LIMIT THEOREMS FOR HIGHER RANK ACTIONS ON HEISENBERG NILMANIFOLDS

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Abstract. The main result of this paper is a construction of finitely additive measures for higher rank abelian actions on Heisenberg nilmanifolds. Under a full measure set of Diophantine conditions for the generators of the action, we construct *Bufetov functional* on rectangles on $2g+1$ dimensional Heisenberg manifold. We prove that deviation of the ergodic integral of higher rank actions is described by the asymptotic of Bufetov functionals for a sufficiently smooth function. As a corollary, the distribution of normalized ergodic integrals to have variance 1, converges along certain subsequences to a non-degenerate compactly supported measure on the real line.

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Date: February 20, 2022.
2010 *Mathematics Subject Classification.* 37A17, 37A20, 37A44, 60F05.
Key words and phrases. Higher rank abelian actions, Limit theorems, Heisenberg groups.

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1. Introduction

1.1. **Introduction.** The asymptotic behavior and limiting distribution of ergodic averages of translation flow were studied by Alexander Bufetov in the series of his works [Buf14a, Buf10, Buf14b]. He constructed finitely-additive Hölder measures and cocycles over translation flows that are known as *Bufetov functionals*. Such functionals are constructed to derive the deviation of ergodic integrals of translation flows by providing a new proof of the celebrated work of G. Forni [For02]. Bufetov noticed a duality between his finitely-additive Hölder measures and Forni's invariant distributions, which played a crucial role in his work of cohomological equations. Following these observations, the duality between such functionals and the space of invariant distributions was constructed analogously for other parabolic flows and higher rank actions (e.g. [Buf13, BS13, BF14, FK20b, For20a]). Bufetov also proved the probabilistic results for normalized ergodic integrals as a part of his main theorem. Furthermore, it turned out that his construction of functional is also closely related to limit shapes of ergodic sums for interval exchange transformation [MMY10, MUY20].

In this paper, our main results are on effective equidistributions for higher rank abelian actions and limit theorems on Heisenberg nilmanifolds. We firstly introduce the construction of the Bufetov functional for higher rank abelian actions. The main argument is based on the renormalizations and induction argument from the work of Cosentino and Flaminio [CF15]. However, in our case, we extended their constructions to the rectangular domain and allowed general products of intervals. We assume that the functional exists for a full measure set of frame α under a sufficient Diophantine condition obtained from recurrent conditions (see §2.3.2). Likewise, we also constructed the duality between functionals and invariant currents in [CF15], enabling the deviation formula. As a corollary, we prove existence of the limit distributions of (normalized) ergodic integrals for higher rank actions. More specifically, as a random variable, normalized ergodic integrals converge in distribution along certain subsequences to a non-degenerate, compactly supported measure on the real line.

As a minor application, it is not surprising that the related result for exponential sums appeared from connections between the Heisenberg group and theta series in several contexts. For Heisenberg flows, as a first return map on a transverse torus is obtained as a skew-translation, its limit theorem of theta sums was studied by J. Griffin J. Marklof and Cellarosi-Marklof [GM14, CM16]. It is recently generalized by Forni and Kanigowski [FK20b]. Likewise, we obtain limit theorems for theta series on Siegel half spaces which generalize results [GG04, Mar99] (see [Tol78, MM07, MNN07] for a general introduction).

Furthermore, L^2-lower bound of Bufetov functional on a transverse torus and its analyticity on the higher dimensional rectangular domain follows. This part contributes that our functional does not vanish and extends to the complex domain in the weighted space of fast-decaying coefficient of functionals. From these results, polynomial type of lower bounds for the analytic function is obtained on certain sub-level sets. Such polynomial estimates on the measure are derived by real analyticity of a functional along the leaves of a foliation transverse to the actions based on results of [Bru99]. This tool was already devised by Forni and Kanigowski [FK20b] to study the bound of correlations for time-changes of nilflows. Unfortunately, it turned out that we can not obtain similar results because the time-changes of higher rank abelian actions on Heisenberg nilmanifolds are all trivial. It follows from the triviality of the first cohomology group (see [CF15, Theorem 3.16]) thus, they are conjugated to the linear action and never mixing.

As a rank 1-action, mixing properties for time changes of Heisenberg nilflows were firstly studied in [AFU11]. Then, it was extended in the dense set of non-trivial time changes for any uniquely ergodic nilflows on general nilmanifolds [Rav18, AFRU21]. As a special case, on time-changes of Heisenberg flows, multiple mixing [FK20a] was proved by Ratner property. In $\mathbb{Z}^k$-actions, mixing of shapes for automorphisms on nilmanifold is proved in [GS14, GS15].

Comprehensive studies of spatial and temporal limit theorems for horocycle flows have been carried out in the work of D. Dolgopyat and O. Sarig [DS17]. Recent work of Ravotti [Rav21] proved spatial and temporal limit theorems for horocycle flows in alternative ways by using Ratner's argument. His work did not rely on the methods from invariant distributions of solving the cohomological equations. However, it is still unknown if temporal (or weaker) limit theorems for Heisenberg flows can be proved, and it will be a possible subject of further works.

1.2. **Definitions and statement of results.** We review definitions about Heisenberg manifold and its moduli space.

1.2.1. *Heisenberg manifold.* Let $g \geq 1$ and H^g be the standard $2g+1$ dimensional Heisenberg group and set $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$ a discrete and co-compact subgroup of H^g. We shall call it standard lattice of H^g and the quotient $M := \mathsf{H}^g/\Gamma$ will be called *Heisenberg manifold*. Lie algebra $\mathfrak{h}^g = Lie(\mathsf{H}^g)$ is equipped with a basis $(X_1, \cdots, X_g, Y_1, \cdots, Y_g, Z)$ satisfying canonical commutation relations

$$[X_i, Y_j] = \delta_{ij} Z, \quad \text{where } \delta_{ij} = 1 \text{ if } i = j, \text{ and } \delta_{ij} = 0 \text{ otherwise.} \tag{1}$$

For $1 \leq d \leq g$, let $\mathsf{P} := \mathsf{P}^d$ be a subgroup of H^g where its Lie algebra $\mathfrak{p} := Lie(\mathsf{P}^d)$ is generated by $\{X_1, \cdots, X_d\}$. For any $\alpha \in Sp_{2g}(\mathbb{R})$, set $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$ for $1 \leq i \leq d$. We define a parametrization of the subgroup $\mathsf{P}^{d,\alpha} = \alpha^{-1}(\mathsf{P}^d)$ according to

$$\mathsf{P}_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \cdots + x_d X_d^\alpha), \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

By central extension of $\mathbb{R}^{2g}$ by $\mathbb{R}$, we have an exact sequence

$$0 \to Z(\mathsf{H}^g) \to \mathsf{H}^g \to \mathbb{R}^{2g} \to 0.$$

The natural projection $pr : M \to \mathsf{H}^g/(\Gamma Z(\mathsf{H}^g))$ maps M onto a $2g$-dimensional torus $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

1.2.2. *Moduli space.* The group of automorphisms of H^g that are trivial on the center is denoted by $Aut_0(\mathsf{H}^g) = Sp_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$. Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to $Sp_{2g}(\mathbb{R})$. We regard $Sp_{2g}(\mathbb{R})$ as the *deformation space* of the standard Heisenberg manifold M and call the quotient $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ the *moduli space* of (standard) Heisenberg manifold.

Siegel modular variety is a double coset space $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ where K_g is a maximal compact subgroup $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$ of $Sp_{2g}(\mathbb{R})$. For $\alpha \in Sp_{2g}(\mathbb{R})$, we denote $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$ by its projection on the moduli space $\mathfrak{M}_g$ and write $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$ the projection of α to the Siegel modular variety Σ_g.

Double coset $K_g \backslash Sp_{2g}(\mathbb{R})/1_{2g}$ is identifed to the Siegel upper half space $\mathfrak{H}_g := \{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$. *Siegel upper half space* of genus g is complex manifold of symmetric complex $g \times g$ matrices $Z = X + iY$ with positive-definite symmetric imaginary part $\Im(Z) = Y$ and arbitrary (symmetric) real part X. Denote by $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$.

1.2.3. *Representation.* We write the Hilbert sum decomposition of

$$L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n \tag{2}$$

into closed H^g-invariant subspaces. Set $f = \sum_{n \in \mathbb{Z}} f_n \in L^2(M)$ and $f_n \in H_n$ where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi \iota n K t) f\}$$

for some fixed $K > 0$. The center $Z(\mathsf{H}^g)$ has spectrum $2\pi\mathbb{Z}\backslash\{0\}$ and the space $L^2(M)$ splits as Hilbert sum of H^g-module H_n, which is equivalent to irreducible representation π.

By Stone-Von Neumann theorem, the unitary irreducible representation π of the Heisenberg group of non-zero central parameter $K > 0$ is unitarily equivalent to the Schrödinger representation π. By differentiating the Schrödinger representation, we obtain a representation of the Lie algebra $\mathfrak{h}^g$ on Schwartz space $\mathscr{S}(\mathbb{R}^g) \subset L^2(\mathbb{R}^g)$ (as a C^∞-vector). This is called *infinitesimal derived representation* $d\pi_*$ with parameter $n \in \mathbb{Z}$, and for each $k = 1, 2, \cdots, g$,

$$d\pi_*(X_k) = \frac{\partial}{\partial x_k}, \quad d\pi_*(Y_k) = 2\pi \iota n K x_k, \quad d\pi_*(Z) = 2\pi \iota n K$$

acts on $L^2(\mathbb{R}^g) \simeq L^2(H_n)$.

Given a basis (V_i) of the Lie algebra, we set a Laplacian $\Delta = -\sum_i V_i^2$ and define L^2-Sobolev norm $\|f\|_s^2 = \langle f, (1+\Delta)^s f\rangle$ where $\langle \cdot, \cdot \rangle$ is an ordinary inner product. For $s > 0$, the *Sobolev space* $W^s(M)$ is defined by a completion of $C^\infty(M)$ equipped with the norm $\|\cdot\|_s$. The Sobolev space $W^s(M) = \bigoplus_{n \in \mathbb{Z}} W^s(H_n)$ decomposes to closed H^g-invariant subspaces $W^s(H_n) = W^s(M) \cap H_n$.

1.2.4. *Renormalization flow.* Denote diagonal matrix $\delta_i = diag(d_1, \cdots, d_g)$ with $d_i = 1$, $d_k = 0$ if $k \neq i$. Then, for each $1 \leq i \leq g$, we denote $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in \mathfrak{sp}_{2g} = Lie(Sp_{2g}(\mathbb{R}))$.

Any such $\hat{\delta}_i$ generate a one-parameter subgroup of automorphism (renormalization flow) $r_i^t := e^{t\hat{\delta}_i}$. We denote (rank d) renormalization actions $r_{\mathbf{t}} := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$ for $\mathbf{t} = (t_1, \cdots, t_d)$ and $1 \leq i_1, \cdots, i_d \leq g$. We also write the corresponding automorphism

$$\exp(\boldsymbol{t}\hat{\delta}(d)) : (x, y, z) \mapsto (e^{\boldsymbol{t}\hat{\delta}} x, e^{-\boldsymbol{t}\hat{\delta}} y, z) \tag{3}$$

of the Heisenberg group (see §2.2.3 for its use).

Main results. One of the main objects in this paper is to construct finitely-additive measures defined on the space of rectangles on Heisenberg manifold M. We state our results beginning with an overview of Bufetov functional.

Definition 1.1. For $(m, \mathbf{T}) \in M \times \mathbb{R}^d_+$, denote the *standard rectangle* for action P,

$$\Gamma^X_{\mathbf{T}}(m) = \{\mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m) \mid \boldsymbol{t} \in U(\mathbf{T}) = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]\}. \tag{4}$$

Let $\mathsf{Q}^{d,Y}_y := \exp(y_1 Y_1 + \cdots + y_d Y_d),\ y = (y_1, \cdots, y_d) \in \mathbb{R}^d$ be an action generated by elements Y_i of standard basis. Let $\phi^Z_z := \exp(zZ),\ z \in \mathbb{R}$ be a flow generated by the central element Z.

Definition 1.2. Let $\mathfrak{R}$ be the collection of the *generalized rectangles* in M. For any $1 \leq j \leq d \leq g$ and $\boldsymbol{t} = (t_1, \cdots, t_d)$, we set

$$\mathfrak{R} := \bigcup_{1 \leq i \leq d} \ \bigcup_{(y,z) \in \mathbb{R}^j \times \mathbb{R}} \ \bigcup_{(m,\mathbf{T}) \in M \times \mathbb{R}^d_+} \{(\phi^Z_{t_i z}) \circ \mathsf{Q}^{j,Y}_y \circ \mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m) \mid \boldsymbol{t} \in U(\mathbf{T})\}. \tag{5}$$

Theorem 1.3. *For any irreducible representation H, there exists a finitely-additive measure $\hat{\beta}_H(\Gamma) \in \mathbb{C}$ defined on every standard d-rectangle Γ such that the following holds:*

(1) (Additive property) *For any decomposition of disjoint rectangles $\Gamma = \bigcup_{i=1}^n \Gamma_i$ or whose intersections have zero measure,*
$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^n \hat{\beta}_H(\alpha, \Gamma_i).$$
(2) (Scaling property) *For $\boldsymbol{t} \in \mathbb{R}^d$,*
$$\hat{\beta}_H(r_{\boldsymbol{t}}(\alpha), \Gamma) = e^{-(t_1 + \cdots + t_d)/2} \hat{\beta}_H(\alpha, \Gamma).$$
(3) (Invariance property) *For any action $\mathsf{Q}^{j,Y}_\tau$ generated by Y_i's for $\tau \in \mathbb{R}^j_+$ and $1 \leq j \leq d$,*
$$\hat{\beta}_H(\alpha, (\mathsf{Q}^{j,Y}_\tau)_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$
(4) (Bounded property) *For any rectangle $\Gamma \in \mathfrak{R}$, there exists a constant $C(\Gamma) > 0$ such that for $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$,*
$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma) (\int_\Gamma |\hat{X}|)^{d/2}.$$

Corollary 1.4. *The functional $\hat{\beta}_H$ defined on standard rectangle $\Gamma^X_{\boldsymbol{T}}$ extends to the class $\mathfrak{R}$.*

Remark. *By the additive property of functionals on rectangles, it is plausible to extend the shape of domain to a general boundary by approximating with rectangles. However, this method may cause limitations by having a weaker estimate of equidistributions. It may be also interesting to compare the methods of Shah [S09a, S09b] and to obtain equidistribution results for smooth (or Lipschitz) boundary in our settings.*

Let us consider arbitrary two d-standard rectangles $U(\mathbf{T}_1)$ and $U(\mathbf{T}_2)$. For convenience, these rectangles are translated to intersect at only one vertex as in the figure. Without loss of generality, assume that d distinct faces of $U(\mathbf{T}_1)$ emanate

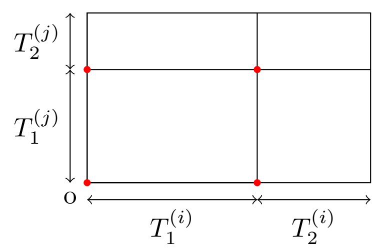

FIGURE 1. Illustration of the rectangles $U(\mathbf{T}_1)$ and $U(\mathbf{T}_2)$ on i, j-th coordinate.

from the origin and $U(\mathbf{T}_1)\cap U(\mathbf{T}_2) = \{(\mathbf{T}_1^{(l)})_{1\leq l\leq d}\}$. Denote by $\mathbf{P}(\mathbf{T}_1)$ a collection of 2^d vertices $v = (v^{(1)}, v^{(2)}, \cdots, v^{(d)}) \in \mathbb{R}_+^d$ of $U(\mathbf{T}_1)$.

Then we define a vector $\mathbf{T}_v = (\mathbf{T}_v^{(l)}) \in \mathbb{R}_+^d$ associated with v given by

$$\mathbf{T}_v^{(l)} := \begin{cases} \mathbf{T}_2^{(l)} & \text{if} \quad v^{(l)} = \mathbf{T}_1^{(l)} \\ \mathbf{T}_1^{(l)} & \text{if} \quad v^{(l)} = 0. \end{cases}$$

Corollary 1.5. *Let us denote*

$$\beta_H(\alpha, m, \boldsymbol{T}) := \hat{\beta}_H(\alpha, \Gamma_{\boldsymbol{T}}^X(m)). \tag{6}$$

The function β_H satisfies the following properties:

(1) (Cocycle property) *For all $(m, \boldsymbol{T}_1, \boldsymbol{T}_2) \in M \times \mathbb{R}_+^d \times \mathbb{R}_+^d$,*

$$\beta_H(\alpha, m, \boldsymbol{T}_1 + \boldsymbol{T}_2) = \sum_{v\in \mathbf{P}(\boldsymbol{T}_1)} \beta_H(\alpha, \mathsf{P}_v^{d,\alpha}(m), \boldsymbol{T}_v).$$

(2) (Scaling property) *For all $m \in M$ and $\boldsymbol{t} = (t_1, \cdots, t_d)$,*

$$\beta_H(r_{\boldsymbol{t}}(\alpha), m, \boldsymbol{T}) = e^{-(t_1+\cdots+t_d)/2}\beta_H(\alpha, m, \boldsymbol{T}).$$

(3) (Bounded property) *Let us denote the largest length of edges of $U(\boldsymbol{T})$ by $T_{max} = \max_{1\leq i\leq d} \boldsymbol{T}^{(i)}$. Then there exists a constant $C_H > 0$ such that*

$$\beta_H(\alpha, m, \boldsymbol{T}) \leq C_H T_{max}^{d/2}.$$

(4) (Orthogonality) *For all $\boldsymbol{T} \in \mathbb{R}^d$, $\beta_H(\alpha, \cdot, \boldsymbol{T})$ belongs to an irreducible component, i.e,*

$$\beta_H(\alpha, \cdot, \boldsymbol{T}) \in H \subset L^2(M).$$

By representation theory introduced in §1.2.3, for any $f = \sum_H f_H \in W^s(M)$, define a *Bufetov cocycle* associated to f as a sum

$$\beta^f(\alpha, m, \boldsymbol{T}) = \sum_H D_\alpha^H(f)\beta_H(\alpha, m, \boldsymbol{T}). \tag{7}$$

Notation. Let $\hat{X}_i^\alpha$ be a 1-form dual to the vector field X_i^α, in the sense that $\hat{X}_i^\alpha(X_i^\alpha) = 1$ and $\hat{X}_i^\alpha(X_j^\alpha) = 0$ if $i \neq j$ on M. Given a Jordan region U and a point $m \in M$, set $\mathscr{P}_U^{d,\alpha}m$ the *Birkhoff integrals* (currents) associated to the action $\mathsf{P}_x^{d,\alpha}$ given by

$$\left\langle \mathscr{P}_U^{d,\alpha}m, \omega_f \right\rangle := \int_U f(\mathsf{P}_x^{d,\alpha}m)dx_1\cdots dx_d$$

for any degree d $\mathfrak{p}$-form $\omega_f = f\hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$ with a smooth function $f \in C_0^\infty(M)$ with zero averages.

Theorem 1.6. *For all $s > d(d+11)/4 + g + 1$, there exists a constant $C_s > 0$ such that for almost all frequency α and for all $f \in W^s(M)$ and for all $(m, \boldsymbol{T}) \in M \times \mathbb{R}^d$, we have*

$$|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_f \right\rangle - \beta^f(\alpha, m, \boldsymbol{T})| \leq C_s \left\|\omega_f\right\|_s . \tag{8}$$

for $U(\boldsymbol{T}) = [0, T^{(1)}] \times \cdots \times [0, T^{(d)}]$ and $\omega_f = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p} \otimes W^s(M)$.

The family of random variable

$$E_{\boldsymbol{T_n}}(f) := \frac{1}{vol(U(\boldsymbol{T_n}))^{1/2}} \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T_n})}(m), \omega_f \right\rangle$$

is defined where $U(\boldsymbol{T_n})$ is a sequence of rectangles. The point $m \in M$ is distributed accordingly to the probability measure *vol* on M. Our goal is to understand the asymptotic behavior of the probability distributions of $E_{\boldsymbol{T_n}}(f)$ as $U(\boldsymbol{T_n}) \nearrow \mathbb{R}^d$ in a sense of Følner.

Theorem 1.7. *Let $\{\boldsymbol{T_n}\}$ be any sequence such that*

$$\lim_{n\to\infty} r_{\log \boldsymbol{T_n}}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

For every closed form $\omega_f \in \Lambda^d\mathfrak{p} \otimes W^s(M)$ with $s > d(d+11)/4 + g + 1$, which is not a coboundary, the limit distribution of the family of random variables $E_{\boldsymbol{T_n}}(f)$ exists along a subsequence of $\{\boldsymbol{T_n}\}$. In particular, for almost all α, the limit distribution of $E_{\boldsymbol{T_n}}(f)$ has compact support.

We finish the section by giving some remarks on the new adaptation of transfer operator techniques from hyperbolic dynamics. The method stems from the analysis of the transfer operator, firstly treated by P. Gieulietti and Liverani [GL19]. They set up a non-linear flow on the torus and proved asymptotics of ergodic averages with expansions of invariant distributions and eigenvalues of transfer operators called *Ruelle resonances* (see also [AB18, FGL19, For20b, B19] for applications in parabolic flows).

A recent work of L. Simonelli and O. Butterley [BS20] reproved some of the results of Flaminio and Forni [FF06] by analytical methods of hyperbolic theory, not relying on representation theory. Although their work is restricted to the periodic type of α in the moduli space $\mathfrak{M}$ (the flow is only renormalized by partially hyperbolic diffeomorphism, or α is periodic) their methods showed indirect similarities with [FK20b] and our current work.[1] However, it is still not known if it is possible to extend their approach to full measure set of α, requiring construction of renormalization cocycle, so called *Transfer cocycle*. It is also not well-studied how to replace the use of Anisotropic norm, not relying on the result of Faure-Tsujii [FT15].[2]

On higher step nilmanifolds, there does not exist a renormalization flow (moduli space $\mathfrak{M}$ is trivial). It is not possible to construct the Bufetov functionals

[1]For instance, it is plausible to view their construction of functionals obtained by spectral projection as a functional $\hat{\beta}_H$ on each sub-representation H in our setting (see [BS20, §4]).

[2]Furthermore, instead of relying on the methods of Sobolev constant techniques introduced in [FF06], we do not have proper tools on Anisotropic spaces yet.

by the same strategies introduced in here, but other methods in handling non-renormalizable flows are possibly applied (cf. [FF14, FFT16, Kim21]).

Outline of the paper. In section 2, we give basic definitions on higher rank actions, moduli spaces and Sobolev spaces. In section 3, we state main theorem and prove constructions of Bufetov functionals with main properties. In section 4, we prove asymptotic formula of ergodic integrals and limit theorems. In section 5, we prove L^2-lower bound of the Bufetov functionals. In Section 6, analyticity of functional and extensions of domain are provided. In section 7, there exist measure estimates of functionals on the sets where values of functionals are small. This result only holds when frame α is of bounded type.

2. Analysis on Heisenberg manifolds

In this section, we review definitions of Sobolev space, currents, representation and renormalization flows on moduli space.

2.1. **Sobolev space.**

2.1.1. *Sobolev norm.* Given a basis (V_i) of the Lie algebra $\mathfrak{h}^g$, we write a new basis $((\alpha^{-1})_* V_i)$ for $\alpha \in Aut_0(\mathsf{H}^g)$. Similarly, denote by Laplacian $\Delta_\alpha = -\sum_i (\alpha^{-1})_* V_i^2$ with respect to the new basis. For any $s \in \mathbb{R}$ and any C^∞-function $f \in L^2(M)$, *Sobolev norm* is defined by

$$\|f\|_{\alpha,s} = \langle f, (1+\Delta_\alpha)^s f\rangle^{1/2}.$$

Let $W^s_\alpha(M)$ be a completion of $C^\infty(M)$ with the norm above. The dual space of $W^s_\alpha(M)$ is denoted by $W^{-s}_\alpha(M)$ and it is isomorphic to $W^s_\alpha(M)$. Extending it to the exterior algebra, define the Sobolev spaces of the form $\Lambda^d\mathfrak{p} \otimes W^s_\alpha(M)$ of cochains of degree d and we use the same notations for the norm.

2.1.2. *Sobolev bundle.* The group $Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \to Sp_{2g}(\mathbb{R})$$

where

$$(\alpha, \varphi) \mapsto (\alpha,\varphi)\gamma = (\alpha\gamma, \gamma^*\varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \to \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

and the fiber over $[\alpha] \in \mathfrak{M}_g$ is locally identified with the space $W^s_\alpha(M)$.

By invariance of $Sp_{2g}(\mathbb{Z})$ action, the class of (α,φ) is denoted by $[\alpha,\varphi]$ and $Sp_{2g}(\mathbb{Z})$-invariant Sobolev norm is written by

$$\|f\|_{\alpha,s} := \|[\alpha, f]\|_s\,.$$

We denote the bundle of $\mathfrak{p}$-forms of degree j of Sobolev order s by $A^j(\mathfrak{p}, \mathfrak{M}^s)$. The space of continuous linear functional on $A^j(\mathfrak{p}, \mathfrak{M}^s)$ will be called the *space of currents* of dimension j and denoted by $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$. There is a flat bundle of (currents) distribution $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$ whose fiber over $[\alpha]$ is locally identified with the space $W^{-s}_\alpha(M)$. Likewise, the space of Sobolev currents $A_j(\mathfrak{p}, W^{-s}(M))$ of dimension j with order s is identified with $\Lambda^j\mathfrak{p} \otimes W^{-s}(M)$. We write the norm for form ω and currents $\mathscr{D}$ by

$$\|\omega\|_{\alpha,s} := \|[\alpha,\omega]\|_s\,, \quad \|\mathscr{D}\|_{\alpha,s} := \|[\alpha,\mathscr{D}]\|_s\,.$$

2.1.3. *Best Sobolev constant.* The *Sobolev embedding theorem* implies that for any $\alpha \in Sp_{2g}(\mathbb{R})$ and $s > g + 1/2$, there exists a constant $B_s(\alpha)$ such that for any $f \in W^s_\alpha(M)$,

$$B_s(\alpha) := \sup_{f \in W^s_\alpha(M)\setminus\{0\}} \frac{\|f\|_\infty}{\|f\|_{\alpha,s}}. \tag{9}$$

By Lemma 4.4 in [CF15], the *best Sobolev constant* B_s is a $Sp_{2g}(\mathbb{Z})$-modular function on $\mathfrak{H}_g$. Thus, we shall write $B_s([[\alpha]])$ or $B_s([\alpha])$ for $B_s(\alpha)$. By the Sobolev embedding theorem, we have a bound for the Birkhoff integral current.

Lemma 2.1. *[CF15, Lemma 5.5] For any Jordan region $U \subset \mathbb{R}^d$ with Lebesgue measure $|U|$, for any $s > g + 1/2$ and all $m \in M$,*

$$\left\| [\alpha, \mathscr{P}^{d,\alpha}_U m] \right\|_{-s} \leq B_s([[\alpha]])|U|.$$

2.2. **Invariant currents.**

2.2.1. *Identification.* The *boundary operators*

$$\partial : A_j(\mathfrak{p}, W^{-s}(M)) \to A_{j-1}(\mathfrak{p}, W^{-s}(M))$$

are adjoint of the differentials d such that $\langle \partial \mathcal{D}, \omega \rangle = \langle \mathcal{D}, d\omega \rangle$ for any $\omega \in \Lambda^{j-1}\mathfrak{p} \otimes W^s(M)$. A current $\mathcal{D}$ is called *closed* if $\partial \mathcal{D} = 0$.

For $s > 0$, we denote $Z_d(\mathfrak{p}, W^{-s}(M))$ by the space of closed currents of dimension d and Sobolev order s. $I_d(\mathfrak{p}, W^{-s}(M))$ is denoted by the d-dimensional space of P-invariant currents of Sobolev order s. Now we find the relations between these two currents.

Proposition 2.2. *[CF15, Proposition 3.13] For any $s > d/2 = \dim \mathsf{P}/2$, we have $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$ for all $\epsilon > 0$. Additionally,*

- *$I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is one dimensional space if $\dim \mathsf{P} = g$,*
- *$I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is an infinite-dimensional space if $\dim \mathsf{P} < g$,*
- *$I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) = Z_d(\mathfrak{p}, W^{-s}(M))$ for any $1 \leq d \leq g$.*

2.2.2. *Basic currents.* The current B_α is called *basic* if for all $j \in \{i_1, \cdots, i_d\}$,

$$\iota_{X_j} B_\alpha = L_{X_j} B_\alpha = 0.$$

For an irreducible representation H, there exists a unique *basic current* B^H_α (of degree $2g + 1 - d$ and dimension d) associated to invariant distribution D^H_α. It is defined by $B^H_\alpha = D^H_\alpha \eta_X$ and this formula implies that for every d-form ξ,

$$B^H_\alpha(\xi) = D^H_\alpha \left(\frac{\eta_X \wedge \xi}{\omega_{vol}} \right) \tag{10}$$

where $\eta_X := \iota_{X_{i_1}} \cdots \iota_{X_{i_d}} \omega_{vol}$ and ω_{vol} is an invariant volume form (with total unit volume).

The basic current B^H_α belongs to the Sobolev space of currents and it is P-invariant. It follows that for all $s > d/2$, by Sobolev embedding theorem, $B^H_\alpha \in A_j(\mathfrak{p}, W^{-s}_\alpha(M))$ if and only if $D^H_\alpha \in W^{-s}_\alpha(M)$ for all $s > d/2$.

Remark. *The formula (10) yields an isomorphism between the space of basic (closed) currents and invariant distributions (see also [For02, §6.1] and [BF14,*

§2.1]). For any d-dimensional $\mathfrak{p}$-form $\omega_f = f\hat{X}_1 \wedge \cdots \wedge \hat{X}_d$ with $f \in C^\infty(M)$, we will identify currents $\mathcal{D}$ with distribution D by writing

$$\langle D, f\rangle = \langle \mathcal{D}, \omega_f\rangle.$$

2.2.3. *Renormalization.* Let $s > d/2$. Recall the definition of renormalization flow (3) in §1.2.4. For $\omega \in \Lambda^d\mathfrak{p} \otimes W^s(\mathbb{R}^g)$, $\mathcal{D} \in Z_d(\mathfrak{p}, W^{-s}(M))$, and $t \in \mathbb{R}$

$$r_i^t[\alpha, \omega] = [r_i^t\alpha, \omega], \quad r_i^t[\alpha, \mathcal{D}] = [r_i^t\alpha, \mathcal{D}].$$

By Proposition 5.2 in [CF15], the sub-bundle $Z_d(\mathfrak{p}, W^{-s}(M))$ is invariant under the renormalization flows r_i^t. Furthermore, we have

$$\left\| r_1^{t_1} \ldots r_d^{t_d}[\alpha, \mathcal{D}]\right\|_{-s} = e^{-(t_1+\cdots+t_d)/2} \left\|[\alpha, \mathcal{D}]\right\|_{-s}.$$

Let $U_{\mathbf{t}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be unitary operator for $\mathbf{t} = (t_1, \cdots, t_d)$,

$$U_{\mathbf{t}} f(x) = e^{-(t_1+\cdots+t_d)/2} f(e^{t_1}x_1, \cdots e^{t_d}x_d). \tag{11}$$

Then, for any invariant distribution D^H_α,

$$D^H_{r_{\boldsymbol{t}}(\alpha)} = e^{(t_1+\cdots+t_d)/2} D^H_\alpha.$$

By reparametrization of $\mathbb{R}^d$-action $r_{\boldsymbol{t}} = r_1^{t_1} \ldots r_d^{t_d}$,

$$\mathsf{P}_x^{d,(r_1^{t_1}\ldots r_d^{t_d}\alpha)} = \mathsf{P}^{d,\alpha}_{(e^{-t_1}x_1,\ldots,e^{-t_d}x_d)}. \tag{12}$$

Then the Birkhoff integral current also satisfies the following identity

$$\mathscr{P}_U^{d,(r_1^{t_1}\ldots r_d^{t_d}\alpha)} m = e^{(t_1+\cdots+t_d)/2} \mathscr{P}^{d,\alpha}_{(e^{-t_1},\ldots,e^{-t_d})U} m. \tag{13}$$

2.3. **Diophantine condition.**

2.3.1. *Height function.*

Definition 2.3 (Height function)**.** The *height* of a point $Z \in \mathfrak{H}_g$ in Siegel upper half space is the positive number

$$hgt(Z) := \det\Im(Z).$$

The *maximal height function* $Hgt : \Sigma_g \to \mathbb{R}^+$ is the maximal height of a $Sp_{2g}(\mathbb{Z})$ orbit of Z. That is, for the class of $[Z] \in \Sigma_g$,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

By Proposition 4.8 of [CF15], there exists a universal constant $C(s) > 0$ such that the best Sobolev constant satisfies the estimate

$$B_s([[\alpha]]) \le C(s) \cdot (Hgt[[\alpha]])^{1/4}. \tag{14}$$

We rephrase Lemma 4.9 in [CF15] regarding the bound of renormalized height.

Lemma 2.4. *For any $[\alpha] \in \mathfrak{M}_g$ and any $\boldsymbol{t} \in \mathbb{R}^d_+$,*

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \le (det(e^{\boldsymbol{t}\hat{\delta}}))^2 Hgt([[\alpha]]). \tag{15}$$

2.3.2. *Diophantine condition.*

Definition 2.5. An automorphism $\alpha \in Sp_{2g}(\mathbb{R})$ or a point $[\alpha] \in \mathfrak{M}_g$ is $\hat{\delta}(d)$-*Diophantine* of type σ if there exists a $\sigma > 0$ and a constant $C > 0$ such that

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \leq CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{(1-\sigma)}Hgt([[\alpha]]),\ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \tag{16}$$

This states that $\alpha \in Sp_{2g}(\mathbb{R})$ satisfies $\hat{\delta}(d)$-Diophantine if the height of the projection of $\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha$ in the Siegel modular variety Σ_g is bounded by $e^{2(t_1+\cdots+t_d)(1-\sigma)}$. Furthermore,

- $[\alpha] \in \mathfrak{M}_g$ satisfies a $\hat{\delta}(d)$-*Roth condition* if for any $\epsilon > 0$ there exists a constant $C > 0$ such that

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \leq CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{\epsilon}Hgt([[\alpha]]),\ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \tag{17}$$

That is, $\hat{\delta}(d)$-Diophantine of type $0 < \sigma < 1$.

- $[\alpha]$ is of *bounded type* if there exists a constant $C > 0$ such that

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]]) \leq C,\ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \tag{18}$$

Definition 2.6. Let $X = G/\Lambda$ be a homogeneous space equipped with the probability Haar measure μ. A function $\phi : X \to \mathbb{R}$ is said k-DL (distance like) for some exponent $k > 0$ if it is uniformly continuous and if there exist constants $C_1, C_2 > 0$ such that

$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

By the work of Kleinblock and Margulis, a multi-parameter generalization of Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

Theorem 2.7. *[KM99, Theorem 1.9] Let G be a connected semisimple Lie group without compact factors, μ its normalized Haar measure, $\Lambda \subset G$ an irreducible lattice, $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G. Let $\mathfrak{d}_+$ be a non-empty open cone in a d-dimensional subalgebra $\mathfrak{d}$ of $\mathfrak{a}$ $(1 \leq d \leq rank_{\mathbb{R}}(G))$. If $\phi : G/\Lambda \to \mathbb{R}$ is a k-DL function for some $k > 0$, then for μ-almost all $x \in G/\Lambda$ one has*

$$\limsup_{\mathbf{z}\in\mathfrak{d}_+, \mathbf{z}\to\infty} \frac{\phi(\exp(\mathbf{z})x)}{\log\|\mathbf{z}\|} = \frac{d}{k}$$

By Lemma 4.7 of [CF15], the logarithm of Height function is DL-function with exponent $k = \frac{g+1}{2}$ on the Siegel variety Σ_g (and induces on $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$). Hence, we obtain the following proposition.

Proposition 2.8. *Under the assumption $X = \mathfrak{M}_g$ of Theorem 2.7, for $s > g+1/2$, there exists a full measure set $\Omega_g(\hat{\delta})$ and for all $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$*

$$\limsup_{\boldsymbol{t}\in\mathbb{R}^d_+, \boldsymbol{t}\to\infty} \frac{\log Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]])}{\log\|\boldsymbol{t}\|} \leq \frac{2d}{g+1}. \tag{19}$$

Any such $[\alpha]$ satisfies a $\hat{\delta}(d)$-Roth condition (17).

For any $L > 0$ and $1 \leq d \leq g$, let $DC(L)$ denote the set of $[\alpha] \in \mathfrak{M}_g$ such that

$$\int_0^\infty \cdots \int_0^\infty e^{-(t_1+\cdots+t_d)/2}\mathrm{Hgt}([[r_{\mathbf{-t}}(\alpha)]])^{1/4} dt_1 \cdots dt_d \leq L. \tag{20}$$

Let DC denote the union of the sets $DC(L)$ over all $L > 0$. It follows immediately that the set $DC \subset \mathfrak{M}_g$ has full Haar volume.

Remark. *In the work of Cosentino-Flaminio [CF15], Diophantine condition is restricted to one-parameter subgroup* $\exp(-t\hat{\delta})$ *for non-negative (g-dimensional) diagonal directions. This is based on the easy part of theorem in Kleinblock and Margulis [KM99, Theorem 1.7]. Our condition is for* $d-$*dimensional renormalizations actions. We will write the notation* $r_{\boldsymbol{-t}}$ *instead of* $\exp(-\boldsymbol{t}\hat{\delta}(d))$ *for later sections.*

3. Constructions of the functionals

In this section, we construct Bufetov functionals for higher rank actions on the standard rectanglular domain. Firstly, we prove the bound of Birkhoff integrals on rectangles. It consists of similar proofs of Theorem 5.10 in [CF15], but this type of estimate will be used to control remainder estimate (Lemma 3.4) to define functionals under Diophantine condition (Lemma 3.5). Furthermore, we will verify the properties of functionals $\hat{\beta}_H$ and cocycles β_H stated in Theorem 1.3.

3.1. **Remainder estimates.** For any exponent $s > d/2$, Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^{\perp}$. Denote by $\mathcal{I}^{-s}$ and $\mathcal{R}^{-s}$ the corresponding orthogonal projection operator and by $\mathcal{I}^{-s}_{\alpha}$ and $\mathcal{R}^{-s}_{\alpha}$ the restrictions to the fiber over $[\alpha] \in \mathfrak{M}$ for $\alpha \in Sp_{2g}(\mathbb{R})$. In particular, for the current (Birkhoff integrals) $\mathscr{D} = \mathscr{P}^{d,\alpha}_{U} m$, we call $\mathcal{I}^{-s}_{\alpha}(\mathscr{D}) = \mathcal{I}^{-s}[\alpha, \mathscr{D}]$ *boundary* term and $\mathcal{R}^{-s}_{\alpha}(\mathscr{D}) = \mathcal{R}^{-s}[\alpha, \mathscr{D}]$ *remainder* term respectively. Consider the orthogonal projection

$$\mathscr{D} = \mathcal{I}^{-s}_{r_{-\boldsymbol{t}}[\alpha]}(\mathscr{D}) + \mathcal{R}^{-s}_{r_{-\boldsymbol{t}}[\alpha]}(\mathscr{D}). \tag{21}$$

We firstly recall the following estimate of boundary term.

Lemma 3.1. *[CF15, Lemma 5.7] Let* $s > d/2 + 2$. *There exists a constant* $C = C(s) > 0$ *such that for all* $t_i \geq 0$ *for* $1 \leq i \leq d$*, we have*

$$\begin{aligned}\left\|\mathcal{I}^{-s}[\alpha, \mathscr{D}]\right\|_{-s} &\leq e^{-(t_1+\cdots+t_d)/2} \left\|\mathcal{I}^{-s}[r_1^{-t_1}\cdots r_d^{-t_d}\alpha, \mathscr{D}]\right\|_{-s} \\ &\quad + C_1|t_1+\cdots+t_d| \int_0^1 e^{-u(t_1+\cdots+t_d)/2} \left\|\mathcal{R}^{-s}[r_1^{-ut_1}\cdots r_d^{-ut_d}\alpha, \mathscr{D}\right\|_{-(s-2)} du.\end{aligned}$$

By Stokes' theorem, we have the following remainder estimate.

Lemma 3.2. *[CF15, Lemma 5.6] Let* $s > g + d/2 + 1$. *For any non-negative* $s' < s - (d+1)/2$ *and Jordan region* $U \subset \mathbb{R}^d$, *there exists* $C = C(d, g, s, s') > 0$ *such that*

$$\left\|\mathcal{R}^{-s}[\alpha, (\mathscr{P}^{d,\alpha}_{U} m)]\right\|_{-s} \leq C \left\|[\alpha, \partial(\mathscr{P}^{d,\alpha}_{U} m)]\right\|_{-s'}.$$

Quantitative bound of Birkhoff integrals on the square domain was obtained in [CF15], but we need to extend the result to the rectangular shapes for analyticity of functionals in the section §6.

From now on, assume that $s_{d,g} := d(d+11)/4 + g + 1/2$.

Theorem 3.3. *For $s > s_{d,g}$, there exists a constant $C(s,d) > 0$ such that the following holds. For any $t_i > 0$, $m \in M$ and $U_d(t) = [0, e^{t_1}] \times \cdots \times [0, e^{t_d}]$, we have*

(22)
$$\left\|[\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)} m)]\right\|_{-s} \leq C \sum_{k=0}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} \exp\Big(\frac{1}{2}\sum_{l=1}^{d} t_l - \frac{1}{2}\sum_{l=1}^{k} u_{i_l}\Big) \times Hgt\Big([[\prod_{1 \leq j \leq d} r_j^{-t_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha]]\Big)^{1/4} du_{i_1} \cdots du_{i_k}.$$

Proof. We proceed by induction. For $d = 1$, it follows from the Theorem 5.8 in [CF15]. We assume that the result holds for $d-1$. Decompose the current as a sum of boundary and remainder term as in (21).

Step 1. We estimate the boundary term. By Lemma 3.1, renormalize terms with $r^u = r_1^u \cdots r_d^u$. Then, we have

(23)
$$\begin{aligned} \left\|\mathcal{I}^{-s}[\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)} m)]\right\|_{-s} &\leq e^{-(t_1+\cdots+t_d)/2} \left\|\mathcal{I}^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)} m)]\right\|_{-s} \\ &+ C_1(s) \int_0^{t_1+\cdots+t_d} e^{-ud/2} \left\|\mathcal{R}^{-s}[r^{-u}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)} m)]\right\|_{-(s-2)} du \\ &:= (I) + (II). \end{aligned}$$

By renormalization (13) and Lemma 2.1 for unit volume,

$$\begin{aligned} \left\|\mathcal{I}^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)} m)]\right\|_{-s} &= e^{t_1+\cdots+t_d} \left\|\mathcal{I}^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha, (\mathscr{P}^{d,r_1^{-t_1}\cdots r_d^{-t_d}\alpha}_{U_d(0)} m)]\right\|_{-s} \\ &\leq C_2 e^{t_1+\cdots+t_d} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d}\alpha]])^{1/4}. \end{aligned}$$

Hence
$$I \leq C_2 e^{(t_1+\cdots t_d)/2} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d}\alpha]])^{1/4},$$
where the sum corresponds to the first term ($k = 0$) in the statement.

Step 2. To estimate (II),

(24)
$$\begin{aligned} \left\|\mathcal{R}^{-s}[r^{-u}\alpha, (\mathscr{P}^{d,\alpha}_{U_d(t)} m)]\right\|_{-(s-2)} &= \left\|e^{ud}\mathcal{R}^{-s}[r^{-u}\alpha, (\mathscr{P}^{d,r^{-u}\alpha}_{U_d(t-u)} m)]\right\|_{-(s-2)} \\ &\leq C_3(s,s') e^{ud} \left\|[r^{-u}\alpha, \partial(\mathscr{P}^{d,r^{-u}\alpha}_{U_d(t-u)} m)]\right\|_{-s'}. \end{aligned}$$

The boundary $\partial(\mathsf{P}^{d,r^{-t}\alpha}_{U_d})$ is the sum of $2d$ currents of dimension $d-1$. These currents are Birkhoff sums of d face subgroups obtained from $\mathsf{P}^{d,r^{-t}\alpha}_{U_d}$ by omitting one of the base vector fields X_i. It is reduced to $(d-1)$ dimensional shape obtained from $U_d(t-u) := [0, e^{t_1-u}] \times \cdots \times [0, e^{t_d-u}]$. For each $1 \leq j \leq d$, there are Birkhoff sums along $d-1$ dimensional cubes. By induction hypothesis, we add all the $d-1$

dimensional cubes by adding all the terms along j:

(25)

$$\left\|[r^{-u}\alpha, (\mathscr{P}^{d-1,r^{-u}\alpha}_{U_{d-1}(t-u)}m)]\right\|_{-s'} \leq C_4(s',d-1)\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j,\forall l}}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u}$$

$$\exp(\frac{1}{2}\sum_{\substack{l=1\\ l\neq j}}^{d}(t_l-u)-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{\substack{1\leq l\leq d\\ l\neq j}}r_l^{-(t_l-u)}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r^{-u}\alpha)]])^{1/4}du_{i_1}\cdots du_{i_k}.$$

Combining (23) and (24), we obtain the estimate for (II).

$$(II)\leq C_5(s',d-1)\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j}}\int_0^{t_1+\cdots t_d}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u}du_{i_1}\cdots du_{i_k}du$$

$$\times\exp(\frac{1}{2}\sum_{\substack{l=1\\ l\neq j}}^{d}t_l-\frac{1}{2}u-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\leq l\leq d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_j^{-u+t_j}\alpha)]])^{1/4}.$$

Applying the change of variable $u_j=t_j-u$, we obtain

$$(II)\leq C_6(s',d-1)\sum_{j=1}^{d}\sum_{k=1}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j}}$$

$$\times\Big(\int_{-(t_1+\cdots t_d)+t_j}^{t_j}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u}du_{i_1}\cdots du_{i_k}du_j$$

$$\times\exp(\frac{1}{2}(t_1+\cdots+t_d)-\frac{1}{2}u_j-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\leq l\leq d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_j^{-u_j}\alpha)]])^{1/4}\Big).$$

Simplifying multi-summation above, (with $-(t_1+\cdots t_d)+t_j\leq 0$)

$$(II)\leq C_7(s',d)\sum_{k=1}^{d}\sum_{1\leq i_1<\cdots<i_k\leq d}\int_0^{t_{i_k}}\cdots\int_0^{t_{i_1}}du_{i_1}\cdots du_{i_k}$$

$$\times\exp(\frac{1}{2}(t_1+\cdots t_d)-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\leq l\leq d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}\alpha]])^{1/4}.$$

Step 3. (Remainder estimate). The remainder term is obtained from Lemma 3.2 (Stokes' theorem). Following step 2, estimate of remainder reduces to that of $d-1$ form. Combining with the step 1, we have the following

(26)

$$\left\|\mathcal{R}^{-s}[\alpha,\partial(\mathscr{P}^{d,\alpha}_{U_d}m)]\right\|_{-s}\leq C(s)\sum_{i=1}^{d-1}\left\|\mathcal{I}^{-s}[\alpha,(\mathscr{P}^{i,\alpha}_{U_i}m)]\right\|_{-s}+\left\|\mathcal{R}^{-s}[\alpha,(\mathscr{P}^{1,\alpha}_{U_1}m)]\right\|_{-s}$$

where U_i is i-dimensional rectangle. Sum of the boundary terms are absorbed in the bound of $(I)+(II)$. For 1-dimensional remainder with interval Γ_T, the boundary is a 0-dimensional current. Then,

$$\langle\partial(\mathscr{P}^{1,\alpha}_{[0,T]}m),f\rangle=f(\mathsf{P}^{1,\alpha}_{T}m)-f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (9) and (14),

$$\left\|\mathcal{R}^{-s}[\alpha, \partial(\mathscr{P}^{1,\alpha}_{[0,T]}m)]\right\|_{-s} \leq 2B_s([[\alpha]]) \leq C(s)Hgt([[\alpha]])^{1/4}.$$

Then, by inequality (15)

$$\begin{aligned} C(s)Hgt([[\alpha]])^{1/4} &= C(s)Hgt([[r_{\boldsymbol{t}}r_{-\boldsymbol{t}}\alpha]])^{1/4} \\ &\leq C(s)e^{(t_1+\cdots+t_d)/2}Hgt([[r_1^{-t_1}\cdots r_d^{-t_d}\alpha]])^{1/4}. \end{aligned}$$

This implies that remainder term produces one more term like the bound of (I). Therefore, the Theorem follows from combining all the terms (I), (II), and remainder. □

Remark. *It is also reasonable to decompose the rectangle $U_d(t)$ as a union of several squares and renormalize their faces. However, it may also involve computational difficulty and such approach of summing up squares may provide a rough upper bound of ergodic integrals. To obtain a necessary bound for the Lemma 3.4, we rather simply generalized the methods of Theorem 5.10 in [CF15].*

Let us set

$$\mathcal{K}_{\alpha,\mathbf{t},s}(\Gamma) := \left\|\mathcal{R}^{-s}[r_{-\boldsymbol{t}}(\alpha), (\mathscr{P}^{d,r_{-\boldsymbol{t}}(\alpha)}_{U_\Gamma}m)]\right\|_{-(s+1)}. \tag{27}$$

Now we prove the remainder estimate that will be used in (34).

Lemma 3.4. *Let $s > s_{d,g}$. There exists a constant $C(s,\Gamma) > 0$ such that for any rectangle $U_\Gamma = [0, e^{\Gamma_1}] \times \cdots \times [0, e^{\Gamma_d}]$,*

$$\mathcal{K}_{\alpha,\boldsymbol{t},s}(\Gamma) \leq C(s,\Gamma)Hgt([[r_{-\boldsymbol{t}}(\alpha)]])^{1/4}.$$

Proof. By Lemma 3.2, we begin to find the bound of $d-1$ (renormalized) currents. By Theorem 3.3, we obtain the remainder estimate

$$\begin{aligned} \mathcal{K}_{\alpha,t}(\Gamma) \leq C\sum_{k=0}^{d-1}\sum_{1\leq i_1<\cdots<i_k\leq d-1}\int_0^{\Gamma_{i_k}}\cdots\int_0^{\Gamma_{i_1}}\exp(\frac{1}{2}\sum_{l=1}^{d-1}\Gamma_l - \frac{1}{2}\sum_{l=1}^{k}u_{i_l}) \\ \times Hgt([[\prod_{1\leq j\leq d-1}r_j^{-\Gamma_j}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_{-\boldsymbol{t}}(\alpha))]])^{1/4}du_{i_1}\cdots du_{i_k}. \end{aligned}$$

It follows from (15) that for $0 \leq k \leq d-1$,

$$Hgt([[\prod_{1\leq j\leq d-1}r_j^{-\Gamma_j}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_{-\boldsymbol{t}}(\alpha))]])^{1/4} \leq e^{\frac{1}{2}(\sum_{l=1}^{k}u_{i_l} - \sum_{l=1}^{d}\Gamma_l)}Hgt([[r_{-\boldsymbol{t}}(\alpha)]])^{1/4}.$$

Then, we obtain

$$\begin{aligned} \left\|\mathcal{R}^{-s}[r_{-\boldsymbol{t}}(\alpha), (\mathscr{P}^{d,r_{-\boldsymbol{t}}\alpha}_{U_\Gamma}m)]\right\|_{-s} &\leq C(s)\Big(\sum_{k=0}^{d-1}\sum_{1\leq i_1<\cdots<i_k\leq d-1}\prod_{l=1}^{k}\Gamma_{i_l}\Big)Hgt([[r_{-\boldsymbol{t}}(\alpha)]])^{1/4} \\ &\leq C(s,\Gamma)Hgt([[r_{-\boldsymbol{t}}(\alpha)]])^{1/4}. \end{aligned} \tag{28}$$

Therefore, we obtain the conclusion. □

3.2. **Constructions of the functionals.** For fixed $\alpha \in Aut_0(\mathsf{H}^g)$, let $\Pi_H^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \to A_d(\mathfrak{p}, W_\alpha^{-s}(H))$ denote the orthogonal projection on a single irreducible unitary representation H. We further decompose projection operators with basic current $B_\alpha^{-s,H}$ and its remainder $R_\alpha^{-s,H}$ given by

$$\Pi_H^{-s} = \mathscr{B}_\alpha^{-s}(\Gamma) B_\alpha^{-s,H} + R_\alpha^{-s,H}$$

where $\mathscr{B}_{H,\alpha}^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \to \mathbb{C}$ denote the orthogonal component map in the direction of basic current, supported on a single irreducible unitary representation.

The *Bufetov functionals* on a standard rectangle Γ are defined for all $[\alpha] \in DC$ as follows.

Lemma 3.5. *Let $[\alpha] \in DC(L)$. For $s > s_{d,g}$, the limit*

$$\hat{\beta}_H(\alpha, \Gamma) = \lim_{t_1, \ldots, t_d \to \infty} e^{-(t_1 + \cdots + t_d)/2} \mathscr{B}_{H, r_{\text{-}t}(\alpha)}^{-s}(\Gamma)$$

exists and it defines a finitely-additive measure on the set of standard rectangles. Moreover, there exists a constant $C(s, \Gamma) > 0$ such that the following estimate holds:

$$\left\| \Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}_H(\alpha, \Gamma) B_\alpha^H \right\|_{\alpha, -s} \leq C(s, \Gamma)(1 + L). \tag{29}$$

Proof. For simplicity, we omit dependence of H. For every $\mathbf{t} \in \mathbb{R}^d$, we have the following orthogonal splitting:

$$\Pi_{H,\alpha}^{-s}(\Gamma) = \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}},$$

where

$$\mathscr{B}_{\alpha,\mathbf{t}}^{-s} := \mathscr{B}_{H, r_{\text{-}\mathbf{t}}(\alpha)}^{-s}, \; B_{\alpha,\mathbf{t}} := B_{r_{\text{-}\mathbf{t}}(\alpha)}^{-s,H}, \; R_{\alpha,\mathbf{t}} := R_{r_{\text{-}\mathbf{t}}(\alpha)}^{-s,H}.$$

For any $\mathbf{h} \in \mathbb{R}^d$, we have

$$\mathscr{B}_{\alpha,\mathbf{t+h}}^{-s}(\Gamma) B_{\alpha,\mathbf{t+h}} + R_{\alpha,\mathbf{t+h}} = \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}}.$$

By reparametrization (12), we have $B_{\mathbf{t+h}} = e^{-(h_1 + \cdots + h_d)/2} B_{\mathbf{t}}$ and

$$\mathscr{B}_{\alpha,\mathbf{t+h}}^{-s}(\Gamma) = e^{(h_1 + \cdots + h_d)/2} \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) + \mathscr{B}_{\alpha,\mathbf{t+h}}^{-s}(R_{\alpha,\mathbf{t}}) \tag{30}$$

and it follows that

$$\mathscr{B}_{\alpha,\mathbf{t+h}}^{-s}(\Gamma) = e^{h_1/2} \mathscr{B}_{\alpha, t_1, t_2 + h_2, \cdots, t_d + h_d}^{-s}(\Gamma) + \mathscr{B}_{\alpha,\mathbf{t+h}}^{-s}(R_{\alpha,\mathbf{t}}).$$

By differentiating at $h_1 = 0$,

$$\frac{d}{dt_1} \mathscr{B}_{\alpha, t_1, t_2 + h_2, \cdots, t_d + h_d}^{-s}(\Gamma) = \frac{1}{2} \mathscr{B}_{\alpha, t_1, t_2 + h_2, \cdots, t_d + h_d}^{-s}(\Gamma) + [\frac{d}{dh_1} \mathscr{B}_{\alpha,\mathbf{t+h}}^{-s}(R_{\alpha,\mathbf{t}})]_{h_1 = 0}. \tag{31}$$

Therefore, we solve the following first order ODE

$$\frac{d}{dt_1} \mathscr{B}_{\alpha, t_1, t_2 + h_2, \cdots, t_d + h_d}^{-s}(\Gamma) = \frac{1}{2} \mathscr{B}_{\alpha, t_1, t_2 + h_2, \cdots, t_d + h_d}^{-s}(\Gamma) + \mathcal{K}_{\alpha,\boldsymbol{t}}^{(1)}(\Gamma)$$

where

$$\mathcal{K}_{\alpha,\boldsymbol{t},s}^{(1)}(\Gamma) := [\frac{d}{dh_1} \mathscr{B}_{\alpha,\mathbf{t+h}}^{-s}(R_{\alpha,\mathbf{t}})]_{h_1 = 0}.$$

Then, the solution of the differential equation is

$$\begin{aligned}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) &= e^{t_1/2}\left(\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots,t_d+h_d}(\Gamma)+\int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(1)}_{\alpha,\tau,s}(\Gamma)d\tau_1\right)\\ &= e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots,t_d+h_d}(\Gamma)+\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(1)}_{\alpha,\tau,s}(\Gamma)d\tau_1.\end{aligned}$$

Note by reparametrization

$$e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots,t_d+h_d}(\Gamma) = e^{h_2/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3,\cdots,t_d+h_d}(\Gamma)$$

and it is possible to differentiate the previous equation with respect to h_2 again. Then

$$\frac{d}{dt_2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots,t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3,\cdots,t_d+h_d}+\int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(2)}_{\alpha,\tau,s}(\Gamma)d\tau_1 \tag{32}$$

where $\mathcal{K}^{(2)}_{\alpha,\tau,s}(\Gamma) = \frac{d}{dh_2}\mathcal{K}^{(1)}_{\alpha,\tau,s}(\Gamma)$.

Then, the solution of equation (32) is

$$\begin{aligned}&\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots,t_d+h_d}(\Gamma)\\ &= e^{t_2/2}\Big(\mathscr{B}^{-s}_{\alpha,t_1,0,t_3+h_3,\cdots,t_d+h_d}+\int_0^{t_2} e^{-\tau_2/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau,s}(\Gamma)d\tau_1 d\tau_2\Big)\\ &= e^{h_3/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3,\cdots,t_d+h_d}+\int_0^{t_2} e^{(t_2-\tau_2)/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau,s}(\Gamma)d\tau_1 d\tau_2.\end{aligned}$$

Inductively, we solve the first order ODE repeatedly and obtain the following solution

$$\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}(\Gamma) = e^{(t_1+\cdots+t_d)/2}\Big(\mathscr{B}^{-s}_{\alpha,0}+\int_0^{t_d}\cdots\int_0^{t_1} e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}^{(d)}_{\alpha,\tau,s}(\Gamma)d\tau_1\cdots d\tau_d\Big) \tag{33}$$

where

$$\mathcal{K}^{(d)}_{\alpha,\mathbf{t},s}(\Gamma) = [\frac{d}{dh_d}\cdots\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,\mathbf{t+h}}(R_{\alpha,\mathbf{t}})]_{h_d\cdots h_1=0}.$$

Let $\langle\cdot,\cdot\rangle_{\alpha,\boldsymbol{t}}$ denote the inner product in the space of Hilbert current $A_d(\mathfrak{p}, W^{-s}_{r_{\text{-}\mathbf{t}}(\alpha)}(H))$. By the intertwining formula (11),

$$\begin{aligned}\mathscr{B}^{-s}_{\alpha,\boldsymbol{t+h}}(R_{\alpha,\boldsymbol{t}}) &= \langle R_{\alpha,\boldsymbol{t}}, \frac{B_{\alpha,\boldsymbol{t+h}}}{|B_{\alpha,\boldsymbol{t+h}}|^2_{\boldsymbol{t+h}}}\rangle_{\alpha,\boldsymbol{t+h}}\\ &= \langle R_{\alpha,\boldsymbol{t}}\circ U_{-\boldsymbol{h}}, \frac{B_{\alpha,\boldsymbol{t+h}}\circ U_{-\boldsymbol{h}}}{|B_{\alpha,\boldsymbol{t+h}}|^2_{\boldsymbol{t+h}}}\rangle_{\alpha,\boldsymbol{t}}\\ &= \langle R_{\alpha,\boldsymbol{t}}\circ U_{-\boldsymbol{h}}, \frac{B_{\alpha,\boldsymbol{t}}}{|B_{\alpha,\boldsymbol{t}}|^2_{\boldsymbol{t}}}\rangle_{\alpha,\boldsymbol{t}} = \mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}(R_{\alpha,\boldsymbol{t}}\circ U_{-\boldsymbol{h}}).\end{aligned}$$

In the sense of distributions,

$$\begin{aligned}\frac{d}{dh_d}\cdots\frac{d}{dh_1}(R_{\alpha,\boldsymbol{t}}\circ U_{-\boldsymbol{h}}) &= -R_{\alpha,\boldsymbol{t}}\circ(\frac{d}{2}+\sum_{i=1}^d X_i(t))\circ U_{-\boldsymbol{h}}\\ &= [(\sum_{i=1}^d X_i(t)-\frac{d}{2})R_{\alpha,\boldsymbol{t}}]\circ U_{-\boldsymbol{h}}.\end{aligned}$$

Then we compute

$$[\frac{d}{dh_d}\cdots\frac{d}{dh_1}(\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}+\boldsymbol{h}}(R_{\alpha,\boldsymbol{t}}))]_{\boldsymbol{h}=0} = -\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2})R_{\alpha,\boldsymbol{t}}).$$

Recall by (27) that $\mathcal{K}_{\alpha,\mathbf{t},s}(\Gamma) = \left\|\mathcal{R}^{-s}_{\alpha,\boldsymbol{t}}\right\|_{r_{-\boldsymbol{t}}(\alpha),-(s+1)}$. By Proposition 2.8 and Lemma 3.4, for any $\epsilon > 0$

$$|\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2})R_{\alpha,\boldsymbol{t}})| \leq \mathcal{K}_{\alpha,\mathbf{t},s}(\Gamma) \leq C(s,\Gamma)Hgt([[r_{-\boldsymbol{t}}(\alpha)]])^{1/4}. \tag{34}$$

Therefore, the solution of equation (33) exists under Diophantine condition (20) and the following holds:

$$\lim_{t_1,\ldots,t_d\to\infty} e^{-(t_1+\cdots+t_d)/2}\mathscr{B}^{-s}_{\alpha,t}(\Gamma) = \hat{\beta}_H(\alpha,\Gamma).$$

Moreover, the complex number

$$\hat{\beta}_H(\alpha,\Gamma) = \mathscr{B}^{-s}_{\alpha,0} + \int_0^\infty\cdots\int_0^\infty e^{-(\tau_1+\cdots+\tau_d)/2}\mathcal{K}_{\alpha,\tau,s}(\Gamma)d\tau_1\cdots d\tau_d$$

depends continuously on $\alpha \in DC(L)$. Since we have

$$\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}(\alpha,\Gamma)B^H_\alpha = R_0 - \left(\int_0^\infty\cdots\int_0^\infty e^{-(\tau_1+\cdots+\tau_d)/2}\mathcal{K}_{\alpha,\tau,s}(\Gamma)d\tau_1\cdots d\tau_d\right)B^H_\alpha,$$

by Diophantine condition again,

$$\left\|\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}_H(\alpha,\Gamma)B^H_\alpha\right\|_{\alpha,-s} \leq C(s,\Gamma)(1+L).$$

□

3.3. **Proof of Theorem 1.3.** The proof of Theorem 1.3 follows immediately from the refinement to the constructions of Bufetov functionals (see also [BF14, §2.5] for horocycle flows).

Notation. The action of flow $\{r_t\}_{t\in\mathbb{R}}$ on a current $\mathcal{C}$ is defined by pull-back as follows:

$$(r_t^*\mathcal{C})(\omega) = \mathcal{C}(r_{-t}^*\omega), \quad \text{for any smooth form } \omega.$$

Lemma 3.6 (Invariance)**.** *Let $1 \leq d \leq g$. The functional $\hat{\beta}_H$ defined on d-standard rectangle $\Gamma = \Gamma^X_{\boldsymbol{T}}$ is invariant under the action of $(\mathsf{Q}^{j,Y}_y)$ for any $y \in \mathbb{R}^j_+$ and $1 \leq j \leq d$. That is,*

$$\hat{\beta}_H(\alpha,(\mathsf{Q}^{j,Y}_y)_*\Gamma) = \hat{\beta}_H(\alpha,\Gamma).$$

Proof. We will prove the functional $\hat{\beta}_H$ exists with respect to rectangle $(\mathsf{Q}^{j,Y}_y)_*\Gamma$ and prove its invariance under the action $\mathsf{Q}^{j,Y}_y$. It suffices to verify the invariance property under the rank 1 action $\mathsf{Q}^{1,Y}_\tau$ for $\tau \in \mathbb{R}$ since we can apply the statement for d-rank actions repeatedly.

Given a standard d-dimensional rectangle Γ, set $\Gamma_{\mathsf{Q}} := (\mathsf{Q}^{1,Y}_\tau)_*\Gamma$. Let $D(\Gamma,\Gamma_{\mathsf{Q}})$ be the $(d+1)$ dimensional space spanned by trajectories of the action of $\mathsf{Q}^{1,Y}_\tau$ projecting Γ onto Γ_{Q}. Then $D(\Gamma,\Gamma_{\mathsf{Q}})$ is a union of all orbits I of action $\mathsf{Q}^{1,Y}_\tau$ such that the boundary of I, d-dimensional faces, is contained in $\Gamma\cup\Gamma_{\mathsf{Q}}$, and interior of I is disjoint from $\Gamma\cup\Gamma_{\mathsf{Q}}$. $D(\Gamma,\Gamma_{\mathsf{Q}})$ is defined by integration and it is a $(d+1)$-current.

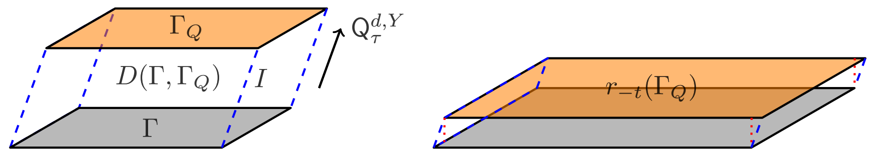

FIGURE 2. Illustration of the standard d-rectangles Γ, Γ_Q, $d+1$ dimensional current $D(\Gamma,\Gamma_{\mathsf{Q}})$ and supports of $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_{\mathsf{Q}})$.

For convenience, let us denote renormalization flow by $r_t := r_i^t$ for some i-th coordinates. Then, $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_{\mathsf{Q}})$ are respectively the support of the currents $r_t^*\Gamma$ and $r_t^*\Gamma_{\mathsf{Q}}$. Thus, we have the following identity

$$r_t^* D(\Gamma,\Gamma_{\mathsf{Q}}) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}})).$$

Since the current $\partial D(\Gamma,\Gamma_{\mathsf{Q}}) - (\Gamma - \Gamma_{\mathsf{Q}})$ is composed of orbits for the action $\mathsf{Q}^{1,Y}_\tau$, it follows that

$$\partial[r_t^* D(\Gamma,\Gamma_{\mathsf{Q}})] - (r_t^*\Gamma - r_t^*\Gamma_{\mathsf{Q}}) = r_t^*[\partial D(\Gamma,\Gamma_{\mathsf{Q}}) - (\Gamma-\Gamma_{\mathsf{Q}})] \to 0. \tag{35}$$

Now, we turn to prove the volume of $D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}}))$ is uniformly bounded for all $t>0$. For any $p\in\Gamma$, set $\tau(p)$ be length of the arc lying in $D_Q := D(\Gamma,\Gamma_{\mathsf{Q}})$, and set $\tau_\Gamma := \sup\{\tau(p) \mid p\in\Gamma\} < \infty$. We write

$$vol_{d+1}(D_Q) = \int_\Gamma \tau dvol_d.$$

Since $vol_d(r_{-t}(\Gamma)) \le e^t vol_d(\Gamma)$,

$$vol_{d+1}(r_{-t}(D_Q)) = \int_{r_{-t}(\Gamma)} \tau dvol_d \le \tau_\Gamma e^{-t} vol_d(r_{-t}(\Gamma)) \le \tau_\Gamma vol_d(\Gamma) < \infty. \tag{36}$$

Note that d-dimensional current $(\mathsf{Q}^{1,Y}_\tau)_*\Gamma - \Gamma$ is equal to the boundary of the $(d+1)$ dimensional current D_Q. By arguments in remainder estimate (or Sobolev embedding theorem),

$$\left\|(\mathsf{Q}^{1,Y}_\tau)_*\Gamma - \Gamma\right\|_{r^{-t}(\alpha),-s} \le C_s\tau B_s([[r_{-t}\alpha]]) \le C_s\tau \mathrm{Hgt}[[r_{-t}\alpha]]^{1/4} \tag{37}$$

is finite for all $t>0$.

Then, by (35), (36) and existence of Bufetov functional $\hat\beta_H(\alpha,\Gamma)$, the last inequality holds:

$$\left\|\mathscr{B}^{-s}_{\alpha,t}((\mathsf{Q}^{1,Y}_\tau)_*\Gamma) - \mathscr{B}^{-s}_{\alpha,t}(\Gamma)\right\|_{\alpha,-s} < \infty.$$

Therefore, by the definition of Bufetov functional in the Lemma 3.5, $\hat\beta_H(\alpha,(\mathsf{Q}^{1,Y}_\tau)_*\Gamma)$ exists and $\hat\beta_H(\alpha,\Gamma)$ is invariant under the action of $\mathsf{Q}^{1,Y}_\tau$. $\square$

Proof of Theorem 1.3. Additive property. It follows from the linearity of projections and limit.

Scaling property. It is immediate from the definition.

Bounded property. By the scaling property, for $r_t = r_1^t\cdots r_d^t$ with $t>0$,

$$\hat\beta_H(\alpha,\Gamma) = e^{dt/2}\hat\beta_H(r_t(\alpha),\Gamma).$$

Choose $t = \log(\int_\Gamma |\hat{X}|)$ and $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$, then Bufetov functional on the rectangle Γ is bounded:

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

Invariance property. Now it follows directly from the Lemma 3.6. □

Notation. We write for $\boldsymbol{T} = (T^{(i)}) \in \mathbb{R}^d$ and $\boldsymbol{t} = (t_1, \cdots, t_d) \in \mathbb{R}^d$,

$$vol(U(\boldsymbol{T})) := \prod_{i=1}^d T^{(i)}, \quad vol(U(\mathbf{t})) = \prod_{i=1}^d t_i. \tag{38}$$

Now we extend the properties of functional $\hat{\beta}_H$ to cocycle β_H.

Proof of Corollary 1.5. Cocycle property of β_H follows from the additive property of $\hat{\beta}_H$. Scaling and bounded properties are immediate. Denote $\boldsymbol{tT} := (t_1 T^{(1)}, \cdots, t_d T^{(d)}) \in \mathbb{R}^d_+$, and we obtain

$$\beta_H(\alpha, m, \boldsymbol{tT}) = vol(U(\boldsymbol{T}))^{1/2} \beta_H(r_{\log \mathbf{T}}(\alpha), m, \mathbf{t}). \tag{39}$$

By Lemma 3.5 and scaling property, we have

$$\begin{aligned} D^H_\alpha(f)\beta_H(\alpha, m, \mathbf{t}) &= \lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \beta_H(r_{-\log \mathbf{t}}(\alpha), m, \boldsymbol{tT}) \\ &= \lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\langle \mathscr{P}^{d, r_{-\log \mathbf{t}}(\alpha)}_{U(\boldsymbol{tT})} m, \omega_{f_H} \right\rangle. \end{aligned}$$

It follows that $\beta_H(\alpha, \cdot, \boldsymbol{t}) \in H$ as a point-wise limit of Birkhoff integrals. This implies orthogonal property. □

For $\gamma \in Sp_{2g}(\mathbb{Z})$, we have

$$\beta_H(\gamma\alpha, \gamma(m), \boldsymbol{T}) = \beta_H(\alpha, m, \boldsymbol{T}).$$

It means that the function $\beta_H(\cdot, m, \boldsymbol{T})$ is well-defined on the moduli space $\mathfrak{M}_g$.

3.4. **Proof of Theorem 1.6.**

We define the excursion function

$$\begin{aligned} E_{\mathfrak{M}}(\alpha, \mathbf{T}) &:= \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1 + \cdots + t_d)/2} \mathrm{Hgt}([[r_{\mathbf{t} - \log \mathbf{T}}(\alpha)]])^{1/4} dt_1 \cdots dt_d \\ &= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1 + \cdots + t_d)/2} \mathrm{Hgt}([[r_{\mathbf{t}}(\alpha)]])^{1/4} dt_1 \cdots dt_d. \end{aligned}$$

Lemma 3.7 (Bound of cocycle)**.** *For any Diophantine $[\alpha] \in DC(L)$ and for any $f \in W^s(M)$ for $s > s_{d,g} + 1/2$, the Bufetov cocycle β^f is defined by a uniformly convergent series:*

$$|\beta^f(\alpha, m, \boldsymbol{tT})| \leq C_s \left(L + vol(U(\boldsymbol{T}))^{1/2}(1 + vol(U(\boldsymbol{t})) + E_{\mathfrak{M}}(\alpha, \boldsymbol{T})) \right) \|f\|_{\alpha, s}.$$

Proof. Recall that by (7),

$$\beta^f(\alpha, m, \boldsymbol{T}) = \sum_H D^H_\alpha(f) \beta_H(\alpha, m, \boldsymbol{T}).$$

It follows from Lemma 3.5 that there exists a constant $C > 0$ such that whenever $[\alpha] \in DC(L)$, then

$$|\beta_H(\alpha, m, \mathbf{t})| \leq C(1 + L + vol(U(\boldsymbol{t}))), \quad (m, \mathbf{t}) \in M \times \mathbb{R}^d_+. \tag{40}$$

By Diophantine condition (20), whenever $[\alpha] \in DC(L)$ then $[r_{\log \mathbf{T}}(\alpha)] \in DC(L_{\mathbf{T}})$ with

$$L_{\mathbf{T}} \leq Lvol(U(\boldsymbol{T}))^{-1/2} + E_{\mathfrak{M}}(\alpha, \mathbf{T}).$$

Thus by (40), we obtain for all $(m, \mathbf{t}) \in M \times \mathbb{R}^d$,

$$|\beta_H(r_{\log \mathbf{T}}(\alpha), m, \mathbf{t})| \leq C(1 + L_{\mathbf{T}} + vol(U(\boldsymbol{t}))).$$

By the scaling property (39), it follows that for all $s > s_{d,g}$ and $q > 1/2$

$$\begin{aligned}|\beta^f(\alpha, m, \boldsymbol{tT})| &\leq C_s vol(U(\boldsymbol{T}))^{1/2}(1 + L + vol(U(\boldsymbol{t}))) \sum_{n\in\mathbb{Z}} \|f_n\|_{\alpha,s} \\ &\leq C_s vol(U(\boldsymbol{T}))^{1/2}(1 + L_{\mathbf{T}} + vol(U(\boldsymbol{t})))\big(\sum_{n\in\mathbb{Z}}(1+n^2)^q\big)^{-1/2} \\ &\times (\sum_{n\in\mathbb{Z}} \left\|(1-Z^2)^{q/2} f_n\right\|^2_{\alpha,s})^{1/2}.\end{aligned}$$

Therefore, for all $s' = s_{d,g} + q$, there exists a constant $C_{s'} > 0$ such that

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_{s'} vol(U(\boldsymbol{T}))^{1/2}\big(1 + L_{\mathbf{T}} + vol(U(\boldsymbol{t}))\big) \|f\|_{\alpha,s'}.$$

□

By Lemma 3.5, 3.7 and identification of the norm for the form ω_f, asymptotic formula on each irreducible component provides the following corollary.

Corollary 3.8. *For all $s > s_{d,g} + 1/2$, there exists a constant $C_s > 0$ such that for all $[\alpha] \in DC(L)$, for all $f \in W^s_\alpha(M)$ and for all $(m, \boldsymbol{T}) \in M \times \mathbb{R}^d$, we have*

$$|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_f \right\rangle - \beta^f(\alpha, m, \boldsymbol{T})| \leq C_s(1 + L) \|\omega_f\|_{\alpha,s} \tag{41}$$

for $U(\boldsymbol{T}) = [0, T^{(1)}] \times \cdots \times [0, T^{(d)}]$ and $\omega_f = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p} \otimes W^s_\alpha(M)$.

Proof of Theorem 1.6. The theorem follows from Corollary 3.8 for $\alpha \in \bigcup_{L>0} DC(L)$. □

4. Limit distributions

In this section, we prove Theorem 1.7, limit distribution of normalized ergodic integrals of higher rank actions on the standard rectangles.

4.1. **Proof of Theorem 1.7.**

Lemma 4.1. *There exists a continuous modular function $\theta_H : Aut_0(\mathsf{H}^g) \to H \subset L^2(M)$ such that for any $\omega_f = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p} \otimes W^s_\alpha(H)$ with $s > d/2$,*

$$\lim_{|U(\boldsymbol{T})|\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\cdot), \omega_f \right\rangle - \theta_H(r_{\log \boldsymbol{T}}(\alpha)) D^H_\alpha(f) \right\|_{L^2(M)} = 0. \tag{42}$$

The family $\{\theta_H(\alpha) \mid \alpha \in Aut_0(\mathsf{H}^g)\}$ has a constant norm in $L^2(M)$.

Proof. By the Fourier transform, the space of smooth vectors and Sobolev space $W^s(H)$ is represented as the Schwartz space $\mathscr{S}^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |(1 + \sum_{i=1}^d (\frac{\partial^2}{\partial u_i^2} + u_i^2))^{s/2} \hat{f}(u)|^2 du < \infty.$$

Let $\boldsymbol{t}, \boldsymbol{u} \in \mathbb{R}^d$. Then we claim for any $f \in \mathscr{S}^s(\mathbb{R}^d)$, there exists a function $\theta(\alpha)(\cdot) \in L^2(\mathbb{R}^d)$ such that

$$\lim_{|U(\boldsymbol{T})|\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} f(\boldsymbol{u}+\boldsymbol{t})d\boldsymbol{t} - \theta_H(r_{\log \boldsymbol{T}}(\alpha))(\boldsymbol{u})Leb(f) \right\|_{L^2(\mathbb{R}^d, d\boldsymbol{u})} = 0.$$

This is equivalent to the statement (42). By the standard Fourier transform on $\mathbb{R}^d$, equivalently

$$\lim_{|U(\boldsymbol{T})|\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} e^{i\boldsymbol{t}\cdot\hat{\boldsymbol{u}}} \hat{f}(\hat{\boldsymbol{u}})dt - \hat{\theta}_H(r_{\log \boldsymbol{T}}(\alpha))(\hat{\boldsymbol{u}})\hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{\boldsymbol{u}})} = 0.$$

For $\chi \in L^2(\mathbb{R}^d, d\hat{\boldsymbol{u}})$ and $\hat{\boldsymbol{u}} = (\hat{u}_j)_{1\le j\le d}$, we denote

$$\chi_j(\hat{\boldsymbol{u}}) = \frac{e^{i\hat{u}_j}-1}{i\hat{u}_j}, \ \chi(\hat{\boldsymbol{u}}) = \prod_{j=1}^d \chi_j(\hat{\boldsymbol{u}}).$$

Let $\hat{\theta}(\alpha)(\hat{\boldsymbol{u}}) := \chi(\hat{\boldsymbol{u}})$ for all $\hat{\boldsymbol{u}} \in \mathbb{R}^d$. Now we will compute $\theta(r_{\log \boldsymbol{T}}(\alpha))$. By intertwining formula (11) for $\mathbf{T} \in \mathbb{R}^d$,

$$U_{\mathbf{T}}(f)(\hat{\boldsymbol{u}}) = \prod_{i=1}^d (T^{(i)})^{1/2} f(\boldsymbol{T}\hat{\boldsymbol{u}}), \text{ for } \boldsymbol{T}\hat{\boldsymbol{u}} = (T^{(1)}\hat{u}_1, \cdots, T^{(d)}\hat{u}_d).$$

Then, for all $\alpha \in Aut_0(\mathsf{H}^g)$,

$$\hat{\theta}(r_{\log \boldsymbol{T}}(\alpha))(\hat{\boldsymbol{u}}) = U_{\mathbf{T}}(\chi)(\hat{\boldsymbol{u}}) = vol(U(\boldsymbol{T}))^{1/2}\chi(\boldsymbol{T}\hat{\boldsymbol{u}}).$$

The function $\theta(\alpha)$ is defined by inverse Fourier transform of $\hat{\theta}(\alpha)$ and

$$\|\theta_H(\alpha)\|_H = \|\theta(\alpha)\|_{L^2(\mathbb{R}^d)} = \left\|\hat{\theta}(\alpha)\right\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{\boldsymbol{u}})\|_{L^2(\mathbb{R}^d, d\hat{u})} = C > 0.$$

By integration,

$$\begin{aligned}(43) \quad \int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{i\boldsymbol{t}\cdot\hat{\boldsymbol{u}}} \hat{f}(\hat{\boldsymbol{u}})dt &= vol(U(\boldsymbol{T}))\chi(\boldsymbol{T}\hat{\boldsymbol{u}})\hat{f}(\hat{\boldsymbol{u}}) \\ &= vol(U(\boldsymbol{T}))\chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{\boldsymbol{u}}) - \hat{f}(0)) + vol(U(\boldsymbol{T}))^{1/2}\hat{\theta}(r_{\log \boldsymbol{T}}(\alpha))(\hat{\boldsymbol{u}})\hat{f}(0).\end{aligned}$$

Then the claim reduces to the following:

$$\limsup_{|U(\boldsymbol{T})|\to\infty} \left\| vol(U(\boldsymbol{T}))^{1/2}\chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{\boldsymbol{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If $f \in \mathscr{S}^s(\mathbb{R}^g)$ with $s > d/2$, function $\hat{f} \in C^0(\mathbb{R}^d)$ and bounded. Thus, by Dominated convergence theorem and change of variables,

$$\left\| vol(U(\boldsymbol{T}))^{1/2}\chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{\boldsymbol{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{\boldsymbol{u}})} = \left\| \chi(\nu)(\hat{f}(\frac{\boldsymbol{\nu}}{\boldsymbol{T}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\boldsymbol{\nu})} \to 0.$$

□

Corollary 4.2. *For any $s > d/2$, $\alpha \in Aut_0(\mathsf{H}^g)$ and $f \in W^s_\alpha(H)$, there exists a constant $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})|\to\infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_f \right\rangle \right\|_{L^2(M)} = C|D^H_\alpha(f)|.$$

From Corollary 4.2, we derive the following limit result for the L^2-norm of Bufetov functionals.

Corollary 4.3. *For every irreducible component H and $[\alpha] \in DC$, there exists $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})|\to\infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \|\beta_H(\alpha, \cdot, \boldsymbol{T})\|_{L^2(M)} = C.$$

Proof. By the normalization of invariant distribution in Sobolev space, for any $\alpha \in Aut_0(\mathsf{H}^g)$, there exists a function $f_\alpha^H \in W_\alpha^s(H)$ such that $D_\alpha(f_\alpha^H) = \left\|f_\alpha^H\right\|_s = 1$. For all $[\alpha] \in DC(L)$, by asymptotic formula (41) for $f = f_H$,

$$\left| \left\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha} m, \omega_f \right\rangle - \beta^f(\alpha, m, \boldsymbol{T})\right| \leq C_s(1 + L).$$

Therefore, L^2-estimate follows from Corollary 4.2. □

A relation between the Bufetov functional and the modular function θ_H is established below.

Corollary 4.4. *For every irreducible component $H \subset L^2(M)$, the following holds. For any $L > 0$ and any $r_{\mathbf{t}}$-invariant probability measure μ supported on $DC(L) \subset \mathfrak{M}_g$,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H(\alpha)(\cdot), \quad \textit{for } \mu\textit{-almost all } [\alpha] \in \mathfrak{M}_g.$$

Proof. By Theorem 1.6 and Lemma 4.1, there exists a constant $C > 0$ such that for all $[\alpha] \in supp(\mu) \subset DC(L)$ and $\mathbf{T} \in \mathbb{R}_+^d$, we have

$$\lim_{|U(\boldsymbol{T})|\to\infty} \|\beta_H(r_{\log \boldsymbol{T}}(\alpha), \cdot, 1) - \theta_H(r_{\log \boldsymbol{T}}(\alpha))\|_{L^2(M)} \leq \frac{C_\mu}{vol(U(\boldsymbol{T}))^{1/2}}. \tag{44}$$

By Luzin's theorem, for any $\delta > 0$ there exists a compact subset $E(\delta) \subset \mathfrak{M}$ such that we have the measure bound $\mu(\mathfrak{M}\backslash E(\delta)) < \delta$ and the function $\beta_H(\alpha, \cdot, 1) \in L^2(M)$ depends continuously on $[\alpha] \in E(\delta)$. By Poincaré recurrence, there is a full measure set $F \subset \mathfrak{M}$ for $\mathbb{R}^d$-action. Denote by a full measure set $E'(\delta) = E(\delta) \cap F \subset E(\delta)$.

For every $\alpha_0 \in E'(\delta)$, there is a divergent sequence $(\boldsymbol{t}_n)$ such that $\{r_{\boldsymbol{t_n}}(\alpha_0)\} \subset E(\delta)$ and $\lim_{n\to\infty} r_{\boldsymbol{t}_n}(\alpha_0) = \alpha_0$. By continuity of θ_H and β_H at α_0, we have

$$\begin{aligned} \|\beta_H(\alpha_0, \cdot, 1) - \theta_H(\alpha_0)\|_{L^2(M)} \\ = \lim_{n\to\infty} \|\beta_H(r_{\boldsymbol{t}_n}(\alpha_0), \cdot, 1) - \theta_H(r_{\boldsymbol{t}_n}(\alpha_0))\|_{L^2(M)} = 0. \end{aligned} \tag{45}$$

Then $\beta_H(\alpha, \cdot, 1) = \theta_H(\alpha) \in L^2(M)$ for all $\alpha \in E'(\delta)$. It follows that the set where the equality (45) fails has a measure less than any $\delta > 0$, thus the identity holds for μ-almost all $\alpha \in Aut_0(\mathsf{H}^g)$. □

For all $\alpha \in Aut_0(\mathsf{H}^g)$, smooth function $f \in W^s(M)$ for $s > s_{d,g} + 1/2$ decompose as an infinite sum, and the functional θ^f is defined by a convergent series

$$\theta^f(\alpha) := \sum_H D_\alpha^H(f)\theta_H(\alpha). \tag{46}$$

Hence, the modular function $\theta^f : Aut_0(\mathsf{H}^g) \to L^2(M)$ is continuous.

The following result is an extension of Lemma 4.1 to an asymptotic formula.

Lemma 4.5. *For all $\alpha \in Aut_0(\mathsf{H}^g)$, $f \in W^s(M)$ and $s > s_{d,g} + 1/2$,*

$$\lim_{n\to\infty} \left\| \frac{1}{vol(U(\boldsymbol{T_n}))^{1/2}} \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T_n})} m, \omega_f \right\rangle - \theta^f(r_{\log \boldsymbol{T_n}}(\alpha)) \right\|_{L^2(M)} = 0.$$

We summarize our results on limit distributions for higher rank actions.

Theorem 4.6 (Theorem 1.7)**.** *Let $(\boldsymbol{T_n})$ be any sequence such that*

$$\lim_{n\to\infty} r_{\log \boldsymbol{T_n}}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_{d,g} + 1/2$, which is not a coboundary, the limit distribution of the family of random variables $E_{\boldsymbol{T_n}}(f)$ exists along a subsequence of $\{\boldsymbol{T_n}\}$ and is equal to the distribution of the function $\theta^f(\alpha_\infty) = \beta(\alpha_\infty, \cdot, 1) \in L^2(M)$.

If $\alpha_\infty \in DC$, then $\theta^f(\alpha_\infty)$ is a bounded function on M, and the limit distribution has compact support.

Proof of Theorem 1.7. Since $\alpha_\infty \in \mathfrak{M}_g$, the existence of limit follows from the Lemma 4.5 and Theorem 4.6. $\square$

A relation with Birkhoff integrals and theta sum was introduced in [CF15, §5.3], and as an application, we derive the limit theorem of theta sums.

Corollary 4.7. *Let $\mathscr{Q}[x] = x^\top \mathscr{Q} x$ be the quadratic forms defined by $g \times g$ real matrix $\mathscr{Q}$, where $\alpha = \begin{pmatrix} I & 0 \\ \mathscr{Q} & I \end{pmatrix} \in Sp_{2g}(\mathbb{R})$ and $\ell(x) = \ell^\top x$ is the linear form defined by $\ell \in \mathbb{R}^g$. Then the theta sum*

$$\Theta(\mathscr{Q}, \ell; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0,N]} \exp(2\pi\iota(\mathscr{Q}[n] + \ell(n)))$$

has a limit distribution and it has compact support.

5. L^2-lower bounds

In this section we prove L^2- lower bounds of ergodic integrals on transverse torus.

5.1. **Structure of return map.** The polarized Heisenberg group $\mathsf{H}^g_{pol} \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ is equipped with the group law $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx')$. Reduced standard Heisenberg group is defined by quotient

$$\mathsf{H}^g_{red} := \mathsf{H}^g_{pol}/(\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}/\frac{1}{2}\mathbb{Z}.$$

Then the Reduced standard lattice is $\Gamma^g_{red} = \mathbb{Z}^g \times \mathbb{Z}^g \times \{0\} \subset \mathsf{H}^g_{red}$ and the quotient $\mathsf{H}^g_{red}/\Gamma^g_{red}$ is isomorphic to the standard Heisenberg manifold $M = \mathsf{H}^g/\Gamma$.

Given the standard frame (X_i, Y_i, Z), $(g+1)$-dimensional (transverse) torus is denoted by

$$\mathbb{T}^{g+1}_\Gamma := \{\Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

Now, we consider a return map of $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}^{g+1}_\Gamma$ on the coordinates in reduced Heisenberg group. For $x = (x_1, \cdots, x_g) \in \mathbb{R}^g$, we write in the coordinate of H^g_{red} for convenience

$$\exp(x_1 X^\alpha_1 + \cdots + x_g X^\alpha_g) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta, \text{ and } w \in \mathbb{R}^d.$$

Then by group law,

$$\exp(x_1X_1^\alpha + \cdots + x_gX_g^\alpha) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x)$$

and given $(n, m, 0) \in \mathsf{\Gamma}^g_{red}$,

$$\exp(x_1X_1^\alpha+\cdots+x_gX_g^\alpha)\cdot(0,y,z)\cdot(n,m,0) = \exp(x_1'X_1^\alpha+\cdots+x_g'X_g^\alpha)\cdot(0,y',z') \tag{47}$$

if and only if

$$x_\alpha' = x_\alpha + n, y' = y + (x_\beta - x_\beta') + m \text{ and } z' = z + (w - w') \cdot x + n^\top(y + x_\beta).$$

Assume $\langle X_i^\alpha, X_j\rangle \neq 0$ for all i, j, and we write the first return time for $\mathsf{P}^{d,\alpha}$ action

$$t_{Ret} = (t_{Ret,1}, \cdots, t_{Ret,d}) \in \mathbb{R}^d$$

on transverse torus $\mathbb{T}^{g+1}_{\mathsf{\Gamma}}$. We denote the domain for return time $U(t_{Ret}) = [0, t_{Ret,1}] \times \cdots \times [0, t_{Ret,d}]$. Return map of action $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}^{g+1}_{\mathsf{\Gamma}}$ has a form of skew-shift

$$A_{\rho,\tau}(y, z) = (y + \rho, z + v \cdot y + \tau) \text{ on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z} \tag{48}$$

for some non-zero vectors $\rho, v \in \mathbb{R}^g$ and $\tau \in \mathbb{R}$.

Furthermore, $A_{\rho,\tau} = A_{d,\rho,\tau} \circ \cdots \circ A_{1,\rho,\tau}$ decomposes with commuting linear skew-shifts

$$A_{i,\rho,\tau}(y, z) = (y + \rho_i, z + v_i \cdot y + \tau_i) \text{ on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z} \tag{49}$$

for some $\rho_i, v_i \in \mathbb{R}^g$ and $v_i \in \mathbb{R}^g$. For each $j \neq k$, it is easily verified that

$$A_{j,\rho,\tau} \circ A_{k,\rho,\tau} = A_{k,\rho,\tau} \circ A_{j,\rho,\tau}.$$

Given pair $(\mathbf{m}, n) \in \mathbb{Z}^g_{K|n|} \times \mathbb{Z}$, let $H_{(\mathbf{m},n)}$ denote the corresponding factor and $C^\infty(H_{(\mathbf{m},n)})$ be a subspace of smooth functions on $H_{(\mathbf{m},n)}$. Denote $\{e_{\mathbf{m},n} \mid (\mathbf{m}, n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z}\}$ the basis of characters on $\mathbb{T}^{g+1}_{\mathsf{\Gamma}}$ and for all $(y, z) \in \mathbb{T}^g \times \mathbb{T}$,

$$e_{\mathbf{m},n}(y, z) := \exp[2\pi\iota(\mathbf{m} \cdot y + nKz)].$$

For each $A_{i,\rho,\sigma}$, we set $\boldsymbol{v_i} = (v_{i1}, \cdots, v_{ig}) \in \mathbb{Z}^g_{K|n|}$. Then the orbit can be identified with the following dual orbit

$$\begin{aligned}\mathcal{O}_{A_i}(\boldsymbol{m}, n) &= \{(\boldsymbol{m} + (nj_i)\boldsymbol{v_i}, n),\ j_i \in \mathbb{Z}\}\\ &= \{(m_1 + (nv_{i1})j_i, \cdots, m_g + (nv_{ig})j_i, n),\ j_i \in \mathbb{Z}\}.\end{aligned}$$

If $n = 0$, the orbit $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$ of $(\mathbf{m}, 0)$ is reduced to a single element. If $n \neq 0$, then the dual orbit $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$ of $(\mathbf{m}, n)$ for higher rank actions is described as follows:

$$\mathcal{O}_A(\boldsymbol{m}, n) = \{(m_k + n\sum_{i=1}^{d}(v_{ik}j_i), n)_{1\le k\le d} : j = (j_1, \cdots j_d) \in \mathbb{Z}^d\}.$$

It follows that every A-orbit for rank $\mathbb{R}^d$-action (or A_i-orbit) can be labeled uniquely by a pair $(\mathbf{m}, n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z}\backslash\{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. Thus, the subspace of functions with non-zero central characters splits as a direct sum of components $H_{(\mathbf{m},n)}$

$$L^2(\mathbb{T}^{g+1}_{\mathsf{\Gamma}}) = \bigoplus_{\omega\in\mathcal{O}_A} H_\omega, \quad \text{where } H_\omega = \bigoplus_{(\mathbf{m},n)\in\omega} \mathbb{C}e_{(\mathbf{m},n)}.$$

5.2. **Higher cohomology for $\mathbb{Z}^d$-action of skew-shifts.** In this subsection, we will find a relations with the return map for $\mathbb{Z}^d$ action of $\mathsf{P}^{d,\alpha}$ on the torus $\mathbb{T}^{g+1}_{\Gamma}$ and obstructions to cohomological equation $\omega = d\Omega$.

We will restrict our interest to the following cocycle equation

$$\varphi(x,t) = \mathfrak{D}\Phi(x,t), \quad x \in \mathbb{T}^g, \ t \in \mathbb{Z}^d \tag{50}$$

where d-cocycle $\varphi : \mathbb{T}^{g+1}_{\Gamma} \times \mathbb{Z}^d \to \mathbb{R}$, $(d-1)$ cochain $\Phi : \mathbb{T}^{g+1}_{\Gamma} \to \mathbb{R}^d$, $\Phi = (\Phi_1, \cdots, \Phi_d)$, and $\mathfrak{D}$ is coboundary operator

$$\mathfrak{D}\Phi = \sum_{i=1}^{d} (-1)^{i+1} \Delta_i \Phi_i$$

where $\Delta_i \Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$.

In the work of Katok [KK95, §2], he proved the existence of solutions for cocycle equations by studying the dual equations $\hat{\varphi} = \mathfrak{D}\hat{\Phi}$ in the space of Fourier coefficients (dual orbit). We apply this result to our A-orbit and verify the invariant distributions (or currents) explicitly.

Proposition 5.1. *[KK95, Proposition 2.2] A dual cocycle $\hat{\varphi}$ satisfies cocycle equation (50) if and only if $\sum_{j\in\mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j = 0$.*

For fixed $(\mathbf{m}, n) \in \mathbb{Z}^g \times \mathbb{Z}$, we denote an obstruction of cohomological equation restricted to the A-orbit of $(\mathbf{m}, n)$ by $\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j\in\mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j$. Since A^j is composition of commuting toral automorphisms, we obtain the following generalized formula (see [AFU11, §5] and [Ka03, §11] for rank 1 map on $\mathbb{T}^2$).

Lemma 5.2. *There exists a distributional obstruction to the existence of a smooth solution $\varphi \in C^\infty(H_{(\mathbf{m},n)})$ of the cohomological equation (50). A generator of the space of invariant distribution $\mathcal{D}_{m,n}$ is given by*

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) := e^{-2\pi\iota \sum_{i=1}^{d}[(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}$$

if $(a, b) = (m_k + K\sum_{i=1}^{d}(v_{ik}j_i), n)_{1\le k\le g}$ and 0 otherwise.

Proof. From previous observation, there exists an obstruction

$$\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j\in\mathbb{Z}^d} \int_{\mathbb{T}^{g+1}_{\Gamma}} \varphi(x,y)\overline{e_{\mathbf{m},n} \circ A^j_{\rho,\tau}}dxdy. \tag{51}$$

By direct computation, for fixed $j = (j_1, \cdots j_d)$,

$$e_{\mathbf{m},n} \circ A^j_{\rho,\tau}(y,z) = \prod_{i=1}^{d} \left(e^{2\pi\iota[(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}\right)\left(e^{2\pi\iota(\mathbf{m}\cdot y + K(z + n\sum_{k=1}^{d}(v_{ik}j_i)y_k))}\right).$$

Then, we choose $\hat{\varphi} = e_{a,b}$ for $(a, b) = (m_k + K\sum_{i=1}^{d}(v_{ik}j_i), n)_{1\le k\le g}$ in the nontrivial orbit $(n \neq 0)$,

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi\iota \sum_{i=1}^{d}[(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}. \tag{52}$$

$\square$

5.3. **Changes of coordinates.** For any frame $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$ and any $m \in M$, denote a transverse cylinder

$$\mathscr{C}_{\alpha,m} := \{m \exp(\sum_{i=1}^g y_i' Y_i^\alpha + z'Z) \mid (y', z') \in U(t_{Ret}^{-1}) \times \mathbb{T}\} \subset M.$$

For any $\xi \in \mathbb{T}_\Gamma^{g+1}$, let $\xi' \in \mathscr{C}_{\alpha,m}$ denote first intersection of the orbit $\{\mathsf{P}_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}_+^d\}$ with transverse cylinder $\mathscr{C}_{\alpha,m}$. Then, there exists a first return time to the cylinder $t(\xi) = (t_1(\xi), \cdots, t_d(\xi)) \in \mathbb{R}_+^d$ such that the map $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \to \mathscr{C}_{\alpha,m}$ is defined by

$$\xi' = \Phi_{\alpha,m}(\xi) = \mathsf{P}_{t(\xi)}^{d,\alpha}(\xi), \quad \forall \xi \in \mathbb{T}_\Gamma^{g+1}.$$

Let (y, z) and (y', z') denote the coordinates on $\mathbb{T}_\Gamma^{g+1}$ and $\mathscr{C}_{\alpha,m}$ respectively, given by the exponential map

$$(y, z) \to \xi_{y,z} := \Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ), \quad (y', z') \to m \exp(\sum_{i=1}^g y_i' Y_i^\alpha + z'Z).$$

Recall that if $\alpha \in Sp_{2g}(\mathbb{R})$, then for $1 \leq i, j \leq g$ there exist matrices $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$ such that

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2g}(\mathbb{R}),$$

satisfying $A^t D - C^t B = I_{2g}$, $C^t A = A^t C$, $D^t B = B^t D$, and $\det(A) \neq 0$. Set

$$X_i^\alpha = \sum_{j=1}^g (a_{ij} X_j + b_{ij} Y_j) + w_i Z \quad \text{ and } Y_i^\alpha = \sum_{j=1}^g (c_{ij} X_j + d_{ij} Y_j) + v_i Z.$$

Let $x = \Gamma \exp(\sum_{i=1}^d y_{x,i} Y_i + z_x Z) \exp(\sum_{i=1}^d t_{x,i} X_i)$, for some $(y_x, z_x) \in \mathbb{T}^d \times \mathbb{R}/K\mathbb{Z}$ and $t_x = (t_{x,i}) \in [0,1)^d$. Then, the map $\Phi_{\alpha,x} : \mathbb{T}_\Gamma^{g+1} \to \mathscr{C}_{\alpha,x}$ is defined by $\Phi_{\alpha,x}(y, z) = (y', z')$ where

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_g' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix}, \tag{53}$$

and $z' = z + P(\alpha, x, y)$ for some degree 4 polynomial P.

Therefore, the map $\Phi_{\alpha,x}$ is invertible with

$$\Phi_{\alpha,x}^*(dy_1' \wedge \cdots dy_g' \wedge dz') = \frac{1}{det(A)} dy_1 \wedge \cdots dy_g \wedge dz.$$

Since $A^t D - C^t B = I_{2g}$, by direct computation we obtain return time, we have

$$\begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}. \tag{54}$$

Then,

$$\|t(\xi)\| \leq \max_i |t_i(\xi)|^g \leq \max_i |\sum_{j=1}^g d_{ij}t_{x,i} + c_{ij}(y_i - y_{x,i})|^g \leq \max_i \|Y_i^\alpha\|^g .$$

5.4. **L^2-lower bound of functional.** We will prove the bounds for square mean of integrals along leaves of foliations of the torus $\mathbb{T}_\Gamma^{g+1}$.

Lemma 5.3. *There exists a constant $C > 0$ such that for all $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$, and for every irreducible component $H := H_n$ of central parameter $n \neq 0$, there exists a function f_H such that*

$$|f_H|_{L^\infty(H)} \leq Cvol(U(t_{Ret}))^{-1}|\mathcal{D}_\alpha^H(f_H)|,$$

$$\|f_H\|_{\alpha,s} \leq Cvol(U(t_{Ret}))^{-1}|\mathcal{D}_\alpha^H(f_H)|\Big(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|\Big)^s(1+n^2)^{s/2}$$

where $\|Y\| := \max_{1\leq i\leq g}\|Y_i^\alpha\|$ and $\Sigma(t_{Ret}) = \sum_{i=1}^g t_{Ret,i}$.

On rectangular domain $U(\boldsymbol{T})$, for all $m \in \mathbb{T}_\Gamma^{g+1}$ and $T^{(i)} \in \mathbb{Z}_{t_{Ret,i}}$

$$\Big\|\Big\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y}m), \omega_H\Big\rangle\Big\|_{L^2(\mathbb{T}^g,dy)} = |\mathcal{D}_\alpha^H(f_H)|\left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2}. \tag{55}$$

In addition, whenever $H \perp H' \subset L^2(M)$ the functions

$$\Big\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y}m), \omega_H\Big\rangle \quad \textit{and} \quad \Big\langle \mathscr{P}_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y}m), \omega_{H'}\Big\rangle$$

are orthogonal in $L^2(\mathbb{T}^g, dy)$.

Proof. The operator $I_\alpha : L^2(M) \to L^2(\mathbb{T}_\Gamma^{g+1})$ is defined by

$$f \to I_\alpha(f) := \int_{U(t_{Ret})} f \circ \mathsf{P}_s^{d,\alpha}(\cdot)ds. \tag{56}$$

Then operator I_α is surjective linear map of $L^2(M)$ onto $L^2(\mathbb{T}_\Gamma^{g+1})$ with a right inverse R_α^χ defined as follows. Let $\chi \in C_0^\infty(0,1)^g$ be any function of jointly integrable with integral 1. For any $F \in L^2(\mathbb{T}_\Gamma^{g+1})$, let $R_\alpha^\chi(F) \in L^2(M)$ be a function defined by

$$R_\alpha^\chi(F)(\mathsf{P}_v^{d,\alpha}(x)) = \frac{1}{vol(U(t_{Ret}))}\chi(\frac{v}{t_{Ret}})F(x),\ (x,v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows that there exists a constant $C_\chi > 0$ such that

$$\begin{aligned} \|R_\alpha^\chi(F)\|_{\alpha,s} &\leq C_\chi vol(U(t_{Ret}))^{-1}(1 + \sum_{i=1}^g t_{Ret,i}^{-1}\|Y_i^\alpha\|)^s\|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})} \\ &\leq C_\chi vol(U(t_{Ret}))^{-1}\Big(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|\Big)^s\|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}. \end{aligned} \tag{57}$$

As explained in §5.1, the space $L^2(\mathbb{T}_\Gamma^{g+1})$ decompose as a direct sum of irreducible subspaces invariant under the action of each $A_{j,\rho,\sigma}$. It follows that the subspace of functions with non-zero central character can be split as direct sum of components

$H_{(\mathbf{m},n)}$ with $(\mathbf{m},n)\in\mathbb{Z}^g_{|n|}\times\mathbb{Z}\backslash\{0\}$ with $\mathbf{m}=(m_1,\cdots,m_g)$. For a function $F\in H_{(\mathbf{m},n)}$, it is characterized by Fourier expansion

$$F=\sum_{j\in\mathbb{Z}^d}F_je_{A^j(\boldsymbol{m},n)}=\sum_{j\in\mathbb{Z}^d}F_je_{(m_k+K\sum_{i=1}^d(v_{ik}j_i),n)}.$$

Then, we can choose $f_H:=R^\chi_\alpha(e_{\mathbf{m},n})\in C^\infty(H)$ such that

$$|D_\alpha(f^H_\alpha)|=|D_{\mathbf{m},n}(e_{\mathbf{m},n})|=1,\tag{58}$$

$$\int_{U(t_{Ret})}f_H\circ\mathsf{P}^{d,\alpha}_t(y,z)dt=e_{\mathbf{m},n}(y,z),\ \text{for }(y,z)\in\mathbb{T}^{g+1}_\Gamma.\tag{59}$$

Therefore, it follows from (57) that

$$|f_H|_{L^\infty(H)}\le C_\chi vol(U(t_{Ret}))^{-1},$$

$$\|f_H\|_{\alpha,s}\le Cvol(U(t_{Ret}))^{-1}|\mathcal{D}^H_\alpha(f_H)|\Big(1+\frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|\Big)^s(1+n^2)^{s/2}.$$

Moreover, since $\{e_{\mathbf{m},n}\circ A^j_{\rho,\tau}\}_{j\in\mathbb{Z}^d}\subset L^2(\mathbb{T}^g_\Gamma,dy)$ is orthonormal, we verify

$$\begin{aligned}\left\|\left\langle\mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\mathsf{Q}^{g,Y}_yx),\omega_H\right\rangle\right\|_{L^2(\mathbb{T}^g,dy)}&=\left\|\sum_{j_d=0}^{[\frac{T^{(d)}}{t_{Ret,d}}]-1}\cdots\sum_{j_1=0}^{[\frac{T^{(1)}}{t_{Ret,1}}]-1}e_{\mathbf{m},n}\circ A^j_{\rho,\tau}\right\|_{L^2(\mathbb{T}^g,dy)}\\&=\left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2}.\end{aligned}$$

□

Recall that

$$L^2(M)=\bigoplus_{n\in\mathbb{Z}}H_n:=\bigoplus_{n\in\mathbb{Z}}\bigoplus_{i=1}^{\mu(n)}H_{i,n}\tag{60}$$

where $H_n=\bigoplus_{i=1}^{\mu(n)}H_{i,n}$ is irreducible representation with a central parameter n and $\mu(n)$ is countable by Howe-Richardson multiplicity formula.

For any infinite dimensional vector $\mathbf{c}:=(c_{i,n})\in\ell^2$, let $\beta_\mathbf{c}$ denote Bufetov functional

$$\beta_\mathbf{c}=\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mu(n)}c_{i,n}\beta^{i,n}.$$

By orthogonal property and from Corollary 4.3, the function $\beta_\mathbf{c}(\alpha,\cdot,\mathbf{T})\in L^2(M)$ for all $(\alpha,\mathbf{T})\in Aut_0(\mathsf{H}^g)\times\mathbb{R}^d_+$. Furthermore,

$$\|\beta_\mathbf{c}(\alpha,\cdot,\mathbf{T})\|^2_{L^2(M)}=\sum_{n\in\mathbb{Z}}\sum_{i=1}^{\mu(n)}|c_{i,n}|^2\left\|\beta^{i,n}(\alpha,\cdot,\mathbf{T})\right\|^2_{L^2(H_{i,n})}\le C^2|\mathbf{c}|^2_{\ell^2}vol(U(\boldsymbol{T})).$$

For any $\mathbf{c}:=(c_{i,n})$, let $|\mathbf{c}|_s$ denote the norm defined by

$$|\mathbf{c}|^2_s=\sum_{n\in\mathbb{Z}\backslash\{0\}}\sum_{i=1}^{\mu(n)}(1+K^2n^2)^s|c_{i,n}|^2.\tag{61}$$

Lemma 5.4. *For any $s > s_{d,g} + 1/2$, there exists a constant $C_s > 0$ such that for all $\alpha \in DC(L)$, for all $\mathbf{c} \in \ell^2$, for all $z \in \mathbb{T}$ and all $T > 0$,*

$$\left| \|\beta_{\boldsymbol{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})\|_{L^2(\mathbb{T}^g,dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\boldsymbol{c}|_0 \right|$$
$$\leq C_s(vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1+L)\Big(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|\Big)^s |\boldsymbol{c}|_s.$$

Proof. By Lemma 5.2, there exists a function $f_{i,n} \in C^\infty(H_{i,n})$ with $|\mathcal{D}^{i,n}(f_{i,n})| = 1$. Let $f_{\mathbf{c}} = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n} \in C^\infty(M)$, summing up all the functions on irreducibles. Then by the estimates in the Lemma 5.3 and (61),

$$|f_{\mathbf{c}}|_{L^\infty(M)} \leq C|\mathbf{c}|_{\ell^1}. \tag{62}$$

$$\|f_{\mathbf{c}}\|_{\alpha,s} \leq C vol(U(t_{Ret}))^{-1}\Big(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|\Big)^s |\mathbf{c}|_s. \tag{63}$$

By orthogonality,

$$\left\|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} \circ \mathsf{Q}^{g,Y}_y, \omega_{\mathbf{c}} \right\rangle\right\|_{L^2(\mathbb{T}^g,dy)} = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\mathbf{c}|_0.$$

By the estimate in Lemma 5.3 for each $f_{i,n}$, for every $z \in \mathbb{T}$ and all $\mathbf{T} \in \mathbb{R}^d_+$, we have

$$\left\|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle\right\|_{L^2(\mathbb{T}^g,dy)} \leq 2|f_{\mathbf{c}}|_{L^\infty(M)} \|Y\|.$$

Let $U(\mathbf{T}_{Ret}) = [0, \mathbf{T}_{Ret,1}] \times \cdots \times [0, \mathbf{T}_{Ret,g}]$ where $\mathbf{T}_{Ret,i} := t_{Ret,i}([T^{(i)}/t_{Ret,i}] + 1)$. Then,

$$\left\|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathscr{P}^{d,\alpha}_{U(\mathbf{T}_{Ret})}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle\right\|_{L^2(\mathbb{T}^g,dy)} \leq vol(U(t_{Ret}))|f_{\mathbf{c}}|_{L^\infty(M)}.$$

Therefore, there exists a constant $C' > 0$ such that

$$\left| \left\|\left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle\right\|_{L^2(\mathbb{T}^g,dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\mathbf{c}|_0 \right|$$
$$\leq C' vol(U(t_{Ret}))|\mathbf{c}|_{\ell^1}.$$

For all $s > s_{d,g} + 1/2$, by asymptotic property of Theorem 1.6, for some constant $C_s > 0$,

$$\left| \left\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega \right\rangle - \beta_H(\alpha, m, \boldsymbol{T})\mathcal{D}^H_\alpha(f_H) \right| \leq C_s(1+L)\|f\|_{\alpha,s}.$$

Applying $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$ and combining bounds on the function $f_{\mathbf{c}}$ with (62),

$$\left| \|\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})\|_{L^2(\mathbb{T}^g,dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\mathbf{c}|_0 \right|$$
$$\leq C' vol(U(t_{Ret}))|\mathbf{c}|_{\ell^1} + C_s vol(U(t_{Ret}))^{-1}(1+L)\|f_{\mathbf{c}}\|_{\alpha,s}$$
$$\leq C'_s(vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1+L)\big(1 + \frac{\Sigma(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|\big)^s |\mathbf{c}|_s.$$

Therefore, we derive the estimates in the statement. ☐

6. Analyticity of functionals

In this section we prove that for all $\alpha \in DC$, the Bufetov functionals on any square are real analytic.

6.1. **Analyticity.** By the orthogonal property of Bufetov cocycle β_H on an irreducible component $H = H_n$ with central parameter $n \in \mathbb{Z}\backslash\{0\}$, the following statement is immediate. For any $(m, \mathbf{T}) \in M \times \mathbb{R}^d_+$ and $t \in \mathbb{R}$,

$$\beta_H(\alpha, \phi^Z_t(m), \mathbf{T}) = e^{2\pi\iota Knt}\beta_H(\alpha, m, \mathbf{T}). \tag{64}$$

Definition 6.1. For every $t \in \mathbb{R}$, $1 \le i \le d$, and $m \in M$, the *stretched (in direction of Z) rectangle* is denoted by

$$[\Gamma_{\boldsymbol{T}}]^Z_{i,t}(m) := \{(\phi^Z_{ts_i}) \circ \mathsf{P}^{d,\alpha}_{\boldsymbol{s}}(m) \mid \boldsymbol{s} \in U(\mathbf{T})\}. \tag{65}$$

For $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}^d$, let us denote by the standard rectangle $\Gamma_{\mathbf{T}}(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma_d(s_d))$ for $\gamma_i(s_i) = \exp(s_i X_i)$. Similarly, we also write the stretched rectangle

$$[\Gamma_{\boldsymbol{T}}]^Z_{i,t}(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma^Z_{i,t}(s_i), \cdots \gamma_d(s_d)) \tag{66}$$

where $\gamma^Z_{i,t}(s_i) := \phi^Z_{ts_i}(\gamma_i(s_i))$ is a stretched curve.

Definition 6.2. The *restricted rectangle* $\Gamma_{T,i,s}$ of the standard rectangle $\Gamma_{\mathbf{T}}$ is defined as a restriction on i-th coordinate given by

$$\Gamma_{T,i,s}(\mathbf{s}) := \Gamma_{\mathbf{T}}|_{U_{T,i,s}}(\mathbf{s}),$$

where $U_{T,i,s} = [0, T^{(1)}] \times \cdots \times \underbrace{[0, s]}_{\text{i-th}} \cdots \times [0, T^{(d)}]$ for some $0 < s \le T^{(i)}$.

Lemma 6.3. *For fixed elements (X_i, Y_i, Z) satisfying commutation relation (1), the following formula for rank 1 action holds:*

$$\hat{\beta}_H(\alpha, [\Gamma_{\boldsymbol{T}}]^Z_{i,t}) = e^{2\pi\iota tnKT^{(i)}}\hat{\beta}_H(\alpha, \Gamma_{\boldsymbol{T}}) - 2\pi\iota nKt\int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\hat{\beta}_H(\alpha, \Gamma_{T,i,s})ds_i.$$

Proof. Let $\alpha = (X_i, Y_i, Z)$ and ω be d-form supported on a single irreducible representation H. We obtain following the formula for stretches of curve $\gamma^Z_{i,t}$ (see [FK20b, §4 and Lemma 9.1]),

$$\frac{d\gamma^Z_{i,t}}{ds_i} = D\phi^Z_{ts_i}(\frac{d\gamma_i}{ds_i}) + tZ \circ \gamma^Z_{i,t}.$$

It follows that pairing is given by

$$\begin{aligned}\langle[\Gamma_{\mathbf{T}}]^Z_{i,t}, \omega\rangle &= \int_{U(\boldsymbol{T})} \omega\Big(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma^Z_{i,t}}{ds_i}(s_i), \cdots, \frac{d\gamma_d}{ds_d}(s_d)\Big)d\mathbf{s} \\ &= \int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega\Big(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d)\Big)] + \iota_Z\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t}(\mathbf{s})d\mathbf{s}\end{aligned}$$

Denote $d-1$ dimensional triangle $U_{d-1}(\boldsymbol{T})$ with $U(\boldsymbol{T}) = U_{d-1}(\boldsymbol{T}) \times [0, T^{(i)}]$. Integration by parts for a fixed i-th integral gives

$$\int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$
$$= e^{2\pi\iota nKtT^{(i)}} \int_{U(\boldsymbol{T})} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$
$$- 2\pi\iota nKt \int_0^{T^{(i)}} e^{2\pi\iota nKts_i} \int_{U_{d-1}(\boldsymbol{T})} \left(\int_0^{s_i} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_i}{ds_i}(r) \cdots, \frac{d\gamma_d}{ds_d}(s_d))]dr \right) d\mathbf{s}.$$

Then, we have the following formula

$$\langle [\Gamma_{\mathbf{T}}]^Z_{i,t}, \omega \rangle = e^{2\pi\iota nKtT^{(i)}} \langle [\Gamma_{\mathbf{T}}], \omega \rangle - 2\pi\iota nKt \int_0^{T^{(i)}} e^{2\pi\iota nKts_i} \langle \Gamma_{T,i,s}, \omega \rangle ds_i$$
$$+ \int_{U(\boldsymbol{T})} (\iota_Z \omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s}) d\mathbf{s}.$$

Since the action of $\mathsf{P}^{d,X}_{\mathbf{t}}$ for $\mathbf{t} \in \mathbb{R}^d$ is identity on the center Z,

$$\lim_{t_1,\ldots,t_d \to \infty} e^{-(t_1+\cdots t_d)/2} \int_{U(\boldsymbol{T})} (\iota_Z (\mathsf{P}^{d,X}_{\mathbf{t}})^* \omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s}) d\mathbf{s} = 0.$$

Thus, in particular concerning $d = 1$, it follows by the definition of Bufetov functional (Lemma 3.5), the statement holds. $\square$

Here we define a restricted vector $\mathbf{T}_{i,s}$ of $\mathbf{T} = (T^{(1)}, \cdots, T^{(d)}) \in \mathbb{R}^d$. For fixed $i \in [1, d]$, pick $s_i \in [0, T^{(i)}]$ such that $\mathbf{T}_{i,s} \in \mathbb{R}^d$ is a vector with its coordinates

$$T^{(j)}_{i,s} = \begin{cases} T^{(j)} & \text{if} \quad j \neq i \\ s_i & \text{if} \quad j = i. \end{cases}$$

Similarly, $\mathbf{T}_{i_1,\cdots,i_k,s}$ is a vector whose $i_1, \cdots, i_k$-th coordinates are replaced by $s_{i_1}, \cdots, s_{i_k}$.

Lemma 6.4. *For $m \in M$ and $y_i \in \mathbb{R}$, the following property holds:*

$$\beta_H(\alpha, \phi^{Y_i}_{y_i}(m), \boldsymbol{T}) =$$
$$e^{-2\pi\iota y_i nKT^{(i)}} \beta_H(\alpha, m, \boldsymbol{T}) + 2\pi\iota nK y_i \int_0^{T^{(i)}} e^{-2\pi\iota y_i nKs_i} \beta_H(\alpha, m, \boldsymbol{T}_{i,s}) ds_i.$$

Proof. By definition (4), (66) and commutation relation (1), it follows that

$$\phi^{Y_i}_{y_i}(\Gamma^X_{\mathbf{T}}(m)) = [\Gamma^X_{\mathbf{T}}(\phi^{Y_i}_{y_i}(m))]^Z_{i,t}.$$

By the invariance property of Bufetov functional and Lemma 6.3,

$$\beta_H(\alpha, m, \mathbf{T}) = \hat{\beta}_H(\alpha, \phi^{Y_i}_{y_i}(\Gamma^X_{\mathbf{T}}(m))$$
$$= e^{2\pi\iota y_i nKT^{(i)}} \hat{\beta}_H(\alpha, \Gamma^X_{\mathbf{T}}(\phi^{Y_i}_{y_i}(m)) - 2\pi\iota nK y_i \int_0^{T^{(i)}} e^{2\pi\iota nK y_i s_i} \hat{\beta}_H(\alpha, \Gamma^X_{T,i,s}(\phi^{Y_i}_{s_i}(m)) ds_i$$
$$= e^{2\pi\iota y_i nKT^{(i)}} \beta_H(\alpha, \phi^{Y_i}_{y_i}(m), \mathbf{T}) - 2\pi\iota nK y_i \int_0^{T^{(i)}} e^{2\pi\iota nK y_i s_i} \beta_H(\alpha, \phi^{Y_i}_{s_i}(m), \mathbf{T}_{i,s}) ds_i.$$

Then the statement follows immediately. $\square$

We extend previous Lemma 6.4 to higher rank actions by induction argument.

Lemma 6.5 (Rank-d action)**.** *For $m \in M$ and $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, the following identity for the cocycle β_H holds:*

(67)
$$\begin{aligned}
&\beta_H(\alpha, \mathsf{Q}^{d,Y}_y(m), \boldsymbol{T}) = e^{-2\pi\iota \sum_{j=1}^d y_j nKT^{(j)}} \beta_H(\alpha, m, \boldsymbol{T}) \\
&+ \sum_{k=1}^d \sum_{1\le i_1 < \cdots < i_k \le d} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)})} \\
&\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})} \beta_H(\alpha, m, \boldsymbol{T}_{i_1,\cdots,i_k,s}) ds_{i_k} \cdots ds_{i_1}.
\end{aligned}$$

Proof. We verified that the statement works for $d = 1$ in Lemma 6.4. Assume that (67) holds for rank $d-1$ action $\mathsf{Q}^{d-1,Y}_{y'}$ by induction hypothesis. For convenience, we write

$$\mathsf{Q}^{d,Y}_y(m) = \phi^{Y_d}_{y_d} \circ \mathsf{Q}^{d-1,Y}_{y'}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 6.4,

$$\begin{aligned}
\beta_H(\alpha, \mathsf{Q}^{d,Y}_y(m), \mathbf{T}) &= e^{-2\pi\iota y_d nKT^{(d)}} \beta_H(\alpha, \mathsf{Q}^{d-1,Y}_{y'}(m), \mathbf{T}) \\
&\quad + 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d} \beta_H(\alpha, \mathsf{Q}^{d-1,Y}_{y'}(m), \mathbf{T}_{d,s}) ds_d \qquad (68)\\
&:= I + II.
\end{aligned}$$

Firstly, by induction hypothesis,

$$\begin{aligned}
I &= e^{-2\pi\iota y_d nKT^{(d)}} \Bigg(e^{-2\pi\iota \sum_{j=1}^{d-1} y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \\
&+ \sum_{k=1}^{d-1} \sum_{1\le i_1 < \cdots < i_k \le d-1} \prod_{j=1}^k (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l\notin\{i_1,\cdots,i_k\}} y_l T^{(l)} + y_d T^{(d)})} \\
&\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,s}) ds_{i_k} \cdots ds_{i_1} \Bigg) \\
&= e^{-2\pi\iota \sum_{j=1}^d y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T}) + III.
\end{aligned}$$

Then term III contains 0 to $d-1$th iterated integrals on the restricted rectangles (from 0 to $d-1$-th) containing $e^{-2\pi\iota nKy_dT^{(d)}}$ outside of iterated integrals.

For the second part, we apply induction hypothesis again for restricted rectangle $\mathbf{T}_{d,s}$. Then,

$$\begin{aligned}II &= 2\pi\iota nKy_d\int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d}\big[e^{-2\pi\iota\sum_{j=1}^{d-1}y_j nKT^{(j)}}\beta_H(\alpha,m,\mathbf{T}_{d,s})]ds_d\\ &+\sum_{k=1}^{d-1}\sum_{1\le i_1<\cdots<i_k\le d-1}(2\pi\iota nKy_d)\prod_{j=1}^{k}(2\pi\iota nKy_{i_j})e^{-2\pi\iota nK(\sum_{l\notin\{i_1,\cdots,i_k\}}y_lT^{(l)})}\\ &\times\int_0^{T^{(d)}}\left(\int_0^{T^{(i_1)}}\cdots\int_0^{T^{(i_k)}}ds_{i_k}\cdots ds_{i_1}\right)ds_d\\ &\times e^{-2\pi\iota nK(y_{i_1}s_{i_1}+\cdots+y_{i_k}s_{i_k}+y_ds_d)}\beta_H(\alpha,m,\mathbf{T}_{i_1,\cdots,i_k,d,s}).\end{aligned}$$

The term II consist of 1 to d-th iterated integrals on the (from 1 to d-th) restricted rectangles containing $e^{-2\pi\iota nKy_ds_d}$ inside of iterated integrals. Thus, by rearranging terms II and III, we obtain all the terms in the expression (67). □

6.2. **Extensions of domain.** In this subsection, the domain of functionals defined on standard rectangle $\Gamma^X_{\mathbf{T}}$ extends to the class $\mathfrak{R}$ (see Definition 1.1). Furthermore, the functional associated with the analytic norm extends to holomorphic function on a complex domain.

Proof of Corollary 1.4. Firstly, we can extend our functional to class $(\mathsf{Q}^{d,Y}_y)_*\Gamma^X_{\mathbf{T}}$ for any $y\in\mathbb{R}^d$ by invariance property (Lemma 3.6). Similarly, by Lemma 6.3, Bufetov functional defined on the standard rectangles extends to the class of generalized rectangle $(\phi^Z_{t_iz})\circ\mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m)$. Since the flow generated by Z commutes with other actions P and Q, for any standard rectangle $\Gamma=\Gamma(m)$ with a fixed point $m\in M$, we have $(\phi^Z_z)_*\Gamma(m)=\Gamma(\phi^Z_z(m))$ for any $z\in\mathbb{R}$. Therefore, by combining with the invariance under the action Q from Lemma 3.6, the domain of Bufetov functional extends to the class $\mathfrak{R}$. □

For any $R>0$, the *analytic norm* defined for all $\mathbf{c}\in\ell^2$ as

$$\|\mathbf{c}\|_{\omega,R}=\sum_{n\neq 0}\sum_{i=1}^{\mu(n)}e^{nR}|c_{i,n}|.$$

Let Ω_R denote the subspace of $\mathbf{c}\in\ell^2$ such that $\|\mathbf{c}\|_{\omega,R}$ is finite.

Lemma 6.6. *For $\boldsymbol{c}\in\Omega_R$ and $\boldsymbol{T}\in\mathbb{R}^d_+$, the function*

$$\beta_{\boldsymbol{c}}(\alpha,\mathsf{Q}^{d,Y}_y\circ\phi^Z_z(m),\boldsymbol{T}),\ (y,z)\in\mathbb{R}^d\times\mathbb{T}$$

extends to a holomorphic function in the domain

$$D_{R,T}:=\{(y,z)\in\mathbb{C}^d\times\mathbb{C}/\mathbb{Z}\mid\sum_{i=1}^{d}|Im(y_i)|T^{(i)}+|Im(z)|<\frac{R}{2\pi K}\}.\tag{69}$$

The following bound holds: for any $R'<R$ there exists a constant $C>0$ such that, for all $(y,z)\in D_{R',T}$ we have

$$\begin{aligned}&|\beta_{\boldsymbol{c}}(\alpha,\mathsf{Q}^{d,Y}_y\circ\phi^Z_z(m),\boldsymbol{T})|\\ &\quad\le C_{R,R'}\|\mathbf{c}\|_{\omega,R}(L+vol(U(\boldsymbol{T}))^{1/2}(1+E_M(a,\boldsymbol{T}))(1+K\sum_{i=1}^{d}|Im(y_i)|T^{(i)}).\end{aligned}$$

Proof. By Lemma 6.5 and (64),

$$\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}) = e^{(z-2\pi\iota\sum_{j=1}^d y_j nKT^{(j)})}\beta_H(\alpha, m, \mathbf{T})$$
$$+\sum_{k=1}^d \sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k (2\pi\iota nKy_{i_j})e^{-2\pi\iota nK(\sum_{l\notin\{i_1,\cdots,i_k\}} y_l T^{(l)})}$$
$$\times e^{2\pi\iota nKz}\int_0^{T^{(i_1)}}\cdots\int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1}+\cdots+y_{i_k}s_{i_k})}\beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,s})ds_{i_k}\cdots ds_{i_1}.$$

As a consequence, by Lemma 3.7 for each variable $(y_i, z)\in\mathbb{C}\times\mathbb{C}/\mathbb{Z}$, Then for the rank d-action, by induction, for $(y,z)\in\mathbb{C}^d\times\mathbb{C}/\mathbb{Z}$ we have

$$|\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y}\circ\phi_z^Z(x), \mathbf{T})|$$
$$\le (L+vol(U(\boldsymbol{T})))^{1/2}(1+E_M(a,\boldsymbol{T}))\Bigg(C_1\sum_{n\neq 0}\sum_{i=1}^{\mu(n)} e^{nR}|c_{i,n}|e^{2\pi|Im(z-\sum_{i=1}^d T^{(i)}y_i)|nK}$$
$$+\sum_{k=1}^d C_k\Big(\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k(|Im(y_{i_j})|T^{(i_j)})$$
$$\times\sum_{n\neq 0}\sum_{i=1}^{\mu(n)} n|c_{i,n}|e^{2\pi(|Im(z)|+\sum_{j=1}^k T^{(i_j)}|Im(y_{i_j})|)nK}\Big)\Bigg).$$

Therefore, the functional $\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y}\circ\phi_z^Z(m), \mathbf{T})$ is bounded by a series of holomorphic functions on $\mathbb{C}^d\times\mathbb{C}/\mathbb{Z}$ and it converges uniformly on compact subsets of domain $D_{R,T}$. Thus it is holomorphic on the set. □

7. Measure estimation for bounded-type

In this section, we prove a measure estimation of Bufetov functional with bounded-type α. This result is a generalization of §11 in [FK20b].

Let $\mathcal{O}_r$ denote the space of holomorphic functions on the ball $B_{\mathbb{C}}(0,r)\subset\mathbb{C}^n$.

Theorem 7.1. *[Bru99, Theorem 1.9] For any $f\in\mathcal{O}_r$, there is a constant $d := d_f(r) > 0$ such that for any convex set $D\subset B_{\mathbb{R}}(0,1) := B_{\mathbb{C}}(0,1)\cap\mathbb{R}^n$, for any measurable subset $U\subset D$*

$$\sup_D |f| \le \left(\frac{4nLeb(D)}{Leb(U)}\right)^d \sup_U |f|.$$

We say that a holomorphic function f defined in a disk is *p-valent* if it assumes no value more than p-times there. We also say that f is *0-valent* if it is a constant.

Definition 7.2. [Bru99, Def 1.6] Let $\mathcal{L}_t$ denote the set of one-dimensional complex affine spaces $L\subset\mathbb{C}^n$ such that $L\cap B_{\mathbb{C}}(0,t)\neq\emptyset$. For $f\in\mathcal{O}_r$, the number

$$\nu_f(t) := \sup_{L\in\mathcal{L}_t}\{\text{valency of } f \mid L\cap B_{\mathbb{C}}(0,t)\neq\emptyset\}$$

is called the *valency* of f in $B_{\mathbb{C}}(0,t)$.

By Proposition 1.7 of [Bru99], for any $f\in\mathcal{O}_r$ with finite valency $\nu_f(t)$ for any $t\in[1,r)$, there is a constant $c := c(r) > 0$ such that

$$d_f(r)\le c\nu_f(\frac{1+r}{2}). \tag{70}$$

Lemma 7.3. *[FK20b, Lemma 10.3] Let $R > r > 1$. For any normal family $\mathcal{F} \subset \mathcal{O}_R$, assume that no functions in $\overline{\mathcal{F}} = \emptyset$ are constant along a one-dimensional complex line. Then we have*

$$\sup_{f \in \mathcal{F}} \nu_f(r) < \infty.$$

Lemma 7.4. *Let $L > 0$ and $\mathscr{B} \subset DC(L)$ be a bounded subset. Given $R > 0$, for all $\boldsymbol{c} \in \Omega_R$ and all $\boldsymbol{T}^{(i)} > 0$, denote $\mathcal{F}(\boldsymbol{c}, \boldsymbol{T})$ by the family of real analytic functions of the variable $y \in [0,1)^d$ and*

$$\mathcal{F}(\boldsymbol{c}, \boldsymbol{T}) := \{\beta_{\boldsymbol{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}) \mid (\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}\}.$$

Then there exist $\boldsymbol{T}_{\mathscr{B}} := (\boldsymbol{T}^{(i)}_{\mathscr{B}})$ and $\rho_{\mathscr{B}} > 0$, such that for every $(R, \boldsymbol{T})$ with $R/\boldsymbol{T}^{(i)} \geq \rho_{\mathscr{B}}$, $\boldsymbol{T}^{(i)} \geq \boldsymbol{T}^{(i)}_{\mathscr{B}}$ and for all $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, we have

$$\sup_{f \in \mathcal{F}(\boldsymbol{c}, \boldsymbol{T})} \nu_f < \infty. \tag{71}$$

Proof. Since $\mathscr{B} \subset \mathfrak{M}$ is bounded, for each time $t_i \in \mathbb{R}$ and $1 \leq i \leq g$,

$$0 < t^{min}_{\mathscr{B}} = \min_i \inf_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} \leq \max_i \sup_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} = t^{max}_{\mathscr{B}} < \infty.$$

For any $\alpha \in \mathscr{B}$ and $x \in M$, the map $\Phi_{\alpha,x} : [0,1)^d \times \mathbb{T} \to \prod_{i=1}^d [0, t_{\alpha,i}) \times \mathbb{T}$ in (53) extends to a complex analytic diffeomorphism $\hat{\Phi}_{\alpha,x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$. By Lemma 6.6, it follows that for fixed $z \in \mathbb{T}$, real analytic function $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})$ extends to a holomorphic function defined on a region

$$H_{\alpha,m,R,t} := \{y \in \mathbb{C}^d \mid \sum_{i=1}^d |Im(y_i)| \leq h_{\alpha,m,R,t}\}.$$

By boundedness of the set $\mathscr{B} \subset \mathfrak{M}$, it follows that

$$\inf_{(\alpha,x) \in \mathscr{B} \times M} h_{\alpha,m,R,t} := h_{R,T} > 0.$$

We remark that the function $h_{\alpha,m,R,t}$ and its lower bound $h_{R,T}$ can be obtained from the formula (53) for the polynomial $\Phi_{\alpha,x}$ and the definition of the domain $D_{R,T}$ in the formula (69).

For every $r > 1$, there exists $\rho_{\mathscr{B}} > 1$ such that for every $(R, \mathbf{T})$ with $R/\mathbf{T}^{(i)} > \rho_{\mathscr{B}}$,

$$\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \in \mathcal{O}_r \tag{72}$$

as a function of $y \in \mathbb{T}^d$.

By Lemma 6.6, the family $\mathcal{F}(c, \mathbf{T})$ is uniformly bounded and normal. By Lemma 5.4 for the non-zero L^2-lower bound of functionals, for sufficiently large pair $\mathbf{T}$, no sequence from $\mathcal{F}(c, \mathbf{T})$ can converge to a constant. Therefore, by Lemma 7.3 for the family $\mathcal{F} = \mathcal{F}(c, \mathbf{T})$, the main statement follows. □

We derive measure estimates of Bufetov functionals on the rectangular domain.

Lemma 7.5. *Let $\alpha \in DC$ such that the forward orbit of $\mathbb{R}^d$-action $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t} \in \mathbb{R}^d_+}$ is contained in a compact set of $\mathfrak{M}_g$. There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}^d_+$ such that, for every $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, $T \geq T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{m \in M \mid |\beta_{\boldsymbol{c}}(\alpha, m, \boldsymbol{T})| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

Proof. Since $\alpha \in DC$ and the orbit $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t}\in\mathbb{R}_+}$ is contained in a compact set, there exists $L > 0$ such that $r_{\mathbf{t}}[\alpha] \in DC(L)$ for all $\boldsymbol{t} \in \mathbb{R}^d_+$. Then, we choose $\mathbf{T}_0 \in \mathbb{R}^d$ and $R > 0$ from the conclusion of Lemma 7.4. By the scaling property,

$$\beta_{\mathbf{c}}(\alpha, m, \boldsymbol{T}) = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(\boldsymbol{T_0}))}\right)^{1/2} \beta_{\mathbf{c}}(g_{\log(\boldsymbol{T}/\boldsymbol{T_0})}[\alpha], m, \boldsymbol{T}_0).$$

By Fubini's theorem, it suffices to estimate

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}).$$

Let $\delta^{-1} := c(r)\sup_{f\in\mathcal{F}(\mathbf{c},\mathbf{T}_0)} \nu_f(\frac{1+r}{2}) < \infty$ as in (70) and (71). By Lemma 5.4, we have

$$\inf_{(\alpha,x,z)\in\mathscr{B}\times M\times\mathbb{T}} \sup_{y\in[0,1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| > 0$$

so that the functional is not trivial. By Theorem 7.1 for the unit ball $D = B_{\mathbb{R}}(0,1)$ and setting

$$U = \{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\},$$

by the bound in (70) for $d_f(r)$, there exists a constant $C > 0$ such that for all $\epsilon > 0$ and $(\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}$,

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}) \leq C\epsilon^\delta.$$

Then the statement follows from the Fubini theorem. □

Corollary 7.6. *Let α be as in the previous Lemma 7.5. There exist $R, C, \delta > 0$ and $\boldsymbol{T}_0 \in \mathbb{R}^d_+$ such that, for every $\boldsymbol{c} \in \Omega_R\backslash\{0\}$, $\boldsymbol{T} \geq \boldsymbol{T}_0$ and for every $\epsilon > 0$, we have*

$$vol\left(\{m \in M \mid |\langle \mathscr{P}^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_{\boldsymbol{c}}\rangle| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}\right) \leq C\epsilon^\delta.$$

Acknowledgement. The author deeply appreciates Giovanni Forni for valuable suggestions to improve the draft. He acknowledges Rodrigo Treviño and Corinna Ulcigrai for giving several comments. He is also grateful to Oliver Butterley, Jacky Jia Chong, Krzysztof Frączek, Osama Khail, and Lucia D. Simonelli for fruitful discussions. This work was initiated when the author visited the Institut de Mathematiques de Jussieu-Paris Rive Gauche in Paris, France. He acknowledges invitation and hospitality during the visit. Lastly, the author is thankful to the referee for helpful comments and suggestions for improvement in the presentation of this work. This research was partially supported by the NSF grant DMS 1600687 and by the Centre of Excellence "Dynamics, mathematical analysis and artificial intelligence" at Nicolaus Copernicus University in Toruń.

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