LIMIT THEOREMS FOR HIGHER RANK ACTIONS ON HEISENBERG NILMANIFOLDS

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ABSTRACT. The main result of this paper is a construction of finitely additive measures for higher rank action on Heisenberg nilmanifold. Under a full measure set of Diophantine condition of the generators of the action, we construct *Bufetov functional* on rectangles on $2g + 1$ dimensional Heisenberg manifold. We prove the deviation of ergodic integral of higher rank actions by its asymptotics to Bufetov functionals for a sufficiently smooth function. In this paper, we derive limit distribution which proves normalized ergodic integrals to have variance 1 converges in distribution to a nondegenerate compactly supported measure on the real line.

CONTENTS

1. INTRODUCTION

1.1. **Introduction.** The asymptotic behavior and limiting distribution of ergodic averages of translation flow were studied by Alexander Bufetov in the series of works [Buf14a, Buf10, Buf14b]. He constructed finitely-additive Hölder measures and cocycles over translation flows that are known as *Bufetov functionals*. The construction of such functionals is used to derive deviation of ergodic averages and results on probabilistic behavior of ergodic averages of translation flows. Interestingly, it was discovered that there was a duality between the Bufetov functionals and invariant distributions which plays a key role in the work of G. Forni [For02]. Following these observations, the duality between such functionals and a space of invariant distributions, called Flaminio-Forni space, is observed analogously for other parabolic flows (see [BF14, FK20b, For20a]).

In this paper, our main results are on limit distributions of higher rank abelian actions. We firstly introduce the Bufetov functional for higher rank abelian actions under bounded type of Diophantine conditions.

This is a key idea used in Cosentino and Flaminio in [CF15], but we extend their constructions to rectangular shape and derive the deviation of ergodic averages for higher rank actions. Likewise, this explains the duality between Bufetov functionals and invariant currents appeared in [CF15]. The crucial part is handling the estimate of deviation ergodic averages on (stretched) rectangles, and this enables to derive our main theorems (see [Rav20, For20a]).

As a corollary, we prove there exists a limit distributions of (normalized) ergodic integrals of abelian actions with variance 1. More specifically, normalized ergodic integrals converge in distribution to a non-degenerate, compactly supported measure on the real line. This is certain form of Bufetov functionals.

This generalizes a limit theorem for theta series on Siegel half spaces, introduced by Götze and Gordin [GG04] and Marklof [Mar99]. (See [Tol78, MM07, MNN07] for general introduction and nilflow case [GM14, CM16].) In the last section, the analyticity of Bufetov functional on higher dimensional rectangular domain is proved and we obtain polynomial type of lower bounds of sub-level sets of analytic function. Such polynomial estimates on the measure are derived by real analyticity of a functional along the leaves of a foliation transverse to the actions based on results of [Bru99].

Mixing properties of time changes of Heisenberg nilflows were firstly studied in [AFU11]. Then, it was extended in the dense set of non-trivial time changes for any uniquely ergodic nilflows on general nilmanifolds [Rav18, AFRU21]. As a special example, on time-changes of Heisenberg flows, decay of correlation [FK20b] and multiple mixing [FK20a] was proved by analyticity of Bufetov functional and Ratner property, respectively. However, in the higher rank setting, time-changes of higher rank abelian actions on Heisenberg nilmanifold are all trivial (by triviality of first cohomology group, see [CF15, Theorem 3.16]). Thus, they are conjugated to the linear action and never mixing. In $\mathbb{Z}^k$-actions, mixing of shapes for automorphisms on nilmanifold is proved in [GS14, GS15].

1.2. **Definition and statement of results.** We review definitions about Heisenberg manifold and its moduli space.

1.2.1. *Heisenberg manifold.* Let H^g be the standard $2g + 1$ dimensional Heisenberg group and set $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$ a discrete and co-compact subgroup of H^g. We shall call it standard lattice of H^g and the quotient $M := \mathsf{H}^g/\Gamma$ will be called *Heisenberg manifold.* Lie algebra $\mathfrak{h}^g = Lie(\mathsf{H}^g)$ is equipped with a basis $(X_1, \cdots, X_g, Y_1, \cdots Y_g, Z)$ satisfying canonical commutation relations

(1) $[X_i, Y_j] = \delta_{ij} Z$,   where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.

For $1 \leq d \leq g$, let P^d be a subgroup of H^g where its Lie algebra $\mathfrak{p}_d = Lie(\mathsf{P}^d)$ is generated by $\{X_1, \cdots, X_d\}$. For any $\alpha \in Sp_{2g}(\mathbb{R})$, set $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$ for $1 \leq i \leq d$. We define parametrization of the subgroup $\mathsf{P}^{d,\alpha}$ by

$$\mathsf{P}_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \cdots + x_d X_d^\alpha), \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

By central extension of $\mathbb{R}^{2g}$ by $\mathbb{R}$, we have an exact sequence

$$0 \to Z(H^g) \to H^g \to \mathbb{R}^{2g} \to 0.$$

The natural projection map $pr : M \to H^g/(\Gamma Z(H^g))$ maps M onto a $2g$-dimensional torus $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

1.2.2. *Moduli space.* The group of automorphisms of H^g that are trivial on the center is $Aut_0(\mathsf{H}^g) = Sp_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$. Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to $Sp_{2g}(\mathbb{R})$. We call *moduli space* of the standard Heisenbeg manifold the quotient $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$. We regard $Sp_{2g}(\mathbb{R})$ as the deformation space of the standard Heisenberg manifold M and $\mathfrak{M}_g$ as the moduli space of M.

Siegel modular variety is a double coset space $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ where K_g is a maximal compact subgroup $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$ of $Sp_{2g}(\mathbb{R})$. For $\alpha \in Sp_{2g}(\mathbb{R})$, we denote $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$ by its projection on the moduli space $\mathfrak{M}_g$ and write $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$ the projection of α to the Siegel modular variety Σ_g.

Double coset $K_g \backslash Sp_{2g}(\mathbb{R})/1_{2g}$ is identifed to the Siegel upper half space $\mathfrak{H}_g := \{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$. *Siegel upper half space* of genus g is complex manifold of symmetric complex $g \times g$ matrices $Z = X + iY$ with positive definite symmetric imaginary part $\Im(Z) = Y$ and arbitrary real part X. Denote by $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$.

1.2.3. *Sobolev bundles.* Given a basis (V_i) of Lie algebra $\mathfrak{h}^g$, let $\Delta = -\sum V_i^2$ denote Laplacian via the standard basis. Similarly, denote Δ_α Laplacian defined by the basis $(\alpha^{-1})_* V_i$. For any $s \in \mathbb{R}$ and any C^∞ function $f \in L^2(M)$,

$$\|f\|_{\alpha,s} = \langle f, (1 + \Delta_\alpha)^s f \rangle^{1/2}.$$

Let $W_\alpha^s(M)$ be a completion of $C^\infty(M)$ with the norm above and denote $W_\alpha^{-s}(M)$ by its dual space. Extending it to the exterior algebra, define the Sobolev spaces $\Lambda^d \mathfrak{p} \otimes W^s(M)$ of cochains of degree d and we use the same notations for the norms.

The group $Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \to Sp_{2g}(\mathbb{R})$$

where

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma = (\alpha\gamma, \gamma^*\varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \to \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

the fiber over $[\alpha] \in \mathfrak{M}_g$ is locally identified with the space $W^s_\alpha(M)$.

By invariance of $Sp_{2g}(\mathbb{Z})$ action, we denote the class (α, φ) by $[\alpha, \varphi]$ and write $Sp_{2g}(\mathbb{Z})$-invariant Sobolev norm

$$\|(\alpha, f)\|_s := \|f\|_{\alpha, s}.$$

1.2.4. *Representation.* We write the Hilbert sum decomposition of

$$(2) \qquad\qquad\qquad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n$$

into closed H^g-invariant subspaces. Set $f = \sum_{n \in \mathbb{Z}} f_n \in L^2(M)$ and $f_n \in H_n$ where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi \iota n K t)f\}$$

for some fixed $K > 0$. For $s > 0$, Sobolev space $W^s(M) = \bigoplus_{n \in \mathbb{Z}} W^s(H_n)$ decomposes to closed H^g-invariant subspaces $W^s(H_n) = W^s(M) \cap H_n$. The center $Z(H^g)$ has spectrum $2\pi\mathbb{Z}\backslash\{0\}$ and the space splits as Hilbert sum of H^g-module H_n, which is equivalent to irreducible representation π.

By Stone-Von Neumann theorem, the unitary irreducible representation π of the Heisenberg group of non-zero central parameter $K > 0$ is unitarily equivalent to the Schrödinger representation. Infinitesimal derived representation $d\pi_*$ with parameter $n \in \mathbb{Z}$ for each $k = 1, 2, \cdots, g$

$$d\pi_*(X_k) = \frac{\partial}{\partial x_k}, \quad d\pi_*(Y_k) = 2\pi \iota n K x_k, \quad d\pi_*(Z) = 2\pi \iota n K$$

acts on $L^2(\mathbb{R}^g) \simeq L^2(H_n)$.

1.2.5. *Renormalization flow.* Denote diagonal matrix $\delta_i = diag(d_1, \cdots, d_g)$ with $d_i = 1$, $d_k = 0$ if $k \neq i$. Then, for each $1 \leq i \leq g$, we denote $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in \mathfrak{sp}_{2g}$.

Any such $\hat{\delta}_i$ generate a one-parameter subgroup of automorphism $r_i^t = e^{t\hat{\delta}_i}$. We denote (rank d) renormalization flow $r_{\mathbf{t}} := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$ for $\mathbf{t} = (t_1, \cdots, t_d)$.

Main results. One of the main objects in this paper is to construct finitely-additive measures defined on the space of rectangles on Heisenberg manifold M. We state our results beginning with an overview of Bufetov functional.

Definition 1.1. For $(m, \mathbf{T}) \in M \times \mathbb{R}^d_+$, denote the *standard rectangle* for action P,

$$(3) \qquad\qquad \Gamma^X_{\mathbf{T}}(m) = \{\mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m) \mid \boldsymbol{t} \in U(\mathbf{T}) = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]\}.$$

Let $\mathsf{Q}^{d,Y}_y := \exp(y_1 Y_1 + \cdots + y_d Y_d)$, $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$ be the action generated by elements Y_i of standard basis. Let $\phi^Z_z := \exp(zZ)$ be a flow generated by central element Z.

Definition 1.2. Let $\mathfrak{R}$ be the collection of the *generalized rectangles* in M. For each $1 \leq d \leq g$ and $\boldsymbol{t} = (t_1, \cdots, t_d)$,

$$(4) \qquad \mathfrak{R} := \bigcup_{1 \leq i \leq d} \bigcup_{(y,z) \in \mathbb{R}^d \times \mathbb{R}} \bigcup_{(m,\mathbf{T}) \in M \times \mathbb{R}^d_+} \{(\phi^Z_{t_i z}) \circ \mathsf{Q}^{d,Y}_y \circ \mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m) \mid \boldsymbol{t} \in U(\mathbf{T})\}.$$

Theorem 1.3. *For any irreducible representation H, there exists a measure $\hat{\beta}_H(\Gamma) \in \mathbb{C}$ for every rectangle $\Gamma \in \mathfrak{R}$ such that the following holds:*

(1) *(Additive property) For any decomposition of disjoint rectangles $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$ or whose intersections have zero measure,*

$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^{n} \hat{\beta}_H(\alpha, \Gamma_i).$$

(2) *(Scaling property) For $\boldsymbol{t} \in \mathbb{R}^d$,*

$$\hat{\beta}_H(r_{\boldsymbol{t}}[\alpha], \Gamma) = e^{-(t_1 + \cdots t_d)/2} \hat{\beta}_H(\alpha, \Gamma).$$

(3) *(Invariance property) For any action $\mathsf{Q}_\tau^{d,Y}$ generated by Y_i's and $\tau \in \mathbb{R}_+^d$,*

$$\hat{\beta}_H(\alpha, (\mathsf{Q}_\tau^{d,Y})_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

(4) *(Bounded property) For any rectangle $\Gamma \in \mathfrak{R}$, there exists a constant $C(\Gamma) > 0$ such that for $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$,*

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

For arbitrary rectangle $U_{\mathbf{T}} = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]$, pick $\mathbf{T}'^{(i)} \in [0, \mathbf{T}^{(i)}]$ for each i to decompose $U_{\mathbf{T}}$ into 2^d sub-rectangles. We write $\mathbf{P}(\mathbf{T}')$ collection of 2^d vertices $v = (v^{(1)}, v^{(2)}, \cdots, v^{(d)})$ where $v^{(i)} \in \{0, \mathbf{T}'^{(i)}\}$. Let $U_{\mathbf{T},v}$ be a rectangle whose sides $\boldsymbol{I}_v = (I^{(1)}, I^{(2)}, \cdots, I^{(d)}) \in \mathbb{R}_+^d$ where

$$I^{(l)} = \begin{cases} T^{(l)} - T'^{(l)} & \text{if } v^{(l)} = T'^{(l)} \\ T'^{(l)} & \text{if } v^{(l)} = 0. \end{cases}$$

Then, we have $U_{\mathbf{T}} = \bigcup_{v \in \mathbf{P}(\mathbf{T}')} U_{\mathbf{T},v}$.

Theorem 1.4. *Let us denote $\beta_H(\alpha, m, \boldsymbol{T}) := \hat{\beta}_H(\alpha, \Gamma_{\boldsymbol{T}}^X(m))$. The function β_H satisfies following properties:*

(1) *(Cocycle property) For all $(m, \boldsymbol{T}_1, \boldsymbol{T}_2) \in M \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\beta_H(\alpha, m, \boldsymbol{T}_1 + \boldsymbol{T}_2) = \sum_{v \in \mathbf{P}(\boldsymbol{T}_1)} \beta_H(\alpha, \mathsf{P}_v^{d,\alpha}(m), \boldsymbol{I}_v).$$

(2) *(Scaling property) For all $m \in M$,*

$$\beta_H(r_{\boldsymbol{t}}\alpha, m, \boldsymbol{T}) = e^{(t_1 + \cdots t_d)/2} \beta_H(\alpha, m, \boldsymbol{T}).$$

(3) *(Bounded proprety) Let us denote largest length of side $T_{max} = \max_i \boldsymbol{T}^{(i)}$. Then, there exists a constant $C_H > 0$ such that*

$$\beta_H(\alpha, m, \boldsymbol{T}) \leq C_H T_{max}^{d/2}.$$

(4) *(Orthogonality) For all $[\alpha] \in DC$ and all $\boldsymbol{T} \in \mathbb{R}^d$, bounded function $\beta_H(\alpha, \cdot, \boldsymbol{T})$ belongs to the irreducible component, i.e,*

$$\beta_H(\alpha, \cdot, \boldsymbol{T}) \in H \subset L^2(M).$$

By representation theory introduced (2), for any $f \in W^s(M)$ has a decomposition

$$f = \sum_H f_H$$

and we define *Bufetov cocycle* associated to f (or form ω) as the sum

$$(5) \qquad \beta^f(\alpha, m, \boldsymbol{T}) = \sum_H D_\alpha^H(f)\beta_H(\alpha, m, \boldsymbol{T}).$$

Remark. *For convenience, identification of distributions of form ω and function f were used. The formula (5) yields a duality between the space of basic (closed) currents and invariant distributions. In the similar setting for horocycle flow, refer [BF14, Cor 1.2, p.10].*

Given a Jordan region U and a point $m \in M$, set $P_U^{d,\alpha}m$ the Birkhoff sums associated to some $m \in M$ for the action $\mathsf{P}_x^{d,\alpha}$ given by

$$\left\langle P_U^{d,\alpha}m, \omega \right\rangle := \int_U f(\mathsf{P}_x^{d,\alpha}m)dx_1 \cdots dx_d$$

for any degree $\mathfrak{p}$-form $\omega = f\hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$, with $f \in C_0^\infty(M)$ (smooth function with zero averages).

Let the family of random variable

$$E_{T_n}(f) := \frac{1}{vol(U(T_n))^{1/2}} \left\langle P_{U(T_n)}^{d,\alpha}(\cdot), \omega_f \right\rangle,$$

and we are interested in asymptotic behavior of the probability distributions of $E_{T_n}(f)$. Our goal is to understand the asymptotics of E_{T_n}.

Theorem 1.5. *For every closed form $\omega_f \in \Lambda^d\mathfrak{p} \otimes W^s(M)$ with $s > s_{d,g} = d(d + 11)/4 + g + 1/2$, which is not a coboundary, the limit distribution of the family of random variables $E_{T_n}(f)$ exists, and for almost all frequency α, it has compact support on the real line.*

Open problem. Our work leaves open questions.

Problem 1. Can we prove measure estimation for full measure set of α?

Problem 2. Can we construct the Bufetov functionals on higher step nilmanifolds?

Since moduli space is trivial and there does not exist renormalization flows, we could not obtain the results in the same method. However, we still hope other methods in handling non-renormalizable flows are possibly applied (Cf. [FF14, FFT16, Kim21]).

Outline of the paper. In section 2, we give basic definitions on Higher rank actions, moduli spaces and Sobolev spaces. In section 3, we state main theorem and prove constructions of Bufetov functionals with main properties. In section 4, we prove asymptotic formula of Birkhoff sums and its limit theorems. In section 5 and 6, we prove L^2-lower bound and analyticity of Bufetov functional. In section 7, there is a measure estimates of Bufetov functionals on the sets where they are small. This result only holds when frame α is of bounded type.

2. Analysis on Heisenberg manifolds

In this section, we will recall definitions of currents, representation and renormalization on moduli space.

2.1. **Height function.**

Definition 2.1. The height of a point Z in Siegel upper half space $\mathfrak{H}_g$ is the positive number

$$hgt(Z) := \det \Im(Z).$$

The height function $Hgt : \Sigma_g \to \mathbb{R}^+$ to be the maximal height of a $Sp_{2g}(\mathbb{Z})$ orbit. That is, for the class of $[Z] \in \Sigma_g$,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

2.1.1. *Best Sobolev constant.* The *Sobolev embedding theorem* implies that for any $\alpha \in Sp_{2g}(\mathbb{R})$ and $s > g + 1/2$, there exists a constant $B_s(\alpha)$ such that for any $f \in W_\alpha^s(M)$,

$$\|f\|_\infty \leq B_s(\alpha) \|f\|_{s,\alpha}.$$

The *best Sobolev constant* is defined as the function on the group of automorphism

$$(6) \qquad B_s(\alpha) = \sup_{f \in W_\alpha^s(M)} \frac{\|f\|_\infty}{\|f\|_{s,\alpha}}.$$

By Proposition 4.8 of [CF15], there exists a universal constant $C(s) > 0$ such that the best Sobolev constant satisfies the estimate

$$(7) \qquad B_s([[\alpha]]) \leq C(s) \cdot (Hgt[[\alpha]])^{1/4}.$$

From the Sobolev embedding theorem and the definition of the best Sobolev constant, we have the following bound.

Lemma 2.2. *[CF15, Lemma 5.5] For any Jordan region $U \subset \mathbb{R}^d$ with Lebesgue measure $|U|$, for any $s > g + 1/2$ and all $m \in M$,*

$$\left\| [\alpha, (P_U^{d,\alpha} m)] \right\|_{-s} \leq B_s(\alpha)|U|.$$

2.2. **Diophantine condition.** We recall Diophantine condition introduced in Flaminio and Cosentino's work ([CF15, Definition 4.10]).

Let $\exp t\hat{\delta}(d)$ be the subgroup of $Sp_{2g}(\mathbb{R})$ defined by $\exp(t\hat{\delta}(d))X_i = e^{t_i}X_i$, for $i = 1, \cdots, d$, and $\exp(t\hat{\delta}(d))X_i = X_i$ for $i = d + 1, \cdots g$. We also denote $r_t = \exp t\hat{\delta}(d)$.

Definition 2.3. An automorphism $\alpha \in Sp_{2g}(\mathbb{R})$ or a point $[\alpha] \in \mathfrak{M}_g$ is $\hat{\delta}(d)$-*Diophantine* of type σ if there exists a $\sigma > 0$ and a constant $C > 0$ such that

$$(8) \qquad Hgt([[\exp(-t\hat{\delta}(d))\alpha]]) \leq C Hgt([[\exp(-t\hat{\delta}(d))]])^{(1-\sigma)} Hgt([[\alpha]]), \ \forall t \in \mathbb{R}_+^d.$$

This states that $\alpha \in Sp_{2g}(\mathbb{R})$ satisfies a $\hat{\delta}(d)$-Diophantine if the height of the projection of $\exp(-t\hat{\delta}(d))\alpha$ in the Siegel modular variety Σ_g is bounded by $e^{2(t_1 + \cdots t_d)(1-\sigma)}$. Furthermore,

- $[\alpha] \in \mathfrak{M}_g$ satisfies a $\hat{\delta}$-*Roth condition* if for any $\epsilon > 0$ there exists a constant $C > 0$ such that

$$(9) \qquad Hgt([[\exp(-t\hat{\delta}(d))\alpha]]) \leq C Hgt([[\exp(-t\hat{\delta}(d))]])^\epsilon Hgt([[\alpha]]), \ \forall t \in \mathbb{R}_+^d.$$

That is, $\hat{\delta}(d)$-Diophantine of type $0 < \sigma < 1$.

- $[\alpha]$ is of *bounded type* if there exists a constant $C > 0$ such that

$$(10) \qquad Hgt([[\exp(-t\hat{\delta}(d))]]) \leq C, \ \forall t \in \mathbb{R}_+^d.$$

Definition 2.4. Let $X = G/\Lambda$ be a homogeneous space equipped with the probability Haar measure μ. A function $\phi : X \to \mathbb{R}$ is said k-DL (distance like) for some exponent $k > 0$ if it is uniformly continuous and if there exist constants $C_1, C_2 > 0$ such that
$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

By the work of Margulis and Kleinblock, a generalization of Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

Theorem 2.5. *[KM99, Theorem 1.9] Let G be a connected semisimple Lie group without compact factors, μ its normalized Haar measure, $\Lambda \subset G$ an irreducible lattice, $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G. Let $\mathfrak{d}_+$ be a non-empty open cone in a d-dimensional subalgebra $\mathfrak{d}$ of $\mathfrak{a}$. If $\phi : G/\Lambda \to \mathbb{R}$ is a k-DL function for some $k > 0$, then for μ-almost all $x \in G/\Lambda$ one has*
$$\limsup_{\mathbf{z} \in \mathfrak{d}_+, \mathbf{z} \to \infty} \frac{\phi(\exp(\mathbf{z})x)}{\log \|\mathbf{z}\|} = \frac{d}{k}$$

By Lemma 4.7 of [CF15], the logarithm of Height function is DL-function with exponent $k = \frac{g+1}{2}$ on the Siegel variety Σ_g (and induces on $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$). Hence, we obtain the following proposition.

Proposition 2.6. *Under the assumption $X = \mathfrak{M}_g$ of Theorem 2.5, for $s > g + 1/2$, there exists a full measure set $\Omega_g(\hat{\delta})$ and for all $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$*

$$(11) \qquad \limsup_{\mathbf{t} \to \infty} \frac{\log Hgt([[\exp(-\mathbf{t}\hat{\delta}(d))\alpha]])}{\log \|\mathbf{t}\|} \leq \frac{2d}{g+1}.$$

Any such $[\alpha]$ satisfies a $\hat{\delta}$-Roth condition (9).

For any $L > 0$ and $1 \leq d \leq g$, let $DC(L)$ denote the set of $[\alpha] \in \mathfrak{M}_g$ such that

$$(12) \qquad \int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots t_d)/2} \mathrm{Hgt}([[r_{-\mathbf{t}}\alpha]])^{1/4} dt_1 \cdots dt_d \leq L.$$

Let DC denote the union of the sets DC(L) over all $L > 0$. It follows immediately that the set $DC \subset \mathfrak{M}_g$ has full Haar volume.

Remark. *We simply rewrite Cosentino-Flaminio's notation to facilitate decomposition of actions $r_{\mathbf{t}} = \exp \mathbf{t}\hat{\delta}(d)$ and to avoid abusing notation r_t in the following next sections. Therefore, our choice of t_i in $\mathbf{t} = (t_i) \in \mathbb{R}^d$ is restrictive since $\mathbf{t}$ lies in an open cone in a d-dimensional subspace. I.e, there always exists non-zero constant c_j such that $t_i = c_j t_j$ for each $j \neq i$ when $\mathbf{t}$ tends to infinity.*

2.3. **Invariant currents.** We denote the bundle of $\mathfrak{p}$-forms of degree j of Sobolev order s by $A^j(\mathfrak{p}, \mathfrak{M}^s)$. Similarly, there is a flat bundle of distribution $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$ whose fiber over $[\alpha]$ is locally identified with the space $W_\alpha^{-s}(M)$ normed by $\|\cdot\|_{\alpha, -s}$.

In the following, we set $\omega^{d,\alpha} = dX_1^\alpha \wedge \cdots \wedge dX_d^\alpha$ a top dimensional $\mathfrak{p}$-form and identify d dimensional currents $\mathcal{D}$ with distributions, for any $f \in C^\infty(M)$
$$\langle \mathcal{D}, f \rangle := \langle \mathcal{D}, f\omega^{d,\alpha} \rangle.$$

For $s > 0$, we denote $D \in Z_d(\mathfrak{p}, W^{-s}(M))$ a closed P-invariant currents of dimension d and Sobolev order s. Then, from formal identities, $\langle D, X_i^\alpha(f) \rangle = 0$ for all test function f and $i \in [1, d]$.

Proposition 2.7. *[CF15, Prop 3.13] For any $s > d/2$ with $d = \dim \mathsf{P}$, denote $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ the space of P-invariant currents of Sobolev order s, which coincides with the space of closed currents of dimension d.*

- *It is one dimensional space if $\dim \mathsf{P} = g$, or an infinite-dimensional space if $\dim \mathsf{P} < g$. We have $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$ for all $\epsilon > 0$.*

Let $\omega \in \Lambda^d \mathfrak{p} \otimes W^s(\mathbb{R}^g)$ with $s > (d+1)/2$. Then, ω admits a primitive Ω if and only if $T(\omega) = 0$ for all $T \in I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. We may have $\Omega \in \Lambda^{d-1}\mathfrak{p}' \otimes W^t(\mathbb{R}^g)$ for any $t < s - (d+1)/2$.

2.3.1. *Renormalization.* By definition of renormalization flow, for $\omega \in \Lambda^d \mathfrak{p} \otimes W^s(\mathbb{R}^g)$ and $\mathcal{D} \in Z_d(\mathfrak{p}, W^{-s}(M))$, we have

$$r_i^t[\alpha, \omega] = [r_i^t \alpha, \omega], \quad r_i^t[\alpha, \mathcal{D}] = [r_i^t \alpha, \mathcal{D}].$$

Let $U_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be unitary operator for $\mathbf{t} = (t_1, \cdots, t_d)$,

$$(13) \qquad U_t f(x) = e^{-(t_1 + \cdots + t_d)/2} f(e^{t_1} x_1, \cdots e^{t_d} x_d).$$

Then, for any invariant currents D_α^H,

$$D_{r_t(\alpha)}^H = e^{(t_1 + \cdots + t_d)/2} D_\alpha^H.$$

Then, the action of $\mathbb{R}^d$ is defined by reparametrization

$$(14) \qquad \mathsf{P}_x^{d,(r_1^{t_1} \cdots r_d^{t_d} \alpha)} = \mathsf{P}_{(e^{-t_1} x_1, \ldots, e^{-t_d} x_d)}^{d,\alpha}$$

and the Birkhoff sum satisfies the following identity

$$(15) \qquad P_U^{d,(r_1^{t_1} \cdots r_d^{t_d} \alpha)} m = e^{(t_1 + \cdots + t_d)/2} P_{(e^{-t_1}, \ldots, e^{-t_d})U}^{d,\alpha} m.$$

We rephrase Lemma 4.9 in [CF15] regarding the bound of renormalized height.

Lemma 2.8. *For any $[\alpha] \in \mathfrak{M}_g$ and any $t \geq 0$,*

$$(16) \qquad Hgt([[\exp(t\hat{\delta}(d))\alpha]]) \leq (det(e^{t\delta}))^2 Hgt([[\alpha]]).$$

3. Constructions of the functionals

For an irreducible representation H, there exists *basic current* B_α^H associated to D_α^H. The current is *basic* in the sense that for all $j \in \{i_1, \cdots, i_d\}$,

$$\iota_{X_j} B_\alpha^H = L_{X_j} B_\alpha^H = 0.$$

The basic current B_α^H is defined as $B_\alpha^H = D_\alpha^H \eta_X$. The formula implies that for every d-form ξ,

$$B_\alpha^H(\xi) = D_\alpha^H \left(\frac{\eta_X \wedge \xi}{\omega} \right)$$

where $\eta_X := \iota_{X_{i_1}} \cdots \iota_{X_{i_d}} \omega$ and ω is an invariant volume form.

The basic current B_α^H belongs to a dual Sobolev space of currents. We write any smooth d-form $\xi = \sum \xi^{(i)} \hat{X}_i$, where $\hat{X}_i \in \Lambda^d \mathfrak{p}'$. It follows that the space of smooth d-form is identified to the product of $C^\infty(M)$ by isomorphism $\xi \longrightarrow \xi^{(i)}$. By isomorphism, we define Sobolev space of currents $\Omega_\alpha^s(M)$ and their dual spaces of currents $\Omega_\alpha^{-s}(M)$.

By Sobolev embedding theorem, for every rectangle Γ, the current $\Gamma \in \Omega_\alpha^{-s}(M)$ for $s > (2d+1)/2$. Then, all basic currents $B_\alpha^H \in \Omega_\alpha^{-s}(M)$ for all $s > d/2$ since $D_\alpha^H \in \Omega_\alpha^{-s}(M)$ for all $s > d/2$.

3.1. **Constructions of the functionals.** For any exponent $s > d/2$, Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^{\perp}$. Denote by I^{-s} and R^{-s} the corresponding orthogonal projection operator and by I_{α}^{-s} and R_{α}^{-s} the restrictions to the fiber over $[\alpha] \in \mathfrak{M}$ for $\alpha \in Sp_{2g}(\mathbb{R})$. In particular, for the Birkhoff sums $D = P_U^{d,\alpha} m$, we call $I_{\alpha}^{-s}(D) = I^{-s}[\alpha, D]$ boundary term and $R_{\alpha}^{-s}(D) = R^{-s}[\alpha, D]$ remainder term respectively. Consider the orthogonal projection

$$(17) \qquad\qquad D = I_{r_{-t}[\alpha]}^{-s}(D) + R_{r_{-t}[\alpha]}^{-s}(D).$$

For fixed α, let $\Pi_H^{-s} : A_d(\mathfrak{p}, W_{\alpha}^{-s}(M)) \to A_d(\mathfrak{p}, W_{\alpha}^{-s}(H))$ denote the orthogonal projection on a single irreducible unitary representation. We further decompose projection operators with

$$\Pi_H^{-s} = \mathscr{B}_{\alpha}^{-s}(\Gamma) B_{\alpha}^{-s,H} + R_{\alpha}^{-s,H}$$

where $\mathscr{B}_{H,\alpha}^{-s} : A_d(\mathfrak{p}, W_{\alpha}^{-s}(M)) \to \mathbb{C}$ denote the orthogonal component map of P-invariant currents (closed), supported on a single irreducible unitary representation.

The *Bufetov functionals* on rectangles $\Gamma \in \mathfrak{R}$ are defined for all $\alpha \in DC$ as follows.

Lemma 3.1. *Let* $\alpha \in DC(L)$*. For* $s > s_d = d(d+11)/4 + g + 1/2$*, the limit*

$$\hat{\beta}_H(\alpha, \Gamma) = \lim_{t_1,\ldots,t_d \to \infty} e^{-(t_1 + \cdots t_d)/2} \mathscr{B}_{H,r_{-t}[\alpha]}^{-s}(\Gamma)$$

exists and define a finitely-additive finite measure on the standard rectangle (3) $\Gamma := \Gamma_T^X(m)$ *for* $m \in M$*. Then there exists a constant* $C(s, \Gamma) > 0$ *such that the following estimate holds:*

$$(18) \qquad\qquad |\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B_{\alpha}^H|_{\alpha,-s} \leq C(s, \Gamma)(1 + L).$$

Proof. For simplicity, we omit dependence of H. For every $\mathbf{t} \in \mathbb{R}^d$, we have the following orthogonal splitting:

$$\Pi_{H,\alpha}^{-s}(\Gamma) = \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}},$$

where

$$\mathscr{B}_{\alpha,\mathbf{t}}^{-s} := \mathscr{B}_{H,r_{-\mathbf{t}}[\alpha]}^{-s}, \ B_{\alpha,\mathbf{t}} := B_{r_{-\mathbf{t}}[\alpha]}^{-s,H}, \ R_{\alpha,\mathbf{t}} := R_{r_{-\mathbf{t}}[\alpha]}^{-s,H}.$$

For any $\mathbf{h} \in \mathbb{R}^d$, we have

$$\mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}+\mathbf{h}} + R_{\alpha,\mathbf{t}+\mathbf{h}} = \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}}.$$

By reparametrization (14), we have $B_{\mathbf{t}+\mathbf{h}} = e^{-(h_1 + \cdots h_d)/2} B_{\mathbf{t}}$ and

$$(19) \qquad \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{(h_1 + \cdots h_d)/2} \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) + \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}})$$

and it follows that

$$(20) \qquad \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{h_1/2} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) + \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}}).$$

By differentiating at $h_1 = 0$,

(21)
$$\frac{d}{dt_1} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) + [\frac{d}{dh_1} \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}})]_{h_1=0}.$$

Therefore, we solve the following first order ODE

$$\frac{d}{dt_1}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) + \mathcal{K}^{(1)}_{\alpha,t}(\Gamma)$$

where

$$\mathcal{K}^{(1)}_{\alpha,t}(\Gamma) = [\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,t+h}(R_{\alpha,t})]_{h_1=0}.$$

Then, the solution of the differential equation is

$$\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots t_d+h_d}(\Gamma) = e^{t_1/2}[\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) + \int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)d\tau_1]$$

$$= e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) + \int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)d\tau_1.$$

Note by reparametrization

$$e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) = e^{h_2/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3\cdots t_d+h_d}(\Gamma)$$

and it is possible to differentiate the previous equation with respect to h_2 again. Then

$$(22)\quad \frac{d}{dt_2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1.$$

where $\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma) = \frac{d}{dh_2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)$.

Then, the solution of (22) is

$$\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma)$$

$$= e^{t_2/2}[\mathscr{B}^{-s}_{\alpha,t_1,0,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_2} e^{-\tau_2/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2]$$

$$= e^{h_3/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3,\cdots t_d+h_d} + \int_0^{t_2} e^{(t_2-\tau_2)/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2.$$

Inductively, we solve first order ODE repeatedly and obtain the following solution

$$(23)\quad \mathscr{B}^{-s}_{\alpha,t}(\Gamma) = e^{(t_1+\cdots t_d)/2}\left(\mathscr{B}^{-s}_{\alpha,0} + \int_0^{t_d}\cdots\int_0^{t_1} e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma)d\tau_1\cdots d\tau_d\right)$$

where

$$\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma) = [\frac{d}{dh_d}\cdots\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,t+h}(R_{\alpha,t})]_{h_d\cdots h_1=0}.$$

Let $\langle\cdot,\cdot\rangle_t$ denote the inner product in Hilbert space $\Omega^{-s}_{r_{-t}[\alpha]}$. From the intertwining formula,

$$\mathscr{B}^{-s}_{\alpha,t+h}(R_{\alpha,t}) = \langle R_{\alpha,t}, \frac{B_{\alpha,t+h}}{|B_{\alpha,t+h}|^2_{t+h}}\rangle_{\alpha,t+h}$$

$$= \langle R_{\alpha,t}\circ U_{-h}, \frac{B_{\alpha,t+h}\circ U_{-h}}{|B_{\alpha,t+h}|^2_{t+h}}\rangle$$

$$= \langle R_{\alpha,t}\circ U_{-h}, \frac{B_{\alpha,t}}{|B_{\alpha,t}|^2_t}\rangle = \mathscr{B}^{-s}_{\alpha,t}(R_{\alpha,t}\circ U_{-h}).$$

In the sense of distributions,

$$\frac{d}{dh_d} \cdots \frac{d}{dh_1}(R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}) = -R_{\alpha,\boldsymbol{t}} \circ (\frac{d}{2} + \sum_{i=1}^{d} X_i(t)) \circ U_{-\boldsymbol{h}}$$

$$= [(\sum_{i=1}^{d} X_i(t) - \frac{d}{2})R_{\alpha,\boldsymbol{t}}] \circ U_{-\boldsymbol{h}}$$

and we compute derivative term of (21) in representation,

$$[\frac{d}{dh_d} \cdots \frac{d}{dh_1}(\mathscr{B}_{\alpha,\boldsymbol{t}+\boldsymbol{h}}^{-s}(R_{\alpha,t}))]_{\boldsymbol{h}=0} = -\mathscr{B}_{\alpha,\boldsymbol{t}}^{-s}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2})R_{\alpha,\boldsymbol{t}}).$$

Set $\mathcal{K}_{\alpha,t}(\Gamma) = |R_{\alpha,\boldsymbol{t}}|_{r_{-\boldsymbol{t}}\alpha,-(s+1)}$ with a bounded non-negative function, then by Lemma 3.5, we claim that

$$(24) \qquad |\mathscr{B}_{\alpha,\boldsymbol{t}}^{-s}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2})R_{\alpha,\boldsymbol{t}}))| \leq \mathcal{K}_{\alpha,\tau}(\Gamma) \leq C(s,\Gamma)Hgt([r_{-\boldsymbol{t}}\alpha])^{1/4}.$$

Therefore, the solution of equation (23) exists under Diophantine condition (12) and the following holds:

$$\lim_{t_d \to \infty} \cdots \lim_{t_1 \to \infty} e^{-(t_1+\cdots t_d)/2}\mathscr{B}_{\alpha,t}^{-s}(\Gamma) = \hat{\beta}_H(\alpha,\Gamma).$$

Moreover, the complex number

$$\hat{\beta}_H(\alpha,\Gamma) = \mathscr{B}_{\alpha,0}^{-s} + \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}_{\alpha,\tau}(\Gamma)d\tau_1 \cdots d\tau_d$$

depends continuously on $\alpha \in DC(L)$. Since we have

$$\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha,\Gamma)B_\alpha^H = R_0 - \left(\int_0^\infty \cdots \int_0^\infty e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}_{\alpha,\tau}(\Gamma)d\tau_1 \cdots d\tau_d\right)B_\alpha^H,$$

by Diophantine condition again,

$$|\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha,\Gamma)B_\alpha^H|_{\alpha,-s} \leq C(s,\Gamma)(1+L).$$

$\square$

Remark. *We do not need dependence of time for solving ODE. Only necessary part is DC.*

3.2. **Remainder estimates.** In this subsection, we obtain an estimate for the remainder term which is used in the proof of Lemma 3.1. Firstly, we prove the bound of Birkhoff sum of rectangles.

Lemma 3.2. *[CF15, Lemma 5.7] Let $s > d/2 + 2$. There exists a constant $C = C(s) > 0$ such that for all $t_i \geq 0$ for $1 \leq i \leq d$, we have*

$$\left\|I^{-s}[\alpha,D]\right\|_{-s} \leq e^{-(t_1+\cdots+t_d)/2}\left\|I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha,D]\right\|_{-s}$$

$$+ C_1|t_1 + \cdots + t_d| \int_0^1 e^{-u(t_1+\cdots+t_d)/2}\left\|R^{-s}[r_1^{-ut_1} \cdots r_d^{-ut_d}\alpha,D]\right\|_{-(s-2)} du.$$

By Stokes' theorem, we have the following remainder estimate.

Lemma 3.3. *[CF15, Lemma 5.6] For any non-negative $s' < s - (d+1)/2$ and Jordan region $U \subset \mathbb{R}^d$, there exists $C = C(d, g, s, s') > 0$ such that*

$$\left\| R^{-s}[\alpha, (P_U^{d,\alpha} m)] \right\|_{-s} \le C \left\| [\alpha, \partial(P_U^{d,\alpha} m)] \right\|_{-s'}.$$

In [CF15], quantitative bound of Birkhoff sums on square domain was obtained. Now we extend the result on rectangular shapes.

Theorem 3.4. *For $s > s_d$, there exists a constant $C(s, d) > 0$ such that the following holds. For any $t_i > 0$, $m \in M$ and $U_d(t) = [0, e^{t_1}] \times \cdots \times [0, e^{t_d}]$, we have*

(25)

$$\left\| [\alpha, (P_{U_d(t)}^{d,\alpha} m)] \right\|_{-s} \le C \sum_{k=0}^{d} \sum_{1 \le i_1 < \cdots < i_k \le d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} \exp(\frac{1}{2} \sum_{l=1}^{d} t_l - \frac{1}{2} \sum_{l=1}^{k} u_{i_l})$$

$$\times Hgt([[\prod_{1 \le j \le d} r_j^{-t_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha]])^{1/4} du_{i_1} \cdots du_{i_k}.$$

Proof. We prove by induction. For $d = 1$, it follows from the Theorem 5.8 in [CF15]. We assume that the result holds for $d-1$. Decompose the current as a sum of boundary and remainder term as in (17).

Step 1. We estimate the boundary term. By Lemma 3.2, renormalize terms with $r^u = r_1^u \cdots r_d^u$. Then, we have

$$\left\| I^{-s}[\alpha, (P_{U_d(t)}^{d,\alpha} m)] \right\|_{-s} \le e^{-(t_1 + \cdots + t_d)/2} \left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{U_d(t)}^{d,\alpha} m)] \right\|_{-s}$$

(26)

$$+ C_1(s) \int_0^{t_1 + \cdots + t_d} e^{-ud/2} \left\| R^{-s}[r^{-u} \alpha, (P_{U_d(t)}^{d,\alpha} m)] \right\|_{-(s-2)} du$$

$$:= (I) + (II).$$

By renormalization (15) and Lemma 2.2 for unit volume,

$$\left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{U_d(t)}^{d,\alpha} m)] \right\|_{-s} = e^{t_1 + \cdots + t_d} \left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{U_d(0)}^{d, r_1^{-t_1} \cdots r_d^{-t_d} \alpha} m)] \right\|_{-s}$$

$$\le C_2 e^{t_1 + \cdots + t_d} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d} \alpha]])^{1/4}.$$

Hence

$$I \le C_2 e^{(t_1 + \cdots t_d)/2} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d} \alpha]])^{1/4},$$

where the sum corresponds to the first term ($k = 0$) in the statement.

Step 2. To estimate (II),

(27)

$$\left\| R^{-s}[r^{-u} \alpha, (P_{U_d(t)}^{d,\alpha} m)] \right\|_{-(s-2)} = \left\| e^{ud} R^{-s}[r^{-u} \alpha, (P_{U_d(t-u)}^{d, r^{-u} \alpha} m)] \right\|_{-(s-2)}$$

$$\le C_3(s, s') e^{ud} \left\| [r^{-u} \alpha, \partial(P_{U_d(t-u)}^{d, r^{-u} \alpha} m)] \right\|_{-s'}.$$

The boundary $\partial(\mathsf{P}_{U_d}^{d, r^{-t} \alpha})$ is the sum of $2d$ currents of dimension $d - 1$. These currents are Birkhoff sums of d face subgroups obtained from $\mathsf{P}_{U_d}^{d, r^{-t} \alpha}$ by omitting one of the base vector fields X_i. It is reduced to $(d-1)$ dimensional shape obtained from $U_d(t - u) := [0, e^{t_1 - u}] \times \cdots [0, e^{t_d - u}]$. For each $1 \le j \le d$, there are Birkhoff

sums along $d-1$ dimensional cubes. By induction hypothesis, we add all the $d-1$ dimensional cubes by adding all the terms along j:

(28)

$$\left\|\left[r^{-u}\alpha,(P_{U_{d-1}(t-u)}^{d-1,r^{-u}\alpha}m)\right]\right\|_{-s'} \le C_4(s',d-1)\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\le i_1<\cdots<i_k\le d \\ i_l\neq j,\forall l}}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u}$$

$$\exp(\frac{1}{2}\sum_{\substack{l=1 \\ l\neq j}}^{d}(t_l-u)-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{\substack{1\le l\le d \\ l\neq j}}r_l^{-(t_l-u)}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r^{-u}\alpha)]])^{1/4}du_{i_1}\cdots du_{i_k}.$$

Combining (26) and (27), we obtain the estimate for (II).

$$(II)\le C_5(s',d-1)\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\le i_1<\cdots<i_k\le d \\ i_l\neq j}}\int_0^{t_1+\cdots t_d}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u}du_{i_1}\cdots du_{i_k}du$$

$$\times\exp(\frac{1}{2}\sum_{\substack{l=1 \\ l\neq j}}^{d}t_l-\frac{1}{2}u-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\le l\le d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_j^{-u+t_j}\alpha)]])^{1/4}.$$

Applying the change of variable $u_j=t_j-u$, we obtain

$$(II)\le C_6(s',d-1)\sum_{j=1}^{d}\sum_{k=1}^{d-1}\sum_{\substack{1\le i_1<\cdots<i_k\le d \\ i_l\neq j}}\int_{-(t_1+\cdots t_d)+t_j}^{t_j}\int_0^{t_{i_k}-u}\cdots\int_0^{t_{i_1}-u}du_{i_1}\cdots du_{i_k}du_j$$

$$\times\exp(\frac{1}{2}(t_1+\cdots t_d)-\frac{1}{2}u_j-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\le l\le d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_j^{-u_j}\alpha)]])^{1/4}.$$

Simplifying multi-summation above, (with $-(t_1+\cdots t_d)+t_j\le 0$)

$$(II)\le C_7(s',d)\sum_{k=1}^{d}\sum_{1\le i_1<\cdots<i_k\le d}\int_0^{t_{i_k}}\cdots\int_0^{t_{i_1}}du_{i_1}\cdots du_{i_k}$$

$$\times\exp(\frac{1}{2}(t_1+\cdots t_d)-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\le l\le d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}\alpha]])^{1/4}.$$

Step 3. (Remainder estimate). The remainder term is obtained from Lemma 3.3 (Stokes' theorem). Following step 2, estimate of remainder reduces to that of $d-1$ form. Combining with the step 1, we have the following
(29)

$$\left\|R^{-s}[\alpha,\partial(P_{U_d}^{d,\alpha}m)]\right\|_{-s}\le C(s)\sum_{i=1}^{d-1}\left\|I^{-s}[\alpha,(P_{U_i}^{i,\alpha}m)]\right\|_{-s}+\left\|R^{-s}[\alpha,(P_{U_1}^{1,\alpha}m)]\right\|_{-s}$$

where U_i is i-dimensional rectangle. Sum of the boundary terms are absorbed in the bound of $(I)+(II)$. For 1-dimensional remainder with interval Γ_T, the boundary is a 0-dimensional current. Then,

$$\langle\partial(P_{U_T}^{1,\alpha}m),f\rangle=f(\mathsf{P}_{U_T}^{1,\alpha}m)-f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (6) and (7),

$$\left\| R^{-s}[\alpha, \partial(P_{U_T}^{1,\alpha}m)] \right\|_{-s} \leq 2B_s([[\alpha]]) \leq C(s)Hgt([[\alpha]])^{1/4}.$$

Then, by inequality (16)

$$C(s)Hgt([[\alpha]])^{1/4} = C(s)Hgt([[r_{\boldsymbol{t}}r_{-\boldsymbol{t}}\alpha]])^{1/4}$$
$$\leq C(s)e^{(t_1+\cdots t_d)/2}Hgt([[r_1^{-t_1}\cdots r_d^{-t_d}\alpha]])^{1/4}.$$

This implies that remainder term produces one more term like the bound of (I). Therefore, the theorem follows from combining all the terms $(I), (II)$, and remainder. $\qquad\square$

Now we prove the remainder estimate used in (24).

Lemma 3.5. *Let $s > s_d$. There exists a constant $C(s,\Gamma) > 0$ such that for any rectangle $U_\Gamma = [0, e^{\Gamma_1}] \times \cdots \times [0, e^{\Gamma_d}]$,*

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C(s,\Gamma)Hgt([r_{-\boldsymbol{t}}\alpha])^{1/4}.$$

Proof. Recall that $\mathcal{K}_{\alpha,t}(\Gamma) = \left\| R^{-s}[r_{-\boldsymbol{t}}[\alpha], (P_{U_\Gamma}^{d,r_{-\boldsymbol{t}}\alpha}m)] \right\|_{-(s+1)}$ and by Lemma 3.3, it is equivalent to prove to find the bound of $d-1$ currents. By Theorem 3.4, we obtain the remainder estimate.

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C\sum_{k=0}^{d-1}\sum_{1\leq i_1<\cdots<i_k\leq d-1}\int_0^{\Gamma_{i_k}}\cdots\int_0^{\Gamma_{i_1}}\exp(\frac{1}{2}\sum_{l=1}^{d-1}\Gamma_l - \frac{1}{2}\sum_{l=1}^{k}u_{i_l})$$
$$\times Hgt([[\prod_{1\leq j\leq d-1}r_j^{-\Gamma_j}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_{-\boldsymbol{t}}\alpha)]])^{1/4}du_{i_1}\cdots du_{i_k}.$$

It follows from (16) that for $0 \leq k \leq d-1$,

$$Hgt([[\prod_{1\leq j\leq d-1}r_j^{-\Gamma_j}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_{-\boldsymbol{t}}\alpha)]])^{1/4} \leq e^{\frac{1}{2}(\sum_{l=1}^{k}u_{i_l}-\sum_{l=1}^{d}\Gamma_l)}Hgt([[r_{-\boldsymbol{t}}\alpha]])^{1/4}.$$

Then, we obtain

(30)
$$\left\| R^{-s}[r_{-\boldsymbol{t}}\alpha, (P_{U_\Gamma}^{d,r_{-\boldsymbol{t}}\alpha}m)] \right\|_{-s} \leq C\Big(\sum_{k=0}^{d-1}\sum_{1\leq i_1<\cdots<i_k\leq d-1}\prod_{l=1}^{k}\Gamma_{i_l}\Big)Hgt([[r_{-\boldsymbol{t}}\alpha]])^{1/4}.$$

Setting $C(s,\Gamma) = C\Big(\sum_{k=0}^{d-1}\sum_{1\leq i_1<\cdots<i_k\leq d-1}\prod_{l=1}^{k}\Gamma_{i_l}\Big)$, we obtain the conclusion. $\qquad\square$

3.3. **Extensions of domain.** Now we extend the domain of Bufetov functional defined on standard rectangle $\Gamma_{\boldsymbol{T}}^X$ to the class $\mathfrak{R}$.

Lemma 3.6. *Let $1 \leq d \leq g$. The functional $\hat{\beta}$ defined on standard rectangle $\Gamma_{\boldsymbol{T}}^X$ extends to the class $(\mathsf{Q}_y^{d,Y})_*\Gamma_{\boldsymbol{T}}^X$ for any $y \in \mathbb{R}^d$.*

Proof. First, we prove the functional $\hat{\beta}$ exists and invariant under the action $\mathsf{Q}_y^{d,Y}$. It suffices to verify that $\hat{\beta}$ is invariant under the rank 1 action $\mathsf{Q}_\tau^{1,Y}$ for $\tau \in \mathbb{R}$.

Given a standard rectangle Γ, set $\Gamma_{\mathsf{Q}} := (\mathsf{Q}_\tau^{1,Y})_*\Gamma$. Let $D(\Gamma, \Gamma_{\mathsf{Q}})$ be the $(d+1)$ dimensional space spanned by the trajectories of the action of $\mathsf{Q}_\tau^{1,Y}$ projecting
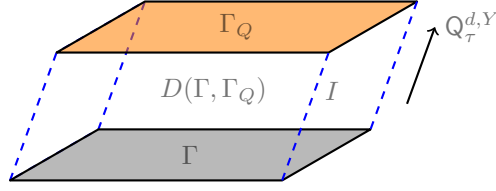
FIGURE 1. Illustration of the standard d-rectangles Γ, Γ_Q and $D(\Gamma, \Gamma_Q)$ as a $d+1$ dimensional current.

Γ onto Γ_Q. Then $D(\Gamma, \Gamma_Q)$ is union of all orbits I of action $Q_\tau^{1,Y}$ such that the boundary of I, d-dimensional faces, is contained in $\Gamma \cup \Gamma_Q$, and interior of I is disjoint from $\Gamma \cup \Gamma_Q$.

By definition, denote $r_t := r_i^t$. Then, $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_Q)$ are respectively the support of the currents $r_t^*\Gamma$ and $r_t^*\Gamma_Q$. Thus, we have the following identity

$$r_t^* D(\Gamma, \Gamma_Q) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q)).$$

Since the currents $\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)$ is composed of orbits of the action of $Q_\tau^{1,Y}$, it follows that

$$(31) \qquad \partial[r_t^* D(\Gamma, \Gamma_Q)] - (r_t^*\Gamma - r_t^*\Gamma_Q) = r_t^*[\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)] \to 0.$$

Now, we turn to prove the volume of $D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q))$ is uniformly bounded for all $t > 0$. For any $p \in \Gamma$, set $\tau(p)$ be length of the arc lying in $D_Q := D(\Gamma, \Gamma_Q)$, and set $\tau_\Gamma := \sup\{\tau(p) \mid p \in \Gamma\} < \infty$. We write

$$vol_{d+1}(D_Q) = \int_\Gamma \tau dvol_d.$$

Since $vol_d(r_{-t}\Gamma) \le e^t vol_d(\Gamma)$,

$$(32) \qquad vol_{d+1}(r_{-t}D_Q) = \int_{r_{-t}\Gamma} \tau dvol_d \le \tau_\Gamma e^{-t} vol_d(r_{-t}\Gamma) \le \tau_\Gamma vol_d(\Gamma) < \infty.$$

Note that d-dimensional current $(Q_\tau^{1,Y})_*\Gamma - \Gamma$ is equal to the boundary of the $(d+1)$ dimensional current D_Q. By arguments in remainder estimate (or Sobolev embedding theorem),

$$(33) \qquad |(Q_\tau^{1,Y})_*\Gamma - \Gamma|_{r^{-t}\alpha, -s} \le C_s \tau B_s([[r^{-t}\alpha]]) \le C_s \tau \text{Hgt}[[r^{-t}\alpha]]^{1/4}$$

is finite for all $t > 0$.

Then, by (31), (32) and existence of Bufetov functional (from Diophantine condition), the last inequality holds

$$|\mathscr{B}_{\alpha,t}^{-s}((Q_\tau^{1,Y})_*\Gamma) - \mathscr{B}_{\alpha,t}^{-s}(\Gamma)|_{\alpha, -s} < \infty.$$

Therefore, by definition of Bufetov functional in the Lemma 3.1, it is extended and invariant under the action of $Q_\tau^{1,Y}$. □

Similarly, Bufetov functional defined on the standard rectangles extends to the class of generalized rectangle $(\phi_{t_i z}^Z) \circ P_t^{d,\alpha}(m)$ (see (4)). We postpone its proof by Lemma 6.3. Since the flow generated by Z commutes with other actions P and Q, combining with the invariance under the action Q from Lemma 3.6, the domain of Bufetov functional extends to the class $\mathfrak{R}$.

Proof of Theorem 1.3. Additive property. It follows from the linearity of projections and limit. *Scaling property* is immediate from the definition.

Bounded property. By the scaling property,

$$\hat{\beta}(\alpha, \Gamma) = e^{dt/2}\hat{\beta}_H(r^t[\alpha], \Gamma).$$

Choose $t = \log(\int_\Gamma |\hat{X}|)$ and $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$, then uniform bound of Bufetov functional on bounded size of rectangles,

$$|\hat{\beta}(\alpha, \Gamma)| \le C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

Invariance property. It follows directly from the Lemma 3.6. $\qquad\qquad\square$

3.4. **Bound of functionals.** We define the excursion function

$$E_{\mathfrak{M}}(\alpha, \mathbf{T}) := \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1+\cdots t_d)/2}\mathrm{Hgt}([[r_{\mathbf{t}-\log \mathbf{T}}\alpha]])^{1/4}dt_1 \cdots dt_d$$

$$= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1+\cdots t_d)/2}\mathrm{Hgt}([[r_{\mathbf{t}}\alpha]])^{1/4}dt_1 \cdots dt_d.$$

Lemma 3.7. *For any Diophantine $[\alpha] \in DC(L)$ and for any $f \in W^s(M)$ for $s > s_d + 1/2$, the Bufetov functional β^f is defined by a uniformly convergent series.*

$$|\beta^f(\alpha, m, \mathbf{tT})| \le C_s(L + \prod_{i=1}^d (T^{(i)})^{1/2}(1 + \prod_{i=1}^d t_i + E_{\mathfrak{M}}(\alpha, \mathbf{T})) \|\omega\|_{\alpha, \mathbf{s}}$$

for $\omega = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p} \otimes W^s(M)$ and $\mathbf{tT} = (t_1 T^{(1)}, \cdots, t_d T^{(d)})$.

Proof. It follows from Lemma 3.1 that there exists a constant $C > 0$ such that

$$|\beta_H(\alpha, m, \mathbf{t})| \le C(1 + L + \prod_{i=1}^d t_i).$$

By the scaling property,

$$\beta_H(\alpha, m, \mathbf{tT}) = \prod_{i=1}^d (T^{(i)})^{1/2}\beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t}).$$

By Diophantine condition (12), whenever $\alpha \in DC(L)$ then $r_{\log \mathbf{T}}[\alpha] \in DC(L_T)$ with

$$L_T \le L \prod_{i=1}^d (T^{(i)})^{-1/2} + E_{\mathfrak{M}}(\alpha, \mathbf{T}).$$

Thus for all $(m, \mathbf{t}) \in M \times \mathbb{R}^d$,

$$|\beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t})| \le C(1 + L_T + \prod_{i=1}^d t_i).$$

It follows that for all $s > 1/2$, we have

$$|\beta^f(\alpha, m, \mathbf{tT})| \le C_s \prod_{i=1}^d (T^{(i)})^{1/2}(1 + L + \prod_{i=1}^d t_i) \sum_{n \in \mathbb{Z}} \|\omega_n\|_{\alpha, s}$$

$$\le C_s \prod_{i=1}^d (T^{(i)})^{1/2}(1 + L_T + \prod_{i=1}^d t_i)(\sum_{n \in \mathbb{Z}}(1 + n^2)^{-s'})^{-1/2}(\sum_{n \in \mathbb{Z}} \left\| (1 - Z^2)^{s'/2}\omega_n \right\|_{\alpha, s}^2)^{1/2}.$$

Therefore, for all $s' > s_d$, there exists a constant $C_{s,s'} > 0$ such that

$$|\beta^f(\alpha, m, \mathbf{t}\mathbf{T})| \leq C_{r,r'} \prod_{i=1}^{d} (T^{(i)})^{1/2} (1 + L_T + \prod_{i=1}^{d} t_i) \|\omega\|_{\alpha, s+s'}.$$

$\square$

This lemma implies that all the properties of Bufetov functionals β_H associated to a single irreducible component are extended to β^f for any $f \in W^s(M), \forall s > s_d + 1/2$. From the lemma above, we can also derive the bounded property for the cocycle $\beta(\alpha, m, \mathbf{T})$.

Corollary 3.8. *For all $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for almost all frequency α and for all $f \in W^s(M)$ and for all $(m, \mathbf{T}) \in M \times \mathbb{R}^d$, we have*

$$(34) \qquad |\left\langle P_{U(\mathbf{T})}^{d,\alpha} m, \omega \right\rangle - \beta^f(\alpha, m, \mathbf{T})| \leq C_s(1 + L) \|\omega\|_{\alpha, s}.$$

for $U(\mathbf{T}) = [0, T^{(1)}] \times \cdots [0, T^{(d)}]$ and $\omega = f\omega^{d,\alpha} \in \Lambda^d \mathfrak{p} \otimes W^s(M)$.

Proof. By Lemma 3.1 and 3.7, asymptotic formula (18) on each irreducible component provides the proof of Corollary 3.8. $\square$

4. Limit distributions

In this section, we prove Theorem 1.5, limit distribution of Birkhoff sums of higher rank actions on rectangles.

4.1. Limiting distributions.

Lemma 4.1. *There exists a continuous modular function $\theta_H : Aut_0(\mathsf{H}^g) \to H \subset L^2(M)$ such that*

$$\lim_{|U(\mathbf{T})| \to \infty} \left\| \frac{1}{vol(U(\mathbf{T}))^{1/2}} \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\cdot), \omega_f \right\rangle - \theta_H(r_{\log \mathbf{T}}[\alpha]) D_\alpha^H(f) \right\|_{L^2(M)} = 0.$$

The family $\{\theta_H(\alpha) \mid \alpha \in Aut_0(\mathsf{H}^g)\}$ has a constant norm in $L^2(M)$.

Proof. By Fourier transform, the space of smooth vectors and Sobolev space $W^s(H)$ is represented as the Schwartz space $\mathscr{S}^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |(1 + \sum_i \frac{\partial^2}{\partial u_i^2} + \sum_i u_i^2)^{s/2} \hat{f}(u)|^2 du < \infty.$$

Let $\boldsymbol{t}, \boldsymbol{u} \in \mathbb{R}^d$. Then we claim for any $f \in \mathscr{S}^s(\mathbb{R}^d)$, there exists a function $\theta([\alpha])(\cdot) \in L^2(\mathbb{R}^d)$ such that

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} f(\boldsymbol{u} + \boldsymbol{t}) d\boldsymbol{t} - \theta_H(r_{\log \boldsymbol{T}}[\alpha])(u) Leb(f) \right\|_{L^2(\mathbb{R}^d, d\boldsymbol{u})} = 0.$$

Equivalently,

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} e^{i\boldsymbol{t}\cdot\hat{\boldsymbol{u}}} \hat{f}(\hat{\boldsymbol{u}}) dt - \hat{\theta}_H(r_{\log \boldsymbol{T}}[\alpha])(\hat{u}) \hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = 0.$$

For $\chi \in L^2(\mathbb{R}^d, d\hat{u})$ and $\hat{u} = (\hat{u}_j)_{1 \leq j \leq d}$, we denote

$$\chi_j(\hat{u}) = \frac{e^{i\hat{u}_j} - 1}{i\hat{u}_j}, \ \chi(\hat{u}) = \prod_{j=1}^{d} \chi_j(\hat{u}).$$

Let $\hat{\theta}([\alpha])(\hat{u}) := \chi(\hat{u})$ for all $\hat{u} \in \mathbb{R}^d$. By intertwining formula (13) for $\mathbf{T} \in \mathbb{R}^d$,

$$U_{\mathbf{T}}(f)(\hat{u}) = \prod_{i=1}^{d}(T^{(i)})^{1/2} f(\boldsymbol{T}\hat{\boldsymbol{u}}), \ \text{for } \boldsymbol{T}\hat{\boldsymbol{u}} = (T^{(1)}\hat{u}_1, \cdots, T^{(d)}\hat{u}_d).$$

Then, for all $\alpha \in Aut_0(\mathsf{H}^g)$,

$$\hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{u}) = U_{\mathbf{T}}(\chi)(\hat{u}) = \prod_{i=1}^{d}(T^{(i)})^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}}).$$

The function $\theta[\alpha]$ is defined by Fourier inverse transform

$$\|\theta_H(\alpha)\|_H = \|\theta(\alpha)\|_{L^2(\mathbb{R}^d)} = \left\|\hat{\theta}(\alpha)\right\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{u})\|_{L^2(\mathbb{R}^d, d\hat{u})} = C > 0.$$

By integration,

$$(35) \quad \int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{i\boldsymbol{t} \cdot \hat{\boldsymbol{u}}} \hat{f}(\hat{u}) dt = (\prod_{i=1}^{d} T^{(i)}) \chi(\boldsymbol{T}\hat{\boldsymbol{u}}) \hat{f}(\hat{u})$$

$$= (\prod_{i=1}^{d} T^{(i)}) \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) + (\prod_{i=1}^{d} T^{(i)})^{1/2} \hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{u}) \hat{f}(0).$$

We note that $\prod_{i=1}^{d} T^{(i)} = vol(U(\boldsymbol{T}))$. Then the claim reduces to the following:

$$\limsup_{vol(U(\boldsymbol{T})) \to \infty} \left\| vol(U(\boldsymbol{T}))^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If $f \in \mathscr{S}^s(\mathbb{R}^g)$ with $s > d/2$, function $\hat{f} \in C^0(\mathbb{R}^d)$ and bounded. Thus, by Dominated convergence theorem and change of variables,

$$\left\| vol(U(\boldsymbol{T}))^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = \left\| \chi(\nu)(\hat{f}(\frac{\nu}{\boldsymbol{T}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\nu)} \to 0.$$

$\square$

Corollary 4.2. *For any $s > d/2$, $\alpha \in Aut_0(\mathsf{H}^g)$ and $f \in W^s(H)$, there exists a constant $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha} m, \omega_f \right\rangle \right\|_{L^2(M)} = C |D_\alpha^H(f)|.$$

From Corollary 4.2, we derive the following limit result for the L^2-norm of Bufetov functionals.

Corollary 4.3. *For every irreducible component H and $\alpha \in DC$, there exists $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \|\beta_H(\alpha, \cdot, \boldsymbol{T})\|_{L^2(M)} = C.$$

Proof. By the normalization of invariant distribution in Sobolev space $W^s(M)$, there exists a function $f_\alpha^H \in W^s(H)$ such that $D_\alpha(f_\alpha^H) = \left\|f_\alpha^H\right\|_s = 1$. For all $\alpha \in DC(L)$, by asymptotic formula (34),

$$\left|\left\langle P_{U(\boldsymbol{T})}^{d,\alpha}m, \omega\right\rangle - \beta^f(\alpha, m, \boldsymbol{T})\right| \leq C_s(1+L).$$

Therefore, L^2-estimate follows from Corollary 4.2. $\qquad\qquad\qquad\qquad\square$

A relation between the Bufetov functional and the modular function θ_H is established below.

Corollary 4.4. *For any $L > 0$ and invariant probability measure supported on $DC(L) \subset \mathfrak{M}_g$,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H([\alpha]), \quad \text{for } \mu\text{-almost all } [\alpha] \in \mathfrak{M}_g.$$

Proof. By Theorem 3.8 and Lemma 4.1, there exists a constant $C > 0$ such that for all $\alpha \in supp(\mu) \subset DC(L)$, for all $T > 0$ we have

$$(36) \qquad \lim_{|U(\boldsymbol{T})|\to\infty} \|\beta_H(r_{\log \boldsymbol{T}}[\alpha], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T}}[\alpha])\|_{L^2(M)} \leq \frac{C_\mu}{vol(U(\boldsymbol{T}))^{1/2}}.$$

By Luzin's theorem, for any $\delta > 0$ there exists a compact subset $E(\delta) \subset \mathfrak{M}$ such that we have the measure bound $\mu(\mathfrak{M}\backslash E(\delta)) < \delta$ and the function $\beta_H(\alpha, \cdot, 1) \in L^2(M)$ depends continuously on $[\alpha] \in E(\delta)$. By Poincare recurrence, there is a full measure subset $E'(\delta) \subset E(\delta)$ of $\mathbb{R}^d$-action.

For every $\alpha_0 \in E'(\delta)$, there is diverging sequence (t_n) such that $\{r^{t_n}(\alpha_0)\} \subset E(\delta)$ and $\lim_{n\to\infty} r^{t_n}(\alpha_0) = (\alpha_0)$. By continuity of θ_H and β_H at $[\alpha_0]$, we have

$$(37) \quad \|\beta_H([\alpha_0], \cdot, 1) - \theta_H([\alpha_0])\|_{L^2(M)}$$
$$= \lim_{n\to\infty} \|\beta_H(r_{\log \boldsymbol{T_n}}[\alpha_0], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T_n}}[\alpha_0])\|_{L^2(M)} = 0.$$

Thus, we have $\beta_H([\alpha], \cdot, 1) = \theta_H([\alpha])$ for $[\alpha] \in E'(\delta)$. It follows that the set of equality fails has less than any $\delta > 0$, thus the identity holds for almost all $[\alpha]$. $\quad\square$

For all $\alpha \in Aut_0(\mathsf{H}^g)$, general smooth function $f \in W^s(M)$ for $s > s_d + 1/2$, f decompose an infinite sum, and the functional θ^f is defined by a convergent series.

$$(38) \qquad\qquad\qquad \theta^f(\alpha) = \sum_H D_\alpha^H(f)\theta_H(\alpha).$$

The following result is an extention to general asymptotic theorem from Corollary 4.4.

Theorem 4.5. *For all $\alpha \in Aut_0(\mathsf{H}^g)$, $f \in W^s(M)$ and $s > s_d + 1/2$,*

$$\lim_{n\to\infty} \left\|\frac{1}{vol(U(T_n))^{1/2}}\left\langle P_{U(T_n)}^{d,\alpha}m, \omega_f\right\rangle - \theta^f(r_{\log \boldsymbol{T_n}}\alpha)\right\|_{L^2(M)} = 0.$$

4.2. Proof of Theorem 1.5. By Theorem 4.5, we summarize our results on limit distributions for higher rank actions.

Theorem 4.6. *Let (T_n) be any sequence such that*

$$\lim_{n\to\infty} r_{\log \boldsymbol{T_n}}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_d + 1/2$, which is not a coboundary, the limit distribution of the family of random variables

$$E_{T_n}(f) := \frac{1}{vol(U(T_n))^{1/2}} \left\langle P_{U(T_n)}^{d,\alpha}(\cdot), \omega_f \right\rangle$$

exists and is equal to the distribution of the function $\theta^f(\alpha_\infty) = \beta(\alpha, \cdot, 1) \in L^2(M)$. If $\alpha_\infty \in DC$, then $\theta^f(\alpha_\infty)$ is bounded function on M, and the limit distribution has compact support.

Proof of Theorem 1.5. Since $\alpha_\infty \in \mathfrak{M}_g$, the existence of limit follows from Corollary 4.4 and Lemma 4.5. $\square$

A relation with Birkhoff sum and theta sum was introduced in [CF15, §5.3], and as an applications, we derive limit theorem of theta sums.

Corollary 4.7. *Let $\mathscr{Q}[x] = x^\top \mathscr{Q} x$ be the quadratic forms defined by $g \times g$ real matrix $\mathscr{Q}$, $\alpha = \begin{pmatrix} I & 0 \\ \mathscr{Q} & I \end{pmatrix} \in Sp_{2g}(\mathbb{R})$, $\ell(x) = \ell^\top x$ be the linear form defined by $l \in \mathbb{R}^g$. Then, Theta sum*

$$\Theta(\mathscr{Q}, l; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0,N]} e(\mathscr{Q}[n] + \ell(n))$$

has limit distribution and it has compact support.

5. L^2-LOWER BOUNDS

In this section we prove bounds for the square mean of ergodic integrals along the leaves of foliations of the torus into circles transverse to central direction.

5.1. **Structure of return map.**
Let $\mathbb{T}_\Gamma^{g+1}$ denote $(g+1)$-dimensional torus with standard frame (X_i, Y_i, Z) with

$$\mathbb{T}_\Gamma^{g+1} := \{\Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

It is convenient to work with the polarized Heisenberg group. Set $H_{pol}^g \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ equipped with the group law $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx')$.

Definition 5.1. Reduced standard Heisenberg group H_{red}^g is defined by quotient $H_{pol}^g / (\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}/\frac{1}{2}\mathbb{Z}$. Reduced standard latice Γ_{red}^g is $\mathbb{Z}^g \times \mathbb{Z}^g \times \{0\}$ and the quotient $H_{red}^g / \Gamma_{red}^g$ is isomorhic to standard Heisenberg manifold H^g / Γ.

Now, we consider return map of $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}_\Gamma^{g+1}$. For $x = (x_1, \cdots, x_g) \in \mathbb{R}^g$,

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta \in \mathbb{R}^d.$$

In H_{red}^g,

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x).$$

Then, given $(n, m, 0) \in \Gamma_{red}^g$,

(39) $\quad \exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) \cdot (n, m, 0) = \exp(x_1' X_1^\alpha + \cdots + x_g' X_g^\alpha) \cdot (0, y', z')$

if and only if $x_\alpha' = x_\alpha + n$, $y' = y + (x_\beta - x_\beta') + m$ and $z' = z + (w - w') \cdot x + n(y + x_\beta)$.

Assume $\langle X_i^\alpha, X_j \rangle \neq 0$ for all i, j, and we write first return time $t_{Ret} = (t_{Ret,1}, \cdots t_{Ret,g})$ for $\mathsf{P}^{d,\alpha}$ on transverse torus $\mathbb{T}_\Gamma^{g+1}$. We denote domain for return time $U(t_{Ret}) = [0, t_{Ret,1}] \times \cdots [0, t_{Ret,g}]$. Return map of action $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}_\Gamma^{g+1}$ has a form of skew-shift

$$(40) \qquad A_{\rho,\tau}(y,z) = (y + \rho, z + v \cdot y + \tau_i) \quad \text{on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z}.$$

From computation of each rank 1 action, for each $1 \leq i \leq d$, it is a composition of commuting linear skew-shift

$$(41) \qquad A_{i,\rho,\tau}(y,z) = (y + \rho_i, z + v_i \cdot y + \tau_i) \quad \text{on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z}$$

for some constant $\rho_i, \sigma_i \in \mathbb{R}^g$ and $v_i \in \mathbb{R}^g$. For each $j \neq k$,

$$A_{j,\rho,\tau} \circ A_{k,\rho,\tau} = A_{k,\rho,\tau} \circ A_{j,\rho,\tau}.$$

Given pair $(\mathbf{m}, n) \in \mathbb{Z}_{K|n|}^g \times \mathbb{Z}$, let $H_{(\mathbf{m},n)}$ denote the corresponding factor and $C^\infty(H_{(m,n)})$ be subspace of smooth function on $H_{(m,n)}$. Denote $\{e_{\mathbf{m},n} \mid (\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z}\}$ the basis of characters of $\mathbb{T}_\Gamma^{g+1}$ and for all $(y,z) \in \mathbb{T}^g \times \mathbb{T}$,

$$e_{\mathbf{m},n}(y,z) := \exp[2\pi\iota(\mathbf{m} \cdot y + nKz)].$$

For each $A_{i,\rho,\sigma}$ and $v_i = (v_{i1}, \cdots v_{id})$, the orbit can be identified with the following dual orbit

$$\mathcal{O}_{A_i}(\boldsymbol{m}, n) = \{(\boldsymbol{m} + (nj_i)v_i, n), \ j_i \in \mathbb{Z}\}$$
$$= \{(m_1 + (nv_{i1})j_i, \cdots, m_d + (nv_{id})j_i, n), \ j_i \in \mathbb{Z}\}.$$

If $n = 0$, the orbit $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$ of $(m, 0)$ is reduced to a single element. If $n \neq 0$, then the dual orbit $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$ of $(\mathbf{m}, n)$ for higher rank actions is described as follows:

$$\mathcal{O}_A(\boldsymbol{m}, n) = \{(m_k + n\sum_{i=1}^d (v_{ik}j_i), n)_{1 \leq k \leq d} : j = (j_1, \cdots j_d) \in \mathbb{Z}^d\}.$$

It follows that every A-orbit for rank $\mathbb{R}^d$-action (or A^j-orbit) can be labeled uniquely by a pair $(\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z}\backslash\{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. Thus, the subspace of functions with non-zero central character can be split as a direct sum of components $H_{(m,n)}$ with $\mathbf{m} \in \mathbb{Z}^g, n \in \mathbb{Z}\backslash\{0\}$. Then,

$$L^2(\mathbb{T}_\Gamma^{g+1}) = \bigoplus_{\omega \in \mathcal{O}_A} H_\omega.$$

Now we proceed to the consideration of the higher cohomology problem which appears in the space of Fourier coefficients.

5.2. **Higher cohomology for $\mathbb{Z}^d$-action of skew-shifts.** We consider a $\mathbb{Z}^d$ action of return map $\mathsf{P}^{d,\alpha}$ on torus $\mathbb{T}_\Gamma^{g+1}$. By identification of cochain complex on torus, it is equivalent to consider the following cohomological equation for degree d form ω,

$$(42) \qquad \omega = d\Omega \iff \varphi(x,t) = D\Phi(x,t), \quad x \in \mathbb{T}^g, \ t \in \mathbb{Z}^d.$$

We restrict our interest of d-cocycle $\varphi : \mathbb{T}_\Gamma^{g+1} \times \mathbb{Z}^d \to \mathbb{R}$ with $\Phi : \mathbb{T}_\Gamma^{g+1} \to \mathbb{R}^d$, $\Phi = (\Phi_1, \cdots \Phi_d)$ and $\mathcal{D}$ is coboundary operator $\mathcal{D}\Phi = \sum_{i=1}^d (-1)^{i+1}\Delta_i \Phi_i$ where $\Delta_i \Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$. The following proposition is the generalization to the argument of [KK95, Prop 2.2]. Let us denote $A^j = A_{1,\rho,\tau}^{j_1} \circ \cdots \circ A_{d,\rho,\tau}^{j_d}$.

Proposition 5.2. *A cocycle $\hat{\varphi}$ satisfies cohomological equation* (42) *if and only if* $\sum_{j \in \mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j = 0$ *for* $j = (j_1, \cdots, j_d) \in \mathbb{Z}^d$.

Proof. We consider a dual equation

(43)
$$\hat{\varphi} = \mathcal{D}\hat{\Phi}.$$

Let us use the following notation:

$$(\delta_i \hat{\varphi})(m_1, \cdots, m_d) = \delta(m_i)\hat{\varphi}(m_1, \cdots, m_d), \quad \text{and} \quad \delta(0) = 1, \text{ otherwise } 0.$$

$$(\Sigma_i \hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=-\infty}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

$$(\Sigma_i^+ \hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=m_i}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

$$(\Sigma_i^- \hat{\varphi})(m_1, \cdots, m_d) = -\sum_{j=-\infty}^{m_i-1} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

It is clear that $\Sigma_i^- - \Sigma_i^+ = \Sigma_i$ and $\Sigma_i^+ \hat{\varphi} = \Sigma_i^- \hat{\varphi}$ if and only if $\Sigma_i \hat{\varphi} = 0$.

Note that

(44)
$$\Sigma_i^+ \Delta_i = \Sigma_i^- \Delta_i = id, \quad \Delta_i \Sigma_i^+ = \Delta_i \Sigma_i^- = id.$$

By direct calculation of Fourier coefficient, $\Sigma_i(\hat{\varphi} - \delta_i \Sigma_i \hat{\varphi}) = 0$. Let $\hat{\Phi}_i(\hat{\varphi}) = \Sigma_i^-(\hat{\varphi} - \delta_i \Sigma_i \hat{\varphi})$, then $\hat{\Phi}_i(\hat{\varphi})$ vanishes at ∞. By (44),

$$\hat{\varphi} - \delta_i \Sigma_i \hat{\varphi} = \Delta_i \hat{\Phi}_i(\hat{\varphi}),$$

and we can proceed this by induction

$$\hat{\varphi} - \Sigma_{1,\cdots,d}\hat{\varphi} = \sum_{i=1}^{d}(\delta_1 \cdots \delta_{i-1}\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi} - \delta_1 \cdots \delta_i \Sigma_1 \cdots \Sigma_i \hat{\varphi})$$

$$= \sum_{i=1}^{d}(\delta_1 \cdots \delta_{i-1}\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi} - \delta_i \Sigma_i(\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi}))$$

$$= \sum_{i=1}^{d}(-1)^{i+1}\Delta_i \hat{\Phi}_i(\hat{\varphi})$$

where

$$\hat{\Phi}_i(\hat{\varphi}) = (-1)^{i+1}\Sigma_i^- \delta_1 \cdots \delta_{i-1}(\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi} - \delta_i \Sigma_i(\Sigma_1 \cdots \Sigma_{i-1}\hat{\varphi})).$$

Thus, $\hat{\Phi}_i$ is a solution of (43) if and only if $\Sigma_1 \cdots \Sigma_d \hat{\varphi} = 0$. $\qquad\square$

For fixed $(\mathbf{m}, n) \in \mathbb{Z}^g \times \mathbb{Z}$, we denote obstruction of cohomological equation restricted to the orbit of $(\mathbf{m}, n)$ by $\mathcal{D}_{m,n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j$.

Lemma 5.3. *There exists a distributional obstruction to the existence of a smooth solution $\varphi \in C^\infty(H_{(\mathbf{m},n)})$ of the cohomological equation* (42).

A generator of the space of invariant distribution $\mathcal{D}_{m,n}$ has form of

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) := \begin{cases} e^{-2\pi\iota \sum_{i=1}^{d}[(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]} & \text{if } (a,b) = (m_k + K\sum_{i=1}^{d}(v_{ik}j_i), n)_{1 \le k \le g} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From previous observation, there exists obstruction

$$(45) \qquad \mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}_\Gamma^{g+1}} \varphi(x,y)\overline{e_{\mathbf{m},n} \circ A_{\rho,\tau}^j}dxdy.$$

By direct computation, for fixed $j = (j_1, \cdots j_d)$,

$$e_{\mathbf{m},n} \circ A_{\rho,\tau}^j(y,z) = \prod_{i=1}^d \left(e^{2\pi\iota[(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}\right)\left(e^{2\pi\iota(\mathbf{m}\cdot y + K(z + n\sum_{k=1}^d (v_{ik}j_i)y_k))}\right).$$

Then, we choose $\varphi = e_{a,b}$ for $(a,b) = (m_k + K\sum_{i=1}^d (v_{ik}j_i), n)_{1 \le k \le g}$ in the non-trivial orbit $(n \neq 0)$,

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi\iota\sum_{i=1}^d [(\mathbf{m}\cdot\rho_i + nK\tau_i)j_i + nK\tau_i\binom{j_i}{2}]}.$$

$\square$

5.3. Changes of coordinates.
For any frame $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$, denote transverse cylinder for any $m \in M$,

$$\mathcal{C}_{\alpha,m} := \{m\exp(\sum_{i=1}^g y_i'Y_i^\alpha + z'Z) \mid (y',z') \in U(t_{Ret}^{-1}) \times \mathbb{T}\}.$$

Let $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \to \mathcal{C}_{\alpha,m}$ denote the maps: for any $\xi \in \mathbb{T}_\Gamma^{g+1}$, let $\xi' \in \mathcal{C}_{\alpha,m}$ denote first intersection of the orbit $\{\mathsf{P}_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}_+^d\}$ with transverse cylinder $\mathcal{C}_{\alpha,m}$. Then, there exists a first return time to the cylinder $t(\xi) = (t_1(\xi), \cdots, t_d(\xi)) \in \mathbb{R}_+^d$ such that

$$\xi' = \Phi_{\alpha,m}(\xi) = \mathsf{P}_{t(\xi)}^{d,\alpha}(\xi), \quad \forall \xi \in \mathbb{T}_\Gamma^{g+1}.$$

Let (y,z) and (y',z') denote the coordinates on $\mathbb{T}_\Gamma^{g+1}$ and $\mathcal{C}_{\alpha,m}$ given by the exponential map respectively,

$$(y,z) \to \xi_{y,z} := \Gamma\exp(\sum_{i=1}^g y_iY_i + zZ), \quad (y',z') \to m\exp(\sum_{i=1}^g y_i'Y_i^\alpha + z'Z).$$

Recall that if $\alpha \in Sp_{2g}(\mathbb{R})$, then for $1 \le i,j \le g$ there exist matrices $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$ such that

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2g}(\mathbb{R}),$$

satisfying $A^tD - C^tB = I_{2g}$, $C^tA = A^tC$, $D^tB = B^tD$, and $\det(A) \neq 0$. Set

$$X_i^\alpha = \sum_j a_{ij}X_j + b_{ij}Y_j + w_iZ \quad \text{and} \quad Y_i^\alpha = \sum_j c_{ij}X_j + d_{ij}Y_j + v_iZ.$$

Let $x = \Gamma\exp(\sum_{i=1}^d y_{x,i}Y_i + z_xZ)\exp(\sum_{i=1}^d t_{x,i}X_i)$, for some $(y_x, z_x) \in \mathbb{T}^d \times \mathbb{R}/K\mathbb{Z}$ and $t_x = (t_{x,i}) \in [0,1)^d$. Then, the map $\Phi_{\alpha,x} : \mathbb{T}_\Gamma^{g+1} \to \mathcal{C}_{\alpha,m}$ is defined by $\Phi_{\alpha,x}(y,z) = (y',z')$ where

$$(46) \qquad \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_g' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix},$$

and $z' = z + P(\alpha, x, y)$ for some degree 4 polynomial P.

Therefore, the map $\Phi_{\alpha,x}$ is invertible with

$$\Phi_{\alpha,x}^*(dy_1' \wedge \cdots dy_g' \wedge dz') = \frac{1}{det(A)} dy_1 \wedge \cdots dy_g \wedge dz.$$

Since $A^t D - C^t B = I_{2g}$, by direct computation we obtain return time, we have

$$(47) \quad \begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}.$$

Then,

$$\|t(\xi)\| \leq \max_i |t_i(\xi)|^g \leq \max_i |\sum_{j=1}^g d_{ij} t_{x,i} + c_{ij}(y_i - y_{x,i})|^g \leq \max_i \|Y_i^\alpha\|^g.$$

5.4. L^2-lower bound of functional.

We will prove bounds for the square mean of integrals along foliations of the torus $\mathbb{T}_\Gamma^{g+1}$ into torus $\{\xi \exp(\sum_{i=1}^g y_i Y_i) \mid y_i \in \mathbb{T}\}_{\xi \in \mathbb{T}_\Gamma^{g+1}}$.

Lemma 5.4. *There exists a constant $C > 0$ such that for all $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$, and for every irreducible component H of central parameter $n \neq 0$, there exists a function f_H such that*

$$|f_H|_{L^\infty(H)} \leq C vol(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)|,$$

$$|f_H|_{\alpha,s} \leq C vol(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)|(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s (1 + n^2)^{s/2}$$

where $\|Y\| := \max_{1 \leq i \leq g} \|Y_i^\alpha\|$ and $T(t_{Ret}) = \sum_{i=1}^g t_{Ret,i}$.

On rectangular domain $U(\boldsymbol{T})$, for all $m \in \mathbb{T}_\Gamma^{g+1}$ and $T^{(i)} \in \mathbb{Z}_{t_{Ret,i}}$,

$$(48) \quad \left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y} m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = |\mathcal{D}_\alpha^H(f_H)| \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2}.$$

In addition, whenever $H \perp H' \subset L^2(M)$ the functions

$$\left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y} m), \omega_H \right\rangle \quad and \quad \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y} m), \omega_{H'} \right\rangle$$

are orthogonal in $L^2(\mathbb{T}^g, dy)$.

Proof. As explained in §5.1, the space $L^2(\mathbb{T}_\Gamma^{d+1})$ decompose as a direct sum of irreducible subspaces invariant under the action of each $A_{j,\rho,\sigma}$. It follows that the subspace of functions with non-zero central character can be split as direct sum of components $H_{(\mathbf{m},n)}$ with $(\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \backslash \{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. For $F \in H_{(\mathbf{m},n)}$, the function is characterized by Fourier expansion

$$F = \sum_{j \in \mathbb{Z}^d} F_j e_{A^j(\boldsymbol{m},n)} = \sum_{j \in \mathbb{Z}^d} F_j e_{(m_k + K \sum_{i=1}^d (v_{ik} j_i), n)}.$$

Then, by Lemma 5.3,

$$(49) \quad \mathcal{D}_{(\mathbf{m},n)}(e_{A^j(\boldsymbol{m},n)}) = e^{-2\pi \iota \sum_{i=1}^d [(m \cdot \rho_i + nK\tau_i) j_i + nK\tau_i \binom{j_i}{2})]}.$$

For any irreducible representation $H := H_n$ with central parameter $n \neq 0$, there exists $\mathbf{m} \in \mathbb{Z}_{|n|}^d$ such that the operator I_α maps the space H onto $H_{(\mathbf{m},n)}$. The operator $I_\alpha : L^2(M) \to L^2(\mathbb{T}_\Gamma^{g+1})$ is defined

$$(50) \qquad\qquad f \to I_\alpha(f) := \int_{U(t_{Ret})} f \circ \mathsf{P}_x^{d,\alpha}(\cdot)dx.$$

Then, operator I_α is surjective linear map of $L^2(M)$ onto $L^2(\mathbb{T}_\Gamma^{g+1})$ with right inverse defined as follows:

Let $\chi \in C_0^\infty(0,1)^g$ be any function of jointly integrable with integral 1. For any $F \in L^2(\mathbb{T}_\Gamma^{g+1})$, let $R_\alpha^\chi(F) \in L^2(M)$ be the function defined by

$$R_\alpha^\chi(F)(\mathsf{P}_v^{d,\alpha}(m)) = \frac{1}{vol(U(t_{Ret}))}\chi(\frac{v}{t_{Ret}})F(m), \ (m,v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows from the definition that there exists a constant $C_\chi > 0$ such that

$$|R_\alpha^\chi(F)|_{\alpha,s} \leq C_\chi vol(U(t_{Ret}))^{-1}(1 + \sum_{i=1}^g t_{Ret,i}^{-1}\|Y_i^\alpha\|)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}$$

$$\leq C_\chi vol(U(t_{Ret}))^{-1}(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}.$$

Choose $f_H := R_\alpha^\chi(e_{m,n}) \in C^\infty(H)$ such that $I_\alpha(f_H) = e_{m,n}$ and

$$(51) \qquad\qquad \int_{U(t_{Ret})} f_H \circ \mathsf{P}_t^{d,\alpha}(y,z)dt = e_{m,n}(y,z), \ \text{for } (y,z) \in \mathbb{T}_\Gamma^{g+1}.$$

For P-action, the space of invariant currents $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-s}(\mathbb{R}^g)$ for $s > d/2 + \epsilon$ for all $\epsilon > 0$. That is, by normalization of invariant distributions in the Sobolev space, for any irreducible components $H = H_n$, there exists a non-unique function f_α^H such that

$$(52) \qquad\qquad\qquad D_\alpha(f_\alpha^H) = \left\|f_\alpha^H\right\|_s = 1.$$

By (49), we have $|\mathcal{D}_H(f_H)| = |\mathcal{D}_{(m,n)}(e_{m,n})| = 1$. Therefore, it follows that

$$|f_H|_{L^\infty(H)} \leq C_\chi vol(U(t_{Ret}))^{-1}$$

$$|f_H|_{\alpha,s} \leq C vol(U(t_{Ret}))^{-1}|D_\alpha^H(f_H)|(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s(1 + n^2)^{s/2}.$$

Moreover, since $\{e_{m,n} \circ A_{\rho,\tau}^j\}_{j\in\mathbb{Z}^d} \subset L^2(\mathbb{T}_\Gamma^g, dy)$ is orthonormal, we verify

$$\left\|\left\langle P_{U(\mathbf{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y}m), \omega_H\right\rangle\right\|_{L^2(\mathbb{T}^g,dy)} = \left\|\sum_{j_d=0}^{[\frac{T^{(d)}}{t_{Ret,d}}]-1} \cdots \sum_{j_1=0}^{[\frac{T^{(1)}}{t_{Ret,1}}]-1} e_{m,n} \circ A_{\rho,\tau}^j\right\|_{L^2(\mathbb{T}^g,dy)}$$

$$= \left(\frac{vol(U(\mathbf{T}))}{vol(U(t_{Ret}))}\right)^{1/2}.$$

$\square$

Recall that

$$(53) \qquad\qquad L^2(M) = \bigoplus_{n\in\mathbb{Z}} H_n := \bigoplus_{n\in\mathbb{Z}} \bigoplus_{i=1}^{\mu(n)} H_{i,n}$$

where $H_n = \bigoplus_{i=1}^{\mu(n)} H_{i,n}$ is irreducible representation with central parameter n and $\mu(n)$ is countable by Howe-Richardson multiplicity formula.

For any infinite dimensional vector $\mathbf{c} := (c_{i,n}) \in l^2$, let $\beta_{\mathbf{c}}$ denote Bufetov functional

$$\beta_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} \beta^{i,n}.$$

For any $\mathbf{c} := (c_{i,n})$, let $|\mathbf{c}|_s$ denote the norm defined as

$$|\mathbf{c}|_s^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{i=1}^{\mu(n)} (1 + K^2 n^2)^s |c_{i,n}|^2.$$

From Corollary 4.3,

$$\|\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T})\|_{L^2(M)}^2 \le C^2 |\mathbf{c}|_{l^2}^2 vol(U(\mathbf{T})).$$

Lemma 5.5. *For any $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for all $\alpha \in DC(L)$, for all $\mathbf{c} \in l^2$, for all $z \in \mathbb{T}$ and all $T > 0$,*

$$(54) \quad \left| \|\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T})\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\mathbf{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right|$$

$$\le C_s (vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1 + L)(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.$$

Proof. By (53), for every $n \ne 0$, there exists a function $f_{i,n} \in C^\infty(H)$ with $D(f_{i,n}) = 1$. Let $f_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n}$, by adding functions on all irreducibles. Then, by Lemma 5.4,

$$(55) \qquad |f_{\mathbf{c}}|_{L^\infty(M)} \le C|\mathbf{c}|_{l^1}$$

$$(56) \qquad |f_{\mathbf{c}}|_{\alpha,s} \le Cvol(U(t_{Ret}))^{-1}(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.$$

By orthogonality,

$$\left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha} \circ \mathsf{Q}_y^{g,Y}, \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = \left(\frac{vol(U(\mathbf{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0.$$

From the estimation for each $f_{i,n}$ in Lemma 5.4, for every $z \in \mathbb{T}$ and all $T > 0$, we have

$$\left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \le 2|f_{\mathbf{c}}|_{L^\infty(M)} \|Y\|.$$

Let $T_{\alpha,i} = t_{Ret,i}([T/t_{Ret,i}] + 1)$ and $U(t_\alpha) = [0, T_{\alpha,1}] \times \cdots \times [0, T_{\alpha,g}]$. Then,

$$\left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle P_{U(t_\alpha)}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \le vol(U(t_{Ret}))|f_{\mathbf{c}}|_{L^\infty(M)}.$$

Therefore, for some constant $C' > 0$ such that

$$\left| \left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\mathbf{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \le C' vol(U(t_{Ret}))|\mathbf{c}|_{l^1}.$$

For all $s > s_d + 1/2$, by asymptotic property of Theorem 3.8, there exists constant $C_s > 0$ such that

$$\left| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha} m, \omega \right\rangle - \beta_H(\alpha, m, \boldsymbol{T}) D_\alpha^H(f_H) \right| \leq C_s(1+L) \left\| f \right\|_{\alpha,s}.$$

Applying $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$ and combining bounds on the function $f_{\mathbf{c}}$ with (55),

$$\left| \left\| \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}) \right\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right|$$

$$\leq C' vol(U(t_{Ret})) |\mathbf{c}|_{l^1} + C_s vol(U(t_{Ret}))^{-1} (1+L) |f_{\mathbf{c}}|_{\alpha,s}$$

$$\leq C'_s (vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1+L)(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \left\| Y \right\|)^s |\mathbf{c}|_s.$$

Therefore, we derive the estimates in the statement. $\qquad\square$

6. ANALYTICITY OF FUNCTIONALS

In this section we will prove that for all $\alpha \in DC$, the Bufetov functionals on any square are real analytic.

Definition 6.1. For every $t \in \mathbb{R}$, $1 \leq i \leq d$, and $m \in M$, the *stretched (in direction of Z) rectangle* is denoted by

$$(57) \qquad\qquad [\Gamma_T]_{i,t}^Z(m) := \{(\phi_{ts_i}^Z) \circ \mathsf{P}_{\boldsymbol{s}}^{d,\alpha}(m) \mid \boldsymbol{s} \in U(\mathbf{T})\}.$$

Recall Definition 1.1 for standard rectangles. For $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}^d$, let us denote $\Gamma_{\mathbf{T}}(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma_d(s_d))$ for $\gamma_i(s_i) = \exp(s_i X_i)$. Similarly, we also write

$$(58) \qquad\qquad [\Gamma_T]_{i,t}^Z(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma_{i,t}^Z(s_i), \cdots \gamma_d(s_d))$$

where $\gamma_{i,t}^Z(s_i) := \phi_{ts_i}^Z(\gamma_i(s_i))$ is a stretched curve.

Definition 6.2. The *restriction* $\Gamma_{T,i,s}$ of the rectangle $\Gamma_{\mathbf{T}}$ is defined on restricted domain $U_{T,i,s} = [0, T^{(1)}] \times \cdots \times [0, s] \cdots \times [0, T^{(d)}]$ for $s \leq T^{(i)}$ as following.

$$\Gamma_{T,i,s}(\mathbf{s}) := \Gamma_{\mathbf{T}}(\mathbf{s}), \; \mathbf{s} \in U_{T,i,s}.$$

Recall the orthogonal property on a irreducible component H (central parameter $n \in \mathbb{Z} \backslash \{0\}$). For any $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$ and $t \in \mathbb{R}$,

$$(59) \qquad\qquad \beta_H(\alpha, \phi_t^Z(m), \mathbf{T}) = e^{2\pi\iota Knt} \beta_H(\alpha, m, \mathbf{T}).$$

We obtain the following lemma for stretched rectangle by applying orthogonal property.

Lemma 6.3. *For fixed elements (X_i, Y_i, Z) satisfying commutation relation (1), the following formula for rank 1 action holds:*

$$\hat{\beta}_H(\alpha, [\Gamma_T]_{i,t}^Z) = e^{2\pi\iota t n K T^{(i)}} \hat{\beta}_H(\alpha, \Gamma_T) - 2\pi\iota n K t \int_0^{T^{(i)}} e^{2\pi\iota n K t s_i} \hat{\beta}_H(\alpha, \Gamma_{T,i,s}) ds_i.$$

Proof. Let $\alpha = (X_i, Y_i, Z)$ and ω be d-form supported on a single irreducible representation H. We obtain following the formula for stretches of curve $\gamma_{i,t}^Z$ (see [FK20b, §4, Lemma 9.1]),

$$\frac{d\gamma_{i,t}^Z}{ds_i} = D\phi_{ts_i}^Z(\frac{d\gamma_i}{ds_i}) + tZ \circ \gamma_{i,t}^Z.$$

It follows that pairing is given by

$$\langle[\Gamma_{\mathbf{T}}]^Z_{i,t}, \omega\rangle = \int_{U(\mathbf{T})} \omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma^Z_{i,t}}{ds_i}(s_i), \cdots, \frac{d\gamma_d}{ds_d}(s_d))d\mathbf{s}$$

$$= \int_{U(\mathbf{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))] + \iota_Z\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t}(\mathbf{s})d\mathbf{s}$$

Denote $d-1$ dimensional triangle $U_{d-1}(\mathbf{T})$ with $U(\mathbf{T}) = U_{d-1}(\mathbf{T}) \times [0, T^{(i)}]$. Integration by parts for a fixed i-th integral gives

$$\int_{U(\mathbf{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$

$$= e^{2\pi\iota nKtT^{(i)}} \int_{U(\mathbf{T})} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$

$$- 2\pi\iota nKt \int_0^{T^{(i)}} e^{2\pi\iota nKts_i} \int_{U_{d-1}(\mathbf{T})} \left(\int_0^{s_i} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_i}{ds_i}(r) \cdots, \frac{d\gamma_d}{ds_d}(s_d))]dr\right)d\mathbf{s}.$$

Then, we have the following formula

$$\langle[\Gamma_{\mathbf{T}}]^Z_{i,t}, \omega\rangle = e^{2\pi\iota nKtT^{(i)}}\langle[\Gamma_{\mathbf{T}}], \omega\rangle - 2\pi\iota nKt \int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\langle\Gamma_{T,i,s}, \omega\rangle ds_i$$

$$+ \int_{U(\mathbf{T})}(\iota_Z\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s}.$$

Since the action of $\mathsf{P}^{d,X}_{\mathbf{t}}$ for $\mathbf{t} \in \mathbb{R}^d$ is identity on the center Z,

$$\lim_{t_d\to\infty} \cdots \lim_{t_1\to\infty} e^{-(t_1+\cdots t_d)/2} \int_{U(\mathbf{T})} (\iota_Z(\mathsf{P}^{d,X}_{\mathbf{t}})^*\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s} = 0.$$

Thus, it follows by definition of Bufetov functional, the statement holds. $\qquad\square$

Here we define a restricted vector $\mathbf{T}_{i,s}$ of $\mathbf{T} = (T^{(1)}, \cdots, T^{(d)}) \in \mathbb{R}^d$. For fixed i, pick $s_i \in [0, T^{(i)}]$ such that $\mathbf{T}_{i,s} \in \mathbb{R}^d$ is a vector with its coordinates

$$T^{(j)}_{i,s} = \begin{cases} T^{(j)} & \text{if} \quad j \neq i \\ s_i & \text{if} \quad j = i. \end{cases}$$

Similarly, $\mathbf{T}_{i_1,\cdots,i_k,s}$ is a vector with $i_1, \cdots i_k$ coordinates replaced by $s_{i_1}, \cdots, s_{i_k}$.

Lemma 6.4. *Let $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. The following equality holds for each rank-1 action.*

(60) $\quad \beta_H(\alpha, \phi^{Y_i}_{y_i}(m), \mathbf{T}) =$

$$e^{-2\pi\iota y_i nKT^{(i)}}\beta_H(\alpha, m, \mathbf{T}) + 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{-2\pi\iota y_i nKs_i}\beta_H(\alpha, m, \mathbf{T}_{i,s})ds_i.$$

Proof. By definition (3), (58) and commutation relation (1), it follows that

$$\phi^{Y_i}_{y_i}(\Gamma^X_{\mathbf{T}}(m)) = [\Gamma^X_{\mathbf{T}}(\phi^{Y_i}_{y_i}(m))]^Z_{i,t}.$$

By the invariance property of Bufetov functional and Lemma 6.3,

$$\beta_H(\alpha, m, \mathbf{T}) = \hat{\beta}_H(\alpha, \phi_{y_i}^{Y_i}(\Gamma_T^X(m)))$$

$$= e^{2\pi\iota y_i nKT^{(i)}} \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))) - 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{2\pi\iota nKy_i s_i} \hat{\beta}_H(\alpha, \Gamma_{T,i,s}^X(\phi_{s_i}^{Y_i}(m))) ds_i$$

$$= e^{2\pi\iota y_i nKT^{(i)}} \beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) - 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{2\pi\iota nKy_i s_i} \beta_H(\alpha, \phi_{s_i}^{Y_i}(m), \mathbf{T}_{i,s}) ds_i.$$

Then statement follows immediately.   □

We extend previous lemma to higher rank actions by induction argument.

Lemma 6.5. *The following equality holds for rank-d action.*

$$\beta_H(\alpha, \mathsf{Q}_y^{d,Y}(m), \boldsymbol{T}) = e^{-2\pi\iota \sum_{j=1}^d y_j nKT^{(j)}} \beta_H(\alpha, m, \boldsymbol{T})$$

$$+ \sum_{k=1}^d \sum_{1 \le i_1 < \cdots < i_k \le d} \prod_{j=1}^k (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1, \cdots, i_k\}} y_l T^{(l)})}$$

$$\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})} \beta_H(\alpha, m, \boldsymbol{T}_{i_1, \cdots, i_k, s}) ds_{i_k} \cdots ds_{i_1}$$

Proof. Assume inductive hypothesis works for rank $d-1$. For convenience, we write

$$\mathsf{Q}_y^{d,Y}(m) = \phi_{y_d}^{Y_d} \circ \mathsf{Q}_{y'}^{d-1,Y}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 6.4,

$$\beta_H(\alpha, \mathsf{Q}_y^{d,Y}(m), \mathbf{T}) = e^{-2\pi\iota y_d nKT^{(d)}} \beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T})$$

(61)
$$+ 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d} \beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T}_{d,s}) ds_d$$

$$:= I + II$$

Firstly, by induction hypothesis

$$I = e^{-2\pi\iota \sum_{j=1}^{d-1} y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T})$$

$$+ \sum_{k=1}^{d-1} \sum_{1 \le i_1 < \cdots < i_k \le d-1} \prod_{j=1}^k (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1, \cdots, i_k\}} y_l T^{(l)} + y_d T^{(d)})}$$

$$\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \cdots, i_k, s}) ds_{i_k} \cdots ds_{i_1}$$

which contains 0 to $d-1$th iterated integrals containing $e^{-2\pi\iota nKy_d T^{(d)}}$ outside of iterated integrals.

For the second part, we apply induction hypothesis again for restricted rectangle $\mathbf{T}_{d,s}$. Then,

$$
II = 2\pi\iota n K y_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d n K s_d}\big[e^{-2\pi\iota \sum_{j=1}^{d-1} y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}_{d,s})\big] ds_d
$$

$$
+ \sum_{k=1}^{d-1} \sum_{1 \le i_1 < \cdots < i_k \le d-1} (2\pi\iota n K y_d) \prod_{j=1}^{k} (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)})}
$$

$$
\times \int_0^{T^{(d)}} \left(\int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} ds_{i_k} \cdots ds_{i_1} \right) ds_d
$$

$$
\times e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \cdots + y_{i_k} s_{i_k} + y_d s_d)} \beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,d,s}).
$$

The term II consist of 1 to d-th iterated integrals containing $e^{-2\pi\iota n K y_d s_d}$ inside of iterated integrals. Thus, combining these two terms, we prove the statement. $\quad\square$

For any $R > 0$, the *analytic norm* defined for all $\mathbf{c} \in l^2$ as

$$
\|\mathbf{c}\|_{\omega, R} = \sum_{n \ne 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}|.
$$

Let Ω_R denote the subspace of $\mathbf{c} \in \ell^2$ such that $\|c\|_{\omega,R}$ is finite.

Lemma 6.6. *For $\mathbf{c} \in \Omega_R$, any $\alpha \in DC(L)$ and $\mathbf{T} \in \mathbb{R}_+^d$, the function*

$$
\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}), \ (y,z) \in \mathbb{R}^d \times \mathbb{T}
$$

extends to a holomorphic function in the domain

$$
(62) \qquad D_{R,T} := \{ (y,z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \mid \sum_{i=1}^d |Im(y_i)| T^{(i)} + |Im(z)| < \frac{R}{2\pi K} \}.
$$

The following bound holds: for any $R' < R$ there exists a constant $C > 0$ such that, for all $(y,z) \in D_{R',T}$ we have

$$
|\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})|
$$

$$
\le C_{R,R'} \|c\|_{\omega,R} (L + vol(U(\mathbf{T}))^{1/2}(1 + E_M(a, \mathbf{T}))(1 + K \sum_{i=1}^d |Im(y_i)| T^{(i)}).
$$

Proof. By Lemma 6.5 and (59),

$$
\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}) = e^{(z - 2\pi\iota \sum_{j=1}^d y_j n K T^{(j)})} \beta_H(\alpha, m, \mathbf{T})
$$

$$
+ \sum_{k=1}^d \sum_{1 \le i_1 < \cdots < i_k \le d} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)})}
$$

$$
\times e^{2\pi\iota n K z} \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \cdots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,s}) ds_{i_k} \cdots ds_{i_1}.
$$

As a consequence, by Lemma 3.7 for each variable $(y_i, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$, Then for the rank d action, by induction, for $(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ we have

$$|\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(x), \mathbf{T})|$$

$$\leq (L + vol(U(\mathbf{T}))^{1/2}(1 + E_M(a, \mathbf{T}))[C_1 \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR}|c_{i,n}|e^{2\pi|Im(z - \sum_{i=1}^{d} T^{(i)}y_i)|nK}$$

$$+ \sum_{k=1}^{d} C_k \left(\sum_{1 \leq i_1 < \cdots < i_k \leq d} \prod_{j=1}^{k} (|Im(y_{i_j})|T^{(i_j)}) \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} n|c_{i,n}|e^{2\pi(|Im(z)| + \sum_{j=1}^{k} T^{(i_j)}|Im(y_{i_j})|)nK} \right)].$$

Therefore, the function $\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})$ is bounded by a series of holomorphic functions on $\mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ and it converges uniformly on compact subsets of domain $D_{R,T}$. $\qquad\square$

7. Measure estimation for bounded-type

In this section, we prove a measure estimation of Bufetov functional with bounded-type α. This result is based on generalization of §11 in [FK20b].

Let $\mathcal{O}_r$ denote the space of holomorphic functions on the ball $B_{\mathbb{C}}(0, r) \subset \mathbb{C}^n$. We recall the *Chebyshev degree*, the best constant $d_f(r)$ stated in the following theorem.

Theorem 7.1. *[Bru99, Theorem 1.9] For any $f \in \mathcal{O}_r$, there is a constant $d := d_f(r) > 0$ such that for any convex set $D \subset B_{\mathbb{R}}(0,1) := B_{\mathbb{C}}(0,1) \cap \mathbb{R}^n$, for any measurable subset $U \subset D$*

$$\sup_D |f| \leq \left(\frac{4nLeb(D)}{Leb(U)} \right)^d \sup_U |f|.$$

Let $\mathcal{L}_t$ denote the set of one-dimensional complex affine spaces $L \subset \mathbb{C}^n$ such that $L \cap B_{\mathbb{C}}(0, t) \neq \emptyset$.

Definition 7.2. [Bru99, Def 1.6] Let $f \in \mathcal{O}_r$. The number

$$\nu_f(t) := \sup_{L \in \mathcal{L}_t} \{\text{valency of } f \mid L \cap B_{\mathbb{C}}(0, t) \neq \emptyset\}$$

is called the *valency* of f in $B_{\mathbb{C}}(0, t)$.

By Proposition 1.7 of [Bru99], for any $f \in \mathcal{O}_r$, and the valency $\nu_f(t)$ is finite and any $t \in [1, r)$ there is a constant $c := c(r) > 0$ such that

$$(63) \qquad\qquad d_f(r) \leq c\nu_f(\frac{1+r}{2}).$$

Lemma 7.3. *Let $L > 0$ and $\mathscr{B} \subset DC(L)$ be a bounded subset. Given $R > 0$, for all $\mathbf{c} \in \Omega_R$ and all $\mathbf{T}^{(i)} > 0$, denote $\mathcal{F}(\mathbf{c}, \mathbf{T})$ by the family of real analytic functions of the variable $y \in [0, 1)^d$*

$$\mathcal{F}(\mathbf{c}, \mathbf{T}) := \{\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \mid (\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}\}.$$

Then there exist $\mathbf{T}_{\mathscr{B}} := (\mathbf{T}_{\mathscr{B}}^{(i)})$ and $\rho_{\mathscr{B}} > 0$, such that for every $(R, \mathbf{T})$ with $R/\mathbf{T}^{(i)} \geq \rho_{\mathscr{B}}$, $\mathbf{T}^{(i)} \geq \mathbf{T}_{\mathscr{B}}^{(i)}$ and for all $\mathbf{c} \in \Omega_R \backslash \{0\}$, we have

$$\sup_{f \in \mathcal{F}(\mathbf{c}, \mathbf{T})} \nu_f < \infty.$$

Proof. Since $\mathscr{B} \subset \mathfrak{M}$ is bounded, for each time $t_i \in \mathbb{R}$ and $1 \leq i \leq g$,

$$0 < t_{i,\mathscr{B}}^{min} = \min_i \inf_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} \leq \max_i \sup_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} = t_{\mathscr{B}}^{max} < \infty.$$

For any $\alpha \in \mathscr{B}$ and $x \in M$, the map $\Phi_{\alpha,x} : [0,1)^d \times \mathbb{T} \to \prod_{i=1}^{d}[0, t_{\alpha,i}) \times \mathbb{T}$ in (46) extends to a complex analytic diffeomorphism $\hat{\Phi}_{\alpha,x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$. By Lemma 6.6, it follows that for fixed $z \in \mathbb{T}$, real analytic function $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})$ extends to a holomorphic function defined on a region

$$H_{\alpha,m,R,t} := \{ y \in \mathbb{C}^d \mid \sum_{i=1}^{d} |Im(y_i)| \leq h_{\alpha,m,R,t} \}.$$

By boundedness of the set $\mathscr{B} \subset \mathfrak{M}$, it follows that

$$\inf_{(\alpha,x) \in \mathscr{B} \times M} h_{\alpha,m,R,t} := h_{R,T} > 0.$$

We remark that the function $h_{\alpha,m,R,t}$ and its lower bound $h_{R,T}$ can be obtained from the formula (46) for the polynomial $\Phi_{\alpha,x}$ and the definition of the domain $D_{R,T}$ in formula (62).

For every $r > 1$, there exists $\rho_{\mathscr{B}} > 1$ such that for every R and $\mathbf{T}$ with $R/\mathbf{T}^{(i)} > \rho_{\mathscr{B}}$,

$$(64) \qquad\qquad \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \in \mathcal{O}_r$$

as a function of $y \in \mathbb{T}^d$.

By Lemma 6.6, the family $\mathcal{F}(c, \mathbf{T})$ is uniformly bounded and normal. By Lemma 5.5, for sufficiently large pair $\mathbf{T}$, no sequence from $\mathcal{F}(c, \mathbf{T})$ can converge to a constant. Recall that by Lemma 10.3 of [FK20b], for any normal family $\mathcal{F} \subset \mathcal{O}_R$, no function is constant along a one-dimensional complex line. Therefore, combining with (64), the main statement follows. $\qquad \square$

We derive measure estimates of Bufetov functionals on the rectangular domain.

Lemma 7.4. *Let* $\alpha \in DC$ *such that the forward orbit of* $\mathbb{R}^d$-*action* $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t} \in \mathbb{R}_+^d}$ *is contained in a compact set of* $\mathfrak{M}_g$. *There exist* $R, C, \delta > 0$ *and* $T_0 \in \mathbb{R}_+^d$ *such that, for every* $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, $T \geq T_0$ *and for every* $\epsilon > 0$, *we have*

$$vol(\{m \in M \mid |\beta_{\boldsymbol{c}}(\alpha, m, \boldsymbol{T})| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

Proof. Since $\alpha \in DC$ and the orbit $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t} \in \mathbb{R}_+}$ is contained in a compact set, there exists $L > 0$ such that $r_{\boldsymbol{t}}(\alpha) \in DC(L)$ for all $\boldsymbol{t} \in \mathbb{R}_+^d$. Then, we choose $\mathbf{T}_0 \in \mathbb{R}^d$ from the conclusion of Lemma 7.3. By the scaling property of Bufetov functionals,

$$\beta_{\mathbf{c}}(\alpha, m, \boldsymbol{T}) = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(\boldsymbol{T_0}))} \right)^{1/2} \beta_{\mathbf{c}}(g_{\log(\boldsymbol{T}/\boldsymbol{T_0})}[\alpha], m, \boldsymbol{T_0}).$$

By Fubini's theorem, it suffices to estimate

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T_0})| \leq \epsilon\}).$$

Let $\delta^{-1} = c(r) \sup_{f \in \mathcal{F}(c,T_0)} v_f(\frac{1+r}{2}) < \infty$. Since by Lemma 5.5, we have

$$\inf_{(a,x,z) \in \mathscr{B} \times M \times \mathbb{T}} \sup_{y \in [0,1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T_0})| > 0.$$

By Theorem 7.1, for unit ball $D = B_{\mathbb{R}}(0,1)$ and $U = \{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}$ and bound in formula (63), there exists a constant $C > 0$ and $\delta > 0$ such that for all $\epsilon > 0$ and $(\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}$,

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}) \leq C\epsilon^{\delta}.$$

Then the statement follows from Fubini theorem. $\qquad\qquad\square$

Corollary 7.5. *Let α be as in the previous Lemma 7.4. There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}_+^d$ such that, for every $\mathbf{c} \in \Omega_R \backslash \{0\}$, $T \geq T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{x \in M \mid \left|\langle P_{U(\boldsymbol{T})}^{d,\alpha} m, \omega_{\mathbf{c}} \rangle\right| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

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