# MATH 315 (Winter 2026)

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February 1, 2026

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# Chapter 1

# Introduction

## 1.1 A simple first-order ODE

A simple first-order ODE has the form:

$$y' = \frac{dy}{dx} = ky \quad k \in \mathbb{R},\ k \in \mathbb{C}$$

To solve for the function $y$, we can divide both sides by $y$ then integrate both sides:

$$\begin{aligned}
\int \frac{dy}{dx} \cdot \frac{1}{y} dx &= \int k\ dx \\
\ln|y| &= kx + C \\
y &= e^{kx+C} \\
y &= e^{kx}e^{C} \\
\therefore \quad & \boxed{y = Ce^{kx}}
\end{aligned}$$

If we apply an **initial condition**, such as $y(0) = y_0$, we can solve for the constant $C$:

$$\begin{aligned}
&y(0) = Ce^{k\cdot 0} = C = y_0 \\
&\therefore y = y_0 e^{kx}
\end{aligned}$$

**Example 1.1.1**

Find the general solution of $\frac{dy}{dx} = 2y + 1$.

***Solution:*** We can rearrange the equation as follows:

$$\begin{aligned}
\frac{dy}{dx} \cdot \frac{1}{2y+1} &= 1 \\
\int \frac{1}{2y+1}\ dy &= \int 1\ dx \\
\frac{1}{2}\ln|2y+1| &= x + C \\
\ln|2y+1| &= 2x + C' \\
2y + 1 &= e^{2x+C'} \\
2y + 1 &= Ce^{2x} \\
y &= \frac{Ce^{2x} - 1}{2}
\end{aligned}$$

We can verify this solution by differentiating it:

$$\frac{dy}{dx} = \frac{1}{2} \cdot C \cdot 2e^{2x} = Ce^{2x}$$
$$2y + 1 = 2 \cdot \frac{Ce^{2x} - 1}{2} + 1 = Ce^{2x}$$
$$\therefore \frac{dy}{dx} = 2y + 1$$

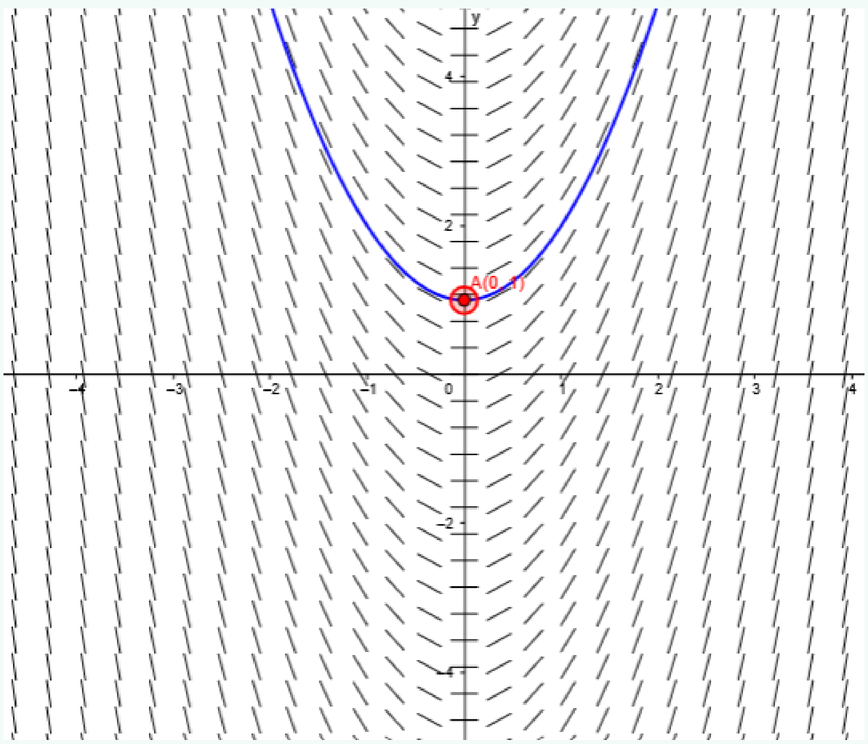
## 1.2 Graphical interpretation of a first-order ODE

Given some equation of the form $y' = F(x, y)$, we could read this as "the slope of the function $y$ at the point $(x, y)$ is given by $F(x, y)$".

If $F(x, y) > 0$, then the function is increasing at that point; if $F(x, y) < 0$, then the function is decreasing at that point. At every point of the $xy$-plane, we can compute the slope of the solution curve that passes through that point. We call this the **slope field** of the differential equation.

**Example 1.2.1**

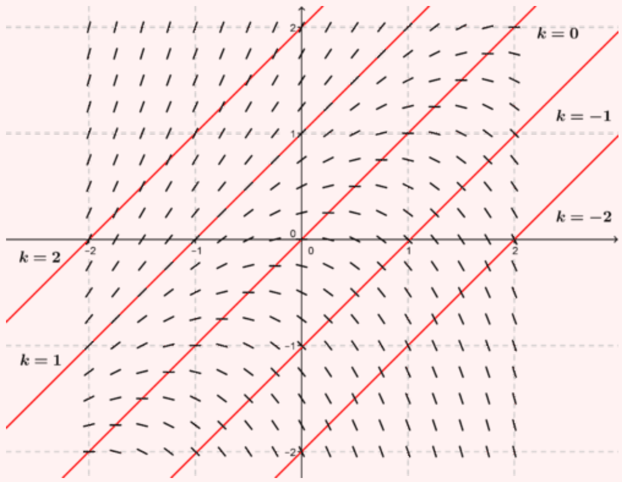
Given the differential equation $\frac{dy}{dx} = 2x$, we can draw a slope field by evaluating the slope at every $(x, y)$ point.
This produces a slope field that looks like this:




The general solution of this differential equation is $y = x^2 + C$.
We can draw a curve that is tangential to the slopes at every point to find a solution curve (or **integral curve**).
If we pick an initial condition, such as $y(0) = 1$, we can approximate the solution curve by following the slopes in the slope field.

**Definition 1.2.1: Isoclines**

Isoclines are curves along which the slope of the solution curve is constant.
For the differential equation $\frac{dy}{dx} = F(x, y)$, the isocline for slope $m$ is given by the equation $F(x, y) = m$.




## 1.3 Numerical approximation

### 1.3.1 Euler's method

Most differential equations cannot be solved analytically.

We define a function $f(x) = \dot{x}$, where $\dot{x}$ is the derivative of $x$ with respect to $t$.

Recall the limit definition of the derivative:

$$\dot{x} = \lim_{h \to 0} \frac{x(t + h) - x(t)}{h}$$

Then for some sufficiently small $h$, we can approximate:

$$\dot{x} \approx \frac{x(t + h) - x(t)}{h}$$

Since we have that $\dot{x} = f(x)$, we can rearrange this to get:

$$\boxed{x(t + h) \approx x(t) + hf(x(t))}$$

This is known as **Euler's method**.

We can also write it in discrete form as:

$$\boxed{x_{m+1} = x_m + hf(x_m)}$$

where $x_m = x(t_m)$ and $t_m = t_0 + mh$.

The Euler backward method looks like:

$$\boxed{x_{m+1} = x_m + hf(x_{m+1})}$$

However, in practice, it is often difficult to solve for $x_{m+1}$ in this equation. Rather, the improved Euler method uses the average of the slopes at the beginning and end of the interval. Then using our approximation of $x_{m+1}$ from the Euler forward method, we could average the "beginning" of the step and the "end" of the step as follows:

$$\tilde{x}_{m+1} = x_m + hf(x_m)$$
$$x_{m+1} = x_m + \frac{1}{2}h\left(f(x_m) + f(\tilde{x}_{m+1})\right)$$

$$\boxed{x_{m+1} = x_m + \frac{h}{2}\left(f(x_m) + f\left(x_m + hf(x_m)\right)\right)}$$

### 1.3.2 Local and global error

The **global error** could be written as:

$$E = |x(t_N) - x_N|$$

where $x(t_N)$ is the exact solution at time $t_N$ and $x_N$ is the numerical approximation at time $t_N$.
Let the final time be $T = N \cdot h$, where $h$ is the step size and $N$ is the number of steps taken.
The **local error** at step $m$ is defined as:

$$E_m = |x(t_m) - x_m|$$

To analyze the error, we can use Taylor's theorem to expand $x(t_{m+1})$ around $t_m$:

$$x(t_{m+1}) = x(t_m) + h\dot{x}(t_m) + \frac{h^2}{2}\ddot{x}(t_m)$$

Then for **Euler's forward method**, we have:

$$x(t_{m+1}) = x(t_m) + hf(x(t_m)) + e$$

Then we can bound $e$ using the second derivative term:

In other words, by Taylor's theorem, the error $e$ at each step is given by the $h^2$ term involving the second derivative, plus even smaller terms involving higher derivatives and higher powers of $h$. That is,

$$|e| \leqslant \text{(a constant)} \times h^2 \times \max_{t\in[0,T]} |\ddot{x}(t)| + \text{(smaller terms involving higher derivatives and higher powers of } h\text{)}$$

where the constant is $1/2$ in this case for the $h^2$ term.

So $e$ (the local error) is dominated by the $h^2$ term, but in reality, there are also even smaller contributions from higher derivatives (like $\dddot{x}$) and higher powers of $h$.

To bound the global error, we add up the local errors over all $N$ steps. This means the total (global) error $E$.

$$\begin{aligned} E &\leqslant \sum_{i=1}^{N} |e_i| \\ &\leqslant \frac{T}{h} \cdot h^2 \cdot \max_{t\in[0,T]} |\ddot{x}(t)| \\ &= T \cdot h \cdot \max_{t\in[0,T]} |\ddot{x}(t)| \end{aligned}$$

So the global error $E$ is proportional to $h$ for Euler's forward method.

**Definition 1.3.1**

If a numerical method has a one-step error of size $h^{p+1}$, then we say the method is of order $p$.
The global error is then of size $h^p$.

# Chapter 2

# Techniques for first-order ODEs

## 2.1 Separation of variables

The standard form of a 1st order ODE is given by:

$$\overline{R}(t)\dot{x} + \overline{P}(t)x = \overline{q}(t) \quad \overline{R}(t) \neq 0$$

If we let:

$$p = \frac{\overline{P}(t)}{\overline{R}(t)} \quad , \quad q = \frac{\overline{q}(t)}{\overline{R}(t)}$$

Then we obtain the equation in the form:

$$\boxed{\dot{x} + p(t)x = q(t)}$$

**Definition 2.1.1: Homogeneous equation**

An ODE is said to be **homogeneous** if $q(t) = 0$ for all $t$ in the interval of interest.

Intuitively, the system evolves with no additional forcing or input beyond its initial conditions. A homogeneous equation is also **separable** if it can be expressed in the form:

$$\frac{\dot{x}}{x} = f(t)$$

**Definition 2.1.2: Separable equation**

An ODE is said to be **separable** if it can be expressed in the form:

$$\frac{dx}{dt} = g(t)h(x)$$

where $g(t)$ is a function of $t$ only and $h(x)$ is a function of $x$ only.

We can easily solve separable equations by separating the variables and integrating both sides.

**Example 2.1.1**

Solve the following ODE:

$$\dot{x} + p(t)x = 0$$

***Solution:*** We can rewrite the equation as:

$$\frac{\dot{x}}{x} = -p(t)$$

Then we separate the variables:

$$\frac{1}{x}\frac{dx}{dt} = -p(t)$$
$$\frac{1}{x}dx = -p(t)dt$$

Now we integrate both sides:

$$\int \frac{1}{x}dx = \int -p(t)dt$$
$$\ln|x| = -\int p(t)dt + C$$
$$|x| = e^{-\int p(t)dt + C}$$
$$x(t) = \pm e^C e^{-\int p(t)dt}$$
$$x(t) = Ke^{-\int p(t)dt} \quad , \quad K = \pm e^C$$

Thus, the general solution to the ODE is:

$$\boxed{x(t) = Ke^{-\int p(t)dt}}$$

### 2.1.1 Steps to solve separable equations

1. Rewrite the ODE in the form $\frac{dx}{dt} = g(t)h(x)$.
2. Separate the variables to obtain $\frac{1}{h(x)}dx = g(t)dt$.
3. Integrate both sides: $\int \frac{1}{h(x)}dx = \int g(t)dt$.
4. Solve for $x(t)$.
5. (Optional) Apply any initial conditions to solve for constants of integration.

**Definition 2.1.3: Homogeneous solution**

In general, the solution to a homogeneous ODE is called the **homogeneous solution**.

$$\boxed{X_h(t) = e^{-\int p(t)dt}}$$

Then:

$$x(t) = KX_h(t)$$

for some constant $K$.

**Example 2.1.2** (Newton's law of cooling)

The change in temperature $u$ over time $t$ is proportional to the difference in temperature between the object and its surroundings.
Let $u_0$ be the initial temperature of the object and $T_{\text{ext}}$ be the external temperature. Let $k$ be the proportionality constant. Then:

$$\dot{u} = -k(u - T_{\text{ext}})$$

***Solution:*** We can rewrite the equation as:

$$\dot{u} + ku = kT_{\text{ext}}$$

This is a first-order linear ODE. We can separate the variables:

$$\begin{aligned}\frac{du}{dt} &= -k(u - T_{\text{ext}}) \\ \frac{1}{u - T_{\text{ext}}} du &= -k dt\end{aligned}$$

Then, we integrate both sides:

$$\begin{aligned}\int \frac{1}{u - T_{\text{ext}}} du &= \int -k dt \\ \ln |u - T_{\text{ext}}| &= -kt + C \\ |u - T_{\text{ext}}| &= e^{-kt+C} \\ u - T_{\text{ext}} &= \pm e^C e^{-kt} \\ u(t) &= \pm e^C e^{-kt} + T_{\text{ext}} \\ u(t) &= K e^{-kt} + T_{\text{ext}} \quad , \quad K = \pm e^C\end{aligned}$$

To find $K$, we use the initial condition $u(0) = u_0$:

$$\begin{aligned}u(0) &= K e^{-k \cdot 0} + T_{\text{ext}} \\ u_0 &= K + T_{\text{ext}} \\ K &= u_0 - T_{\text{ext}}\end{aligned}$$

Therefore, the solution to the ODE is:

$$\boxed{u(t) = (u_0 - T_{\text{ext}}) e^{-kt} + T_{\text{ext}}}$$

Intuitively, this can also be used to model warming as well as cooling. This depends on the sign of $(u_0 - T_{\text{ext}})$.

## 2.2 Integrating factors

We are given an equation:

$$\dot{x} + p(t)x = q(t)$$

The method of integrating factors can be thought of intuitively as "undoing" the product rule of differentiation. Specifically, we would like to transform the left-hand side of the equation into the derivative of a product of two functions $u(t)$ and $x(t)$:

$$\frac{d}{dt}[u(t)x(t)] = u(t)\dot{x}(t) + \dot{u}(t)x(t)$$

Looking at our original equation, we'd like to have some $\mu(t)$ to multiply throughout the equation such that:

$$\begin{aligned}\mu(t)\dot{x} + \mu(t)p(t)x &= u(t)\dot{x} + \dot{u}(t)x \\ &= \frac{d}{dt}[u(t)x(t)]\end{aligned}$$

This means we need to have:

$$\begin{cases}\mu(t) = u(t) \\ \mu(t)p(t) = \dot{u}(t)\end{cases}$$

Solving for $\mu(t)$, we have:

$$\begin{aligned}
\dot{u} &= up(t) \\
\frac{du}{dt} &= up(t) \\
\frac{1}{u}du &= p(t)dt \\
\int \frac{1}{u}du &= \int p(t)dt \\
\ln|u| &= \int p(t)dt + C \\
u(t) = e^{\int p(t)dt+C} &= e^C e^{\int p(t)dt}
\end{aligned}$$

We can ignore the constant as it will cancel out later.

**Definition 2.2.1: Integrating factor**

The **integrating factor** for the first-order linear ODE $\dot{x} + p(t)x = q(t)$ is given by:

$$\boxed{\mu(t) = e^{\int p(t)dt}}$$

Also, this is the reciprocal of the homogeneous solution $X_h(t)$ from separation of variables:

$$\mu(t) = X_h^{-1}(t)$$

### 2.2.1 Steps to solve using integrating factors

1. Start with the standard form of the ODE: $\dot{x} + p(t)x = q(t)$.
2. Compute the integrating factor: $\mu(t) = e^{\int p(t)dt}$.
3. Multiply both sides of the ODE by the integrating factor $\mu(t)$.
4. Recognize that the left-hand side is now the derivative of the product:
$$\frac{d}{dt}\left[\mu(t)x(t)\right] = \mu(t)q(t)$$
5. Integrate both sides with respect to $t$:
$$\mu(t)x(t) = \int \mu(t)q(t)dt + C$$
6. Solve for $x(t)$:
$$x(t) = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)dt + C\right)$$
7. (Optional) Apply any initial conditions to solve for the constant $C$.

**Example 2.2.1**

Solve the initial value problem:

$$t\dot{x} + 2x = t^2 - t + 1, \quad x(1) = \frac{1}{2}$$

***Solution:***

1. Standard form: $\dot{x} + \frac{2}{t}x = t - 1 + \frac{1}{t}$
2. Integrating factor:

$$\begin{aligned} \mu(t) &= e^{\int \frac{2}{t} dt} \\ &= e^{2 \ln |t|} \\ &= t^2 \end{aligned}$$

3. Multiply through by $\mu(t)$:

$$t^2 \dot{x} + 2tx = t^3 - t^2 + t$$

4. Left-hand side as derivative:

$$\frac{d}{dt}\left[t^2 x\right] = t^3 - t^2 + t$$

5. Integrate both sides:

$$\begin{aligned} t^2 x &= \int (t^3 - t^2 + t) dt + C \\ &= \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C \end{aligned}$$

6. Solve for $x(t)$:

$$x(t) = \frac{1}{t^2}\left(\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C\right) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$$

7. Apply initial condition $x(1) = \frac{1}{2}$:

$$\begin{aligned} \frac{1}{2} &= \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \\ C &= \frac{1}{12} \end{aligned}$$

$$\boxed{x(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}}$$

## 2.3 Substitution technique

Consider the differential equation:

$$\dot{y} = F(\frac{y}{x})$$

We can eliminate the $x$ in the denominator to have a function of a single variable by substituting:

$$v = \frac{y}{x} \implies y = vx$$

Then rearranging:

$$\begin{aligned} \dot{y} &= F(v) \\ v + x\dot{v} &= F(v) \\ x\dot{v} &= F(v) - v \\ x \cdot \frac{dv}{dx} &= F(v) - v \\ \frac{dv}{dx} &= \frac{F(v) - v}{x} \\ \frac{dv}{F(v) - v} &= \frac{dx}{x} \quad \text{(separable equation)} \end{aligned}$$

### 2.3.1 Steps to solve using substitution technique

1. Identify the substitution: $v = \frac{y}{x}$, which implies $y = vx$.
2. Differentiate $y$ with respect to $x$ using the product rule:
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$
3. Substitute $\frac{dy}{dx}$ and $y$ into the original ODE to express it in terms of $v$ and $x$.
4. Rearrange the equation to isolate $\frac{dv}{dx}$.
5. Separate the variables to obtain an equation of the form:
$$\frac{dv}{F(v) - v} = \frac{dx}{x}$$
6. Integrate both sides to find $v(x)$.
7. Substitute back to find $y(x)$ using $y = vx$.
8. (Optional) Apply any initial conditions to solve for constants of integration.

We can now solve this separable equation for $v(x)$, then substitute back to find $y(x)$.

**Example 2.3.1** (Tutorial 3 problem 1)
Solve the initial value problem:
$$\begin{cases} \dot{y} = \frac{x^2+3y^2}{2xy} \\ y(-2) = 6 \end{cases}$$

***Solution:*** We rewrite so we can make some substitution for $u = \frac{y}{x}$:

$$\begin{aligned} \dot{y} &= \frac{x^2 + 3y^2}{2xy} \\ &= \frac{x^2}{2xy} + \frac{3y^2}{2xy} \\ &= \frac{x}{2y} + \frac{3y}{2x} \\ &= \frac{1}{2 \cdot \frac{y}{x}} + \frac{3}{2} \cdot \frac{y}{x} \\ &= \frac{1}{2u} + \frac{3}{2}u \end{aligned}$$

Recall that $u = \frac{y}{x} \implies y = ux$. Then $\frac{dy}{dx} = u + x\frac{du}{dx}$. Substituting this in:

$$\begin{aligned}
u + x\frac{du}{dx} &= \frac{1}{2u} + \frac{3}{2}u \\
x\frac{du}{dx} &= \frac{1}{2u} + \frac{3}{2}u - u \\
x\ du &= \left(\frac{1}{2u} + \frac{1}{2}u\right) dx \\
\frac{1}{\frac{1}{2u} + \frac{1}{2}u}\ du &= \frac{dx}{x}
\end{aligned}$$

Note that this is effectively transforming the original equation into the separable form $\frac{dv}{F(v)-v} = \frac{dx}{x}$. Now we integrate by partial fractions:

$$\begin{aligned}
\int \frac{1}{\frac{1}{2u} + \frac{1}{2}u}\ du &= \int \frac{dx}{x} \\
\int \frac{2u}{1+u^2}\ du &= \ln|x| + C \\
\int \frac{1}{1+u^2}\ d(u^2) &= \ln|x| + C \\
\ln|1+u^2| &= \ln|x| + C \\
1 + u^2 &= K|x|\ , \quad K = e^C 1 + u^2 && = Bx \quad B = \pm K
\end{aligned}$$

Then substituting back for $u = \frac{y}{x}$:

$$\begin{aligned}
1 + \left(\frac{y}{x}\right)^2 &= Bx \\
1 + \frac{y^2}{x^2} &= Bx \\
y^2 &= Bx^3 - x^2
\end{aligned}$$

Rearranging gives the general solution:

$$y^2 = Bx^3 - x^2$$

We can solve for $y$:

$$\boxed{y = \pm\sqrt{Bx^3 - x^2}}$$

Now using the initial condition $y(-2) = 6$ to solve for $B$:

$$\begin{aligned}
6 &= \pm\sqrt{B(-2)^3 - (-2)^2} \\
36 &= -8B - 4 \\
40 &= -8B \\
B &= -5
\end{aligned}$$

We must choose the positive root since $y(-2) = 6 > 0$. Thus the particular solution is:

$$\boxed{y = \sqrt{-5x^3 - x^2}}$$

The domain is $-5x^3 - x^2 \geqslant 0 \implies x^2(-5x - 1) \geqslant 0 \implies x \leqslant -\frac{1}{5}$.

## 2.4 Exact equations

We have solely dealt with first-order linear and separable differential equations so far. However, some ODEs will have functions of both $x$ and $y$ that prevent separation.

**Example 2.4.1**

Solve the following ODE:

$$(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

***Solution:*** Now consider some function $\Psi(x, y)$ (don't worry about how we got it yet) such that:

$$\Psi(x, y) = y^2 + (x^2 + 1)y - 3x^3$$

If we compute the partial derivatives, we find:

$$\Psi_x = 2xy - 9x^2$$
$$\Psi_y = 2y + x^2 + 1$$

These expressions appear in our original equation. Using the chain rule for partial derivatives:

$$\frac{d}{dx}\Psi(x, y(x)) = \Psi_x \frac{dx}{dx} + \Psi_y \frac{dy}{dx}$$
$$\frac{d}{dx}\Psi(x, y(x)) = \Psi_x + \Psi_y \frac{dy}{dx}$$

Then we can rewrite the original ODE as:

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0$$

Thus, we have:

$$\frac{d}{dx}\Psi(x, y) = 0$$

Integrating both sides with respect to $x$ gives:

$$\Psi(x, y) = C$$

So the general solution to the ODE is:

$$\boxed{y^2 + (x^2 + 1)y - 3x^3 = C}$$

We are therefore concerned with obtaining a method to find such a function $\Psi(x, y)$ for a given ODE.

**Definition 2.4.1: Exact equation**

An ODE of the form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is said to be an **exact equation** if there exists a function $\Psi(x, y)$ such that:

$$\Psi_x = M(x, y)$$
$$\Psi_y = N(x, y)$$

If an ODE is exact, we have:

$$\boxed{\frac{d}{dx}\Psi(x, y) = \Psi_x + \Psi_y \frac{dy}{dx} = 0}$$

Then the solution to the ODE is given implicitly by:

$$\boxed{\Psi(x, y) = C}$$

If $\Psi(x, y)$ is continuously differentiable, then we have that:

$$(\Psi_x)_y = (\Psi_y)_x$$

Then the equation is only exact if:

$$\boxed{M_y = N_x}$$

### 2.4.1 Steps to solve exact equations

1. Verify that the equation is exact by checking if $M_y = N_x$.
2. Integrate $M(x, y)$ with respect to $x$ to find $\Psi(x, y)$ up to a function of $y$:
$$\Psi(x,y) = \int M(x,y)\,dx + h(y)$$
3. Differentiate $\Psi(x, y)$ with respect to $y$ and set it equal to $N(x, y)$ to solve for $h(y)$:
$$\frac{\partial \Psi}{\partial y} = N(x,y)$$
4. Substitute $h(y)$ back into $\Psi(x, y)$.
5. (Optional) Apply the initial condition if given back into:
$$\Psi(x,y) = C$$

**Note:-**

If it is easier, you can also integrate $N(x, y)$ with respect to $y$ first, then differentiate with respect to $x$ to find the function of $x$.

**Example 2.4.2**

Solve the following initial value problem:
$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3$$

***Solution:***

1. Find and verify $M_y$ and $N_x$:
$$\begin{aligned} M(x,y) &= 2xy - 9x^2 \\ N(x,y) &= 2y + x^2 + 1 \\ M_y(x,y) &= 2x \\ N_x(x,y) &= 2x \end{aligned}$$
Since $M_y = N_x$, the equation is exact.
2. Integrate $M(x, y)$ with respect to $x$:
$$\begin{aligned} \Psi(x,y) &= \int (2xy - 9x^2)\,dx + h(y) \\ &= x^2y - 3x^3 + h(y) \end{aligned}$$
3. Differentiate $\Psi(x, y)$ with respect to $y$ and set equal to $N(x, y)$:
$$\begin{aligned} \frac{\partial \Psi}{\partial y} &= x^2 + h'(y) \\ x^2 + h'(y) &= 2y + x^2 + 1 \\ h'(y) &= 2y + 1 \end{aligned}$$
Integrating gives:
$$h(y) = y^2 + y + K$$

4. Substitute $h(y)$ back into $\Psi(x,y)$:

$$\Psi(x,y) = x^2y - 3x^3 + y^2 + y + K$$

5. Apply the initial condition $y(0) = -3$:

$$\begin{aligned}\Psi(0,-3) &= 0^2 \cdot (-3) - 3 \cdot 0^3 + (-3)^2 + (-3) + K \\ &= 0 + 0 + 9 - 3 + K \\ &= 6 + K\end{aligned}$$

Setting this equal to $C$, we have:

$$C = 6 + K$$

Thus, the particular solution is:

$$\boxed{x^2y - 3x^3 + y^2 + y = 6}$$

6. We could also solve for $y$ explicitly using the quadratic formula:

$$\begin{aligned}y^2 + (x^2+1)y + (-3x^3 - 6) = 0 \\ y = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(-3x^3-6)}}{2} \\ y = \frac{-(x^2+1) \pm \sqrt{x^4 + 2x^2 + 1 + 12x^3 + 24}}{2} \\ y = \frac{-(x^2+1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}\end{aligned}$$

We must now choose the correct sign for the initial condition $y(0) = -3$, since $-3$ is less than $\frac{-(0^2+1)}{2} = -\frac{1}{2}$. Thus we choose the negative root:

$$\boxed{y = \frac{-(x^2+1) - \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}}$$