Ergodic theory and Dynamical systems

Deviation of ergodic averages of higher step nilflows.

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August 16, 2021

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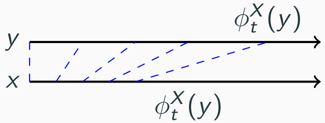
Dynamical systems

Dynamical systems is the study of the long-term behavior (evolution) in complex systems.

- Let $\phi_t^X : M \to M$ be a flow, i.e. one parameter family of transformations.
- The trajectory $\{\phi_t^X(x), t \geq 0\}$ of a point $x \in M$ is called orbit.
- A flow is called parabolic if nearby orbits diverge polynomially, i.e $D\phi_X^t = O(t^d)$.



Example

Area-preserving flow, Horocycle flow, Nilflow.

Motivation

Definition

A flow is ergodic (w.r.t probability measure μ) if for any invariant set A,

$$\mu(\phi_t^X(A)) = \mu(A) \Longrightarrow \mu(A) = 0 \text{ or } 1.$$

By Birkhoff ergodic theorem, for a.e $x \in M$

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt - \int_M f \; d\mu \right| \to 0, \quad \text{as } T \to \infty. \tag{1}$$

We call an orbit of x is equidistributed with respect to μ if (1) holds.

- Effective equidistribution means finding the error bound on the speed of convergence of ergodic averages.

Motivation

For parabolic flows, there were not many results until mid of 90's.

- M. Ratner('86), M. Burger('90), Sarnak('88) proved results for deviations of ergodic averages for horocycle flow on negative constant curvature.
- A. Zorich proved deviation of ergodic averages for interval exchange maps ('97).
- Forni proved the deviation of area-preserving flows ('02) based on his work on cohomological equation for compact surface of higher genus ('97).
- Marmi-Moussa-Yoccoz ('06) solved cohomological equations for interval exchange maps.

Related results

Deviation of ergodic averages for parabolic flows:

- Area-preserving flow (Forni 02)
- Horocycle flow (Flaminio-Forni 03)
- Heisenberg nilflow (Flaminio-Forni 06)

- Higher rank actions on Heisenberg manifold (Flaminio-Cosentino 15)

- Higher step nilflow (Quasi-abelian) (Flaminio-Forni 14)
- Twisted horocycle flow (Flaminio-Forni-Tanis 15)
- Twisted translation flow (Forni 19)

Bufetov's perspective

A. Bufetov proved deviation of ergodic integrals in terms of finitely-additive Hölder measure. It is called Bufetov functional or cocycle and used in proving limit theorems of parabolic flows.

- Translation flow (Bufetov 14) $\Longleftarrow$ (Forni '02)
- Horocycle flow (Bufetov-Forni 14) $\Longleftarrow$ (Flaminio-Forni '03)
- Heisenberg nilflow (Forni-Kanigowski '20) $\Longleftarrow$ (Flaminio-Forni '06)
- Higher rank actions on Heisenberg $\Longleftarrow$ (Cosentino-Flaminio '15)

- Self-similar tiling (Bufetov-Solomyak 13)
- Interval exchange transformation (Klimenko 19)

* Limit shape for IET : Marmi-Moussa-Yoccoz ('14) / M-Ulcigrai-Y ('20)

Background: nilflows

Setting

Let G be connected, simply connected nilpotent Lie group. There exists lower central series

$$G = G^{(1)} \supset G^{(2)} \cdots \supset G^{(k)} = \{e\}.$$

Equivalently, for Lie algebra $\mathfrak{g}$

$$\mathfrak{g} \supset \mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}^{(k)} = [\cdots[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0.$$

- We call step the smallest number k that satisfies $G^{(k)} = \{e\}$.
- A nilmanifold is the quotient $M = \Gamma \backslash G$ of a nilpotent Lie group by lattice $\Gamma < G$.
- $\mathfrak{S}$ is generator of $\mathfrak{g}$ if the smallest sub-algebras containing $\mathfrak{S}$ is $\mathfrak{g}$.

Examples

Definition (Heisenberg Lie algebra)

A Lie algebra $\mathfrak{g}$ is called Heisenberg if $\mathfrak{g} = span\{X, Y, Z\}$ with $[X, Y] = Z \in \mathfrak{Z}(\mathfrak{g})$.

$\mathfrak{Z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0, \text{for any Y} \in \mathfrak{g}\}$ is center of $\mathfrak{g}$.

Denote Heisenberg nilmanifold by $M := G/\Gamma$.

$$M := H_3(\mathbb{R})/H_3(\mathbb{Z}) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{R}/\mathbb{Z}$$

Relations with torus

Definition (Nilflows)

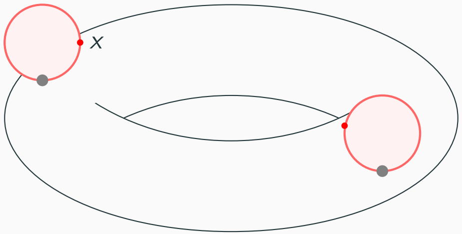
Nilflow ϕ_t^X is defined by one-parameter subgroup such that

$$\phi_t^X(x) = x\exp(tX), \quad x \in M, \quad X \in \mathfrak{g}.$$

Heisenberg nilflow can be understood on torus.

$$\phi_t^X(x) = x\exp((\alpha_1 X + \alpha_2 Y + \alpha_3 Z))$$

There exists a projection $p_1 : M \to \mathbb{T}^2 = \exp(aX + bY)/\Gamma$.



Relations with torus

There exists a projection $p_2 : M \to \mathbb{T}^2 = \exp(bY + cZ)$.

$$r(y, z) = (y + \alpha, z + y + \beta)$$

Definition (Quasi-abelian Lie algebra)

A k-step quasi-abelian filiform nilpotent Lie group G is expressed by its Lie algebra $\mathfrak{g} = span\{X, Y_i\}$ whose descending central sequence has length k with $[X, Y_i] = Y_{i+1}$ for $1 \leq i \leq k-1$.

- This idea can be generalized to higher dimensional $\mathbb{T}^d$.
- First return map to the torus can be written as a skew shift.
- In general nilmanifolds, we do not have a nice return map.

Properties of nilflows

Definition (Nilflow)

Let M be nilmanifold and $\mathfrak{g}$ be its Lie algebras. Nilflow ϕ_t^X is defined by

$$\phi_t^X(x) = x\exp(tX), \quad x \in M, \quad X \in \mathfrak{g}.$$

Theorem (L.Green, L. Auslander, F.Hahn)

The followings are equivalent:

- *The nilflow* $(\phi_t^X)_{t\in\mathbb{R}}$ *is ergodic;*
- *Uniquely ergodic;*
- *Minimal, i.e. all orbits are dense;*
- *The projected flow is an irrational flow on a base torus.*

Note: $G^{ab} = G/[G,G] \simeq \mathbb{R}^n$ is abelianization and there always exists projection $p : M \to \mathbb{T}^n$.

Mixing

Nilflow is never mixing but it is recently proved that mixing property is obtained by its time changes. (Shearing property)

- Heisenberg nilflow [Avila-Forni-Ulcigrai 11']
- Filiform [Ravotti 19']
- Higher step nilmanifolds. [AFUR '21]
- Multiple mixing on Heisenberg nilflow [Forni-Kanigowski '20]
- Decay of correlation on Heisenberg nilflow [F-K '20]

It is open for quantitative mixing and higher order mixing on general nilmanifolds.

Effective equidistribution of higher step nilflows

Results on nilmanifolds

Theorem (Green-Tao '12)

If the projected linear toral flow has a Diophantine frequency, then there exists a constant $C > 0$ and exponent $\delta \in (0, 1)$ such that for all Lipschitz function f on nilmanifold M,

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt - \int_M f \; dvol \right| \leq C \|f\|_{Lip} T^{-\delta}.$$

- Effective equidstribution of Heisenberg nilflows was obtained by Flaminio-Forni ('06). There is an upper bound $T^{-1/2+\epsilon}$ for all $x \in M$.
- Butterley-Simonelli ('20) proved deviation of ergodic averages of Heisenberg nilflows using transfer operator.
- Our question is whether it is possible to prove effective equidistribution of ergodic averages on any higher step nilmanifolds.

Results on nilmanifolds

Theorem (Flaminio-Forni '14)

Let (ϕ_X^t) be a nilflow on k-step filiform nilmanifold M which projects to a linear toral flow on $\mathbb{T}^2$ with Diophantine frequency of exponent $\nu \in [1, k/2]$. For every $\epsilon > 0$, there exists a positive measurable function $K_\epsilon(x) \in L^p(M)$ with $p \in [1, 2)$ such that

$$\left| \frac{1}{T} \int_0^T f \circ \phi_X^t(x) dt - \int_M f \ vol \right| \leq C_s K_\epsilon(x) T^{-\frac{2}{3k(k-1)}+\epsilon} \|f\|_s$$

holds for every function $f \in W^s(M)$ and for almost all $x \in M$ and $T \geq 1$.

Applications in Weyl sums

Definition (Weyl sum)

$$W_N(a_k, \cdots, a_0) = \sum_{n=0}^{N-1} e^{2\pi i P(n)}, \quad P(n) = \sum_{i=0}^{k} a_i n^i$$

- The result on Heisenberg nilflows provides the bound of Weyl sums for quadratic polynomials (theta sum).
- It generalized upper bound on the asymptotic behavior of theta sums in the work of Fiedler, Jurkat and Körner.
- Wooley ('14) obtained the (quadratic) bound of Weyl sum.

Strictly triangular nilmaniofold

Definition

A step 3 strictly triangular Lie algebra $\mathfrak{g}$ is a nilpotent Lie algebra with its basis $\mathcal{F} = \{X_1, X_2, X_3, Y_1, Y_2, Z\}$ with commutation relations:

$$[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_2$$

$$[X_1, Y_2] = Z = [Y_1, X_3]$$

$$H_6(\mathbb{R}) := \begin{pmatrix} 1 & x_1 & y_1 & z \\ 0 & 1 & x_2 & y_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_i, y_j, z \in \mathbb{R}.$$

This model can be generalized to k step triangular.

Main result - step 3 triangular

Theorem

Let (ϕ_t^X) be a nilflow on 3-step triangular nilmanifold M, and projected flow $(\phi_t^{\overline{X}})$ satisfies Roth-type Diophantine frequency. For every $\epsilon > 0$ and $s > 29$ there exists $C > 0$ such that the following holds: for every function $f \in W^s(M)$ and for all $x \in M$ and $T \geq 1$,

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt - \int_M f \ dvol \right| \leq C_s T^{-\frac{1}{12}+\epsilon} \left\| f \right\|_s .$$

Definition (Roth type)

For $\alpha = (1, \alpha_2, \cdots, \alpha_d)$ and $m \in \mathbb{Z}^d$, α satisfies Roth type Diophantine condition if for all $\epsilon > 0$, the following hold with exponent $\nu = 1 + \epsilon$.

$$|m \cdot \alpha| \geq \frac{C(\alpha, \epsilon)}{\left\| m \right\|^{\nu}}$$

Main result - Generalization

- Quesiton: Can we extend this result to the k-step triangular nilmanifold?
  Yes.

Definition (Transversality condition)

Let $\mathfrak{g}$ be nilpotent Lie algebra satisfying *transversality condition* if there exists a basis of $\mathfrak{g}$ such that

$$\langle \mathfrak{G} \rangle \oplus \mathrm{Ran}(\mathrm{ad}_X) + C_{\mathfrak{I}}(X) = \mathfrak{g}$$

$\mathfrak{G}$ is a set of generator, and $C_{\mathfrak{I}}(X)$ is centralizer in codimension 1 ideal $\mathfrak{I}$.

Main theorem

Theorem

Let (ϕ_t^X) be a nilflow on k-step nilmanifold M with $\mathfrak{g}$ satisfying transversality conditions. For every $\epsilon > 0$, there exists $K_\epsilon \in L^p(M)$ and $1 < p < 2$ such that the following holds: for every function $f \in C^\infty(M)$ and for almost all $x \in M$ and $T \geq 1$,

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt - \int_M f \ dvol \right| \leq K_\epsilon(x) T^{-\frac{1}{3S(n,k)}+\epsilon} \|f\|_s .$$

n : the number of generators of $\mathfrak{g}$.

- k-step Quasi-abelian filiform (Flaminio-Forni 14) : $T^{-\frac{2}{3k(k+1)}+\epsilon}$
- Step 3 triangular (K 17) : $T^{-\frac{1}{12}+\epsilon}$ for all $x \in M$
- Triangular (K 19) : $T^{-\frac{1}{3(k-1)(k^2+k-3)}+\epsilon}$

Main steps of the proof.

Cohomological equation

Definition (Cohomological equation.)

The following functional equation is called Cohomological equation.

$$L_X u = Xu = f$$

Definition (Invariant distribution.)

Given cohomological equation $Xu = f$, we define X-invariant distribution D such that the following holds:

$$XD(f) = D(Xf) = 0, \quad \forall f \in C^\infty(M)$$

- Invariant distributions are obstructions to solving cohomological equations.
- Flaminio-Forni (07) proved solutions of cohomological equations for nilflows.

Cohomological equation - example

Example

On $\mathbb{T}^2$, $X = \alpha_1 \partial_x + \alpha_2 \partial_y$

$$Xu = f$$

By Fourier series, for $k = (k_1, k_2)$

$$(2\pi \iota k \cdot \alpha) \hat{u}(k_1, k_2) = \hat{f}(k_1, k_2).$$

If $k \cdot \alpha \neq 0$, then

$$\hat{u}(k_1, k_2) = \frac{\hat{f}(k_1, k_2)}{(2\pi \iota k \cdot \alpha)}.$$

If $k = (0, 0)$, then it is necessary to have

$$\hat{f}(0, 0) = 0 \iff D(f) = 0.$$

Sketch of the proof

Forni's observation: In ergodic averages, we view it as a distribution

$$\gamma_x^T(f) := \frac{1}{T}\int_0^T f\circ\phi_t^X(x)dt = D(T)(f) + R(T)(f)$$

Example

On $\mathbb{T}^2$, $D(T) = \mu$.

If f is coboundary, (assuming zero averages)

$$\left|\frac{1}{T}\int_0^T f\circ\phi_t^X(x)dt\right| = \frac{1}{T}\left|u\circ\phi_T^X(x) - u(x)\right| \leq \frac{2}{T}\|u\|_\infty$$

By Harmonic analysis, we obtain

$$\|u\|_\infty \leq C\|f\|_{C^r(\mathbb{T}^2)}.$$

Sobolev norm

Definition (Sobolev norm)

We denote $f \in W^s(M)$ if

$$\|f\|_s = \sum_{i+j \leq s} \left\|X^i Y^j f\right\|_{L^2(M)}$$

Theorem (Sobolev embedding theorem)

For any $s > g + 1/2$, there exists a constant $B_s > 0$ such that for any $f \in W^s(M)$,

$$\|f\|_\infty \leq B_s \|f\|_s,$$

where $B_s = \sup_{f \in W^s} \frac{\|f\|_\infty}{\|f\|_s}$ is called best Sobolev constant.

Sobolev embedding theorem

Theorem (Sobolev embedding)

Let $s > dim(M)/2$. For all $f \in W^s(M)$,

$$|\gamma_x^T(f)| \leq B_s(\mathcal{F}) \, \|f\|_s \, ,$$

where $B_s(\mathcal{F})$ is called Sobolev constant.

- To replace the bound of $B_s(\mathcal{F})$ in terms of time T, it is inevitable to rescale the frame in time.
- In Heisenberg case, we call renormalization flow

$$g_t : \alpha = (X, Y, Z) \mapsto (e^t X, e^{-t} Y, Z).$$

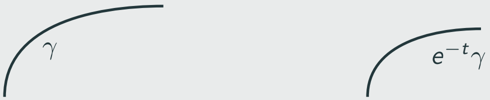
- It reduces controlling the bound of invariant distributions and remainders (backward iteration)

$$\gamma_x^T(f) = \sum_{i \in \mathbb{N}} C_{D_i}(x, T) D_i(T)(f) + R(T)(f).$$

Renormalization

Example (d=1)

$$\gamma_x^T(f) = \frac{1}{T}\int_0^T f \circ \phi_s^X(x)ds = \frac{1}{e^{-t}T}\int_0^{e^{-t}T} f \circ \phi_s^{e^tX}(x)ds.$$



- However, there is no renormalization flows on higher step nilmanifolds. (No more Sobolev constant!)
- Instead, we rescale each vector field and it behaves like renormalization. This is called the rescaling method.

Sobolev trace theorem

Lemma (Sobolev trace theorem)

For any $s > \dim(M)/2$*, there is a constant* $C_s > 0$ *such that the following holds.*

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt \right| \leq \frac{C_s}{T^{\frac{1}{2}} w_{\mathcal{F}}(x, T)^{\frac{1}{2}}} \|f\|_s .$$

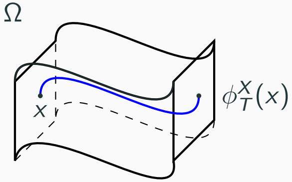
- Bound of averaged width is reduced to ergodic averages of close return function and it ends up estimation of invariant distribution.
- We call a point $x \in M$ is 'Good' if averaged width is not too small. We prove the set of Good points has a full measure in M.

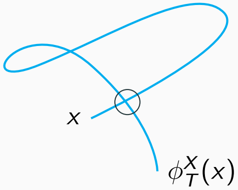
Main lemma

Estimation of width is reduced to return orbits on transverse manifold.

- Set $O_{x,T} := \{\Omega \subset \mathbb{R} \times [-1/2, 1/2]^a \mid [0, T] \times \{0\} \subset \Omega\}$
- Let $w_{\mathcal{F}}(x, T)$ be averaged width

$$w_{\mathcal{F}(t)}(x, T) := \sup_{\Omega \subset O_{x,T}} \left(\frac{1}{T} \int_0^T \frac{ds}{w_\Omega(s)} \right)^{-1}.$$





Remark

Final remark. There are still many issues left to control the average width in higher step case.

- Lack of good return map: the measure of the set of close return (almost periodic) in the transverse manifold should be small.
- Transversality condition is necessary since the set of close return can be too large.

  $$\langle \mathfrak{G} \rangle \oplus \mathrm{Ran}(\mathrm{ad}_X) + C_{\mathfrak{I}}(X) = \mathfrak{g}$$

  $\mathfrak{G}$ is a set of generator, and $C_{\mathfrak{I}}(X)$ is centralizer in codimension 1 ideal $\mathfrak{I}$.
- It is conjectured that desired bound for higher step should work for all $x \in M$.