

# LIMIT THEOREMS FOR HIGHER RANK ACTIONS ON HEISENBERG NILMANIFOLDS

MINSUNG KIM\*

ABSTRACT. The main result of this paper is a construction of finitely additive measures for higher rank action on Heisenberg nilmanifold. Under a full measure set of Diophantine condition of the generators of the action, we construct *Bufetov functional* on rectangles on  $2g + 1$  dimensional Heisenberg manifold. We prove the deviation of ergodic integral of higher rank actions by its asymptotics to Bufetov functionals for a sufficiently smooth function. In this paper, we derive limit distribution which proves normalized ergodic integrals to have variance 1 converges in distribution to a nondegenerate compactly supported measure on the real line.

## CONTENTS

1. Introduction	2
1.1. Introduction	2
1.2. Definition and statement of results	3
Main results	5
2. Analysis on Heisenberg manifolds	7
2.1. Invariant currents	7
2.2. Best Sobolev constant	7
2.3. Renormalization	8
2.4. Diophantine condition	8
3. Constructions of the functionals	9
3.1. Constructions of the functionals	10
3.2. Remainder estimates	12
3.3. Extensions of domain	15
3.4. Bound of functionals	17
4. Limit distributions	18
4.1. Limiting distributions	18
4.2. Proof of Theorem 1.5	20
5. $L^2$ -lower bounds	21
5.1. Structure of return map	21
5.2. Higher cohomology for $\mathbb{Z}^d$ -action of skew-shifts	22
5.3. Changes of coordinates	24
5.4. $L^2$ -lower bound of functional.	25
6. Analyticity of functionals	28
7. Measure estimation for bounded-type	32

---

*Date:* July 16, 2021.

*2010 Mathematics Subject Classification.* 37A17, 37A20, 37A44, 60F05.

*Key words and phrases.* Higher rank abelian actions, Limit theorem, Heisenberg groups.

## 1. INTRODUCTION

**1.1. Introduction.** The asymptotic behavior and limiting distribution of ergodic averages of translation flow was studied by Alexander Bufetov in the series of works [Buf14a, Buf10, Buf14b]. He constructed finitely-additive Hölder measures and cocycles over translation flows that are known as *Bufetov functionals*. The construction of such functional is used to derive deviation of ergodic averages and results on probabilistic behavior of ergodic averages of translation flows. Interestingly, it is discovered that there is a duality between the Bufetov functionals and invariant distributions which plays a key role in the work of G. Forni [For02]. Following these observation, such functionals and duality (or bijective correspondence) between the space of invariant distributions called Flaminio-Forni space are also constructed in the sense of other parabolic flows: horocycle flows [FF03, BF14] and Heisenberg nilflows [FF06, FK20b].

In this paper, our main results are on limit distributions of higher rank abelian actions. We firstly introduce the Bufetov functional for higher rank abelian actions under bounded type of Diophantine conditions. Our main argument is based on the renormalization argument for higher rank actions by induction argument. This is a key idea used in Cosentino and Flaminio in [CF15], but we extend their constructions to rectangular shape and derive the deviation of ergodic averages of higher rank actions. Likewise, this explains the duality between Bufetov functionals and invariant currents appeared in [CF15]. The crucial part is handling the estimation of deviation ergodic averages on (stretched) rectangles, and this enables to derive our main theorems. (See. e.g [Rav19, For20].)

As a corollary, we prove there exists a limit distributions of (normalized) ergodic integrals of abelian actions with variance 1. More specifically, for almost all limit of normalized ergodic integrals of converges in distribution to a nondegenerate compactly supported measure on the real line, which is certain form of Bufetov functionals. This generalizes a limit theorem for theta series on Siegel half spaces, which introduced in the works of Götze and Gordin [GG04] and Marklof [Mar99]. (See [Tol78, MM07, MNN07] for general introduction and nilflow case [GM14, CM16].) In the last section, the analyticity of Bufetov functional on higher dimensional rectangular domain is proved and we obtain polynomial type of lower bounds of sub-level sets of analytic function. Such polynomial estimates on the measure are derived by real analyticity of a functional along the leaves of a foliation transverse to the actions based on results of [Bru99].

It is natural question to ask if we can obtain any mixing properties of time-changes of actions. In rank 1 case, mixing property of time-changes of nilflows has been studied. Nilflows are never mixing but time-changes destroys the elliptic behavior on the base torus, which are sensitive in perturbations. Mixing properties of time changes of Heisenberg nilflows was firstly studied in [AFU11]. Then, it was extended in the dense set of non-trivial time changes for any uniquely ergodic nilflows on general nilmanifolds [Rav18, AFRU21]. As a special example, on time-changes of Heisenberg flows, decay of correlation [FK20b] and multiple mixing [FK20a] was proved by analyticity of Bufetov functional and Ratner property, respectively.

However, in the higher rank setting, time-changes of higher rank abelian actions on Heisenberg nilmanifold are all trivial (by triviality of first cohomology group, see [CF15, Theorem 3.16]). Thus, it is conjugate to the linear action and never mixing. In  $\mathbb{Z}^k$ -actions, mixing of shapes for automorphisms on nilmanifold is proved in [GS14, GS15].

It is also interesting question if it is possible to construct the Bufetov functionals on higher step nilmanifolds ( $s > 2$ ). Since moduli space is trivial and there is no known renormalization flows, we could not obtain the results in the same method. However, we still hope other methods in handling non-renormalizable flows may be possibly applied. (Cf. [FF14, FFT16, Kim20])

**1.2. Definition and statement of results.** We review definitions about Heisenberg manifold and its moduli space.

**1.2.1. Heisenberg manifold.** Let  $H^g$  be standard  $2g + 1$  dimensional Heisenberg group and set  $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$  a discrete and co-compact subgroup of  $H^g$ . We shall call it standard lattice of  $H^g$  and the quotient  $M := H^g/\Gamma$  will be called *Heisenberg manifold*. Lie algebra  $\mathfrak{h}^g = \text{Lie}(H^g)$  of  $H^g$  is equipped with basis  $(X_1, \dots, X_g, Y_1, \dots, Y_g, Z)$  satisfying canonical commutation relation

$$(1) \quad [X_i, Y_j] = \delta_{ij}Z.$$

For  $1 \leq d \leq g$ , let  $P^d < H^g$  be the subgroup with Lie algebra  $\mathfrak{p}$  generated by  $(X_1, \dots, X_d)$  and for any  $\alpha \in Sp_{2g}(\mathbb{R})$ , set  $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$  for  $1 \leq i \leq d$ . We define parametrization of the subgroup  $\alpha^{-1}(P^d)$

$$P_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \dots + x_d X_d^\alpha), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

By central extension of  $\mathbb{R}^{2g}$  by  $\mathbb{R}$ , we have an exact sequence

$$0 \rightarrow Z(H^g) \rightarrow H^g \rightarrow \mathbb{R}^{2g} \rightarrow 0.$$

The natural projection map  $pr : M \rightarrow H^g/(\Gamma Z(H^g))$  maps  $M$  onto a  $2g$ -dimensional torus  $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ .

**1.2.2. Moduli space.** The group of automorphisms of  $H^g$  that are trivial on the center is  $Aut_0(H^g) = Sp_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$ . Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to  $Sp_{2g}(\mathbb{R})$ . We call *moduli space* of the standard Heisenberg manifold the quotient  $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ . We regard  $Sp_{2g}(\mathbb{R})$  as the deformation space of the standard Heisenberg manifold  $M$  and  $\mathfrak{M}_g$  as the moduli space of  $M$ .

*Siegel modular variety* is double coset space  $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R}) / Sp_{2g}(\mathbb{Z})$  where  $K_g$  is maximal compact subgroup  $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$  of  $Sp_{2g}(\mathbb{R})$ . For  $\alpha \in Sp_{2g}(\mathbb{R})$ , we denote  $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$  its projection on the moduli space  $\mathfrak{M}_g$  and write  $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$  the projection of  $\alpha$  to the Siegel modular variety  $\Sigma_g$ .

Double coset  $K_g \backslash Sp_{2g}(\mathbb{R}) / 1_{2g}$  is identified to the Siegel upper half space  $\mathfrak{H}_g := \{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$ . *Siegel upper half space* of genus  $g$  is complex manifold of symmetric complex  $g \times g$  matrices  $Z = X + iY$  with positive definite symmetric imaginary part  $\Im(Z) = Y$  and arbitrary real part  $X$ . We note  $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$ .

1.2.3. *Sobolev bundles.* Given basis  $(V_i)$  of Lie algebra, let  $\Delta = -\sum V_i^2$  denote Laplacian via the standard basis. Similarly, denote  $\Delta_\alpha$  Laplacian defined by the basis  $(\alpha^{-1})_* V_i$ . For any  $s \in \mathbb{R}$  and any  $C^\infty$  function  $f \in L^2(M)$ ,

$$\|f\|_{\alpha,s} = \langle f, (1 + \Delta_\alpha)^s f \rangle^{1/2}.$$

Let  $W_\alpha^s(M)$  be the completion of  $C^\infty(M)$  with above norm and denote  $W_\alpha^{-s}(M)$  its dual space. Extending it to the exterior algebra, define the Sobolev spaces  $\Lambda^d \mathfrak{p} \otimes W^s(M)$  of cochains of degree  $d$  and we use the same notations for the norms.

The group  $Sp_{2g}(\mathbb{Z})$  acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \rightarrow Sp_{2g}(\mathbb{R})$$

where

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma = (\alpha\gamma, \gamma^* \varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \rightarrow \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

the fiber over  $[\alpha] \in \mathfrak{M}_g$  is locally identified with the space  $W_\alpha^s(M)$ .

By invariance of  $Sp_{2g}(\mathbb{Z})$  action, we denote the class  $(\alpha, \varphi)$  by  $[\alpha, \varphi]$  and write  $Sp_{2g}(\mathbb{Z})$ -invariant Sobolev norm

$$\|(\alpha, f)\|_s := \|f\|_{\alpha,s}.$$

1.2.4. *Representation.* We write Hilbert sum decomposition

$$(2) \quad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n$$

into closed  $H^g$ -invariant subspaces. For some fixed  $K > 0$ , we write  $f = \sum_{n \in \mathbb{Z}} f_n \in L^2(M)$ ,  $f_n \in H_n$  where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi i n K t)f\}.$$

We also have  $W^s(M) = \bigoplus_i W^s(H_i)$  of  $W^s(M)$  into closed  $H^g$ -invariant subspaces  $W^s(H_i) = W^s(M) \cap H_i$ . The center  $Z(H^g)$  has spectrum  $2\pi\mathbb{Z} \setminus \{0\}$  the space splits as Hilbert sum of  $H^g$ -module  $H_i$ , which is equivalent to irreducible representation  $\pi$ .

By Stone-Von Neumann theorem, for  $\alpha = (X_i, Y_i, Z)$ , the unitary irreducible representation  $\pi$  of the Heisenberg group of non-zero central parameter  $K$ , is unitarily equivalent to Schrödinger representation. For infinitesimal representation with parameter  $n$  for  $k = 1, 2, \dots, g$

$$D\pi(X_k) = \frac{\partial}{\partial x_k}, \quad D\pi(Y_k) = 2\pi i n K x_k, \quad D\pi(Z) = 2\pi i n K.$$

1.2.5. *Renormalization flow.* Denote diagonal matrix  $\delta_i = \text{diag}(d_1, \dots, d_g)$  with  $d_i = 1$ ,  $d_k = 0$  if  $k \neq i$ . Then, for each  $1 \leq i \leq g$ , we denote  $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in \mathfrak{sp}_{2g}$ .

Any such  $\hat{\delta}_i$  generate a one-parameter subgroup of automorphism  $r_i^t = e^{t\hat{\delta}_i}$ . We denote (rank  $d$ ) renormalization flow  $r_{\mathbf{t}} := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$  for  $\mathbf{t} = (t_1, \dots, t_d)$ .

**Main results.** One of the main objects of this paper is to construct finitely-additive measures defined on the space of rectangles on Heisenberg manifold  $M$ . We state our results beginning with an overview of Bufetov functional.

**Definition 1.1.** For  $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$ , denote the *standard rectangle* for action  $\mathbf{P}$ ,

$$(3) \quad \Gamma_{\mathbf{T}}^X(m) = \{\mathbf{P}_{\mathbf{t}}^{d,\alpha}(m) \mid \mathbf{t} \in U(\mathbf{T}) = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]\}.$$

Let  $\mathbf{Q}_y^{d,Y} := \exp(y_1 Y_1 + \cdots + y_d Y_d)$ ,  $y = (y_1, \dots, y_d) \in \mathbb{R}^d$  be the action generated by elements  $Y_i$  of standard basis. Set  $\phi_z^Z := \exp(zZ)$  is the flow generated by central element  $Z$ .

**Definition 1.2.** Let  $\mathfrak{R}$  be the collection of the *generalized rectangles* in  $M$ . For each  $1 \leq d \leq g$  and  $\mathbf{t} = (t_1, \dots, t_d)$ ,

$$(4) \quad \mathfrak{R} := \bigcup_{1 \leq i \leq d} \bigcup_{(y,z) \in \mathbb{R}^d \times \mathbb{R}} \bigcup_{(m, \mathbf{T}) \in M \times \mathbb{R}_+^d} \{(\phi_{t_i z}^Z) \circ \mathbf{Q}_y^{d,Y} \circ \mathbf{P}_{\mathbf{t}}^{d,\alpha}(m) \mid \mathbf{t} \in U(\mathbf{T})\}.$$

**Theorem 1.3.** For any irreducible representation  $H$ , there exists a measure  $\hat{\beta}_H(\Gamma) \in \mathbb{C}$  for every rectangle  $\Gamma \in \mathfrak{R}$  such that the following holds:

- (1) (Additive property) For any decomposition of disjoint rectangles  $\Gamma = \bigcup_{i=1}^n \Gamma_i$  or whose intersections have zero measure,

$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^n \hat{\beta}_H(\alpha, \Gamma_i).$$

- (2) (Scaling property) For  $\mathbf{t} \in \mathbb{R}^d$ ,

$$\hat{\beta}_H(r_{\mathbf{t}}[\alpha], \Gamma) = e^{-(t_1 + \cdots + t_d)/2} \hat{\beta}_H(\alpha, \Gamma).$$

- (3) (Invariance property) For any action  $\mathbf{Q}_{\tau}^{d,Y}$  generated by  $Y_i$ 's and  $\tau \in \mathbb{R}_+^d$ ,

$$\hat{\beta}_H(\alpha, (\mathbf{Q}_{\tau}^{d,Y})_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

- (4) (Bounded property) For any rectangle  $\Gamma \in \mathfrak{R}$ , there exists a constant  $C(\Gamma) > 0$  such that for  $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$ ,

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma) \left( \int_{\Gamma} |\hat{X}| \right)^{d/2}.$$

For arbitrary rectangle  $U_{\mathbf{T}} = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]$ , pick  $\mathbf{T}'^{(i)} \in [0, \mathbf{T}^{(i)}]$  for each  $i$  to decompose  $U_{\mathbf{T}}$  into  $2^d$  sub-rectangles. We write  $\mathbf{P}(\mathbf{T}')$  collection of  $2^d$  vertices  $v = (v^{(1)}, v^{(2)}, \dots, v^{(d)})$  where  $v^{(i)} \in \{0, \mathbf{T}'^{(i)}\}$ . Let  $U_{\mathbf{T},v}$  be a rectangle whose sides  $\mathbf{I}_v = (I^{(1)}, I^{(2)}, \dots, I^{(d)}) \in \mathbb{R}_+^d$  where

$$I^{(l)} = \begin{cases} T^{(l)} - T'^{(l)} & \text{if } v^{(l)} = T'^{(l)} \\ T'^{(l)} & \text{if } v^{(l)} = 0. \end{cases}$$

Then, we have  $U_{\mathbf{T}} = \bigcup_{v \in \mathbf{P}(\mathbf{T}')} U_{\mathbf{T},v}$ .

**Theorem 1.4.** Let us denote  $\beta_H(\alpha, m, \mathbf{T}) := \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(m))$ . The function  $\beta_H$  satisfies following properties:

- (1) (Cocycle property) For all  $(m, \mathbf{T}_1, \mathbf{T}_2) \in M \times \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\beta_H(\alpha, m, \mathbf{T}_1 + \mathbf{T}_2) = \sum_{v \in \mathbf{P}(\mathbf{T}_1)} \beta_H(\alpha, \mathbf{P}_v^{d,\alpha}(m), \mathbf{I}_v).$$

(2) (Scaling property) For all  $m \in M$ ,

$$\beta_H(r_{\mathbf{t}}\alpha, m, \mathbf{T}) = e^{(t_1 + \dots + t_d)/2} \beta_H(\alpha, m, \mathbf{T}).$$

(3) (Bounded property) Let us denote largest length of side  $T_{max} = \max_i T^{(i)}$ . Then, there exists a constant  $C_H > 0$  such that

$$\beta_H(\alpha, m, \mathbf{T}) \leq C_H T_{max}^{d/2}.$$

(4) (Orthogonality) For all  $[\alpha] \in DC$  and all  $\mathbf{T} \in \mathbb{R}^d$ , bounded function  $\beta_H(\alpha, \cdot, \mathbf{T})$  belongs to the irreducible component, i.e.,

$$\beta_H(\alpha, \cdot, \mathbf{T}) \in H \subset L^2(M).$$

By representation theory introduced (2), for any  $f \in W^s(M)$  has a decomposition

$$f = \sum_H f_H$$

and we define *Bufetov cocycle* associated to  $f$  (or form  $\omega$ ) as the sum

$$(5) \quad \beta^f(\alpha, m, \mathbf{T}) = \sum_H D_\alpha^H(f) \beta_H(\alpha, m, \mathbf{T}).$$

**Remark.** For convenience, identification of distributions of form  $\omega$  and function  $f$  were used. The formula (5) yields a duality between the space of basic (closed) currents and invariant distributions. In the similar setting for horocycle flow, refer [BF14, Cor 1.2, p.10].

Given a Jordan region  $U$  and a point  $m \in M$ , set  $P_U^{d,\alpha} m$  the Birkhoff sums associated to some  $m \in M$  for the action  $\mathbb{P}_x^{d,\alpha}$  given by

$$\langle P_U^{d,\alpha} m, \omega \rangle := \int_U f(\mathbb{P}_x^{d,\alpha} m) dx_1 \cdots dx_d$$

for any degree  $\mathbf{p}$ -form  $\omega = f \hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$ , with  $f \in C_0^\infty(M)$  (smooth function with zero averages).

Let the family of random variable

$$E_{T_n}(f) := \frac{1}{\text{vol}(U(T_n))^{1/2}} \langle P_{U(T_n)}^{d,\alpha}(\cdot), \omega_f \rangle,$$

and we are interested in asymptotic behavior of the probability distributions of  $E_{T_n}(f)$ . Our goal is to understand the asymptotics of  $E_{T_n}$ .

**Theorem 1.5.** For every closed form  $\omega_f \in \Lambda^{\mathbf{d}\mathbf{p}} \otimes W^s(M)$  with  $s > s_{d,g} = d(d+11)/4 + g + 1/2$ , which is not a coboundary, the limit distribution of the family of random variables  $E_{T_n}(f)$  exists, and for almost all frequency  $\alpha$ , it has compact support on the real line.

*Outlines of the paper.* In section 2, we give basic definitions on Higher rank actions, moduli spaces and Sobolev spaces. In section 3, we state main theorem and prove constructions of Bufetov functionals with main properties. In section 4, we prove asymptotic formula of Birkhoff sums and its limit theorems. In section 5 and 6, we prove  $L^2$ -lower bound and analyticity of Bufetov functional. In section 7, there is a measure estimates of Bufetov functionals on the sets where they are small. This result only holds when frame  $\alpha$  is of bounded type.

## 2. ANALYSIS ON HEISENBERG MANIFOLDS

In this section, we will recall definitions of currents, representation and renormalization on moduli space.

**2.1. Invariant currents.** We denote the bundle of  $\mathfrak{p}$ -forms of degree  $j$  of Sobolev order  $s$  by  $A^j(\mathfrak{p}, \mathfrak{M}^s)$ . Similarly, there is a flat bundle of distribution  $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$  whose fiber over  $[\alpha]$  is locally identified with the space  $W_{\alpha}^{-s}(M)$  normed by  $\|\cdot\|_{\alpha, -s}$ .

In the following, we set  $\omega^{d, \alpha} = dX_1^{\alpha} \wedge \cdots \wedge dX_d^{\alpha}$  a top dimensional  $\mathfrak{p}$ -form and identify  $d$  dimensional currents  $\mathcal{D}$  with distributions, for any  $f \in C^{\infty}(M)$

$$\langle \mathcal{D}, f \rangle := \langle \mathcal{D}, f\omega^{d, \alpha} \rangle.$$

**Definition 2.1.** For  $s > 0$ , we denote  $D \in Z_d(\mathfrak{p}, W^{-s}(M))$  a closed  $\mathbf{P}$ -invariant currents of dimension  $d$  and Sobolev order  $s$ . Then, from formal identities,  $\langle D, X_i^{\alpha}(f) \rangle = 0$  for all test function  $f$  and  $i \in [1, d]$ .

By [CF15, Prop 3.13], for any  $s > d/2$  with  $d = \dim \mathbf{P}$ , denote  $I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g))$  the space of  $\mathbf{P}$ -invariant currents of Sobolev order  $s$ , which coincides with the space of closed currents of dimension  $d$ .

- It is one dimensional space if  $\dim \mathbf{P} = g$ , or an infinite-dimensional space if  $\dim \mathbf{P} < g$ . We have  $I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$  for all  $\epsilon > 0$ .

Let  $\omega \in \Lambda^d \mathfrak{p} \otimes W^s(\mathbb{R}^g)$  with  $s > (d+1)/2$ . Then,  $\omega$  admits a primitive  $\Omega$  if and only if  $T(\omega) = 0$  for all  $T \in I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g))$ . We may have  $\Omega \in \Lambda^{d-1} \mathfrak{p}' \otimes W^t(\mathbb{R}^g)$  for any  $t < s - (d+1)/2$ .

**2.2. Best Sobolev constant.** The *Sobolev embedding theorem* implies that for any  $\alpha \in Sp_{2g}(\mathbb{R})$  and  $s > g + 1/2$ , there exists a constant  $B_s(\alpha)$  such that for any  $f \in W_{\alpha}^s(M)$ ,

$$\|f\|_{\infty} \leq B_s(\alpha) \|f\|_{s, \alpha}.$$

The *best Sobolev constant* is defined as the function on the group of automorphism

$$(6) \quad B_s(\alpha) = \sup_{f \in W_{\alpha}^s(M)} \frac{\|f\|_{\infty}}{\|f\|_{s, \alpha}}.$$

By Proposition 4.8 of [CF15], there exists a universal constant  $C(s) > 0$  such that the *best Sobolev constant* satisfies the estimate

$$(7) \quad B_s([\alpha]) \leq C(s) \cdot (Hgt([\alpha]))^{1/4}.$$

From the Sobolev embedding theorem and the definition of the best Sobolev constant, we have the following bound.

**Lemma 2.2.** [CF15, Lemma 5.5] *For any Jordan region  $U \subset \mathbb{R}^d$  with Lebesgue measure  $|U|$ , for any  $s > g + 1/2$  and all  $m \in M$ ,*

$$\left\| [\alpha, (P_U^{d, \alpha} m)] \right\|_{-s} \leq B_s(\alpha) |U|.$$

**2.3. Renormalization.** By definition of renormalization flow, for  $\omega \in \Lambda^d \mathfrak{p} \otimes W^s(\mathbb{R}^g)$  and  $\mathcal{D} \in Z_d(\mathfrak{p}, W^{-s}(M))$ , we have

$$r_i^t[\alpha, \omega] = [r_i^t \alpha, \omega], \quad r_i^t[\alpha, \mathcal{D}] = [r_i^t \alpha, \mathcal{D}].$$

Let  $U_t : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  be unitary operator for  $\mathbf{t} = (t_1, \dots, t_d)$ ,

$$(8) \quad U_t f(x) = e^{-(t_1 + \dots + t_d)/2} f(e^{t_1} x_1, \dots, e^{t_d} x_d).$$

Then, for any invariant currents  $D_\alpha^H$ ,

$$D_{r_t(\alpha)}^H = e^{(t_1 + \dots + t_d)/2} D_\alpha^H.$$

Then, the action of  $\mathbb{R}^d$  is defined by reparametrization

$$(9) \quad \mathbf{P}_x^{d, (r_1^{t_1} \dots r_d^{t_d} \alpha)} = \mathbf{P}_{(e^{-t_1} x_1, \dots, e^{-t_d} x_d)}^{d, \alpha}$$

and the Birkhoff sum satisfies the following identity

$$(10) \quad P_U^{d, (r_1^{t_1} \dots r_d^{t_d} \alpha)} m = e^{(t_1 + \dots + t_d)/2} P_{(e^{-t_1}, \dots, e^{-t_d})U}^{d, \alpha} m.$$

#### 2.4. Diophantine condition.

**Definition 2.3.** The height of a point  $Z$  in Siegel upper half space  $\mathfrak{H}_g$  is the positive number

$$hgt(Z) := \det \Im(Z).$$

The height function  $Hgt : \Sigma_g \rightarrow \mathbb{R}^+$  to be the maximal height of a  $Sp_{2g}(\mathbb{Z})$  orbit. That is, for the class of  $[Z] \in \Sigma_g$ ,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

Let  $\exp \hat{\mathbf{t}}\hat{\delta}(d)$  be the subgroup of  $Sp_{2g}(\mathbb{R})$  defined by  $\exp(\hat{\mathbf{t}}\hat{\delta}(d))X_i = e^{t_i}X_i$ , for  $i = 1, \dots, d$ , and  $\exp(\hat{\mathbf{t}}\hat{\delta}(d))X_i = X_i$  for  $i = d+1, \dots, g$ . We also denote  $r_{\mathbf{t}} = \exp \hat{\mathbf{t}}\hat{\delta}(d)$ .

**Lemma 2.4.** [CF15, Lemma 4.9] For any  $[\alpha] \in \mathfrak{M}_g$  and any  $t \geq 0$ ,

$$(11) \quad Hgt([\exp(\hat{\mathbf{t}}\hat{\delta}(d))\alpha]) \leq (\det(e^{\hat{\mathbf{t}}\hat{\delta}}))^2 Hgt([\alpha]).$$

**Definition 2.5.** [CF15, Definition 4.10] We say that an automorphism  $\alpha \in Sp_{2g}(\mathbb{R})$  or a point  $[\alpha] \in \mathfrak{M}_g$  is  $\hat{\delta}(d)$ -Diophantine of type  $\sigma$  if there exists a  $\sigma > 0$  and a constant  $C > 0$  such that

$$(12) \quad Hgt([\exp(-\hat{\mathbf{t}}\hat{\delta}(d))\alpha]) \leq CHgt([\exp(-\hat{\mathbf{t}}\hat{\delta}(d))])^{(1-\sigma)} Hgt([\alpha]), \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

This states that  $\alpha \in Sp_{2g}(\mathbb{R})$  satisfies a  $\hat{\delta}(d)$ -Diophantine if the height of the projection of  $\exp(-\hat{\mathbf{t}}\hat{\delta}(d))\alpha$  in the Siegel modular variety  $\Sigma_g$  is bounded by  $e^{2(t_1 + \dots + t_d)(1-\sigma)}$ .

-  $[\alpha] \in \mathfrak{M}_g$  satisfies a  $\hat{\delta}$ -Roth condition if for any  $\epsilon > 0$  there exists a constant  $C > 0$  such that

$$(13) \quad Hgt([\exp(-\hat{\mathbf{t}}\hat{\delta}(d))\alpha]) \leq CHgt([\exp(-\hat{\mathbf{t}}\hat{\delta}(d))])^\epsilon Hgt([\alpha]), \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

That is,  $\hat{\delta}(d)$ -Diophantine of type  $0 < \sigma < 1$ .

-  $[\alpha]$  is of *bounded type* if there exists a constant  $C > 0$  such that

$$(14) \quad Hgt([\exp(-\hat{\mathbf{t}}\hat{\delta}(d))]) \leq C, \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$



For  $1 \leq d \leq g$ , according to Margulis-Kleinblock [KM99], a generalization of Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

**Definition 2.6.** Let  $X = G/\Lambda$  be a homogeneous space equipped with the probability Haar measure  $\mu$ . A function  $\phi : X \rightarrow \mathbb{R}$  is said  $k$ -DL (distance like) for some exponent  $k > 0$  if it is uniformly continuous and if there exist constants  $C_1, C_2 > 0$  such that

$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

Theorem 1.9 of [KM99] states the following.

**Proposition 2.7.** *Let  $G$  be a connected semisimple Lie group without compact factors,  $\mu$  its normalized Haar measure,  $\Lambda \subset G$  an irreducible lattice,  $\mathfrak{a}$  a Cartan subalgebra of the Lie algebra of  $G$ . Let  $\mathfrak{d}_+$  be a nonempty open cone in a  $d$ -dimensional subalgebra  $\mathfrak{d}$  of  $\mathfrak{a}$ . If  $\phi : G/\Lambda \rightarrow \mathbb{R}$  is a  $k$ -DL function for some  $k > 0$ , then for  $\mu$ -almost all  $x \in G/\Lambda$  one has*

$$\limsup_{\mathbf{z} \in \mathfrak{d}_+, \|\mathbf{z}\| \rightarrow \infty} \frac{\phi(\exp(\mathbf{z})x)}{\log \|\mathbf{z}\|} = \frac{d}{k}$$

By Lemma 4.7 of [CF15], logarithm of Height function is DL-function with exponent  $k = \frac{g+1}{2}$  on the Siegel variety  $\Sigma_g$  (and induces on  $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ ). Hence, we obtain the following proposition.

**Proposition 2.8.** *Under the assumption  $X = \mathfrak{M}_g$  of Proposition 2.7, for  $s > g + 1/2$ , there exists a full measure set  $\Omega_g(\hat{\delta})$  and for all  $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$*

$$(15) \quad \limsup_{t \rightarrow \infty} \frac{\log Hgt([\exp(-t\hat{\delta}(d))\alpha])}{\log \|t\|} \leq \frac{2d}{g+1}.$$

Any such  $[\alpha]$  satisfies a  $\hat{\delta}$ -Roth condition (13).

For any  $L > 0$  and  $1 \leq d \leq g$ , let  $DC(L)$  denote the set of  $[\alpha] \in \mathfrak{M}_g$  such that

$$(16) \quad \int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots + t_d)/2} Hgt([r_{-\mathbf{t}}\alpha])^{1/4} dt_1 \cdots dt_d \leq L.$$

Let  $DC$  denote the union of the sets  $DC(L)$  over all  $L > 0$ . It follows immediately that the set  $DC \subset \mathfrak{M}_g$  has full Haar volume.

### 3. CONSTRUCTIONS OF THE FUNCTIONALS

For an irreducible representation  $H$ , there exists *basic current*  $B_\alpha^H$  associated to  $D_\alpha^H$ . The current is *basic* in the sense that for all  $j \in \{i_1, \dots, i_d\}$ ,

$$\iota_{X_j} B_\alpha^H = L_{X_j} B_\alpha^H = 0.$$

The basic current  $B_\alpha^H$  is defined as  $B_\alpha^H = D_\alpha^H \eta_X$ . The formula implies that for every  $d$ -form  $\xi$ ,

$$B_\alpha^H(\xi) = D_\alpha^H \left( \frac{\eta_X \wedge \xi}{\omega} \right)$$

where  $\eta_X := \iota_{X_{i_1}} \cdots \iota_{X_{i_d}} \omega$  and  $\omega$  is an invariant volume form.

The basic current  $B_\alpha^H$  belongs to a dual Sobolev space of currents. We write any smooth  $d$ -form  $\xi = \sum \xi^{(i)} \hat{X}_i$ , where  $\hat{X}_i \in \Lambda^d \mathfrak{p}'$ . It follows that the space of smooth  $d$ -form is identified to the product of  $C^\infty(M)$  by isomorphism  $\xi \rightarrow \xi^{(i)}$ .

By isomorphism, we define Sobolev space of currents  $\Omega_\alpha^s(M)$  and their dual spaces of currents  $\Omega_\alpha^{-s}(M)$ .

By Sobolev embedding theorem, for every rectangle  $\Gamma$ , the current  $\Gamma \in \Omega_\alpha^{-s}(M)$  for  $s > (2d+1)/2$ . Then, all basic currents  $B_\alpha^H \in \Omega_\alpha^{-s}(M)$  for all  $s > d/2$  since  $D_\alpha^H \in \Omega_\alpha^{-s}(M)$  for all  $s > d/2$ .

**3.1. Constructions of the functionals.** For any exponent  $s > d/2$ , Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where  $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^\perp$ . Denote by  $I^{-s}$  and  $R^{-s}$  the corresponding orthogonal projection operator and by  $I_\alpha^{-s}$  and  $R_\alpha^{-s}$  the restrictions to the fiber over  $[\alpha] \in \mathfrak{M}$  for  $\alpha \in Sp_{2g}(\mathbb{R})$ . In particular, for the Birkhoff sums  $D = P_U^{d,\alpha} m$ , we call  $I_\alpha^{-s}(D) = I^{-s}[\alpha, D]$  boundary term and  $R_\alpha^{-s}(D) = R^{-s}[\alpha, D]$  remainder term respectively. Consider the orthogonal projection

$$(17) \quad D = I_{r-\mathbf{t}[\alpha]}^{-s}(D) + R_{r-\mathbf{t}[\alpha]}^{-s}(D).$$

For fixed  $\alpha$ , let  $\Pi_H^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \rightarrow A_d(\mathfrak{p}, W_\alpha^{-s}(H))$  denote the orthogonal projection on a single irreducible unitary representation. We further decompose projection operators with

$$\Pi_H^{-s} = \mathcal{B}_\alpha^{-s}(\Gamma) B_\alpha^{-s,H} + R_\alpha^{-s,H}$$

where  $\mathcal{B}_{H,\alpha}^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \rightarrow \mathbb{C}$  denote the orthogonal component map of P-invariant currents (closed), supported on a single irreducible unitary representation.

The *Bufetov functionals* on rectangles  $\Gamma \in \mathfrak{R}$  are defined for all  $\alpha \in DC$  as follows.

**Lemma 3.1.** *Let  $\alpha \in DC(L)$ . For  $s > s_d = d(d+11)/4 + g + 1/2$ , the limit*

$$\hat{\beta}_H(\alpha, \Gamma) = \lim_{t_d \rightarrow \infty} \cdots \lim_{t_1 \rightarrow \infty} e^{-(t_1 + \cdots + t_d)/2} \mathcal{B}_{H,r_d[\alpha]}^{-s}(\Gamma)$$

*exists and define a finitely-additive finite measure on the standard rectangle (3)  $\Gamma := \Gamma_T^X(m)$  for  $m \in M$ . There exists constant  $C(s, \Gamma) > 0$  such that the following estimate holds:*

$$(18) \quad |\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B_\alpha^H|_{\alpha, -s} \leq C(s, \Gamma)(1 + L).$$

*Proof.* For simplicity, we omit dependence of  $H$ . For every  $\mathbf{t} \in \mathbb{R}^d$ , we have the following orthogonal splitting:

$$\Pi_{H,\alpha}^{-s}(\Gamma) = \mathcal{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}},$$

where

$$\mathcal{B}_{\alpha,\mathbf{t}}^{-s} := \mathcal{B}_{H,r-\mathbf{t}[\alpha]}^{-s}, \quad B_{\alpha,\mathbf{t}} := B_{r-\mathbf{t}[\alpha]}^{-s,H}, \quad R_{\alpha,\mathbf{t}} := R_{r-\mathbf{t}[\alpha]}^{-s,H}.$$

For any  $\mathbf{h} \in \mathbb{R}^d$ , we have

$$\mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}+\mathbf{h}} + R_{\alpha,\mathbf{t}+\mathbf{h}} = \mathcal{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}}.$$

By reparametrization (9),  $B_{\mathbf{t}+\mathbf{h}} = e^{-(h_1 + \cdots + h_d)/2} B_{\mathbf{t}}$ ,

$$(19) \quad \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{(h_1 + \cdots + h_d)/2} \mathcal{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) + \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}})$$

and it follows that

$$(20) \quad \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{h_1/2} \mathcal{B}_{\alpha,t_1,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) + \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}}).$$

By differentiating at  $h_1 = 0$ ,

$$(21) \quad \frac{d}{dt_1} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \left[ \frac{d}{dh_1} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) \right]_{h_1=0}.$$

Therefore, we solve the following first order ODE

$$\frac{d}{dt_1} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \mathcal{K}_{\alpha, \mathbf{t}}^{(1)}(\Gamma)$$

where

$$\mathcal{K}_{\alpha, \mathbf{t}}^{(1)}(\Gamma) = \left[ \frac{d}{dh_1} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) \right]_{h_1=0}.$$

Then, the solution of the differential equation is

$$\begin{aligned} \mathcal{B}_{\alpha, t_1, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) &= e^{t_1/2} [\mathcal{B}_{\alpha, 0, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{-\tau_1/2} \mathcal{K}_{\alpha, \tau}^{(1)}(\Gamma) d\tau_1] \\ &= e^{t_1/2} \mathcal{B}_{\alpha, 0, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha, \tau}^{(1)}(\Gamma) d\tau_1. \end{aligned}$$

Note by reparametrization

$$e^{t_1/2} \mathcal{B}_{\alpha, 0, t_2+h_2, \dots, t_d+h_d}^{-s}(\Gamma) = e^{h_2/2} \mathcal{B}_{\alpha, t_1, t_2, t_3+h_3, \dots, t_d+h_d}^{-s}(\Gamma)$$

and it is possible to differentiate the previous equation with respect to  $h_2$  again.

Then

$$(22) \quad \frac{d}{dt_2} \mathcal{B}_{\alpha, t_1, t_2, \dots, t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha, t_1, t_2, t_3+h_3, \dots, t_d+h_d}^{-s} + \int_0^{t_1} e^{-\tau_1/2} \mathcal{K}_{\alpha, \tau}^{(2)}(\Gamma) d\tau_1.$$

where  $\mathcal{K}_{\alpha, \tau}^{(2)}(\Gamma) = \frac{d}{dh_2} \mathcal{K}_{\alpha, \tau}^{(1)}(\Gamma)$ .

Then, the solution of (22) is

$$\begin{aligned} \mathcal{B}_{\alpha, t_1, t_2, \dots, t_d+h_d}^{-s}(\Gamma) &= e^{t_2/2} [\mathcal{B}_{\alpha, t_1, 0, t_3+h_3, \dots, t_d+h_d}^{-s} + \int_0^{t_2} e^{-\tau_2/2} \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha, \tau}^{(2)}(\Gamma) d\tau_1 d\tau_2] \\ &= e^{h_3/2} \mathcal{B}_{\alpha, t_1, t_2, t_3, \dots, t_d+h_d}^{-s} + \int_0^{t_2} e^{(t_2-\tau_2)/2} \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha, \tau}^{(2)}(\Gamma) d\tau_1 d\tau_2. \end{aligned}$$

Inductively, we solve first order ODE repeatedly and obtain the following solution

$$(23) \quad \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(\Gamma) = e^{(t_1+\dots+t_d)/2} \left( \mathcal{B}_{\alpha, 0}^{-s} + \int_0^{t_d} \dots \int_0^{t_1} e^{-(\tau_1+\dots+\tau_d)/2} \mathcal{K}_{\alpha, \tau}^{(d)}(\Gamma) d\tau_1 \dots d\tau_d \right)$$

where

$$\mathcal{K}_{\alpha, \tau}^{(d)}(\Gamma) = \left[ \frac{d}{dh_d} \dots \frac{d}{dh_1} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) \right]_{h_d \dots h_1=0}.$$

Let  $\langle \cdot, \cdot \rangle_t$  denote the inner product in Hilbert space  $\Omega_{r, \mathbf{t}[\alpha]}^{-s}$ . From the intertwining formula,

$$\begin{aligned} \mathcal{B}_{\alpha, \mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}}) &= \langle R_{\alpha, \mathbf{t}}, \frac{B_{\alpha, \mathbf{t}+\mathbf{h}}}{|B_{\alpha, \mathbf{t}+\mathbf{h}}|_{\mathbf{t}+\mathbf{h}}^2} \rangle_{\alpha, \mathbf{t}+\mathbf{h}} \\ &= \langle R_{\alpha, \mathbf{t}} \circ U_{-\mathbf{h}}, \frac{B_{\alpha, \mathbf{t}+\mathbf{h}} \circ U_{-\mathbf{h}}}{|B_{\alpha, \mathbf{t}+\mathbf{h}}|_{\mathbf{t}+\mathbf{h}}^2} \rangle \\ &= \langle R_{\alpha, \mathbf{t}} \circ U_{-\mathbf{h}}, \frac{B_{\alpha, \mathbf{t}}}{|B_{\alpha, \mathbf{t}}|_{\mathbf{t}}^2} \rangle = \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(R_{\alpha, \mathbf{t}} \circ U_{-\mathbf{h}}). \end{aligned}$$

In the sense of distribution,

$$\begin{aligned} \frac{d}{dh_d} \cdots \frac{d}{dh_1} (R_{\alpha,t} \circ U_{-\mathbf{h}}) &= -R_{\alpha,t} \circ \left( \frac{d}{2} + \sum_{i=1}^d X_i(t) \right) \circ U_{-\mathbf{h}} \\ &= \left[ \left( \sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha,t} \right] \circ U_{-\mathbf{h}} \end{aligned}$$

and we compute derivative term of (21) in representation,

$$\left[ \frac{d}{dh_d} \cdots \frac{d}{dh_1} (\mathcal{B}_{\alpha,t+\mathbf{h}}^{-s}(R_{\alpha,t})) \right]_{\mathbf{h}=0} = -\mathcal{B}_{\alpha,t}^{-s} \left( \left( \sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha,t} \right).$$

Set  $\mathcal{K}_{\alpha,t}(\Gamma) = |R_{\alpha,t}|_{r_{-\mathbf{t}}\alpha, -(s+1)}$  with a bounded non-negative function, then by Lemma 3.5, we claim that

$$(24) \quad |\mathcal{B}_{\alpha,t}^{-s} \left( \left( \sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha,t} \right)| \leq \mathcal{K}_{\alpha,\tau}(\Gamma) \leq C(s, \Gamma) \text{Hgt}([r_{-\mathbf{t}}\alpha])^{1/4}.$$

Therefore, the solution of equation (23) exists under Diophantine condition (16) and the following holds:

$$\lim_{t_d \rightarrow \infty} \cdots \lim_{t_1 \rightarrow \infty} e^{-(t_1 + \cdots + t_d)/2} \mathcal{B}_{\alpha,t}^{-s}(\Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

Moreover, by continuity, the complex number

$$\hat{\beta}_H(\alpha, \Gamma) = \mathcal{B}_{\alpha,0}^{-s} + \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots + \tau_d)/2} \mathcal{K}_{\alpha,\tau}(\Gamma) d\tau_1 \cdots d\tau_d$$

depends on  $\alpha \in DC(L)$ . Since we have

$$\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B_\alpha^H = R_0 - \left( \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots + \tau_d)/2} \mathcal{K}_{\alpha,\tau}(\Gamma) d\tau_1 \cdots d\tau_d \right) B_\alpha^H,$$

by Diophantine condition again,

$$|\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B_\alpha^H|_{\alpha, -s} \leq C(s, \Gamma)(1 + L).$$

□

**3.2. Remainder estimates.** In this subsection, we obtain estimate for remainder term which is used in Lemma 3.1. Firstly, we prove the bound of Birkhoff sum of rectangles.

**Lemma 3.2.** [CF15, Lemma 5.7] *Let  $s > d/2 + 2$ . There exists a constant  $C = C(s) > 0$  such that for all  $t_i \geq 0$  for  $1 \leq i \leq d$ , we have*

$$\begin{aligned} \|I^{-s}[\alpha, D]\|_{-s} &\leq e^{-(t_1 + \cdots + t_d)/2} \|I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, D]\|_{-s} \\ &\quad + C_1 |t_1 + \cdots + t_d| \int_0^1 e^{-u(t_1 + \cdots + t_d)/2} \|R^{-s}[r_1^{-ut_1} \cdots r_d^{-ut_d} \alpha, D]\|_{-(s-2)} du. \end{aligned}$$

By Stokes' theorem, we have the following remainder estimate.

**Lemma 3.3.** [CF15, Lemma 5.6] *For any non-negative  $s' < s - (d+1)/2$  and Jordan region  $U \subset \mathbb{R}^d$ , there exists  $C = C(d, g, s, s') > 0$  such that*

$$\|R^{-s}[\alpha, (P_U^{d,\alpha} m)]\|_{-s} \leq C \left\| [\alpha, \partial(P_U^{d,\alpha} m)] \right\|_{-s'}.$$

In [CF15], quantitative bound of Birkhoff sums on square domain was obtained. Now we extend the result on rectangular shapes.

**Theorem 3.4.** *For  $s > s_d$ , there exists a constant  $C(s, d) > 0$  such that the following holds. For any  $t_i > 0$ ,  $m \in M$  and  $U_d(t) = [0, e^{t_1}] \times \cdots \times [0, e^{t_d}]$ , we have*

$$(25) \quad \begin{aligned} \left\| [\alpha, (P_{U_d(t)}^{d, \alpha} m)] \right\|_{-s} &\leq C \sum_{k=0}^d \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^d t_l - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \\ &\quad \times Hgt\left(\left[\prod_{1 \leq j \leq d} r_j^{-t_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}} \alpha\right]\right)^{1/4} du_{i_1} \cdots du_{i_k}. \end{aligned}$$

*Proof.* We prove by induction. For  $d = 1$ , it follows from the theorem 5.8 in [CF15]. We assume that the result holds for  $d-1$ . Decompose the current as a sum of boundary and remainder term as in (17).

*Step 1.* We estimate the boundary term. By Lemma 3.2, renormalize terms with  $r^u = r_1^u \cdots r_d^u$ . Then, we have

$$(26) \quad \begin{aligned} \left\| I^{-s} [\alpha, (P_{U_d(t)}^{d, \alpha} m)] \right\|_{-s} &\leq e^{-(t_1 + \cdots + t_d)/2} \left\| I^{-s} [r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{U_d(t)}^{d, \alpha} m)] \right\|_{-s} \\ &+ C_1(s) \int_0^{t_1 + \cdots + t_d} e^{-ud/2} \left\| R^{-s} [r^{-u} \alpha, (P_{U_d(t)}^{d, \alpha} m)] \right\|_{-(s-2)} du \\ &:= (I) + (II). \end{aligned}$$

By renormalization (10) and Lemma 2.2 for unit volume,

$$\begin{aligned} \left\| I^{-s} [r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{U_d(t)}^{d, \alpha} m)] \right\|_{-s} &= e^{t_1 + \cdots + t_d} \left\| I^{-s} [r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{U_d(0)}^{d, r_1^{-t_1} \cdots r_d^{-t_d} \alpha} m)] \right\|_{-s} \\ &\leq C_2 e^{t_1 + \cdots + t_d} Hgt([r_1^{-t_1} \cdots r_d^{-t_d} \alpha])^{1/4}. \end{aligned}$$

Hence

$$I \leq C_2 e^{(t_1 + \cdots + t_d)/2} Hgt([r_1^{-t_1} \cdots r_d^{-t_d} \alpha])^{1/4},$$

where the sum corresponds to the first term ( $k = 0$ ) in the statement.

*Step 2.* To estimate (II),

$$(27) \quad \begin{aligned} \left\| R^{-s} [r^{-u} \alpha, (P_{U_d(t)}^{d, \alpha} m)] \right\|_{-(s-2)} &= \left\| e^{ud} R^{-s} [r^{-u} \alpha, (P_{U_d(t-u)}^{d, r^{-u} \alpha} m)] \right\|_{-(s-2)} \\ &\leq C_3(s, s') e^{ud} \left\| [r^{-u} \alpha, \partial(P_{U_d(t-u)}^{d, r^{-u} \alpha} m)] \right\|_{-s'}. \end{aligned}$$

The boundary  $\partial(P_{U_d}^{d, r^{-t} \alpha})$  is the sum of  $2d$  currents of dimension  $d-1$ . These currents are Birkhoff sums of  $d$  face subgroups obtained from  $P_{U_d}^{d, r^{-t} \alpha}$  by omitting one of the base vector fields  $X_i$ . It is reduced to  $(d-1)$  dimensional shape obtained from  $U_d(t-u) := [0, e^{t_1-u}] \times \cdots \times [0, e^{t_d-u}]$ . For each  $1 \leq j \leq d$ , there are Birkhoff sums along  $d-1$  dimensional cubes. By induction hypothesis, we add all the  $d-1$

dimensional cubes by adding all the terms along  $j$ :

(28)

$$\begin{aligned} & \left\| [r^{-u} \alpha, (P_{U_{d-1}(t-u)}^{d-1, r^{-u} \alpha} m)] \right\|_{-s'} \leq C_4(s', d-1) \sum_{j=1}^d \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq d \\ i_l \neq j, \forall l}} \int_0^{t_{i_k}-u} \dots \int_0^{t_{i_1}-u} \\ & \exp\left(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^d (t_l - u) - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{\substack{1 \leq l \leq d \\ l \neq j}} r_l^{-(t_l-u)} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r^{-u} \alpha)\right]\right)^{1/4} du_{i_1} \dots du_{i_k}. \end{aligned}$$

Combining (26) and (27), we obtain the estimate for (II).

$$\begin{aligned} (II) & \leq C_5(s', d-1) \sum_{j=1}^d \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq d \\ i_l \neq j}} \int_0^{t_1+\dots+t_d} \int_0^{t_{i_k}-u} \dots \int_0^{t_{i_1}-u} du_{i_1} \dots du_{i_k} du \\ & \times \exp\left(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^d t_l - \frac{1}{2} u - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_j^{-u+t_j} \alpha)\right]\right)^{1/4}. \end{aligned}$$

Applying the change of variable  $u_j = t_j - u$ , we obtain

$$\begin{aligned} (II) & \leq C_6(s', d-1) \sum_{j=1}^d \sum_{k=1}^{d-1} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq d \\ i_l \neq j}} \int_{-(t_1+\dots+t_d)+t_j}^{t_j} \int_0^{t_{i_k}-u} \dots \int_0^{t_{i_1}-u} du_{i_1} \dots du_{i_k} du_j \\ & \times \exp\left(\frac{1}{2} (t_1 + \dots + t_d) - \frac{1}{2} u_j - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_j^{-u_j} \alpha)\right]\right)^{1/4}. \end{aligned}$$

Simplifying multi-summation above, (with  $-(t_1 + \dots + t_d) + t_j \leq 0$ )

$$\begin{aligned} (II) & \leq C_7(s', d) \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \int_0^{t_{i_k}} \dots \int_0^{t_{i_1}} du_{i_1} \dots du_{i_k} \\ & \times \exp\left(\frac{1}{2} (t_1 + \dots + t_d) - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} \alpha\right]\right)^{1/4}. \end{aligned}$$

*Step 3. (Remainder estimate).* The remainder term is obtained from Lemma 3.3 (Stokes' theorem). Following step 2, estimate of remainder reduces to that of  $d-1$  form. Combining with the step 1, we have the following

(29)

$$\left\| R^{-s}[\alpha, \partial(P_{U_d}^{d, \alpha} m)] \right\|_{-s} \leq C(s) \sum_{i=1}^{d-1} \left\| I^{-s}[\alpha, (P_{U_i}^{i, \alpha} m)] \right\|_{-s} + \left\| R^{-s}[\alpha, (P_{U_1}^{1, \alpha} m)] \right\|_{-s}$$

where  $U_i$  is  $i$ -dimensional rectangle. Sum of the boundary terms are absorbed in the bound of (I) + (II). For 1-dimensional remainder with interval  $\Gamma_T$ , the boundary is a 0-dimensional current. Then,

$$\langle \partial(P_{U_T}^{1, \alpha} m), f \rangle = f(P_{U_T}^{1, \alpha} m) - f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (6) and (7),

$$\left\| R^{-s}[\alpha, \partial(P_{U_T}^{1,\alpha} m)] \right\|_{-s} \leq 2B_{s'}([\alpha]) \leq C(s)Hgt([\alpha])^{1/4}.$$

Then, by inequality (11)

$$Hgt([\alpha])^{1/4} \leq C e^{(t_1+\dots+t_d)/2} Hgt([r_1^{-t_1} \dots r_d^{-t_d} \alpha])^{1/4}.$$

This implies that remainder term produces one more term like the bound of (I). Therefore, the theorem follows from combining all the terms (I), (II), and remainder.  $\square$

Now we prove the remainder estimate used in (24).

**Lemma 3.5.** *Let  $s > s_d$ . There exists a constant  $C(s) > 0$  such that for any rectangle  $U_\Gamma = [0, e^{\Gamma_1}] \times \dots \times [0, e^{\Gamma_d}]$ ,*

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C(s, \Gamma) Hgt([r_{-t}\alpha])^{1/4}.$$

*Proof.* Recall that  $\mathcal{K}_{\alpha,t}(\Gamma) = \left\| R^{-s}[r_{-t}\alpha, (P_{U_\Gamma}^{d,r-t\alpha} m)] \right\|_{-(s+1)}$  and by Lemma 3.3, it is equivalent to prove to find the bound of  $d-1$  currents. By theorem 3.4, we obtain the remainder estimate.

$$\begin{aligned} \mathcal{K}_{\alpha,t}(\Gamma) &\leq C \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \int_0^{\Gamma_{i_k}} \dots \int_0^{\Gamma_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^{d-1} \Gamma_l - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \\ &\quad \times Hgt\left(\left[ \prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_{-t}\alpha) \right]\right)^{1/4} du_{i_1} \dots du_{i_k}. \end{aligned}$$

It follows from (11) that for  $0 \leq k \leq d-1$ ,

$$Hgt\left(\left[ \prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_{-t}\alpha) \right]\right)^{1/4} \leq e^{\frac{1}{2}(\sum_{l=1}^k u_{i_l} - \sum_{l=1}^{d-1} \Gamma_l)} Hgt([r_{-t}\alpha])^{1/4}.$$

Then, we obtain

(30)

$$\left\| R^{-s}[r_{-t}\alpha, (P_{U_\Gamma}^{d,r-t\alpha} m)] \right\|_{-s} \leq C \left( \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \prod_{l=1}^k \Gamma_{i_l} \right) Hgt([r_{-t}\alpha])^{1/4}.$$

Setting  $C(s, \Gamma) = C \left( \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \prod_{l=1}^k \Gamma_{i_l} \right)$ , we obtain the conclusion.  $\square$

**3.3. Extensions of domain.** Now we extend the domain of Bufetov functional defined on standard rectangle  $\Gamma_{\mathbf{T}}^X$  to the class  $\mathfrak{R}$ .

**Lemma 3.6.** *The functional  $\hat{\beta}$  defined on standard rectangle  $\Gamma_{\mathbf{T}}^X$  extends to the class  $(Q_y^{d,Y})_* \Gamma_{\mathbf{T}}^X$  for any  $y \in \mathbb{R}^d$ .*

*Proof.* First, we prove the functional  $\hat{\beta}$  exists and invariant under the action  $Q_y^{d,Y}$ . It suffices to verify that  $\hat{\beta}$  is invariant under the rank 1 action  $Q_\tau^{1,Y}$  for  $\tau \in \mathbb{R}$ .

Given a standard rectangle  $\Gamma$ , set  $\Gamma_Q := (Q_\tau^{1,Y})_* \Gamma$ . Let  $D(\Gamma, \Gamma_Q)$  be the  $(d+1)$  dimensional space spanned by the trajectories of the action of  $Q_\tau^{1,Y}$  projecting  $\Gamma$  onto  $\Gamma_Q$ . Then  $D(\Gamma, \Gamma_Q)$  is union of all orbits I of action  $Q_\tau^{1,Y}$  such that the

boundary of  $I$ ,  $d$ -dimensional faces, is contained in  $\Gamma \cup \Gamma_Q$ , and interior of  $I$  is disjoint from  $\Gamma \cup \Gamma_Q$ .

By definition, denote  $r_t := r_t^i$ . Then,  $r_{-t}(\Gamma)$  and  $r_{-t}(\Gamma_Q)$  are respectively the support of the currents  $r_t^* \Gamma$  and  $r_t^* \Gamma_Q$ . Thus, we have the following identity

$$r_t^* D(\Gamma, \Gamma_Q) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q)).$$

Since the currents  $\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)$  is composed of orbits of the action of  $\mathbf{Q}_\tau^{1,Y}$ , it follows that

$$(31) \quad \partial[r_t^* D(\Gamma, \Gamma_Q)] - (r_t^* \Gamma - r_t^* \Gamma_Q) = r_t^* [\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)] \rightarrow 0.$$

Now, we turn to prove the volume of  $D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q))$  is uniformly bounded for all  $t > 0$ . For any  $p \in \Gamma$ , set  $\tau(p)$  be length of the arc lying in  $D := D(\Gamma, \Gamma_Q)$ , and set  $\tau_\Gamma := \sup\{\tau(p) \mid p \in \Gamma\} < \infty$ . We write

$$\text{vol}_{d+1}(D) = \int_\Gamma \tau \text{vol}_d.$$

Since  $\text{vol}_d(r_{-t}\Gamma) \leq e^t \text{vol}_d(\Gamma)$ ,

$$(32) \quad \text{vol}_{d+1}(r_{-t}D) = \int_{r_{-t}\Gamma} \tau \text{vol}_d \leq \tau_\Gamma e^{-t} \text{vol}_d(r_{-t}\Gamma) \leq \tau_\Gamma \text{vol}_d(\Gamma) < \infty.$$

Note that current  $(\mathbf{Q}_\tau^{1,Y})_* \Gamma - \Gamma$  is equal to the boundary of a  $(d+1)$  dimensional current  $D$ . By arguments in remainder estimate (or Sobolev embedding theorem),

$$(33) \quad |(\mathbf{Q}_\tau^{1,Y})_* \Gamma - \Gamma|_{r^{-t}\alpha, -s} \leq C_s \tau B_s([r^{-t}\alpha]) \leq C_s \tau \text{Hgt}([r^{-t}\alpha])^{1/4}$$

is finite for all  $t > 0$ .

Then, by (31), (32) and existence of Bufetov functional (from Diophantine condition), the last inequality holds

$$|\mathcal{B}_{\alpha,t}^{-s}((\mathbf{Q}_\tau^{1,Y})_* \Gamma) - \mathcal{B}_{\alpha,t}^{-s}(\Gamma)|_{\alpha, -s} < \infty.$$

Therefore, by definition of Bufetov functional in the Lemma 3.1, it is extended and invariant under the action of  $\mathbf{Q}_\tau^{1,Y}$ .  $\square$

Similarly, Bufetov functional defined on the standard rectangles extends to the class of generalized rectangle  $(\phi_{t_{iz}}^Z) \circ \mathbf{P}_t^{d,\alpha}(m)$  (see (4)). We postpone its proof by Lemma 6.3. Since the flow generated by  $Z$  commutes with other actions  $\mathbf{P}$  and  $\mathbf{Q}$ , combining with the invariance under the action  $\mathbf{Q}$  from Lemma 3.6, the domain of Bufetov functional extends to the class  $\mathfrak{R}$ .

*Proof of theorem 1.3. Additive property.* It follows from linearity of projections and limit. It is immediate to derive *scaling property* from the definition.

*Bounded property.* By scaling property,

$$\hat{\beta}(\alpha, \Gamma) = e^{dt/2} \hat{\beta}_H(r^t[\alpha], \Gamma).$$

Choose  $t = \log(\int_\Gamma |\hat{X}|)$  and  $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$ , then uniform bound of Bufetov functional on bounded size of rectangles,

$$|\hat{\beta}(\alpha, \Gamma)| \leq C(\Gamma) \left( \int_\Gamma |\hat{X}| \right)^{d/2}.$$

*Invariance property.* It follows directly from the Lemma 3.6.  $\square$



**3.4. Bound of functionals.** We define excursion function

$$\begin{aligned} E_{\mathfrak{M}}(\alpha, \mathbf{T}) &:= \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1+\cdots+t_d)/2} \text{Hgt}([r_{\mathbf{t}-\log \mathbf{T}} \alpha])^{1/4} dt_1 \cdots dt_d \\ &= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1+\cdots+t_d)/2} \text{Hgt}([r_{\mathbf{t}} \alpha])^{1/4} dt_1 \cdots dt_d. \end{aligned}$$

**Lemma 3.7.** *For any Diophantine  $[\alpha] \in DC(L)$  and for any  $f \in W^s(M)$  for  $s > s_d + 1/2$ , the Bufetov functional  $\beta^f$  is defined by a uniformly convergent series.*

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_s (L + \prod_{i=1}^d (T^{(i)})^{1/2} (1 + \prod_{i=1}^d t_i + E_{\mathfrak{M}}(\alpha, \mathbf{T})) \|\omega\|_{\alpha, s}$$

for  $\omega = f\omega^{d, \alpha} \in \Lambda^d \mathfrak{p} \otimes W^s(M)$  and  $\mathbf{tT} = (t_1 T^{(1)}, \dots, t_d T^{(d)})$ .

*Proof.* It follows from Lemma 3.1 that there exists a constant  $C > 0$  such that

$$|\beta_H(\alpha, m, \mathbf{t})| \leq C(1 + L + \prod_{i=1}^d t_i).$$

By exact scaling property,

$$\beta_H(\alpha, m, \mathbf{tT}) = \prod_{i=1}^d (T^{(i)})^{1/2} \beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t}).$$

By Diophantine condition (16), whenever  $\alpha \in DC(L)$  then  $r_{\log \mathbf{T}}[\alpha] \in DC(L_T)$  with

$$L_T \leq L \prod_{i=1}^d (T^{(i)})^{-1/2} + E_{\mathfrak{M}}(\alpha, \mathbf{T}).$$

Thus for all  $(m, \mathbf{t}) \in M \times \mathbb{R}^d$ ,

$$|\beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t})| \leq C(1 + L_T + \prod_{i=1}^d t_i).$$

It follows that for all  $s > 1/2$ , we have

$$\begin{aligned} |\beta^f(\alpha, m, \mathbf{tT})| &\leq C_s \prod_{i=1}^d (T^{(i)})^{1/2} (1 + L + \prod_{i=1}^d t_i) \sum_{n \in \mathbb{Z}} \|\omega_n\|_{\alpha, s} \\ &\leq C_s \prod_{i=1}^d (T^{(i)})^{1/2} (1 + L_T + \prod_{i=1}^d t_i) \left( \sum_{n \in \mathbb{Z}} (1 + n^2)^{-s'} \right)^{-1/2} \left( \sum_{n \in \mathbb{Z}} \left\| (1 - Z^2)^{s'/2} \omega_n \right\|_{\alpha, s}^2 \right)^{1/2}. \end{aligned}$$

Therefore, for all  $s' > s_d$ , there exists a constant  $C_{s, s'} > 0$  such that

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_{s, s'} \prod_{i=1}^d (T^{(i)})^{1/2} (1 + L_T + \prod_{i=1}^d t_i) \|\omega\|_{\alpha, s+s'}.$$

□

This lemma implies that all properties of the Bufetov functionals associated to a single irreducible component  $\beta_H$  can be extended to the Bufetov functionals  $\beta^f$  for any  $f \in W^s(M)$ . From this, we can derive bounded property for the cocycles  $\beta(\alpha, m, \mathbf{T})$  with respect to  $m$  along the orbits of actions in time  $\mathbf{T} \in \mathbb{R}^d$  respectively.

**Corollary 3.8.** *For all  $s > s_d + 1/2$ , there exists a constant  $C_s > 0$  such that for almost all frequency  $\alpha$  and for all  $f \in W^s(M)$  and for all  $(m, \mathbf{T}) \in M \times \mathbb{R}^d$ , we have*

$$(34) \quad |\langle P_{U(\mathbf{T})}^{d,\alpha} m, \omega \rangle - \beta^f(\alpha, m, \mathbf{T})| \leq C_s(1+L) \|\omega\|_{\alpha,s}.$$

for  $U(\mathbf{T}) = [0, T^{(1)}] \times \cdots [0, T^{(d)}]$  and  $\omega = f\omega^{d,\alpha} \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ .

*Proof.* By Lemma 3.1 and 3.7, asymptotic formula (18) on each irreducible provides proof of Corollary 3.8.  $\square$

#### 4. LIMIT DISTRIBUTIONS

In this section, we prove Theorem 1.5, limit distribution of Birkhoff sums of higher rank actions on squares.

##### 4.1. Limiting distributions.

**Lemma 4.1.** *There exists a continuous modular function  $\theta_H : \text{Aut}_0(\mathbf{H}^g) \rightarrow H \subset L^2(M)$  such that*

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \langle P_{U(\mathbf{T})}^{d,\alpha}(\cdot), \omega_f \rangle - \theta_H(r_{\log \mathbf{T}}[\alpha]) D_\alpha^H(f) \right\|_{L^2(M)} = 0.$$

The family  $\{\theta_H(\alpha) \mid \alpha \in \text{Aut}_0(\mathbf{H}^g)\}$  has a constant norm in  $L^2(M)$ .

*Proof.* By Fourier transform, the space of smooth vectors and Sobolev space  $W^s(H)$  is represented as Schwartz-type space  $\mathcal{S}^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$  such that

$$\int_{\mathbb{R}^d} |(1 + \sum_i \frac{\partial^2}{\partial u_i^2} + \sum_i u_i^2)^{s/2} \hat{f}(u)|^2 du < \infty.$$

Let  $\mathbf{t}, \mathbf{u} \in \mathbb{R}^d$ . Then we claim for any  $f \in \mathcal{S}^s(\mathbb{R}^d)$ , there exists  $\theta[\alpha] \in L^2(\mathbb{R}^d)$  such that

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \int_{U(\mathbf{T})} f(\mathbf{u} + \mathbf{t}) d\mathbf{t} - \theta_H(r_{\log \mathbf{T}}[\alpha]) \text{Leb}(f) \right\|_{L^2(\mathbb{R}^d, d\mathbf{u})} = 0.$$

Equivalently,

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{it \cdot \hat{\mathbf{u}}} \hat{f}(\hat{\mathbf{u}}) dt - \hat{\theta}_H(r_{\log \mathbf{T}}[\alpha]) \hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{\mathbf{u}})} = 0.$$

For  $\chi \in L^2(\mathbb{R}^d, d\hat{\mathbf{u}})$ , we denote

$$\chi_j(\hat{u}) = \frac{e^{i\hat{u}_j} - 1}{i\hat{u}_j}, \quad \chi(\hat{u}) = \prod_{j=1}^d \chi_j(\hat{u}_j).$$

Let  $\hat{\theta}[\alpha](\hat{u}) := \chi(\hat{u})$  for all  $\hat{u} \in \mathbb{R}^d$ . By intertwining formula (8), for  $\mathbf{T} \in \mathbb{R}^d$  and  $\mathbf{u} \in \mathbb{R}^d$  we have

$$U_{\mathbf{T}}(f)(\hat{u}) = \prod_{i=1}^d (T^{(i)})^{1/2} f(\mathbf{T}\hat{\mathbf{u}}), \text{ for } \mathbf{T}\hat{\mathbf{u}} = (T^{(1)}\hat{u}_1, \dots, T^{(d)}\hat{u}_d).$$

Then, for all  $\alpha \in \text{Aut}_0(\mathbf{H}^g)$ ,

$$\hat{\theta}(r_{\log \mathbf{T}}[\alpha])(\hat{u}) = U_{\mathbf{T}}(\chi)(\hat{u}) = \prod_{i=1}^d (T^{(i)})^{1/2} \chi(\mathbf{T}\hat{u}).$$

The function  $\theta[\alpha]$  is defined by Fourier inverse transform

$$\|\theta_H(\alpha)\|_H = \|\theta(\alpha)\|_{L^2(\mathbb{R}^d)} = \|\hat{\theta}(\alpha)\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{u})\|_{L^2(\mathbb{R}^d, d\hat{u})} = C > 0.$$

By integration,

$$\begin{aligned} \int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{it \cdot \hat{u}} \hat{f}(\hat{u}) dt &= \left( \prod_{i=1}^d T^{(i)} \right) \chi(\mathbf{T}\hat{u}) \hat{f}(\hat{u}) \\ &= \left( \prod_{i=1}^d T^{(i)} \right) \chi(\mathbf{T}\hat{u}) (\hat{f}(\hat{u}) - \hat{f}(0)) + \left( \prod_{i=1}^d T^{(i)} \right)^{1/2} \hat{\theta}(r_{\log \mathbf{T}}[\alpha])(\hat{u}) \hat{f}(0). \end{aligned}$$

We note that  $\prod_{i=1}^d T^{(i)} = \text{vol}(U(\mathbf{T}))$ . Then the claim reduces to the following:

$$\limsup_{\text{vol}(U(\mathbf{T})) \rightarrow \infty} \left\| \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{u}) (\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If  $f \in \mathcal{S}^s(\mathbb{R}^g)$  with  $s > d/2$ , function  $\hat{f} \in C^0(\mathbb{R}^d)$  and bounded. Thus, by Dominated convergence theorem,

$$\left\| \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{u}) (\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = \left\| \chi(\nu) \left( \hat{f}\left(\frac{\nu}{\mathbf{T}}\right) - \hat{f}(0) \right) \right\|_{L^2(\mathbb{R}^d, d\nu)} \rightarrow 0.$$

□

**Corollary 4.2.** *For any  $s > d/2$ ,  $\alpha \in \text{Aut}_0(\mathbf{H}^g)$  and  $f \in W^s(H)$ , there exists a constant  $C > 0$  such that*

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha} m, \omega_f \right\rangle \right\|_{L^2(M)} = C |D_\alpha^H(f)|.$$

From Corollary 4.2, we derive the following limit result for the  $L^2$  norm of Bufetov functionals.

**Corollary 4.3.** *For irreducible component  $H$  and  $\alpha \in DC$ , there exists  $C > 0$  such that*

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \|\beta_H(\alpha, \cdot, \mathbf{T})\|_{L^2(M)} = C.$$

*Proof.* By the normalization of invariant distribution in Sobolev space  $W^s(M)$ , there exists a function  $f_\alpha^H \in W^s(H)$  such that  $D_\alpha(f_\alpha^H) = \|f_\alpha^H\|_s = 1$ . For all  $\alpha \in DC(L)$ , by asymptotic formula (34),

$$\left| \left\langle P_{U(\mathbf{T})}^{d,\alpha} m, \omega \right\rangle - \beta^f(\alpha, m, \mathbf{T}) \right| \leq C_s(1 + L).$$

Therefore,  $L^2$ -estimate follows from Corollary 4.2. □

A relation between the Bufetov functional and the modular function  $\theta_H$  is established below.

**Corollary 4.4.** *For any  $L > 0$  and invariant probability measure supported on  $DC(L) \subset \mathfrak{M}_g$ ,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H([\alpha]), \quad \text{for } \mu\text{-almost all } [\alpha] \in \mathfrak{M}_g.$$

*Proof.* By Theorem 3.8 and Lemma 4.1, there exists a constant  $C > 0$  such that for all  $\alpha \in \text{supp}(\mu) \subset DC(L)$ , for all  $T > 0$  we have

$$(35) \quad \lim_{|U(\mathbf{T})| \rightarrow \infty} \|\beta_H(r_{\log \mathbf{T}}[\alpha], \cdot, 1) - \theta_H(r_{\log \mathbf{T}}[\alpha])\|_{L^2(M)} \leq \frac{C_\mu}{\text{vol}(U(\mathbf{T}))^{1/2}}.$$

By Luzin's theorem, for any  $\delta > 0$  there exists a compact subset  $E(\delta) \subset \mathfrak{M}$  such that we have the measure bound  $\mu(\mathfrak{M} \setminus E(\delta)) < \delta$  and the function  $\beta_H(\alpha, \cdot, 1) \in L^2(M)$  depends continuously on  $[\alpha] \in E(\delta)$ . By Poincare recurrence, there is a full measure subset  $E'(\delta) \subset E(\delta)$  of  $\mathbb{R}^d$ -action.

For every  $\alpha_0 \in E'(\delta)$ , there is diverging sequence  $(t_n)$  such that  $\{r^{t_n}(\alpha_0)\} \subset E(\delta)$  and  $\lim_{n \rightarrow \infty} r^{t_n}(\alpha_0) = (\alpha_0)$ . By continuity of  $\theta_H$  and  $\beta_H$  at  $[\alpha_0]$ , we have

$$(36) \quad \begin{aligned} & \|\beta_H([\alpha_0], \cdot, 1) - \theta_H([\alpha_0])\|_{L^2(M)} \\ &= \lim_{n \rightarrow \infty} \|\beta_H(r_{\log \mathbf{T}_n}[\alpha_0], \cdot, 1) - \theta_H(r_{\log \mathbf{T}_n}[\alpha_0])\|_{L^2(M)} = 0. \end{aligned}$$

Thus, we have  $\beta_H([\alpha], \cdot, 1) = \theta_H([\alpha])$  for  $[\alpha] \in E'(\delta)$ . It follows that the set of equality fails has less than any  $\delta > 0$ , thus the identity holds for almost all  $[\alpha]$ .  $\square$

For all  $\alpha \in \text{Aut}_0(\mathcal{H}^g)$ , general smooth function  $f \in W^s(M)$  for  $s > s_d + 1/2$ ,  $f$  decompose an infinite sum, and the functional  $\theta^f$  is defined by a convergent series.

$$(37) \quad \theta^f(\alpha) = \sum_H D_\alpha^H(f) \theta_H(\alpha).$$

The following result is an extension to general asymptotic theorem from Corollary 4.4.

**Theorem 4.5.** *For all  $\alpha \in \text{Aut}_0(\mathcal{H}^g)$ ,  $f \in W^s(M)$  and  $s > s_d + 1/2$ ,*

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}_n))^{1/2}} \left\langle P_{U(\mathbf{T}_n)}^{d, \alpha} m, \omega_f \right\rangle - \theta^f(r_{\log \mathbf{T}_n} \alpha) \right\|_{L^2(M)} = 0.$$

**4.2. Proof of Theorem 1.5.** By theorem 4.5, we summarize our results on limit distributions for higher rank actions.

**Theorem 4.6.** *Let  $(\mathbf{T}_n)$  be any sequence such that*

$$\lim_{n \rightarrow \infty} r_{\log \mathbf{T}_n}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

*For every closed form  $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$  with  $s > s_d + 1/2$ , which is not a coboundary, the limit distribution of the family of random variables*

$$E_{\mathbf{T}_n}(f) := \frac{1}{\text{vol}(U(\mathbf{T}_n))^{1/2}} \left\langle P_{U(\mathbf{T}_n)}^{d, \alpha}(\cdot), \omega_f \right\rangle$$

*exists and is equal to the distribution of the function  $\theta^f(\alpha_\infty) = \beta(\alpha, \cdot, 1) \in L^2(M)$ . If  $\alpha_\infty \in DC$ , then  $\theta^f(\alpha_\infty)$  is bounded function on  $M$ , and the limit distribution has compact support.*

*Proof of theorem 1.5.* Since  $\alpha_\infty \in \mathfrak{M}_g$ , the existence of limit follows from Corollary 4.4 and Lemma 4.5.  $\square$

A relation with Birkhoff sum and theta sum was introduced in [CF15, §5.3], and as an applications, we derive limit theorem of theta sums.

**Corollary 4.7.** *Let  $\mathcal{Q}[x] = x^\top \mathcal{Q} x$  be the quadratic forms defined by  $g \times g$  real matrix  $\mathcal{Q}$ ,  $\alpha = \begin{pmatrix} I & 0 \\ \mathcal{Q} & I \end{pmatrix} \in Sp_{2g}(\mathbb{R})$ ,  $\ell(x) = \ell^\top x$  be the linear form defined by  $\ell \in \mathbb{R}^g$ . Then, Theta sum*

$$\Theta(\mathcal{Q}, \ell; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0, N]} e(\mathcal{Q}[n] + \ell(n))$$

*has limit distribution and it has compact support.*

## 5. $L^2$ -LOWER BOUNDS

In this section we prove bounds for the square mean of ergodic integrals along the leaves of foliations of the torus into circles transverse to central direction.

**5.1. Structure of return map.** Let  $\mathbb{T}_\Gamma^{g+1}$  denote  $(g+1)$ -dimensional torus with standard frame  $(X_i, Y_i, Z)$  with

$$\mathbb{T}_\Gamma^{g+1} := \{\Gamma \exp(\sum_{i=1}^g y_i Y_i + z Z) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

It is convenient to work with the polarized Heisenberg group. Set  $H_{pol}^g \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$  equipped with the group law  $(x, y, z) \cdot (x', y', z') = (x+x', y+y', z+z'+yx')$ .

**Definition 5.1.** Reduced standard Heisenberg group  $H_{red}^g$  is defined by quotient  $H_{pol}^g / (\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R} / \frac{1}{2}\mathbb{Z}$ . Reduced standard lattice  $\Gamma_{red}^g$  is  $\mathbb{Z}^g \times \mathbb{Z}^g \times \{0\}$  and the quotient  $H_{red}^g / \Gamma_{red}^g$  is isomorphic to standard Heisenberg manifold  $H^g / \Gamma$ .

Now, we consider return map of  $\mathbf{P}^{d,\alpha}$  on  $\mathbb{T}_\Gamma^{g+1}$ . For  $x = (x_1, \dots, x_g) \in \mathbb{R}^g$ ,

$$\exp(x_1 X_1^\alpha + \dots + x_g X_g^\alpha) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta \in \mathbb{R}^d.$$

In  $H_{red}^g$ ,

$$\exp(x_1 X_1^\alpha + \dots + x_g X_g^\alpha) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x).$$

Then, given  $(n, m, 0) \in \Gamma_{red}^g$ ,

$$(38) \quad \exp(x_1 X_1^\alpha + \dots + x_g X_g^\alpha) \cdot (0, y, z) \cdot (n, m, 0) = \exp(x'_1 X_1^\alpha + \dots + x'_g X_g^\alpha) \cdot (0, y', z')$$

if and only if  $x'_\alpha = x_\alpha + n$ ,  $y' = y + (x_\beta - x'_\beta) + m$  and  $z' = z + (w - w') \cdot x + n(y + x_\beta)$ .

Assume  $\langle X_i^\alpha, X_j \rangle \neq 0$  for all  $i, j$ , and we write first return time  $t_{Ret} = (t_{Ret,1}, \dots, t_{Ret,g})$  for  $\mathbf{P}^{d,\alpha}$  on transverse torus  $\mathbb{T}_\Gamma^{g+1}$ . We denote domain for return time  $U(t_{Ret}) = [0, t_{Ret,1}] \times \dots \times [0, t_{Ret,g}]$ . Return map of action  $\mathbf{P}^{d,\alpha}$  on  $\mathbb{T}_\Gamma^{g+1}$  has a form of skew-shift

$$(39) \quad A_{\rho,\tau}(y, z) = (y + \rho, z + v \cdot y + \tau_i) \quad \text{on } \mathbb{R}^g / \mathbb{Z}^g \times \mathbb{R} / K^{-1}\mathbb{Z}.$$

From computation of each rank 1 action, for each  $1 \leq i \leq d$ , it is a composition of commuting linear skew-shift

$$(40) \quad A_{i,\rho,\tau}(y, z) = (y + \rho_i, z + v_i \cdot y + \tau_i) \quad \text{on } \mathbb{R}^g / \mathbb{Z}^g \times \mathbb{R} / K^{-1}\mathbb{Z}$$

for some constant  $\rho_i, \sigma_i \in \mathbb{R}^g$  and  $v_i \in \mathbb{R}^g$ . For each  $j \neq k$ ,

$$A_{j,\rho,\tau} \circ A_{k,\rho,\tau} = A_{k,\rho,\tau} \circ A_{j,\rho,\tau}.$$

Given pair  $(\mathbf{m}, n) \in \mathbb{Z}_{K|n|}^g \times \mathbb{Z}$ , let  $H_{(\mathbf{m},n)}$  denote the corresponding factor and  $C^\infty(H_{(\mathbf{m},n)})$  be subspace of smooth function on  $H_{(\mathbf{m},n)}$ . Denote  $\{e_{\mathbf{m},n} \mid (\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z}\}$  the basis of characters of  $\mathbb{T}_\Gamma^{g+1}$  and for all  $(y, z) \in \mathbb{T}^g \times \mathbb{T}$ ,

$$e_{\mathbf{m},n}(y, z) := \exp[2\pi i(\mathbf{m} \cdot y + nKz)].$$

For each  $A_{i,\rho,\sigma}$  and  $v_i = (v_{i1}, \dots, v_{id})$ , the orbit can be identified with the following dual orbit

$$\begin{aligned} \mathcal{O}_{A_i}(\mathbf{m}, n) &= \{(\mathbf{m} + (nj_i)v_i, n), j_i \in \mathbb{Z}\} \\ &= \{(m_1 + (nv_{i1})j_i, \dots, m_d + (nv_{id})j_i, n), j_i \in \mathbb{Z}\}. \end{aligned}$$

If  $n = 0$ , the orbit  $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$  of  $(\mathbf{m}, 0)$  is reduced to a single element. If  $n \neq 0$ , then the dual orbit  $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$  of  $(\mathbf{m}, n)$  for higher rank actions is described as follows:

$$\mathcal{O}_A(\mathbf{m}, n) = \{(m_k + n \sum_{i=1}^d (v_{ik}j_i), n)_{1 \leq k \leq d} : j = (j_1, \dots, j_d) \in \mathbb{Z}^d\}.$$

It follows that every A-orbit for rank  $\mathbb{R}^d$ -action (or  $A^j$ -orbit) can be labeled uniquely by a pair  $(\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \setminus \{0\}$  with  $\mathbf{m} = (m_1, \dots, m_g)$ . Thus, the subspace of functions with non-zero central character can be split as a direct sum of components  $H_{(\mathbf{m},n)}$  with  $\mathbf{m} \in \mathbb{Z}^g, n \in \mathbb{Z} \setminus \{0\}$ . Then,

$$L^2(\mathbb{T}_\Gamma^{g+1}) = \bigoplus_{\omega \in \mathcal{O}_A} H_\omega.$$

Now we proceed to the consideration of the higher cohomology problem which appears in the space of Fourier coefficients.

**5.2. Higher cohomology for  $\mathbb{Z}^d$ -action of skew-shifts.** We consider a  $\mathbb{Z}^d$  action of return map  $\mathcal{P}^{d,\alpha}$  on torus  $\mathbb{T}_\Gamma^{g+1}$ . By identification of cochain complex on torus, it is equivalent to consider the following cohomological equation for degree  $d$  form  $\omega$ ,

$$(41) \quad \omega = d\Omega \iff \varphi(x, t) = D\Phi(x, t), \quad x \in \mathbb{T}^g, t \in \mathbb{Z}^d.$$

We restrict our interest of  $d$ -cocycle  $\varphi : \mathbb{T}_\Gamma^{g+1} \times \mathbb{Z}^d \rightarrow \mathbb{R}$  with  $\Phi : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathbb{R}^d$ ,  $\Phi = (\Phi_1, \dots, \Phi_d)$  and  $D$  is coboundary operator  $D\Phi = \sum_{i=1}^d (-1)^{i+1} \Delta_i \Phi_i$  where  $\Delta_i \Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$ . The following proposition is the generalization to the argument of [KK95, Prop 2.2]. Let us denote  $A^j = A_{1,\rho,\tau}^{j_1} \circ \dots \circ A_{d,\rho,\tau}^{j_d}$ .

**Proposition 5.2.** *A cocycle  $\hat{\varphi}$  satisfies cohomological equation (41) if and only if  $\sum_{j \in \mathbb{Z}^d} \hat{\varphi}(\mathbf{m}, n) \circ A^j = 0$  for  $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$ .*

*Proof.* We consider dual equation

$$(42) \quad \hat{\varphi} = D\hat{\Phi}.$$

Let us denote the following notation:

$$(\delta_i \hat{\varphi})(m_1, \dots, m_d) = \delta(m_i) \hat{\varphi}(m_1, \dots, m_d), \text{ and } \delta(0) = 1, \text{ otherwise } 0.$$

$$\begin{aligned}
(\Sigma_i \hat{\varphi})(m_1, \dots, m_d) &= \sum_{j=-\infty}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \dots A_i^j \dots A_d^{m_d}) \\
(\Sigma_i^+ \hat{\varphi})(m_1, \dots, m_d) &= \sum_{j=m_i}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \dots A_i^j \dots A_d^{m_d}) \\
(\Sigma_i^- \hat{\varphi})(m_1, \dots, m_d) &= - \sum_{j=-\infty}^{m_i-1} \hat{\varphi} \circ (A_1^{m_1} \dots A_i^j \dots A_d^{m_d})
\end{aligned}$$

It is clear that  $\Sigma_i^- - \Sigma_i^+ = \Sigma_i$  and  $\Sigma_i^+ \hat{\varphi} = \Sigma_i^- \hat{\varphi}$  if and only if  $\Sigma_i \hat{\varphi} = 0$ .  
Note that

$$(43) \quad \Sigma_i^+ \Delta_i = \Sigma_i^- \Delta_i = id, \quad \Delta_i \Sigma_i^+ = \Delta_i \Sigma_i^- = id.$$

By direct calculation of Fourier coefficient,  $\Sigma_i(\hat{\varphi} - \delta_i \Sigma_i \hat{\varphi}) = 0$ . Let  $\hat{\Phi}_i(\hat{\varphi}) = \Sigma_i^-(\hat{\varphi} - \delta_i \Sigma_i \hat{\varphi})$ , then  $\hat{\Phi}_i(\hat{\varphi})$  vanishes at  $\infty$ . By (43),

$$\hat{\varphi} - \delta_i \Sigma_i \hat{\varphi} = \Delta_i \hat{\Phi}_i(\hat{\varphi}).$$

We can proceed this by induction.

$$\begin{aligned}
\hat{\varphi} - \Sigma_{1,\dots,d} \hat{\varphi} &= \sum_{i=1}^d (\delta_1 \dots \delta_{i-1} \Sigma_1 \dots \Sigma_{i-1} \hat{\varphi} - \delta_1 \dots \delta_i \Sigma_1 \dots \Sigma_i \hat{\varphi}) \\
&= \sum_{i=1}^d (\delta_1 \dots \delta_{i-1} \Sigma_1 \dots \Sigma_{i-1} \hat{\varphi} - \delta_i \Sigma_i (\Sigma_1 \dots \Sigma_{i-1} \hat{\varphi})) \\
&= \sum_{i=1}^d (-1)^{i+1} \Delta_i \hat{\Phi}_i(\hat{\varphi})
\end{aligned}$$

where

$$\hat{\Phi}_i(\hat{\varphi}) = (-1)^{i+1} \Sigma_i^- \delta_1 \dots \delta_{i-1} (\Sigma_1 \dots \Sigma_{i-1} \hat{\varphi} - \delta_i \Sigma_i (\Sigma_1 \dots \Sigma_{i-1} \hat{\varphi}))$$

and  $\hat{\Phi}_i(\hat{\varphi})$  vanishes at  $\infty$ . Thus,  $\hat{\Phi}_i$  is a solution of (42) if and only if  $\Sigma_1 \dots \Sigma_d \hat{\varphi} = 0$ .  $\square$

For fixed  $(\mathbf{m}, n) \in \mathbb{Z}^g \times \mathbb{Z}$ , we denote obstruction of cohomological equation restricted to the orbit of  $(\mathbf{m}, n)$  by  $\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j$ .

**Lemma 5.3.** *There exists a distributional obstruction to the existence of a smooth solution  $\varphi \in C^\infty(H_{(\mathbf{m},n)})$  of the cohomological equation (41).*

*A generator of the space of invariant distribution  $\mathcal{D}_{\mathbf{m},n}$  has form of*

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) := \begin{cases} e^{-2\pi i \sum_{i=1}^d [(\mathbf{m} \cdot \rho_i + n K \tau_i) j_i + n K \tau_i (\frac{j_i}{2})]} & \text{if } (a,b) = (m_k + K \sum_{i=1}^d (v_{ik} j_i), n)_{1 \leq k \leq g} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* From previous observation, there exists obstruction

$$(44) \quad \mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}_\Gamma^{g+1}} \varphi(x, y) \overline{e_{\mathbf{m},n} \circ A_{\rho,\tau}^j} dx dy.$$

By direct computation, for fixed  $j = (j_1, \dots, j_d)$ ,

$$e_{\mathbf{m},n} \circ A_{\rho,\tau}^j(y, z) = \prod_{i=1}^d \left( e^{2\pi i [(\mathbf{m} \cdot \rho_i + nK\tau_i)j_i + nK\tau_i \binom{j_i}{2}]} \right) \left( e^{2\pi i (\mathbf{m} \cdot y + K(z + n \sum_{k=1}^d (v_{ik}j_i)y_k))} \right).$$

Then, we choose  $\varphi = e_{a,b}$  for  $(a, b) = (m_k + K \sum_{i=1}^d (v_{ik}j_i), n)_{1 \leq k \leq g}$  in the non-trivial orbit ( $n \neq 0$ ),

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi i \sum_{i=1}^d [(\mathbf{m} \cdot \rho_i + nK\tau_i)j_i + nK\tau_i \binom{j_i}{2}]}.$$

□

**5.3. Changes of coordinates.** For any frame  $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$ , denote transverse cylinder for any  $m \in M$ ,

$$\mathcal{C}_{\alpha,m} := \{m \exp(\sum_{i=1}^g y'_i Y_i^\alpha + z'Z) \mid (y', z') \in U(t_{Ret}^{-1}) \times \mathbb{T}\}.$$

Let  $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathcal{C}_{\alpha,m}$  denote the maps: for any  $\xi \in \mathbb{T}_\Gamma^{g+1}$ , let  $\xi' \in \mathcal{C}_{\alpha,m}$  denote first intersection of the orbit  $\{\mathbf{P}_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}_+^d\}$  with transverse cylinder  $\mathcal{C}_{\alpha,m}$ . Then, there exists first return time to cylinder  $t(\xi) = (t_1(\xi), \dots, t_d(\xi)) \in \mathbb{R}_+^d$  such that

$$\xi' = \Phi_{\alpha,m}(\xi) = \mathbf{P}_{t(\xi)}^{d,\alpha}(\xi), \quad \forall \xi \in \mathbb{T}_\Gamma^{g+1}.$$

Let  $(y, z)$  and  $(y', z')$  denote the coordinates on  $\mathbb{T}_\Gamma^{g+1}$  and  $\mathcal{C}_{\alpha,m}$  given by the exponential map respectively,

$$(y, z) \rightarrow \xi_{y,z} := \Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ), \quad (y', z') \rightarrow m \exp(\sum_{i=1}^g y'_i Y_i^\alpha + z'Z).$$

Recall that if  $\alpha \in Sp_{2g}(\mathbb{R})$ , then for  $1 \leq i, j \leq g$  there exist matrices  $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$  such that

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2g}(\mathbb{R}),$$

satisfying  $A^t D - C^t B = I_{2g}$ ,  $C^t A = A^t C$ ,  $D^t B = B^t D$ , and  $\det(A) \neq 0$ . Set

$$X_i^\alpha = \sum_j a_{ij} X_j + b_{ij} Y_j + w_i Z \quad \text{and} \quad Y_i^\alpha = \sum_j c_{ij} X_j + d_{ij} Y_j + v_i Z.$$

Let  $x = \Gamma \exp(\sum_{i=1}^d y_{x,i} Y_i + z_x Z) \exp(\sum_{i=1}^d t_{x,i} X_i)$ , for some  $(y_x, z_x) \in \mathbb{T}^d \times \mathbb{R}/K\mathbb{Z}$  and  $t_x = (t_{x,i}) \in [0, 1)^d$ . Then, the map  $\Phi_{\alpha,x} : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathcal{C}_{\alpha,m}$  is defined by  $\Phi_{\alpha,x}(y, z) = (y', z')$  where

$$(45) \quad \begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix},$$

and  $z' = z + P(\alpha, x, y)$  for some degree 4 polynomial  $P$ .

Therefore, the map  $\Phi_{\alpha,x}$  is invertible with

$$\Phi_{\alpha,x}^*(dy'_1 \wedge \cdots \wedge dy'_g \wedge dz') = \frac{1}{\det(A)} dy_1 \wedge \cdots \wedge dy_g \wedge dz.$$



Since  $A^t D - C^t B = I_{2g}$ , by direct computation we obtain return time, we have

$$(46) \quad \begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}.$$

Then,

$$\|t(\xi)\| \leq \max_i |t_i(\xi)|^g \leq \max_i \left| \sum_{j=1}^g d_{ij} t_{x,i} + c_{ij} (y_i - y_{x,i}) \right|^g \leq \max_i \|Y_i^\alpha\|^g.$$

**5.4.  $L^2$ -lower bound of functional.** We will prove bounds for the square mean of integrals along foliations of the torus  $\mathbb{T}_\Gamma^{g+1}$  into torus  $\{\xi \exp(\sum_{i=1}^g y_i Y_i) \mid y_i \in \mathbb{T}\}_{\xi \in \mathbb{T}_\Gamma^{g+1}}$ .

**Lemma 5.4.** *There exists a constant  $C > 0$  such that for all  $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$ , and for every irreducible component  $H$  of central parameter  $n \neq 0$ , there exists a function  $f_H$  such that*

$$|f_H|_{L^\infty(H)} \leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)|,$$

$$|f_H|_{\alpha,s} \leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)| (1 + \frac{T(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|)^s (1 + n^2)^{s/2}$$

where  $\|Y\| := \max_{1 \leq i \leq g} \|Y_i^\alpha\|$  and  $T(t_{Ret}) = \sum_{i=1}^g t_{Ret,i}$ .

On rectangular domain  $U(\mathbf{T})$ , for all  $m \in \mathbb{T}_\Gamma^{g+1}$  and  $T^{(i)} \in \mathbb{Z}_{t_{Ret,i}}$ ,

$$(47) \quad \left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = |\mathcal{D}_\alpha^H(f_H)| \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2}.$$

In addition, whenever  $H \perp H' \subset L^2(M)$  the functions

$$\left\langle P_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} m), \omega_H \right\rangle \quad \text{and} \quad \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} m), \omega_{H'} \right\rangle$$

are orthogonal in  $L^2(\mathbb{T}^g, dy)$ .

*Proof.* As explained in §5.1, the space  $L^2(\mathbb{T}_\Gamma^{d+1})$  decompose as a direct sum of irreducible subspaces invariant under the action of each  $A_{j,\rho,\sigma}$ . It follows that the subspace of functions with non-zero central character can be split as direct sum of components  $H_{(\mathbf{m},n)}$  with  $(\mathbf{m},n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \setminus \{0\}$  with  $\mathbf{m} = (m_1, \dots, m_g)$ . For  $F \in H_{(\mathbf{m},n)}$ , the function is characterized by Fourier expansion

$$F = \sum_{j \in \mathbb{Z}^d} F_j e^{A^j(\mathbf{m},n)} = \sum_{j \in \mathbb{Z}^d} F_j e^{(m_k + K \sum_{i=1}^d (v_{ik} j_i), n)}.$$

Then, by Lemma 5.3,

$$(48) \quad \mathcal{D}_{(\mathbf{m},n)}(e_{A^j(\mathbf{m},n)}) = e^{-2\pi i \sum_{i=1}^d [(m \cdot \rho_i + n K \tau_i) j_i + n K \tau_i \binom{j_i}{2}]}.$$

For any irreducible representation  $H := H_n$  with central parameter  $n \neq 0$ , there exists  $\mathbf{m} \in \mathbb{Z}_{|n|}^d$  such that the operator  $I_\alpha$  maps the space  $H$  onto  $H_{(\mathbf{m},n)}$ . The operator  $I_\alpha : L^2(M) \rightarrow L^2(\mathbb{T}_\Gamma^{g+1})$  is defined

$$(49) \quad f \rightarrow I_\alpha(f) := \int_{U(t_{Ret})} f \circ P_x^{d,\alpha}(\cdot) dx.$$

Then, operator  $I_\alpha$  is surjective linear map of  $L^2(M)$  onto  $L^2(\mathbb{T}_\Gamma^{g+1})$  with right inverse defined as follows:

Let  $\chi \in C_0^\infty(0, 1)^g$  be any function of jointly integrable with integral 1. For any  $F \in L^2(\mathbb{T}_\Gamma^{g+1})$ , let  $R_\alpha^\chi(F) \in L^2(M)$  be the function defined by

$$R_\alpha^\chi(F)(P_v^{d,\alpha}(m)) = \frac{1}{\text{vol}(U(t_{Ret}))} \chi\left(\frac{v}{t_{Ret}}\right) F(m), \quad (m, v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows from the definition that there exists a constant  $C_\chi > 0$  such that

$$\begin{aligned} |R_\alpha^\chi(F)|_{\alpha,s} &\leq C_\chi \text{vol}(U(t_{Ret}))^{-1} \left(1 + \sum_{i=1}^g t_{Ret,i}^{-1} \|Y_i^\alpha\|^s\right) \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})} \\ &\leq C_\chi \text{vol}(U(t_{Ret}))^{-1} \left(1 + \frac{T(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|^s\right) \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}. \end{aligned}$$

Choose  $f_H := R_\alpha^\chi(e_{m,n}) \in C^\infty(H)$  such that  $I_\alpha(f_H) = e_{m,n}$  and

$$(50) \quad \int_{U(t_{Ret})} f_H \circ P_t^{d,\alpha}(y, z) dt = e_{m,n}(y, z), \quad \text{for } (y, z) \in \mathbb{T}_\Gamma^{g+1}.$$

For P-action, the space of invariant currents  $I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g)) \subset W^{-s}(\mathbb{R}^g)$  for  $s > d/2 + \epsilon$  for all  $\epsilon > 0$ . That is, by normalization of invariant distributions in the Sobolev space, for any irreducible components  $H = H_n$ , there exists a non-unique function  $f_\alpha^H$  such that

$$(51) \quad D_\alpha(f_\alpha^H) = \|f_\alpha^H\|_s = 1.$$

By (48), we have  $|\mathcal{D}_H(f_H)| = |\mathcal{D}_{(m,n)}(e_{m,n})| = 1$ . Therefore, it follows that

$$|f_H|_{L^\infty(H)} \leq C_\chi \text{vol}(U(t_{Ret}))^{-1}$$

$$|f_H|_{\alpha,s} \leq C \text{vol}(U(t_{Ret}))^{-1} |D_\alpha^H(f_H)| \left(1 + \frac{T(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|^s (1+n^2)^{s/2}\right).$$

Moreover, since  $\{e_{m,n} \circ A_{\rho,\tau}^j\}_{j \in \mathbb{Z}^d} \subset L^2(\mathbb{T}_\Gamma^g, dy)$  is orthonormal, we verify

$$\begin{aligned} \left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} &= \left\| \sum_{jd=0}^{[\frac{T(d)}{t_{Ret,d}}]-1} \cdots \sum_{j_1=0}^{[\frac{T(1)}{t_{Ret,1}}]-1} e_{m,n} \circ A_{\rho,\tau}^j \right\|_{L^2(\mathbb{T}^g, dy)} \\ &= \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2}. \end{aligned}$$

□

Recall that

$$(52) \quad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n := \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i=1}^{\mu(n)} H_{i,n}$$

where  $H_n = \bigoplus_{i=1}^{\mu(n)} H_{i,n}$  is irreducible representation with central parameter  $n$  and  $\mu(n)$  is countable by Howe-Richardson multiplicity formula.

For any infinite dimensional vector  $\mathbf{c} := (c_{i,n}) \in l^2$ , let  $\beta_{\mathbf{c}}$  denote Bufetov functional

$$\beta_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} \beta^{i,n}.$$

For any  $\mathbf{c} := (c_{i,n})$ , let  $|\mathbf{c}|_s$  denote the norm defined as

$$|\mathbf{c}|_s^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{i=1}^{\mu(n)} (1 + K^2 n^2)^s |c_{i,n}|^2.$$

From Corollary 4.3,

$$\|\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T})\|_{L^2(M)}^2 \leq C^2 |\mathbf{c}|_{l^2}^2 \text{vol}(U(\mathbf{T})).$$

**Lemma 5.5.** *For any  $s > s_d + 1/2$ , there exists a constant  $C_s > 0$  such that for all  $\alpha \in DC(L)$ , for all  $\mathbf{c} \in l^2$ , for all  $z \in \mathbb{T}$  and all  $T > 0$ ,*

$$(53) \quad \left| \|\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T})\|_{L^2(\mathbb{T}^g, dy)} - \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \\ \leq C_s (\text{vol}(U(t_{Ret})) + \text{vol}(U(t_{Ret}))^{-1}) (1 + L) \left( 1 + \frac{T(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|^s \right) |\mathbf{c}|_s.$$

*Proof.* By (52), for every  $n \neq 0$ , there exists a function  $f_{i,n} \in C^\infty(H)$  with  $D(f_{i,n}) = 1$ . Let  $f_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n}$ , by adding functions on all irreducibles. Then, by Lemma 5.4,

$$(54) \quad \|f_{\mathbf{c}}\|_{L^\infty(M)} \leq C |\mathbf{c}|_{l^1}$$

$$(55) \quad \|f_{\mathbf{c}}\|_{\alpha,s} \leq C \text{vol}(U(t_{Ret}))^{-1} \left( 1 + \frac{T(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|^s \right) |\mathbf{c}|_s.$$

By orthogonality,

$$\left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha} \circ \mathbb{Q}_y^{g,Y}, \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0.$$

From the estimation for each  $f_{i,n}$  in Lemma 5.4, for every  $z \in \mathbb{T}$  and all  $T > 0$ , we have

$$\left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq 2 \|f_{\mathbf{c}}\|_{L^\infty(M)} \|Y\|.$$

Let  $T_{\alpha,i} = t_{Ret,i}([T/t_{Ret,i}] + 1)$  and  $U(t_{\alpha}) = [0, T_{\alpha,1}] \times \cdots \times [0, T_{\alpha,g}]$ . Then,

$$\left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle P_{U(t_{\alpha})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq \text{vol}(U(t_{Ret})) \|f_{\mathbf{c}}\|_{L^\infty(M)}.$$

Therefore, for some constant  $C' > 0$  such that

$$\left| \left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} - \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \leq C' \text{vol}(U(t_{Ret})) |\mathbf{c}|_{l^1}.$$

For all  $s > s_d + 1/2$ , by asymptotic property of Theorem 3.8, there exists constant  $C_s > 0$  such that

$$\left| \left\langle P_{U(\mathbf{T})}^{d,\alpha} m, \omega \right\rangle - \beta_H(\alpha, m, \mathbf{T}) D_{\alpha}^H(f_H) \right| \leq C_s (1 + L) \|f\|_{\alpha,s}.$$

Applying  $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$  and combining bounds on the function  $f_{\mathbf{c}}$  with (54),

$$\begin{aligned} & \left| \left\| \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}) \right\|_{L^2(\mathbb{T}^g, dy)} - \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{\text{Ret}}))} \right)^{1/2} |\mathbf{c}|_0 \right| \\ & \leq C' \text{vol}(U(t_{\text{Ret}})) |\mathbf{c}|_{l^1} + C_s \text{vol}(U(t_{\text{Ret}}))^{-1} (1+L) |f_{\mathbf{c}}|_{\alpha, s} \\ & \leq C'_s (\text{vol}(U(t_{\text{Ret}})) + \text{vol}(U(t_{\text{Ret}}))^{-1}) (1+L) \left( 1 + \frac{T(t_{\text{Ret}})}{\text{vol}(U(t_{\text{Ret}}))} \|Y\|^s \right) |\mathbf{c}|_s. \end{aligned}$$

Therefore, we derive the estimates in the statement.  $\square$

## 6. ANALYTICITY OF FUNCTIONALS

In this section we will prove that for all  $\alpha \in DC$ , the Bufetov functionals on any square are real analytic.

**Definition 6.1.** For every  $t \in \mathbb{R}$ ,  $1 \leq i \leq d$ , and  $m \in M$ , the *stretched (in direction of  $Z$ ) rectangle* is denoted by

$$(56) \quad [\Gamma_T]_{i,t}^Z(m) := \{(\phi_{ts_i}^Z) \circ \mathbf{P}_{\mathbf{s}}^{d, \alpha}(m) \mid \mathbf{s} \in U(\mathbf{T})\}.$$

Recall definition 1.1 for standard rectangles. For  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}^d$ , let us denote  $\Gamma_{\mathbf{T}}(\mathbf{s}) := (\gamma_1(s_1), \dots, \gamma_d(s_d))$  for  $\gamma_i(s_i) = \exp(s_i X_i)$ . Similarly, we also write

$$(57) \quad [\Gamma_T]_{i,t}^Z(\mathbf{s}) := (\gamma_1(s_1), \dots, \gamma_{i,t}^Z(s_i), \dots, \gamma_d(s_d))$$

where  $\gamma_{i,t}^Z(s_i) := \phi_{ts_i}^Z(\gamma_i(s_i))$  is a stretched curve.

**Definition 6.2.** The *restriction*  $\Gamma_{T,i,s}$  of the rectangle  $\Gamma_{\mathbf{T}}$  is defined on restricted domain  $U_{T,i,s} = [0, T^{(1)}] \times \dots \times [0, s] \times \dots \times [0, T^{(d)}]$  for  $s \leq T^{(i)}$  as following.

$$\Gamma_{T,i,s}(\mathbf{s}) := \Gamma_{\mathbf{T}}(\mathbf{s}), \quad \mathbf{s} \in U_{T,i,s}.$$

Recall the orthogonal property on a irreducible component  $H$  (central parameter  $n \in \mathbb{Z} \setminus \{0\}$ ). For any  $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$  and  $t \in \mathbb{R}$ ,

$$(58) \quad \beta_H(\alpha, \phi_t^Z(m), \mathbf{T}) = e^{2\pi i K n t} \beta_H(\alpha, m, \mathbf{T}).$$

We obtain the following lemma for stretched rectangle by applying orthogonal property.

**Lemma 6.3.** For fixed elements  $(X_i, Y_i, Z)$  satisfying commutation relation (1), the following formula for rank 1 action holds:

$$\hat{\beta}_H(\alpha, [\Gamma_T]_{i,t}^Z) = e^{2\pi i n K T^{(i)}} \hat{\beta}_H(\alpha, \Gamma_T) - 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \hat{\beta}_H(\alpha, \Gamma_{T,i,s}) ds_i.$$

*Proof.* Let  $\alpha = (X_i, Y_i, Z)$  and  $\omega$  be  $d$ -form supported on a single irreducible representation  $H$ . We obtain following the formula for stretches of curve  $\gamma_{i,t}^Z$  (see [FK20b, §4, Lemma 9.1]),

$$\frac{d\gamma_{i,t}^Z}{ds_i} = D\phi_{ts_i}^Z \left( \frac{d\gamma_i}{ds_i} \right) + tZ \circ \gamma_{i,t}^Z.$$

It follows that pairing is given by

$$\begin{aligned}\langle [\Gamma_{\mathbf{T}}]_{i,t}^Z, \omega \rangle &= \int_{U(\mathbf{T})} \omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_{i,t}^Z}{ds_i}(s_i), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) d\mathbf{s} \\ &= \int_{U(\mathbf{T})} e^{2\pi i n K t s_i} \left[ \omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] + \iota_Z \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z(\mathbf{s}) d\mathbf{s}\end{aligned}$$

Denote  $d-1$  dimensional triangle  $U_{d-1}(\mathbf{T})$  with  $U(\mathbf{T}) = U_{d-1}(\mathbf{T}) \times [0, T^{(i)}]$ . Integration by parts for a fixed  $i$ -th integral gives

$$\begin{aligned}& \int_{U(\mathbf{T})} e^{2\pi i n K t s_i} \left[ \omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] d\mathbf{s} \\ &= e^{2\pi i n K t T^{(i)}} \int_{U(\mathbf{T})} \left[ \omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] d\mathbf{s} \\ &\quad - 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \int_{U_{d-1}(\mathbf{T})} \left( \int_0^{s_i} \left[ \omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_i}{ds_i}(r), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) \right] dr \right) d\mathbf{s}.\end{aligned}$$

Then, we have the following formula

$$\begin{aligned}\langle [\Gamma_{\mathbf{T}}]_{i,t}^Z, \omega \rangle &= e^{2\pi i n K t T^{(i)}} \langle [\Gamma_{\mathbf{T}}], \omega \rangle - 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \langle \Gamma_{T,i,s}, \omega \rangle ds_i \\ &\quad + \int_{U(\mathbf{T})} (\iota_Z \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z)(\mathbf{s}) d\mathbf{s}.\end{aligned}$$

Since the action of  $\mathbf{P}_{\mathbf{t}}^{d,X}$  for  $\mathbf{t} \in \mathbb{R}^d$  is identity on the center  $Z$ ,

$$\lim_{t_d \rightarrow \infty} \dots \lim_{t_1 \rightarrow \infty} e^{-(t_1 + \dots + t_d)/2} \int_{U(\mathbf{T})} (\iota_Z(\mathbf{P}_{\mathbf{t}}^{d,X})^* \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z)(\mathbf{s}) d\mathbf{s} = 0.$$

Thus, it follows by definition of Bufetov functional, the statement holds.  $\square$

Here we define a restricted vector  $\mathbf{T}_{i,s}$  of  $\mathbf{T} = (T^{(1)}, \dots, T^{(d)}) \in \mathbb{R}^d$ . For fixed  $i$ , pick  $s_i \in [0, T^{(i)}]$  such that  $\mathbf{T}_{i,s} \in \mathbb{R}^d$  is a vector with its coordinates

$$T_{i,s}^{(j)} = \begin{cases} T^{(j)} & \text{if } j \neq i \\ s_i & \text{if } j = i. \end{cases}$$

Similarly,  $\mathbf{T}_{i_1, \dots, i_k, s}$  is a vector with  $i_1, \dots, i_k$  coordinates replaced by  $s_{i_1}, \dots, s_{i_k}$ .

**Lemma 6.4.** *Let  $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ . The following equality holds for each rank-1 action.*

$$(59) \quad \beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) = e^{-2\pi i y_i n K T^{(i)}} \beta_H(\alpha, m, \mathbf{T}) + 2\pi i n K y_i \int_0^{T^{(i)}} e^{-2\pi i y_i n K s_i} \beta_H(\alpha, m, \mathbf{T}_{i,s}) ds_i.$$

*Proof.* By definition (3), (57) and commutation relation (1), it follows that

$$\phi_{y_i}^{Y_i}(\Gamma_{\mathbf{T}}^X(m)) = [\Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))]_{i,t}^Z.$$

By the invariance property of Bufetov functional and Lemma 6.3,

$$\begin{aligned}
\beta_H(\alpha, m, \mathbf{T}) &= \hat{\beta}_H(\alpha, \phi_{y_i}^{Y_i}(\Gamma_T^X(m))) \\
&= e^{2\pi\iota y_i n K T^{(i)}} \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))) - 2\pi\iota n K y_i \int_0^{T^{(i)}} e^{2\pi\iota n K y_i s_i} \hat{\beta}_H(\alpha, \Gamma_{T, i, s}^X(\phi_{s_i}^{Y_i}(m))) ds_i \\
&= e^{2\pi\iota y_i n K T^{(i)}} \beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) - 2\pi\iota n K y_i \int_0^{T^{(i)}} e^{2\pi\iota n K y_i s_i} \beta_H(\alpha, \phi_{s_i}^{Y_i}(m), \mathbf{T}_{i, s}) ds_i.
\end{aligned}$$

Then statement follows immediately.  $\square$

We extend previous lemma to higher rank actions by induction argument.

**Lemma 6.5.** *The following equality holds for rank- $d$  action.*

$$\begin{aligned}
\beta_H(\alpha, \mathbf{Q}_y^{d, Y}(m), \mathbf{T}) &= e^{-2\pi\iota \sum_{j=1}^d y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \\
&+ \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\
&\times \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1}
\end{aligned}$$

*Proof.* Assume inductive hypothesis works for rank  $d-1$ . For convenience, we write

$$\mathbf{Q}_y^{d, Y}(m) = \phi_{y_d}^{Y_d} \circ \mathbf{Q}_{y'}^{d-1, Y}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 6.4,

$$\begin{aligned}
\beta_H(\alpha, \mathbf{Q}_y^{d, Y}(m), \mathbf{T}) &= e^{-2\pi\iota y_d n K T^{(d)}} \beta_H(\alpha, \mathbf{Q}_{y'}^{d-1, Y}(m), \mathbf{T}) \\
(60) \quad &+ 2\pi\iota n K y_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d n K s_d} \beta_H(\alpha, \mathbf{Q}_{y'}^{d-1, Y}(m), \mathbf{T}_{d, s}) ds_d \\
&:= I + II
\end{aligned}$$

Firstly, by induction hypothesis

$$\begin{aligned}
I &= e^{-2\pi\iota \sum_{j=1}^{d-1} y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \\
&+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)} + y_d T^{(d)})} \\
&\times \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1}
\end{aligned}$$

which contains 0 to  $d-1$ th iterated integrals containing  $e^{-2\pi\iota n K y_d T^{(d)}}$  outside of iterated integrals.

For the second part, we apply induction hypothesis again for restricted rectangle  $\mathbf{T}_{d,s}$ . Then,

$$\begin{aligned} II &= 2\pi\iota n K y_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d n K s_d} [e^{-2\pi\iota \sum_{j=1}^{d-1} y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}_{d,s})] ds_d \\ &+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} (2\pi\iota n K y_d) \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\ &\times \int_0^{T^{(d)}} \left( \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} ds_{i_k} \dots ds_{i_1} \right) ds_d \\ &\times e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k} + y_d s_d)} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, d, s}). \end{aligned}$$

The term  $II$  consist of 1 to  $d$ -th iterated integrals containing  $e^{-2\pi\iota n K y_d s_d}$  inside of iterated integrals. Thus, combining these two terms, we prove the statement.  $\square$

For any  $R > 0$ , the *analytic norm* defined for all  $\mathbf{c} \in l^2$  as

$$\|\mathbf{c}\|_{\omega, R} = \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}|.$$

Let  $\Omega_R$  denote the subspace of  $\mathbf{c} \in \ell^2$  such that  $\|\mathbf{c}\|_{\omega, R}$  is finite.

**Lemma 6.6.** *For  $\mathbf{c} \in \Omega_R$ , any  $\alpha \in DC(L)$  and  $\mathbf{T} \in \mathbb{R}_+^d$ , the function*

$$\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}), \quad (y, z) \in \mathbb{R}^d \times \mathbb{T}$$

*extends to a holomorphic function in the domain*

$$(61) \quad D_{R,T} := \{(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \mid \sum_{i=1}^d |\operatorname{Im}(y_i)| T^{(i)} + |\operatorname{Im}(z)| < \frac{R}{2\pi K}\}.$$

*The following bound holds: for any  $R' < R$  there exists a constant  $C > 0$  such that, for all  $(y, z) \in D_{R',T}$  we have*

$$\begin{aligned} &|\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})| \\ &\leq C_{R,R'} \|\mathbf{c}\|_{\omega, R} (L + \operatorname{vol}(U(\mathbf{T})))^{1/2} (1 + E_M(a, \mathbf{T})) (1 + K \sum_{i=1}^d |\operatorname{Im}(y_i)| T^{(i)}). \end{aligned}$$

*Proof.* By Lemma 6.5 and (58),

$$\begin{aligned} \beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}) &= e^{(z - 2\pi\iota \sum_{j=1}^d y_j n K T^{(j)})} \beta_H(\alpha, m, \mathbf{T}) \\ &+ \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\ &\times e^{2\pi\iota n K z} \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1}. \end{aligned}$$

As a consequence, by Lemma 3.7 for each variable  $(y_i, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$ , Then for the rank  $d$  action, by induction, for  $(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$  we have

$$\begin{aligned} & |\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(x), \mathbf{T})| \\ & \leq (L + \text{vol}(U(\mathbf{T}))^{1/2}(1 + E_M(a, \mathbf{T}))[C_1 \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}| e^{2\pi |Im(z - \sum_{i=1}^d T^{(i)} y_i)| nK} \\ & + \sum_{k=1}^d C_k \left( \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (|Im(y_{i_j})| T^{(i_j)}) \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} n |c_{i,n}| e^{2\pi (|Im(z)| + \sum_{j=1}^k T^{(i_j)} |Im(y_{i_j})|) nK} \right)]. \end{aligned}$$

Therefore, the function  $\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})$  is bounded by a series of holomorphic functions on  $\mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$  and it converges uniformly on compact subsets of domain  $D_{R,T}$ .  $\square$

## 7. MEASURE ESTIMATION FOR BOUNDED-TYPE

In this section, we prove measure estimation of Bufetov functional under bounded-type case (14). This result is generalization of §11 of [FK20b].

Let  $\mathcal{O}_r$  denote the space of holomorphic functions on the ball  $B_{\mathbb{C}}(0, r) \subset \mathbb{C}^n$ . We recall the *Chebyshev degree*, the best constant  $d_f(r)$  stated in the following theorem and estimation of *valency*.

**Theorem 7.1.** [Bru99, Thm 1.9] *For any  $f \in \mathcal{O}_r$ , there is a constant  $d := d_f(r) > 0$  such that for any convex set  $D \subset B_{\mathbb{R}}(0, 1) := B_{\mathbb{C}}(0, 1) \cap \mathbb{R}^n$ , for any measurable subset  $U \subset D$*

$$\sup_D |f| \leq \left( \frac{4n \text{Leb}(D)}{\text{Leb}(U)} \right)^d \sup_U |f|.$$

Let  $\mathcal{L}_t$  denote the set of one-dimensional complex affine spaces  $L \subset \mathbb{C}^n$  such that  $L \cap B_{\mathbb{C}}(0, t) \neq \emptyset$ .

**Definition 7.2.** [Bru99, Def 1.6] Let  $f \in \mathcal{O}_r$ . The number

$$\nu_f(t) := \sup \{ \text{valency of } f \mid L \cap B_{\mathbb{C}}(0, t) \neq \emptyset \}$$

is called the *valency* of  $f$  in  $B_{\mathbb{C}}(0, t)$ .

By [Bru99, Prop 1.7], for any  $f \in \mathcal{O}_r$ , and the valency  $\nu_f(t)$  is finite and any  $t \in [1, r)$  there is a constant  $c := c(r) > 0$  such that

$$(62) \quad d_f(r) \leq c \nu_f\left(\frac{1+r}{2}\right).$$

**Lemma 7.3.** *Let  $L > 0$  and  $\mathcal{B} \subset DC(L)$  be a bounded subset. Given  $R > 0$ , for all  $c \in \Omega_R$  and all  $\mathbf{T}^{(i)} > 0$ , let  $\mathcal{F}(c, \mathbf{T})$  denote the family of real analytic functions of the variable  $y \in [0, 1)^d$  defined as*

$$\mathcal{F}(c, \mathbf{T}) := \{ \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}) \mid (\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T} \}.$$

*There exists  $\mathbf{T}_{\mathcal{B}} := (\mathbf{T}_{\mathcal{B}}^{(i)})$  and  $\rho_{\mathcal{B}} > 0$ , such that for every  $(R, \mathbf{T})$  such that  $R/\mathbf{T}^{(i)} \geq \rho_{\mathcal{B}}$  and  $\mathbf{T}^{(i)} \geq \mathbf{T}_{\mathcal{B}}^{(i)}$ , and for all  $c \in \Omega_R \setminus \{0\}$ , we have*

$$\sup_{f \in \mathcal{F}(c, \mathbf{T})} \nu_f < \infty.$$



*Proof.* Since  $\mathcal{B} \subset \mathfrak{M}$  is bounded, for each time  $t_i \in \mathbb{R}$  and  $1 \leq i \leq g$ ,

$$0 < t_{i,\mathcal{B}}^{\min} = \min_i \inf_{\alpha \in \mathcal{B}} t_{Ret,i,\alpha} \leq \max_i \sup_{\alpha \in \mathcal{B}} t_{Ret,i,\alpha} = t_{\mathcal{B}}^{\max} < \infty.$$

For any  $\alpha \in \mathcal{B}$  and  $x \in M$ , the map  $\Phi_{\alpha,x} : [0, 1]^d \times \mathbb{T} \rightarrow \prod_{i=1}^d [0, t_{\alpha,i}) \times \mathbb{T}$  in (45) extends to a complex analytic diffeomorphism  $\hat{\Phi}_{\alpha,x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ . By Lemma 6.6, it follows that for fixed  $z \in \mathbb{T}$ , real analytic function  $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T})$  extends to a holomorphic function defined on a region

$$H_{\alpha,m,R,t} := \{y \in \mathbb{C}^d \mid \sum_{i=1}^d |Im(y_i)| \leq h_{\alpha,m,R,t}\}.$$

By boundedness of the set  $\mathcal{B} \subset \mathfrak{M}$ , it follows that

$$\inf_{(\alpha,x) \in \mathcal{B} \times M} h_{\alpha,m,R,t} := h_{R,T} > 0.$$

We remark that the function  $h_{\alpha,m,R,t}$  and its lower bound  $h_{R,T}$  can be obtained from the formula (45) for the polynomial  $\Phi_{\alpha,x}$  and the definition of the domain  $D_{R,T}$  in formula (61).

For every  $r > 1$ , there exists  $\rho_{\mathcal{B}} > 1$  such that, for every  $R$  and  $\mathbf{T}$  with  $R/\mathbf{T}^{(i)} > \rho_{\mathcal{B}}$ , then as a function of  $y \in \mathbb{T}^d$

$$\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \in \mathcal{O}_r.$$

Then, by Lemma 6.6, the family  $\mathcal{F}(c, \mathbf{T})$  is uniformly bounded and normal. By Lemma 5.5, for sufficiently large pair  $\mathbf{T}$ , no sequence from  $\mathcal{F}(c, \mathbf{T})$  can converge to a constant. By Lemma 10.3 of [FK20b], for any normal family  $\mathcal{F} \subset \mathcal{O}_R$  such that no functions is constant along a one-dimensional complex line, hence the statement follows.  $\square$

We derive measure estimates of Bufetov functionals on the rectangular domain.

**Lemma 7.4.** *Let  $\alpha \in DC$  such that the forward orbit of  $\mathbb{R}^d$ -action  $\{r_t[\alpha]\}_{t \in \mathbb{R}_+^d}$  is contained in a compact set of  $M$ . There exist  $R, C, \delta > 0$  and  $T_0 \in \mathbb{R}_+^d$  such that, for every  $\mathbf{c} \in \Omega_R \setminus \{0\}$ ,  $T \geq T_0$  and for every  $\epsilon > 0$ , we have*

$$vol(\{m \in M \mid |\beta_{\mathbf{c}}(\alpha, m, \mathbf{T})| \leq \epsilon vol(U(\mathbf{T}))^{1/2}\}) \leq C\epsilon^\delta.$$

*Proof.* Since  $\alpha \in DC$  and the orbit  $\{r_t[\alpha]\}_{t \in \mathbb{R}_+^d}$  is contained in a compact set, there exists  $L > 0$  such that  $r_t(\alpha) \in DC(L)$  for all  $t \in \mathbb{R}_+^d$ . Then, we choose  $\mathbf{T}_0 \in \mathbb{R}^d$  from conclusion of Lemma 7.3. By scaling property of Bufetov functionals,

$$\beta_{\mathbf{c}}(\alpha, m, \mathbf{T}) = \left( \frac{vol(U(\mathbf{T}))}{vol(U(\mathbf{T}_0))} \right)^{1/2} \beta_{\mathbf{c}}(g_{\log(\mathbf{T}/\mathbf{T}_0)}[\alpha], m, \mathbf{T}_0).$$

By Fubini's theorem, it suffices to estimate

$$Leb(\{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}_0)| \leq \epsilon\}).$$

Let  $\delta^{-1} = c(r) \sup_{f \in \mathcal{F}(c, T_0)} v_f(\frac{1+r}{2}) < \infty$ . Since by Lemma 5.5, we have

$$\inf_{(a,x,z) \in \mathcal{B} \times M \times \mathbb{T}} \sup_{y \in [0,1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}_0)| > 0.$$

By theorem 7.1, for unit ball  $D = B_{\mathbb{R}}(0, 1)$  and  $U = \{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\}$  and bound in formula (62), there exists a constant  $C > 0$  and  $\delta > 0$  such that for all  $\epsilon > 0$  and  $(\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}$ ,

$$\text{Leb}(\{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\}) \leq C\epsilon^{\delta}.$$

Then the statement follows from Fubini theorem.  $\square$

**Corollary 7.5.** *Let  $\alpha$  be as in the previous Lemma 7.4. There exist  $R, C, \delta > 0$  and  $T_0 \in \mathbb{R}_+^d$  such that, for every  $\mathbf{c} \in \Omega_R \setminus \{0\}$ ,  $T \geq T_0$  and for every  $\epsilon > 0$ , we have*

$$\text{vol}(\{x \in M \mid |\langle P_{U(\mathbf{T})}^{d, \alpha} m, \omega_{\mathbf{c}} \rangle| \leq \epsilon \text{vol}(U(\mathbf{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

**Acknowledgement.** The author highly appreciates Giovanni Forni for discussions and various suggestions to improve the draft. He acknowledges Rodrigo Treviño and Corinna Ulcigrai for giving several comments. He also thanks Osama Khail and Jacky Jia Chong for helpful discussion. This work was initiated when the author visited the Institut de Mathématiques de Jussieu-Paris Rive Gauche in Paris, France. He acknowledges invitation and hospitality during the visit. This research was partially supported by the NSF grant DMS 1600687 and by the Centre of Excellence “Dynamics, mathematical analysis and artificial intelligence” at Nicolaus Copernicus University in Toruń.

## REFERENCES

- [AFRU21] Artur Avila, Giovanni Forni, Davide Ravotti, and Corinna Ulcigrai, *Mixing for smooth time-changes of general nilflows*, *Advances in Mathematics* **385** (2021), 107759.
- [AFU11] Artur Avila, Giovanni Forni, and Corinna Ulcigrai, *Mixing for the time-changes of heisenberg nilflows*, *Journal of Differential Geometry* **89** (2011), no. 3, 369–410.
- [BF14] Alexander Bufetov and Giovanni Forni, *Limit theorems for horocycle flows*, *Ann. Sci. Éc. Norm. Supér. (4)* **47** (2014), no. 5, 851–903. MR 3294619
- [Bru99] Alexander Brudnyi, *On local behavior of analytic functions*, *Journal of Functional Analysis* **2** (1999), no. 169, 481–493.
- [Buf10] Alexander I Bufetov, *Hölder cocycles and ergodic integrals for translation flows on flat surfaces*, *Electronic Research Announcements* **17** (2010), 34–42.
- [Buf14a] ———, *Finitely-additive measures on the asymptotic foliations of a markov compactum*, *Moscow Mathematical Journal* **14** (2014), no. 2, 205–224.
- [Buf14b] Alexander I. Bufetov, *Limit theorems for translation flows*, *Ann. of Math. (2)* **179** (2014), no. 2, 431–499. MR 3152940
- [CF15] Salvatore Cosentino and Livio Flaminio, *Equidistribution for higher-rank Abelian actions on Heisenberg nilmanifolds*, *J. Mod. Dyn.* **9** (2015), 305–353. MR 3426826
- [CM16] Francesco Cellarosi and Jens Marklof, *Quadratic weyl sums, automorphic functions and invariance principles*, *Proceedings of the London Mathematical Society* **113** (2016), no. 6, 775–828.
- [FF03] Livio Flaminio and Giovanni Forni, *Invariant distributions and time averages for horocycle flows*, *Duke Math. J.* **119** (2003), no. 3, 465–526. MR 2003124
- [FF06] ———, *Equidistribution of nilflows and applications to theta sums*, *Ergodic Theory Dynam. Systems* **26** (2006), no. 2, 409–433. MR 2218767
- [FF14] ———, *On effective equidistribution for higher step nilflows*, *arXiv preprint arXiv:1407.3640* (2014).
- [FFT16] Livio Flaminio, Giovanni Forni, and James Tanis, *Effective equidistribution of twisted horocycle flows and horocycle maps*, *Geom. Funct. Anal.* **26** (2016), no. 5, 1359–1448. MR 3568034
- [FK20a] Giovanni Forni and Adam Kanigowski, *Multiple mixing and disjointness for time changes of bounded-type heisenberg nilflows*, *Journal de l’École polytechnique—Mathématiques* **7** (2020), 63–91.
- [FK20b] ———, *Time-changes of heisenberg nilflows*, *Asterisque* (2020), no. 416, 253–299.

- [For02] Giovanni Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Annals of Mathematics **155** (2002), no. 1, 1–103.
- [For20] ———, *On the equidistribution of unstable curves for pseudo-anosov diffeomorphisms of compact surfaces*, arXiv preprint arXiv:2007.03144 (2020).
- [GG04] Friedrich Götze and Mikhail Gordin, *Limiting distributions of theta series on siegel half-spaces*, St. Petersburg Mathematical Journal **15** (2004), no. 1, 81–102.
- [GM14] JORY GRIFFIN and JENS MARKLOF, *Limit theorems for skew translations.*, Journal of Modern Dynamics **8** (2014), no. 2.
- [GS14] Alexander Gorodnik and Ralf Spatzier, *Exponential mixing of nilmanifold automorphisms*, Journal d'Analyse Mathématique **123** (2014), no. 1, 355–396.
- [GS15] ———, *Mixing properties of commuting nilmanifold automorphisms*, Acta Mathematica **215** (2015), no. 1, 127–159.
- [Kim20] Minsung Kim, *Effective equidistribution for generalized higher step nilflows*, arXiv preprint arXiv:2001.09789 (2020).
- [KK95] Anatole Katok and Svetlana Katok, *Higher cohomology for abelian groups of toral automorphisms*, Ergodic Theory and Dynamical Systems **15** (1995), no. 3, 569–592.
- [KM99] D. Y. Kleinbock and G. A. Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math. **138** (1999), no. 3, 451–494. MR 1719827
- [Mar99] Jens Marklof, *Limit theorems for theta sums*, Duke mathematical journal **97** (1999), no. 1, 127–154.
- [MM07] David Mumford and C Musili, *Tata lectures on theta. i (modern birkhäuser classics)*, Birkhäuser Boston Incorporated, 2007.
- [MNN07] David Mumford, Madhav Nori, and Peter Norman, *Tata lectures on theta iii*, vol. 43, Springer, 2007.
- [Rav18] Davide Ravotti, *Mixing for suspension flows over skew-translations and time-changes of quasi-abelian filiform nilflows*, Ergodic Theory and Dynamical Systems (2018), 1–30.
- [Rav19] ———, *Quantitative equidistribution of horocycle push-forwards of transverse arcs*, arXiv preprint arXiv:1910.03187 (2019).
- [Sul82] Dennis Sullivan, *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics*, Acta Math. **149** (1982), no. 3-4, 215–237. MR 688349
- [Tol78] R Tolimieri, *Heisenberg manifolds and theta functions*, Transactions of the American Mathematical Society **239** (1978), 293–319.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY,  
CHOPINA 12/18, 87-100 TORUŃ, POLAND  
Email address: minsungdream@gmail.com