LIMIT THEOREMS FOR HIGHER RANK ACTIONS ON HEISENBERG NILMANIFOLDS

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ABSTRACT. The main result of this paper is a construction of finitely additive measures for higher rank action on Heisenberg nilmanifold. Under a full measure set of Diophantine condition of the generators of the action, we construct *Bufetov functional* on rectangles on $2g + 1$ dimensional Heisenberg manifold. We prove the deviation of ergodic integral of higher rank actions by its asymptotics to Bufetov functionals for a sufficiently smooth function. In this paper, we derive limit distribution which proves normalized ergodic integrals to have variance 1 converges in distribution to a nondegenerate compactly supported measure on the real line.

CONTENTS

1. Introduction

1.1. **Introduction.** The asymptotic behavior and limiting distribution of ergodic averages of translation flow was studied by Alexander Bufetov in the series of works [Buf09, Buf10, Buf14]. He constructed finitely-additive Hölder measures and cocycles over translation flows that are known as *Bufetov functionals*. The construction of such functional is used to derive deviation of ergodic averages and results on probabilistic behavior of ergodic averages of translation flows. Interestingly, it is discovered that there is a duality between the Bufetov functionals and invariant distributions which plays a key role in the work of G. Forni [For02]. Following these observation, such functionals and duality (or bijective correspondence) between the space of invariant distributions called Flaminio-Forni space are also constructed in the sense of other parabolic flows: horocycle flows [FF03, BF14] and Heisenberg nilflows [FF06, FK17].

In this paper, our main results are on limit distributions of higher rank abelian actions. We firstly introduce the Bufetov functional for higher rank abelian actions under bounded type of Diophantine conditions. Our main argument is based on the renormalization argument for higher rank actions by induction argument. This is a key idea used in Cosentino and Flaminio in [CF15], but we extend their constructions to rectangular shape and derive the deviation of ergodic averages of higher rank actions. Likewise, this explains the duality between Bufetov functionals and invariant currents appeared in [CF15]. The crucial part is handling the estimation of deviation ergodic averages on (stretched) rectangles, and this enables to derive our main theorems. (See. e.g [Rav19, For].)

As a corollary, we can prove there exists a limit distributions of (normalized) ergodic integrals of abelian actions with variance 1. More specifically, for almost all limit of normalized ergodic integrals of converges in distribution to a nondegenerate compactly supported measure on the real line, which is certain form of Bufetov functionals. This generalizes a limit theorem for theta series on Siegel half spaces, which introduced in the works of Götze and Gordin [GG04] and Marklof [Mar99]. (See [Tol78, MM07, MNN07] for general introduction and nilflow case [GM14, CM16].)

A possible application of this paper will be proving the mixing of time-changes of abelian higher rank actions and estimate on the (polynomial) decay of correlation. This is realized by the analyticity of Bufetov functional on higher dimensional rectangular domain and polynomial type of lower bounds of sub-level sets of analytic function. Such polynomial estimates on the measure are derived by real analyticity of a functional along the leaves of a foliation transverse to the actions based on results of [Bru99]. Work in progress of the author [Kim] indicates that the *shearing* method developed in the proof of mixing for a dense set of time-changes for uniquely ergodic nilflows is also a crucial step in that direction.

In rank 1 case, mixing property of time-changes of nilflows has been studied recently. Nilflows are never mixing but time-changes destroys the elliptic behavior

on the base torus, which are sensitive in perturbations. Mixing property of time changes of Heisenberg nilflows was firstly studied in [AFU11]. Then, it was extended in the dense set of non-trivial time changes for any uniquely ergodic nilflows on general nilmanifolds [Rav18, AFRU19]. As a special example, on time-changes of Heisenberg flows, decay of correlation [FK17] and multiple mixing [FK20] was proved by analyticity of Bufetov functional and Ratner property, respectively.

In the higher rank setting, mixing of $\mathbb{Z}^k$-actions (or mixing of shapes) for automorphisms on nilmanifold is proved in [GS14, GS15]. Related results of $\mathbb{R}^k$-actions, however, are not well-known on higher step nilmanifolds. It is also interesting question if it is possible to construct Bufetov functionals on higher step nilmanifolds ($s > 2$). Since moduli space is trivial and there is no known renormalization flows, we could not obtain the results in the desired method. However, we still hope other methods in handling non-renormalizable flows may be possibly applied. (Cf. [FF14, FFT16, Kim20])

1.2. **Definition and statement of results.** We review definitions about Heisenberg manifold and its moduli space.

1.2.1. *Heisenberg manifold.* Let H^g be standard $2g + 1$ dimensional Heisenberg group and set $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$ a discrete and co-compact subgroup of H^g. We shall call it standard lattice of H^g and the quotient $M := \mathsf{H}^g/\Gamma$ will be called *Heisenberg manifold.* Lie algebra $\mathfrak{h}^g = Lie(\mathsf{H}^g)$ of H^g is equipped with basis $(X_1, \cdots, X_g, Y_1, \cdots Y_g, Z)$ satisfying canonical commutation relation

$$[X_i, Y_j] = \delta_{ij} Z. \tag{1}$$

For $1 \leq d \leq g$, let $\mathsf{P}^d < \mathsf{H}^g$ be the subgroup with Lie algebra $\mathfrak{p}$ generated by $(X_1, \cdots, X_d)$ and for any $\alpha \in Sp_{2g}(\mathbb{R})$, set $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$ for $1 \leq i \leq d$. We define parametrization of the subgroup $\alpha^{-1}(\mathsf{P}^d)$

$$\mathsf{P}_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \cdots + x_d X_d^\alpha), \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

By central extension of $\mathbb{R}^{2g}$ by $\mathbb{R}$, we have an exact sequence

$$0 \to Z(H^g) \to H^g \to \mathbb{R}^{2g} \to 0.$$

The natural projection map $pr : M \to H^g/(\Gamma Z(H^g))$ maps M onto a $2g$-dimensional torus $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

1.2.2. *Moduli space.* The group of automorphisms of H^g that are trivial on the center is $Aut_0(\mathsf{H}^g) = Sp(2g, \mathbb{R}) \ltimes \mathbb{R}^{2g}$. Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to $Sp(2g, \mathbb{R})$. We call *moduli space* of the standard Heisenbeg manifold the quotient $\mathfrak{M}_g = Sp(2g, \mathbb{R})/Sp(2g, \mathbb{Z})$. We regard $Sp(2g, \mathbb{R})$ as the deformation space of the the standard Heisenbeg manifold M and $\mathfrak{M}_g$ as the moduli space of M.

Siegel modular variety is double coset space $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ where K_g is maximal compact subgroup $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$ of $Sp_{2g}(\mathbb{R})$.

For $\alpha \in Sp_{2g}(\mathbb{R})$, we denote $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$ its projection on the moduli space $\mathfrak{M}_g$ and write $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$ the projection of α to the Siegel modular variety Σ_g. Double coset $K_g \backslash Sp_{2g}(\mathbb{R})/1_{2g}$ is identifed with $\mathfrak{H}_g := \{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$ *Siegel upper half space* of genus g, and consequently $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$.

1.2.3. *Sobolev bundles.* Given basis (V_i) of Lie algebra, let $\Delta = -\sum V_i^2$ denote Laplacian via the standard basis. Similarly, denote Δ_α Laplacian defined by the basis $(\alpha^{-1})_* V_i$. For any $s \in \mathbb{R}$ and any C^∞ function $f \in L^2(M)$,

$$\|f\|_{\alpha,s} = \langle f, (1 + \Delta_\alpha)^s f\rangle^{1/2}.$$

Let $W_\alpha^s(M)$ be the completion of $C^\infty(M)$ with above norm and denote $W_\alpha^{-s}(M)$ its dual space. Extending it to the exterior algebra, define the Sobolev spaces $\Lambda^d \mathfrak{p} \otimes W^s(M)$ of cochains of degree d, and use the same notations for the norms.

The group $Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \to Sp_{2g}(\mathbb{R}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \to \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

the fiber over $[\alpha] \in \mathfrak{M}_g$ is locally identified with the space $W_\alpha^s(M)$.

$Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles by

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma = (\alpha\gamma, \gamma^*\varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

By invariance of $Sp_{2g}(\mathbb{Z})$ action, we denote the class (α, φ) by $[\alpha, \varphi]$ and write $Sp_{2g}(\mathbb{Z})$-invariant Sobolev norm

$$\|(\alpha, f)\|_s := \|f\|_{\alpha,s}.$$

Main results. One of the main objects of this paper is a space of finitely-additive measures defined on the space of all squares on Heisenberg manifold M. We state our results beginning with an overview of Bufetov functional.

Theorem 1.1. *Let $\hat{B}_H(M)$ be the space of all functionals $\hat{\beta}_H$ on irreducible representation H. Assign every rectangle $\Gamma \subset M$ a complex number $\hat{\beta}_H(\Gamma) \in \mathbb{C}$ so that the following holds:*

(1) (Additive property) *For any decomposition of disjoint rectangles $\Gamma = \bigcup_{i=1}^n \Gamma_i$ or those intersections have zero measure,*

$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^n \hat{\beta}_H(\alpha, \Gamma_i).$$

(2) (Scaling property) *For $\boldsymbol{t} \in \mathbb{R}^d$,*

$$\hat{\beta}_H(r_{\boldsymbol{t}}[\alpha], \Gamma) = e^{-(t_1 + \cdots t_d)/2} \hat{\beta}_H(\alpha, \Gamma).$$

(3) (Invariance property) *For any action $\mathsf{Q}_\tau^{d,Y}$ generated by Y_i's and $\tau \in \mathbb{R}_+^d$,*

$$\hat{\beta}_H(\alpha, (\mathsf{Q}_\tau^{d,Y})_*\Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

(4) (Bounded proprety) *For any rectangle Γ, there exists a constant $C(\Gamma) > 0$ such that for $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$,*

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

Definition 1.2. For $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$, denote the rectangle of action of P,

$$(2) \qquad \Gamma_{\mathbf{T}}^X(m) = \{\mathsf{P}_{\boldsymbol{t}}^{d,\alpha}(m) \mid \boldsymbol{t} \in [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]\}$$

and define $\beta_H(\alpha, m, \mathbf{T}) := \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(m))$.

For arbitrary rectangle $\Gamma = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]$, pick $\mathbf{T}'^{(i)} \in [0, \mathbf{T}^{(i)}]$ for each i to decompose Γ into 2^d sub-rectangles. We write $\mathbf{P}(\mathbf{T}')$ collection of 2^d vertices $v = (v^{(1)}, v^{(2)}, \cdots, v^{(d)})$ where $v^{(i)} \in \{0, \mathbf{T}'^{(i)}\}$. Let Γ_v be a rectangle whose sides $\mathbf{I}_v = (I^{(1)}, I^{(2)}, \cdots, I^{(d)}) \in \mathbb{R}_+^d$ where

$$I^{(l)} = \begin{cases} T^{(l)} - T'^{(l)} & \text{if} \quad v^{(l)} = T'^{(l)} \\ T'^{(l)} & \text{if} \quad v^{(l)} = 0. \end{cases}$$

Then, we have $\Gamma = \bigcup_{v \in \mathbf{P}(\mathbf{T}')} \Gamma_v$.

Theorem 1.3. *The function β_H satisfies following properties:*

(1) (Cocycle property) *For all $(m, \mathbf{T}_1, \mathbf{T}_2) \in M \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\beta_H(\alpha, m, \mathbf{T}_1 + \mathbf{T}_2) = \sum_{v \in \mathbf{P}(\mathbf{T}_1)} \beta_H(\alpha, \mathsf{P}_v^{d,\alpha}(m), \mathbf{I}_v).$$

(2) (Scaling property) *For all $m \in M$,*

$$\beta_H(r_{\mathbf{t}}\alpha, m, \mathbf{T}) = e^{(t_1 + \cdots t_d)/2} \beta_H(\alpha, m, \mathbf{T}).$$

(3) (Bounded proprety) *Let us denote largest length of side $T_{max} = \max_i \mathbf{T}^{(i)}$. Then, there exists a constant $C_H > 0$ such that*

$$\beta_H(\alpha, m, \mathbf{T}) \leq C_H T_{max}^{d/2}.$$

(4) (Orthogonality) *For all $[\alpha] \in DC$ and all $\mathbf{T} \in \mathbb{R}^d$, bounded function $\beta_H(\alpha, \cdot, \mathbf{T})$ belongs to the irreducible component, i.e,*

$$\beta_H(\alpha, \cdot, \mathbf{T}) \in H \subset L^2(M).$$

By representation theory introduced (4), for any $f \in W^s(M)$ has a decomposition

$$f = \sum_H f_H$$

and we define *Bufetov cocycle* associated to f (or form ω) as the sum

$$(3) \qquad \beta^f(\alpha, m, \mathbf{T}) = \sum_H D_\alpha^H(f)\beta_H(\alpha, m, \mathbf{T}).$$

Remark. *For convenience, identification of distributions of form ω and function f were used. The formula (3) yields a duality between the space of Basic (closed) currents and invariant distributions. In similar setting for horocycle flow, refer [BF14, Cor 1.2, p.10].*

Given a Jordan region U and a point $m \in M$, set $P_U^{d,\alpha}m$ the Birkhoff sums associated to some $m \in M$ for the action $\mathsf{P}_x^{d,\alpha}$ given by

$$\left\langle P_U^{d,\alpha}m, \omega \right\rangle := \int_U f(\mathsf{P}_x^{d,\alpha}m) dx_1 \cdots dx_d$$

for any degree $\mathfrak{p}$-form $\omega = f\hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$, with $f \in C_0^\infty(M)$ (smooth function with zero averages).

Let the family of random variable

$$E_{T_n}(f) := \frac{1}{vol(U(T_n))^{1/2}} \left\langle P_{U(T_n)}^{d,\alpha}(\cdot), \omega_f \right\rangle,$$

and we are interested in asymptotic behavior of the probability distributions of $E_{T_n}(f)$. Our goal is to understand the asymptotics of E_{T_n}.

Theorem 1.4. *For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_{d,g} = d(d + 11)/4 + g + 1/2$, which is not a coboundary, the limit distribution of the family of random variables $E_{T_n}(f)$ exists, and for almost all frequency α, it has compact support on the real line.*

Outlines of the paper. In section 2, we give basic definitions on Higher rank actions, moduli spaces and Sobolev spaces. In section 3, we state main theorem and prove constructions of Bufetov functionals with main properties. In section 4, we prove asymptotic formula of Birkhoff integrals and its limit theorems. In section 5 and 6, we prove L^2-lower bound and analyticity of Bufetov functional. In section 7, there is a measure estimates of Bufetov functionals on the sets where they are small. This result only holds when frame α is of bounded type.

2. Analysis on Heisenberg manifolds

In this section, we will recall definitions of currents, representation and renormalization on moduli space.

2.1. **Invariant currents.** We denote the bundle of $\mathfrak{p}$-forms of degree j of Sobolev order s by $A^j(\mathfrak{p}, \mathfrak{M}^s)$. Similarly, there is a flat bundle of distribution $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$ whose fiber over $[\alpha]$ is locally identified with the space $W_\alpha^{-s}(M)$ normed by $\|\cdot\|_{\alpha,-s}$.

In the following, we set $\omega^{d,\alpha} = dX_1^\alpha \wedge \cdots \wedge dX_d^\alpha$ a top dimensional $\mathfrak{p}$-form and identify d dimensional currents $\mathcal{D}$ with distributions, for any $f \in C^\infty(M)$

$$\langle \mathcal{D}, f \rangle := \langle \mathcal{D}, f\omega^{d,\alpha} \rangle.$$

Definition 2.1. For $s > 0$, we denote $D \in Z_d(\mathfrak{p}, W^{-s}(M))$ a closed P-invariant currents of dimension d and Sobolev order s. Then, from formal identities, $\langle D, X_i^\alpha(f) \rangle = 0$ for all test function f and $i \in [1, d]$.

By [CF15, Prop 3.13], for any $s > d/2$ with $d = \dim \mathsf{P}$, denote $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ the space of P-invariant currents of Sobolev order s, which coincides with the space of closed currents of dimension d.

- It is one dimensional space if $\dim \mathsf{P} = g$, or an infinite-dimensional space if $\dim \mathsf{P} < g$. We have $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$ for all $\epsilon > 0$.

Let $\omega \in \Lambda^d \mathfrak{p}' \otimes W^s(\mathbb{R}^g)$ with $s > (d+1)/2$. Then, ω admits a primitive Ω if and only if $T(\omega) = 0$ for all $T \in I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. We may have $\Omega \in \Lambda^{d-1}\mathfrak{p}' \otimes W^t(\mathbb{R}^g)$ for any $t < s - (d+1)/2$.

2.2. **Representation.** We write Hilbert sum decomposition

$$(4) \qquad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n$$

into closed H^g-invariant subspaces. For some fixed $K > 0$, we write $f = \sum_{n \in \mathbb{Z}} f_n \in L^2(M)$, $f_n \in H_n$ where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi \iota n K t)f\}.$$

We also have $W^s(M) = \bigoplus_i W^s(H_i)$ of $W^s(M)$ into closed H^g-invariant subspaces $W^s(H_i) = W^s(M) \cap H_i$. The center $Z(H^g)$ has spectrum $2\pi\mathbb{Z}\backslash\{0\}$ the space splits as Hilburt sum of H^g-module H_i, which is equivalent to irreducible representation π.

Theorem 2.2. *[Stone-Von Neumann] For* $\alpha = (X_i, Y_i, Z)$, *the unitary irreducible representation* π *of the Heisenberg group of non-zero central parameter* K, *is unitarily equivalent to Schrödinger representation. For infinitesimal representation with parameter* n *for* $k = 1, 2, \cdots, g$

$$D\pi(X_k) = \frac{\partial}{\partial x_k}, \quad D\pi(Y_k) = 2\pi \iota n K x_k, \quad D\pi(Z) = 2\pi \iota n K.$$

2.3. **Best Sobolev constant.** The *Sobolev embedding theorem* implies that for any $\alpha \in Sp(2g, \mathbb{R})$ and $s > g + 1/2$, there exists a constant $B_s(\alpha)$ such that for any $f \in W_\alpha^s(M)$,

$$\|f\|_\infty \leq B_s(\alpha) \|f\|_{s,\alpha}.$$

The *best Sobolev constant* is defined as the function on the group of automorphism

$$(5) \qquad B_s(\alpha) = \sup_{f \in W_\alpha^s(M)} \frac{\|f\|_\infty}{\|f\|_{s,\alpha}}.$$

By Proposition 4.8 of [CF15], there exists a universal constant $C(s) > 0$ such that the *best Sobolev constant* satisfies the estimate

$$(6) \qquad B_s([[\alpha]]) \leq C(s) \cdot (Hgt[[\alpha]])^{1/4}.$$

From the Sobolev embedding theorem and the definition of the best Sobolev constant, we have the following bound.

Lemma 2.3. *[CF15, Lemma 5.5] For any Jordan region* $U \subset \mathbb{R}^d$ *with Lebesgue measure* $|U|$, *for any* $s > g + 1/2$ *and all* $m \in M$,

$$\left\| [\alpha, (P_U^{d,\alpha} m)] \right\|_{-s} \leq B_s(\alpha) |U|.$$

2.4. **Renormalization.** Any such $\hat{\delta}_i$ generate a one-parameter subgroup of automorphism $r_i^t = e^{t\hat{\delta}_i}$. We denote renormalization flow $r_{\mathbf{t}} := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$ and

$$r_i^t[\alpha, \omega] = [r_i^t \alpha, \omega], \quad r_i^t[\alpha, \mathcal{D}] = [r_i^t \alpha, \mathcal{D}].$$

Let $U_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be unitary operator for $t = (t_1, \cdots, t_d)$,

$$(7) \qquad U_t f(x) = e^{-(t_1 + \cdots + t_d)/2} f(e^{t_1} x_1, \cdots e^{t_d} x_d).$$

That is, for invariant currents D_α^H,

$$D_{r_t(\alpha)}^H = e^{(t_1 + \cdots + t_d)/2} D_\alpha^H.$$

Then, the action of $\mathbb{R}^d$ defined by their parametrization is

$$(8) \qquad \mathsf{P}_x^{d,(r_1^{t_1} \cdots r_d^{t_d} \alpha)} = \mathsf{P}_{(e^{-t_1} x_1, \ldots, e^{-t_d} x_d)}^{d,\alpha}$$

and the Birkhoff sum satisfy identities

$$(9) \qquad P_U^{d,(r_1^{t_1} \cdots r_d^{t_d} \alpha)} m = e^{(t_1 + \cdots + t_d)/2} P_{(e^{-t_1}, \ldots, e^{-t_d})U}^{d,\alpha} m.$$

2.5. **Diophantine condition.** Diagonal matrix $\delta_i = diag(d_1, \cdots, d_g)$ with $d_i = 1$, $d_k = 0$ if $k \neq i$. For each $1 \leq i \leq g$, we denote $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in \mathfrak{sp}_{2g}$.

We define height function $Hgt : \Sigma_g \to \mathbb{R}^+$ to be the maximal height of a $Sp_{2g}(\mathbb{Z})$ orbit. The height of $W \in \mathfrak{H}_g$ is the positive number $hgt(W) := det\Im(W)$ and for the class of $[W] \in \Sigma_g$,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

For detailed information, refer to [CF15, §4.3 - 4.4]. Let $\delta = diag(\delta_1, \cdots, \delta_g)$. By Lemma 4.9 of [CF15], for any $[\alpha] \in \mathfrak{M}_g$ and any $t \geq 0$,

(10) $$Hgt([[\exp(t\hat{\delta}(d))\alpha]]) \leq (\det(e^{t\delta}))^2 Hgt([[\alpha]]).$$

Let $\exp \boldsymbol{t}\hat{\delta}(d)$ be the subgroup of $Sp_{2g}(\mathbb{R})$ defined by $\exp(\boldsymbol{t}\hat{\delta}(d))X_i = e^{t_i}X_i$, for $i = 1, \cdots, d$, and $\exp(\boldsymbol{t}\hat{\delta}(d))X_i = X_i$ for $i = d+1, \cdots g$. We also denote $r_{\boldsymbol{t}} = \exp \boldsymbol{t}\hat{\delta}(d)$.

Definition 2.4. We say that an automorphism $\alpha \in Sp_{2g}(\mathbb{R})$ or a point $[\alpha] \in \mathfrak{M}_g$ is $\hat{\delta}(d)$-*Diophantine* of type σ if there exists a $\sigma > 0$ and a constant $C > 0$ such that

(11) $$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \leq CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{(1-\sigma)} Hgt([[\alpha]]), \ \forall \boldsymbol{t} \in \mathbb{R}_+^d.$$

This states that $\alpha \in Sp_{2g}(\mathbb{R})$ satisfies a $\hat{\delta}(d)$-Diophantine if the height of the projection of $\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha$ in the Siegel modular variety Σ_g is bounded by $e^{2(t_1 + \cdots t_d)(1-\sigma)}$.

- $[\alpha] \in \mathfrak{M}_g$ satisfies a $\hat{\delta}$-*Roth condition* if for any $\epsilon > 0$ there exists a constant $C > 0$ such that

(12) $$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \leq CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{\epsilon} Hgt([[\alpha]]), \ \forall \boldsymbol{t} \in \mathbb{R}_+^d.$$

That is, $\hat{\delta}(d)$-Diophantine of type $0 < \sigma < 1$.

- $[\alpha]$ is of *bounded type* if there exists a constant $C > 0$ such that

(13) $$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]]) \leq C, \ \forall \boldsymbol{t} \in \mathbb{R}_+^d.$$

For $1 \leq d \leq g$, according to Margulis-Kleinblock [KM99], a generalization of Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

Definition 2.5. Let $X = G/\Lambda$ be a homogeneous space equipped with the probability Haar measure μ. A function $\phi : X \to \mathbb{R}$ is said k-DL (distance like) for some exponent $k > 0$ if it is uniformly continuous and if there exist constants $C_1, C_2 > 0$ such that

$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

Theorem 1.9 of [KM99] states the following.

Proposition 2.6. *Let G be a connected semisimple Lie group without compact factors, μ its normalized Haar measure, $\Lambda \subset G$ an irreducible lattice, $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G. Let $\mathfrak{d}_+$ be a nonempty open cone in a d-dimensional subalgebra $\mathfrak{d}$ of $\mathfrak{a}$. If $\phi : G/\Lambda \to \mathbb{R}$ is a k-DL function for some $k > 0$, then for μ-almost all $x \in G/\Lambda$ one has*

$$\limsup_{\mathbf{z} \in \mathfrak{d}_+, \mathbf{z} \to \infty} \frac{\phi(\exp(\mathbf{z})x)}{\log \|\mathbf{z}\|} = \frac{d}{k}$$

By Lemma 4.7 of [CF15], logarithm of Height function is DL-function with exponent $k = \frac{g+1}{2}$ on the Siegel variety Σ_g (and induces on $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$). Hence, we obtain the following proposition.

Proposition 2.7. *Under the assumption $X = \mathfrak{M}_g$ of Proposition 2.6, for $s > g + 1/2$, there exists a full measure set $\Omega_g(\hat{\delta})$ and for all $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$*

$$(14) \qquad \limsup_{\boldsymbol{t} \to \infty} \frac{\log Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]])}{\log \|\boldsymbol{t}\|} \leq \frac{2d}{g+1}.$$

Any such $[\alpha]$ satisfies a $\hat{\delta}$-Roth condition (12).

For any $L > 0$ and $1 \leq d \leq g$, let $\mathrm{DC}(L)$ denote the set of $[\alpha] \in \mathfrak{M}_g$ such that

$$(15) \qquad \int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots t_d)/2} Hgt([[r_{-\boldsymbol{t}}\alpha]])^{1/4} dt_1 \cdots dt_d \leq L.$$

Let DC denote the union of the sets DC(L) over all $L > 0$. It follows immediately that the set $DC \subset \mathfrak{M}_g$ has full Haar volume.

3. Constructions of the functionals

Let Γ be any d-dimensional rectangles, which defines a current, a continuous functional on d-forms. For an irreducible representation H, there exists *basic current* B_α^H associated to D_α^H. Let ω be invariant volume form and let $\eta_X := \iota_{X_{i_1}} \cdots \iota_{X_{i_d}} \omega$. The basic current B_α^H is defined as

$$B_\alpha^H = D_\alpha^H \eta_X.$$

For every d-form ξ, the formula implies

$$B_\alpha^H(\xi) = D_\alpha^H \left(\frac{\eta_X \wedge \xi}{\omega} \right).$$

The current is *basic* in the sense that for all $j \in \{i_1, \cdots, i_d\}$,

$$\iota_{X_j} B_\alpha^H = L_{X_j} B_\alpha^H = 0.$$

The basic current B_α^H belongs to a dual Sobolev space of currents. We write any smooth d-form $\xi = \sum \xi^{(i)} \hat{X}_i$, where $\hat{X}_i \in \Lambda^d \mathfrak{p}'$. It follows that the space of smooth d-form is identified to the product of $C^\infty(M)$ by isomorphism $\xi \longrightarrow \xi^{(i)}$. By isomorphism, we define Sobolev space of currents $\Omega_\alpha^s(M)$ and their dual spaces of currents $\Omega_\alpha^{-s}(M)$.

By Sobolev embedding theorem, for every rectangle Γ, the current $\Gamma \in \Omega_\alpha^{-s}(M)$ for $s > (2d+1)/2$. Then, all basic currents $B_\alpha^H \in \Omega_\alpha^{-s}(M)$ for all $s > d/2$ since $D_\alpha^H \in \Omega_\alpha^{-s}(M)$ for all $s > d/2$.

3.1. Constructions of the functionals.
For any exponent $s > d/2$, Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^\perp$. Denote by I^{-s} and R^{-s} the corresponding orthogonal projection operator and by I_α^{-s} and R_α^{-s} the restrictions to the fiber over $[\alpha] \in \mathfrak{M}$ for $\alpha \in Sp(2g, \mathbb{R})$. In particular, for the Birkhoff averages $D = P_U^{d,\alpha} m$, we call $I_\alpha^{-s}(D) = I^{-s}[\alpha, D]$ boundary term and $R_\alpha^{-s}(D) = R^{-s}[\alpha, D]$ remainder term respectively. Consider the orthogonal projection

$$(16) \qquad D = I_{r_{-\boldsymbol{t}}[\alpha]}^{-s}(D) + R_{r_{-\boldsymbol{t}}[\alpha]}^{-s}(D).$$

For fixed α, let $\Pi_H^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \to A_d(\mathfrak{p}, W_\alpha^{-s}(H))$ denote the orthogonal projection on a single irreducible unitary representation. We further decompose projection operators with

$$\Pi_H^{-s} = \mathscr{B}_\alpha^{-s}(\Gamma) B_\alpha^{-s,H} + R_\alpha^{-s,H}$$

where $\mathscr{B}_{H,\alpha}^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \to \mathbb{C}$ denote the orthogonal component map of P-invariant currents (closed), supported on a single irreducible unitary representation.

The *Bufetov functionals* on rectangles Γ are defined for all $\alpha \in DC$ as follows.

Lemma 3.1. *Let* $\alpha \in DC(L)$. *For* $s > s_d = d(d+11)/4 + g + 1/2$ *and every rectangle* Γ *on* M, *the limit*

$$\hat{\beta}_H(\alpha, \Gamma) = \lim_{t_d \to \infty} \cdots \lim_{t_1 \to \infty} e^{-(t_1 + \cdots t_d)/2} \mathscr{B}_{H, r_{-t}[\alpha]}^{-s}(\Gamma)$$

exists and define a finitely-additive finite measure on the space of rectangles. There exists constant $C(s, \Gamma) > 0$ *such that the following estimate holds:*

$$(17) \qquad |\Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B_\alpha^H|_{\alpha, -s} \le C(s, \Gamma)(1 + L).$$

Proof. For simplicity, we omit dependence of H. For every $\mathbf{t} \in \mathbb{R}^d$, we have the following orthogonal splitting:

$$\Pi_{H,\alpha}^{-s}(\Gamma) = \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}},$$

where

$$\mathscr{B}_{\alpha,\mathbf{t}}^{-s} := \mathscr{B}_{H, r_{-\mathbf{t}}[\alpha]}^{-s}, \ B_{\alpha,\mathbf{t}} := B_{r_{-\mathbf{t}}[\alpha]}^{-s,H}, \ R_{\alpha,\mathbf{t}} := R_{r_{-\mathbf{t}}[\alpha]}^{-s,H}.$$

For any $\mathbf{h} \in \mathbb{R}^d$, we have

$$\mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}+\mathbf{h}} + R_{\alpha,\mathbf{t}+\mathbf{h}} = \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}}.$$

By reparametrization (8), $B_{\mathbf{t}+\mathbf{h}} = e^{-(h_1 + \cdots h_d)/2} B_{\mathbf{t}}$,

$$(18) \qquad \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{(h_1 + \cdots h_d)/2} \mathscr{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) + \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}})$$

and it follows that

$$(19) \qquad \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{h_1/2} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) + \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}}).$$

By differentiating at $h_1 = 0$,

$$(20)$$
$$\frac{d}{dt_1} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) + [\frac{d}{dh_1} \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}})]_{h_1=0}.$$

Therefore, we solve the following first order ODE

$$\frac{d}{dt_1} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}^{-s}(\Gamma) + \mathcal{K}_{\alpha,\mathbf{t}}^{(1)}(\Gamma)$$

where

$$\mathcal{K}_{\alpha,\mathbf{t}}^{(1)}(\Gamma) = [\frac{d}{dh_1} \mathscr{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}})]_{h_1=0}.$$

Then, the solution of the differential equation is

$$\mathscr{B}_{\alpha,t_1,t_2+h_2,\cdots t_d+h_d}^{-s}(\Gamma) = e^{t_1/2}[\mathscr{B}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{-\tau_1/2} \mathcal{K}_{\alpha,\tau}^{(1)}(\Gamma) d\tau_1]$$

$$= e^{t_1/2} \mathscr{B}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha,\tau}^{(1)}(\Gamma) d\tau_1.$$

Note by reparametrization

$$e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) = e^{h_2/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3\cdots t_d+h_d}(\Gamma)$$

and it is possible to differentiate the previous equation with respect to h_2 again. Then

$$(21) \quad \frac{d}{dt_2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1.$$

where $\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma) = \frac{d}{dh_2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)$.

Then, the solution of (21) is

$$\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma)$$

$$= e^{t_2/2}[\mathscr{B}^{-s}_{\alpha,t_1,0,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_2} e^{-\tau_2/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2]$$

$$= e^{h_3/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3,\cdots t_d+h_d} + \int_0^{t_2} e^{(t_2-\tau_2)/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2.$$

Inductively, we solve first order ODE repeatedly and obtain the following solution

$$(22) \quad \mathscr{B}^{-s}_{\alpha,\mathbf{t}}(\Gamma) = e^{(t_1+\cdots t_d)/2}\Big(\mathscr{B}^{-s}_{\alpha,0} + \int_0^{t_d}\cdots\int_0^{t_1} e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma)d\tau_1\cdots d\tau_d\Big)$$

where

$$\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma) = [\frac{d}{dh_d}\cdots\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}})]_{h_d\cdots h_1=0}.$$

Let $\langle\cdot,\cdot\rangle_t$ denote the inner product in Hilbert space $\Omega^{-s}_{r_{-\mathbf{t}}[\alpha]}$. From the intertwining formula,

$$\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}}) = \langle R_{\alpha,\mathbf{t}}, \frac{B_{\alpha,\mathbf{t}+\mathbf{h}}}{|B_{\alpha,\mathbf{t}+\mathbf{h}}|^2_{\mathbf{t}+\mathbf{h}}}\rangle_{\alpha,\mathbf{t}+\mathbf{h}}$$

$$= \langle R_{\alpha,\mathbf{t}}\circ U_{-\mathbf{h}}, \frac{B_{\alpha,\mathbf{t}+\mathbf{h}}\circ U_{-\mathbf{h}}}{|B_{\alpha,\mathbf{t}+\mathbf{h}}|^2_{\mathbf{t}+\mathbf{h}}}\rangle$$

$$= \langle R_{\alpha,\mathbf{t}}\circ U_{-\mathbf{h}}, \frac{B_{\alpha,\mathbf{t}}}{|B_{\alpha,\mathbf{t}}|^2_{\mathbf{t}}}\rangle = \mathscr{B}^{-s}_{\alpha,\mathbf{t}}(R_{\alpha,\mathbf{t}}\circ U_{-\mathbf{h}}).$$

In the sense of distribution,

$$\frac{d}{dh_d}\cdots\frac{d}{dh_1}(R_{\alpha,\mathbf{t}}\circ U_{-\mathbf{h}}) = -R_{\alpha,\mathbf{t}}\circ(\frac{d}{2} + \sum_{i=1}^d X_i(t))\circ U_{-\mathbf{h}}$$

$$= [(\sum_{i=1}^d X_i(t) - \frac{d}{2})R_{\alpha,\mathbf{t}}]\circ U_{-\mathbf{h}}$$

and we compute derivative term of (20) in representation,

$$[\frac{d}{dh_d}\cdots\frac{d}{dh_1}(\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,t}))]_{\mathbf{h}=0} = -\mathscr{B}^{-s}_{\alpha,\mathbf{t}}((\sum_{i=1}^d X_i(t) - \frac{d}{2})R_{\alpha,\mathbf{t}}).$$

Set $\mathcal{K}_{\alpha,t}(\Gamma) = |R_{\alpha,\mathbf{t}}|_{r_{-\mathbf{t}}\alpha,-(s+1)}$ with a bounded non-negative function, then by Lemma 3.5, we claim that

$$|\mathscr{B}^{-s}_{\alpha,\mathbf{t}}((\sum_{i=1}^d X_i(t) - \frac{d}{2})R_{\alpha,\mathbf{t}}))| \le \mathcal{K}_{\alpha,\tau}(\Gamma) \le C(s,\Gamma)Hgt([r_{-\mathbf{t}}\alpha])^{1/4}.$$

Therefore, the solution of equation (22) exists under Diophantine condition (15) and the following holds:

$$\lim_{t_d \to \infty} \cdots \lim_{t_1 \to \infty} e^{-(t_1 + \cdots t_d)/2} \mathscr{B}^{-s}_{\alpha,t}(\Gamma) = \hat{\beta}_H(\alpha, \Gamma)$$

Moreover, by continuity, the complex number

$$\hat{\beta}_H(\alpha, \Gamma) = \mathscr{B}^{-s}_{\alpha,0} + \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots \tau_d)/2} \mathcal{K}_{\alpha,\tau}(\Gamma) d\tau_1 \cdots d\tau_d$$

depends on $\alpha \in DC(L)$. Since we have

$$\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B^H_\alpha = R_0 - \left(\int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \cdots \tau_d)/2} \mathcal{K}_{\alpha,\tau}(\Gamma) d\tau_1 \cdots d\tau_d \right) B^H_\alpha,$$

by Diophantine condition again,

$$|\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B^H_\alpha|_{\alpha,-s} \leq C(s, \Gamma)(1 + L).$$

$\square$

3.2. **Remainder estimates.** In this subsection, we obtain estimate for remainder term which is used in Lemma 3.1. Firstly, we prove the bound of Birkhoff sum of rectangles.

Lemma 3.2. *[CF15, Lemma 5.7] Let $s > d/2 + 2$. There exists a constant $C = C(s) > 0$ such that for all $t_i \geq 0$ for $1 \leq i \leq d$, we have*

$$\left\| I^{-s}[\alpha, D] \right\|_{-s} \leq e^{-(t_1 + \cdots + t_d)/2} \left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, D] \right\|_{-s}$$

$$+ C_1 |t_1 + \cdots + t_d| \int_0^1 e^{-u(t_1 + \cdots + t_d)/2} \left\| R^{-s}[r_1^{-ut_1} \cdots r_d^{-ut_d} \alpha, D] \right\|_{-(s-2)} du.$$

By Stokes' theorem, we have the following remainder estimate.

Lemma 3.3. *[CF15, Lemma 5.6] For any non-negative $s' < s - (d+1)/2$ and Jordan region $\Gamma \subset \mathbb{R}^d$, there exists $C = C(d, g, s, s') > 0$ such that*

$$\left\| R^{-s}[\alpha, (P_\Gamma^{d,\alpha} m)] \right\|_{-s} \leq C \left\| [\alpha, \partial(P_\Gamma^{d,\alpha} m)] \right\|_{-s'}.$$

Here we prove quantitative bound of Birkhoff averages of higher rank actions on rectangle. (Cf. [CF15, theorem 5.10]).

Theorem 3.4. *For $s > s_d$, there exists a constant $C(s, d) > 0$ such that the following holds.*

For all $t_i > 0$, $m \in M$ and $\Gamma_d(t) = [0, e^{t_1}] \times \cdots \times [0, e^{t_d}]$, we have

(23)

$$\left\| [\alpha, (P^{d,\alpha}_{\Gamma_d(t)} m)] \right\|_{-s} \leq C \sum_{k=0}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^{d} t_l - \frac{1}{2} \sum_{l=1}^{k} u_{i_l} \right)$$

$$\times Hgt([[\prod_{1 \leq j \leq d} r_j^{-t_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha]])^{1/4} du_{i_1} \cdots du_{i_k}.$$

Proof. We prove by induction. For $d = 1$, it follows from the theorem 5.8 in [CF15]. We assume that the result holds for $d-1$. Decompose the current as a sum of boundary and remainder term as in (16).

Step 1. We estimate the boundary term. By Lemma 3.2, renormalize terms with $r^u = r_1^u \cdots r_d^u$. Then, we have

$$
\left\| I^{-s}[\alpha, (P_{\Gamma_d(t)}^{d,\alpha} m)] \right\|_{-s} \leq e^{-(t_1 + \cdots + t_d)/2} \left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{\Gamma_d(t)}^{d,\alpha} m)] \right\|_{-s}
$$

$$
(24) \quad + C_1(s) \int_0^{t_1 + \cdots + t_d} e^{-ud/2} \left\| R^{-s}[r^{-u}\alpha, (P_{\Gamma_d(t)}^{d,\alpha} m)] \right\|_{-(s-2)} du
$$

$$
:= (I) + (II)
$$

By renormalization (9) and Lemma 2.3 for unit volume,

$$
\left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{\Gamma_d(t)}^{d,\alpha} m)] \right\|_{-s} = e^{t_1 + \cdots + t_d} \left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (P_{\Gamma_d(0)}^{d, r_1^{-t_1} \cdots r_d^{-t_d}\alpha} m)] \right\|_{-s}
$$

$$
\leq C_2 e^{t_1 + \cdots + t_d} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d} \alpha]])^{1/4}
$$

Hence,

$$
I \leq C_2 e^{(t_1 + \cdots t_d)/2} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d} \alpha]])^{1/4},
$$

where the sum corresponds to the first term $(k = 0)$ in the statement.

Step 2. To estimate (II),

$$
(25) \quad \left\| R^{-s}[r^{-u}\alpha, (P_{\Gamma_d(t)}^{d,\alpha} m)] \right\|_{-(s-2)} = \left\| e^{ud} R^{-s}[r^{-u}\alpha, (P_{\Gamma_d(t-u)}^{d, r^{-u}\alpha} m)] \right\|_{-(s-2)}
$$

$$
\leq C_3(s, s') e^{ud} \left\| [r^{-u}\alpha, \partial(P_{\Gamma_d(t-u)}^{d, r^{-u}\alpha} m)] \right\|_{-s'}.
$$

The boundary $\partial(\mathsf{P}_\Gamma^{d, r^{-t}\alpha})$ is the sum of $2d$ currents of dimension $d-1$. These currents are Birkhoff sums of d face subgroups obtained from $\mathsf{P}_\Gamma^{d, r^{-t}\alpha}$ by omitting one of the base vector fields X_i. It is reduced to $(d-1)$ dimensional shape obtained from $\Gamma_d(t-u) := [0, e^{t_1-u}] \times \cdots [0, e^{t_d-u}]$. For each $1 \leq j \leq d$, there are Birkhoff sums along $d-1$ dimensional cubes. By induction hypothesis, we add all the $d-1$ dimensional cubes by adding all the terms along j:

(26)

$$
\left\| [r^{-u}\alpha, (P_{U_{d-1}(t-u)}^{d-1, r^{-u}\alpha} m)] \right\|_{-s'} \leq C_4(s', d-1) \sum_{j=1}^{d} \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq d \\ i_l \neq j, \forall l}} \int_0^{t_{i_k} - u} \cdots \int_0^{t_{i_1} - u}
$$

$$
\exp\left(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^{d} (t_l - u) - \frac{1}{2} \sum_{l=1}^{k} u_{i_l} \right) Hgt\left(\left[\left[\prod_{\substack{1 \leq l \leq d \\ l \neq j}} r_l^{-(t_l - u)} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} (r^{-u}\alpha) \right]\right] \right)^{1/4} du_{i_1} \cdots du_{i_k}.
$$

Combining (24) and (25), we obtain the estimate for (II).

$$(II) \leq C_5(s', d-1) \sum_{j=1}^{d} \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq d \\ i_l \neq j}} \int_0^{t_1 + \cdots t_d} \int_0^{t_{i_k} - u} \cdots \int_0^{t_{i_1} - u} du_{i_1} \cdots du_{i_k} du$$

$$\times \exp(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^{d} t_l - \frac{1}{2} u - \frac{1}{2} \sum_{l=1}^{k} u_{i_l}) Hgt([[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} (r_j^{-u+t_j} \alpha)]])^{1/4}.$$

Applying the change of variable $u_j = t_j - u$, we obtain

$$(II) \leq C_6(s', d-1) \sum_{j=1}^{d} \sum_{k=1}^{d-1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq d \\ i_l \neq j}} \int_{-(t_1 + \cdots t_d) + t_j}^{t_j} \int_0^{t_{i_k} - u} \cdots \int_0^{t_{i_1} - u} du_{i_1} \cdots du_{i_k} du_j$$

$$\times \exp(\frac{1}{2}(t_1 + \cdots t_d) - \frac{1}{2} u_j - \frac{1}{2} \sum_{l=1}^{k} u_{i_l}) Hgt([[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} (r_j^{-u_j} \alpha)]])^{1/4}.$$

Simplifying multi-summation above, (with $-(t_1 + \cdots t_d) + t_j \leq 0$)

$$(II) \leq C_7(s', d) \sum_{k=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} du_{i_1} \cdots du_{i_k}$$

$$\times \exp(\frac{1}{2}(t_1 + \cdots t_d) - \frac{1}{2} \sum_{l=1}^{k} u_{i_l}) Hgt([[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha]])^{1/4}.$$

Step 3. Remainder estimate.

The remainder term is obtained from Lemma 3.3 (Stokes theorem). Following step 2, estimate of remainder reduces to that of $d-1$ form. Combining with the step 1, we have the following
(27)
$$\left\| R^{-s}[\alpha, \partial(P_{\Gamma_d}^{d,\alpha} m)] \right\|_{-s} \leq C(s) \sum_{i=1}^{d-1} \left\| I^{-s}[\alpha, (P_{\Gamma_i}^{i,\alpha} m)] \right\|_{-s} + \left\| R^{-s}[\alpha, (P_{\Gamma_1}^{1,\alpha} m)] \right\|_{-s}$$

where Γ_i is i-dimensional rectangle. Sum of boundary terms are absorbed in the bound of $(I) + (II)$. For 1-dimensional remainder with interval Γ_T, the boundary is a 0-dimensional current. Then,

$$\langle \partial(P_{\Gamma_T}^{1,\alpha} m), f \rangle = f(\mathsf{P}_{\Gamma_T}^{1,\alpha} m) - f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (5) and (6),

$$\left\| R^{-s}[\alpha, \partial(P_{\Gamma_T}^{1,\alpha} m)] \right\|_{-s} \leq 2 B_{s'}([[\alpha]]) \leq C(s) Hgt([[\alpha]])^{1/4}.$$

Then, by inequality (10)

$$Hgt([[\alpha]])^{1/4} \leq C e^{(t_1 + \cdots t_d)/2} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d} \alpha]])^{1/4}.$$

This implies that remainder term produces one more term like the bound of (I). Therefore, the theorem follows from combining all the terms $(I), (II)$, and remainder. $\square$

Now we prove the estimate for constructing Bufetov functionals.

Lemma 3.5. *Let $s > s_d$. There exists a constant $C(s) > 0$ such that for any rectangle $\Gamma_d = [0, e^{\Gamma_1}] \times \cdots \times [0, e^{\Gamma_d}]$ and all $m \in M$,*

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C(s, \Gamma) Hgt([r_{-t}\alpha])^{1/4}.$$

Proof. Recall that $\mathcal{K}_{\alpha,t}(\Gamma) = \left\| R^{-s}[r_{-t}[\alpha], (P_\Gamma^{d, r^{-t}\alpha}m)] \right\|_{-(s+1)}$ and by Lemma 3.3, it is equivalent to prove to find the bound of $d - 1$ currents. By theorem 3.4, we obtain the remainder estimate.

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \int_0^{\Gamma_{i_k}} \cdots \int_0^{\Gamma_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^{d-1} \Gamma_l - \frac{1}{2} \sum_{l=1}^{k} u_{i_l}\right)$$

$$\times Hgt([[\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}}(r_{-t}\alpha)]])^{1/4} du_{i_1} \cdots du_{i_k}.$$

It follows from (10) that for $0 \leq k \leq d - 1$,

$$Hgt([[\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}}(r_{-t}\alpha)]])^{1/4} \leq e^{\frac{1}{2}(\sum_{l=1}^{k} u_{i_l} - \sum_{l=1}^{d} \Gamma_l)} Hgt([[r_{-t}\alpha]])^{1/4}.$$

Then, we obtain

(28)

$$\left\| R^{-s}[r_{-t}\alpha, (P_{\Gamma_d}^{d, r^{-t}\alpha}m)] \right\|_{-s} \leq C\left(\sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \prod_{l=1}^{k} \Gamma_{i_l}\right) Hgt([[r_{-t}\alpha]])^{1/4}.$$

Setting $C(s, \Gamma) = C\left(\sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \prod_{l=1}^{k} \Gamma_{i_l}\right)$, we obtain the conclusion. $\square$

3.3. **Proof of theorem 1.1.** The main theorem follows from refinement of the method of [BF14, FK17] to the d-form and currents.

Proof of theorem 1.1. Additive property follows from linearity of projections and limit and it is immediate to derive *Scaling property* from definition.

Bounded property. By scaling property,

$$\hat{\beta}(\alpha, \Gamma) = e^{dt/2} \hat{\beta}_H(r^t[\alpha], \Gamma).$$

Choose $t = \log(\int_\Gamma |\hat{X}|)$ and $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$, then uniform bound of Bufetov functional on bounded size of rectangles,

$$|\hat{\beta}(\alpha, \Gamma)| \leq C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

Invariance property. It suffices to check invariance property of rank 1 action $\mathsf{Q}_\tau^{1,Y}$ for $\tau \in \mathbb{R}$. Let Y_i be generator of $\mathsf{Q}_\tau^{1,Y}$ and pick X_i with $[X_i, Y_i] = Z$.

Proof. Given rectangle Γ, set $\Gamma_\mathsf{Q} := (\mathsf{Q}_\tau^{1,Y})_* \Gamma$. Let $D(\Gamma, \Gamma_\mathsf{Q})$ be the $(d + 1)$ dimensional space spanned by the trajectories of the action of $\mathsf{Q}_\tau^{1,Y}$ projecting Γ onto Γ_Q. Then $D(\Gamma, \Gamma_\mathsf{Q})$ is union of all orbits I of action $\mathsf{Q}_\tau^{1,Y}$ such that the boundary of I, d-dimensional faces, is contained in $\Gamma \cup \Gamma_\mathsf{Q}$, and interior of I is disjoint from $\Gamma \cup \Gamma_\mathsf{Q}$.

By definition, denote $r_t := r_i^t$. Then, $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_{\mathsf{Q}})$ are respectively the support of the currents $r_t^*\Gamma$ and $r_t^*\Gamma_{\mathsf{Q}}$. Thus, we have the following identity

$$r_t^* D(\Gamma, \Gamma_{\mathsf{Q}}) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}})).$$

Since the currents $\partial D(\Gamma, \Gamma_{\mathsf{Q}}) - (\Gamma - \Gamma_{\mathsf{Q}})$ is composed of orbits of the action of $\mathsf{Q}_\tau^{1,Y}$, it follows that

$$(29) \qquad \partial[r_t^* D(\Gamma, \Gamma_{\mathsf{Q}})] - (r_t^*\Gamma - r_t^*\Gamma_{\mathsf{Q}}) = r_t^*[\partial D(\Gamma, \Gamma_{\mathsf{Q}}) - (\Gamma - \Gamma_{\mathsf{Q}})] \to 0.$$

Now, we turn to prove the volume of $D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}}))$ is uniformly bounded for all $t > 0$. For any $p \in \Gamma$, set $\tau(p)$ be length of the arc lying in $D := D(\Gamma, \Gamma_{\mathsf{Q}})$, and set $\tau_\Gamma := \sup\{\tau(p) \mid p \in \Gamma\} < \infty$. We write

$$vol_{d+1}(D) = \int_\Gamma \tau dvol_d.$$

Since $vol_d(r_{-t}\Gamma) \le e^t vol_d(\Gamma)$,

$$(30) \qquad vol_{d+1}(r_{-t}D) = \int_{r_{-t}\Gamma} \tau dvol_d \le \tau_\Gamma e^{-t} vol_d(r_{-t}\Gamma) \le \tau_\Gamma vol_d(\Gamma) < \infty.$$

Note that current $(\mathsf{Q}_\tau^{1,Y})_*\Gamma - \Gamma$ is equal to the boundary of a $(d+1)$ dimensional current D. By arguments in remainder estimate (or Sobolev embedding theorem),

$$(31) \qquad |(\mathsf{Q}_\tau^{1,Y})_*\Gamma - \Gamma|_{r^{-t}\alpha,-s} \le C_s \tau B_s([[r^{-t}\alpha]]) \le C_s \tau \mathrm{Hgt}[[r^{-t}\alpha]]^{1/4}$$

is finite for all $t > 0$.

Then, by (29), (30) and existence of Bufetov functional (from Diophantine condition), the last inequality holds

$$|\mathscr{B}_{\alpha,t}^{-s}((\mathsf{Q}_\tau^{1,Y})_*\Gamma) - \mathscr{B}_{\alpha,t}^{-s}(\Gamma)|_{\alpha,-s} < \infty.$$

Therefore, by definition of Bufetov functional, we prove the invariance property. $\square$

3.4. **Bound of functionals.** We define excursion function

$$E_{\mathfrak{M}}(\alpha, \mathbf{T}) := \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1+\cdots t_d)/2} \mathrm{Hgt}([[r_{\mathbf{t}-\log \mathbf{T}}\alpha]])^{1/4} dt_1 \cdots dt_d$$

$$= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1+\cdots t_d)/2} \mathrm{Hgt}([[r_{\mathbf{t}}\alpha]])^{1/4} dt_1 \cdots dt_d.$$

Denote $\boldsymbol{tT} = (t_1 T^{(1)}, \cdots, t_d T^{(d)})$ and $\mathbf{t} = (t_1, \cdots, t_d)$.

Lemma 3.6. *For any Diophantine $[\alpha] \in DC(L)$ and for any $f \in W^s(M)$ for $s > s_d + 1/2$, the Bufetov functional β^f is defined by a uniformly convergent series.*

$$|\beta^f(\alpha, m, \boldsymbol{tT})| \le C_s (L + \prod_{i=1}^d (T^{(i)})^{1/2}(1 + \prod_{i=1}^d t_i + E_{\mathfrak{M}}(\alpha, \mathbf{T})) \|\omega\|_{\alpha,\mathbf{s}}$$

for $\omega = f\omega^{d,\alpha} \in \Lambda^d \mathfrak{p} \otimes W^s(M)$.

Proof. It follows from Lemma 3.1 that there exists a constant $C > 0$ such that

$$|\beta_H(\alpha, m, \mathbf{t})| \le C(1 + L + \prod_{i=1}^d t_i).$$

By exact scaling property,

$$\beta_H(\alpha, m, \boldsymbol{t}\boldsymbol{T}) = \prod_{i=1}^{d}(T^{(i)})^{1/2}\beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t}).$$

By Diophantine condition (15), whenever $\alpha \in DC(L)$ then $r_{\log \mathbf{T}}[\alpha] \in DC(L_T)$ with

$$L_T \leq L\prod_{i=1}^{d}(T^{(i)})^{-1/2} + E_{\mathfrak{M}}(\alpha, \mathbf{T}).$$

Thus for all $(m, \mathbf{t}) \in M \times \mathbb{R}^d$,

$$|\beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t})| \leq C(1 + L_T + \prod_{i=1}^{d}t_i).$$

It follows that for all $s > 1/2$, we have

$$|\beta^f(\alpha, m, \boldsymbol{t}\boldsymbol{T})| \leq C_s\prod_{i=1}^{d}(T^{(i)})^{1/2}(1 + L + \prod_{i=1}^{d}t_i)\sum_{n\in\mathbb{Z}}\|\omega_n\|_{\alpha,s}$$

$$\leq C_s\prod_{i=1}^{d}(T^{(i)})^{1/2}(1 + L_T + \prod_{i=1}^{d}t_i)(\sum_{n\in\mathbb{Z}}(1 + n^2)^{-s'})^{-1/2}(\sum_{n\in\mathbb{Z}}\left\|(1 - Z^2)^{s'/2}\omega_n\right\|_{\alpha,s}^2)^{1/2}.$$

Therefore, for all $s' > s_d$, there exists a constant $C_{s,s'} > 0$ such that

$$|\beta^f(\alpha, m, \mathbf{t}\mathbf{T})| \leq C_{r,r'}\prod_{i=1}^{d}(T^{(i)})^{1/2}(1 + L_T + \prod_{i=1}^{d}t_i)\|\omega\|_{\alpha,s+s'}.$$

$\square$

This lemma implies that all properties of the Bufetov functionals associated to a single irreducible component β_H can be extended to the Bufetov functionals β^f for any $f \in W^s(M)$. From this, we can derive bounded property for the cocycles $\beta(\alpha, m, \mathbf{T})$ with respect to m along the orbits of actions in time $\mathbf{T} \in \mathbb{R}^d$ respectively.

Corollary 3.7. *For all $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for almost all frequency α and for all $f \in W^s(M)$ and for all $(m, \boldsymbol{T}) \in M \times \mathbb{R}^d$, we have*

$$\text{(32)} \qquad |\left\langle P_{U(\boldsymbol{T})}^{d,\alpha}m, \omega \right\rangle - \beta^f(\alpha, m, \boldsymbol{T})| \leq C_s(1 + L)\|\omega\|_{\alpha,s}.$$

for $U(\boldsymbol{T}) = [0, T^{(1)}] \times \cdots [0, T^{(d)}]$ and $\omega = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p} \otimes W^s(M)$.

Proof. By Lemma 3.1 and 3.6, asymptotic formula (17) on each irreducible provides proof of Corollary 3.7. $\square$

4. Limit distributions

In this section, we prove Theorem 1.4, limit distribution of Birkhoff sums of higher rank actions on squares.

4.1. **Limiting distributions.**

Lemma 4.1. *There exists a continuous modular function $\theta_H : Aut_0(\mathsf{H}^g) \to H \subset L^2(M)$ such that*

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\cdot), \omega_f \right\rangle - \theta_H(r_{\log \boldsymbol{T}}[\alpha]) D_\alpha^H(f) \right\|_{L^2(M)} = 0.$$

The family $\{\theta_H(\alpha) \mid \alpha \in Aut_0(\mathsf{H}^g)\}$ has a constant norm in $L^2(M)$.

Proof. By Fourier transform, the space of smooth vectors and Sobolev space $W^s(H)$ is represented as Schwartz-type space $\mathscr{S}^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |(1 + \sum_i \frac{\partial^2}{\partial u_i^2} + \sum_i u_i^2)^{s/2} \hat{f}(u)|^2 du < \infty.$$

Let $\boldsymbol{t}, \boldsymbol{u} \in \mathbb{R}^d$. Then we claim for any $f \in \mathscr{S}^s(\mathbb{R}^d)$, there exists $\theta[\alpha] \in L^2(\mathbb{R}^d)$ such that

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} f(\boldsymbol{u} + \boldsymbol{t}) d\boldsymbol{t} - \theta_H(r_{\log \boldsymbol{T}}[\alpha]) Leb(f) \right\|_{L^2(\mathbb{R}^d, d\boldsymbol{u})} = 0.$$

Equivalently,

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{i\boldsymbol{t} \cdot \hat{\boldsymbol{u}}} \hat{f}(\hat{\boldsymbol{u}}) dt - \hat{\theta}_H(r_{\log \boldsymbol{T}}[\alpha]) \hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = 0.$$

For $\chi \in L^2(\mathbb{R}^d, d\hat{u})$, we denote

$$\chi_j(\hat{u}) = \frac{e^{i\hat{u}_j} - 1}{i\hat{u}_j}, \ \chi(\hat{u}) = \prod_{j=1}^d \chi_j(\hat{u}).$$

Let $\hat{\theta}[\alpha](\hat{u}) := \chi(\hat{u})$ for all $\hat{u} \in \mathbb{R}^d$. Then, by intertwining formula, for $\mathbf{T} \in \mathbb{R}^d$ and $\mathbf{u} \in \mathbb{R}^d$,

$$U_{\mathbf{T}}(f)(\hat{u}) = \prod_{i=1}^d (T^{(i)})^{1/2} f(\boldsymbol{T}\hat{\boldsymbol{u}}), \ \text{for } \boldsymbol{T}\hat{\boldsymbol{u}} = (T^{(1)} \hat{u}_1, \cdots, T^{(d)} \hat{u}_d).$$

Then, for all $\alpha \in A$,

$$\hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{u}) = U_{\mathbf{T}}(\chi)(\hat{u}) = \prod_{i=1}^d (T^{(i)})^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}}).$$

The function $\theta[\alpha]$ is defined by Fourier inverse transform

$$\|\theta_H(\alpha)\|_H = \|\theta(\alpha)\|_{L^2(\mathbb{R}^d)} = \left\| \hat{\theta}(\alpha) \right\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{u})\|_{L^2(\mathbb{R}^d, d\hat{u})} = C > 0.$$

By integration,

$$\int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{i\boldsymbol{t} \cdot \hat{\boldsymbol{u}}} \hat{f}(\hat{u}) dt = (\prod_{i=1}^d T^{(i)}) \chi(\boldsymbol{T}\hat{\boldsymbol{u}}) \hat{f}(\hat{u})$$

$$= (\prod_{i=1}^d T^{(i)}) \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) + (\prod_{i=1}^d T^{(i)})^{1/2} \hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{u}) \hat{f}(0).$$

We note that $\prod_{i=1}^{d} T^{(i)} = vol(U(\boldsymbol{T}))$. Then the claim reduces to the following:

$$\limsup_{vol(U(T))\to\infty} \left\| vol(U(\boldsymbol{T}))^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If $f \in \mathscr{S}^s(\mathbb{R}^g)$ with $s > d/2$, function $\hat{f} \in C^0(\mathbb{R}^d)$ and bounded. Thus, by Dominated convergence theorem,

$$\left\| vol(U(\boldsymbol{T}))^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = \left\| \chi(\nu)(\hat{f}(\frac{\nu}{\boldsymbol{T}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\nu)} \to 0.$$

$\square$

Corollary 4.2. *There exists a constant $C > 0$ such that for any $s > d/2$, for any $\alpha \in A$ and $f \in W^s(H)$, we have*

$$\lim_{|U(\boldsymbol{T})|\to\infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha} m, \omega_f \right\rangle \right\|_{L^2(M)} = C|D_\alpha^H(f)|.$$

From Corollary 4.2, we derive the following limit result for the L^2 norm of Bufetov functionals.

Corollary 4.3. *For irreducible component H and $\alpha \in DC$, there exists $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})|\to\infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\| \beta_H(\alpha, \cdot, \boldsymbol{T}) \right\|_{L^2(M)} = C.$$

Proof. By the normalization of invariant distribution in Sobolev space $W^s(M)$, there exists a function $f_\alpha^H \in W^s(H)$ such that $D_\alpha(f_\alpha^H) = \left\| f_\alpha^H \right\|_s = 1$. For all $\alpha \in DC(L)$, by asymptotic formula (32),

$$\left| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha} m, \omega \right\rangle - \beta^f(\alpha, m, \boldsymbol{T}) \right| \le C_s(1 + L).$$

Therefore, L^2-estimate follows from Corollary 4.2. $\square$

A relation between the Bufetov functional and the modular function θ_H is established below.

Corollary 4.4. *For any $L > 0$ and invariant probability measure supported on $DC(L) \subset \mathfrak{M}_g$,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H([\alpha]), \quad \text{for } \mu\text{-almost all } [\alpha] \in \mathfrak{M}_g.$$

Proof. By Theorem 3.7 and Lemma 4.1, there exists a constant $C > 0$ such that for all $\alpha \in supp(\mu) \subset DC(L)$, for all $T > 0$ we have

$$(33) \quad \lim_{|U(\boldsymbol{T})|\to\infty} \| \beta_H(r_{\log \boldsymbol{T}}[\alpha], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T}}[\alpha]) \|_{L^2(M)} \le \frac{C_\mu}{vol(U(\boldsymbol{T}))^{1/2}}.$$

By Luzin's theorem, for any $\delta > 0$ there exists a compact subset $E(\delta) \subset \mathfrak{M}$ such that we have the measure bound $\mu(\mathfrak{M}\backslash E(\delta)) < \delta$ and the function $\beta_H(\alpha, \cdot, 1) \in L^2(M)$ depends continuously on $[\alpha] \in E(\delta)$. By Poincare recurrence, there is a full measure subset $E'(\delta) \subset E(\delta)$ of $\mathbb{R}^d$-action.

For every $\alpha_0 \in E'(\delta)$, there is diverging sequence (t_n) such that $\{r^{t_n}(\alpha_0)\} \subset E(\delta)$ and $\lim_{n\to\infty} r^{t_n}(\alpha_0) = (\alpha_0)$. By continuity of θ_H and β_H at $[\alpha_0]$, we have

$$(34) \quad \| \beta_H([\alpha_0], \cdot, 1) - \theta_H([\alpha_0]) \|_{L^2(M)}$$
$$= \lim_{n\to\infty} \| \beta_H(r_{\log \boldsymbol{T_n}}[\alpha_0], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T_n}}[\alpha_0]) \|_{L^2(M)} = 0.$$

Thus, we have $\beta_H([\alpha], \cdot, 1) = \theta_H([\alpha])$ for $[\alpha] \in E'(\delta)$. It follows that the set of equality fails has less than any $\delta > 0$, thus the identity holds for almost all $[\alpha]$. $\square$

For all $\alpha \in Aut_0(\mathsf{H}^g)$, general smooth function $f \in W^s(M)$ for $s > s_d + 1/2$, f decompose an infinite sum, and the functional θ^f is defined by a convergent series.

$$(35) \qquad \theta^f(\alpha) = \sum_H D_\alpha^H(f)\theta_H(\alpha).$$

The following result is an extention to general asymptotic theorem from Corollary 4.4.

Theorem 4.5. *For all $\alpha \in Aut_0(\mathsf{H}^g)$, and for all $f \in W^s(M)$ for $s > s_d + 1/2$,*

$$\lim_{n\to\infty} \left\| \frac{1}{vol(U(T_n))^{1/2}} \left\langle P_{U(T_n)}^{d,\alpha}m, \omega_f \right\rangle - \theta^f(r_{\log T_n}\alpha) \right\|_{L^2(M)} = 0.$$

4.2. **Proof of Theorem 1.4.** By theorem 4.5, we summarize our results on limit distributions for higher rank actions.

Theorem 4.6. *Let (T_n) be any sequence such that*

$$\lim_{n\to\infty} r_{\log T_n}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_d + 1/2$, which is not a coboundary, the limit distribution of the family of random variables

$$E_{T_n}(f) := \frac{1}{vol(U(T_n))^{1/2}} \left\langle P_{U(T_n)}^{d,\alpha}(\cdot), \omega_f \right\rangle$$

exists and is equal to the distribution of the function $\theta^f(\alpha_\infty) = \beta(\alpha, \cdot, 1) \in L^2(M)$. If $\alpha_\infty \in DC$, then $\theta^f(\alpha_\infty)$ is bounded function on M, and the limit distribution has compact support.

Proof of theorem 1.4. Since $\alpha_\infty \in \mathfrak{M}_g$, the existence of limit follows from Corollary 4.4 and Lemma 4.5. $\square$

A relation with Birkhoff sum and theta sum was introduced in [CF15, §5.3], and as an applications, we derive limit theorem of theta sums.

Corollary 4.7. *Let $\mathscr{Q}[x] = x^\top \mathscr{Q}x$ be the quadratic forms defined by $g \times g$ real matrix $\mathscr{Q}$, $\alpha = \begin{pmatrix} I & 0 \\ \mathscr{Q} & I \end{pmatrix} \in Sp_{2g}(\mathbb{R})$, $\ell(x) = \ell^\top x$ be the linear form defined by $l \in \mathbb{R}^g$. Then, Theta sum*

$$\Theta(\mathscr{Q}, l; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0,N]} e(\mathscr{Q}[n] + \ell(n))$$

has limit distribution and it has compact support.

5. L^2-LOWER BOUNDS

In this section we prove bounds for the square mean of ergodic integrals along the leaves of foliations of the torus into circles transverse to central direction.

5.1. **Structure of return map.** Let $\mathbb{T}_\Gamma^{g+1}$ denote $(g+1)$-dimensional torus with standard frame (X_i, Y_i, Z) with

$$\mathbb{T}_\Gamma^{g+1} := \{\Gamma \exp(\sum_{i=1}^g y_i Y_i + zZ) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

It is convenient to work with the polarized Heisenberg group. Set $H_{pol}^g \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ equipped with the group law $(x, y, z) \cdot (x', y', z') = (x+x', y+y', z+z'+yx')$.

Definition 5.1. Reduced standard Heisenberg group H_{red}^g is defined by quotient $H_{pol}^g/(\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}/\frac{1}{2}\mathbb{Z}$. Reduced standard latice Γ_{red}^g is $\mathbb{Z}^g \times \mathbb{Z}^g \times \{0\}$ and the quotient H_{red}^g/Γ_{red}^g is isomorhic to standard Heisenberg manifold H^g/Γ.

Now, we consider return map of $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}_\Gamma^{g+1}$. For $x = (x_1, \cdots, x_g) \in \mathbb{R}^g$,

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta \in \mathbb{R}^d.$$

In H_{red}^g,

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x).$$

Then, given $(n, m, 0) \in \Gamma_{red}^g$,

$$(36) \quad \exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) \cdot (n, m, 0) = \exp(x_1' X_1^\alpha + \cdots + x_g' X_g^\alpha) \cdot (0, y', z')$$

if and only if $x_\alpha' = x_\alpha + n$, $y' = y + (x_\beta - x_\beta') + m$ and $z' = z + (w - w') \cdot x + n(y + x_\beta)$.

Assume $\langle X_i^\alpha, X_j \rangle \neq 0$ for all i, j, and we write first return time $t_{Ret} = (t_{Ret,1}, \cdots t_{Ret,g})$ for $\mathsf{P}^{d,\alpha}$ on transverse torus $\mathbb{T}_\Gamma^{g+1}$. We denote domain for return time $U(t_{Ret}) = [0, t_{Ret,1}] \times \cdots [0, t_{Ret,g}]$. Return map of action $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}_\Gamma^{g+1}$ has a form of skew-shift

$$(37) \qquad A_{\rho,\tau}(y, z) = (y + \rho, z + v \cdot y + \tau_i) \quad \text{on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z}.$$

From computation of each rank 1 action, for each $1 \leq i \leq d$, it is a composition of commuting linear skew-shift

$$(38) \qquad A_{i,\rho,\tau}(y, z) = (y + \rho_i, z + v_i \cdot y + \tau_i) \quad \text{on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z}$$

for some constant $\rho_i, \sigma_i \in \mathbb{R}^g$ and $v_i \in \mathbb{R}^g$. For each $j \neq k$,

$$A_{j,\rho,\tau} \circ A_{k,\rho,\tau} = A_{k,\rho,\tau} \circ A_{j,\rho,\tau}.$$

Given pair $(\mathbf{m}, n) \in \mathbb{Z}_{K|n|}^g \times \mathbb{Z}$, let $H_{(\mathbf{m},n)}$ denote the corresponding factor and $C^\infty(H_{(m,n)})$ be subspace of smooth function on $H_{(m,n)}$. Denote $\{e_{\mathbf{m},n} \mid (\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z}\}$ the basis of characters of $\mathbb{T}_\Gamma^{g+1}$ and for all $(y, z) \in \mathbb{T}^g \times \mathbb{T}$,

$$e_{\mathbf{m},n}(y, z) := \exp[2\pi\iota(\mathbf{m} \cdot y + nKz)].$$

For each $A_{i,\rho,\sigma}$ and $v_i = (v_{i1}, \cdots v_{id})$, the orbit can be identified with the following dual orbit

$$\mathcal{O}_{A_i}(\boldsymbol{m}, n) = \{(\boldsymbol{m} + (nj_i)v_i, n), \; j_i \in \mathbb{Z}\}$$
$$= \{(m_1 + (nv_{i1})j_i, \cdots, m_d + (nv_{id})j_i, n), \; j_i \in \mathbb{Z}\}.$$

If $n = 0$, the orbit $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$ of $(m, 0)$ is reduced to a single element. If $n \neq 0$, then the dual orbit $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$ of $(\mathbf{m}, n)$ for higher rank actions is described as follows:

$$\mathcal{O}_A(\boldsymbol{m}, n) = \{(m_k + n \sum_{i=1}^{d} (v_{ik} j_i), n)_{1 \leq k \leq d} : j = (j_1, \cdots j_d) \in \mathbb{Z}^d\}.$$

It follows that every A-orbit for rank $\mathbb{R}^d$-action (or A^j-orbit) can be labeled uniquely by a pair $(\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \backslash \{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. Thus, the subspace of functions with non-zero central character can be split as a direct sum of components $H_{(m,n)}$ with $\mathbf{m} \in \mathbb{Z}^g, n \in \mathbb{Z} \backslash \{0\}$. Then,

$$L^2(\mathbb{T}_\Gamma^{g+1}) = \bigoplus_{\omega \in \mathcal{O}_A} H_\omega.$$

Now we proceed to the cosideration of the higher cohomology problem which appears in the space of Fourier coefficients.

5.2. **Higher cohomology for $\mathbb{Z}^d$-action of skew-shifts.** We consider a $\mathbb{Z}^d$ action of return map $\mathsf{P}^{d,\alpha}$ on torus $\mathbb{T}_\Gamma^{g+1}$. By identification of cochain complex on torus, it is equivalent to consider the following cohomological equation for degree d form ω,

(39) $\omega = d\Omega \iff \varphi(x, t) = D\Phi(x, t), \quad x \in \mathbb{T}^g, \ t \in \mathbb{Z}^d.$

We restrict our interest of d-cocycle $\varphi : \mathbb{T}_\Gamma^{g+1} \times \mathbb{Z}^d \to \mathbb{R}$ with $\Phi : \mathbb{T}_\Gamma^{g+1} \to \mathbb{R}^d$, $\Phi = (\Phi_1, \cdots \Phi_d)$ and D is coboundary operator $D\Phi = \sum_{i=1}^{d} (-1)^{i+1} \Delta_i \Phi_i$ where $\Delta_i \Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$. The following proposition is the generalization to the argument of [KK95, Prop 2.2]. Let us denote $A^j = A_{1,\rho,\tau}^{j_1} \circ \cdots \circ A_{d,\rho,\tau}^{j_d}$.

Proposition 5.2. *A cocycle $\hat{\varphi}$ satisfies cohomological equation* (39) *if and only if* $\sum_{j \in \mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m}, n)} \circ A^j = 0$ *for* $j = (j_1, \cdots, j_d) \in \mathbb{Z}^d$.

Proof. We consider dual equation

(40) $\hat{\varphi} = D\hat{\Phi}.$

Let us denote the following notation:

$$(\delta_i \hat{\varphi})(m_1, \cdots, m_d) = \delta(m_i) \hat{\varphi}(m_1, \cdots, m_d), \text{ and } \delta(0) = 1, \text{ otherwise } 0.$$

$$(\Sigma_i \hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=-\infty}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^{j} \cdots A_d^{m_d})$$

$$(\Sigma_i^+ \hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=m_i}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^{j} \cdots A_d^{m_d})$$

$$(\Sigma_i^- \hat{\varphi})(m_1, \cdots, m_d) = - \sum_{j=-\infty}^{m_i-1} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^{j} \cdots A_d^{m_d})$$

It is clear that $\Sigma_i^- - \Sigma_i^+ = \Sigma_i$ and $\Sigma_i^+ \hat{\varphi} = \Sigma_i^- \hat{\varphi}$ if and only if $\Sigma_i \hat{\varphi} = 0$.
Note that

(41) $\Sigma_i^+ \Delta_i = \Sigma_i^- \Delta_i = id, \quad \Delta_i \Sigma_i^+ = \Delta_i \Sigma_i^- = id.$

By direct calculation of Fourier coefficient, $\Sigma_i(\hat{\varphi} - \delta_i\Sigma_i\hat{\varphi}) = 0$. Let $\hat{\Phi}_i(\hat{\varphi}) = \Sigma_i^-(\hat{\varphi} - \delta_i\Sigma_i\hat{\varphi})$, then $\hat{\Phi}_i(\hat{\varphi})$ vanishes at ∞. By (41),

$$\hat{\varphi} - \delta_i\Sigma_i\hat{\varphi} = \Delta_i\hat{\Phi}_i(\hat{\varphi}).$$

We can proceed this by induction.

$$\hat{\varphi} - \Sigma_{1,\cdots,d}\hat{\varphi} = \sum_{i=1}^{d}(\delta_1\cdots\delta_{i-1}\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi} - \delta_1\cdots\delta_i\Sigma_1\cdots\Sigma_i\hat{\varphi})$$

$$= \sum_{i=1}^{d}(\delta_1\cdots\delta_{i-1}\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi} - \delta_i\Sigma_i(\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi}))$$

$$= \sum_{i=1}^{d}(-1)^{i+1}\Delta_i\hat{\Phi}_i(\hat{\varphi})$$

where

$$\hat{\Phi}_i(\hat{\varphi}) = (-1)^{i+1}\Sigma_i^-\delta_1\cdots\delta_{i-1}(\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi} - \delta_i\Sigma_i(\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi}))$$

and $\hat{\Phi}_i(\hat{\varphi})$ vanishes at ∞. Thus, $\hat{\Phi}_i$ is a solution of (40) if and only if $\Sigma_1\cdots\Sigma_d\hat{\varphi} = 0$. $\qquad\square$

For fixed $(\mathbf{m}, n) \in \mathbb{Z}^g \times \mathbb{Z}$, we denote obstruction of cohomological equation restricted to the orbit of $(\mathbf{m}, n)$ by $\mathcal{D}_{m,n}(\varphi) = \sum_{j\in\mathbb{Z}^d}\hat{\varphi}_{(\mathbf{m},n)} \circ A^j$.

Lemma 5.3. *There exists a distributional obstruction to the existence of a smooth solution $\varphi \in C^\infty(H_{(\mathbf{m},n)})$ of the cohomological equation (39).*

A generator of the space of invariant distribution $\mathcal{D}_{m,n}$ has form of

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) := \begin{cases} e^{-2\pi\iota\sum_{i=1}^{d}[(\mathbf{m}\cdot\rho_i+nK\tau_i)j_i+nK\tau_i\binom{j_i}{2}]} & \text{if } (a,b) = (m_k + K\sum_{i=1}^{d}(v_{ik}j_i), n)_{1\le k\le g} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From previous observation, there exists obstruction

(42) $$\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j\in\mathbb{Z}^d}\int_{\mathbb{T}_{\Gamma}^{g+1}}\varphi(x,y)\overline{e_{\mathbf{m},n}\circ A^j_{\rho,\tau}}dxdy.$$

By direct computation, for fixed $j = (j_1, \cdots j_d)$,

$$e_{\mathbf{m},n}\circ A^j_{\rho,\tau}(y,z) = \prod_{i=1}^{d}\left(e^{2\pi\iota[(\mathbf{m}\cdot\rho_i+nK\tau_i)j_i+nK\tau_i\binom{j_i}{2}]}\right)\left(e^{2\pi\iota(\mathbf{m}\cdot y+K(z+n\sum_{k=1}^{d}(v_{ik}j_i)y_k))}\right).$$

Then, we choose $\varphi = e_{a,b}$ for $(a,b) = (m_k + K\sum_{i=1}^{d}(v_{ik}j_i), n)_{1\le k\le g}$ in the non-trivial orbit $(n \ne 0)$,

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi\iota\sum_{i=1}^{d}[(\mathbf{m}\cdot\rho_i+nK\tau_i)j_i+nK\tau_i\binom{j_i}{2}]}.$$

$\qquad\square$

We conclude this section by introducing the theory of unitary representations.

(43) $$L^2(M) = \bigoplus_{n\in\mathbb{Z}}H_n := \bigoplus_{n\in\mathbb{Z}}\bigoplus_{i=1}^{\mu(n)}H_{i,n}$$

where $H_n = \bigoplus_{i=1}^{\mu(n)} H_{i,n}$ is irreducible representation with central parameter n and $\mu(n)$ is countable by Howe-Richardson multiplicity formula. By direct calculation, we obtain $\mu(n) = |n|^d$ for all $n \neq 0$.

For P-action, the space of invariant currents $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-s}(\mathbb{R}^g)$ for $s > d/2 + \epsilon$ for all $\epsilon > 0$. That is, by normalization of invariant distributions in the Sobolev space, for any irreducible components $H = H_n$ and α, there exists a non-unique function f_α^H such that

$$(44) \qquad\qquad D_\alpha(f_\alpha^H) = \left\| f_\alpha^H \right\|_s = 1.$$

5.3. **Changes of coordinates.** For any frame $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$, denote transverse cylinder for any $m \in M$,

$$\mathscr{C}_{\alpha,m} := \{ m \exp(\sum_{i=1}^g y_i' Y_i^\alpha + z' Z) \mid (y', z') \in U(t_{Ret}^{-1}) \times \mathbb{T} \}.$$

Let $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \to \mathscr{C}_{\alpha,m}$ denote the maps: for any $\xi \in \mathbb{T}_\Gamma^{g+1}$, let $\xi' \in \mathscr{C}_{\alpha,m}$ denote first intersection of the orbit $\{ \mathsf{P}_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}_+^d \}$ with transverse cylinder $\mathscr{C}_{\alpha,m}$. Then, there exists first return time to cylinder $t(\xi) = (t_1(\xi), \cdots, t_d(\xi)) \in \mathbb{R}_+^d$ such that

$$\xi' = \Phi_{\alpha,m}(\xi) = \mathsf{P}_{t(\xi)}^{d,\alpha}(\xi), \quad \forall \xi \in \mathbb{T}_\Gamma^{g+1}.$$

Let (y, z) and (y', z') denote the coordinates on $\mathbb{T}_\Gamma^{g+1}$ and $\mathscr{C}_{\alpha,m}$ given by the exponential map respectively,

$$(y, z) \to \xi_{y,z} := \Gamma \exp(\sum_{i=1}^g y_i Y_i + z Z), \quad (y', z') \to m \exp(\sum_{i=1}^g y_i' Y_i^\alpha + z' Z).$$

For $1 \leq i, j \leq g$ and matrix $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$, set

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2g, \mathbb{R}),$$

satisfying $A^t D - C^t B = I_{2g}$, $C^t A = A^t C$, $D^t B = B^t D$, and $\det(A) \neq 0$.

Recall that $X_i^\alpha = \sum_j a_{ij} X_j + b_{ij} Y_j + w_i Z$ and $Y_i^\alpha = \sum_j c_{ij} X_j + d_{ij} Y_j + v_i Z$ with $\det(A) \neq 0$. Let $x = \Gamma \exp(\sum_{i=1}^d y_{x,i} Y_i + z_x Z) \exp(\sum_{i=1}^d t_{x,i} X_i)$, for some $(y_x, z_x) \in \mathbb{T}^d \times \mathbb{R}/K\mathbb{Z}$ and $t_x = (t_{x,i}) \in [0, 1)^d$.

Then, the map $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \to \mathscr{C}_{\alpha,m}$ is defined by $\Phi_{\alpha,x}(y, z) = (y', z')$ where

$$(45) \qquad \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_g' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix}.$$

and $z' = z + P(\alpha, x, y)$ for some degree 4 polynomial P.

Therefore, the map $\Phi_{\alpha,x}$ is invertible with

$$\Phi_{\alpha,x}^*(dy_1' \wedge \cdots dy_g' \wedge dz') = \frac{1}{\det(A)} dy_1 \wedge \cdots dy_g \wedge dz.$$

Since $A^t D - C^t B = I_{2g}$, by direct computation we obtain return time

$$
(46) \qquad
\begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix}
=
\begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{gg} \end{bmatrix}
\begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix}
+
\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix}
\begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}
$$

and we have

$$
\|t(\xi)\| \le \max_i |t_i(\xi)|^g \le \max_i |\sum_{j=1}^{g} d_{ij} t_{x,i} + c_{ij}(y_i - y_{x,i})|^g \le \max_i \|Y_i^\alpha\|^g .
$$

5.4. L^2-lower bound of functional.

We will prove bounds for the square mean of integrals along foliations of the torus $\mathbb{T}_\Gamma^{g+1}$ into torus $\{\xi \exp(\sum_{i=1}^{g} y_i Y_i) \mid y_i \in \mathbb{T}\}_{\xi \in \mathbb{T}_\Gamma^{g+1}}$. Let $Q_y^{g,Y} := \exp(y_1 Y_1 + \cdots + y_g Y_g)$, $y = (y_1, \cdots, y_g) \in \mathbb{R}^g$ be the action generated by elements of standard basis Y_i.

Lemma 5.4. *There exists a constant $C > 0$ such that for all $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$, and for every irreducible component H of central parameter $n \ne 0$, there exists a function f_H such that*

$$
|f_H|_{L^\infty(H)} \le C vol(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)|
$$

$$
|f_H|_{\alpha,s} \le C vol(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)| (1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s (1 + n^2)^{s/2}
$$

where $\|Y\| := \max_{1 \le i \le g} \|Y_i^\alpha\|$ and $T(t_{Ret}) = \sum_{i=1}^{g} t_{Ret,i}$.

On rectangular domain $U(\mathbf{T})$, for all $m \in \mathbb{T}_\Gamma^{g+1}$ and $T^{(i)} \in \mathbb{Z}_{t_{Ret,i}}$,

$$
(47) \qquad \left\| \left\langle P_{U(\mathbf{T})}^{d,\alpha}(Q_y^{g,Y} m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = |\mathcal{D}_\alpha^H(f_H)| \left(\frac{vol(U(\mathbf{T}))}{vol(U(t_{Ret}))} \right)^{1/2} .
$$

In addition, whenever $H \perp H' \subset L^2(M)$ the functions

$$
\left\langle P_{U(\mathbf{T})}^{d,\alpha}(Q_y^{g,Y} m), \omega_H \right\rangle \quad and \quad \left\langle P_{U(\mathbf{T})}^{d,\alpha}(Q_y^{g,Y} m), \omega_{H'} \right\rangle
$$

are orthogonal in $L^2(\mathbb{T}^g, dy)$.

Proof. As explained in §5.1, the space $L^2(\mathbb{T}_\Gamma^{d+1})$ decompose as a direct sum of irreducible subspaces invariant under the action of each $A_{j,\rho,\sigma}$. It follows that the subspace of functions with non-zero central character can be split as direct sum of components $H_{(\mathbf{m},n)}$ with $(\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \setminus \{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. For $F \in H_{(\mathbf{m},n)}$, the function is characterized by Fourier expansion

$$
F = \sum_{j \in \mathbb{Z}^d} F_j e_{A^j(\mathbf{m},n)} = \sum_{j \in \mathbb{Z}^d} F_j e_{(m_k + K \sum_{i=1}^{d}(v_{ik} j_i), n)}.
$$

Then, by Lemma 5.3,

$$
(48) \qquad \mathcal{D}_{(\mathbf{m},n)}(e_{A^j(\mathbf{m},n)}) = e^{-2\pi \iota \sum_{i=1}^{d} [(m \cdot \rho_i + nK\tau_i) j_i + nK\tau_i \binom{j_i}{2})]}.
$$

For any irreducible representation $H := H_n$ with central parameter $n \ne 0$, there exists $\mathbf{m} \in \mathbb{Z}_{|n|}^d$ such that the operator I_α maps the space H onto $H_{(\mathbf{m},n)}$. The

operator $I_\alpha : L^2(M) \to L^2(\mathbb{T}_\Gamma^{g+1})$ is defined

(49)
$$f \to I_\alpha(f) := \int_{U(t_{Ret})} f \circ \mathsf{P}_x^{d,\alpha}(\cdot) dx.$$

Then, operator I_α is surjective linear map of $L^2(M)$ onto $L^2(\mathbb{T}_\Gamma^{g+1})$ with right inverse defined as follows:

Let $\chi \in C_0^\infty(0,1)^g$ be any function of jointly integrable with integral 1. For any $F \in L^2(\mathbb{T}_\Gamma^{g+1})$, let $R_\alpha^\chi(F) \in L^2(M)$ be the function defined by

$$R_\alpha^\chi(F)(\mathsf{P}_v^{d,\alpha}(m)) = \frac{1}{vol(U(t_{Ret}))} \chi(\frac{v}{t_{Ret}}) F(m), \ (m,v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows from the definition that there exists a constant $C_\chi > 0$ such that

$$|R_\alpha^\chi(F)|_{\alpha,s} \le C_\chi vol(U(t_{Ret}))^{-1} (1 + \sum_{i=1}^g t_{Ret,i}^{-1} \|Y_i^\alpha\|)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}$$

$$\le C_\chi vol(U(t_{Ret}))^{-1} (1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}.$$

Choose $f_H := R_\alpha^\chi(e_{m,n}) \in C^\infty(H)$ such that $I_\alpha(f_H) = e_{m,n}$ and

(50)
$$\int_{U(t_{Ret})} f_H \circ \mathsf{P}_t^{d,\alpha}(y,z) dt = e_{m,n}(y,z), \text{ for } (y,z) \in \mathbb{T}_\Gamma^{g+1}.$$

By (44) and (48), we have $|\mathcal{D}_H(f_H)| = |\mathcal{D}_{(m,n)}(e_{m,n})| = 1$. Therefore, it follows that

$$|f_H|_{L^\infty(H)} \le C_\chi vol(U(t_{Ret}))^{-1}$$

$$|f_H|_{\alpha,s} \le C vol(U(t_{Ret}))^{-1} |D_\alpha^H(f_H)| (1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s (1+n^2)^{s/2}.$$

Moreover, since $\{e_{m,n} \circ A_{\rho,\tau}^j\}_{j \in \mathbb{Z}^d} \subset L^2(\mathbb{T}_\Gamma^g, dy)$ is orthonormal, we verify

$$\left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\mathsf{Q}_y^{g,Y} m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g,dy)} = \left\| \sum_{j_d=0}^{[\frac{T^{(d)}}{t_{Ret,d}}]-1} \cdots \sum_{j_1=0}^{[\frac{T^{(1)}}{t_{Ret,1}}]-1} e_{m,n} \circ A_{\rho,\tau}^j \right\|_{L^2(\mathbb{T}^g,dy)}$$

$$= \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2}.$$

$\square$

For any infinite dimensional vector $\mathbf{c} := (c_{i,n}) \in l^2$, let $\beta_{\mathbf{c}}$ denote Bufetov functional

$$\beta_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} \beta^{i,n}.$$

For any $\mathbf{c} := (c_{i,n})$, let $|\mathbf{c}|_s$ denote the norm defined as

$$|\mathbf{c}|_s^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{i=1}^{\mu(n)} (1 + K^2 n^2)^s |c_{i,n}|^2.$$

From Corollary 4.3,

$$\|\beta_{\mathbf{c}}(\alpha, \cdot, \boldsymbol{T})\|_{L^2(M)}^2 \le C^2 |\mathbf{c}|_{l^2}^2 vol(U(\boldsymbol{T})).$$

Lemma 5.5. *For any $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for all $\alpha \in DC(L)$, for all $\mathbf{c} \in l^2$, for all $z \in \mathbb{T}$ and all $T > 0$,*

$$
(51) \quad \left| \left\| \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}) \right\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right|
$$

$$
\leq C_s (vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1 + L)(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.
$$

Proof. By (43), for every $n \neq 0$, there exists a function $f_{i,n} \in C^\infty(H)$ with $D(f_{i,n}) = 1$. Let $f_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n}$, by adding functions on all irreducibles. Then, by Lemma 5.4,

$$
(52) \qquad |f_{\mathbf{c}}|_{L^\infty(M)} \leq C|\mathbf{c}|_{l^1}
$$

$$
(53) \qquad |f_{\mathbf{c}}|_{\alpha,s} \leq Cvol(U(t_{Ret}))^{-1}(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.
$$

By orthogonality,

$$
\left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha} \circ \mathsf{Q}_y^{g,Y}, \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0.
$$

From the estimation for each $f_{i,n}$ in Lemma 5.4, for every $z \in \mathbb{T}$ and all $T > 0$, we have

$$
\left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq 2|f_{\mathbf{c}}|_{L^\infty(M)} \|Y\|.
$$

Let $T_{\alpha,i} = t_{Ret,i}([T/t_{Ret,i}] + 1)$ and $U(t_\alpha) = [0, T_{\alpha,1}] \times \cdots \times [0, T_{\alpha,g}]$. Then,

$$
\left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle P_{U(t_\alpha)}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq vol(U(t_{Ret}))|f_{\mathbf{c}}|_{L^\infty(M)}.
$$

Therefore, for some constant $C' > 0$ such that

$$
\left| \left\| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \leq C' vol(U(t_{Ret}))|\mathbf{c}|_{l^1}.
$$

For all $s > s_d + 1/2$, by asymptotic property of Theorem 3.7, there exists constant $C_s > 0$ such that

$$
\left| \left\langle P_{U(\boldsymbol{T})}^{d,\alpha} m, \omega \right\rangle - \beta_H(\alpha, m, \boldsymbol{T}) D_\alpha^H(f_H) \right| \leq C_s(1 + L) \|f\|_{\alpha,s}.
$$

Applying $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$ and combining bounds on the function $f_{\mathbf{c}}$ with (52),

$$
\left| \left\| \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}) \right\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right|
$$

$$
\leq C' vol(U(t_{Ret}))|\mathbf{c}|_{l^1} + C_s vol(U(t_{Ret}))^{-1}(1 + L)|f_{\mathbf{c}}|_{\alpha,s}
$$

$$
\leq C'_s (vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1 + L)(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.
$$

Therefore, we derive the estimates in the statement. $\qquad\square$

6. Analyticity of functionals

In this section we will prove that for all $\alpha \in DC$, the Bufetov functionals on any square are real analytic.

We recall the orthogonal property on a irreducible component H (central parameter $n \in \mathbb{Z}\backslash\{0\}$) : for any $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$ and $t \in \mathbb{R}$,

$$(54) \qquad \beta_H(\alpha, \phi_t^Z(m), \mathbf{T}) = e^{2\pi\iota Knt}\beta_H(\alpha, m, \mathbf{T}).$$

Let $\Gamma_{\mathbf{T}} : U(\boldsymbol{T}) \to M$ be a C^1 parametrized rectangle with coordinate of $\mathbf{s} = (s_1, \cdots, s_d)$. Recall Definition 1.2 and we write $\Gamma_{\mathbf{T}}$ with action of $\mathsf{P}_{\mathbf{s}}^{d,X}$ generated by standard frame X_i. Set $\Gamma_{\mathbf{T}} = (\gamma_1(s_1), \cdots, \gamma_d(s_d))$ for $\gamma_i(s_i) = \exp(s_i X_i)$.

For every $t \in \mathbb{R}$, define stretched rectangle by

$$(55) \qquad [\Gamma_T]_{i,t}^Z(\mathbf{s}) = (\gamma_1(s_1), \cdots, \gamma_{i,t}^Z(s_i), \cdots \gamma_d(s_d))$$

with a new curve $\gamma_{i,t}^Z(s_i) := \phi_{ts_i}^Z(\gamma_i(s_i))$.

We write restriction of a rectangle map Γ_T by the following way: For $\mathbf{s} = (s_1, \cdots, s_d)$, set $\Gamma_{T,i,s}(\mathbf{s}) := \Gamma_T(\mathbf{s})$ is defined with its coordinate $s_i \in [0, s]$ for fixed i and $s_j \in [0, T^{(j)}]$ for all $j \neq i$.

Lemma 6.1. *For a fixed generator* (X_i, Y_i, Z) *satisfying commutation relation (1), the following formula for rank 1 action holds:*

$$\hat{\beta}_H(\alpha, [\Gamma_T]_{i,t}^Z) = e^{2\pi\iota tnKT^{(i)}}\hat{\beta}_H(\alpha, \Gamma_T) - 2\pi\iota nKt \int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\hat{\beta}_H(\alpha, \Gamma_{T,i,s})ds_i.$$

Proof. Let $\alpha = (X_i, Y_i, Z)$ and ω be d-form supported on a single irreducible representation H. We obtain following the formula for stretches of curve $\gamma_{i,t}^Z$ (see [FK17, §4, Lemma 9.1]),

$$\frac{d\gamma_{i,t}^Z}{ds_i} = D\phi_{ts_i}^Z(\frac{d\gamma_i}{ds_i}) + tZ \circ \gamma_{i,t}^Z.$$

It follows that pairing is given by

$$\langle[\Gamma_{\mathbf{T}}]_{i,t}^Z, \omega\rangle = \int_{U(\boldsymbol{T})} \omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_{i,t}^Z}{ds_i}(s_i), \cdots, \frac{d\gamma_d}{ds_d}(s_d))d\mathbf{s}$$

$$= \int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))] + \iota_Z\omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z(\mathbf{s})d\mathbf{s}$$

Denote $d-1$ dimensional triangle $U_{d-1}(\boldsymbol{T})$ with $U(\boldsymbol{T}) = U_{d-1}(\boldsymbol{T}) \times [0, T^{(i)}]$. Integration by parts for a fixed i-th integral gives

$$\int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$

$$= e^{2\pi\iota nKtT^{(i)}} \int_{U(\boldsymbol{T})} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$

$$- 2\pi\iota nKt \int_0^{T^{(i)}} e^{2\pi\iota nKts_i} \int_{U_{d-1}(\boldsymbol{T})} \left(\int_0^{s_i} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_i}{ds_i}(r) \cdots, \frac{d\gamma_d}{ds_d}(s_d))]dr\right) d\mathbf{s}.$$

Then, we have the following formula

$$\langle[\Gamma_{\mathbf{T}}]^Z_{i,t},\omega\rangle = e^{2\pi\iota nKtT^{(i)}}\langle[\Gamma_{\mathbf{T}}],\omega\rangle - 2\pi\iota nKt\int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\langle\Gamma_{T,i,s},\omega\rangle ds_i$$

$$+ \int_{U(\boldsymbol{T})}(\iota_Z\omega\circ[\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s}.$$

Since the action of $\mathsf{P}^{d,X}_{\mathbf{t}}$ for $\mathbf{t}\in\mathbb{R}^d$ is identity on the center Z,

$$\lim_{t_d\to\infty}\cdots\lim_{t_1\to\infty}e^{-(t_1+\cdots t_d)/2}\int_{U(\boldsymbol{T})}(\iota_Z(\mathsf{P}^{d,X}_{\mathbf{t}})^*\omega\circ[\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s} = 0.$$

Thus, it follows by definition of Bufetov functional, the statement holds. $\qquad\square$

Here we define a restricted vector $\mathbf{T}_{i,s}$ of $\mathbf{T} = (T^{(1)},\cdots,T^{(d)})\in\mathbb{R}^d$. For fixed i, pick $s_i\in[0,T^{(i)}]$ such that $\mathbf{T}_{i,s}\in\mathbb{R}^d$ is a vector with its coordinates

$$T^{(j)}_{i,s} = \begin{cases} T^{(j)} & \text{if} \quad j\neq i \\ s_i & \text{if} \quad j = i. \end{cases}$$

Similarly, $\mathbf{T}_{i_1,\cdots,i_k,s}$ is a vector with $i_1,\cdots i_k$ coordinates replaced by $s_{i_1},\cdots,s_{i_k}$.

Lemma 6.2. *Let* $y = (y_1,\cdots,y_d)\in\mathbb{R}^d$. *The following equality holds for each rank-1 action.*

(56) $\quad \beta_H(\alpha,\phi^{Y_i}_{y_i}(m),\boldsymbol{T}) =$

$$e^{-2\pi\iota y_i nKT^{(i)}}\beta_H(\alpha,m,\boldsymbol{T}) + 2\pi\iota nKy_i\int_0^{T^{(i)}} e^{-2\pi\iota y_i nKs_i}\beta_H(\alpha,m,\boldsymbol{T}_{i,s})ds_i.$$

Proof. Let $\Gamma^X_T(m) = \mathsf{P}^{d,X}_{\mathbf{s}}(m)$ for all $\mathbf{s}\in U(\boldsymbol{T})$. By definition (55) and commutation relation (1), it follows that

$$\phi^{Y_i}_{y_i}(\Gamma^X_{\mathbf{T}}(m)) = [\Gamma^X_{\mathbf{T}}(\phi^{Y_i}_{y_i}(m))]^Z_{i,t}.$$

By the invariance property of Bufetov functional and Lemma 6.1,

$$\beta_H(\alpha,m,\mathbf{T}) = \hat\beta_H(\alpha,\phi^{Y_i}_{y_i}(\Gamma^X_T(m)))$$

$$= e^{2\pi\iota y_i nKT^{(i)}}\hat\beta_H(\alpha,\Gamma^X_{\mathbf{T}}(\phi^{Y_i}_{y_i}(m))) - 2\pi\iota nKy_i\int_0^{T^{(i)}} e^{2\pi\iota nKy_i s_i}\hat\beta_H(\alpha,\Gamma^X_{T,i,s}(\phi^{Y_i}_{s_i}(m)))ds_i$$

$$= e^{2\pi\iota y_i nKT^{(i)}}\beta_H(\alpha,\phi^{Y_i}_{y_i}(m),\mathbf{T}) - 2\pi\iota nKy_i\int_0^{T^{(i)}} e^{2\pi\iota nKy_i s_i}\beta_H(\alpha,\phi^{Y_i}_{s_i}(m),\mathbf{T}_{i,s})ds_i.$$

Then statement follows immediately. $\qquad\square$

We extend previous lemma for higher rank actions by induction.

Lemma 6.3. *The following equality holds for rank-d action.*

$$\beta_H(\alpha,\mathsf{Q}^{d,Y}_y(m),\boldsymbol{T}) = e^{-2\pi\iota\sum_{j=1}^d y_j nKT^{(j)}}\beta_H(\alpha,m,\boldsymbol{T})$$

$$+ \sum_{k=1}^d\sum_{1\le i_1<\cdots<i_k\le d}\prod_{j=1}^k(2\pi\iota nKy_{i_j})e^{-2\pi\iota nK(\sum_{l\notin\{i_1,\cdots,i_k\}}y_l T^{(l)})}$$

$$\times\int_0^{T^{(i_1)}}\cdots\int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1}+\cdots+y_{i_k}s_{i_k})}\beta_H(\alpha,m,\boldsymbol{T}_{i_1,\cdots,i_k,s})ds_{i_k}\cdots ds_{i_1}$$

Proof. Assume inductive hypothesis works for rank $d-1$. For convenience, we write

$$\mathsf{Q}_y^{d,Y}(m) = \phi_{y_d}^{Y_d} \circ \mathsf{Q}_{y'}^{d-1,Y}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 6.2,

$$\beta_H(\alpha, \mathsf{Q}_y^{d,Y}(m), \mathbf{T}) = e^{-2\pi\iota y_d nKT^{(d)}} \beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T})$$

(57)
$$+ 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d} \beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T}_{d,s}) ds_d$$

$$:= I + II$$

Firstly, by induction hypothesis

$$I = e^{-2\pi\iota \sum_{j=1}^{d-1} y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T})$$

$$+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \prod_{j=1}^{k} (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)} + y_d T^{(d)})}$$

$$\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1} s_{i_1} + \cdots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,s}) ds_{i_k} \cdots ds_{i_1}$$

which contains 0 to $d-1$th iterated integrals containing $e^{-2\pi\iota nKy_d T^{(d)}}$ outside of iterated integrals.

For the second part, we apply induction hypothesis again for restricted rectangle $\mathbf{T}_{d,s}$. Then,

$$II = 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d} \left[e^{-2\pi\iota \sum_{j=1}^{d-1} y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T}_{d,s}) \right] ds_d$$

$$+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} (2\pi\iota nKy_d) \prod_{j=1}^{k} (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)})}$$

$$\times \int_0^{T^{(d)}} \left(\int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} ds_{i_k} \cdots ds_{i_1} \right) ds_d$$

$$\times e^{-2\pi\iota nK(y_{i_1} s_{i_1} + \cdots + y_{i_k} s_{i_k} + y_d s_d)} \beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,d,s}).$$

The term II consist of 1 to d-th iterated integrals containing $e^{-2\pi\iota nKy_d s_d}$ inside of iterated integrals. Thus, combining these two terms, we prove the statement. $\square$

For any $R > 0$, the *analytic norm* defined for all $\mathbf{c} \in l^2$ as

$$\|\mathbf{c}\|_{\omega,R} = \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}|.$$

Let Ω_R denote the subspace of $\mathbf{c} \in \ell^2$ such that $\|c\|_{\omega,R}$ is finite.

Lemma 6.4. *For $\mathbf{c} \in \Omega_R$, any $\alpha \in DC(L)$ and $\mathbf{T} \in \mathbb{R}_+^d$, the function*

$$\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}), \ (y, z) \in \mathbb{R}^d \times \mathbb{T}$$

extends to a holomorphic function in the domain

$$D_{R,T} := \{(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \mid \sum_{i=1}^{d} |Im(y_i)| T^{(i)} + |Im(z)| < \frac{R}{2\pi K}\}.$$

The following bound holds: for any $R' < R$ there exists a constant $C > 0$ such that, for all $(y, z) \in D_{R',T}$ we have

$$|\beta_{\boldsymbol{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \boldsymbol{T})|$$

$$\leq C_{R,R'} \, \|c\|_{\omega,R} \, (L + vol(U(\boldsymbol{T}))^{1/2}(1 + E_M(a, \boldsymbol{T}))(1 + K \sum_{i=1}^d |Im(y_i)|T^{(i)})$$

Proof. By Lemma 6.3 and (54),

$$\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}) = e^{(z - 2\pi\iota \sum_{j=1}^d y_j n K T^{(j)})} \beta_H(\alpha, m, \mathbf{T})$$

$$+ \sum_{k=1}^d \sum_{1 \leq i_1 < \cdots < i_k \leq d} \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \cdots, i_k\}} y_l T^{(l)})}$$

$$\times e^{2\pi\iota n K z} \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \cdots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \cdots, i_k, s}) ds_{i_k} \cdots ds_{i_1}$$

As a consequence, by Lemma 3.6 for each variable $(y_i, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$, Then for the rank d action, by induction, for $(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ we have

$$|\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(x), \mathbf{T})|$$

$$\leq (L + vol(U(\boldsymbol{T}))^{1/2}(1 + E_M(a, \boldsymbol{T}))[C_1 \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}| e^{2\pi|Im(z - \sum_{i=1}^d T^{(i)} y_i)|nK}$$

$$+ \sum_{k=1}^d C_k \left(\sum_{1 \leq i_1 < \cdots < i_k \leq d} \prod_{j=1}^k (|Im(y_{i_j})|T^{(i_j)}) \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} n |c_{i,n}| e^{2\pi(|Im(z)| + \sum_{j=1}^k T^{(i_j)} |Im(y_{i_j})|)nK} \right)].$$

Therefore, the function $\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})$ is bounded by a series of holomorphic functions on $\mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ and it converges uniformly on compact subsets of domain $D_{R,T}$. $\qquad \square$

7. Measure estimation for bounded-type

Bufetov functionals were constructed for $\alpha \in A$ under Diophantine condition, which depends on backward orbit under renormalization in the moduli space $\mathfrak{M}_g$. In this section, we prove measure estimation of Bufetov functional under bounded-type case (13). This result is generalization of §11 of [FK17].

Let $\mathcal{O}_r$ denote thee space of holomorphic functions on the ball $B_{\mathbb{C}}(0, r) \subset \mathbb{C}^n$. We recall the *Chebyshev degree*, the best constant $d_f(r)$ stated in the following theorem and estimation of *valency*.

Theorem 7.1. *[Bru99, Thm 1.9] For any $f \in \mathcal{O}_r$, there is a constant $d := d_f(r) > 0$ such that for any convex set $D \subset B_{\mathbb{R}}(0, 1) := B_{\mathbb{C}}(0, 1) \cap \mathbb{R}^n$, for any measurable subset $U \subset D$*

$$\sup_D |f| \leq \left(\frac{4n Leb(D)}{Leb(U)} \right)^d \sup_U |f|.$$

Let $\mathcal{L}_t$ denote the set of one-dimensional complex affine spaces $L \subset \mathbb{C}^n$ such that $L \cap B_{\mathbb{C}}(0, t) \neq \emptyset$.

Definition 7.2. [Bru99, Def 1.6] Let $f \in \mathcal{O}_r$. The number

$$\nu_f(t) := \sup\{\text{valency of } f \mid L \cap B_{\mathbb{C}}(0, t) \neq \emptyset\}$$

is called the valency of f in $B_{\mathbb{C}}(0, t)$.

By [Bru99, Prop 1.7], for any $f \in \mathcal{O}_r$, and the valency $\nu_f(t)$ is finite and any $t \in [1, r)$ there is a constant $c := c(r) > 0$ such that

$$(58) \qquad\qquad\qquad d_f(r) \leq c\nu_f(\frac{1+r}{2}).$$

Lemma 7.3. *Let $L > 0$ and $\mathcal{B} \subset DC(L)$ be a bounded subset. Given $R > 0$, for all $c \in \Omega_R$ and all $\boldsymbol{T}^{(i)} > 0$, let $\mathcal{F}(c, \boldsymbol{T})$ denote the family of real analytic functions of the variable $y \in [0, 1)^d$ defined as*

$$\mathcal{F}(c, \boldsymbol{T}) := \{\beta_{\boldsymbol{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \boldsymbol{T}) \mid (\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}\}.$$

There exists $\boldsymbol{T}_{\mathcal{B}} := (\boldsymbol{T}_{\mathcal{B}}^{(i)})$ and $\rho_{\mathcal{B}} > 0$, such that for every $(R, \boldsymbol{T})$ such that $R/\boldsymbol{T}^{(i)} \geq \rho_{\mathcal{B}}$ and $\boldsymbol{T}^{(i)} \geq \boldsymbol{T}_{\mathcal{B}}^{(i)}$, and for all $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, we have

$$\sup_{f \in \mathcal{F}(c, T)} \nu_f < \infty.$$

Proof. Since $\mathcal{B} \subset \mathfrak{M}$ is bounded, for each time $t_i \in \mathbb{R}$ and $1 \leq i \leq g$,

$$0 < t_{i, \mathcal{B}}^{min} = \min_i \inf_{\alpha \in \mathcal{B}} t_{Ret, i, \alpha} \leq \max_i \sup_{\alpha \in \mathcal{B}} t_{Ret, i, \alpha} = t_{\mathcal{B}}^{max} < \infty.$$

For any $\alpha \in \mathcal{B}$ and $x \in M$, the map $\Phi_{\alpha, x} : [0, 1)^d \times \mathbb{T} \to \prod_{i=1}^d [0, t_{\alpha, i}) \times \mathbb{T}$ in (45) extends to a complex analytic diffeomorphism $\hat{\Phi}_{\alpha, x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$. By Lemma 6.4, it follows that for fixed $z \in \mathbb{T}$, real analytic function $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \boldsymbol{T})$ extends to a holomorphic function defined on a region

$$H_{\alpha, m, R, t} = \{y \in \mathbb{C}^d \mid \sum_{i=1}^d |Im(y_i)| \leq h_{\alpha, m, R, t}\}.$$

By boundedness of the set $\mathcal{B} \subset \mathfrak{M}$, it follows that

$$\inf_{(\alpha, x) \in \mathcal{B} \times M} h_{\alpha, m, R, t} := h_{R, T} > 0.$$

For every $r > 1$, there exists $\rho_{\mathcal{B}} > 1$ such that, for every R and $\mathbf{T}$ with $R/\mathbf{T}^{(i)} > \rho_{\mathcal{B}}$, then as a function of $y \in \mathbb{T}^d$

$$\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}) \in \mathcal{O}_r.$$

Then, by Lemma 6.4, the family $\mathcal{F}(c, \mathbf{T})$ is uniformly bounded and normal. By Lemma 5.5, for sufficiently large pair $\mathbf{T}$, no sequence from $\mathcal{F}(c, \mathbf{T})$ can converge to a constant. By Lemma 10.3 of [FK17], for any normal family $\mathcal{F} \subset \mathcal{O}_R$ such that no functions is constant along a one-dimensional complex line, hence the statement follows. $\qquad\square$

We derive measure estimates of Bufetov functionals on the rectangular domain.

Lemma 7.4. *Let $\alpha \in DC$ such that the forward orbit of $\mathbb{R}^d$-action $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t} \in \mathbb{R}_+^d}$ is contained in a compact set of M. There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}_+^d$ such that, for every $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, $T \geq T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{m \in M \mid |\beta_{\boldsymbol{c}}(\alpha, m, \boldsymbol{T})| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

Proof. Since $\alpha \in DC$ and the orbit $\{r_t[\alpha]\}_{t \in \mathbb{R}_+}$ is contained in a compact set, there exists $L > 0$ such that $r_t(\alpha) \in DC(L)$ for all $t \in \mathbb{R}_+^d$. Then, we choose $T_0 \in \mathbb{R}^d$ from conclusion of Lemma 7.3. By scaling property of Bufetov functionals,

$$\beta_{\mathbf{c}}(\alpha, m, T) = \left(\frac{vol(U(T))}{vol(U(T_0))} \right)^{1/2} \beta_{\mathbf{c}}(g_{\log(T/T_0)}[\alpha], m, T_0).$$

By Fubini's theorem, it suffices to estimate

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), T_0)| \le \epsilon\}).$$

Let $\delta^{-1} = c(r) \sup_{f \in \mathcal{F}(c,T_0)} v_f(\frac{1+r}{2}) < \infty$. Since by Lemma 5.5, we have

$$\inf_{(a,x,z) \in \mathscr{B} \times M \times \mathbb{T}} \sup_{y \in [0,1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), T_0)| > 0.$$

By theorem 7.1, for unit ball $D = B_{\mathbb{R}}(0,1)$ and $U = \{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), T_0)| \le \epsilon\}$ and bound in formula (58), there exists a constant $C > 0$ and $\delta > 0$ such that for all $\epsilon > 0$ and $(\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}$,

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), T_0)| \le \epsilon\}) \le C\epsilon^{\delta}.$$

Then statement follows from Fubini theorem. $\square$

Corollary 7.5. *Let α be as in the previous Lemma 7.4. There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}_+^d$ such that, for every $\mathbf{c} \in \Omega_R \backslash \{0\}$, $T \ge T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{x \in M \mid \left| \langle P_{U(T)}^{d,\alpha} m, \omega_{\mathbf{c}} \rangle \right| \le \epsilon vol(U(T))^{1/2}\}) \le C\epsilon^{\delta}.$$

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