

1 **LIMIT THEOREMS FOR HIGHER RANK ACTIONS ON**  
2 **HEISENBERG NILMANIFOLDS**

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ABSTRACT. The main result of this paper is a construction of finitely additive measures for higher rank abelian actions on Heisenberg nilmanifolds. Under a full measure set of Diophantine condition for the generators of the action, we construct *Bufetov functional* on rectangles on  $2g + 1$  dimensional Heisenberg manifold. We prove that the deviation of ergodic integral of higher rank actions is described by the asymptotics of Bufetov functionals for a sufficiently smooth function. As a corollary, the distribution of normalized ergodic integrals to have variance 1, converges along certain subsequences to a non-degenerate compactly supported measure on the real line.

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## 3 1. INTRODUCTION

4 **1.1. Introduction.** The asymptotic behavior and limiting distribution of ergodic  
5 averages of translation flow were studied by Alexander Bufetov in the series of his  
6 works [Buf14a, Buf10, Buf14b]. He constructed finitely-additive Hölder measures  
7 and cocycles over translation flows that are known as *Bufetov functionals*. Such  
8 functionals are constructed to derive the deviation of ergodic integrals of translation  
9 flows by providing a new proof of the celebrated work of G. Forni [For02]. Bufetov  
10 noticed a duality between his finitely-additive Hölder measures and Forni's invariant  
11 distributions, which played a crucial role in his work of cohomological equations.  
12 Following these observations, the duality between such functionals and the space  
13 of invariant distributions was constructed analogously for other parabolic flows  
14 and higher rank actions (e.g. [Buf13, BS13, BF14, FK20b, For20a]). Bufetov also  
15 proved the probabilistic results for normalized ergodic integrals as a part of his  
16 main theorem. Furthermore, it turned out that his construction of functional is also  
17 closely related to limit shapes of ergodic sums for interval exchange transformation  
18 [MMY10, MUY20].

19 In this paper, our main results are on effective equidistributions for higher rank  
20 abelian actions and limit theorems on Heisenberg nilmanifolds. We firstly introduce  
21 the construction of the Bufetov functional for higher rank abelian actions. The  
22 main argument is based on the renormalizations and induction argument from the  
23 work of Cosentino and Flaminio [CF15]. However, in our case, we extended their  
24 constructions to the rectangular domain and allowed general products of intervals.  
25 We assume that the functional exists for a full measure set of frame  $\alpha$  under a  
26 sufficient Diophantine condition obtained from recurrent conditions (see §2.3.2).  
27 Likewise, we also constructed the duality between functionals and invariant currents  
28 in [CF15], enabling the deviation formula. As a corollary, we prove existence of  
29 the limit distributions of (normalized) ergodic integrals for higher rank actions.  
30 More specifically, as a random variable, normalized ergodic integrals converge in  
31 distribution along certain subsequences to a non-degenerate, compactly supported  
32 measure on the real line.

33 As a minor application, it is not surprising that the related result for exponential  
34 sums appeared from connections between the Heisenberg group and theta series in  
35 several contexts. For Heisenberg flows, as a first return map on a transverse torus  
36 is obtained as a skew-translation, its limit theorem of theta sums was studied by J.  
37 Griffin J. Marklof and Cellarosi-Marklof [GM14, CM16]. It is recently generalized by  
38 Forni and Kanigowski [FK20b]. Likewise, we obtain limit theorems for theta series  
39 on Siegel half spaces which generalize results [GG04, Mar99] (see [Tol78, MM07,  
40 MNN07] for a general introduction).

41 Furthermore,  $L^2$ -lower bound of Bufetov functional on a transverse torus and  
42 its analyticity on the higher dimensional rectangular domain follows. This part  
43 contributes that our functional does not vanish and extends to the complex domain  
44 in the weighted space of fast-decaying coefficient of functionals. From these results,

polynomial type of lower bounds for the analytic function is obtained on certain sub-level sets. Such polynomial estimates on the measure are derived by real analyticity of a functional along the leaves of a foliation transverse to the actions based on results of [Bru99]. This tool was already devised by Forni and Kanigowski [FK20b] to study the bound of correlations for time-changes of nilflows. Unfortunately, it turned out that we can not obtain similar results because the time-changes of higher rank abelian actions on Heisenberg nilmanifolds are all trivial. It follows from the triviality of the first cohomology group (see [CF15, Theorem 3.16]) thus, they are conjugated to the linear action and never mixing.

As a rank 1-action, mixing properties for time changes of Heisenberg nilflows were firstly studied in [AFU11]. Then, it was extended in the dense set of non-trivial time changes for any uniquely ergodic nilflows on general nilmanifolds [Rav18, AFRU21]. As a special case, on time-changes of Heisenberg flows, multiple mixing [FK20a] was proved by Ratner property. In  $\mathbb{Z}^k$ -actions, mixing of shapes for automorphisms on nilmanifold is proved in [GS14, GS15].

Comprehensive studies of spatial and temporal limit theorems for horocycle flows have been carried out in the work of D. Dolgopyat and O. Sarig [DS17]. Recent work of Ravotti [Rav21] proved spatial and temporal limit theorems for horocycle flows in alternative ways by using Ratner's argument. His work did not rely on the methods from invariant distributions of solving the cohomological equations. However, it is still unknown if temporal (or weaker) limit theorems for Heisenberg flows can be proved, and it will be a possible subject of further works.

**1.2. Definitions and statement of results.** We review definitions about Heisenberg manifold and its moduli space.

**1.2.1. Heisenberg manifold.** Let  $g \geq 1$  and  $H^g$  be the standard  $2g + 1$  dimensional Heisenberg group and set  $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$  a discrete and co-compact subgroup of  $H^g$ . We shall call it standard lattice of  $H^g$  and the quotient  $M := H^g/\Gamma$  will be called *Heisenberg manifold*. Lie algebra  $\mathfrak{h}^g = Lie(H^g)$  is equipped with a basis  $(X_1, \dots, X_g, Y_1, \dots, Y_g, Z)$  satisfying canonical commutation relations

$$(1) \quad [X_i, Y_j] = \delta_{ij}Z, \quad \text{where } \delta_{ij} = 1 \text{ if } i = j, \text{ and } \delta_{ij} = 0 \text{ otherwise.}$$

For  $1 \leq d \leq g$ , let  $P := P^d$  be a subgroup of  $H^g$  where its Lie algebra  $\mathfrak{p} := Lie(P^d)$  is generated by  $\{X_1, \dots, X_d\}$ . For any  $\alpha \in Sp_{2g}(\mathbb{R})$ , set  $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$  for  $1 \leq i \leq d$ . We define a parametrization of the subgroup  $P^{d,\alpha} = \alpha^{-1}(P^d)$  according to

$$P_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \dots + x_d X_d^\alpha), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

By central extension of  $\mathbb{R}^{2g}$  by  $\mathbb{R}$ , we have an exact sequence

$$0 \rightarrow Z(H^g) \rightarrow H^g \rightarrow \mathbb{R}^{2g} \rightarrow 0.$$

The natural projection  $pr : M \rightarrow H^g/(\Gamma Z(H^g))$  maps  $M$  onto a  $2g$ -dimensional torus  $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ .

**1.2.2. Moduli space.** The group of automorphisms of  $H^g$  that are trivial on the center is denoted by  $Aut_0(H^g) = Sp_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$ . Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to  $Sp_{2g}(\mathbb{R})$ . We regard  $Sp_{2g}(\mathbb{R})$  as the *deformation space* of the standard Heisenberg manifold  $M$  and call the quotient  $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$  the *moduli space* of (standard) Heisenberg manifold.

1 *Siegel modular variety* is a double coset space  $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R}) / Sp_{2g}(\mathbb{Z})$  where  
 2  $K_g$  is a maximal compact subgroup  $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$  of  $Sp_{2g}(\mathbb{R})$ . For  $\alpha \in$   
 3  $Sp_{2g}(\mathbb{R})$ , we denote  $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$  by its projection on the moduli space  $\mathfrak{M}_g$   
 4 and write  $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$  the projection of  $\alpha$  to the Siegel modular variety  
 5  $\Sigma_g$ .

6 Double coset  $K_g \backslash Sp_{2g}(\mathbb{R}) / 1_{2g}$  is identified to the Siegel upper half space  $\mathfrak{H}_g :=$   
 7  $\{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$ . *Siegel upper half space* of genus  $g$  is complex manifold  
 8 of symmetric complex  $g \times g$  matrices  $Z = X + iY$  with positive-definite symmetric  
 9 imaginary part  $\Im(Z) = Y$  and arbitrary (symmetric) real part  $X$ . Denote by  
 10  $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$ .

11 1.2.3. *Representation.* We write the Hilbert sum decomposition of

$$(2) \quad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n$$

into closed  $H^g$ -invariant subspaces. Set  $f = \sum_{n \in \mathbb{Z}} f_n \in L^2(M)$  and  $f_n \in H_n$  where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi i n K t)f\}$$

12 for some fixed  $K > 0$ . The center  $Z(H^g)$  has spectrum  $2\pi\mathbb{Z} \setminus \{0\}$  and the space  
 13  $L^2(M)$  splits as Hilbert sum of  $H^g$ -module  $H_n$ , which is equivalent to irreducible  
 14 representation  $\pi$ .

By Stone-Von Neumann theorem, the unitary irreducible representation  $\pi$  of the  
 Heisenberg group of non-zero central parameter  $K > 0$  is unitarily equivalent to the  
 Schrödinger representation  $\pi$ . By differentiating the Schrödinger representation, we  
 obtain a representation of the Lie algebra  $\mathfrak{h}^g$  on Schwartz space  $\mathcal{S}(\mathbb{R}^g) \subset L^2(\mathbb{R}^g)$   
 (as a  $C^\infty$ -vector). This is called *infinitesimal derived representation*  $d\pi_*$  with pa-  
 rameter  $n \in \mathbb{Z}$ , and for each  $k = 1, 2, \dots, g$ ,

$$d\pi_*(X_k) = \frac{\partial}{\partial x_k}, \quad d\pi_*(Y_k) = 2\pi i n K x_k, \quad d\pi_*(Z) = 2\pi i n K$$

15 acts on  $L^2(\mathbb{R}^g) \simeq L^2(H_n)$ .

16 Given a basis  $(V_i)$  of the Lie algebra, we set a Laplacian  $\Delta = -\sum_i V_i^2$  and define  
 17  $L^2$ -Sobolev norm  $\|f\|_s^2 = \langle f, (1 + \Delta)^s f \rangle$  where  $\langle \cdot, \cdot \rangle$  is an ordinary inner product.  
 18 For  $s > 0$ , the *Sobolev space*  $W^s(M)$  is defined by a completion of  $C^\infty(M)$  equipped  
 19 with the norm  $\|\cdot\|_s$ . The Sobolev space  $W^s(M) = \bigoplus_{n \in \mathbb{Z}} W^s(H_n)$  decomposes to  
 20 closed  $H^g$ -invariant subspaces  $W^s(H_n) = W^s(M) \cap H_n$ .

21 1.2.4. *Renormalization flow.* Denote diagonal matrix  $\delta_i = \text{diag}(d_1, \dots, d_g)$  with  
 22  $d_i = 1$ ,  $d_k = 0$  if  $k \neq i$ . Then, for each  $1 \leq i \leq g$ , we denote  $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in$   
 23  $\mathfrak{sp}_{2g} = \text{Lie}(Sp_{2g}(\mathbb{R}))$ .

24 Any such  $\hat{\delta}_i$  generate a one-parameter subgroup of automorphism (renormaliza-  
 25 tion flow)  $r_i^t := e^{t\hat{\delta}_i}$ . We denote (rank  $d$ ) renormalization actions  $r_t := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$   
 26 for  $t = (t_1, \dots, t_d)$  and  $1 \leq i_1, \dots, i_d \leq g$ . We also write the corresponding auto-  
 27 morphism

$$(3) \quad \exp(t\hat{\delta}(d)) : (x, y, z) \mapsto (e^{t\hat{\delta}}x, e^{-t\hat{\delta}}y, z)$$

28 of the Heisenberg group (see §2.2.3 for its use).

1 **Main results.** One of the main objects in this paper is to construct finitely-  
 2 additive measures defined on the space of rectangles on Heisenberg manifold  $M$ .  
 3 We state our results beginning with an overview of Bufetov functional.

4 **Definition 1.1.** For  $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$ , denote the *standard rectangle* for action  $\mathbf{P}$ ,

$$(4) \quad \Gamma_{\mathbf{T}}^X(m) = \{P_{\mathbf{t}}^{d,\alpha}(m) \mid \mathbf{t} \in U(\mathbf{T}) = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]\}.$$

5 Let  $\mathbf{Q}_y^{d,Y} := \exp(y_1 Y_1 + \cdots + y_d Y_d)$ ,  $y = (y_1, \dots, y_d) \in \mathbb{R}^d$  be an action generated  
 6 by elements  $Y_i$  of standard basis. Let  $\phi_z^Z := \exp(zZ)$ ,  $z \in \mathbb{R}$  be a flow generated  
 7 by the central element  $Z$ .

8 **Definition 1.2.** Let  $\mathfrak{R}$  be the collection of the *generalized rectangles* in  $M$ . For  
 9 any  $1 \leq j \leq d \leq g$  and  $\mathbf{t} = (t_1, \dots, t_d)$ , we set

$$(5) \quad \mathfrak{R} := \bigcup_{1 \leq i \leq d} \bigcup_{(y,z) \in \mathbb{R}^j \times \mathbb{R}} \bigcup_{(m,\mathbf{T}) \in M \times \mathbb{R}_+^d} \{(\phi_{t_{iz}}^Z) \circ \mathbf{Q}_y^{j,Y} \circ P_{\mathbf{t}}^{d,\alpha}(m) \mid \mathbf{t} \in U(\mathbf{T})\}.$$

10 **Theorem 1.3.** For any irreducible representation  $H$ , there exists a finitely-additive  
 11 measure  $\hat{\beta}_H(\Gamma) \in \mathbb{C}$  defined on every standard  $d$ -rectangle  $\Gamma$  such that the following  
 12 holds:

(1) (Additive property) For any decomposition of disjoint rectangles  $\Gamma = \bigcup_{i=1}^n \Gamma_i$   
 or whose intersections have zero measure,

$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^n \hat{\beta}_H(\alpha, \Gamma_i).$$

(2) (Scaling property) For  $\mathbf{t} \in \mathbb{R}^d$ ,

$$\hat{\beta}_H(r_{\mathbf{t}}(\alpha), \Gamma) = e^{-(t_1 + \cdots + t_d)/2} \hat{\beta}_H(\alpha, \Gamma).$$

(3) (Invariance property) For any action  $\mathbf{Q}_{\tau}^{j,Y}$  generated by  $Y_i$ 's for  $\tau \in \mathbb{R}_+^j$   
 and  $1 \leq j \leq d$ ,

$$\hat{\beta}_H(\alpha, (\mathbf{Q}_{\tau}^{j,Y})_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

(4) (Bounded property) For any rectangle  $\Gamma \in \mathfrak{R}$ , there exists a constant  $C(\Gamma) > 0$   
 such that for  $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$ ,

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma) \left( \int_{\Gamma} |\hat{X}| \right)^{d/2}.$$

13 **Corollary 1.4.** The functional  $\hat{\beta}_H$  defined on standard rectangle  $\Gamma_{\mathbf{T}}^X$  extends to the  
 14 class  $\mathfrak{R}$ .

15 **Remark.** By the additive property of functionals on rectangles, it is plausible to  
 16 extend the domain of functionals to general boundary by approximations. However,  
 17 this method may cause a limitation by having a weaker estimate of equidistribution.  
 18 It may be also interesting to compare the methods of Shah [S09a, S09b] and to obtain  
 19 equidistribution results for smooth boundary in our settings.

20 For arbitrary intersecting two  $d$ -rectangles  $U_{\mathbf{T}_1}$  and  $U_{\mathbf{T}_2}$ , consider their  $d$  inter-  
 21 secting points on the boundary, denoted by  $v = (v^{(1)}, v^{(2)}, \dots, v^{(d)})$ . For each  $1 \leq$   
 22  $l \leq d$ , set  $\mathbf{T}_{\min}^{(l)} := \min\{\mathbf{T}_1^{(l)}, \mathbf{T}_2^{(l)}\}$  and consider a rectangle  $U(\mathbf{T}_{\min})$  with its sides  
 23  $\mathbf{T}_{\min}^{(l)}$ . Denote by  $\mathbf{P}(\mathbf{T}_1, \mathbf{T}_2)$  a collection of  $2^d$  vertices  $v = (v^{(1)}, v^{(2)}, \dots, v^{(d)}) \in \mathbb{R}_+^d$   
 24 of  $U(\mathbf{T}_{\min})$ .

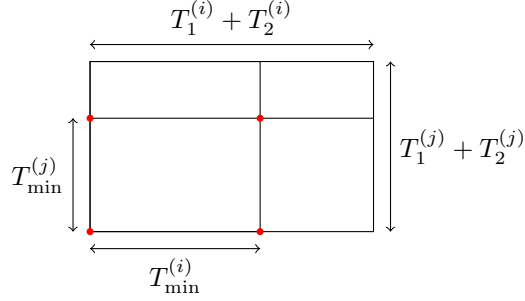


FIGURE 1. Illustration of the rectangles  $U_{\mathbf{T}_1}$  and  $U_{\mathbf{T}_2}$  with  $i, j$ -th coordinate and intersection points  $v$ .

We define a vector  $\mathbf{T}_v := (\mathbf{T}_v^{(i)})_i$  associated with  $v$  given by

$$\mathbf{T}_v^{(i)} := \begin{cases} (T_1^{(j)} + T_2^{(j)}) - T_{\min}^{(i)} & \text{if } v^{(l)} = T_{\min}^{(i)} \\ T_{\min}^{(i)} & \text{if } v^{(l)} = 0. \end{cases}$$

**Corollary 1.5.** *Let us denote*

$$(6) \quad \beta_H(\alpha, m, \mathbf{T}) := \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(m)).$$

*The function  $\beta_H$  satisfies the following properties:*

(1) (Cocycle property) *For all  $(m, \mathbf{T}_1, \mathbf{T}_2) \in M \times \mathbb{R}_+^d \times \mathbb{R}_+^d$ ,*

$$\beta_H(\alpha, m, \mathbf{T}_1 + \mathbf{T}_2) = \sum_{v \in \mathbf{P}(\mathbf{T}_1, \mathbf{T}_2)} \beta_H(\alpha, \mathbf{P}_v^{d, \alpha}(m), \mathbf{T}_v).$$

(2) (Scaling property) *For all  $m \in M$  and  $\mathbf{t} = (t_1, \dots, t_d)$ ,*

$$\beta_H(r_{\mathbf{t}}(\alpha), m, \mathbf{T}) = e^{-(t_1 + \dots + t_d)/2} \beta_H(\alpha, m, \mathbf{T}).$$

(3) (Bounded property) *Let us denote the largest length of side  $T_{\max} = \max_{1 \leq i \leq d} T^{(i)}$ .*

*Then there exists a constant  $C_H > 0$  such that*

$$\beta_H(\alpha, m, \mathbf{T}) \leq C_H T_{\max}^{d/2}.$$

(4) (Orthogonality) *For all  $\mathbf{T} \in \mathbb{R}^d$ ,  $\beta_H(\alpha, \cdot, \mathbf{T})$  belongs to the irreducible component, i.e.,*

$$\beta_H(\alpha, \cdot, \mathbf{T}) \in H \subset L^2(M).$$

By representation theory introduced in §1.2.3, for any  $f = \sum_H f_H \in W^s(M)$ ,  
define a *Bufetov cocycle* associated to  $f$  as a sum

$$(7) \quad \beta^f(\alpha, m, \mathbf{T}) = \sum_H D_{\alpha}^H(f) \beta_H(\alpha, m, \mathbf{T}).$$

*Notation.* Given a Jordan region  $U$  and a point  $m \in M$ , set  $\mathcal{P}_U^{d, \alpha} m$  the *Birkhoff integrals* (currents) associated to the action  $\mathbf{P}_x^{d, \alpha}$  given by

$$\langle \mathcal{P}_U^{d, \alpha} m, \omega_f \rangle := \int_U f(\mathbf{P}_x^{d, \alpha} m) dx_1 \cdots dx_d$$

1 for any degree  $d$   $\mathfrak{p}$ -form  $\omega_f = f \hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$  with a smooth function  $f \in$   
 2  $C_0^\infty(M)$  with zero averages. We denote by 1-form  $\hat{X}_i^\alpha$  satisfying  $\hat{X}_i^\alpha(X_i^\alpha) = 1$   
 3 and  $\hat{X}_i^\alpha(X_j^\alpha) = 0$  if  $i \neq j$ .

4 **Theorem 1.6.** *For all  $s > d(d+11)/4 + g + 1$ , there exists a constant  $C_s > 0$  such*  
 5 *that for almost all frequency  $\alpha$  and for all  $f \in W^s(M)$  and for all  $(m, \mathbf{T}) \in M \times \mathbb{R}^d$ ,*  
 6 *we have*

$$(8) \quad |\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} m, \omega_f \rangle - \beta^f(\alpha, m, \mathbf{T})| \leq C_s \|\omega_f\|_s.$$

7 for  $U(\mathbf{T}) = [0, T^{(1)}] \times \cdots \times [0, T^{(d)}]$  and  $\omega_f = f \omega^{d,\alpha} \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ .

Let the family of random variable

$$E_{\mathbf{T}_n}(f) := \frac{1}{\text{vol}(U(\mathbf{T}_n))^{1/2}} \langle \mathcal{P}_{U(\mathbf{T}_n)}^{d,\alpha}(m), \omega_f \rangle$$

8 where  $U(\mathbf{T}_n)$  is a sequence of rectangles. The point  $m \in M$  is distributed ac-  
 9 cordingly to the probability measure  $\text{vol}$  on  $M$ . Our goal is to understand the  
 10 asymptotic behavior of the probability distributions of  $E_{\mathbf{T}_n}(f)$  as  $U(\mathbf{T}_n) \nearrow \mathbb{R}^d$  in  
 11 a sense of Følner.

**Theorem 1.7.** *Let  $\{\mathbf{T}_n\}$  be any sequence such that*

$$\lim_{n \rightarrow \infty} r_{\log \mathbf{T}_n}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

12 *For every closed form  $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$  with  $s > d(d+11)/4 + g + 1$ , which is not*  
 13 *a coboundary, the limit distribution of the family of random variables  $E_{\mathbf{T}_n}(f)$  exists*  
 14 *along a subsequence of  $\{\mathbf{T}_n\}$ . In particular, for almost all  $\alpha$ , the limit distribution*  
 15 *of  $E_{\mathbf{T}_n}(f)$  has compact support.*

16 We finish the section by giving some remarks on the new adaptation of transfer  
 17 operator techniques from hyperbolic dynamics. The method stems from the analysis  
 18 of the transfer operator, firstly treated by P. Gieulietti and Liverani [GL19]. They  
 19 set up a non-linear flows on the torus and proved asymptotics of ergodic averages  
 20 with expansions of invariant distributions and eigenvalues of transfer operators  
 21 called *Ruelle resonances* (see also [AB18, FGL19, For20b, B19] for applications in  
 22 parabolic flows).

23 A recent work of L. Simonelli and O. Butterley [BS20] reproved some of the re-  
 24 sults of Flaminio and Forni [FF06] by analytical methods of hyperbolic theory, not  
 25 relying on representation theory. Although their work is restricted to the periodic  
 26 type of  $\alpha$  in the moduli space  $\mathfrak{M}$  (the flow is only renormalized by partially hyper-  
 27 bolic diffeomorphism, or  $\alpha$  is periodic) their methods showed indirect similarities  
 28 with [FK20b] and our current work.<sup>1</sup> However, it is still not known if it is possible to  
 29 extend their approach to full measure set of  $\alpha$ , requiring construction of renormal-  
 30 ization cocycle, so called *Transfer cocycle*. It is also not well-studied how to replace  
 31 the use of Anisotropic norm, not relying on the result of Faure-Tsujii [FT15].<sup>2</sup>

32 **Open problem.** Our work leaves open questions.

<sup>1</sup>For instance, it is plausible to view their construction of functionals obtained by spectral projection as a functional  $\hat{\beta}_H$  on each sub-representation  $H$  in our setting (see [BS20, §4]).

<sup>2</sup>Furthermore, it is unknown how to set Diophantine conditions for the recurrence of frequency  $\alpha$ . Instead of relying on the methods in Sobolev constant techniques introduced in [FF06], we do not have proper tools for Anisotropic spaces.

1 *Problem 1.* Can we prove measure estimation of  $\beta^f$  in §7 for the full measure  
2 set of  $\alpha$ ?

3 *Problem 2.* Can the Bufetov functionals defined on the set of rectangles be  
4 extended to the general (Lipschitz) domain?

5 *Problem 3.* Can we prove the deviation of ergodic averages for higher rank  
6 abelian actions by adopting Gieulietti-Liverani approach?

7 On higher step nilmanifolds, there does not exist a renormalization flow (mod-  
8 uli space  $\mathfrak{M}$  is trivial). It is not possible to construct the Bufetov functionals  
9 by the same strategies introduced in here, but other methods in handling non-  
10 renormalizable flows are possibly applied (cf. [FF14, FFT16, Kim21]).

11 *Outline of the paper.* In section 2, we give basic definitions on Higher rank  
12 actions, moduli spaces and Sobolev spaces. In section 3, we state main theorem  
13 and prove constructions of Bufetov functionals with main properties. In section 4,  
14 we prove asymptotic formula of Birkhoff integrals and its limit theorems. In section  
15 5, we prove  $L^2$ -lower bound of the Bufetov functionals. In Section 6, analyticity of  
16 functional and extensions of domain are provided. In section 7, there exist measure  
17 estimates of functionals on the sets where values of functionals are small. This  
18 result only holds when frame  $\alpha$  is of bounded type.

## 19 2. ANALYSIS ON HEISENBERG MANIFOLDS

20 In this section, we review definitions of Sobolev space, currents, representation  
21 and renormalization flows on moduli space.

### 22 2.1. Sobolev space.

2.1.1. *Sobolev norm.* Given a basis  $(V_i)$  of the Lie algebra  $\mathfrak{h}^g$ , we write a new basis  
23  $((\alpha^{-1})_* V_i)$  for  $\alpha \in \text{Aut}_0(\mathbf{H}^g)$ . Similarly, denote by Laplacian  $\Delta_\alpha = -\sum_i (\alpha^{-1})_* V_i^2$   
24 with respect to the new basis. For any  $s \in \mathbb{R}$  and any  $C^\infty$ -function  $f \in L^2(M)$ ,  
25 *Sobolev norm* is defined by

$$\|f\|_{\alpha,s} = \langle f, (1 + \Delta_\alpha)^s f \rangle^{1/2}.$$

23 Let  $W_\alpha^s(M)$  be a completion of  $C^\infty(M)$  with the norm above. The dual space of  
24  $W_\alpha^s(M)$  is denoted by  $W_\alpha^{-s}(M)$  and it is isomorphic to  $W_\alpha^s(M)$ . Extending it to the  
25 exterior algebra, define the Sobolev spaces of the form  $\Lambda^d \mathfrak{p} \otimes W_\alpha^s(M)$  of cochains  
26 of degree  $d$  and we use the same notations for the norm.

2.1.2. *Sobolev bundle.* The group  $Sp_{2g}(\mathbb{Z})$  acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \rightarrow Sp_{2g}(\mathbb{R})$$

where

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma = (\alpha\gamma, \gamma^* \varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \rightarrow \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

27 and the fiber over  $[\alpha] \in \mathfrak{M}_g$  is locally identified with the space  $W_\alpha^s(M)$ .

By invariance of  $Sp_{2g}(\mathbb{Z})$  action, the class of  $(\alpha, \varphi)$  is denoted by  $[\alpha, \varphi]$  and  
 $Sp_{2g}(\mathbb{Z})$ -invariant Sobolev norm is written by

$$\|f\|_{\alpha,s} := \|[\alpha, f]\|_s.$$



We denote the bundle of  $\mathfrak{p}$ -forms of degree  $j$  of Sobolev order  $s$  by  $A^j(\mathfrak{p}, \mathfrak{M}^s)$ . The space of continuous linear functional on  $A^j(\mathfrak{p}, \mathfrak{M}^s)$  will be called the *space of currents* of dimension  $j$  and denoted by  $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$ . There is a flat bundle of (currents) distribution  $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$  whose fiber over  $[\alpha]$  is locally identified with the space  $W_{\alpha}^{-s}(M)$ . Likewise, the space of Sobolev currents  $A_j(\mathfrak{p}, W^{-s}(M))$  of dimension  $j$  with order  $s$  is identified with  $\Lambda^j \mathfrak{p} \otimes W^{-s}(M)$ . We write the norm for form  $\omega$  and currents  $\mathcal{D}$  by

$$\|\omega\|_{\alpha,s} := \|[\alpha, \omega]\|_s, \quad \|\mathcal{D}\|_{\alpha,s} := \|[\alpha, \mathcal{D}]\|_s.$$

2.1.3. *Best Sobolev constant.* The *Sobolev embedding theorem* implies that for any  $\alpha \in Sp_{2g}(\mathbb{R})$  and  $s > g + 1/2$ , there exists a constant  $B_s(\alpha)$  such that for any  $f \in W_{\alpha}^s(M)$ ,

$$(9) \quad B_s(\alpha) := \sup_{f \in W_{\alpha}^s(M) \setminus \{0\}} \frac{\|f\|_{\infty}}{\|f\|_{\alpha,s}}.$$

By Lemma 4.4 in [CF15], the *best Sobolev constant*  $B_s$  is a  $Sp_{2g}(\mathbb{Z})$ -modular function on  $\mathfrak{H}_g$ . Thus, we shall write  $B_s([\alpha])$  or  $B_s([\alpha])$  for  $B_s(\alpha)$ . By the Sobolev embedding theorem, we have a bound for the Birkhoff integral current.

**Lemma 2.1.** [CF15, Lemma 5.5] *For any Jordan region  $U \subset \mathbb{R}^d$  with Lebesgue measure  $|U|$ , for any  $s > g + 1/2$  and all  $m \in M$ ,*

$$\left\| [\alpha, \mathcal{D}_U^{d,\alpha} m] \right\|_{-s} \leq B_s([\alpha])|U|.$$

## 2.2. Invariant currents.

2.2.1. *Identification.* The *boundary operators*

$$\partial : A_j(\mathfrak{p}, W^{-s}(M)) \rightarrow A_{j-1}(\mathfrak{p}, W^{-s}(M))$$

are adjoint of the differentials  $d$  such that  $\langle \partial \mathcal{D}, \omega \rangle = \langle \mathcal{D}, d\omega \rangle$  for any  $\omega \in \Lambda^{j-1} \mathfrak{p} \otimes W^s(M)$ . A current  $\mathcal{D}$  is called *closed* if  $\partial \mathcal{D} = 0$ .

For  $s > 0$ , we denote  $Z_d(\mathfrak{p}, W^{-s}(M))$  by the space of closed currents of dimension  $d$  and Sobolev order  $s$ .  $I_d(\mathfrak{p}, W^{-s}(M))$  is denoted by the  $d$ -dimensional space of  $\mathbb{P}$ -invariant currents of Sobolev order  $s$ . Now we find the relations between these two currents.

**Proposition 2.2.** [CF15, Proposition 3.13] *For any  $s > d/2 = \dim \mathbb{P}/2$ , we have  $I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$  for all  $\epsilon > 0$ . Additionally,*

- $I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g))$  is one dimensional space if  $\dim \mathbb{P} = g$ ,
- $I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g))$  is an infinite-dimensional space if  $\dim \mathbb{P} < g$ ,
- $I_d(\mathfrak{p}, \mathcal{S}(\mathbb{R}^g)) = Z_d(\mathfrak{p}, W^{-s}(M))$  for any  $1 \leq d \leq g$ .

2.2.2. *Basic currents.* The current  $B_{\alpha}$  is called *basic* if for all  $j \in \{i_1, \dots, i_d\}$ ,

$$\iota_{X_j} B_{\alpha} = L_{X_j} B_{\alpha} = 0.$$

For an irreducible representation  $H$ , there exists a unique *basic current*  $B_{\alpha}^H$  (of degree  $2g + 1 - d$  and dimension  $d$ ) associated to invariant distribution  $D_{\alpha}^H$ . It is defined by  $B_{\alpha}^H = D_{\alpha}^H \eta_X$  and this formula implies that for every  $d$ -form  $\xi$ ,

$$(10) \quad B_{\alpha}^H(\xi) = D_{\alpha}^H \left( \frac{\eta_X \wedge \xi}{\omega_{vol}} \right)$$

1 where  $\eta_X := \iota_{X_{i_1}} \cdots \iota_{X_{i_d}} \omega_{vol}$  and  $\omega_{vol}$  is an invariant volume form (with total unit  
2 volume).

3 The basic current  $B_\alpha^H$  belongs to the Sobolev space of currents and it is P-  
4 invariant. It follows that for all  $s > d/2$ , by Sobolev embedding theorem,  $B_\alpha^H \in$   
5  $A_j(\mathfrak{p}, W_\alpha^{-s}(M))$  if and only if  $D_\alpha^H \in W_\alpha^{-s}(M)$  for all  $s > d/2$ .

**Remark.** The formula (10) yields a duality between the space of basic (closed)  
currents and invariant distributions (see also [For02, §6.1] and [BF14, §2.1]). For  
any  $d$ -dimensional  $\mathfrak{p}$ -form  $\omega_f = f \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$  with  $f \in C^\infty(M)$ , we will identify  
currents  $\mathcal{D}$  with distribution  $D$  by writing

$$\langle D, f \rangle = \langle \mathcal{D}, \omega_f \rangle.$$

2.2.3. *Renormalization.* Let  $s > d/2$ . Recall the definition of renormalization flow  
(3) in §1.2.4. For  $\omega \in \Lambda^d \mathfrak{p} \otimes W^s(\mathbb{R}^g)$ ,  $\mathcal{D} \in Z_d(\mathfrak{p}, W^{-s}(M))$ , and  $t \in \mathbb{R}$

$$r_i^t[\alpha, \omega] = [r_i^t \alpha, \omega], \quad r_i^t[\alpha, \mathcal{D}] = [r_i^t \alpha, \mathcal{D}].$$

6 By Proposition 5.2 in [CF15], the sub-bundle  $Z_d(\mathfrak{p}, W^{-s}(M))$  is invariant under  
7 the renormalization flows  $r_i^t$ . Furthermore, we have

$$\|r_1^{t_1} \cdots r_d^{t_d} [\alpha, \mathcal{D}]\|_{-s} = e^{-(t_1 + \cdots + t_d)/2} \|[\alpha, \mathcal{D}]\|_{-s}.$$

8 Let  $U_{\mathbf{t}} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  be unitary operator for  $\mathbf{t} = (t_1, \dots, t_d)$ ,

$$(11) \quad U_{\mathbf{t}} f(x) = e^{-(t_1 + \cdots + t_d)/2} f(e^{t_1} x_1, \dots, e^{t_d} x_d).$$

Then, for any invariant distribution  $D_\alpha^H$ ,

$$D_{r_{\mathbf{t}}(\alpha)}^H = e^{(t_1 + \cdots + t_d)/2} D_\alpha^H.$$

9 By reparametrization of  $\mathbb{R}^d$ -action  $r_{\mathbf{t}} = r_1^{t_1} \cdots r_d^{t_d}$ ,

$$(12) \quad P_x^{d, (r_1^{t_1} \cdots r_d^{t_d} \alpha)} = P_{(e^{-t_1} x_1, \dots, e^{-t_d} x_d)}^{d, \alpha}.$$

10 Then the Birkhoff integral current also satisfies the following identity

$$(13) \quad \mathcal{P}_U^{d, (r_1^{t_1} \cdots r_d^{t_d} \alpha)} m = e^{(t_1 + \cdots + t_d)/2} \mathcal{P}_{(e^{-t_1}, \dots, e^{-t_d})U}^{d, \alpha} m.$$

## 11 2.3. Diophantine condition.

### 12 2.3.1. Height function.

**Definition 2.3** (Height function). The *height* of a point  $Z \in \mathfrak{H}_g$  in Siegel upper  
half space is the positive number

$$hgt(Z) := \det \Im(Z).$$

The *maximal height function*  $Hgt : \Sigma_g \rightarrow \mathbb{R}^+$  is the maximal height of a  $Sp_{2g}(\mathbb{Z})$   
orbit of  $Z$ . That is, for the class of  $[Z] \in \Sigma_g$ ,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

13 By Proposition 4.8 of [CF15], there exists a universal constant  $C(s) > 0$  such  
14 that the best Sobolev constant satisfies the estimate

$$(14) \quad B_s([\alpha]) \leq C(s) \cdot (Hgt([\alpha]))^{1/4}.$$

15 We rephrase Lemma 4.9 in [CF15] regarding the bound of renormalized height.

1 **Lemma 2.4.** For any  $[\alpha] \in \mathfrak{M}_g$  and any  $\mathbf{t} \in \mathbb{R}_+^d$ ,

$$(15) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))\alpha]) \leq (\det(e^{\mathbf{t}\hat{\delta}}))^2 Hgt([\alpha]).$$

2 2.3.2. *Diophantine condition.*

3 **Definition 2.5.** An automorphism  $\alpha \in Sp_{2g}(\mathbb{R})$  or a point  $[\alpha] \in \mathfrak{M}_g$  is  $\hat{\delta}(d)$ -  
4 *Diophantine* of type  $\sigma$  if there exists a  $\sigma > 0$  and a constant  $C > 0$  such that

$$(16) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))\alpha]) \leq CHgt([\exp(-\mathbf{t}\hat{\delta}(d))])^{(1-\sigma)} Hgt([\alpha]), \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

5 This states that  $\alpha \in Sp_{2g}(\mathbb{R})$  satisfies a  $\hat{\delta}(d)$ -Diophantine if the height of the projec-  
6 tion of  $\exp(-\mathbf{t}\hat{\delta}(d))\alpha$  in the Siegel modular variety  $\Sigma_g$  is bounded by  $e^{2(t_1+\dots+t_d)(1-\sigma)}$ .  
7 Furthermore,

8 -  $[\alpha] \in \mathfrak{M}_g$  satisfies a  $\hat{\delta}(d)$ -*Roth condition* if for any  $\epsilon > 0$  there exists a constant  
9  $C > 0$  such that

$$(17) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))\alpha]) \leq CHgt([\exp(-\mathbf{t}\hat{\delta}(d))])^\epsilon Hgt([\alpha]), \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

10 That is,  $\hat{\delta}(d)$ -Diophantine of type  $0 < \sigma < 1$ .

11 -  $[\alpha]$  is of *bounded type* if there exists a constant  $C > 0$  such that

$$(18) \quad Hgt([\exp(-\mathbf{t}\hat{\delta}(d))]) \leq C, \quad \forall \mathbf{t} \in \mathbb{R}_+^d.$$

**Definition 2.6.** Let  $X = G/\Lambda$  be a homogeneous space equipped with the proba-  
bility Haar measure  $\mu$ . A function  $\phi : X \rightarrow \mathbb{R}$  is said  $k$ -DL (distance like) for some  
exponent  $k > 0$  if it is uniformly continuous and if there exist constants  $C_1, C_2 > 0$   
such that

$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

12 By the work of Kleinblock and Margulis, a multi-parameter generalization of  
13 Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

**Theorem 2.7.** [KM99, Theorem 1.9] Let  $G$  be a connected semisimple Lie group  
without compact factors,  $\mu$  its normalized Haar measure,  $\Lambda \subset G$  an irreducible  
lattice,  $\mathfrak{a}$  a Cartan subalgebra of the Lie algebra of  $G$ . Let  $\mathfrak{d}_+$  be a non-empty open  
cone in a  $d$ -dimensional subalgebra  $\mathfrak{d}$  of  $\mathfrak{a}$  ( $1 \leq d \leq \text{rank}_{\mathbb{R}}(G)$ ). If  $\phi : G/\Lambda \rightarrow \mathbb{R}$  is  
a  $k$ -DL function for some  $k > 0$ , then for  $\mu$ -almost all  $x \in G/\Lambda$  one has

$$\limsup_{\mathbf{z} \in \mathfrak{d}_+, \mathbf{z} \rightarrow \infty} \frac{\phi(\exp(\mathbf{z})x)}{\log \|\mathbf{z}\|} = \frac{d}{k}$$

14 By Lemma 4.7 of [CF15], the logarithm of Height function is DL-function with ex-  
15 ponent  $k = \frac{g+1}{2}$  on the Siegel variety  $\Sigma_g$  (and induces on  $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ ).  
16 Hence, we obtain the following proposition.

17 **Proposition 2.8.** Under the assumption  $X = \mathfrak{M}_g$  of Theorem 2.7, for  $s > g+1/2$ ,  
18 there exists a full measure set  $\Omega_g(\hat{\delta})$  and for all  $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$

$$(19) \quad \limsup_{\mathbf{t} \in \mathbb{R}_+^d, \mathbf{t} \rightarrow \infty} \frac{\log Hgt([\exp(-\mathbf{t}\hat{\delta}(d))\alpha])}{\log \|\mathbf{t}\|} \leq \frac{2d}{g+1}.$$

19 Any such  $[\alpha]$  satisfies a  $\hat{\delta}(d)$ -Roth condition (17).

For any  $L > 0$  and  $1 \leq d \leq g$ , let  $DC(L)$  denote the set of  $[\alpha] \in \mathfrak{M}_g$  such that

$$(20) \quad \int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots + t_d)/2} \text{Hgt}([r_{-\mathbf{t}}(\alpha)])^{1/4} dt_1 \cdots dt_d \leq L.$$

Let  $DC$  denote the union of the sets  $DC(L)$  over all  $L > 0$ . It follows immediately that the set  $DC \subset \mathfrak{M}_g$  has full Haar volume.

**Remark.** In the work of Cosentino-Flaminio [CF15], Diophantine condition is restricted to one-parameter subgroup  $\exp(-t\hat{\delta})$  for non-negative ( $g$ -dimensional) diagonal directions. This is based on the easy part of theorem in Kleinblock and Margulis [KM99, Theorem 1.7]. Our condition is for  $d$ -dimensional renormalizations actions. We will write the notation  $r_{-\mathbf{t}}$  instead of  $\exp(-t\hat{\delta}(d))$  for later sections.

### 3. CONSTRUCTIONS OF THE FUNCTIONALS

In this section, we construct Bufetov functionals for higher rank actions on the standard rectangular domain. Firstly, we prove the bound of Birkhoff integrals on rectangles. It consists of similar proofs of Theorem 5.10 in [CF15], but this type of estimate is used to control remainder estimate (Lemma 3.4) to define functionals under Diophantine condition (Lemma 3.5). Furthermore, we will verify the properties of functionals  $\hat{\beta}_H$  and cocycles  $\beta_H$  stated in Theorem 1.3.

**3.1. Remainder estimates.** For any exponent  $s > d/2$ , Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where  $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^\perp$ . Denote by  $\mathcal{I}^{-s}$  and  $\mathcal{R}^{-s}$  the corresponding orthogonal projection operator and by  $\mathcal{I}_\alpha^{-s}$  and  $\mathcal{R}_\alpha^{-s}$  the restrictions to the fiber over  $[\alpha] \in \mathfrak{M}$  for  $\alpha \in Sp_{2g}(\mathbb{R})$ . In particular, for the current (Birkhoff integrals)  $\mathcal{D} = \mathcal{P}_U^{d,\alpha} m$ , we call  $\mathcal{I}_\alpha^{-s}(\mathcal{D}) = \mathcal{I}^{-s}[\alpha, \mathcal{D}]$  boundary term and  $\mathcal{R}_\alpha^{-s}(\mathcal{D}) = \mathcal{R}^{-s}[\alpha, \mathcal{D}]$  remainder term respectively. Consider the orthogonal projection

$$(21) \quad \mathcal{D} = \mathcal{I}_{r_{-\mathbf{t}}[\alpha]}^{-s}(\mathcal{D}) + \mathcal{R}_{r_{-\mathbf{t}}[\alpha]}^{-s}(\mathcal{D}).$$

We firstly recall the following estimate of boundary term.

**Lemma 3.1.** [CF15, Lemma 5.7] Let  $s > d/2 + 2$ . There exists a constant  $C = C(s) > 0$  such that for all  $t_i \geq 0$  for  $1 \leq i \leq d$ , we have

$$\begin{aligned} \|\mathcal{I}^{-s}[\alpha, \mathcal{D}]\|_{-s} &\leq e^{-(t_1 + \cdots + t_d)/2} \|\mathcal{I}^{-s}[r_1^{-t_1} \cdots r_d^{-t_d} \alpha, \mathcal{D}]\|_{-s} \\ &\quad + C_1 |t_1 + \cdots + t_d| \int_0^1 e^{-u(t_1 + \cdots + t_d)/2} \|\mathcal{R}^{-s}[r_1^{-ut_1} \cdots r_d^{-ut_d} \alpha, \mathcal{D}]\|_{-(s-2)} du. \end{aligned}$$

By Stokes' theorem, we have the following remainder estimate.

**Lemma 3.2.** [CF15, Lemma 5.6] Let  $s > g + d/2 + 1$ . For any non-negative  $s' < s - (d+1)/2$  and Jordan region  $U \subset \mathbb{R}^d$ , there exists  $C = C(d, g, s, s') > 0$  such that

$$\|\mathcal{R}^{-s}[\alpha, (\mathcal{P}_U^{d,\alpha} m)]\|_{-s} \leq C \|\alpha, \partial(\mathcal{P}_U^{d,\alpha} m)\|_{-s'}.$$

Quantitative bound of Birkhoff integrals on the square domain was obtained in [CF15], but we need to extend the result to the rectangular shapes for analyticity of functionals in the section §6.

From now on, assume that  $s_{d,g} := d(d+11)/4 + g + 1/2$ .

**Theorem 3.3.** *For  $s > s_{d,g}$ , there exists a constant  $C(s, d) > 0$  such that the following holds. For any  $t_i > 0$ ,  $m \in M$  and  $U_d(t) = [0, e^{t_1}] \times \cdots \times [0, e^{t_d}]$ , we have*

(22)

$$\begin{aligned} \left\| [\alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)] \right\|_{-s} &\leq C \sum_{k=0}^d \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^d t_l - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \\ &\quad \times Hgt\left(\left[\prod_{1 \leq j \leq d} r_j^{-t_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}} \alpha\right]\right)^{1/4} du_{i_1} \cdots du_{i_k}. \end{aligned}$$

*Proof.* We proceed by induction. For  $d = 1$ , it follows from the Theorem 5.8 in [CF15]. We assume that the result holds for  $d - 1$ . Decompose the current as a sum of boundary and remainder term as in (21).

*Step 1.* We estimate the boundary term. By Lemma 3.1, renormalize terms with  $r^u = r_1^u \cdots r_d^u$ . Then, we have

$$\begin{aligned} \left\| \mathcal{I}^{-s} [\alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)] \right\|_{-s} &\leq e^{-(t_1 + \cdots + t_d)/2} \left\| \mathcal{I}^{-s} [r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)] \right\|_{-s} \\ (23) \quad &+ C_1(s) \int_0^{t_1 + \cdots + t_d} e^{-ud/2} \left\| \mathcal{R}^{-s} [r^{-u} \alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)] \right\|_{-(s-2)} du \\ &:= (I) + (II). \end{aligned}$$

By renormalization (13) and Lemma 2.1 for unit volume,

$$\begin{aligned} \left\| \mathcal{I}^{-s} [r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)] \right\|_{-s} &= e^{t_1 + \cdots + t_d} \left\| \mathcal{I}^{-s} [r_1^{-t_1} \cdots r_d^{-t_d} \alpha, (\mathcal{P}_{U_d(0)}^{d, r_1^{-t_1} \cdots r_d^{-t_d} \alpha} m)] \right\|_{-s} \\ &\leq C_2 e^{t_1 + \cdots + t_d} Hgt([r_1^{-t_1} \cdots r_d^{-t_d} \alpha])^{1/4}. \end{aligned}$$

Hence

$$I \leq C_2 e^{(t_1 + \cdots + t_d)/2} Hgt([r_1^{-t_1} \cdots r_d^{-t_d} \alpha])^{1/4},$$

where the sum corresponds to the first term ( $k = 0$ ) in the statement.

*Step 2.* To estimate (II),

$$\begin{aligned} \left\| \mathcal{R}^{-s} [r^{-u} \alpha, (\mathcal{P}_{U_d(t)}^{d,\alpha} m)] \right\|_{-(s-2)} &= \left\| e^{ud} \mathcal{R}^{-s} [r^{-u} \alpha, (\mathcal{P}_{U_d(t-u)}^{d, r^{-u} \alpha} m)] \right\|_{-(s-2)} \\ (24) \quad &\leq C_3(s, s') e^{ud} \left\| [r^{-u} \alpha, \partial(\mathcal{P}_{U_d(t-u)}^{d, r^{-u} \alpha} m)] \right\|_{-s'}. \end{aligned}$$

The boundary  $\partial(\mathcal{P}_{U_d}^{d, r^{-t} \alpha})$  is the sum of  $2d$  currents of dimension  $d - 1$ . These currents are Birkhoff sums of  $d$  face subgroups obtained from  $\mathcal{P}_{U_d}^{d, r^{-t} \alpha}$  by omitting one of the base vector fields  $X_i$ . It is reduced to  $(d - 1)$  dimensional shape obtained from  $U_d(t - u) := [0, e^{t_1 - u}] \times \cdots \times [0, e^{t_d - u}]$ . For each  $1 \leq j \leq d$ , there are Birkhoff sums along  $d - 1$  dimensional cubes. By induction hypothesis, we add all the  $d - 1$

dimensional cubes by adding all the terms along  $j$ :

(25)

$$\begin{aligned} & \left\| [r^{-u} \alpha, (\mathcal{P}_{U_{d-1}(t-u)}^{d-1, r^{-u} \alpha} m)] \right\|_{-s'} \leq C_4(s', d-1) \sum_{j=1}^d \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq d \\ i_l \neq j, \forall l}} \int_0^{t_{i_k}-u} \dots \int_0^{t_{i_1}-u} \\ & \exp\left(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^d (t_l - u) - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{\substack{1 \leq l \leq d \\ l \neq j}} r_l^{-(t_l-u)} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r^{-u} \alpha)\right]\right)^{1/4} du_{i_1} \dots du_{i_k}. \end{aligned}$$

1 Combining (23) and (24), we obtain the estimate for (II).

$$\begin{aligned} (II) & \leq C_5(s', d-1) \sum_{j=1}^d \sum_{k=0}^{d-1} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq d \\ i_l \neq j}} \int_0^{t_1+\dots+t_d} \int_0^{t_{i_k}-u} \dots \int_0^{t_{i_1}-u} du_{i_1} \dots du_{i_k} du \\ & \times \exp\left(\frac{1}{2} \sum_{\substack{l=1 \\ l \neq j}}^d t_l - \frac{1}{2} u - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_j^{-u+t_j} \alpha)\right]\right)^{1/4}. \end{aligned}$$

Applying the change of variable  $u_j = t_j - u$ , we obtain

$$\begin{aligned} (II) & \leq C_6(s', d-1) \sum_{j=1}^d \sum_{k=1}^{d-1} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq d \\ i_l \neq j}} \\ & \times \left( \int_{-(t_1+\dots+t_d)+t_j}^{t_j} \int_0^{t_{i_k}-u} \dots \int_0^{t_{i_1}-u} du_{i_1} \dots du_{i_k} du_j \right. \\ & \times \exp\left(\frac{1}{2}(t_1 + \dots + t_d) - \frac{1}{2} u_j - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} (r_j^{-u_j} \alpha)\right]\right)^{1/4} \Big). \end{aligned}$$

Simplifying multi-summation above, (with  $-(t_1 + \dots + t_d) + t_j \leq 0$ )

$$\begin{aligned} (II) & \leq C_7(s', d) \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \int_0^{t_{i_k}} \dots \int_0^{t_{i_1}} du_{i_1} \dots du_{i_k} \\ & \times \exp\left(\frac{1}{2}(t_1 + \dots + t_d) - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) Hgt\left(\left[\prod_{1 \leq l \leq d} r_l^{-t_l} \prod_{l=1}^k r_{i_l}^{u_{i_l}} \alpha\right]\right)^{1/4}. \end{aligned}$$

2 *Step 3. (Remainder estimate).* The remainder term is obtained from Lemma 3.2  
 3 (Stokes' theorem). Following step 2, estimate of remainder reduces to that of  $d-1$   
 4 form. Combining with the step 1, we have the following  
 (26)

$$\left\| \mathcal{R}^{-s}[\alpha, \partial(\mathcal{P}_{U_d}^{d, \alpha} m)] \right\|_{-s} \leq C(s) \sum_{i=1}^{d-1} \left\| \mathcal{I}^{-s}[\alpha, (\mathcal{P}_{U_i}^{i, \alpha} m)] \right\|_{-s} + \left\| \mathcal{R}^{-s}[\alpha, (\mathcal{P}_{U_1}^{1, \alpha} m)] \right\|_{-s}$$

where  $U_i$  is  $i$ -dimensional rectangle. Sum of the boundary terms are absorbed in the bound of (I) + (II). For 1-dimensional remainder with interval  $\Gamma_T$ , the boundary is a 0-dimensional current. Then,

$$\langle \partial(\mathcal{P}_{[0,T]}^{1, \alpha} m), f \rangle = f(P_T^{1, \alpha} m) - f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (9) and (14),

$$\left\| \mathcal{R}^{-s}[\alpha, \partial(\mathcal{P}_{[0,T]}^{1,\alpha} m)] \right\|_{-s} \leq 2B_s([\alpha]) \leq C(s)Hgt([\alpha])^{1/4}.$$

Then, by inequality (15)

$$\begin{aligned} C(s)Hgt([\alpha])^{1/4} &= C(s)Hgt([r_{\mathbf{t}} r_{-\mathbf{t}} \alpha])^{1/4} \\ &\leq C(s)e^{(t_1 + \dots + t_d)/2} Hgt([r_1^{-t_1} \dots r_d^{-t_d} \alpha])^{1/4}. \end{aligned}$$

- 1 This implies that remainder term produces one more term like the bound of (I).  
 2 Therefore, the Theorem follows from combining all the terms (I), (II), and remain-  
 3 der.  $\square$

4 **Remark.** It is also reasonable to decompose the rectangle  $U_d(t)$  as a union of  
 5 several several squares and renormalize their sides. However, it may also involve  
 6 computational difficulty and such approach of summing up squares may provide a  
 7 rough upper bound. To obtain necessary bound for the Lemma 3.4, we rather simply  
 8 generalized the methods of Theorem 5.10 in [CF15].

9 Let us set

$$(27) \quad \mathcal{K}_{\alpha, \mathbf{t}, s}(\Gamma) := \left\| \mathcal{R}^{-s}[r_{-\mathbf{t}}(\alpha), (\mathcal{P}_{U_\Gamma}^{d, r_{-\mathbf{t}}(\alpha)} m)] \right\|_{-(s+1)}$$

- 10 Now we prove the remainder estimate that will be used in (34).

**Lemma 3.4.** Let  $s > s_{d,g}$ . There exists a constant  $C(s, \Gamma) > 0$  such that for any rectangle  $U_\Gamma = [0, e^{\Gamma_1}] \times \dots \times [0, e^{\Gamma_d}]$ ,

$$\mathcal{K}_{\alpha, \mathbf{t}, s}(\Gamma) \leq C(s, \Gamma)Hgt([r_{-\mathbf{t}}(\alpha)])^{1/4}.$$

*Proof.* By Lemma 3.2, we begin to find the bound of  $d-1$  (renormalized) currents. By Theorem 3.3, we obtain the remainder estimate

$$\begin{aligned} \mathcal{K}_{\alpha, \mathbf{t}}(\Gamma) &\leq C \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \int_0^{\Gamma_{i_k}} \dots \int_0^{\Gamma_{i_1}} \exp\left(\frac{1}{2} \sum_{l=1}^{d-1} \Gamma_l - \frac{1}{2} \sum_{l=1}^k u_{i_l}\right) \\ &\quad \times Hgt\left(\left[\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}}(r_{-\mathbf{t}}(\alpha))\right]\right)^{1/4} du_{i_1} \dots du_{i_k}. \end{aligned}$$

It follows from (15) that for  $0 \leq k \leq d-1$ ,

$$Hgt\left(\left[\prod_{1 \leq j \leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^k r_{i_l}^{u_{i_l}}(r_{-\mathbf{t}}(\alpha))\right]\right)^{1/4} \leq e^{\frac{1}{2}(\sum_{l=1}^k u_{i_l} - \sum_{l=1}^{d-1} \Gamma_l)} Hgt([r_{-\mathbf{t}}(\alpha)])^{1/4}.$$

Then, we obtain

$$\begin{aligned} (28) \quad \left\| \mathcal{R}^{-s}[r_{-\mathbf{t}}(\alpha), (\mathcal{P}_{U_\Gamma}^{d, r_{-\mathbf{t}}(\alpha)} m)] \right\|_{-s} &\leq C(s) \left( \sum_{k=0}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \prod_{l=1}^k \Gamma_{i_l} \right) Hgt([r_{-\mathbf{t}}(\alpha)])^{1/4} \\ &\leq C(s, \Gamma)Hgt([r_{-\mathbf{t}}(\alpha)])^{1/4}. \end{aligned}$$

- 11 Therefore, we obtain the conclusion.  $\square$

**3.2. Constructions of the functionals.** For fixed  $\alpha \in \text{Aut}_0(H^g)$ , let  $\Pi_H^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \rightarrow A_d(\mathfrak{p}, W_\alpha^{-s}(H))$  denote the orthogonal projection on a single irreducible unitary representation  $H$ . We further decompose projection operators with basic current  $B_\alpha^{-s,H}$  and its remainder  $R_\alpha^{-s,H}$  given by

$$\Pi_H^{-s} = \mathcal{B}_\alpha^{-s}(\Gamma) B_\alpha^{-s,H} + R_\alpha^{-s,H}$$

1 where  $\mathcal{B}_{H,\alpha}^{-s} : A_d(\mathfrak{p}, W_\alpha^{-s}(M)) \rightarrow \mathbb{C}$  denote the orthogonal component map in the  
 2 direction of basic current, supported on a single irreducible unitary representation.

3 The *Bufetov functionals* on a standard rectangle  $\Gamma$  are defined for all  $[\alpha] \in DC$   
 4 as follows.

**Lemma 3.5.** *Let  $[\alpha] \in DC(L)$ . For  $s > s_{d,g}$ , the limit*

$$\hat{\beta}_H(\alpha, \Gamma) = \lim_{t_1, \dots, t_d \rightarrow \infty} e^{-(t_1 + \dots + t_d)/2} \mathcal{B}_{H, r_t(\alpha)}^{-s}(\Gamma)$$

5 *exists and it defines a finitely-additive measure on the set of standard rectangles.*  
 6 *Moreover, there exists a constant  $C(s, \Gamma) > 0$  such that the following estimate holds:*

$$(29) \quad \left\| \Pi_{H,\alpha}^{-s}(\Gamma) - \hat{\beta}_H(\alpha, \Gamma) B_\alpha^H \right\|_{\alpha, -s} \leq C(s, \Gamma)(1 + L).$$

*Proof.* For simplicity, we omit dependence of  $H$ . For every  $\mathbf{t} \in \mathbb{R}^d$ , we have the following orthogonal splitting:

$$\Pi_{H,\alpha}^{-s}(\Gamma) = \mathcal{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}},$$

where

$$\mathcal{B}_{\alpha,\mathbf{t}}^{-s} := \mathcal{B}_{H, r_{-\mathbf{t}}(\alpha)}^{-s}, \quad B_{\alpha,\mathbf{t}} := B_{r_{-\mathbf{t}}(\alpha)}^{-s,H}, \quad R_{\alpha,\mathbf{t}} := R_{r_{-\mathbf{t}}(\alpha)}^{-s,H}.$$

For any  $\mathbf{h} \in \mathbb{R}^d$ , we have

$$\mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}+\mathbf{h}} + R_{\alpha,\mathbf{t}+\mathbf{h}} = \mathcal{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}}.$$

7 By reparametrization (12), we have  $B_{\mathbf{t}+\mathbf{h}} = e^{-(h_1 + \dots + h_d)/2} B_{\mathbf{t}}$  and

$$(30) \quad \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{(h_1 + \dots + h_d)/2} \mathcal{B}_{\alpha,\mathbf{t}}^{-s}(\Gamma) + \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}})$$

and it follows that

$$\mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(\Gamma) = e^{h_1/2} \mathcal{B}_{\alpha,t_1,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) + \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}}).$$

8 By differentiating at  $h_1 = 0$ ,

$$(31) \quad \frac{d}{dt_1} \mathcal{B}_{\alpha,t_1,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha,t_1,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) + \left[ \frac{d}{dh_1} \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}}) \right]_{h_1=0}.$$

Therefore, we solve the following first order ODE

$$\frac{d}{dt_1} \mathcal{B}_{\alpha,t_1,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha,t_1,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) + \mathcal{K}_{\alpha,\mathbf{t}}^{(1)}(\Gamma)$$

where

$$\mathcal{K}_{\alpha,\mathbf{t},s}^{(1)}(\Gamma) := \left[ \frac{d}{dh_1} \mathcal{B}_{\alpha,\mathbf{t}+\mathbf{h}}^{-s}(R_{\alpha,\mathbf{t}}) \right]_{h_1=0}.$$



Then, the solution of the differential equation is

$$\begin{aligned}\mathcal{B}_{\alpha,t_1,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) &= e^{t_1/2} \left( \mathcal{B}_{\alpha,0,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{-\tau_1/2} \mathcal{K}_{\alpha,\tau,s}^{(1)}(\Gamma) d\tau_1 \right) \\ &= e^{t_1/2} \mathcal{B}_{\alpha,0,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) + \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha,\tau,s}^{(1)}(\Gamma) d\tau_1.\end{aligned}$$

Note by reparametrization

$$e^{t_1/2} \mathcal{B}_{\alpha,0,t_2+h_2,\dots,t_d+h_d}^{-s}(\Gamma) = e^{h_2/2} \mathcal{B}_{\alpha,t_1,t_2,t_3+h_3,\dots,t_d+h_d}^{-s}(\Gamma)$$

- 1 and it is possible to differentiate the previous equation with respect to  $h_2$  again.  
2 Then

$$(32) \quad \frac{d}{dt_2} \mathcal{B}_{\alpha,t_1,t_2,\dots,t_d+h_d}^{-s}(\Gamma) = \frac{1}{2} \mathcal{B}_{\alpha,t_1,t_2,t_3+h_3,\dots,t_d+h_d}^{-s} + \int_0^{t_1} e^{-\tau_1/2} \mathcal{K}_{\alpha,\tau,s}^{(2)}(\Gamma) d\tau_1$$

- 3 where  $\mathcal{K}_{\alpha,\tau,s}^{(2)}(\Gamma) = \frac{d}{dh_2} \mathcal{K}_{\alpha,\tau,s}^{(1)}(\Gamma)$ .

Then, the solution of equation (32) is

$$\begin{aligned}\mathcal{B}_{\alpha,t_1,t_2,\dots,t_d+h_d}^{-s}(\Gamma) &= e^{t_2/2} \left( \mathcal{B}_{\alpha,t_1,0,t_3+h_3,\dots,t_d+h_d}^{-s} + \int_0^{t_2} e^{-\tau_2/2} \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha,\tau,s}^{(2)}(\Gamma) d\tau_1 d\tau_2 \right) \\ &= e^{h_3/2} \mathcal{B}_{\alpha,t_1,t_2,t_3,\dots,t_d+h_d}^{-s} + \int_0^{t_2} e^{(t_2-\tau_2)/2} \int_0^{t_1} e^{(t_1-\tau_1)/2} \mathcal{K}_{\alpha,\tau,s}^{(2)}(\Gamma) d\tau_1 d\tau_2.\end{aligned}$$

- 4 Inductively, we solve the first order ODE repeatedly and obtain the following  
5 solution

$$(33) \quad \mathcal{B}_{\alpha,t}^{-s}(\Gamma) = e^{(t_1+\dots+t_d)/2} \left( \mathcal{B}_{\alpha,0}^{-s} + \int_0^{t_d} \dots \int_0^{t_1} e^{-(\tau_1+\dots+\tau_d)/2} \mathcal{K}_{\alpha,\tau,s}^{(d)}(\Gamma) d\tau_1 \dots d\tau_d \right)$$

where

$$\mathcal{K}_{\alpha,t,s}^{(d)}(\Gamma) = \left[ \frac{d}{dh_d} \dots \frac{d}{dh_1} \mathcal{B}_{\alpha,t+h}^{-s}(R_{\alpha,t}) \right]_{h_d \dots h_1=0}.$$

Let  $\langle \cdot, \cdot \rangle_{\alpha,t}$  denote the inner product in the space of Hilbert current  $A_d(\mathfrak{p}, W_{r-t(\alpha)}^{-s}(H))$ .  
By the intertwining formula (11),

$$\begin{aligned}\mathcal{B}_{\alpha,t+h}^{-s}(R_{\alpha,t}) &= \langle R_{\alpha,t}, \frac{B_{\alpha,t+h}}{|B_{\alpha,t+h}|_{t+h}^2} \rangle_{\alpha,t+h} \\ &= \langle R_{\alpha,t} \circ U_{-h}, \frac{B_{\alpha,t+h} \circ U_{-h}}{|B_{\alpha,t+h}|_{t+h}^2} \rangle_{\alpha,t} \\ &= \langle R_{\alpha,t} \circ U_{-h}, \frac{B_{\alpha,t}}{|B_{\alpha,t}|_t^2} \rangle_{\alpha,t} = \mathcal{B}_{\alpha,t}^{-s}(R_{\alpha,t} \circ U_{-h}).\end{aligned}$$

In the sense of distributions,

$$\begin{aligned}\frac{d}{dh_d} \dots \frac{d}{dh_1} (R_{\alpha,t} \circ U_{-h}) &= -R_{\alpha,t} \circ \left( \frac{d}{2} + \sum_{i=1}^d X_i(t) \right) \circ U_{-h} \\ &= \left[ \left( \sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha,t} \right] \circ U_{-h}.\end{aligned}$$

Then we compute

$$\left[ \frac{d}{dh_d} \cdots \frac{d}{dh_1} (\mathcal{B}_{\alpha, \mathbf{t} + \mathbf{h}}^{-s}(R_{\alpha, \mathbf{t}})) \right]_{\mathbf{h}=0} = -\mathcal{B}_{\alpha, \mathbf{t}}^{-s} \left( \left( \sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha, \mathbf{t}} \right).$$

- 1 Recall by (27) that  $\mathcal{K}_{\alpha, \mathbf{t}, s}(\Gamma) = \|\mathcal{R}_{\alpha, \mathbf{t}}^{-s}\|_{r_{-\mathbf{t}}(\alpha), -(s+1)}$ . By Proposition 2.8 and  
 2 Lemma 3.4, for any  $\epsilon > 0$

$$(34) \quad |\mathcal{B}_{\alpha, \mathbf{t}}^{-s} \left( \left( \sum_{i=1}^d X_i(t) - \frac{d}{2} \right) R_{\alpha, \mathbf{t}} \right)| \leq \mathcal{K}_{\alpha, \mathbf{t}, s}(\Gamma) \leq C(s, \Gamma) Hgt([r_{-\mathbf{t}}(\alpha)])^{1/4}.$$

Therefore, the solution of equation (33) exists under Diophantine condition (20) and the following holds:

$$\lim_{t_1, \dots, t_d \rightarrow \infty} e^{-(t_1 + \dots + t_d)/2} \mathcal{B}_{\alpha, \mathbf{t}}^{-s}(\Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

Moreover, the complex number

$$\hat{\beta}_H(\alpha, \Gamma) = \mathcal{B}_{\alpha, 0}^{-s} + \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \dots + \tau_d)/2} \mathcal{K}_{\alpha, \tau, s}(\Gamma) d\tau_1 \cdots d\tau_d$$

depends continuously on  $\alpha \in DC(L)$ . Since we have

$$\Pi_{H, \alpha}^{-s}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B_\alpha^H = R_0 - \left( \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1 + \dots + \tau_d)/2} \mathcal{K}_{\alpha, \tau, s}(\Gamma) d\tau_1 \cdots d\tau_d \right) B_\alpha^H,$$

by Diophantine condition again,

$$\left\| \Pi_{H, \alpha}^{-s}(\Gamma) - \hat{\beta}_H(\alpha, \Gamma) B_\alpha^H \right\|_{\alpha, -s} \leq C(s, \Gamma)(1 + L).$$

3

□

- 4 **3.3. Proof of Theorem 1.3.** The proof of Theorem 1.3 follows immediately from  
 5 the refinement to the constructions of Bufetov functionals (see also [BF14, §2.5] for  
 6 horocycle flows).

- 7 *Notation.* The action of flow  $\{r_t\}_{t \in \mathbb{R}}$  on a current  $\mathcal{C}$  is defined by pull-back as  
 8 follows:

$$(r_t^* \mathcal{C})(\omega) = \mathcal{C}(r_{-t}^* \omega), \quad \text{for any smooth form } \omega.$$

**Lemma 3.6** (Invariance). *Let  $1 \leq d \leq g$ . The functional  $\hat{\beta}_H$  defined on  $d$ -standard rectangle  $\Gamma = \Gamma_T^X$  is invariant under the action of  $(Q_y^{j, Y})$  for any  $y \in \mathbb{R}_+^j$  and  $1 \leq j \leq d$ . That is,*

$$\hat{\beta}_H(\alpha, (Q_y^{j, Y})_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

- 9 *Proof.* We will prove the functional  $\hat{\beta}_H$  exists with respect to rectangle  $(Q_y^{j, Y})_* \Gamma$   
 10 and prove its invariance under the action  $Q_y^{j, Y}$ . It suffices to verify the invariance  
 11 property under the rank 1 action  $Q_\tau^{1, Y}$  for  $\tau \in \mathbb{R}$  since we can apply the statement  
 12 for  $d$ -rank actions repeatedly.

- 13 Given a standard  $d$ -dimensional rectangle  $\Gamma$ , set  $\Gamma_Q := (Q_\tau^{1, Y})_* \Gamma$ . Let  $D(\Gamma, \Gamma_Q)$   
 14 be the  $(d+1)$  dimensional space spanned by trajectories of the action of  $Q_\tau^{1, Y}$   
 15 projecting  $\Gamma$  onto  $\Gamma_Q$ . Then  $D(\Gamma, \Gamma_Q)$  is a union of all orbits  $I$  of action  $Q_\tau^{1, Y}$  such  
 16 that the boundary of  $I$ ,  $d$ -dimensional faces, is contained in  $\Gamma \cup \Gamma_Q$ , and interior of  $I$   
 17 is disjoint from  $\Gamma \cup \Gamma_Q$ .  $D(\Gamma, \Gamma_Q)$  is defined by integration and it is a  $(d+1)$ -current.

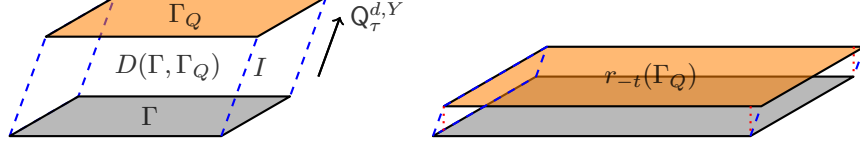


FIGURE 2. Illustration of the standard  $d$ -rectangles  $\Gamma$ ,  $\Gamma_Q$ ,  $d+1$  dimensional current  $D(\Gamma, \Gamma_Q)$  and supports of  $r_{-t}(\Gamma)$  and  $r_{-t}(\Gamma_Q)$ .

For convenience, let us denote renormalization flow by  $r_t := r_i^t$  for some  $i$ -th coordinates. Then,  $r_{-t}(\Gamma)$  and  $r_{-t}(\Gamma_Q)$  are respectively the support of the currents  $r_t^*\Gamma$  and  $r_t^*\Gamma_Q$ . Thus, we have the following identity

$$r_t^*D(\Gamma, \Gamma_Q) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q)).$$

- 1 Since the current  $\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)$  is composed of orbits for the action  $Q_\tau^{1,Y}$ ,  
 2 it follows that

$$(35) \quad \partial[r_t^*D(\Gamma, \Gamma_Q)] - (r_t^*\Gamma - r_t^*\Gamma_Q) = r_t^*[\partial D(\Gamma, \Gamma_Q) - (\Gamma - \Gamma_Q)] \rightarrow 0.$$

Now, we turn to prove the volume of  $D(r_{-t}(\Gamma), r_{-t}(\Gamma_Q))$  is uniformly bounded for all  $t > 0$ . For any  $p \in \Gamma$ , set  $\tau(p)$  be length of the arc lying in  $D_Q := D(\Gamma, \Gamma_Q)$ , and set  $\tau_\Gamma := \sup\{\tau(p) \mid p \in \Gamma\} < \infty$ . We write

$$vol_{d+1}(D_Q) = \int_\Gamma \tau dvol_d.$$

- 3 Since  $vol_d(r_{-t}(\Gamma)) \leq e^t vol_d(\Gamma)$ ,

$$(36) \quad vol_{d+1}(r_{-t}(D_Q)) = \int_{r_{-t}(\Gamma)} \tau dvol_d \leq \tau_\Gamma e^{-t} vol_d(r_{-t}(\Gamma)) \leq \tau_\Gamma vol_d(\Gamma) < \infty.$$

- 4 Note that  $d$ -dimensional current  $(Q_\tau^{1,Y})_*\Gamma - \Gamma$  is equal to the boundary of the  
 5  $(d+1)$  dimensional current  $D_Q$ . By arguments in remainder estimate (or Sobolev  
 6 embedding theorem),

$$(37) \quad \|(Q_\tau^{1,Y})_*\Gamma - \Gamma\|_{r_{-t}(\alpha), -s} \leq C_s \tau B_s([r^{-t}\alpha]) \leq C_s \tau \text{Hgt}([r^{-t}\alpha])^{1/4}$$

- 7 is finite for all  $t > 0$ .

Then, by (35), (36) and existence of Bufetov functional  $\hat{\beta}_H(\alpha, \Gamma)$ , the last inequality holds:

$$\|\mathcal{B}_{\alpha, t}^{-s}((Q_\tau^{1,Y})_*\Gamma) - \mathcal{B}_{\alpha, t}^{-s}(\Gamma)\|_{\alpha, -s} < \infty.$$

- 8 Therefore, by the definition of Bufetov functional in the Lemma 3.5,  $\hat{\beta}_H(\alpha, (Q_\tau^{1,Y})_*\Gamma)$   
 9 exists and  $\hat{\beta}_H(\alpha, \Gamma)$  is invariant under the action of  $Q_\tau^{1,Y}$ .  $\square$

10 *Proof of Theorem 1.3. Additive property.* It follows from the linearity of projections  
 11 and limit.

12 *Scaling property.* It is immediate from the definition.

*Bounded property.* By the scaling property, for  $r_t = r_1^t \cdots r_d^t$  with  $t > 0$ ,

$$\hat{\beta}_H(\alpha, \Gamma) = e^{dt/2} \hat{\beta}_H(r_t(\alpha), \Gamma).$$

Choose  $t = \log(\int_{\Gamma} |\hat{X}|)$  and  $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$ , then Bufetov functional on the rectangle  $\Gamma$  is bounded:

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma) \left( \int_{\Gamma} |\hat{X}| \right)^{d/2}.$$

1 *Invariance property.* Now it follows directly from the Lemma 3.6.  $\square$

2 *Notation.* We write for  $\mathbf{T} = (T^{(i)}) \in \mathbb{R}^d$  and  $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$ ,

$$(38) \quad \text{vol}(U(\mathbf{T})) := \prod_{i=1}^d T^{(i)}, \quad \text{vol}(U(\mathbf{t})) = \prod_{i=1}^d t_i.$$

3 Now we extend the properties of functional  $\hat{\beta}_H$  to cocycle  $\beta_H$ .

4 *Proof of Corollary 1.5.* Cocycle property of  $\beta_H$  follows from the additive property  
5 of  $\hat{\beta}_H$ . Scaling and bounded properties are immediate. Denote  $\mathbf{tT} := (t_1 T^{(1)}, \dots, t_d T^{(d)}) \in$   
6  $\mathbb{R}_+^d$ , and we obtain

$$(39) \quad \beta_H(\alpha, m, \mathbf{tT}) = \text{vol}(U(\mathbf{T}))^{1/2} \beta_H(r_{\log \mathbf{T}}(\alpha), m, \mathbf{t}).$$

By Lemma 3.5 and scaling property, we have

$$\begin{aligned} D_{\alpha}^H(f) \beta_H(\alpha, m, \mathbf{t}) &= \lim_{|\mathbf{U}(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \beta_H(r_{-\log \mathbf{t}}(\alpha), m, \mathbf{tT}) \\ &= \lim_{|\mathbf{U}(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \left\langle \mathcal{P}_{U(\mathbf{tT})}^{d, r_{-\log \mathbf{t}}(\alpha)} m, \omega_{f_H} \right\rangle. \end{aligned}$$

7 It follows that  $\beta_H(\alpha, \cdot, \mathbf{t}) \in H$  as a point-wise limit of Birkhoff integrals. This  
8 implies *Orthogonal property*.  $\square$

9 For  $\gamma \in Sp_{2g}(\mathbb{Z})$ , we have

$$\beta_H(\gamma\alpha, \gamma(m), \mathbf{T}) = \beta_H(\alpha, m, \mathbf{T}).$$

10 It means that the function  $\beta_H(\cdot, m, \mathbf{T})$  is well-defined on the moduli space  $\mathfrak{M}_g$ .

**3.4. Proof of Theorem 1.6.** We define the excursion function

$$\begin{aligned} E_{\mathfrak{M}}(\alpha, \mathbf{T}) &:= \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1 + \cdots + t_d)/2} \text{Hgt}([r_{\mathbf{t} - \log \mathbf{T}}(\alpha)])^{1/4} dt_1 \cdots dt_d \\ &= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1 + \cdots + t_d)/2} \text{Hgt}([r_{\mathbf{t}}(\alpha)])^{1/4} dt_1 \cdots dt_d. \end{aligned}$$

**Lemma 3.7** (Bound of cocycle). *For any Diophantine  $[\alpha] \in DC(L)$  and for any  $f \in W^s(M)$  for  $s > s_{d,g} + 1/2$ , the Bufetov cocycle  $\beta^f$  is defined by a uniformly convergent series:*

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_s \left( L + \text{vol}(U(\mathbf{T}))^{1/2} (1 + \text{vol}(U(\mathbf{t})) + E_{\mathfrak{M}}(\alpha, \mathbf{T})) \right) \|f\|_{\alpha, s}.$$

*Proof.* Recall that by (7),

$$\beta^f(\alpha, m, \mathbf{T}) = \sum_H D_{\alpha}^H(f) \beta_H(\alpha, m, \mathbf{T}).$$

11 It follows from Lemma 3.5 that there exists a constant  $C > 0$  such that whenever  
12  $[\alpha] \in DC(L)$ , then

$$(40) \quad |\beta_H(\alpha, m, \mathbf{t})| \leq C(1 + L + \text{vol}(U(\mathbf{t}))), \quad (m, \mathbf{t}) \in M \times \mathbb{R}_+^d.$$

By Diophantine condition (20), whenever  $[\alpha] \in DC(L)$  then  $[r_{\log \mathbf{T}}(\alpha)] \in DC(L_{\mathbf{T}})$  with

$$L_{\mathbf{T}} \leq L \text{vol}(U(\mathbf{T}))^{-1/2} + E_{\mathfrak{M}}(\alpha, \mathbf{T}).$$

Thus by (40), we obtain for all  $(m, \mathbf{t}) \in M \times \mathbb{R}^d$ ,

$$|\beta_H(r_{\log \mathbf{T}}(\alpha), m, \mathbf{t})| \leq C(1 + L_{\mathbf{T}} + \text{vol}(U(\mathbf{t}))).$$

By the scaling property (39), it follows that for all  $s > s_{d,g}$  and  $q > 1/2$

$$\begin{aligned} |\beta^f(\alpha, m, \mathbf{t}\mathbf{T})| &\leq C_s \text{vol}(U(\mathbf{T}))^{1/2} (1 + L + \text{vol}(U(\mathbf{t}))) \sum_{n \in \mathbb{Z}} \|f_n\|_{\alpha, s} \\ &\leq C_s \text{vol}(U(\mathbf{T}))^{1/2} (1 + L_{\mathbf{T}} + \text{vol}(U(\mathbf{t}))) \left( \sum_{n \in \mathbb{Z}} (1 + n^2)^q \right)^{-1/2} \\ &\quad \times \left( \sum_{n \in \mathbb{Z}} \left\| (1 - Z^2)^{q/2} f_n \right\|_{\alpha, s}^2 \right)^{1/2}. \end{aligned}$$

Therefore, for all  $s' = s_{d,g} + q$ , there exists a constant  $C_{s'} > 0$  such that

$$|\beta^f(\alpha, m, \mathbf{t}\mathbf{T})| \leq C_{s'} \text{vol}(U(\mathbf{T}))^{1/2} (1 + L_{\mathbf{T}} + \text{vol}(U(\mathbf{t}))) \|f\|_{\alpha, s'}.$$

1

□

2 By Lemma 3.5, 3.7 and identification of the norm for the form  $\omega_f$ , asymptotic  
3 formula on each irreducible component provides the following corollary.

4 **Corollary 3.8.** *For all  $s > s_{d,g} + 1/2$ , there exists a constant  $C_s > 0$  such that for*  
5 *all  $[\alpha] \in DC(L)$ , for all  $f \in W_{\alpha}^s(M)$  and for all  $(m, \mathbf{T}) \in M \times \mathbb{R}^d$ , we have*

$$(41) \quad \left| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha} m, \omega_f \right\rangle - \beta^f(\alpha, m, \mathbf{T}) \right| \leq C_s (1 + L) \|\omega_f\|_{\alpha, s}$$

6 for  $U(\mathbf{T}) = [0, T^{(1)}] \times \cdots \times [0, T^{(d)}]$  and  $\omega_f = f \omega^{d, \alpha} \in \Lambda^d \mathfrak{p} \otimes W_{\alpha}^s(M)$ .

7 *Proof of Theorem 1.6.* The theorem follows from Corollary 3.8 for  $\alpha \in \bigcup_{L>0} DC(L)$ .  
8 □

9

#### 4. LIMIT DISTRIBUTIONS

10 In this section, we prove Theorem 1.7, limit distribution of normalized ergodic  
11 integrals of higher rank actions on the standard rectangles.

##### 12 4.1. Proof of Theorem 1.7.

13 **Lemma 4.1.** *There exists a continuous modular function  $\theta_H : \text{Aut}_0(\mathbf{H}^g) \rightarrow H \subset$*   
14  *$L^2(M)$  such that for any  $\omega_f = f \omega^{d, \alpha} \in \Lambda^d \mathfrak{p} \otimes W_{\alpha}^s(H)$  with  $s > d/2$ ,*

$$(42) \quad \lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha}(\cdot), \omega_f \right\rangle - \theta_H(r_{\log \mathbf{T}}(\alpha)) D_{\alpha}^H(f) \right\|_{L^2(M)} = 0.$$

15 *The family  $\{\theta_H(\alpha) \mid \alpha \in \text{Aut}_0(\mathbf{H}^g)\}$  has a constant norm in  $L^2(M)$ .*

*Proof.* By the Fourier transform, the space of smooth vectors and Sobolev space  $W^s(H)$  is represented as the Schwartz space  $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$  such that

$$\int_{\mathbb{R}^d} \left| \left( 1 + \sum_{i=1}^d \left( \frac{\partial^2}{\partial u_i^2} + u_i^2 \right) \right)^{s/2} \hat{f}(u) \right|^2 du < \infty.$$

Let  $\mathbf{t}, \mathbf{u} \in \mathbb{R}^d$ . Then we claim for any  $f \in \mathcal{S}^s(\mathbb{R}^d)$ , there exists a function  $\theta(\alpha)(\cdot) \in L^2(\mathbb{R}^d)$  such that

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \int_{U(\mathbf{T})} f(\mathbf{u} + \mathbf{t}) d\mathbf{t} - \theta_H(r_{\log \mathbf{T}}(\alpha))(\mathbf{u}) \text{Leb}(f) \right\|_{L^2(\mathbb{R}^d, d\mathbf{u})} = 0.$$

This is equivalent to the statement (42). By the standard Fourier transform on  $\mathbb{R}^d$ , equivalently

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \int_{U(\mathbf{T})} e^{i\mathbf{t} \cdot \hat{\mathbf{u}}} \hat{f}(\hat{\mathbf{u}}) d\mathbf{t} - \hat{\theta}_H(r_{\log \mathbf{T}}(\alpha))(\hat{\mathbf{u}}) \hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{\mathbf{u}})} = 0.$$

For  $\chi \in L^2(\mathbb{R}^d, d\hat{\mathbf{u}})$  and  $\hat{\mathbf{u}} = (\hat{u}_j)_{1 \leq j \leq d}$ , we denote

$$\chi_j(\hat{\mathbf{u}}) = \frac{e^{i\hat{u}_j} - 1}{i\hat{u}_j}, \quad \chi(\hat{\mathbf{u}}) = \prod_{j=1}^d \chi_j(\hat{\mathbf{u}}).$$

Let  $\hat{\theta}(\alpha)(\hat{\mathbf{u}}) := \chi(\hat{\mathbf{u}})$  for all  $\hat{\mathbf{u}} \in \mathbb{R}^d$ . Now we will compute  $\theta(r_{\log \mathbf{T}}(\alpha))$ . By intertwining formula (11) for  $\mathbf{T} \in \mathbb{R}^d$ ,

$$U_{\mathbf{T}}(f)(\hat{\mathbf{u}}) = \prod_{i=1}^d (T^{(i)})^{1/2} f(\mathbf{T}\hat{\mathbf{u}}), \quad \text{for } \mathbf{T}\hat{\mathbf{u}} = (T^{(1)}\hat{u}_1, \dots, T^{(d)}\hat{u}_d).$$

Then, for all  $\alpha \in \text{Aut}_0(\mathbf{H}^g)$ ,

$$\hat{\theta}(r_{\log \mathbf{T}}(\alpha))(\hat{\mathbf{u}}) = U_{\mathbf{T}}(\chi)(\hat{\mathbf{u}}) = \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{\mathbf{u}}).$$

The function  $\theta(\alpha)$  is defined by inverse Fourier transform of  $\hat{\theta}(\alpha)$  and

$$\|\theta_H(\alpha)\|_H = \|\theta(\alpha)\|_{L^2(\mathbb{R}^d)} = \|\hat{\theta}(\alpha)\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{\mathbf{u}})\|_{L^2(\mathbb{R}^d, d\hat{\mathbf{u}})} = C > 0.$$

By integration,

$$\begin{aligned} (43) \quad & \int_0^{T^{(d)}} \dots \int_0^{T^{(1)}} e^{i\mathbf{t} \cdot \hat{\mathbf{u}}} \hat{f}(\hat{\mathbf{u}}) d\mathbf{t} = \text{vol}(U(\mathbf{T})) \chi(\mathbf{T}\hat{\mathbf{u}}) \hat{f}(\hat{\mathbf{u}}) \\ & = \text{vol}(U(\mathbf{T})) \chi(\mathbf{T}\hat{\mathbf{u}}) (\hat{f}(\hat{\mathbf{u}}) - \hat{f}(0)) + \text{vol}(U(\mathbf{T}))^{1/2} \hat{\theta}(r_{\log \mathbf{T}}(\alpha))(\hat{\mathbf{u}}) \hat{f}(0). \end{aligned}$$

Then the claim reduces to the following:

$$\limsup_{|U(\mathbf{T})| \rightarrow \infty} \left\| \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{\mathbf{u}}) (\hat{f}(\hat{\mathbf{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If  $f \in \mathcal{S}^s(\mathbb{R}^g)$  with  $s > d/2$ , function  $\hat{f} \in C^0(\mathbb{R}^d)$  and bounded. Thus, by Dominated convergence theorem and change of variables,

$$\left\| \text{vol}(U(\mathbf{T}))^{1/2} \chi(\mathbf{T}\hat{\mathbf{u}}) (\hat{f}(\hat{\mathbf{u}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{\mathbf{u}})} = \left\| \chi(\nu) \left( \hat{f}\left(\frac{\nu}{\mathbf{T}}\right) - \hat{f}(0) \right) \right\|_{L^2(\mathbb{R}^d, d\nu)} \rightarrow 0.$$

1

□

**Corollary 4.2.** *For any  $s > d/2$ ,  $\alpha \in \text{Aut}_0(\mathbf{H}^g)$  and  $f \in W_\alpha^s(H)$ , there exists a constant  $C > 0$  such that*

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha} m, \omega_f \right\rangle \right\|_{L^2(M)} = C |D_\alpha^H(f)|.$$

1 From Corollary 4.2, we derive the following limit result for the  $L^2$ -norm of Bufe-  
 2 tov functionals.

**Corollary 4.3.** *For every irreducible component  $H$  and  $[\alpha] \in DC$ , there exists  $C > 0$  such that*

$$\lim_{|U(\mathbf{T})| \rightarrow \infty} \frac{1}{\text{vol}(U(\mathbf{T}))^{1/2}} \|\beta_H(\alpha, \cdot, \mathbf{T})\|_{L^2(M)} = C.$$

*Proof.* By the normalization of invariant distribution in Sobolev space, for any  $\alpha \in \text{Aut}_0(\mathbb{H}^g)$ , there exists a function  $f_\alpha^H \in W_\alpha^s(H)$  such that  $D_\alpha(f_\alpha^H) = \|f_\alpha^H\|_s = 1$ . For all  $[\alpha] \in DC(L)$ , by asymptotic formula (41) for  $f = f_H$ ,

$$|\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} m, \omega_f \rangle - \beta^f(\alpha, m, \mathbf{T})| \leq C_s(1 + L).$$

3 Therefore,  $L^2$ -estimate follows from Corollary 4.2.  $\square$

4 A relation between the Bufetov functional and the modular function  $\theta_H$  is es-  
 5 tablished below.

**Corollary 4.4.** *For every irreducible component  $H \subset L^2(M)$ , the following holds. For any  $L > 0$  and any  $r_{\mathbf{t}}$ -invariant probability measure  $\mu$  supported on  $DC(L) \subset \mathfrak{M}_g$ ,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H(\alpha)(\cdot), \quad \text{for } \mu\text{-almost all } [\alpha] \in \mathfrak{M}_g.$$

6 *Proof.* By Theorem 1.6 and Lemma 4.1, there exists a constant  $C > 0$  such that  
 7 for all  $[\alpha] \in \text{supp}(\mu) \subset DC(L)$  and  $\mathbf{T} \in \mathbb{R}_+^d$ , we have

$$(44) \quad \lim_{|U(\mathbf{T})| \rightarrow \infty} \|\beta_H(r_{\log \mathbf{T}}(\alpha), \cdot, 1) - \theta_H(r_{\log \mathbf{T}}(\alpha))\|_{L^2(M)} \leq \frac{C_\mu}{\text{vol}(U(\mathbf{T}))^{1/2}}.$$

8 By Luzin's theorem, for any  $\delta > 0$  there exists a compact subset  $E(\delta) \subset \mathfrak{M}$  such that  
 9 we have the measure bound  $\mu(\mathfrak{M} \setminus E(\delta)) < \delta$  and the function  $\beta_H(\alpha, \cdot, 1) \in L^2(M)$   
 10 depends continuously on  $[\alpha] \in E(\delta)$ . By Poincaré recurrence, there is a full measure  
 11 set  $F \subset \mathfrak{M}$  for  $\mathbb{R}^d$ -action. Denote by a full measure set  $E'(\delta) = E(\delta) \cap F \subset E(\delta)$ .

For every  $\alpha_0 \in E'(\delta)$ , there is diverging sequence  $(\mathbf{t}_n)$  such that  $\{r_{\mathbf{t}_n}(\alpha_0)\} \subset E(\delta)$  and  $\lim_{n \rightarrow \infty} r_{\mathbf{t}_n}(\alpha_0) = \alpha_0$ . By continuity of  $\theta_H$  and  $\beta_H$  at  $\alpha_0$ , we have

$$(45) \quad \begin{aligned} \|\beta_H(\alpha_0, \cdot, 1) - \theta_H(\alpha_0)\|_{L^2(M)} \\ = \lim_{n \rightarrow \infty} \|\beta_H(r_{\mathbf{t}_n}(\alpha_0), \cdot, 1) - \theta_H(r_{\mathbf{t}_n}(\alpha_0))\|_{L^2(M)} = 0. \end{aligned}$$

12 Thus, we have  $\beta_H(\alpha, \cdot, 1) = \theta_H(\alpha) \in L^2(M)$  for all  $\alpha \in E'(\delta)$ . It follows that the  
 13 set where the equality (45) fails has a measure less than any  $\delta > 0$ , thus the identity  
 14 holds for  $\mu$ -almost all  $\alpha \in \text{Aut}_0(\mathbb{H}^g)$ .  $\square$

15 For all  $\alpha \in \text{Aut}_0(\mathbb{H}^g)$ , smooth function  $f \in W^s(M)$  for  $s > s_{d,g} + 1/2$  decompose  
 16 as an infinite sum, and the functional  $\theta^f$  is defined by a convergent series

$$(46) \quad \theta^f(\alpha) := \sum_H D_\alpha^H(f) \theta_H(\alpha).$$

17 Hence, the modular function  $\theta^f : \text{Aut}_0(\mathbb{H}^g) \rightarrow L^2(M)$  is continuous.

18 The following result is an extension of Lemma 4.1 to an asymptotic formula.

**Lemma 4.5.** *For all  $\alpha \in \text{Aut}_0(\mathbf{H}^g)$ ,  $f \in W^s(M)$  and  $s > s_{d,g} + 1/2$ ,*

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{\text{vol}(U(\mathbf{T}_n))^{1/2}} \left\langle \mathcal{P}_{U(\mathbf{T}_n)}^{d,\alpha} m, \omega_f \right\rangle - \theta^f(r_{\log \mathbf{T}_n}(\alpha)) \right\|_{L^2(M)} = 0.$$

1 We summarize our results on limit distributions for higher rank actions.

**Theorem 4.6** (Theorem 1.7). *Let  $(\mathbf{T}_n)$  be any sequence such that*

$$\lim_{n \rightarrow \infty} r_{\log \mathbf{T}_n}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

2 *For every closed form  $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$  with  $s > s_{d,g} + 1/2$ , which is not a*  
 3 *coboundary, the limit distribution of the family of random variables  $E_{\mathbf{T}_n}(f)$  ex-*  
 4 *ists along a subsequence of  $\{\mathbf{T}_n\}$  and is equal to the distribution of the function*  
 5  $\theta^f(\alpha_\infty) = \beta(\alpha_\infty, \cdot, 1) \in L^2(M)$ .

6 *If  $\alpha_\infty \in DC$ , then  $\theta^f(\alpha_\infty)$  is a bounded function on  $M$ , and the limit distribution*  
 7 *has compact support.*

8 *Proof of Theorem 1.7.* Since  $\alpha_\infty \in \mathfrak{M}_g$ , the existence of limit follows from the  
 9 Lemma 4.5 and Theorem 4.6.  $\square$

10 A relation with Birkhoff integrals and theta sum was introduced in [CF15, §5.3],  
 11 and as an application, we derive the limit theorem of theta sums.

**Corollary 4.7.** *Let  $\mathcal{Q}[x] = x^\top \mathcal{Q} x$  be the quadratic forms defined by  $g \times g$  real*  
*matrix  $\mathcal{Q}$ , where  $\alpha = \begin{pmatrix} I & 0 \\ \mathcal{Q} & I \end{pmatrix} \in Sp_{2g}(\mathbb{R})$  and  $\ell(x) = \ell^\top x$  is the linear form*  
*defined by  $\ell \in \mathbb{R}^g$ . Then the theta sum*

$$\Theta(\mathcal{Q}, \ell; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0, N]} \exp(2\pi i(\mathcal{Q}[n] + \ell(n)))$$

12 *has a limit distribution and it has compact support.*

## 13 5. $L^2$ -LOWER BOUNDS

14 In this section we prove  $L^2$ - lower bounds of ergodic integrals on transverse torus.

**5.1. Structure of return map.** The polarized Heisenberg group  $\mathbf{H}_{pol}^g \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$  is equipped with the group law  $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx')$ . Reduced standard Heisenberg group is defined by quotient

$$\mathbf{H}_{red}^g := \mathbf{H}_{pol}^g / (\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R} / \frac{1}{2}\mathbb{Z}.$$

15 Then the Reduced standard lattice is  $\Gamma_{red}^g = \mathbb{Z}^g \times \mathbb{Z}^g \times \{0\} \subset \mathbf{H}_{red}^g$  and the quotient  
 16  $\mathbf{H}_{red}^g / \Gamma_{red}^g$  is isomorphic to the standard Heisenberg manifold  $M = \mathbf{H}^g / \Gamma$ .

Given the standard frame  $(X_i, Y_i, Z)$ ,  $(g + 1)$ -dimensional (transverse) torus is denoted by

$$\mathbb{T}_\Gamma^{g+1} := \{\Gamma \exp(\sum_{i=1}^g y_i Y_i + z Z) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

Now, we consider a return map of  $\mathbf{P}^{d,\alpha}$  on  $\mathbb{T}_\Gamma^{g+1}$  on the coordinates in reduced Heisenberg group. For  $x = (x_1, \dots, x_g) \in \mathbb{R}^g$ , we write in the coordinate of  $\mathbf{H}_{red}^g$  for convenience

$$\exp(x_1 X_1^\alpha + \dots + x_g X_g^\alpha) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta, \text{ and } w \in \mathbb{R}^d.$$



Then by group law,

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x)$$

1 and given  $(n, m, 0) \in \Gamma_{red}^g$ ,

$$(47) \quad \exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0, y, z) \cdot (n, m, 0) = \exp(x'_1 X_1^\alpha + \cdots + x'_g X_g^\alpha) \cdot (0, y', z')$$

2 if and only if

$$x'_\alpha = x_\alpha + n, y' = y + (x_\beta - x'_\beta) + m \text{ and } z' = z + (w - w') \cdot x + n^\top (y + x_\beta).$$

Assume  $\langle X_i^\alpha, X_j \rangle \neq 0$  for all  $i, j$ , and we write the first return time for  $P^{d, \alpha}$  action

$$t_{Ret} = (t_{Ret,1}, \dots, t_{Ret,d}) \in \mathbb{R}^d$$

3 on transverse torus  $\mathbb{T}_r^{g+1}$ . We denote the domain for return time  $U(t_{Ret}) =$   
4  $[0, t_{Ret,1}] \times \cdots \times [0, t_{Ret,d}]$ . Return map of action  $P^{d, \alpha}$  on  $\mathbb{T}_r^{g+1}$  has a form of  
5 skew-shift

$$(48) \quad A_{\rho, \tau}(y, z) = (y + \rho, z + v \cdot y + \tau) \text{ on } \mathbb{R}^g / \mathbb{Z}^g \times \mathbb{R} / K^{-1} \mathbb{Z}$$

6 for some non-zero vectors  $\rho, v \in \mathbb{R}^g$  and  $\tau \in \mathbb{R}$ .

7 Furthermore,  $A_{\rho, \tau} = A_{d, \rho, \tau} \circ \cdots \circ A_{1, \rho, \tau}$  decomposes with commuting linear  
8 skew-shifts

$$(49) \quad A_{i, \rho, \tau}(y, z) = (y + \rho_i, z + v_i \cdot y + \tau_i) \text{ on } \mathbb{R}^g / \mathbb{Z}^g \times \mathbb{R} / K^{-1} \mathbb{Z}$$

for some  $\rho_i, v_i \in \mathbb{R}^g$  and  $\tau_i \in \mathbb{R}$ . For each  $j \neq k$ , it is easily verified that

$$A_{j, \rho, \tau} \circ A_{k, \rho, \tau} = A_{k, \rho, \tau} \circ A_{j, \rho, \tau}.$$

Given pair  $(\mathbf{m}, n) \in \mathbb{Z}_{K|n|}^g \times \mathbb{Z}$ , let  $H_{(\mathbf{m}, n)}$  denote the corresponding factor and  $C^\infty(H_{(\mathbf{m}, n)})$  be a subspace of smooth functions on  $H_{(\mathbf{m}, n)}$ . Denote  $\{e_{\mathbf{m}, n} \mid (\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z}\}$  the basis of characters on  $\mathbb{T}_r^{g+1}$  and for all  $(y, z) \in \mathbb{T}^g \times \mathbb{T}$ ,

$$e_{\mathbf{m}, n}(y, z) := \exp[2\pi i(\mathbf{m} \cdot y + nKz)].$$

For each  $A_{i, \rho, \sigma}$ , we set  $\mathbf{v}_i = (v_{i1}, \dots, v_{ig}) \in \mathbb{Z}_{K|n|}^g$ . Then the orbit can be identified with the following dual orbit

$$\begin{aligned} \mathcal{O}_{A_i}(\mathbf{m}, n) &= \{(\mathbf{m} + (nj_i)\mathbf{v}_i, n), j_i \in \mathbb{Z}\} \\ &= \{(m_1 + (nv_{i1})j_i, \dots, m_g + (nv_{ig})j_i, n), j_i \in \mathbb{Z}\}. \end{aligned}$$

If  $n = 0$ , the orbit  $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$  of  $(\mathbf{m}, 0)$  is reduced to a single element. If  $n \neq 0$ , then the dual orbit  $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$  of  $(\mathbf{m}, n)$  for higher rank actions is described as follows:

$$\mathcal{O}_A(\mathbf{m}, n) = \{(m_k + n \sum_{i=1}^d (v_{ik} j_i), n)_{1 \leq k \leq d} : j = (j_1, \dots, j_d) \in \mathbb{Z}^d\}.$$

It follows that every  $A$ -orbit for rank  $\mathbb{R}^d$ -action (or  $A_i$ -orbit) can be labeled uniquely by a pair  $(\mathbf{m}, n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \setminus \{0\}$  with  $\mathbf{m} = (m_1, \dots, m_g)$ . Thus, the subspace of functions with non-zero central characters splits as a direct sum of components  $H_{(\mathbf{m}, n)}$

$$L^2(\mathbb{T}_r^{g+1}) = \bigoplus_{\omega \in \mathcal{O}_A} H_\omega, \quad \text{where } H_\omega = \bigoplus_{(\mathbf{m}, n) \in \omega} \mathbb{C} e_{(\mathbf{m}, n)}.$$

5.2. **Higher cohomology for  $\mathbb{Z}^d$ -action of skew-shifts.** In this subsection, we will find a relations with the return map for  $\mathbb{Z}^d$  action of  $P^{d,\alpha}$  on the torus  $\mathbb{T}_\Gamma^{g+1}$  and obstructions to cohomological equation  $\omega = d\Omega$ .

We will restrict our interest to the following cocycle equation

$$(50) \quad \varphi(x, t) = \mathfrak{D}\Phi(x, t), \quad x \in \mathbb{T}^g, \quad t \in \mathbb{Z}^d$$

where  $d$ -cocycle  $\varphi : \mathbb{T}_\Gamma^{g+1} \times \mathbb{Z}^d \rightarrow \mathbb{R}$ ,  $(d-1)$  cochain  $\Phi : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathbb{R}^d$ ,  $\Phi = (\Phi_1, \dots, \Phi_d)$ , and  $\mathfrak{D}$  is coboundary operator

$$\mathfrak{D}\Phi = \sum_{i=1}^d (-1)^{i+1} \Delta_i \Phi_i$$

where  $\Delta_i \Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$ .

In the work of Katok [KK95, §2], he proved the existence of solutions for cocycle equations by studying the dual equations  $\hat{\varphi} = \mathfrak{D}\hat{\Phi}$  in the space of Fourier coefficients (dual orbit). We apply this result to our  $A$ -orbit and verify the invariant distributions (or currents) explicitly.

**Proposition 5.1.** [KK95, Proposition 2.2] *A cocycle  $\hat{\varphi}$  satisfies cocycle equation (50) if and only if  $\sum_{j \in \mathbb{Z}^d} \hat{\varphi}(\mathbf{m}, n) \circ A^j = 0$ .*

For fixed  $(\mathbf{m}, n) \in \mathbb{Z}^g \times \mathbb{Z}$ , we denote an obstruction of cohomological equation restricted to the  $A$ -orbit of  $(\mathbf{m}, n)$  by  $\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \hat{\varphi}(\mathbf{m}, n) \circ A^j$ . Since  $A^j$  is composition of commuting toral automorphisms, we obtain the following generalized formula (see [AFU11, §5] and [Ka03, §11] for rank 1 map on  $\mathbb{T}^2$ ).

**Lemma 5.2.** *There exists a distributional obstruction to the existence of a smooth solution  $\varphi \in C^\infty(H_{(\mathbf{m},n)})$  of the cohomological equation (50). A generator of the space of invariant distribution  $\mathcal{D}_{\mathbf{m},n}$  is given by*

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) := e^{-2\pi i \sum_{i=1}^d [(\mathbf{m} \cdot \rho_i + nK\tau_i)j_i + nK\tau_i \binom{j_i}{2}]}$$

if  $(a, b) = (m_k + K \sum_{i=1}^d (v_{ik}j_i), n)_{1 \leq k \leq g}$  and 0 otherwise.

*Proof.* From previous observation, there exists an obstruction

$$(51) \quad \mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}_\Gamma^{g+1}} \varphi(x, y) \overline{e_{\mathbf{m},n} \circ A_{\rho,\tau}^j} dx dy.$$

By direct computation, for fixed  $j = (j_1, \dots, j_d)$ ,

$$e_{\mathbf{m},n} \circ A_{\rho,\tau}^j(y, z) = \prod_{i=1}^d \left( e^{2\pi i [(\mathbf{m} \cdot \rho_i + nK\tau_i)j_i + nK\tau_i \binom{j_i}{2}]} \right) \left( e^{2\pi i (\mathbf{m} \cdot y + K(z + n \sum_{k=1}^d (v_{ik}j_i)y_k))} \right).$$

Then, we choose  $\hat{\varphi} = e_{a,b}$  for  $(a, b) = (m_k + K \sum_{i=1}^d (v_{ik}j_i), n)_{1 \leq k \leq g}$  in the non-trivial orbit ( $n \neq 0$ ),

$$(52) \quad \mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi i \sum_{i=1}^d [(\mathbf{m} \cdot \rho_i + nK\tau_i)j_i + nK\tau_i \binom{j_i}{2}]}.$$

**5.3. Changes of coordinates.** For any frame  $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$ , denote transverse cylinder for any  $m \in M$ ,

$$\mathcal{C}_{\alpha,m} := \{m \exp(\sum_{i=1}^g y'_i Y_i^\alpha + z' Z) \mid (y', z') \in U(t_{Ret}^{-1}) \times \mathbb{T}\}.$$

Let  $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathcal{C}_{\alpha,m}$  denote the maps: for any  $\xi \in \mathbb{T}_\Gamma^{g+1}$ , let  $\xi' \in \mathcal{C}_{\alpha,m}$  denote first intersection of the orbit  $\{P_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}_+^d\}$  with transverse cylinder  $\mathcal{C}_{\alpha,m}$ . Then, there exists a first return time to the cylinder  $t(\xi) = (t_1(\xi), \dots, t_d(\xi)) \in \mathbb{R}_+^d$  such that

$$\xi' = \Phi_{\alpha,m}(\xi) = P_{t(\xi)}^{d,\alpha}(\xi), \quad \forall \xi \in \mathbb{T}_\Gamma^{g+1}.$$

Let  $(y, z)$  and  $(y', z')$  denote the coordinates on  $\mathbb{T}_\Gamma^{g+1}$  and  $\mathcal{C}_{\alpha,m}$  given by the exponential map respectively,

$$(y, z) \rightarrow \xi_{y,z} := \Gamma \exp(\sum_{i=1}^g y_i Y_i + z Z), \quad (y', z') \rightarrow m \exp(\sum_{i=1}^g y'_i Y_i^\alpha + z' Z).$$

- 1 Recall that if  $\alpha \in Sp_{2g}(\mathbb{R})$ , then for  $1 \leq i, j \leq g$  there exist matrices  $A =$   
 2  $(a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$  such that

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2g}(\mathbb{R}),$$

satisfying  $A^t D - C^t B = I_{2g}$ ,  $C^t A = A^t C$ ,  $D^t B = B^t D$ , and  $\det(A) \neq 0$ . Set

$$X_i^\alpha = \sum_{j=1}^g (a_{ij} X_j + b_{ij} Y_j) + w_i Z \quad \text{and} \quad Y_i^\alpha = \sum_{j=1}^g (c_{ij} X_j + d_{ij} Y_j) + v_i Z.$$

- 3 Let  $x = \Gamma \exp(\sum_{i=1}^d y_{x,i} Y_i + z_x Z) \exp(\sum_{i=1}^d t_{x,i} X_i)$ , for some  $(y_x, z_x) \in \mathbb{T}^d \times$   
 4  $\mathbb{R}/K\mathbb{Z}$  and  $t_x = (t_{x,i}) \in [0, 1]^d$ . Then, the map  $\Phi_{\alpha,x} : \mathbb{T}_\Gamma^{g+1} \rightarrow \mathcal{C}_{\alpha,m}$  is defined by  
 5  $\Phi_{\alpha,x}(y, z) = (y', z')$  where

$$(53) \quad \begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix},$$

- 6 and  $z' = z + P(\alpha, x, y)$  for some degree 4 polynomial  $P$ .

Therefore, the map  $\Phi_{\alpha,x}$  is invertible with

$$\Phi_{\alpha,x}^*(dy'_1 \wedge \cdots \wedge dy'_g \wedge dz') = \frac{1}{\det(A)} dy_1 \wedge \cdots \wedge dy_g \wedge dz.$$

- 7 Since  $A^t D - C^t B = I_{2g}$ , by direct computation we obtain return time, we have

$$(54) \quad \begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{dg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}.$$

Then,

$$\|t(\xi)\| \leq \max_i |t_i(\xi)|^g \leq \max_i \left| \sum_{j=1}^g d_{ij} t_{x,i} + c_{ij} (y_i - y_{x,i}) \right|^g \leq \max_i \|Y_i^\alpha\|^g.$$

- 1 5.4.  **$L^2$ -lower bound of functional.** We will prove the bounds for square mean  
 2 of integrals along leaves of foliations of the torus  $\mathbb{T}_\Gamma^{g+1}$ .

**Lemma 5.3.** *There exists a constant  $C > 0$  such that for all  $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$ , and for every irreducible component  $H := H_n$  of central parameter  $n \neq 0$ , there exists a function  $f_H$  such that*

$$\|f_H\|_{L^\infty(H)} \leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)|,$$

$$\|f_H\|_{\alpha,s} \leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)| \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))}\right) \|Y\|^s (1+n^2)^{s/2}$$

- 3 where  $\|Y\| := \max_{1 \leq i \leq g} \|Y_i^\alpha\|$  and  $\Sigma(t_{Ret}) = \sum_{i=1}^g t_{Ret,i}$ .

- 4  
 5 On rectangular domain  $U(\mathbf{T})$ , for all  $m \in \mathbb{T}_\Gamma^{g+1}$  and  $T^{(i)} \in \mathbb{Z}_{t_{Ret,i}}$

$$(55) \quad \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = |\mathcal{D}_\alpha^H(f_H)| \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2}.$$

In addition, whenever  $H \perp H' \subset L^2(M)$  the functions

$$\left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} m), \omega_H \right\rangle \quad \text{and} \quad \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\mathbf{Q}_y^{g,Y} m), \omega_{H'} \right\rangle$$

- 6 are orthogonal in  $L^2(\mathbb{T}^g, dy)$ .

- 7 *Proof.* The operator  $I_\alpha : L^2(M) \rightarrow L^2(\mathbb{T}_\Gamma^{g+1})$  is defined by

$$(56) \quad f \rightarrow I_\alpha(f) := \int_{U(t_{Ret})} f \circ \mathbf{P}_s^{d,\alpha}(\cdot) ds.$$

Then operator  $I_\alpha$  is surjective linear map of  $L^2(M)$  onto  $L^2(\mathbb{T}_\Gamma^{g+1})$  with a right inverse  $R_\alpha^X$  defined as follows. Let  $\chi \in C_0^\infty(0,1)^g$  be any function of jointly integrable with integral 1. For any  $F \in L^2(\mathbb{T}_\Gamma^{g+1})$ , let  $R_\alpha^X(F) \in L^2(M)$  be a function defined by

$$R_\alpha^X(F)(\mathbf{P}_v^{d,\alpha}(x)) = \frac{1}{\text{vol}(U(t_{Ret}))} \chi\left(\frac{v}{t_{Ret}}\right) F(x), \quad (x, v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows that there exists a constant  $C_\chi > 0$  such that

$$(57) \quad \begin{aligned} \|R_\alpha^X(F)\|_{\alpha,s} &\leq C_\chi \text{vol}(U(t_{Ret}))^{-1} \left(1 + \sum_{i=1}^g t_{Ret,i}^{-1} \|Y_i^\alpha\|\right)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})} \\ &\leq C_\chi \text{vol}(U(t_{Ret}))^{-1} \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))}\right) \|Y\|^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})}. \end{aligned}$$

As explained in §5.1, the space  $L^2(\mathbb{T}_\Gamma^{g+1})$  decompose as a direct sum of irreducible subspaces invariant under the action of each  $A_{j,\rho,\sigma}$ . It follows that the subspace of functions with non-zero central character can be split as direct sum of components

$H_{(\mathbf{m},n)}$  with  $(\mathbf{m},n) \in \mathbb{Z}_{|n|}^g \times \mathbb{Z} \setminus \{0\}$  with  $\mathbf{m} = (m_1, \dots, m_g)$ . For a function  $F \in H_{(\mathbf{m},n)}$ , it is characterized by Fourier expansion

$$F = \sum_{j \in \mathbb{Z}^d} F_j e^{A^j(\mathbf{m},n)} = \sum_{j \in \mathbb{Z}^d} F_j e^{(m_k + K \sum_{i=1}^d (v_{ik} j_i), n)}.$$

Then, we can choose  $f_H := R_\alpha^\chi(e_{\mathbf{m},n}) \in C^\infty(H)$  such that

$$(58) \quad |D_\alpha(f_\alpha^H)| = |D_{\mathbf{m},n}(e_{\mathbf{m},n})| = 1,$$

$$(59) \quad \int_{U(t_{Ret})} f_H \circ P_t^{d,\alpha}(y,z) dt = e_{\mathbf{m},n}(y,z), \text{ for } (y,z) \in \mathbb{T}_\Gamma^{g+1}.$$

Therefore, it follows from (57) that

$$|f_H|_{L^\infty(H)} \leq C_\chi \text{vol}(U(t_{Ret}))^{-1},$$

$$\|f_H\|_{\alpha,s} \leq C \text{vol}(U(t_{Ret}))^{-1} |\mathcal{D}_\alpha^H(f_H)| \left(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|\right)^s (1+n^2)^{s/2}.$$

Moreover, since  $\{e_{\mathbf{m},n} \circ A_{\rho,\tau}^j\}_{j \in \mathbb{Z}^d} \subset L^2(\mathbb{T}_\Gamma^g, dy)$  is orthonormal, we verify

$$\begin{aligned} \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\mathbb{Q}_y^{g,Y} x), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} &= \left\| \sum_{j_d=0}^{\lfloor \frac{T(d)}{t_{Ret,d}} \rfloor - 1} \cdots \sum_{j_1=0}^{\lfloor \frac{T(1)}{t_{Ret,1}} \rfloor - 1} e_{\mathbf{m},n} \circ A_{\rho,\tau}^j \right\|_{L^2(\mathbb{T}^g, dy)} \\ &= \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2}. \end{aligned}$$

1

□

2 Recall that

$$(60) \quad L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n := \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i=1}^{\mu(n)} H_{i,n}$$

3 where  $H_n = \bigoplus_{i=1}^{\mu(n)} H_{i,n}$  is irreducible representation with central parameter  $n$  and  
 4  $\mu(n)$  is countable by Howe-Richardson multiplicity formula.

For any infinite dimensional vector  $\mathbf{c} := (c_{i,n}) \in \ell^2$ , let  $\beta_{\mathbf{c}}$  denote Bufetov functional

$$\beta_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} \beta_{i,n}^{i,n}.$$

By orthogonal property and from Corollary 4.3, the function  $\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T}) \in L^2(M)$  for all  $(\alpha, \mathbf{T}) \in \text{Aut}_0(\mathbb{H}^g) \times \mathbb{R}_+^d$ . Furthermore,

$$\|\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T})\|_{L^2(M)}^2 = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} |c_{i,n}|^2 \|\beta_{i,n}^{i,n}(\alpha, \cdot, \mathbf{T})\|_{L^2(H_{i,n})}^2 \leq C^2 |\mathbf{c}|_{\ell^2}^2 \text{vol}(U(\mathbf{T})).$$

5 For any  $\mathbf{c} := (c_{i,n})$ , let  $|\mathbf{c}|_s$  denote the norm defined as

$$(61) \quad |\mathbf{c}|_s^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{i=1}^{\mu(n)} (1 + K^2 n^2)^s |c_{i,n}|^2.$$

**Lemma 5.4.** *For any  $s > s_{d,g} + 1/2$ , there exists a constant  $C_s > 0$  such that for all  $\alpha \in DC(L)$ , for all  $\mathbf{c} \in \ell^2$ , for all  $z \in \mathbb{T}$  and all  $T > 0$ ,*

$$\left| \left\| \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \right\|_{L^2(\mathbb{T}^g, dy)} - \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \leq C_s (\text{vol}(U(t_{Ret})) + \text{vol}(U(t_{Ret}))^{-1})(1+L)(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.$$

*Proof.* By Lemma 5.2, there exists a function  $f_{i,n} \in C^\infty(H_{i,n})$  with  $|\mathcal{D}^{i,n}(f_{i,n})| = 1$ . Let  $f_{\mathbf{c}} = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n} \in C^\infty(M)$ , summing up all the functions on irreducibles. Then by the estimates in the Lemma 5.3 and (61),

$$(62) \quad \|f_{\mathbf{c}}\|_{L^\infty(M)} \leq C |\mathbf{c}|_{\ell^1}.$$

$$(63) \quad \|f_{\mathbf{c}}\|_{\alpha,s} \leq C \text{vol}(U(t_{Ret}))^{-1} (1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.$$

By orthogonality,

$$\left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} \circ \mathbf{Q}_y^{g,Y}, \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0.$$

By the estimate in Lemma 5.3 for each  $f_{i,n}$ , for every  $z \in \mathbb{T}$  and all  $\mathbf{T} \in \mathbb{R}_+^d$ , we have

$$\left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq 2 \|f_{\mathbf{c}}\|_{L^\infty(M)} \|Y\|.$$

Let  $U(\mathbf{T}_{Ret}) = [0, \mathbf{T}_{Ret,1}] \times \cdots \times [0, \mathbf{T}_{Ret,g}]$  where  $\mathbf{T}_{Ret,i} := t_{Ret,i}([T^{(i)}/t_{Ret,i}] + 1)$ . Then,

$$\left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle \mathcal{P}_{U(\mathbf{T}_{Ret})}^{d,\alpha}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \leq \text{vol}(U(t_{Ret})) \|f_{\mathbf{c}}\|_{L^\infty(M)}.$$

Therefore, there exists a constant  $C' > 0$  such that

$$\left| \left\| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} - \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \leq C' \text{vol}(U(t_{Ret})) |\mathbf{c}|_{\ell^1}.$$

For all  $s > s_{d,g} + 1/2$ , by asymptotic property of Theorem 1.6, for some constant  $C_s > 0$ ,

$$\left| \left\langle \mathcal{P}_{U(\mathbf{T})}^{d,\alpha} m, \omega \right\rangle - \beta_H(\alpha, m, \mathbf{T}) \mathcal{D}_\alpha^H(f_H) \right| \leq C_s (1+L) \|f\|_{\alpha,s}.$$

Applying  $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$  and combining bounds on the function  $f_{\mathbf{c}}$  with (62),

$$\begin{aligned} & \left| \left\| \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \right\|_{L^2(\mathbb{T}^g, dy)} - \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \\ & \leq C' \text{vol}(U(t_{Ret})) |\mathbf{c}|_{\ell^1} + C_s \text{vol}(U(t_{Ret}))^{-1} (1+L) \|f_{\mathbf{c}}\|_{\alpha,s} \\ & \leq C'_s (\text{vol}(U(t_{Ret})) + \text{vol}(U(t_{Ret}))^{-1})(1+L)(1 + \frac{\Sigma(t_{Ret})}{\text{vol}(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s. \end{aligned}$$

1 Therefore, we derive the estimates in the statement.  $\square$

## 6. ANALYTICITY OF FUNCTIONALS

In this section we will prove that for all  $\alpha \in DC$ , the Bufetov functionals on any square are real analytic.

## 6.1. Analyticity.

**Definition 6.1.** For every  $t \in \mathbb{R}$ ,  $1 \leq i \leq d$ , and  $m \in M$ , the *stretched (in direction of  $Z$ ) rectangle* is denoted by

$$(64) \quad [\Gamma_{\mathbf{T}}]_{i,t}^Z(m) := \{(\phi_{ts_i}^Z) \circ \mathbf{P}_{\mathbf{s}}^{d,\alpha}(m) \mid \mathbf{s} \in U(\mathbf{T})\}.$$

For  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}^d$ , let us denote by the standard rectangle  $\Gamma_{\mathbf{T}}(\mathbf{s}) := (\gamma_1(s_1), \dots, \gamma_d(s_d))$  for  $\gamma_i(s_i) = \exp(s_i X_i)$ . Similarly, we also write the stretched rectangle

$$(65) \quad [\Gamma_{\mathbf{T}}]_{i,t}^Z(\mathbf{s}) := (\gamma_1(s_1), \dots, \gamma_{i,t}^Z(s_i), \dots, \gamma_d(s_d))$$

where  $\gamma_{i,t}^Z(s_i) := \phi_{ts_i}^Z(\gamma_i(s_i))$  is a stretched curve.

**Definition 6.2.** The *restricted rectangle*  $\Gamma_{T,i,s}$  of the standard rectangle  $\Gamma_{\mathbf{T}}$  is defined as a restriction

$$\Gamma_{T,i,s}(\mathbf{s}) := \Gamma_{\mathbf{T}}|_{U_{T,i,s}}(\mathbf{s}),$$

where  $U_{T,i,s} = [0, T^{(1)}] \times \dots \times \underbrace{[0, s]}_{i\text{-th}} \times \dots \times [0, T^{(d)}]$  for some  $0 < s \leq T^{(i)}$ .

Recall the orthogonal property on a irreducible component  $H$  (central parameter  $n \in \mathbb{Z} \setminus \{0\}$ ). For any  $(m, \mathbf{T}) \in M \times \mathbb{R}_+^d$  and  $t \in \mathbb{R}$ ,

$$(66) \quad \beta_H(\alpha, \phi_t^Z(m), \mathbf{T}) = e^{2\pi i K n t} \beta_H(\alpha, m, \mathbf{T}).$$

We obtain the following lemma for stretched rectangle by applying orthogonal property.

**Lemma 6.3.** For fixed elements  $(X_i, Y_i, Z)$  satisfying commutation relation (1), the following formula for rank 1 action holds:

$$\hat{\beta}_H(\alpha, [\Gamma_{\mathbf{T}}]_{i,t}^Z) = e^{2\pi i \iota n K T^{(i)}} \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}) - 2\pi i \iota n K t \int_0^{T^{(i)}} e^{2\pi i \iota n K t s_i} \hat{\beta}_H(\alpha, \Gamma_{T,i,s}) ds_i.$$

*Proof.* Let  $\alpha = (X_i, Y_i, Z)$  and  $\omega$  be  $d$ -form supported on a single irreducible representation  $H$ . We obtain following the formula for stretches of curve  $\gamma_{i,t}^Z$  (see [FK20b, §4 and Lemma 9.1]),

$$\frac{d\gamma_{i,t}^Z}{ds_i} = D\phi_{ts_i}^Z\left(\frac{d\gamma_i}{ds_i}\right) + tZ \circ \gamma_{i,t}^Z.$$

It follows that pairing is given by

$$\begin{aligned} \langle [\Gamma_{\mathbf{T}}]_{i,t}^Z, \omega \rangle &= \int_{U(\mathbf{T})} \omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_{i,t}^Z}{ds_i}(s_i), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right) d\mathbf{s} \\ &= \int_{U(\mathbf{T})} e^{2\pi i \iota n K t s_i} [\omega\left(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d)\right)] + \iota_Z \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z(\mathbf{s}) d\mathbf{s} \end{aligned}$$

Denote  $d - 1$  dimensional triangle  $U_{d-1}(\mathbf{T})$  with  $U(\mathbf{T}) = U_{d-1}(\mathbf{T}) \times [0, T^{(i)}]$ . Integration by parts for a fixed  $i$ -th integral gives

$$\begin{aligned} & \int_{U(\mathbf{T})} e^{2\pi i n K t s_i} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d))] ds \\ &= e^{2\pi i n K t T^{(i)}} \int_{U(\mathbf{T})} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_d}{ds_d}(s_d))] ds \\ &- 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \int_{U_{d-1}(\mathbf{T})} \left( \int_0^{s_i} [\omega(\frac{d\gamma_1}{ds_1}(s_1), \dots, \frac{d\gamma_i}{ds_i}(r) \dots, \frac{d\gamma_d}{ds_d}(s_d))] dr \right) ds. \end{aligned}$$

Then, we have the following formula

$$\begin{aligned} \langle [\Gamma_{\mathbf{T}}]_{i,t}^Z, \omega \rangle &= e^{2\pi i n K t T^{(i)}} \langle [\Gamma_{\mathbf{T}}], \omega \rangle - 2\pi i n K t \int_0^{T^{(i)}} e^{2\pi i n K t s_i} \langle \Gamma_{T,i,s}, \omega \rangle ds_i \\ &+ \int_{U(\mathbf{T})} (\iota_Z \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z)(\mathbf{s}) ds. \end{aligned}$$

Since the action of  $\mathbf{P}_{\mathbf{t}}^{d,X}$  for  $\mathbf{t} \in \mathbb{R}^d$  is identity on the center  $Z$ ,

$$\lim_{t_1, \dots, t_d \rightarrow \infty} e^{-(t_1 + \dots + t_d)/2} \int_{U(\mathbf{T})} (\iota_Z (\mathbf{P}_{\mathbf{t}}^{d,X})^* \omega \circ [\Gamma_{\mathbf{T}}]_{i,t}^Z)(\mathbf{s}) ds = 0.$$

- 1 Thus, in particular concerning  $d = 1$ , it follows by the definition of Bufetov func-  
2 tional (Lemma 3.5), the statement holds.  $\square$

Here we define a restricted vector  $\mathbf{T}_{i,s}$  of  $\mathbf{T} = (T^{(1)}, \dots, T^{(d)}) \in \mathbb{R}^d$ . For fixed  $i \in [1, d]$ , pick  $s_i \in [0, T^{(i)}]$  such that  $\mathbf{T}_{i,s} \in \mathbb{R}^d$  is a vector with its coordinates

$$T_{i,s}^{(j)} = \begin{cases} T^{(j)} & \text{if } j \neq i \\ s_i & \text{if } j = i. \end{cases}$$

- 3 Similarly,  $\mathbf{T}_{i_1, \dots, i_k, s}$  is a vector whose  $i_1, \dots, i_k$ -th coordinates are replaced by  
4  $s_{i_1}, \dots, s_{i_k}$ .

**Lemma 6.4.** *For  $m \in M$  and  $y_i \in \mathbb{R}$ , the following property holds:*

$$\begin{aligned} \beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) &= \\ e^{-2\pi i y_i n K T^{(i)}} \beta_H(\alpha, m, \mathbf{T}) &+ 2\pi i n K y_i \int_0^{T^{(i)}} e^{-2\pi i y_i n K s_i} \beta_H(\alpha, m, \mathbf{T}_{i,s}) ds_i. \end{aligned}$$

*Proof.* By definition (4), (65) and commutation relation (1), it follows that

$$\phi_{y_i}^{Y_i}(\Gamma_{\mathbf{T}}^X(m)) = [\Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))]_{i,t}^Z.$$

By the invariance property of Bufetov functional and Lemma 6.3,

$$\begin{aligned} \beta_H(\alpha, m, \mathbf{T}) &= \hat{\beta}_H(\alpha, \phi_{y_i}^{Y_i}(\Gamma_{\mathbf{T}}^X(m))) \\ &= e^{2\pi i y_i n K T^{(i)}} \hat{\beta}_H(\alpha, \Gamma_{\mathbf{T}}^X(\phi_{y_i}^{Y_i}(m))) - 2\pi i n K y_i \int_0^{T^{(i)}} e^{2\pi i n K y_i s_i} \hat{\beta}_H(\alpha, \Gamma_{T,i,s}^X(\phi_{s_i}^{Y_i}(m))) ds_i \\ &= e^{2\pi i y_i n K T^{(i)}} \beta_H(\alpha, \phi_{y_i}^{Y_i}(m), \mathbf{T}) - 2\pi i n K y_i \int_0^{T^{(i)}} e^{2\pi i n K y_i s_i} \beta_H(\alpha, \phi_{s_i}^{Y_i}(m), \mathbf{T}_{i,s}) ds_i. \end{aligned}$$

- 5 Then statement follows immediately.  $\square$



1 We extend previous Lemma 6.4 to higher rank actions by induction argument.

**Lemma 6.5** (Rank- $d$  action). *For  $m \in M$  and  $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ , the following identity for the cocycle  $\beta_H$  holds:*

$$(67) \quad \begin{aligned} \beta_H(\alpha, Q_y^{d,Y}(m), \mathbf{T}) &= e^{-2\pi i \sum_{j=1}^d y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \\ &+ \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (2\pi i n K y_{i_j}) e^{-2\pi i n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\ &\times \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi i n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1}. \end{aligned}$$

*Proof.* We verified that the statement works for  $d = 1$  in Lemma 6.4. Assume that (67) holds for rank  $d - 1$  action  $Q_{y'}^{d-1,Y}$  by induction hypothesis. For convenience, we write

$$Q_y^{d,Y}(m) = \phi_{y_d}^{Y_d} \circ Q_{y'}^{d-1,Y}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 6.4,

$$(68) \quad \begin{aligned} \beta_H(\alpha, Q_y^{d,Y}(m), \mathbf{T}) &= e^{-2\pi i y_d n K T^{(d)}} \beta_H(\alpha, Q_{y'}^{d-1,Y}(m), \mathbf{T}) \\ &+ 2\pi i n K y_d \int_0^{T^{(d)}} e^{-2\pi i y_d n K s_d} \beta_H(\alpha, Q_{y'}^{d-1,Y}(m), \mathbf{T}_{d,s}) ds_d \\ &:= I + II. \end{aligned}$$

Firstly, by induction hypothesis,

$$\begin{aligned} I &= e^{-2\pi i y_d n K T^{(d)}} \left( e^{-2\pi i \sum_{j=1}^{d-1} y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \right. \\ &+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} \prod_{j=1}^k (2\pi i n K y_{i_j}) e^{-2\pi i n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)} + y_d T^{(d)})} \\ &\times \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi i n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1} \Big) \\ &= e^{-2\pi i \sum_{j=1}^d y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}) + III. \end{aligned}$$

2 Then term  $III$  contains 0 to  $d - 1$ th iterated integrals on the restricted rectangles  
3 (from 0 to  $d - 1$ -th) containing  $e^{-2\pi i n K y_d T^{(d)}}$  outside of iterated integrals.

4 For the second part, we apply induction hypothesis again for restricted rectangle  
5  $\mathbf{T}_{d,s}$ . Then,

$$\begin{aligned}
II &= 2\pi\iota n K y_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d n K s_d} [e^{-2\pi\iota \sum_{j=1}^{d-1} y_j n K T^{(j)}} \beta_H(\alpha, m, \mathbf{T}_{d,s})] ds_d \\
&+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \dots < i_k \leq d-1} (2\pi\iota n K y_d) \prod_{j=1}^k (2\pi\iota n K y_{i_j}) e^{-2\pi\iota n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\
&\times \int_0^{T^{(d)}} \left( \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} ds_{i_k} \dots ds_{i_1} \right) ds_d \\
&\times e^{-2\pi\iota n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k} + y_d s_d)} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, d, s}).
\end{aligned}$$

1 The term  $II$  consist of 1 to  $d$ -th iterated integrals on the (from 1 to  $d$ -th) restricted  
2 rectangles containing  $e^{-2\pi\iota n K y_d s_d}$  inside of iterated integrals. Thus, by rearranging  
3 terms  $II$  and  $III$ , we obtain all the terms in the expression (67).  $\square$

4 **6.2. Extensions of domain.** In this subsection, we extend the domain of Bufetov  
5 functional defined on standard rectangle  $\Gamma_{\mathbf{T}}^X$  to the class  $\mathfrak{R}$  (see Definition 1.1). We  
6 also extend the functional to holomorphic function on a complex domain.

7 *Proof of Corollary 1.4.* Firstly, we can extend our functional to class  $(\mathbb{Q}_y^{d,Y})_* \Gamma_{\mathbf{T}}^X$  for  
8 any  $y \in \mathbb{R}^d$  by invariance property (Lemma 3.6). Similarly, by Lemma 6.3, Bufetov  
9 functional defined on the standard rectangles extends to the class of generalized  
10 rectangle  $(\phi_{t_{iz}}^Z) \circ \mathbf{P}_t^{d,\alpha}(m)$ . Since the flow generated by  $Z$  commutes with other  
11 actions  $\mathbf{P}$  and  $\mathbf{Q}$ , for any standard rectangle  $\Gamma = \Gamma(m)$  with a fixed point  $m \in M$ ,  
12 we have  $(\phi_z^Z)_* \Gamma(m) = \Gamma(\phi_z^Z(m))$  for any  $z \in \mathbb{R}$ . Therefore, by combining with the  
13 invariance under the action  $\mathbf{Q}$  from Lemma 3.6, the domain of Bufetov functional  
14 extends to the class  $\mathfrak{R}$ .  $\square$

For any  $R > 0$ , the *analytic norm* defined for all  $\mathbf{c} \in \ell^2$  as

$$\|\mathbf{c}\|_{\omega, R} = \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}|.$$

15 Let  $\Omega_R$  denote the subspace of  $\mathbf{c} \in \ell^2$  such that  $\|\mathbf{c}\|_{\omega, R}$  is finite.

**Lemma 6.6.** For  $\mathbf{c} \in \Omega_R$  and  $\mathbf{T} \in \mathbb{R}_+^d$ , the function

$$\beta_{\mathbf{c}}(\alpha, \mathbb{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}), \quad (y, z) \in \mathbb{R}^d \times \mathbb{T}$$

16 extends to a holomorphic function in the domain

$$(69) \quad D_{R, \mathbf{T}} := \{(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \mid \sum_{i=1}^d |\operatorname{Im}(y_i)| T^{(i)} + |\operatorname{Im}(z)| < \frac{R}{2\pi K}\}.$$

The following bound holds: for any  $R' < R$  there exists a constant  $C > 0$  such  
that, for all  $(y, z) \in D_{R', \mathbf{T}}$  we have

$$\begin{aligned}
&|\beta_{\mathbf{c}}(\alpha, \mathbb{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})| \\
&\leq C_{R, R'} \|\mathbf{c}\|_{\omega, R} (L + \operatorname{vol}(U(\mathbf{T}))^{1/2} (1 + E_M(a, \mathbf{T})) (1 + K \sum_{i=1}^d |\operatorname{Im}(y_i)| T^{(i)}).
\end{aligned}$$

*Proof.* By Lemma 6.5 and (66),

$$\begin{aligned} \beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}) &= e^{(z-2\pi i \sum_{j=1}^d y_j n K T^{(j)})} \beta_H(\alpha, m, \mathbf{T}) \\ &+ \sum_{k=1}^d \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (2\pi i n K y_{i_j}) e^{-2\pi i n K (\sum_{l \notin \{i_1, \dots, i_k\}} y_l T^{(l)})} \\ &\times e^{2\pi i n K z} \int_0^{T^{(i_1)}} \dots \int_0^{T^{(i_k)}} e^{-2\pi i n K (y_{i_1} s_{i_1} + \dots + y_{i_k} s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \dots, i_k, s}) ds_{i_k} \dots ds_{i_1}. \end{aligned}$$

As a consequence, by Lemma 3.7 for each variable  $(y_i, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$ , Then for the rank  $d$ -action, by induction, for  $(y, z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$  we have

$$\begin{aligned} &|\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(x), \mathbf{T})| \\ &\leq (L + \text{vol}(U(\mathbf{T}))^{1/2} (1 + E_M(a, \mathbf{T}))) \left( C_1 \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}| e^{2\pi |Im(z - \sum_{i=1}^d T^{(i)} y_i)| nK} \right. \\ &+ \sum_{k=1}^d C_k \left( \sum_{1 \leq i_1 < \dots < i_k \leq d} \prod_{j=1}^k (|Im(y_{i_j})| T^{(i_j)}) \right. \\ &\times \left. \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} n |c_{i,n}| e^{2\pi (|Im(z)| + \sum_{j=1}^k T^{(i_j)} |Im(y_{i_j})|) nK} \right) \Bigg). \end{aligned}$$

- 1 Therefore, the functional  $\beta_{\mathbf{c}}(\alpha, \mathbf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})$  is bounded by a series of holo-  
 2 morphic functions on  $\mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$  and it converges uniformly on compact subsets of  
 3 domain  $D_{R,T}$ . Thus it is holomorphic on the set.  $\square$

#### 4 7. MEASURE ESTIMATION FOR BOUNDED-TYPE

5 In this section, we prove a measure estimation of Bufetov functional with bounded-  
 6 type  $\alpha$ . This result is a generalization of §11 in [FK20b].

7 Let  $\mathcal{O}_r$  denote the space of holomorphic functions on the ball  $B_{\mathbb{C}}(0, r) \subset \mathbb{C}^n$ .

**Theorem 7.1.** [Bru99, Theorem 1.9] For any  $f \in \mathcal{O}_r$ , there is a constant  $d := d_f(r) > 0$  such that for any convex set  $D \subset B_{\mathbb{R}}(0, 1) := B_{\mathbb{C}}(0, 1) \cap \mathbb{R}^n$ , for any measurable subset  $U \subset D$

$$\sup_D |f| \leq \left( \frac{4n \text{Leb}(D)}{\text{Leb}(U)} \right)^d \sup_U |f|.$$

- 8 We say that a holomorphic function  $f$  defined in a disk is  $p$ -valent if it assumes  
 9 no value more than  $p$ -times there. We also say that  $f$  is  $\theta$ -valent if it is a constant.

**Definition 7.2.** [Bru99, Def 1.6] Let  $\mathcal{L}_t$  denote the set of one-dimensional complex affine spaces  $L \subset \mathbb{C}^n$  such that  $L \cap B_{\mathbb{C}}(0, t) \neq \emptyset$ . For  $f \in \mathcal{O}_r$ , the number

$$\nu_f(t) := \sup_{L \in \mathcal{L}_t} \{\text{valency of } f \mid L \cap B_{\mathbb{C}}(0, t) \neq \emptyset\}$$

- 10 is called the *valency* of  $f$  in  $B_{\mathbb{C}}(0, t)$ .

- 11 By Proposition 1.7 of [Bru99], for any  $f \in \mathcal{O}_r$  with finite valency  $\nu_f(t)$  for any  
 12  $t \in [1, r)$ , there is a constant  $c := c(r) > 0$  such that

$$(70) \quad d_f(r) \leq c \nu_f\left(\frac{1+r}{2}\right).$$

**Lemma 7.3.** *[FK20b, Lemma 10.3] Let  $R > r > 1$ . For any normal family  $\mathcal{F} \subset \mathcal{O}_R$ , assume that no functions in  $\overline{\mathcal{F}} = \emptyset$  are constant along a one-dimensional complex line. Then we have*

$$\sup_{f \in \mathcal{F}} \nu_f(r) < \infty.$$

**Lemma 7.4.** *Let  $L > 0$  and  $\mathcal{B} \subset DC(L)$  be a bounded subset. Given  $R > 0$ , for all  $\mathbf{c} \in \Omega_R$  and all  $\mathbf{T}^{(i)} > 0$ , denote  $\mathcal{F}(\mathbf{c}, \mathbf{T})$  by the family of real analytic functions of the variable  $y \in [0, 1]^d$  and*

$$\mathcal{F}(\mathbf{c}, \mathbf{T}) := \{\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}) \mid (\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}\}.$$

*Then there exist  $\mathbf{T}_{\mathcal{B}} := (\mathbf{T}_{\mathcal{B}}^{(i)})$  and  $\rho_{\mathcal{B}} > 0$ , such that for every  $(R, \mathbf{T})$  with  $R/\mathbf{T}^{(i)} \geq \rho_{\mathcal{B}}$ ,  $\mathbf{T}^{(i)} \geq \mathbf{T}_{\mathcal{B}}^{(i)}$  and for all  $\mathbf{c} \in \Omega_R \setminus \{0\}$ , we have*

$$(71) \quad \sup_{f \in \mathcal{F}(\mathbf{c}, \mathbf{T})} \nu_f < \infty.$$

*Proof.* Since  $\mathcal{B} \subset \mathfrak{M}$  is bounded, for each time  $t_i \in \mathbb{R}$  and  $1 \leq i \leq g$ ,

$$0 < t_{\mathcal{B}}^{\min} = \min_i \inf_{\alpha \in \mathcal{B}} t_{\text{Ret}, i, \alpha} \leq \max_i \sup_{\alpha \in \mathcal{B}} t_{\text{Ret}, i, \alpha} = t_{\mathcal{B}}^{\max} < \infty.$$

For any  $\alpha \in \mathcal{B}$  and  $x \in M$ , the map  $\Phi_{\alpha, x} : [0, 1]^d \times \mathbb{T} \rightarrow \prod_{i=1}^d [0, t_{\alpha, i}] \times \mathbb{T}$  in (53) extends to a complex analytic diffeomorphism  $\hat{\Phi}_{\alpha, x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ . By Lemma 6.6, it follows that for fixed  $z \in \mathbb{T}$ , real analytic function  $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T})$  extends to a holomorphic function defined on a region

$$H_{\alpha, m, R, t} := \{y \in \mathbb{C}^d \mid \sum_{i=1}^d |\text{Im}(y_i)| \leq h_{\alpha, m, R, t}\}.$$

By boundedness of the set  $\mathcal{B} \subset \mathfrak{M}$ , it follows that

$$\inf_{(\alpha, x) \in \mathcal{B} \times M} h_{\alpha, m, R, t} := h_{R, T} > 0.$$

We remark that the function  $h_{\alpha, m, R, t}$  and its lower bound  $h_{R, T}$  can be obtained from the formula (53) for the polynomial  $\Phi_{\alpha, x}$  and the definition of the domain  $D_{R, T}$  in the formula (69).

For every  $r > 1$ , there exists  $\rho_{\mathcal{B}} > 1$  such that for every  $(R, \mathbf{T})$  with  $R/\mathbf{T}^{(i)} > \rho_{\mathcal{B}}$ ,

$$(72) \quad \beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}) \in \mathcal{O}_r$$

as a function of  $y \in \mathbb{T}^d$ .

By Lemma 6.6, the family  $\mathcal{F}(\mathbf{c}, \mathbf{T})$  is uniformly bounded and normal. By Lemma 5.4 for the non-zero  $L^2$ -lower bound of functionals, for sufficiently large pair  $\mathbf{T}$ , no sequence from  $\mathcal{F}(\mathbf{c}, \mathbf{T})$  can converge to a constant. Therefore, by Lemma 7.3 for the family  $\mathcal{F} = \mathcal{F}(\mathbf{c}, \mathbf{T})$ , the main statement follows.  $\square$

We derive measure estimates of Bufetov functionals on the rectangular domain.

**Lemma 7.5.** *Let  $\alpha \in DC$  such that the forward orbit of  $\mathbb{R}^d$ -action  $\{r_t[\alpha]\}_{t \in \mathbb{R}_+^d}$  is contained in a compact set of  $\mathfrak{M}_g$ . There exist  $R, C, \delta > 0$  and  $T_0 \in \mathbb{R}_+^d$  such that, for every  $\mathbf{c} \in \Omega_R \setminus \{0\}$ ,  $T \geq T_0$  and for every  $\epsilon > 0$ , we have*

$$\text{vol}(\{m \in M \mid |\beta_{\mathbf{c}}(\alpha, m, \mathbf{T})| \leq \epsilon \text{vol}(U(\mathbf{T}))^{1/2}\}) \leq C\epsilon^\delta.$$

*Proof.* Since  $\alpha \in DC$  and the orbit  $\{r_t[\alpha]\}_{t \in \mathbb{R}_+}$  is contained in a compact set, there exists  $L > 0$  such that  $r_t[\alpha] \in DC(L)$  for all  $t \in \mathbb{R}_+^d$ . Then, we choose  $\mathbf{T}_0 \in \mathbb{R}^d$  and  $R > 0$  from the conclusion of Lemma 7.4. By the scaling property,

$$\beta_{\mathbf{c}}(\alpha, m, \mathbf{T}) = \left( \frac{\text{vol}(U(\mathbf{T}))}{\text{vol}(U(\mathbf{T}_0))} \right)^{1/2} \beta_{\mathbf{c}}(g_{\log(\mathbf{T}/\mathbf{T}_0)}[\alpha], m, \mathbf{T}_0).$$

By Fubini's theorem, it suffices to estimate

$$\text{Leb}(\{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\}).$$

Let  $\delta^{-1} := c(r) \sup_{f \in \mathcal{F}(\mathbf{c}, \mathbf{T}_0)} \nu_f(\frac{1+r}{2}) < \infty$  as in (70) and (71). By Lemma 5.4, we have

$$\inf_{(\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}} \sup_{y \in [0, 1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| > 0$$

so that the functional is not trivial. By Theorem 7.1 for the unit ball  $D = B_{\mathbb{R}}(0, 1)$  and setting

$$U = \{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\},$$

by the bound in (70) for  $d_f(r)$ , there exists a constant  $C > 0$  such that for all  $\epsilon > 0$  and  $(\alpha, x, z) \in \mathcal{B} \times M \times \mathbb{T}$ ,

$$\text{Leb}(\{y \in [0, 1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha, x}(\xi_{y, z}), \mathbf{T}_0)| \leq \epsilon\}) \leq C\epsilon^\delta.$$

1 Then the statement follows from the Fubini theorem.  $\square$

**Corollary 7.6.** *Let  $\alpha$  be as in the previous Lemma 7.5. There exist  $R, C, \delta > 0$  and  $\mathbf{T}_0 \in \mathbb{R}_+^d$  such that, for every  $\mathbf{c} \in \Omega_R \setminus \{0\}$ ,  $\mathbf{T} \geq \mathbf{T}_0$  and for every  $\epsilon > 0$ , we have*

$$\text{vol}\left(\{m \in M \mid |\langle \mathcal{P}_{U(\mathbf{T})}^{d, \alpha} m, \omega_{\mathbf{c}} \rangle| \leq \epsilon \text{vol}(U(\mathbf{T}))^{1/2}\}\right) \leq C\epsilon^\delta.$$

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