Homework 4

See due date on Canvas

COE 347

There is absolutely no tolerance for academic misconduct. All assigned material is to be prepared individually. Late assignments will not be accepted, unless under exceptional circumstances at the instructor's full discretion.

Submit your homework electronically as a PDF via Canvas by 11:59pm on the due date.

If you are submitting a scanned copy of your handwritten notes, rather than a typeset document, please take the time to reduce the file to a manageable size by adjusting the resolution.

Objectives

The objective of this homework is to apply the theory of finite difference formulas to the solution of the Poisson equation in one dimension.

Instructions

Consider the following ODE discussed in class

$$u''(x) = f(x) \tag{1}$$

in $x \in [0, 1]$ where $f(x) = (4\pi)^2 \cos(4\pi x)$. Together with appropriate boundary conditions, the ODE above constitutes a boundary value problem, for which you will find numerical solutions.

In all that follows and in your solutions, call N the number of interior grid points, so that $h = 1/(N + 1)$ is the grid spacing and the interior coordinates are $x_i = ih$ with $i = 1, \ldots, N$. Then, u_i is the numerical solution at $x = x_i$, which is an approximation to $\tilde{u}(x_i)$, the exact solution at the same x coordinate. Also, call $x_0 = 0$ and $x_{N+1} = 1$ the two boundaries and $u(0) = u_0$ and $u(1) = u_{N+1}$ the solution at these two locations.

1. **20 pts.** Consider the second order, centered approximation to the second order derivative

   $$u''(x_i)h^2 = u_{i+1} - 2u_i + u_{i-1} + Ch^4 u^{(4)}(x_i) + h.o.t. \tag{2}$$

   where $u^{(4)}(x) = d^4u/dx^4$ is the fourth-order derivative of u. Find the value of the constant C by the method of Taylor's expansion.

   Turn in your derivations and a clearly identified answer to the question about the value of C.

2. **20 pts.** Implement a computer code that solves the ODE in Eq. (1) with boundary conditions $u(0) = 0$ and $u(1) = 2$ with the second order FDF as approximation to $u''(x)$ as in Eq. (2). Obtain a numerical solution with $N = 10$. This is the same case that we discussed in class. You can check your solution against the "exact solution" provided on Canvas in the file "solutionA_N10000.dat" for $N = 10^4$.

Turn in your computer code, the solution in tabular form (one column is x_i and the other is u_i and N rows for $i = 1, \dots, N = 10$), and a plot of the numerical solution together with the "exact solution" that you downloaded from Canvas.

3. **20 pts.** Now obtain a solution with $N = 10^4$, which you will use as if it were the "exact solution". Make sure it matches closely the solution inside the file "solutionA_N10000.dat" that you downloaded from Canvas.

   Define two errors, E and e as follows

$$E = \left[\sum_{i=1}^{N} (u_i - \tilde{u}_i)^2 \right]^{1/2} \tag{3}$$

$$e = \frac{1}{N} \left[\sum_{i=1}^{N} (u_i - \tilde{u}_i)^2 \right]^{1/2} \tag{4}$$

   Compute the errors for $N = \{5, 10, 20, 40, 80, 160, 320, 640, 1280\}$ and the corresponding values of h. Produce a log-log plot that shows E and e versus $1/h$. Fit a function of the form Ch^α and report the value of α for both E and e. What can you conclude about the order of the errors E and e with respect to $h = (N+1)^{-1}$?

   Turn in the one log-log plot, the values of α for E and e, and a short sentence that presents your conclusions.

   Note. In the expressions above, you will have to be careful on how you calculate $\tilde{u}_i$, which is the exact solution at x_i, since all you have is the solution on a very large number of points ($N = 10^4$). So how do you do that? I suggest you interpolate the "exact" solution with $N = 10^4$ onto the x_i location where it's needed to evaluate the errors. However, pay attention that the interpolation error is not larger than the solution error that you are trying to quantify. In order to do so, you need to use an interpolant of order of accuracy higher than 2, for example MATLAB's `interp1(X,Y,'pchip')`.

4. **20 pts.** Using Taylor's expansion method, derive a one-sided second order finite difference formula that approximates $u'(x)$ as

$$u'(x_i)h = au_i + bu_{i+1} + cu_{i+2} + \mathcal{O}(h^3), \tag{5}$$

   by finding the values of the coefficients a, b, and c.

   Turn in your derivations and a clearly identified answer to the question about the values of the coefficients a, b, and c.

5. **20 pts.** Implement a computer code that solves the ODE in Eq. (1) with boundary conditions $u'(0) = 10$ and $u(1) = 2$ with the second order FDF as approximation to $u''(x)$ as in Eq. (2). You will need to use the one sided finite difference formula in Eq. (5) as an approximation of u' at $x = 0$

   Obtain a numerical solution with $N = 10$. Make sure it matches closely the solution inside the file "solutionB_N10000.dat" that you downloaded from Canvas.

   Turn in your computer code, the solution in tabular form (one column is x_i and the other is u_i and N rows for $i = 1, \dots, N = 10$), and a plot of the numerical solution together with the "exact solution" that you downloaded from Canvas.

6. **Extra credit (20 pts).** Consider $N = \{5, 10, 20, 40, 80, 160, 320, 640, 1280\}$ and produce errors (like the ones that you produced above in 3.) for the numerical solution to the ODE with the boundary conditions $u'(0) = 10$ and $u(1) = 2$.

   Now change your one-sided finite difference approximation at $x = 0$ to a first-order formula, such as the forward derivative and recompute the errors again.

   Produce a log-log plot of the errors. Fit a function of the form Ch^{α} and report the value of α for both E and e and both one-sided derivatives. What can you conclude about the order of the errors E and e with respect to $h = (N+1)^{-1}$?

   What can you conclude?

   Turn in the one log-log plot for the set of four errors. I.e. show E and e versus $1/h$ with the second order one-sided derivative and with the first order one-sided derivative at $x = 0$. Turn in the values of the coefficient α for each of the four errors. Turn in a short explanation of the results with your observations.