alpha,s} \\ &\leq C'_s (vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1 + L)\big(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))} \left\| Y \right\| \big)^s |\mathbf{c}|_s . \end{aligned}$$

Therefore, we derive the estimates in the statement. □

6. Analyticity of functionals

In this section we will prove that for all $\alpha \in DC$, the Bufetov functionals on any square are real analytic.

6.1. **Analyticity.**

Definition 6.1. For every $t \in \mathbb{R}$, $1 \leq i \leq d$, and $m \in M$, the *stretched (in direction of Z) rectangle* is denoted by

$$[\Gamma_{\boldsymbol{T}}]^Z_{i,t}(m) := \{(\phi^Z_{ts_i}) \circ \mathsf{P}^{d,\alpha}_{\boldsymbol{s}}(m) \mid \boldsymbol{s} \in U(\mathbf{T})\}. \tag{64}$$

For $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}^d$, let us denote by the standard rectangle $\Gamma_{\mathbf{T}}(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma_d(s_d))$ for $\gamma_i(s_i) = \exp(s_i X_i)$. Similarly, we also write the stretched rectangle

$$[\Gamma_{\boldsymbol{T}}]^Z_{i,t}(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma^Z_{i,t}(s_i), \cdots \gamma_d(s_d)) \tag{65}$$

where $\gamma^Z_{i,t}(s_i) := \phi^Z_{ts_i}(\gamma_i(s_i))$ is a stretched curve.

Definition 6.2. The *restriction* $\Gamma_{T,i,s}$ of the rectangle $\Gamma_{\mathbf{T}}$ is defined on restricted domain $U_{T,i,s} = [0, T^{(1)}] \times \cdots \times [0, s] \cdots \times [0, T^{(d)}]$ for $s \leq T^{(i)}$ as following.

$$\Gamma_{T,i,s}(\mathbf{s}) := \Gamma_{\mathbf{T}}(\mathbf{s}), \ \mathbf{s} \in U_{T,i,s}.$$

Recall the orthogonal property on a irreducible component H (central parameter $n \in \mathbb{Z}\backslash\{0\}$). For any $(m, \mathbf{T}) \in M \times \mathbb{R}^d_+$ and $t \in \mathbb{R}$,

$$\beta_H(\alpha, \phi^Z_t(m), \mathbf{T}) = e^{2\pi\iota Knt}\beta_H(\alpha, m, \mathbf{T}). \tag{66}$$

We obtain the following lemma for stretched rectangle by applying orthogonal property.

Lemma 6.3. *For fixed elements (X_i, Y_i, Z) satisfying commutation relation (1), the following formula for rank 1 action holds:*

$$\hat{\beta}_H(\alpha, [\Gamma_{\boldsymbol{T}}]^Z_{i,t}) = e^{2\pi\iota t nKT^{(i)}}\hat{\beta}_H(\alpha, \Gamma_{\boldsymbol{T}}) - 2\pi\iota nKt\int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\hat{\beta}_H(\alpha, \Gamma_{T,i,s})ds_i.$$

Proof. Let $\alpha = (X_i, Y_i, Z)$ and ω be d-form supported on a single irreducible representation H. We obtain following the formula for stretches of curve $\gamma^Z_{i,t}$ (see [FK20b, §4, Lemma 9.1]),

$$\frac{d\gamma^Z_{i,t}}{ds_i} = D\phi^Z_{ts_i}(\frac{d\gamma_i}{ds_i}) + tZ \circ \gamma^Z_{i,t}.$$

It follows that pairing is given by

$$\begin{aligned}\langle[\Gamma_{\mathbf{T}}]^Z_{i,t}, \omega\rangle &= \int_{U(\boldsymbol{T})} \omega\Big(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma^Z_{i,t}}{ds_i}(s_i), \cdots, \frac{d\gamma_d}{ds_d}(s_d)\Big)d\mathbf{s} \\ &= \int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega\Big(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d)\Big)] + \iota_Z\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t}(\mathbf{s})d\mathbf{s}\end{aligned}$$

Denote $d-1$ dimensional triangle $U_{d-1}(\boldsymbol{T})$ with $U(\boldsymbol{T}) = U_{d-1}(\boldsymbol{T}) \times [0, T^{(i)}]$. Integration by parts for a fixed i-th integral gives

$$\begin{aligned}&\int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s} \\ &= e^{2\pi\iota nKtT^{(i)}}\int_{U(\boldsymbol{T})}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s} \\ &- 2\pi\iota nKt\int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\int_{U_{d-1}(\boldsymbol{T})}\Big(\int_0^{s_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_i}{ds_i}(r)\cdots, \frac{d\gamma_d}{ds_d}(s_d))]dr\Big)d\mathbf{s}.\end{aligned}$$

Then, we have the following formula

$$\begin{aligned}\langle[\Gamma_{\mathbf{T}}]^Z_{i,t}, \omega\rangle = e^{2\pi\iota nKtT^{(i)}}\langle[\Gamma_{\mathbf{T}}], \omega\rangle - 2\pi\iota nKt\int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\langle\Gamma_{T,i,s}, \omega\rangle ds_i \\ + \int_{U(\boldsymbol{T})}(\iota_Z\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s}.\end{aligned}$$

Since the action of $\mathsf{P}^{d,X}_{\mathbf{t}}$ for $\mathbf{t} \in \mathbb{R}^d$ is identity on the center Z,

$$\lim_{t_d\to\infty}\cdots\lim_{t_1\to\infty} e^{-(t_1+\cdots t_d)/2}\int_{U(\boldsymbol{T})}(\iota_Z(\mathsf{P}^{d,X}_{\mathbf{t}})^*\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s} = 0.$$

Thus, in particular concerning $d = 1$, it follows by the definition of Bufetov functional (Lemma 3.5), the statement holds. □

Here we define a restricted vector $\mathbf{T}_{i,s}$ of $\mathbf{T} = (T^{(1)}, \cdots, T^{(d)}) \in \mathbb{R}^d$. For fixed i, pick $s_i \in [0, T^{(i)}]$ such that $\mathbf{T}_{i,s} \in \mathbb{R}^d$ is a vector with its coordinates

$$T_{i,s}^{(j)} = \begin{cases} T^{(j)} & \text{if} \quad j \neq i \\ s_i & \text{if} \quad j = i. \end{cases}$$

Similarly, $\mathbf{T}_{i_1,\cdots,i_k,s}$ is a vector with $i_1, \cdots i_k$ coordinates replaced by $s_{i_1}, \cdots, s_{i_k}$.

Lemma 6.4. *Let $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. The following equality holds for each rank-1 action.*

$$\beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \boldsymbol{T}) = e^{-2\pi\iota y_i nKT^{(i)}}\beta_H(\alpha, m, \boldsymbol{T}) + 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{-2\pi\iota y_i nKs_i}\beta_H(\alpha, m, \boldsymbol{T}_{i,s})ds_i. \tag{67}$$

Proof. By definition (3), (65) and commutation relation (1), it follows that

$$\phi_{y_i}^{Y_i}(\Gamma_{\mathbf{T}}^X(m)) = [\Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))]_{i,t}^Z.$$

By the invariance property of Bufetov functional and Lemma 6.3,

$$\begin{aligned} &\beta_H(\alpha, m, \mathbf{T}) = \hat{\beta}_H(\alpha, \phi_{y_i}^{Y_i}(\Gamma_{\mathbf{T}}^X(m)) \\ &= e^{2\pi\iota y_i nKT^{(i)}}\hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m)) - 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{2\pi\iota nKy_is_i}\hat{\beta}_H(\alpha, \Gamma_{T,i,s}^X(\phi_{s_i}^{Y_i}(m))ds_i \\ &= e^{2\pi\iota y_i nKT^{(i)}}\beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) - 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{2\pi\iota nKy_is_i}\beta_H(\alpha, \phi_{s_i}^{Y_i}(m), \mathbf{T}_{i,s})ds_i. \end{aligned}$$

Then statement follows immediately. $\square$

We extend previous Lemma 6.4 to higher rank actions by induction argument.

Lemma 6.5. *The following equality holds for rank-d action.*

$$\begin{aligned} &\beta_H(\alpha, \mathsf{Q}_y^{d,Y}(m), \boldsymbol{T}) = e^{-2\pi\iota\sum_{j=1}^d y_j nKT^{(j)}}\beta_H(\alpha, m, \boldsymbol{T}) \\ &+ \sum_{k=1}^d \sum_{1\leq i_1<\cdots<i_k\leq d} \prod_{j=1}^k (2\pi\iota nKy_{i_j})e^{-2\pi\iota nK(\sum_{l\notin\{i_1,\cdots,i_k\}} y_lT^{(l)})} \\ &\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1}+\cdots+y_{i_k}s_{i_k})}\beta_H(\alpha, m, \boldsymbol{T}_{i_1,\cdots,i_k,s})ds_{i_k}\cdots ds_{i_1} \end{aligned} \tag{68}$$

Proof. We verified that the statement works for $d = 1$ in Lemma 6.4. Assume that (68) holds for rank $d-1$ action by induction hypothesis. For convenience, we write

$$\mathsf{Q}_y^{d,Y}(m) = \phi_{y_d}^{Y_d} \circ \mathsf{Q}_{y'}^{d-1,Y}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 6.4,

$$\begin{aligned} \beta_H(\alpha, \mathsf{Q}_y^{d,Y}(m), \mathbf{T}) &= e^{-2\pi\iota y_d nKT^{(d)}}\beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T}) \\ &\quad + 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d}\beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T}_{d,s})ds_d \\ &:= I + II \end{aligned} \tag{69}$$

Firstly, by induction hypothesis,

$$
\begin{aligned}
I &= e^{-2\pi\iota y_d nKT^{(d)}}\Bigg(e^{-2\pi\iota\sum_{j=1}^{d-1} y_j nKT^{(j)}}\beta_H(\alpha, m, \mathbf{T}) \\
&+\sum_{k=1}^{d-1}\sum_{1\le i_1<\cdots<i_k\le d-1}\prod_{j=1}^{k}(2\pi\iota nKy_{i_j})e^{-2\pi\iota nK(\sum_{l\notin\{i_1,\cdots,i_k\}} y_l T^{(l)}+y_d T^{(d)})} \\
&\times\int_0^{T^{(i_1)}}\cdots\int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1}+\cdots+y_{i_k}s_{i_k})}\beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,s})ds_{i_k}\cdots ds_{i_1}\Bigg) \\
&= e^{-2\pi\iota\sum_{j=1}^{d} y_j nKT^{(j)}}\beta_H(\alpha, m, \mathbf{T}) + III.
\end{aligned}
$$

Then term III contains 0 to $d-1$th iterated integrals containing $e^{-2\pi\iota nKy_dT^{(d)}}$ outside of iterated integrals.

For the second part, we apply induction hypothesis again for restricted rectangle $\mathbf{T}_{d,s}$. Then,

$$
\begin{aligned}
II &= 2\pi\iota nKy_d\int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d}\big[e^{-2\pi\iota\sum_{j=1}^{d-1} y_j nKT^{(j)}}\beta_H(\alpha, m, \mathbf{T}_{d,s})\big]ds_d \\
&+\sum_{k=1}^{d-1}\sum_{1\le i_1<\cdots<i_k\le d-1}(2\pi\iota nKy_d)\prod_{j=1}^{k}(2\pi\iota nKy_{i_j})e^{-2\pi\iota nK(\sum_{l\notin\{i_1,\cdots,i_k\}} y_l T^{(l)})} \\
&\times\int_0^{T^{(d)}}\left(\int_0^{T^{(i_1)}}\cdots\int_0^{T^{(i_k)}} ds_{i_k}\cdots ds_{i_1}\right)ds_d \\
&\times e^{-2\pi\iota nK(y_{i_1}s_{i_1}+\cdots+y_{i_k}s_{i_k}+y_d s_d)}\beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,d,s}).
\end{aligned}
$$

The term II consist of 1 to d-th iterated integrals containing $e^{-2\pi\iota nKy_ds_d}$ inside of iterated integrals. Thus, by rearranging terms II and III, we obtain expression (68). □

6.2. **Extensions of domain.** In this subsection, we extend the domain of Bufetov functional defined on standard rectangle $\Gamma^X_{\mathbf{T}}$ to the class $\mathfrak{R}$ (see Definition 1.1). We also extend the functional to holomorphic function on a complex domain.

Theorem 6.6. *$\hat{\beta}_H$ defined on standard rectangle $\Gamma^X_{\boldsymbol{T}}$ extends to the class $\mathfrak{R}$.*

Proof. Firstly, we can extend our functional to class $(\mathsf{Q}^{d,Y}_y)_*\Gamma^X_{\mathbf{T}}$ for any $y\in\mathbb{R}^d$ by invariance property (Lemma 3.6). Similarly, by Lemma 6.3, Bufetov functional defined on the standard rectangles extends to the class of generalized rectangle $(\phi^Z_{t_iz})\circ\mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m)$. Since the flow generated by Z commutes with other actions P and Q, for any standard rectangle $\Gamma=\Gamma(m)$ with a fixed point $m\in M$, we have $(\phi^Z_z)_*\Gamma(m)=\Gamma(\phi^Z_z(m))$ for any $z\in\mathbb{R}$. Therefore, by combining with the invariance under the action Q from Lemma 3.6, the domain of Bufetov functional extends to the class $\mathfrak{R}$. □

For any $R>0$, the *analytic norm* defined for all $\mathbf{c}\in\ell^2$ as

$$
\|\mathbf{c}\|_{\omega,R}=\sum_{n\neq 0}\sum_{i=1}^{\mu(n)} e^{nR}|c_{i,n}|.
$$

Let Ω_R denote the subspace of $\mathbf{c}\in\ell^2$ such that $\|c\|_{\omega,R}$ is finite.

Lemma 6.7. *For $\boldsymbol{c} \in \Omega_R$ and $\boldsymbol{T} \in \mathbb{R}^d_+$, the function*

$$\beta_{\boldsymbol{c}}(\alpha, \mathsf{Q}^{d,Y}_y \circ \phi^Z_z(m), \boldsymbol{T}),\ (y,z) \in \mathbb{R}^d \times \mathbb{T}$$

extends to a holomorphic function in the domain

$$D_{R,T} := \{(y,z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \mid \sum_{i=1}^d |Im(y_i)|T^{(i)} + |Im(z)| < \frac{R}{2\pi K}\}. \tag{70}$$

The following bound holds: for any $R' < R$ there exists a constant $C > 0$ such that, for all $(y,z) \in D_{R',T}$ we have

$$\begin{aligned}&|\beta_{\boldsymbol{c}}(\alpha, \mathsf{Q}^{d,Y}_y \circ \phi^Z_z(m), \boldsymbol{T})|\\ &\quad \le C_{R,R'} \left\|c\right\|_{\omega,R} (L + vol(U(\boldsymbol{T}))^{1/2}(1 + E_M(a,\boldsymbol{T}))(1 + K\sum_{i=1}^d |Im(y_i)|T^{(i)}).\end{aligned}$$

Proof. By Lemma 6.5 and (66),

$$\begin{aligned}&\beta_{\mathbf{c}}(\alpha, \mathsf{Q}^{d,Y}_y \circ \phi^Z_z(m), \mathbf{T}) = e^{(z - 2\pi\imath \sum_{j=1}^d y_j nKT^{(j)})}\beta_H(\alpha, m, \mathbf{T})\\ &+ \sum_{k=1}^d \sum_{1 \le i_1 < \cdots < i_k \le d} \prod_{j=1}^k (2\pi\imath nKy_{i_j}) e^{-2\pi\imath nK(\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)})}\\ &\times e^{2\pi\imath nKz} \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\imath nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})}\beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,s}) ds_{i_k}\cdots ds_{i_1}.\end{aligned}$$

As a consequence, by Lemma 3.7 for each variable $(y_i, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$, Then for the rank d-action, by induction, for $(y,z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ we have

$$\begin{aligned}&|\beta_{\mathbf{c}}(\alpha, \mathsf{Q}^{d,Y}_y \circ \phi^Z_z(x), \mathbf{T})|\\ &\le (L + vol(U(\boldsymbol{T}))^{1/2}(1 + E_M(a,\boldsymbol{T}))\Bigg(C_1 \sum_{n \ne 0}\sum_{i=1}^{\mu(n)} e^{nR}|c_{i,n}|e^{2\pi|Im(z - \sum_{i=1}^d T^{(i)}y_i)|nK}\\ &+ \sum_{k=1}^d C_k\Big(\sum_{1 \le i_1 < \cdots < i_k \le d} \prod_{j=1}^k (|Im(y_{i_j})|T^{(i_j)})\\ &\times \sum_{n \ne 0}\sum_{i=1}^{\mu(n)} n|c_{i,n}|e^{2\pi(|Im(z)| + \sum_{j=1}^k T^{(i_j)}|Im(y_{i_j})|)nK}\Big)\Bigg).\end{aligned}$$

Therefore, the functional $\beta_{\mathbf{c}}(\alpha, \mathsf{Q}^{d,Y}_y \circ \phi^Z_z(m), \mathbf{T})$ is bounded by a series of holomorphic functions on $\mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ and it converges uniformly on compact subsets of domain $D_{R,T}$. Thus it is holomorphic on the set. ☐

7. Measure estimation for bounded-type

In this section, we prove a measure estimation of Bufetov functional with bounded-type α. This result is a generalization of §11 in [FK20b].

Let $\mathcal{O}_r$ denote the space of holomorphic functions on the ball $B_{\mathbb{C}}(0,r) \subset \mathbb{C}^n$.

Theorem 7.1. *[Bru99, Theorem 1.9] For any $f \in \mathcal{O}_r$, there is a constant $d := d_f(r) > 0$ such that for any convex set $D \subset B_{\mathbb{R}}(0,1) := B_{\mathbb{C}}(0,1) \cap \mathbb{R}^n$, for any measurable subset $U \subset D$*

$$\sup_D |f| \leq \left(\frac{4nLeb(D)}{Leb(U)}\right)^d \sup_U |f|.$$

We say that a holomorphic function f defined in a disk is *p-valent* if it assumes no value more than p-times there. We also say that f is *0-valent* if it is a constant.

Definition 7.2. [Bru99, Def 1.6] Let $\mathcal{L}_t$ denote the set of one-dimensional complex affine spaces $L \subset \mathbb{C}^n$ such that $L \cap B_{\mathbb{C}}(0,t) \neq \emptyset$. For $f \in \mathcal{O}_r$, the number

$$\nu_f(t) := \sup_{L \in \mathcal{L}_t} \{\text{valency of } f \mid L \cap B_{\mathbb{C}}(0,t) \neq \emptyset\}$$

is called the *valency* of f in $B_{\mathbb{C}}(0,t)$.

By Proposition 1.7 of [Bru99], for any $f \in \mathcal{O}_r$ with finite valency $\nu_f(t)$ for any $t \in [1,r)$, there is a constant $c := c(r) > 0$ such that

$$d_f(r) \leq c\nu_f(\frac{1+r}{2}). \tag{71}$$

Lemma 7.3. *[FK20b, Lemma 10.3] Let $R > r > 1$. For any normal family $\mathcal{F} \subset \mathcal{O}_R$, assume that no functions in $\overline{\mathcal{F}} = \emptyset$ are constant along a one-dimensional complex line. Then we have*

$$\sup_{f \in \mathcal{F}} \nu_f(r) < \infty.$$

Lemma 7.4. *Let $L > 0$ and $\mathscr{B} \subset DC(L)$ be a bounded subset. Given $R > 0$, for all $\boldsymbol{c} \in \Omega_R$ and all $\boldsymbol{T}^{(i)} > 0$, denote $\mathcal{F}(\boldsymbol{c}, \boldsymbol{T})$ by the family of real analytic functions of the variable $y \in [0,1)^d$ and*

$$\mathcal{F}(\boldsymbol{c}, \boldsymbol{T}) := \{\beta_{\boldsymbol{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}) \mid (\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}\}.$$

Then there exist $\boldsymbol{T}_{\mathscr{B}} := (\boldsymbol{T}_{\mathscr{B}}^{(i)})$ and $\rho_{\mathscr{B}} > 0$, such that for every $(R, \boldsymbol{T})$ with $R/\boldsymbol{T}^{(i)} \geq \rho_{\mathscr{B}}$, $\boldsymbol{T}^{(i)} \geq \boldsymbol{T}_{\mathscr{B}}^{(i)}$ and for all $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, we have

$$\sup_{f \in \mathcal{F}(\boldsymbol{c}, \boldsymbol{T})} \nu_f < \infty. \tag{72}$$

Proof. Since $\mathscr{B} \subset \mathfrak{M}$ is bounded, for each time $t_i \in \mathbb{R}$ and $1 \leq i \leq g$,

$$0 < t_{i,\mathscr{B}}^{min} = \min_i \inf_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} \leq \max_i \sup_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} = t_{\mathscr{B}}^{max} < \infty.$$

For any $\alpha \in \mathscr{B}$ and $x \in M$, the map $\Phi_{\alpha,x} : [0,1)^d \times \mathbb{T} \to \prod_{i=1}^d [0, t_{\alpha,i}) \times \mathbb{T}$ in (51) extends to a complex analytic diffeomorphism $\hat{\Phi}_{\alpha,x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$. By Lemma 6.7, it follows that for fixed $z \in \mathbb{T}$, real analytic function $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})$ extends to a holomorphic function defined on a region

$$H_{\alpha,m,R,t} := \{y \in \mathbb{C}^d \mid \sum_{i=1}^d |Im(y_i)| \leq h_{\alpha,m,R,t}\}.$$

By boundedness of the set $\mathscr{B} \subset \mathfrak{M}$, it follows that

$$\inf_{(\alpha,x) \in \mathscr{B} \times M} h_{\alpha,m,R,t} := h_{R,T} > 0.$$

We remark that the function $h_{\alpha,m,R,t}$ and its lower bound $h_{R,T}$ can be obtained from the formula (51) for the polynomial $\Phi_{\alpha,x}$ and the definition of the domain $D_{R,T}$ in formula (70).

For every $r > 1$, there exists $\rho_{\mathscr{B}} > 1$ such that for every R and $\mathbf{T}$ with $R/\mathbf{T}^{(i)} > \rho_{\mathscr{B}}$,

$$\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \in \mathcal{O}_r \tag{73}$$

as a function of $y \in \mathbb{T}^d$.

By Lemma 6.7, the family $\mathcal{F}(c, \mathbf{T})$ is uniformly bounded and normal. By Lemma 5.5 for the non-zero L^2-lower bound of functionals, for sufficiently large pair $\mathbf{T}$, no sequence from $\mathcal{F}(c, \mathbf{T})$ can converge to a constant. Therefore, by Lemma 7.3 for the family $\mathcal{F} = \mathcal{F}(c, \mathbf{T})$, the main statement follows. $\square$

We derive measure estimates of Bufetov functionals on the rectangular domain.

Lemma 7.5. *Let $\alpha \in DC$ such that the forward orbit of $\mathbb{R}^d$-action $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t}\in\mathbb{R}^d_+}$ is contained in a compact set of $\mathfrak{M}_g$. There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}^d_+$ such that, for every $\boldsymbol{c} \in \Omega_R\backslash\{0\}$, $T \geq T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{m \in M \mid |\beta_{\mathbf{c}}(\alpha, m, \boldsymbol{T})| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

Proof. Since $\alpha \in DC$ and the orbit $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t}\in\mathbb{R}_+}$ is contained in a compact set, there exists $L > 0$ such that $r_{\mathbf{t}}(\alpha) \in DC(L)$ for all $\boldsymbol{t} \in \mathbb{R}^d_+$. Then, we choose $\mathbf{T}_0 \in \mathbb{R}^d$ and $R > 0$ from the conclusion of Lemma 7.4. By the scaling property,

$$\beta_{\mathbf{c}}(\alpha, m, \boldsymbol{T}) = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(\boldsymbol{T_0}))}\right)^{1/2} \beta_{\mathbf{c}}(g_{\log(\boldsymbol{T}/\boldsymbol{T_0})}[\alpha], m, \boldsymbol{T}_0).$$

By Fubini's theorem, it suffices to estimate

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}).$$

Let $\delta^{-1} := c(r) \sup_{f\in\mathcal{F}(\mathbf{c},\mathbf{T}_0)} \nu_f(\frac{1+r}{2}) < \infty$ as in (71) and (72). By Lemma 5.5, we have

$$\inf_{(\alpha,x,z)\in\mathscr{B}\times M\times\mathbb{T}} \sup_{y\in[0,1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| > 0$$

so that the functional is not trivial. By Theorem 7.1 for the unit ball $D = B_{\mathbb{R}}(0,1)$ and setting

$$U = \{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\},$$

by the bound in (71) for $d_f(r)$, there exists a constant $C > 0$ such that for all $\epsilon > 0$ and $(\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}$,

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}) \leq C\epsilon^{\delta}.$$

Then the statement follows from the Fubini theorem. $\square$

Corollary 7.6. *Let α be as in the previous Lemma 7.5. There exist $R, C, \delta > 0$ and $\boldsymbol{T}_0 \in \mathbb{R}^d_+$ such that, for every $\boldsymbol{c} \in \Omega_R\backslash\{0\}$, $\boldsymbol{T} \geq \boldsymbol{T}_0$ and for every $\epsilon > 0$, we have*

$$vol\left(\{m \in M \mid |\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_{\boldsymbol{c}}\rangle| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}\right) \leq C\epsilon^{\delta}.$$

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