Numerical Methods for Scalar-Advective Equation

Author: Jiwoong 'Alex' Choi
Date: March 25, 2025

Introduction

In this report, we will apply various numerical methods to solve the **scalar advective equation**:

$$u_t + cu_x = 0.$$

For simplicity, we set the wave speed $c = 1$ and assume the initial condition:

$$u_0(x) = u(x, 0) = \sin(x).$$

We solve this **initial value problem (IVP)** in the domain $x \in [0, 2\pi]$ with **periodic boundary conditions**:

$$u(0, t) = u(2\pi, t).$$

Analytical Solution

The exact solution to this equation is given by:

$$u(x, t) = \sin(x - t).$$

Since the problem is periodic, at integer multiples of $t = 2\pi m$ (where m is a positive integer), the solution returns to its initial condition:

$$u(x, 2\pi m) = \sin(x - 2\pi m) = \sin(x).$$

Computational Setup

We discretize the spatial domain using N **interior grid points**, with a **grid spacing**:

$$h = \frac{2\pi}{N+1}.$$

The grid points are given by:

$$x_i = ih, \quad i = 1, \dots, N.$$

We also define the numerical solution at these points:

$$U_i^n,$$

which represents the approximation of $u(x_i, t_n)$ at **time step** $t_n = nk$, where k is the time step size and n is the discrete time index.

This notebook will explore **different numerical methods** for solving the problem and compare their performance with the analytical solution.

Question 1

1a

$$\textbf{Recall: Implicit Central Method.}$$

$$U_i^{n+1} = U_i^n - \frac{\sigma}{2}\left(U_{i+1}^{n+1} - U_{i-1}^{n+1}\right) \quad (1)$$

Note: $x_1 = 0$ and $x_5 = 2\pi$ suggests $U_0 = U_4$ and $U_6 = U_2$

$$U_1^{n+1} = U_1^n - \frac{\sigma}{2}\left(U_2^{n+1} - U_0^{n+1}\right)$$

$$= U_1^n - \frac{\sigma}{2}\left(U_2^{n+1} - U_4^{n+1}\right) \quad (2)$$

$$U_2^{n+1} = U_2^n - \frac{\sigma}{2}\left(U_3^{n+1} - U_1^{n+1}\right) \quad (3)$$

$$U_3^{n+1} = U_3^n - \frac{\sigma}{2}\left(U_4^{n+1} - U_2^{n+1}\right) \quad (4)$$

$$U_4^{n+1} = U_4^n - \frac{\sigma}{2}\left(U_5^{n+1} - U_3^{n+1}\right) \quad (5)$$

$$U_5^{n+1} = U_5^n - \frac{\sigma}{2}\left(U_6^{n+1} - U_4^{n+1}\right)$$

$$= U_5^n - \frac{\sigma}{2}\left(U_2^{n+1} - U_4^{n+1}\right) \quad (6)$$

$(3), (4), (5)$ are not affected by Periodic Boundary Conditions.

(2) had U_0^{n+1}, which we replaced with U_4^{n+1}.

(6) had U_6^{n+1}, which we replaced with U_2^{n+1}.

Turning these into algebraic form:

i) Move everything to LHS except U_i^n

$$U_1^{n+1} + \frac{\sigma}{2}U_2^{n+1} - \frac{\sigma}{2}U_4^{n+1} = U_1^n$$

$$-\frac{\sigma}{2}U_1^{n+1} + U_2^{n+1} + \frac{\sigma}{2}U_3^{n+1} = U_2^n$$

$$-\frac{\sigma}{2}U_2^{n+1} + U_3^{n+1} + \frac{\sigma}{2}U_4^{n+1} = U_3^n$$

$$-\frac{\sigma}{2}U_3^{n+1} + U_4^{n+1} + \frac{\sigma}{2}U_5^{n+1} = U_4^n$$

$$\frac{\sigma}{2}U_2^{n+1} - \frac{\sigma}{2}U_4^{n+1} + U_5^{n+1} = U_5^n$$

ii) Multiply both sides by 2

$$2U_1^{n+1} + \sigma U_2^{n+1} - \sigma U_4^{n+1} = 2U_1^n$$

$$-\sigma U_1^{n+1} + 2U_2^{n+1} + \sigma U_3^{n+1} = 2U_2^n$$

$$-\sigma U_2^{n+1} + 2U_3^{n+1} + \sigma U_4^{n+1} = 2U_3^n$$

$$-\sigma U_3^{n+1} + 2U_4^{n+1} + \sigma U_5^{n+1} = 2U_4^n$$

$$\sigma U_2^{n+1} - \sigma U_4^{n+1} + 2U_5^{n+1} = 2U_5^n$$

iii) Turn them into matrix-vector form.

$$
\begin{pmatrix} 2 & \sigma & 0 & -\sigma & 0 \\ -\sigma & 2 & \sigma & 0 & 0 \\ 0 & -\sigma & 2 & \sigma & 0 \\ 0 & 0 & -\sigma & 2 & \sigma \\ 0 & \sigma & 0 & -\sigma & 2 \end{pmatrix} \begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \end{pmatrix} = \begin{pmatrix} 2U_1^n \\ 2U_2^n \\ 2U_3^n \\ 2U_4^n \\ 2U_5^n \end{pmatrix}.
$$

1b

Five Equations of Lax-Wendroff Method

$$
\begin{aligned}
U_1^{n+1} &= U_1^n - \frac{\sigma}{2}(U_2^n - U_0^n) + \frac{\sigma^2}{2}(U_2^n - 2U_1^n + U_0^n) \\
&= U_1^n - \frac{\sigma}{2}(U_2^n - U_4^n) + \frac{\sigma^2}{2}(U_2^n - 2U_1^n + U_4^n) \quad (9) \\
U_2^{n+1} &= U_2^n - \frac{\sigma}{2}(U_3^n - U_1^n) + \frac{\sigma^2}{2}(U_3^n - 2U_2^n + U_1^n) \quad (10) \\
U_3^{n+1} &= U_3^n - \frac{\sigma}{2}(U_4^n - U_2^n) + \frac{\sigma^2}{2}(U_4^n - 2U_3^n + U_2^n) \quad (11) \\
U_4^{n+1} &= U_4^n - \frac{\sigma}{2}(U_5^n - U_3^n) + \frac{\sigma^2}{2}(U_5^n - 2U_4^n + U_3^n) \quad (12) \\
U_5^{n+1} &= U_5^n - \frac{\sigma}{2}(U_6^n - U_4^n) + \frac{\sigma^2}{2}(U_6^n - 2U_5^n + U_4^n) \\
&= U_5^n - \frac{\sigma}{2}(U_2^n - U_4^n) + \frac{\sigma^2}{2}(U_2^n - 2U_5^n + U_4^n) \quad (13)
\end{aligned}
$$

Local Truncation Error using Taylor Expansion for Lax-Wendroff method.

(i) Move every term to LHS in (8) and replace U with u

$$
u_i^{n+1} - u_i^n + \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right) - \frac{\sigma^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \neq 0
$$

(ii) Divide by k as LTE is w.r.t. per time-step

$$
\frac{1}{k}\left(u_i^{n+1} - u_i^n\right) + \frac{c}{2h}\left(u_{i+1}^n - u_{i-1}^n\right) - \frac{c^2 k}{2h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) = L_h
$$

(iii) Taylor Expand $u(x,t)$ to $u(x,t+k)$

$$
u_i^{n+1} = u_i^n + u_t k + u_{tt}\frac{k^2}{2} + u_{ttt}\frac{k^3}{6} + u_{tttt}\frac{k^4}{24} + \mathcal{O}(k^5)
$$

(iv) Taylor Expand $u(x,t)$ to $u(x-h,t)$

$$
u_{i-1}^n = u_i^n - u_x h + u_{xx}\frac{h^2}{2} - u_{xxx}\frac{h^3}{6} + u_{xxxx}\frac{h^4}{24} + \mathcal{O}(h^5)
$$

(v) Taylor Expand $u(x,t)$ to $u(x+h,t)$

$$
u_{i+1}^n = u_i^n + u_x h + u_{xx}\frac{h^2}{2} + u_{xxx}\frac{h^3}{6} + u_{xxxx}\frac{h^4}{24} + \mathcal{O}(h^5)
$$

(vi) Substituting Time Derivatives with Spatial Derivatives

$$u_t + Cu_x = 0 \Rightarrow u_t = -Cu_x$$

$$u_{tt} = -Cu_{xt} = -Cu_{tx} = -C(-Cu_{xx}) = C^2 u_{xx}$$

$$u_{ttt} = C^2 u_{xxt} = c^2 u_{txx} = C^2(-Cu_{xxx}) = -C^3 u_{xxx}$$

$$u_{tttt} = -C^3 u_{xxxt} = -C^3 u_{txxx} - C^3(-Cu_{xxxx}) = C^4 u_{xxxx}$$

Expanding the Finite Difference Terms To Make Our Life Easier Later:

$$u_{i+1}^n - u_{i-1}^n = 2u_x h + 2u_{xxx}\frac{h^3}{6}$$

$$u_{i+1}^n + u_{i-1}^n = 2u_i^n + 2u_{xx}\frac{h^2}{2} + 2u_{xxxx}\frac{h^4}{24}$$

Substituting into the LTE Equation:

$$L_h = \frac{1}{k}\left(u_t + C^2 u_{xx}\frac{k^2}{2} - C^3 u_{xxx}\frac{k^3}{6} + C^4 u_{xxxx}\frac{k^4}{24}\right)$$

$$+\frac{C}{2h}\left(2u_x h + 2u_{xxx}\frac{h^3}{6}\right) - \frac{C^2 k}{2h^2}\left(2u_i^n + 2u_{xx}\frac{h^2}{2} + 2u_{xxxx}\frac{h^4}{24} - 2u_i^n\right)$$

Simplifying:

$$L_h = u_t + \frac{C^2 k}{2}u_{xx} - \frac{C^3 k^2}{6}u_{xxx} + \frac{C^4 k^3}{24}u_{xxxx}$$

$$+Cu_x + \frac{Ch^2}{6}u_{xxx} - \frac{C^2 k}{2}u_{xx} - \frac{C^2 k h^2}{24}u_{xxxx}$$

Final Grouping:

$$L_h = \frac{u_{xxx}}{6}(Ch^2 - C^3 k^2) + \frac{u_{xxxx}}{24}(C^4 k^3 - C^2 h^2 k)$$

$$L_h = \frac{h^2 h}{6k}\sigma(1-\sigma^2)u_{xxx} + \frac{h^3 h}{24k}\sigma^2(\sigma^2 - 1)$$

$$L_h = -\frac{h^2 h}{6k}\sigma(1-\sigma^2)u_{xxx} + \frac{h^3 h}{24k}\sigma^2(1-\sigma^2)$$

From our final form, we can see that the **order of LW method is *second*** as our leading error term is associated with u_{xxx} and it has coefficients $-\frac{h^2 h}{6k}$. However, $\frac{h}{k}$ is a constant, thus what determines our order as $h \to 0$ is h^2 value. **Hence, this is a second order method.**

Question 2

Overview of the `wave_solver.py` Script

The `wave_solver.py` script is responsible for numerically solving the **scalar advection equation** using four different numerical methods. Each method applies a different finite-difference scheme to approximate the solution over time. **The solver is attached at the very end of this PDF**.

Implemented Numerical Methods

The script supports the following four schemes:

1. **First-Order Upwind (`forward-upwind`)** – A first-order explicit method that introduces numerical diffusion, making it stable for certain Courant numbers but less accurate.
2. **Implicit-Central (`implicit-central`)** – A second-order implicit method that forms a system of linear equations to solve for the next time step, providing better stability but requiring matrix inversion.
3. **Beam-Warming (`beam-warming`)** – A second-order explicit upwind scheme that reduces numerical diffusion compared to the first-order upwind method.
4. **Lax-Wendroff (`lax-wendroff`)** – A second-order explicit scheme that introduces dispersion effects but provides higher accuracy in smooth regions.

Numerical Approach

- **For the implicit method (`implicit-central`)**, we reformulate the problem as a system of **linear equations** and solve for the solution at the next time step using the solver.
- **For explicit methods (`forward-upwind` , `beam-warming` , and `lax-wendroff`)**, we construct a **dU matrix** (representing the difference between the next time step and the current time step) and directly update the solution using matrix-vector multiplication.

Detailed Explanation of Each Method

First-Order Upwind (FOU) Scheme

The original update equation:

$$U_i^{n+1} = U_i^n - \sigma(U_i^n - U_{i-1}^n)$$

Delta Form

Rearrange to define ΔU_i:

$$\Delta U_i = U_i^{n+1} - U_i^n = -\sigma U_i^n + \sigma U_{i-1}^n, \quad i = 0, \dots, N+1$$

For $i = 0, \dots, N+1$

For $i = 0$, we use periodic boundary condition to change $i = -1$ to $i = -1 + N + 1 = N$:

$$\Delta U_0 = -\sigma U_0^n + \sigma U_{-1}^n = -\sigma U_0^n + \sigma U_N^n$$

$$\Delta U_1 = -\sigma U_1^n + \sigma U_0^n$$

$$\Delta U_2 = -\sigma U_2^n + \sigma U_1^n$$

$$\vdots$$

$$\Delta U_N = -\sigma U_N^n + \sigma U_{N-1}^n$$

$$\Delta U_{N+1} = -\sigma U_{N+1}^n + \sigma U_N^n$$

Matrix-Vector Form

Expressing in matrix notation:

$$\mathbf{\Delta U_i} = A\vec{\mathbf{U}}_i^n$$

where:

$$\begin{bmatrix} -\sigma & 0 & 0 & \dots & \sigma & 0 \\ \sigma & -\sigma & 0 & \dots & 0 & 0 \\ 0 & \sigma & -\sigma & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma & -\sigma \end{bmatrix} \begin{bmatrix} U_0^n \\ U_1^n \\ U_2^n \\ \vdots \\ U_N^n \\ U_{N+1}^n \end{bmatrix}$$

Solving

At each time step, we just continously add ΔU_i to $\vec{\mathbf{U}}_i^n$ to solve for $\vec{\mathbf{U}}_i^{n+1}$

Implicit-Center Scheme

The original Finite Difference Equation (FDE):

$$U_i^{n+1} = U_i^n - \frac{\sigma}{2}(U_{i+1}^{n+1} - U_{i-1}^{n+1})$$

System of Equations

Since this is an implicit method, we must solve a system of equations to compute $\vec{\mathbf{U}}^{n+1}$.

$$U_i^{n+1} + \frac{\sigma}{2}U_{i+1}^{n+1} - \frac{\sigma}{2}U_{i-1}^{n+1} = U_i^n, \quad i = 0, \dots, N+1$$

Rearranging the equation:

$$-\sigma U_{i-1}^{n+1} + 2U_i^{n+1} + \sigma U_{i+1}^{n+1} = 2U_i^n$$

For $i = 0, \dots, N+1$

For $i = 0$, we use periodic boundary condition to change $i = -1$ to $i = -1 + N + 1 = N$:

$$\begin{gathered} -\sigma U_{-1}^{n+1} + 2U_0^{n+1} + \sigma U_1^{n+1} = -\sigma U_N^{n+1} + 2U_0^{n+1} + \sigma U_1^{n+1} = 2U_0^n \\ -\sigma U_0^{n+1} + 2U_1^{n+1} + \sigma U_2^{n+1} = 2U_1^n \\ -\sigma U_1^{n+1} + 2U_2^{n+1} + \sigma U_3^{n+1} = 2U_2^n \\ \vdots \\ -\sigma U_{N-1}^{n+1} + 2U_N^{n+1} + \sigma U_{N+1}^{n+1} = 2U_N^n \end{gathered}$$

For $i = N+1$, we use periodic boundary condition to change $i = N+2$ to $i = 1$:

$$-\sigma U_N^{n+1} + 2U_{N+1}^{n+1} + \sigma U_{N+2}^{n+1} = -\sigma U_N^{n+1} + 2U_{N+1}^{n+1} + \sigma U_1^{n+1} = 2U_{N+1}^n$$

Matrix-Vector Form

Expressing in matrix notation:

$$\mathbf{A}\vec{\mathbf{U}}_i^{n+1} = \vec{\mathbf{U}}_i^n$$

where:

$$\begin{bmatrix} 2 & \sigma & 0 & \dots & -\sigma & 0 \\ -\sigma & 2 & \sigma & \dots & 0 & 0 \\ 0 & -\sigma & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & \sigma \\ 0 & \sigma & 0 & \dots & -\sigma & 2 \end{bmatrix} \begin{bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_N^{n+1} \\ U_{N+1}^{n+1} \end{bmatrix} = \begin{bmatrix} 2U_0^n \\ 2U_1^n \\ 2U_2^n \\ \vdots \\ 2U_N^n \\ 2U_{N+1}^n \end{bmatrix}$$

Solving

At each time step, we solve for $\vec{\mathbf{U}}_i^{n+1}$.

Question 3

3a

To analyze the accuracy of different numerical methods for solving the PDE, we will implement the **First-Order Upwind, Implicit-Central, Beam-Warming, and Lax-Wendroff** schemes. The solutions will be computed for Courant numbers $\sigma = \frac{ck}{h} = 0.25, 0.5, 0.75, 1.25$ and evolved up to $T = 4\pi$

For each Courant number, we will generate a separate figure displaying the numerical solutions obtained from all four methods alongside the exact solution at $T = 4\pi$. This will allow for a direct visual comparison of their accuracy.

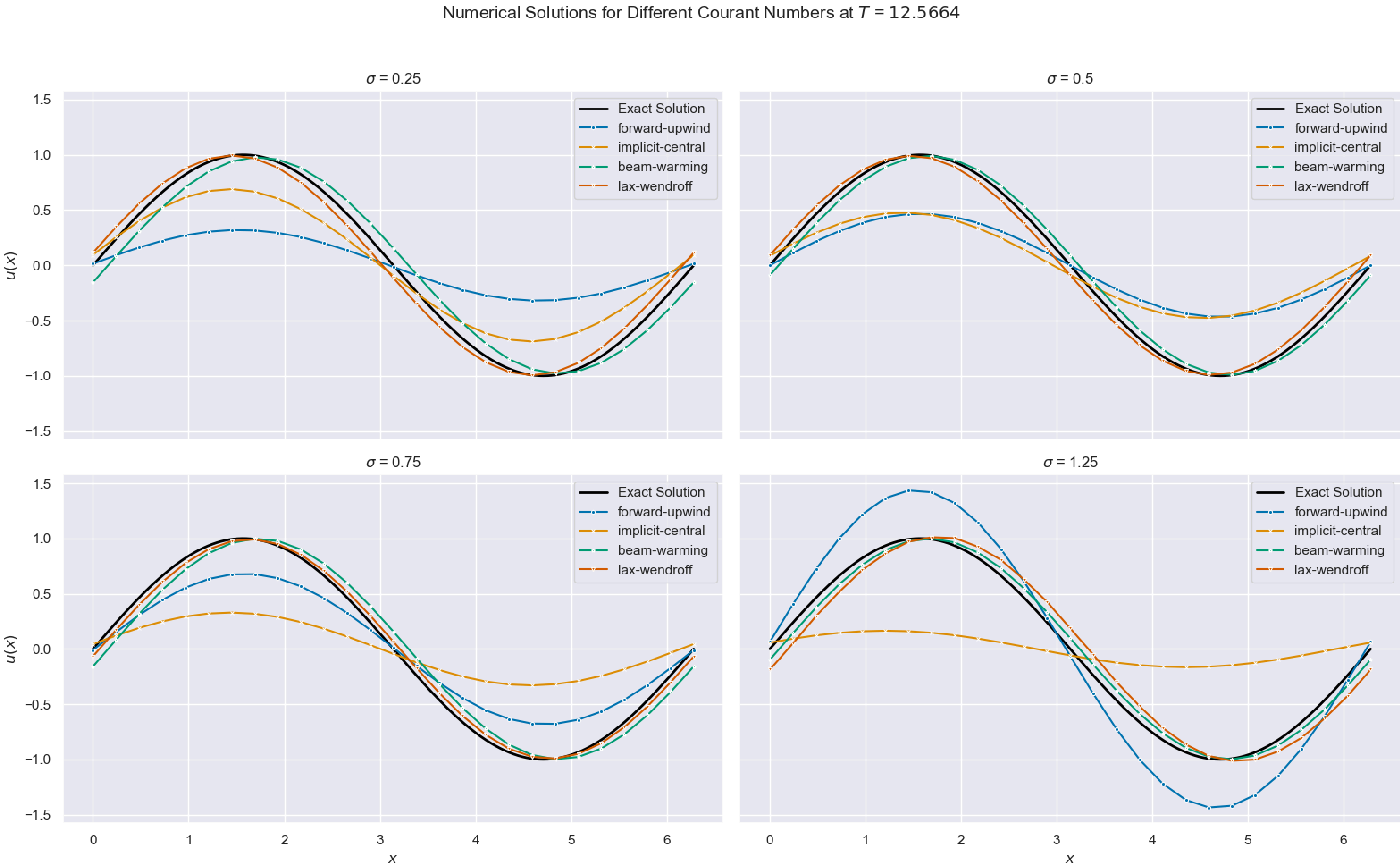
We will then compare and analyze the results by considering the effects of **dissipation errors** (which reduce amplitude) and **dispersion errors** (which cause phase shifts). By examining these numerical artifacts, we will explain how each method performs under different Courant numbers and discuss their strengths and weaknesses.

Plotting Each Numerical Methods Against the Exact Solution for Varying Courant Numbers

In [48]:

```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
from wave_solver import wave_solve

# Define Courant numbers and numerical methods
Courant_nums = [0.25, 0.5, 0.75, 1.25]
methods = ["forward-upwind", "implicit-central", "beam-warming", "lax-wendroff"]
markers = ['o', 'x', '*', 'v']
T = 4 * np.pi  # Final time
n = 25  # Number of interior points
advec_speed = 1  # advective speed
domain = 2 * np.pi  # Spatial domain

# Set up Seaborn with high-contrast color palette
sns.set_theme(style="darkgrid")
palette = sns.color_palette("colorblind", n_colors=len(methods))

# Set up figure with 2x2 subplots
fig, axes = plt.subplots(2, 2, figsize=(16, 10), sharex=True, sharey=True)
fig.suptitle(f"Numerical Solutions for Different Courant Numbers at $T$ = ${T:.4f}$", fontsize=14)

# Fine spatial grid for exact solution
x_exact = np.linspace(0, domain, 2000)  # Very fine grid for smooth exact solution
u_exact = np.sin(x_exact - advec_speed * T)  # Recall exact general sol: u0(x - ct) = u(x,t)

# Iterate over Courant numbers
```

```python
for i, Courant_num in enumerate(Courant_nums):
    ax = axes[i // 2, i % 2]  # Select appropriate subplot
    # Plot exact solution in black
    sns.lineplot(x=x_exact, y=u_exact, linestyle="-", linewidth=2, color = 'black', label="Exact Solution", ax=ax)

    # Solve and plot for each numerical method using Seaborn
    for j, method in enumerate(methods):
        # compute sol
        out = wave_solve(c = advec_speed, L = domain, n = n, Courant = Courant_num, T = T, M = 0, u0 = lambda x: np.sin(x), method = method)
        # plot it
        sns.lineplot(x=out['x'], y=out['U'][:, -1], linewidth=1.5, marker=markers[j], markersize=3, color=palette[j], label=method, ax=ax)

    # Formatting
    ax.set_title(f"$\sigma$ = {Courant_num}")
    ax.set_xlabel("$x$")
    ax.set_ylabel("$u(x)$")
    ax.legend()
    ax.grid(True)

# Adjust layout and show the plot
plt.tight_layout(rect=[0, 0, 1, 0.96])
plt.show()
```



Analysis of Accuracies

For all Courant numbers, as expected, the higher-order $O(h^2)$ **methods** (**Beam-Warming (BW) and Lax-Wendroff (LW)**) align much more closely with the exact solution compared to the **first-order** $O(h^1)$ **methods** (**Forward-Upwind (FW) and Implicit-Central (IC)**). The amplitude errors for BW and LW are minimal and nearly indistinguishable, whereas their phase errors are present but relatively small. Forward-Upwind and Implicit-Central, however, exhibits severe dissipation, completely damping out higher-frequency components.

As expected, **FW and IC methods exhibit dominant amplitude errors** across all Courant numbers. This is consistent with their dissipative leading error term (u_{xx}) in their **Local Truncation Error (LTE)** expressions:

For **Forward-Upwind (FOU):**

$$L_h(x,t) = \frac{h^2}{2k}\sigma(1-\sigma)u_{xx} - \frac{h^3}{6k}\sigma(1-\sigma^2)u_{xxx} + \frac{h^4}{24k}\sigma(1-\sigma^3)u_{xxxx}$$

For **Implicit-Central (IC):**

$$L_h(x,t) = \frac{h^2}{2k}\sigma^2 u_{xx} - \frac{h^3}{6k}\sigma(2\sigma^2+1)u_{xxx} + \frac{h^4}{24k}\sigma^2(3\sigma^2+4)u_{xxxx}$$

Since both methods contain **leading-order dissipative error (u_{xx} term)**, they suppress or increase wave amplitudes, leading to **dissipation** and **energy loss** in the numerical solution.

In contrast, **BW and LW methods exhibit stronger phase errors**, as their **leading error term (u_{xxx}) is dispersive rather than dissipative**:

For **Beam-Warming (BW):**

$$L_h(x,t) = \frac{h^2}{6k}\sigma(1-\sigma)(2-\sigma)u_{xxx} - \frac{h^3}{24k}\sigma(1-\sigma)(2-\sigma)(3+\sigma)u_{xxxx}$$

For **Lax-Wendroff (LW):**

$$L_h(x,t) = -\frac{h^2}{6k}\sigma(1-\sigma^2)u_{xxx} + \frac{h^3}{24k}\sigma^2(1-\sigma^2)u_{xxxx}$$

Since **BW and LW methods primarily contain dispersive errors (u_{xxx} term)**, they shift wave peaks and introduce **phase errors** while better preserving amplitude.

Effect of Courant Number on Stability and Accuracy

For all methods, **Courant numbers of $\sigma = 0.25, 0.5, 0.75$ produce stable solutions** without explosive growth. However, key differences emerge:

- **At $\sigma = 0.75$**, solutions show **higher accuracy**, particularly for **higher-order methods**.
- **At $\sigma = 0.5$**, numerical solutions are more consistent between methods of the same order, making it a good choice for balanced accuracy and stability.
- **At $\sigma = 1.25$**, a **small numerical explosion is observed in the FOU scheme**, which aligns with its **stability constraint of $0 \leq \sigma \leq 1$**. This instability occurs because FOU is only **conditionally stable** under this requirement. Implicit-Central exhibits significant disspiation. Meanwhile, Beam-Warming and Lax-Wendroff remain stable.

3b: Numerical Analysis at $\sigma = 1$

In this section, we investigate the behavior of our numerical methods when the **Courant number is set to $\sigma = 1$**, a special case in numerical advection schemes. Using the **First-Order Upwind, Implicit-Central, Beam-Warming, and Lax-Wendroff** methods, we integrate the PDE up to $T = 4\pi$ and compare the results against the exact solution.

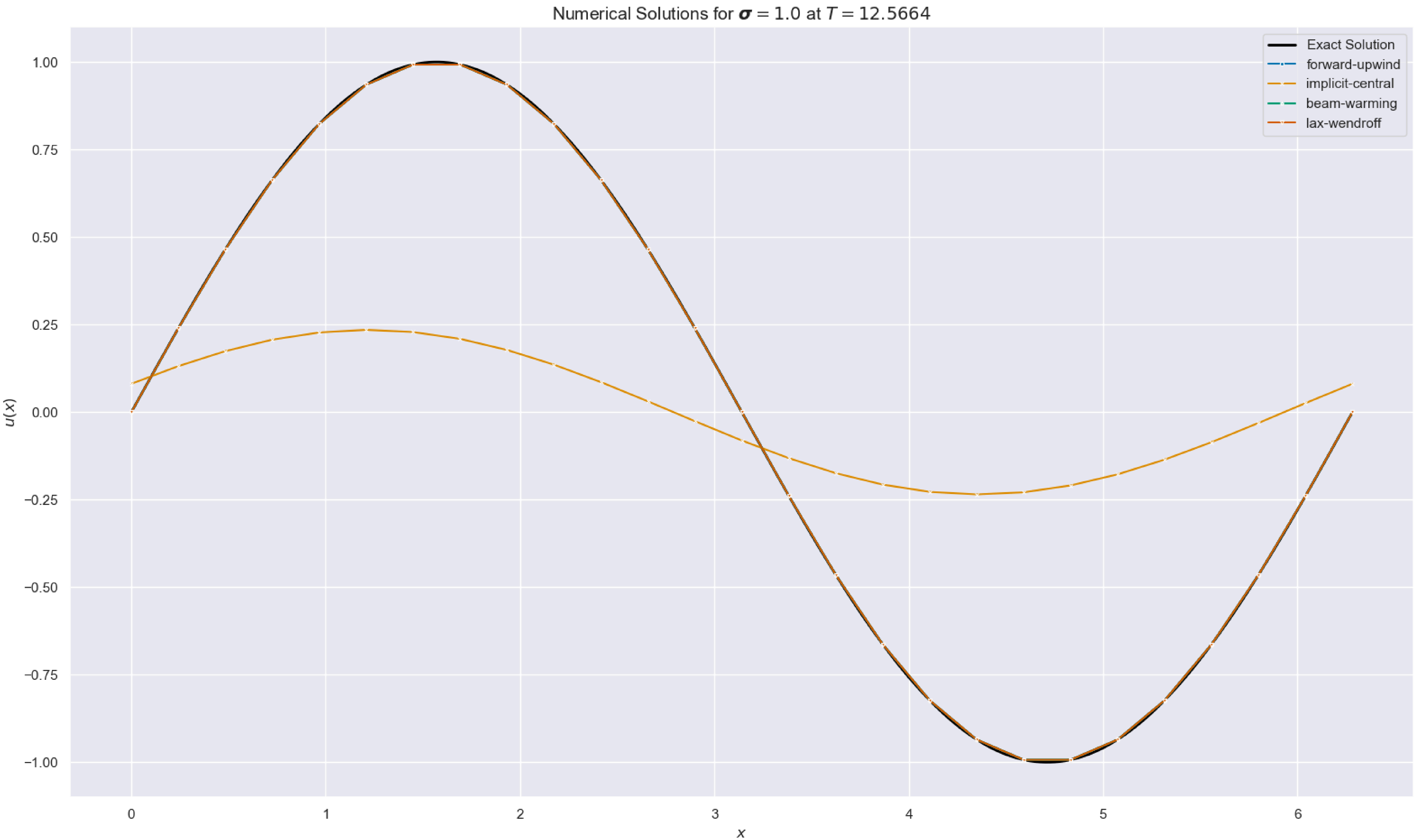
We will generate a **single figure** displaying the numerical solutions from all four methods at $T = 4\pi$ to examine their accuracy and stability at this particular Courant number.

Our analysis will:

- **Compare numerical accuracy** at $\sigma = 1$ relative to previous results at other Courant numbers.
- **Identify dominant numerical errors** (dissipation vs. dispersion) and discuss their effects.
- **Explain stability behavior**, particularly why some methods may be more or less accurate at this special Courant number.
- **Discuss why $\sigma = 1$ is sometimes called the "golden ratio"** in numerical advection schemes.

In [49]:
```
# Set up a figure
plt.figure(figsize=(16, 10))

# Fine spatial grid for exact solution
x_exact = np.linspace(0, domain, 2000)  # Very fine grid for smooth exact solution
u_exact = np.sin(x_exact - advec_speed * T)  # Recall exact general sol: u0(x - ct) = u(x,t)
# plot the exact sol
sns.lineplot(x=x_exact, y=u_exact, linewidth = 2.0, color = 'black', label = 'Exact Solution')

# Iterate over Courant numbers
for i, method in enumerate(methods):
    # compute sol for Courant = 1
```

```
    out = wave_solve(c = advec_speed, L = domain, n = n, Courant = 1.0, T = T, M = 0, u0 = lambda x: np.sin(x), method = method)
    # plot it
    sns.lineplot(x=out['x'], y=out['U'][:, -1], linewidth=1.5, marker=markers[i], markersize=3, color=palette[i], label=method)

# Formatting
plt.title(r"Numerical Solutions for $\boldsymbol{\sigma} = 1.0$ at $T = %.4f$" % T, fontsize=14)
plt.xlabel("$x$")
plt.ylabel("$u(x)$")
plt.legend()
plt.grid(True)
# Adjust layout and show the plot
plt.tight_layout(rect=[0, 0, 1, 0.96])
plt.show()
```



Analysis of Accuracies

From the plot above, we can understand why $\sigma = 1.0$ is considered *golden raito* in numerical schemes for PDEs. Except the implicit-central method, all other numerical methods follow the exact solution extremely closely with almost zero amplitutde and phase errors, almost being indistinguishable. Even the first-order-upwind scheme is stable and very accurate, showing no signs of noticeable amplitude and phase error, unlike from 3a. Similalry, at $\sigma = 1.0$, the noticeable phase errors for BW and LW methods are not visible here.

Explanation of Golden Ratio through LTE expressions.

Let us plug in $\sigma = 1$ to all of our local truncation error expressions and see how we get this extreme accuracy and why IC scheme still has significant amplitude error.

For **Forward-Upwind (FOU):**

$$L_h(x,t) = \frac{h^2}{2k}\sigma(1-\sigma)u_{xx} - \frac{h^3}{6k}\sigma(1-\sigma^2)u_{xxx} + \frac{h^4}{24k}\sigma(1-\sigma^3)u_{xxxx} = \frac{h^2}{2k}(0)u_{xx} - \frac{h^3}{6k}(0)u_{xxx} + \frac{h^4}{24k}(0)u_{xxxx} = \mathbf{0}$$

As you can see, according to our expression, there are no errors when we have Courant number of 1, showing neither disspative nor dispersion error.

For **Implicit-Central (IC):**

$$L_h(x,t) = \frac{h^2}{2k}\sigma^2 u_{xx} - \frac{h^3}{6k}\sigma(2\sigma^2+1)u_{xxx} + \frac{h^4}{24k}\sigma^2(3\sigma^2+4)u_{xxxx} = \frac{h^2}{2k}u_{xx} - \frac{h^3}{6k}(3)u_{xxx} + \frac{h^4}{24k}(7)u_{xxxx} \neq \mathbf{0}$$

Aligning with our plot, IC scheme still exhibits non-zero local truncation error value when $\sigma = 1$.

For **Beam-Warming (BW):**

$$L_h(x,t) = \frac{h^2}{6k}\sigma(1-\sigma)(2-\sigma)u_{xxx} - \frac{h^3}{24k}\sigma(1-\sigma)(2-\sigma)(3+\sigma)u_{xxxx} = \frac{h^2}{6k}(0)u_{xxx} - \frac{h^3}{24k}(0)(4)u_{xxxx} = \mathbf{0}$$

Similr to FOU scheme, our expression leas to the conclusion that BW methos' local truncation error is **zero** due to zero contributions from both amplitude and phase error terms.

For **Lax-Wendroff (LW):**

$$L_h(x,t) = -\frac{h^2}{6k}\sigma(1-\sigma^2)u_{xxx} + \frac{h^3}{24k}\sigma^2(1-\sigma^2)u_{xxxx} = -\frac{h^2}{6k}(0)u_{xxx} + \frac{h^3}{24k}(0)u_{xxxx} = \mathbf{0}$$

Again, for LW scheme, error is zero.

Thus, at $\sigma = 1$, the FOU, BW, and LW methods exhibit near-perfect accuracy due to the cancellation of leading-order error terms, while IC still suffers from dissipation.

Wave Solver Function Script

In [50]:
```
def wave_solve(c: float, L:float, n:int, Courant:float, T:float, M:int, u0, method:str) -> dict:
    """
    Parameters:
        c : float
            Advective speed (assumed positive).
        L : float
            Domain size [0, L].
        n : int
            Number of interior grid points.
        Courant : float
            Courant number.
        T : float
            Final time.
        M : int
            Number of solution snapshots between [0, T]
        u0 : callable
```

```python
            Initial condition function: u0(x_array) -> array of same length.
        method : str
            One of {'forward-upwind', 'implicit-central', 'beam-warming', 'lax-wendroff'}.
`
    Returns
        dict:
            A dictionary with:
                out['h']   : float, spatial grid spacing
                out['k']   : float, base time step size
                out['l']   : int,   total number of time steps taken
                out['x']   : 1D array, spatial grid from 0 to L
                out['TT']  : 1D array, times at which solutions are recorded
                out['U']   : 2D array, solution snapshots; out['U'][:, j] is the solution at time out['TT'][j]
    """
    # Initialize output
    result = {}

    ###############################################
    # 1. Spatial grid
    h = L / (n + 1)  # get the space-step-size
    result['h'] = h

    # generate spatial points to compute solutions about:
    spatial_grid = np.linspace(0, L, n+2)  # n interior points => n+2 total
    result['x'] = spatial_grid
    N = len(spatial_grid)  # number of total points, should be equal to n+2
    exit(f"Total Number of Points (N = {N}) doesn't equal Number of Interior Points (n) + 2 = {n+2}") if N != n+2 else None
    ###############################################
    # 2. Times at which we store solutions
    TT = np.linspace(0, T, M+2)  # from 0 to T, M+2 points
    result['TT'] = TT
    ###############################################
    # 3. ideal Time-step size from user-defined Courant number
    k = Courant * h / c # biggest time-step-size that can be taken while preserving stability
    result['k'] = k
    ###############################################
    # 4. Build/update coeff matrix based on method
    match method.lower():
        case'forward-upwind':
            if c < 0:
                raise ValueError("Please specify a positive advective speed.")
            # Create an N×N matrix A for delta U, for (-U^n_i + U^n_i-1)
            A = -np.diag(np.ones(N)) + np.diag(np.ones(N-1), k=-1)
            # Periodic boundary condition on U(1) = U0 equals U_N
            A[0, n] = 1.0

        case 'implicit-central':
            Courant_stable = min(TT[1] - TT[0], k) * c / h
            # need ot create a Coeff * U_new = U_old system.
            A = np.zeros(shape = (N, N))
            # main diagonals are 2s:
            np.fill_diagonal(A, 2)
            # sub-diagonals are -Courant
            np.fill_diagonal(A[1:], -Courant_stable)
            # super-diags are +Courant
            np.fill_diagonal(A[:-1, 1:], Courant_stable)
            # apply bounary conditions
            A[0, n] = -Courant_stable # for left boundary, U_-1  = U_n
            A[-1,1] = Courant_stable # for right boundary, U_n+2 = U_1

        case 'beam-warming':
            # create N by N matrix
            A = np.zeros(shape=(N, N))
            alpha = (Courant**2 - Courant) / 2 # coeff in front of U_i-2 term
            beta = 2*Courant - Courant**2 # coeff in front of U_i-1 term
            gamma = (Courant ** 2 - 3*Courant) / 2 # coeff. in front of U_i term:
```

```
        # main diagonals
        np.fill_diagonal(A, val = gamma)
        # sub diagonals
        np.fill_diagonal(A[1:], beta)
        # sub-sub diagonal
        np.fill_diagonal(A[2:], alpha)
        # apply Periodic BCs:
        # for the i = 0 or the 1st row:
        ## i = -2 -> n - 1
        A[0, n-1] = alpha
        ## i = -1 -> n
        A[0, n] = beta
        # for the i = 1 or the 2nd row:
        ## i = -1 -> i = n
        A[1, n] = alpha

    case 'lax-wendroff':
        # create N by N matrix
        A = np.zeros(shape=(N, N))
        alpha = (Courant**2 + Courant) / 2 # coeff in front of U_i-1 term
        beta = -Courant ** 2 # coeff in front of U_i term
        gamma = (Courant ** 2 - Courant) / 2 # coeff. in front of U_i+1 term:
        # main diagonals
        np.fill_diagonal(A, val = beta)
        # sub diagonals
        np.fill_diagonal(A[1:], alpha)
        # super diagonal
        np.fill_diagonal(A[:-1, 1:], gamma)
        # apply Periodic BCs:
        # for the i = 0 or the 1st row:
        ## i = -1 -> n
        A[0, n] = alpha
        # for the last row:
        ## i = n+2 -> i = 1
        A[-1, 1] = gamma

    case _:
        raise ValueError("Unknown method.")
################################################
# 5. Solution Calculations
# Prepare storage for solutions at each time in TT
U = np.zeros((N, M+2)) # for N = n+2 points and M + 2 time snap-shots
## indices for use:
num_time_steps = 0   # total number of time steps
j = 0  # accessing solution matrix column
t = 0.0 # time
## compute inital condition on spatial grid:
U_temp = u0(spatial_grid)  # 1D NumPy array for t = 0, j = 0
exit("u0(x) must return an array of length N = n+2.") if len(U_temp) != N else None
U[:, j] = U_temp # initial condition at t = 0 or j = 0
j += 1 # advance to next column.

## Time integration
while (t < TT[-1]): # run until the end time
    # we already computed max 'k' that can be used for stable solution
    # however, user might want to store the sol at smaller increment that is still stable
    # we only use user defined time step size from spatial_grid if it's stable (smaller than theoretical k)
    k_stable = min(TT[j] - t, k)
    Courant_stable = k_stable * c / h
    print(f"Time: {t:.6f}; Courant = {Courant_stable:.6f}; Time step = {k_stable:.6f}")

    # Zero the update for a j-th column of U matrix.
    dU = np.zeros_like(U_temp) # 1d array

    # Method-specific update
    match method.lower():
```

```python
            case 'forward-upwind':
                # Simple explicit Euler + forward-upwind operator
                dU = Courant_stable * (A @ U_temp)
                # Update solution
                U_temp += dU

            case 'implicit-central':
                # solve the new U vector using current U_temp and A matrix
                U_temp = np.linalg.solve(A, 2 * U_temp)

            case 'beam-warming':
                # apply delta U to current U
                dU = A @ U_temp
                # update
                U_temp += dU

            case 'lax-wendroff':
                # apply delta U to current U
                dU = A @ U_temp
                # update
                U_temp += dU
            case _:
                exit("Method is unknwon!")

        # Advance indices
        num_time_steps += 1
        t += k_stable # advance time

        # If we've just hit the next snapshot time that user defined, store the solution.
        if np.isclose(t, TT[j]):
            # Record the solution
            TT[j] = t  # ensure no floating rounding error
            U[:, j] = U_temp
            j += 1 # move to next time step / col in sol mx

    # 7. Finalize outputs
    result['U'] = U
    result['l'] = num_time_steps

    return result
```