MATH 315 (Winter 2026)

Notes by: Emily Wang

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Chapter 1

Introduction

1.1 A simple first-order ODE

An ordinary differential equation (ODE) is an equation involving a function and its derivatives.

A simple first-order ODE has the form:

$$y' = \frac{dy}{dx} = ky \quad k \in \mathbb{R},\ k \in \mathbb{C}$$

To solve for the function y, we can divide both sides by y then integrate both sides:

$$\begin{aligned}
\int \frac{dy}{dx} \cdot \frac{1}{y} dx &= \int k\ dx \\
\ln|y| &= kx + C \\
y &= e^{kx+C} \\
y &= e^{kx}e^{C} \\
\therefore \quad & \boxed{y = Ce^{kx}}
\end{aligned}$$

If we apply an **initial condition**, such as $y(0) = y_0$, we can solve for the constant C:

$$\begin{aligned}
& y(0) = Ce^{k\cdot 0} = C = y_0 \\
& \therefore y = y_0 e^{kx}
\end{aligned}$$

Example 1.1.1

Find the general solution of $\frac{dy}{dx} = 2y + 1$.

Solution: We can rearrange the equation as follows:

$$\begin{aligned}
\frac{dy}{dx} \cdot \frac{1}{2y+1} &= 1 \\
\int \frac{1}{2y+1}\ dy &= \int 1\ dx \\
\frac{1}{2} \ln|2y+1| &= x + C \\
\ln|2y+1| &= 2x + C' \\
2y + 1 &= e^{2x+C'} \\
2y + 1 &= Ce^{2x} \\
y &= \frac{Ce^{2x} - 1}{2}
\end{aligned}$$

We can verify this solution by differentiating it:

$$\frac{dy}{dx} = \frac{1}{2} \cdot C \cdot 2e^{2x} = Ce^{2x}$$
$$2y + 1 = 2 \cdot \frac{Ce^{2x} - 1}{2} + 1 = Ce^{2x}$$
$$\therefore \frac{dy}{dx} = 2y + 1$$

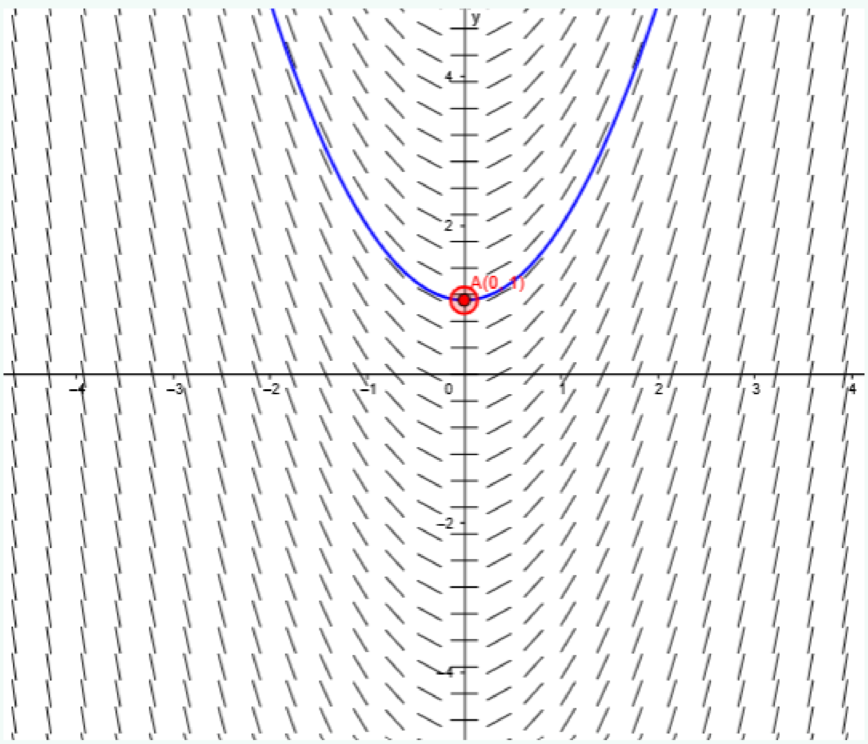
1.2 Graphical interpretation of a first-order ODE

Given some equation of the form $y' = F(x, y)$, we could read this as "the slope of the function y at the point (x, y) is given by $F(x, y)$".

If $F(x, y) > 0$, then the function is increasing at that point; if $F(x, y) < 0$, then the function is decreasing at that point. At every point of the xy-plane, we can compute the slope of the solution curve that passes through that point. We call this the **slope field** of the differential equation.

Example 1.2.1

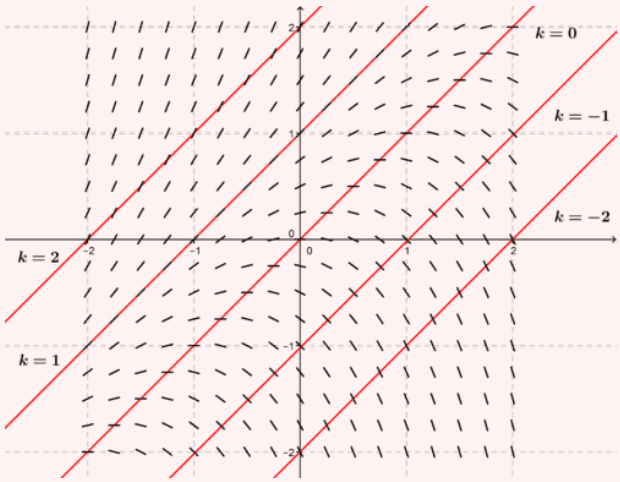
Given the differential equation $\frac{dy}{dx} = 2x$, we can draw a slope field by evaluating the slope at every (x, y) point.
This produces a slope field that looks like this:



The general solution of this differential equation is $y = x^2 + C$.
We can draw a curve that is tangential to the slopes at every point to find a solution curve (or **integral curve**).
If we pick an initial condition, such as $y(0) = 1$, we can approximate the solution curve by following the slopes in the slope field.

Definition 1.2.1: Isoclines

Isoclines are curves along which the slope of the solution curve is constant.
For the differential equation $\frac{dy}{dx} = F(x, y)$, the isocline for slope m is given by the equation $F(x, y) = m$.



1.3 Numerical approximation

1.3.1 Euler's method

Most differential equations cannot be solved analytically.

We define a function $f(x) = \dot{x}$, where $\dot{x}$ is the derivative of x with respect to t.

Recall the limit definition of the derivative:

$$\dot{x} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

Then for some sufficiently small h, we can approximate:

$$\dot{x} \approx \frac{x(t+h) - x(t)}{h}$$

Since we have that $\dot{x} = f(x)$, we can rearrange this to get:

$$\boxed{x(t+h) \approx x(t) + hf(x(t))}$$

This is known as **Euler's method**.

We can also write it in discrete form as:

$$\boxed{x_{m+1} = x_m + hf(x_m)}$$

where $x_m = x(t_m)$ and $t_m = t_0 + mh$.

The Euler backward method looks like:

$$\boxed{x_{m+1} = x_m + hf(x_{m+1})}$$

However, in practice, it is often difficult to solve for x_{m+1} in this equation. Rather, the improved Euler method uses the average of the slopes at the beginning and end of the interval. Then using our approximation of x_{m+1} from the Euler forward method, we could average the "beginning" of the step and the "end" of the step as follows:

$$\begin{aligned}
\tilde{x}_{m+1} &= x_m + hf(x_m) \\
x_{m+1} &= x_m + \frac{1}{2}h\left(f(x_m) + f(\tilde{x}_{m+1})\right)
\end{aligned}$$

$$\boxed{x_{m+1} = x_m + \frac{h}{2}\left(f(x_m) + f\left(x_m + hf(x_m)\right)\right)}$$

1.3.2 Local and global error

The **global error** could be written as:

$$E = |x(t_N) - x_N|$$

where $x(t_N)$ is the exact solution at time t_N and x_N is the numerical approximation at time t_N.
Let the final time be $T = N \cdot h$, where h is the step size and N is the number of steps taken.
The **local error** at step m is defined as:

$$E_m = |x(t_m) - x_m|$$

To analyze the error, we can use Taylor's theorem to expand $x(t_{m+1})$ around t_m:

$$x(t_{m+1}) = x(t_m) + h\dot{x}(t_m) + \frac{h^2}{2}\ddot{x}(t_m)$$

Then for **Euler's forward method**, we have:

$$x(t_{m+1}) = x(t_m) + hf(x(t_m)) + e$$

Then we can bound e using the second derivative term:

In other words, by Taylor's theorem, the error e at each step is given by the h^2 term involving the second derivative, plus even smaller terms involving higher derivatives and higher powers of h. That is,

$$|e| \leqslant (\text{a constant}) \times h^2 \times \max_{t\in[0,T]} |\ddot{x}(t)| + (\text{smaller terms involving higher derivatives and higher powers of } h)$$

where the constant is $1/2$ in this case for the h^2 term.

So e (the local error) is dominated by the h^2 term, but in reality, there are also even smaller contributions from higher derivatives (like $\dddot{x}$) and higher powers of h.

To bound the global error, we add up the local errors over all N steps. This means the total (global) error E.

$$\begin{aligned} E &\leqslant \sum_{i=1}^{N} |e_i| \\ &\leqslant \frac{T}{h} \cdot h^2 \cdot \max_{t\in[0,T]} |\ddot{x}(t)| \\ &= T \cdot h \cdot \max_{t\in[0,T]} |\ddot{x}(t)| \end{aligned}$$

So the global error E is proportional to h for Euler's forward method.

Definition 1.3.1

If a numerical method has a one-step error of size h^{p+1}, then we say the method is of order p.
The global error is then of size h^p.

Chapter 2

Techniques for solving first-order ODEs

2.1 Separation of variables

The standard form of a 1st order ODE is given by:

$$\overline{R}(t)\dot{x} + \overline{P}(t)x = \overline{q}(t) \quad \overline{R}(t) \neq 0$$

If we let:

$$p = \frac{\overline{P}(t)}{\overline{R}(t)} \quad , \quad q = \frac{\overline{q}(t)}{\overline{R}(t)}$$

Then we obtain the equation in the form:

$$\boxed{\dot{x} + p(t)x = q(t)}$$

Definition 2.1.1: Homogeneous equation

An ODE is said to be **homogeneous** if $q(t) = 0$ for all t in the interval of interest.

Intuitively, the system evolves with no additional forcing or input beyond its initial conditions. A homogeneous equation is also **separable** if it can be expressed in the form:

$$\frac{\dot{x}}{x} = f(t)$$

Definition 2.1.2: Separable equation

An ODE is said to be **separable** if it can be expressed in the form:

$$\frac{dx}{dt} = g(t)h(x)$$

where $g(t)$ is a function of t only and $h(x)$ is a function of x only.

We can easily solve separable equations by separating the variables and integrating both sides.

Example 2.1.1

Solve the following ODE:

$$\dot{x} + p(t)x = 0$$

Solution: We can rewrite the equation as:

$$\frac{\dot{x}}{x} = -p(t)$$

Then we separate the variables:

$$\frac{1}{x}\frac{dx}{dt} = -p(t)$$
$$\frac{1}{x}dx = -p(t)dt$$

Now we integrate both sides:

$$\int \frac{1}{x}dx = \int -p(t)dt$$
$$\ln|x| = -\int p(t)dt + C$$
$$|x| = e^{-\int p(t)dt + C}$$
$$x(t) = \pm e^{C}e^{-\int p(t)dt}$$
$$x(t) = Ke^{-\int p(t)dt} \quad , \quad K = \pm e^{C}$$

Thus, the general solution to the ODE is:

$$\boxed{x(t) = Ke^{-\int p(t)dt}}$$

2.1.1 Steps to solve separable equations

1. Rewrite the ODE in the form $\frac{dx}{dt} = g(t)h(x)$.
2. Separate the variables to obtain $\frac{1}{h(x)}dx = g(t)dt$.
3. Integrate both sides: $\int \frac{1}{h(x)}dx = \int g(t)dt$.
4. Solve for $x(t)$.
5. (Optional) Apply any initial conditions to solve for constants of integration.

Definition 2.1.3: Homogeneous solution

In general, the solution to a homogeneous ODE is called the **homogeneous solution**.

$$\boxed{X_h(t) = e^{-\int p(t)dt}}$$

Then:

$$x(t) = KX_h(t)$$

for some constant K.

Example 2.1.2 (Newton's law of cooling)

The change in temperature u over time t is proportional to the difference in temperature between the object and its surroundings.
Let u_0 be the initial temperature of the object and T_{ext} be the external temperature. Let k be the proportionality constant. Then:

$$\dot{u} = -k(u - T_{\text{ext}})$$

Solution: We can rewrite the equation as:

$$\dot{u} + ku = kT_{\text{ext}}$$

This is a first-order linear ODE. We can separate the variables:

$$\begin{aligned}\frac{du}{dt} &= -k(u - T_{\text{ext}}) \\ \frac{1}{u - T_{\text{ext}}} du &= -k dt\end{aligned}$$

Then, we integrate both sides:

$$\begin{aligned}\int \frac{1}{u - T_{\text{ext}}} du &= \int -k dt \\ \ln |u - T_{\text{ext}}| &= -kt + C \\ |u - T_{\text{ext}}| &= e^{-kt+C} \\ u - T_{\text{ext}} &= \pm e^C e^{-kt} \\ u(t) &= \pm e^C e^{-kt} + T_{\text{ext}} \\ u(t) &= Ke^{-kt} + T_{\text{ext}} \quad , \quad K = \pm e^C\end{aligned}$$

To find K, we use the initial condition $u(0) = u_0$:

$$\begin{aligned}u(0) &= Ke^{-k\cdot 0} + T_{\text{ext}} \\ u_0 &= K + T_{\text{ext}} \\ K &= u_0 - T_{\text{ext}}\end{aligned}$$

Therefore, the solution to the ODE is:

$$\boxed{u(t) = (u_0 - T_{\text{ext}})e^{-kt} + T_{\text{ext}}}$$

Intuitively, this can also be used to model warming as well as cooling. This depends on the sign of $(u_0 - T_{\text{ext}})$.

2.2 Integrating factors

We are given an equation:

$$\dot{x} + p(t)x = q(t)$$

The method of integrating factors can be thought of intuitively as "undoing" the product rule of differentiation. Specifically, we would like to transform the left-hand side of the equation into the derivative of a product of two functions $u(t)$ and $x(t)$:

$$\frac{d}{dt}[u(t)x(t)] = u(t)\dot{x}(t) + \dot{u}(t)x(t)$$

Looking at our original equation, we'd like to have some $\mu(t)$ to multiply throughout the equation such that:

$$\begin{aligned}\mu(t)\dot{x} + \mu(t)p(t)x &= u(t)\dot{x} + \dot{u}(t)x \\ &= \frac{d}{dt}[u(t)x(t)]\end{aligned}$$

This means we need to have:

$$\begin{cases}\mu(t) = u(t) \\ \mu(t)p(t) = \dot{u}(t)\end{cases}$$

Solving for $\mu(t)$, we have:

$$\begin{aligned}
\dot{u} &= up(t) \\
\frac{du}{dt} &= up(t) \\
\frac{1}{u}du &= p(t)dt \\
\int \frac{1}{u}du &= \int p(t)dt \\
\ln|u| &= \int p(t)dt + C \\
u(t) = e^{\int p(t)dt+C} &= e^C e^{\int p(t)dt}
\end{aligned}$$

We can ignore the constant as it will cancel out later.

Definition 2.2.1: Integrating factor

The **integrating factor** for the first-order linear ODE $\dot{x} + p(t)x = q(t)$ is given by:

$$\boxed{\mu(t) = e^{\int p(t)dt}}$$

Also, this is the reciprocal of the homogeneous solution $X_h(t)$ from separation of variables:

$$\mu(t) = X_h^{-1}(t)$$

2.2.1 Steps to solve using integrating factors

1. Start with the standard form of the ODE: $\dot{x} + p(t)x = q(t)$.
2. Compute the integrating factor: $\mu(t) = e^{\int p(t)dt}$.
3. Multiply both sides of the ODE by the integrating factor $\mu(t)$.
4. Recognize that the left-hand side is now the derivative of the product:
$$\frac{d}{dt}\left[\mu(t)x(t)\right] = \mu(t)q(t)$$
5. Integrate both sides with respect to t:
$$\mu(t)x(t) = \int \mu(t)q(t)dt + C$$
6. Solve for $x(t)$:
$$x(t) = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)dt + C\right)$$
7. (Optional) Apply any initial conditions to solve for the constant C.

Example 2.2.1

Solve the initial value problem:

$$t\dot{x} + 2x = t^2 - t + 1, \quad x(1) = \frac{1}{2}$$

Solution:

1. Standard form: $\dot{x} + \frac{2}{t}x = t - 1 + \frac{1}{t}$
2. Integrating factor:

$$\begin{aligned}\mu(t) &= e^{\int \frac{2}{t}dt} \\ &= e^{2\ln|t|} \\ &= t^2\end{aligned}$$

3. Multiply through by $\mu(t)$:

$$t^2\dot{x} + 2tx = t^3 - t^2 + t$$

4. Left-hand side as derivative:

$$\frac{d}{dt}\left[t^2 x\right] = t^3 - t^2 + t$$

5. Integrate both sides:

$$\begin{aligned}t^2 x &= \int (t^3 - t^2 + t)dt + C \\ &= \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C\end{aligned}$$

6. Solve for $x(t)$:

$$x(t) = \frac{1}{t^2}\left(\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C\right) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$$

7. Apply initial condition $x(1) = \frac{1}{2}$:

$$\begin{aligned}\frac{1}{2} &= \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \\ C &= \frac{1}{12}\end{aligned}$$

$$\boxed{x(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}}$$

2.3 Substitution technique

Consider the differential equation:

$$\dot{y} = F(\frac{y}{x})$$

We can eliminate the x in the denominator to have a function of a single variable by substituting:

$$v = \frac{y}{x} \implies y = vx$$

Then rearranging:

$$\begin{aligned}\dot{y} &= F(v) \\ v + x\dot{v} &= F(v) \\ x\dot{v} &= F(v) - v \\ x \cdot \frac{dv}{dx} &= F(v) - v \\ \frac{dv}{dx} &= \frac{F(v) - v}{x} \\ \frac{dv}{F(v) - v} &= \frac{dx}{x} \quad \text{(separable equation)}\end{aligned}$$

2.3.1 Steps to solve using substitution technique

1. Identify the substitution: $v = \frac{y}{x}$, which implies $y = vx$.
2. Differentiate y with respect to x using the product rule:
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$
3. Substitute $\frac{dy}{dx}$ and y into the original ODE to express it in terms of v and x.
4. Rearrange the equation to isolate $\frac{dv}{dx}$.
5. Separate the variables to obtain an equation of the form:
$$\frac{dv}{F(v) - v} = \frac{dx}{x}$$
6. Integrate both sides to find $v(x)$.
7. Substitute back to find $y(x)$ using $y = vx$.
8. (Optional) Apply any initial conditions to solve for constants of integration.

We can now solve this separable equation for $v(x)$, then substitute back to find $y(x)$.

Example 2.3.1 (Tutorial 3 problem 1)

Solve the initial value problem:
$$\begin{cases} \dot{y} = \frac{x^2+3y^2}{2xy} \\ y(-2) = 6 \end{cases}$$

Solution: We rewrite so we can make some substitution for $u = \frac{y}{x}$:

$$\begin{aligned} \dot{y} &= \frac{x^2 + 3y^2}{2xy} \\ &= \frac{x^2}{2xy} + \frac{3y^2}{2xy} \\ &= \frac{x}{2y} + \frac{3y}{2x} \\ &= \frac{1}{2 \cdot \frac{y}{x}} + \frac{3}{2} \cdot \frac{y}{x} \\ &= \frac{1}{2u} + \frac{3}{2}u \end{aligned}$$

Recall that $u = \frac{y}{x} \implies y = ux$. Then $\frac{dy}{dx} = u + x\frac{du}{dx}$. Substituting this in:

$$u + x\frac{du}{dx} = \frac{1}{2u} + \frac{3}{2}u$$
$$x\frac{du}{dx} = \frac{1}{2u} + \frac{3}{2}u - u$$
$$x\ du = \left(\frac{1}{2u} + \frac{1}{2}u\right) dx$$
$$\frac{1}{\frac{1}{2u} + \frac{1}{2}u}\ du = \frac{dx}{x}$$

Note that this is effectively transforming the original equation into the separable form $\frac{dv}{F(v)-v} = \frac{dx}{x}$. Now we integrate by partial fractions:

$$\int \frac{1}{\frac{1}{2u} + \frac{1}{2}u}\ du = \int \frac{dx}{x}$$
$$\int \frac{2u}{1+u^2}\ du = \ln|x| + C$$
$$\int \frac{1}{1+u^2}\ d(u^2) = \ln|x| + C$$
$$\ln|1+u^2| = \ln|x| + C$$
$$1 + u^2 = K|x|\ , \quad K = e^C 1 + u^2 \qquad = Bx \quad B = \pm K$$

Then substituting back for $u = \frac{y}{x}$:

$$1 + \left(\frac{y}{x}\right)^2 = Bx$$
$$1 + \frac{y^2}{x^2} = Bx$$
$$y^2 = Bx^3 - x^2$$

Rearranging gives the general solution:

$$y^2 = Bx^3 - x^2$$

We can solve for y:

$$\boxed{y = \pm\sqrt{Bx^3 - x^2}}$$

Now using the initial condition $y(-2) = 6$ to solve for B:

$$6 = \pm\sqrt{B(-2)^3 - (-2)^2}$$
$$36 = -8B - 4$$
$$40 = -8B$$
$$B = -5$$

We must choose the positive root since $y(-2) = 6 > 0$. Thus the particular solution is:

$$\boxed{y = \sqrt{-5x^3 - x^2}}$$

The domain is $-5x^3 - x^2 \geqslant 0 \implies x^2(-5x - 1) \geqslant 0 \implies x \leqslant -\frac{1}{5}$.

2.4 Exact equations

We have solely dealt with first-order linear and separable differential equations so far. However, some ODEs will have functions of both x and y that prevent separation.

Example 2.4.1

Solve the following ODE:

$$(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

Solution: Now consider some function $\Psi(x, y)$ (don't worry about how we got it yet) such that:

$$\Psi(x, y) = y^2 + (x^2 + 1)y - 3x^3$$

If we compute the partial derivatives, we find:

$$\Psi_x = 2xy - 9x^2$$
$$\Psi_y = 2y + x^2 + 1$$

These expressions appear in our original equation. Using the chain rule for partial derivatives:

$$\frac{d}{dx}\Psi(x, y(x)) = \Psi_x\frac{dx}{dx} + \Psi_y\frac{dy}{dx}$$
$$\frac{d}{dx}\Psi(x, y(x)) = \Psi_x + \Psi_y\frac{dy}{dx}$$

Then we can rewrite the original ODE as:

$$\Psi_x + \Psi_y\frac{dy}{dx} = 0$$

Thus, we have:

$$\frac{d}{dx}\Psi(x, y) = 0$$

Integrating both sides with respect to x gives:

$$\Psi(x, y) = C$$

So the general solution to the ODE is:

$$\boxed{y^2 + (x^2 + 1)y - 3x^3 = C}$$

We are therefore concerned with obtaining a method to find such a function $\Psi(x, y)$ for a given ODE.

Definition 2.4.1: Exact equation

An ODE of the form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is said to be an **exact equation** if there exists a function $\Psi(x, y)$ such that:

$$\Psi_x = M(x, y)$$
$$\Psi_y = N(x, y)$$

If an ODE is exact, we have:

$$\boxed{\frac{d}{dx}\Psi(x, y) = \Psi_x + \Psi_y\frac{dy}{dx} = 0}$$

Then the solution to the ODE is given implicitly by:

$$\boxed{\Psi(x, y) = C}$$

If $\Psi(x, y)$ is continuously differentiable, then we have that:

$$(\Psi_x)_y = (\Psi_y)_x$$

Then the equation is only exact if:

$$\boxed{M_y = N_x}$$

2.4.1 Steps to solve exact equations

1. Verify that the equation is exact by checking if $M_y = N_x$.
2. Integrate $M(x, y)$ with respect to x to find $\Psi(x, y)$ up to a function of y:

$$\Psi(x, y) = \int M(x, y)\,dx + h(y)$$

3. Differentiate $\Psi(x, y)$ with respect to y and set it equal to $N(x, y)$ to solve for $h(y)$:

$$\frac{\partial \Psi}{\partial y} = N(x, y)$$

4. Substitute $h(y)$ back into $\Psi(x, y)$.
5. (Optional) Apply the initial condition if given back into:

$$\Psi(x, y) = C$$

Note:-

If it is easier, you can also integrate $N(x, y)$ with respect to y first, then differentiate with respect to x to find the function of x.

Example 2.4.2

Solve the following initial value problem:

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3$$

Solution:

1. Find and verify M_y and N_x:

$$\begin{aligned} M(x, y) &= 2xy - 9x^2 \\ N(x, y) &= 2y + x^2 + 1 \\ M_y(x, y) &= 2x \\ N_x(x, y) &= 2x \end{aligned}$$

   Since $M_y = N_x$, the equation is exact.

2. Integrate $M(x, y)$ with respect to x:

$$\begin{aligned} \Psi(x, y) &= \int (2xy - 9x^2)\,dx + h(y) \\ &= x^2 y - 3x^3 + h(y) \end{aligned}$$

3. Differentiate $\Psi(x, y)$ with respect to y and set equal to $N(x, y)$:

$$\begin{aligned} \frac{\partial \Psi}{\partial y} &= x^2 + h'(y) \\ x^2 + h'(y) &= 2y + x^2 + 1 \\ h'(y) &= 2y + 1 \end{aligned}$$

   Integrating gives:

$$h(y) = y^2 + y + K$$

4. Substitute $h(y)$ back into $\Psi(x, y)$:

$$\Psi(x, y) = x^2 y - 3x^3 + y^2 + y + K$$

5. Apply the initial condition $y(0) = -3$:

$$\begin{aligned}\Psi(0, -3) &= 0^2 \cdot (-3) - 3 \cdot 0^3 + (-3)^2 + (-3) + K \\ &= 0 + 0 + 9 - 3 + K \\ &= 6 + K\end{aligned}$$

Setting this equal to C, we have:

$$C = 6 + K$$

Thus, the particular solution is:

$$\boxed{x^2 y - 3x^3 + y^2 + y = 6}$$

6. We could also solve for y explicitly using the quadratic formula:

$$\begin{aligned}y^2 + (x^2 + 1)y + (-3x^3 - 6) = 0 \\ y &= \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(-3x^3 - 6)}}{2} \\ y &= \frac{-(x^2+1) \pm \sqrt{x^4 + 2x^2 + 1 + 12x^3 + 24}}{2} \\ y &= \frac{-(x^2+1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}\end{aligned}$$

We must now choose the correct sign for the initial condition $y(0) = -3$, since -3 is less than $\frac{-(0^2+1)}{2} = -\frac{1}{2}$. Thus we choose the negative root:

$$\boxed{y = \frac{-(x^2+1) - \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}}$$

2.5 Non-exact equations

Sometimes, we will have that $M_y \neq N_x$. Like with integrating factors, we can multiply through by some function $\mu(x, y)$ to make the equation exact.

Let there be some function $\mu(x, y)$ such that:

$$\mu M + \mu N \dot{y} = 0$$

Then for exactness, we need:

$$(\mu M)_y = (\mu N)_x$$

Expanding both sides using the product rule:

$$\begin{aligned}(\mu M)_y &= \mu_y M + \mu M_y \\ (\mu N)_x &= \mu_x N + \mu N_x\end{aligned}$$

This is a partial differential equation in $\mu(x, y)$. In general, this is difficult to solve. However, if we assume that μ is only a function of x or only a function of y, we can reduce this to an ordinary differential equation.

Definition 2.5.1: Integrating factor for non-exact equations

An **integrating factor** for a non-exact equation of the form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is a function $\mu(x)$ or $\mu(y)$ such that multiplying through by μ makes the equation exact.

1. **Case 1: $\mu = \mu(x)$ (function of only x).** Then if we rearrange for $\frac{d\mu}{dx}$, we have:

$$\begin{aligned}
\mu_y M + \mu M_y &= \mu_x N + \mu N_x \\
0 + \mu M_y &= \frac{d\mu}{dx} N + \mu N_x \\
\mu M_y - \mu N_x &= \frac{d\mu}{dx} N \\
\frac{1}{\mu}\frac{d\mu}{dx} &= \frac{M_y - N_x}{N} \\
\frac{d\mu}{dx} &= \mu \cdot \frac{M_y - N_x}{N}
\end{aligned}$$

   As this is now separable, we can write the integrating factor as:

$$\boxed{\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}}$$

2. **Case 2: $\mu = \mu(y)$ (function of only y).** Then if we rearrange for $\frac{d\mu}{dy}$, we have:

$$\begin{aligned}
\mu_y M + \mu M_y &= \mu_x N + \mu N_x \\
\frac{d\mu}{dy} M + \mu M_y &= 0 + \mu N_x \\
\frac{d\mu}{dy} M &= \mu N_x - \mu M_y \\
\frac{1}{\mu}\frac{d\mu}{dy} &= \frac{N_x - M_y}{M} \\
\frac{d\mu}{dy} &= \mu \cdot \frac{N_x - M_y}{M}
\end{aligned}$$

   Then we can write the integrating factor as:

$$\boxed{\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}}$$

Note:-

This method only works if we can find an integrating factor that is solely a function of x or solely a function of y. So we must first check if either $\frac{M_y - N_x}{N}$ is a function of x only or $\frac{N_x - M_y}{M}$ is a function of y only.

2.5.1 Steps to solve non-exact equations using integrating factors

1. Compute M_y and N_x.
2. Check if $\frac{M_y - N_x}{N}$ is a function of x only or if $\frac{N_x - M_y}{M}$ is a function of y only.
3. If $\frac{M_y - N_x}{N}$ is a function of x only, compute the integrating factor:

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

   If $\frac{N_x - M_y}{M}$ is a function of y only, compute the integrating factor:

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

4. Multiply the original ODE by the integrating factor $\mu(x)$ or $\mu(y)$.

5. Verify that the new equation is exact by checking if the new $M_y = N_x$.
6. Solve the exact equation using the steps for exact equations.
7. (Optional) Apply any initial conditions to solve for constants of integration.

Example 2.5.1

Find a general solution to the ODE:

$$y + (2x - ye^y)\frac{dy}{dx} = 0$$

Solution:

1. We compute M_y and N_x:

$$\begin{aligned} M(x,y) &= y \\ N(x,y) &= 2x - ye^y \\ M_y(x,y) &= 1 \\ N_x(x,y) &= 2 \end{aligned}$$

   Then this is not exact since $M_y \neq N_x$.

2. We check if $\frac{M_y - N_x}{N}$ is a function of x only or if $\frac{N_x - M_y}{M}$ is a function of y only:

$$\begin{aligned} \frac{M_y - N_x}{N} &= \frac{1-2}{2x - ye^y} = \frac{-1}{2x - ye^y} \quad \text{(not a function of } x \text{ only)} \\ \frac{N_x - M_y}{M} &= \frac{2-1}{y} = \frac{1}{y} \quad \text{(function of } y \text{ only)} \end{aligned}$$

3. We compute the integrating factor:

$$\begin{aligned} \mu(y) &= e^{\int \frac{1}{y}\,dy} \\ &= e^{\ln|y|} \\ &= |y| \end{aligned}$$

4. We multiply through by $\mu(y) = |y|$:

$$y^2 + |y|(2x - ye^y)\frac{dy}{dx} = 0$$

5. We verify exactness by computing the new M_y and N_x:

$$\begin{aligned} M(x,y) &= y^2 \\ N(x,y) &= |y|(2x - ye^y) \\ M_y(x,y) &= 2y \\ N_x(x,y) &= 2|y| \end{aligned}$$

   Since $M_y = N_x$, the new equation is exact.

6. We solve the exact equation:

$$\begin{aligned} \Psi(x,y) &= \int y^2\,dx + h(y) \\ &= xy^2 + h(y) \end{aligned}$$

Differentiating with respect to y and setting equal to $N(x, y)$:

$$\begin{aligned}
\Psi_y &= 2xy + h'(y) \\
2xy + h'(y) &= |y|(2x - ye^y) \\
h'(y) &= |y|(2x - ye^y) - 2xy \\
h'(y) &= -|y|ye^y \\
h'(y) &= -y^2e^y \quad \text{(since } y^2 \geqslant 0\text{, we can drop the absolute value)}
\end{aligned}$$

Integrating gives:

$$\begin{aligned}
h(y) &= \int -y^2e^y \, dy \\
&= -(y^2 - 2y + 2)e^y + K
\end{aligned}$$

Substituting back into $\Psi(x, y)$:

$$\Psi(x, y) = xy^2 - (y^2 - 2y + 2)e^y + K$$

Thus, the general solution is given implicitly by:

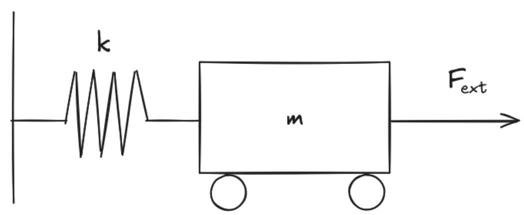
$$\boxed{xy^2 - (y^2 - 2y + 2)e^y = C}$$

Chapter 3

Techniques for solving second-order ODEs

3.1 Classical mechanics: mass-spring system

Consider a mass m attached to a spring with spring constant k. It is also subject to an external force F_{ext}.



Note that velocity is the first derivative of position with respect to time, denoted $\dot{x}(t)$, and acceleration is the second derivative of position with respect to time, denoted $\ddot{x}(t)$.

Hooke's law gives the force exerted by the spring:

$$F_{\text{spring}} = -kx, \quad k > 0$$

Then, by Newton's second law, the total force on the mass is equal to its mass times its acceleration:

$$m\ddot{x} = F_{\text{spring}} + F_{\text{ext}}$$

Rearranging gives the second-order ODE:

$$m\ddot{x} + kx = F_{\text{ext}}$$

We could also apply a damping force (such as friction) proportional to velocity, with damping coefficient b. We call this:

$$F_{\text{damping}} = -b\dot{x}, \quad b > 0$$

Then the ODE becomes:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$$

Definition 3.1.1: Linear ODE

A linear ODE is one of the form:

$$a_m x^{(m)} + a_{m-1} x^{(m-1)} + \cdots + a_1 \dot{x} + a_0 x = q(t)$$

where $a_0, a_1, \ldots, a_m$ are coefficients that may depend on t but not on x or its derivatives, and $q(t)$ is a function of t.

We call this a m-th order linear ODE.

In our mass-spring system, we will have two initial conditions.

1. Initial position: $x(0) = x_0$
2. Initial velocity: $\dot{x}(0) = v_0$

Example 3.1.1 (A special case)

Consider a mass-spring system where $b = 0$ (no damping) and there is no external force ($F_{\text{ext}} = 0$). Solve the ODE:

$$m\ddot{x} + kx = 0$$

with initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

Solution: Without knowing any specific methods, we rearrange the equation:

$$\ddot{x} = -\frac{k}{m}x$$

So we need some function whose second derivative is proportional to the negative of the function itself. We know that sine and cosine functions have this property.

Try $x = \sin(\omega t)$, and differentiate several times:

$$\begin{aligned} \dot{x} &= \omega\cos(\omega t) \\ \ddot{x} &= -\omega^2\sin(\omega t) \end{aligned}$$

Then substituting into the ODE gives:

$$-m\omega^2\sin(\omega t) + k\sin(\omega t) = 0$$

$$\boxed{x(t) = \sin(\omega t), \quad \text{where } \omega = \sqrt{\frac{k}{m}}}$$

This is only a particular solution (you could try with cosine as well, and the general solution is a linear combination of both). Intuitively, the mass will oscillate back and forth indefinitely.

Theorem 3.1.1 Superposition principle (applied to second-order linear ODEs)

For a linear ODE, if $x_1(t)$ and $x_2(t)$ are solutions to the homogeneous equation (i.e., when $q(t) = 0$), then any linear combination of these solutions is also a solution. That is:

$$x(t) = C_1x_1(t) + C_2x_2(t)$$

is also a solution for any constants C_1 and C_2. The functions $x_1(t)$ and $x_2(t)$ are called the **modes** of the system.

Proof: Let $x_1(t)$ and $x_2(t)$ be solutions to the homogeneous equation:

$$a_2\ddot{x} + a_1\dot{x} + a_0x = 0$$

Then, substituting the potential solution $x(t) = C_1x_1(t) + C_2x_2(t)$:

$$\begin{aligned} a_2\ddot{x} + a_1\dot{x} + a_0x &= a_2\left(C_1\ddot{x}_1 + C_2\ddot{x}_2\right) + a_1\left(C_1\dot{x}_1 + C_2\dot{x}_2\right) + a_0\left(C_1x_1 + C_2x_2\right) \\ &= C_1\left(a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0x_1\right) + C_2\left(a_2\ddot{x}_2 + a_1\dot{x}_2 + a_0x_2\right) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0 \end{aligned}$$

Thus, $x(t) = C_1x_1(t) + C_2x_2(t)$ is also a solution. 😃

Example 3.1.2 (General solution)

Find the general solution to the ODE:
$$m\ddot{x} + kx = 0$$

Solution: From the previous example, we know that both $\sin(\omega t)$ and $\cos(\omega t)$ are solutions, where $\omega = \sqrt{\frac{k}{m}}$.
Therefore, by the superposition principle, the general solution is:

$$\boxed{x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)}$$

where C_1 and C_2 are constants determined by the initial conditions.

Note:-

The functions $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent because to transform one into another, you would need to add some factor in the argument (i.e., phase shift). This is not a scalar multiple transformation, so no linear transformation exists between them.

3.2 Equations with constant coefficients

3.2.1 Deriving the characteristic equation

We make the ansatz (educated guess) that the solution is of the form:

$$x(t) = ce^{rt}$$

where r is a constant to be determined. This guess is motivated by the fact that exponentials have the property that their derivatives are proportional to themselves (remember how we worked with the sin and cos functions earlier).

We want to attempt to form a linear combination of such solutions.

Recalling our equation of the form:
$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting our ansatz into the ODE gives:

$$mr^2ce^{rt} + brce^{rt} + kce^{rt} = 0$$

The trivial solution is $c = 0$, but we are interested in non-trivial solutions where $c \neq 0$. Dividing both sides by ce^{rt} (which is never zero) gives the characteristic equation:

$$mr^2 + br + k = 0$$

This is the **characteristic equation** of the ODE. It can be observed that we will obtain two roots, or two **modes**. This makes sense - we expect two linearly independent solutions for a second-order ODE, with the general solution being a linear combination of these two solutions.

3.2.2 Distinct real roots

Example 3.2.1

Find the general solution to the ODE:
$$\ddot{x} + 5\dot{x} + 4x = 0$$

Then, find the specific solution satisfying the initial conditions $x(0) = 2$ and $\dot{x}(0) = -5$.

Solution: We write this as
$$1 \cdot \ddot{x} + 5 \cdot \dot{x} + 4 \cdot x = 0$$

so that we can identify $m = 1$, $b = 5$, and $k = 4$. Then, the characteristic equation is:

$$P(s) = 1 \cdot s^2 + 5 \cdot s + 4 = 0$$

Solving for the roots, we have:

$$r^2 + 5r + 4 = 0$$

Factoring gives:

$$(r + 4)(r + 1) = 0$$

Thus, the roots are $r_1 = -4$ and $r_2 = -1$.

Therefore, the general solution is:

$$\boxed{x(t) = C_1 e^{-4t} + C_2 e^{-t}}$$

where C_1 and C_2 are constants determined by initial conditions.

To find the specific solution, we apply the initial conditions:

$$\begin{aligned}
x(0) &= C_1 e^0 + C_2 e^0 = C_1 + C_2 = 2 \quad (1) \\
\dot{x}(t) &= -4C_1 e^{-4t} - C_2 e^{-t} \\
\dot{x}(0) &= -4C_1 e^0 - C_2 e^0 = -4C_1 - C_2 = -5 \quad (2)
\end{aligned}$$

Solving equations (1) and (2): From (1): $C_2 = 2 - C_1$. Substituting into (2):

$$\begin{aligned}
-4C_1 - (2 - C_1) &= -5 \\
-4C_1 - 2 + C_1 &= -5 \\
-3C_1 - 2 &= -5 \\
-3C_1 &= -3 \\
C_1 &= 1
\end{aligned}$$

Then from (1):

$$C_2 = 2 - 1 = 1$$

Thus, the specific solution is:

$$\boxed{x(t) = e^{-4t} + e^{-t}}$$

With an equation of the form $m\ddot{x} + b\dot{x} + kx = 0$, we can summarize the types of solutions based on the discriminant $D = b^2 - 4mk$:

- **Overdamped** ($D > 0$): Two distinct real roots, leading to solutions that decay exponentially without oscillating.

  $$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- **Critically damped** ($D = 0$): One repeated real root, leading to solutions that decay to zero as quickly as possible without oscillating.

  $$x(t) = C_1 e^{rt} + C_2 t e^{rt}$$

- **Underdamped** ($D < 0$): Complex conjugate roots, leading to **oscillatory** solutions that decay exponentially.

3.2.3 Review of complex numbers

Note:-

The lectures started from the introduction of $i^2 = -1$. These notes skip ahead to the parts more relevant to ODEs.

Definition 3.2.1: Argument of a complex number

The **argument** of a complex number $z = a + bi$ is the angle θ formed with the positive real axis in the complex plane. It is denoted as $\arg(z)$.
The argument can be calculated using the arctangent function:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Some useful properties of complex numbers and their arguments:

$$
\begin{aligned}
|z_1 z_2| &= |z_1||z_2| \\
\arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\
\arg(z^n) &= n \arg(z)
\end{aligned}
$$

We can parametrize a complex number on the unit circle by letting $t = \arg(z)$:

$$z = \cos t + i \sin t$$

Then, differentiating:

$$\dot{z} = -\sin t + i \cos t$$

This then implies a useful identity:

$$\boxed{\dot{z} = iz}$$

Solving this ODE gives:

$$z(t) = z(0)e^{it}$$

Euler's identity states that:

$$\boxed{e^{it} = \cos t + i \sin t}$$

This is useful because it allows us to express trigonometric functions in terms of exponentials.
Another interesting identity arises when we set $t = \pi$:

$$e^{i\pi} + 1 = 0$$

Now, consider a complex number $z = a + bi$. We have that:

$$
\begin{aligned}
\dot{z} &= (a + bi)z \\
z(t) &= z(0)e^{(a+bi)t} \\
&= z(0)e^{at}(\cos bt + i \sin bt)
\end{aligned}
$$

where e^{at} represents the magnitude (scaling) and $\cos bt + i \sin bt$ represents the rotation in the complex plane (argument).
Another useful property of complex numbers:

$$\boxed{\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})}$$

$$\boxed{\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})}$$

where $\overline{z} = a - bi$ is the complex conjugate of $z = a + bi$.
Then we define cos and sin in terms of exponentials:

$$\boxed{\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right)}$$

$$\boxed{\sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right)}$$

3.2.4 Complex roots

Theorem 3.2.1

If z is a complex-valued solution to $m\ddot{z} + b\dot{z} + kz = 0$, then both the real part $\mathrm{Re}(z)$ and the imaginary part

Im(z) are also solutions to the ODE.

Proof: This should not be surprising. If we write $z = u(t) + iv(t)$ where $u(t) = \text{Re}(z)$ and $v(t) = \text{Im}(z)$, then substituting into the ODE gives:

$$\begin{aligned} m\ddot{z} + b\dot{z} + kz &= m(\ddot{u} + i\ddot{v}) + b(\dot{u} + i\dot{v}) + k(u + iv) \\ &= (m\ddot{u} + b\dot{u} + ku) + i(m\ddot{v} + b\dot{v} + kv) \end{aligned}$$

We basically get two separate equations, corresponding to the real and imaginary parts:

$$\begin{aligned} m\ddot{u} + b\dot{u} + ku &= 0 \\ m\ddot{v} + b\dot{v} + kv &= 0 \end{aligned}$$

Since z is a solution, both parts must equal zero. Thus, both Re(z) and Im(z) are solutions to the ODE. 😄

We can observe that an oscillatory solution arises when the roots of the characteristic equation are complex conjugates. To derive this, we use the second-order ODE:

$$r_1 = a + bi, \quad r_2 = a - bi$$

Extracting the real and imaginary parts gives us two linearly independent solutions:

$$\begin{aligned} x_1(t) &= e^{(a+bi)t} = e^{at}\cos(bt) \\ x_2(t) &= e^{(a-bi)t} = e^{at}\sin(bt) \end{aligned}$$

So the general solution is a linear combination of these two solutions (where a determines the exponential decay or growth - the real part, and b determines the frequency of oscillation - the imaginary part):

$$\boxed{x(t) = C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)}$$

We can also represent this as a phase shifted cosine function:

$$\boxed{x(t) = Ae^{at}\cos(bt - \phi)}$$

where A is the amplitude and ϕ is the phase shift, which can be determined from the constants C_1 and C_2. Formulas for A and ϕ are:

$$\boxed{A = \sqrt{C_1^2 + C_2^2}}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right)}$$

The identity used to get this phase shift:

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \frac{C_1}{A}\cos bt + \frac{C_2}{A}\sin bt \end{aligned}$$

Example 3.2.2

Find the general solution to the ODE:

$$\ddot{x} + 4\dot{x} + 5x = 0$$

Solution: We write this as

$$1 \cdot \ddot{x} + 4 \cdot \dot{x} + 5 \cdot x = 0$$

so that we can identify $m = 1$, $b = 4$, and $k = 5$. Then, the characteristic equation is:

$$P(s) = 1 \cdot s^2 + 4 \cdot s + 5 = 0$$

Solving for the roots, we have:

$$r^2 + 4r + 5 = 0$$

Using the quadratic formula:

$$r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

Thus, the roots are $r_1 = -2 + i$ and $r_2 = -2 - i$. Therefore, the general solution is:

$$\boxed{x(t) = e^{-2t}(C_1 \cos t + C_2 \sin t)}$$

We can also express this as a phase shifted cosine function:

$$\boxed{x(t) = Ae^{-2t}\cos(t - \phi)}$$

3.2.5 Repeated roots

Let $P(D)$ be the polynomial differential operator associated with a linear ODE, where Q is a polynomial with no repeated roots. If r is a root of multiplicity m of the characteristic polynomial $P(\lambda)$, then

$$P(\lambda) = (\lambda - r)^m Q(\lambda) \quad \Rightarrow \quad P(D) = (D - r)^m Q(D).$$

Since $Q(r) \neq 0$:

$$(D - r)^m y = 0.$$

Solving successively,

$$\begin{aligned} (D - r)y &= 0 \\ Dy - ry &= 0 \\ \frac{dy}{dt} &= ry \\ y &= e^{rt} \end{aligned}$$

and each additional application of $(D - r)^{-1}$ introduces a factor of t. Hence the m linearly independent solutions are

$$e^{rt},\ te^{rt},\ t^2e^{rt},\ \ldots,\ t^{m-1}e^{rt}.$$

Example 3.2.3

Find the general solution to the ODE:

$$\ddot{x} - 4\dot{x} + 4x = 0$$

Solution: We write this as

$$1 \cdot \ddot{x} - 4 \cdot \dot{x} + 4 \cdot x = 0$$

so that we can identify $m = 1$, $b = -4$, and $k = 4$. Then, the characteristic equation is:

$$P(s) = 1 \cdot s^2 - 4 \cdot s + 4 = 0$$

Solving for the roots, we have:

$$r^2 - 4r + 4 = 0$$

Factoring gives:

$$(r - 2)^2 = 0$$

Thus, there is a repeated root at $r = 2$.

Therefore, the general solution is:

$$\boxed{x(t) = C_1e^{2t} + C_2te^{2t}}$$

3.3 Non-homogeneous equations

We were previous using $m\ddot{x} + b\ddot{x} + kx = 0$ to represent a situation where a mass is attached to a spring with no external damping force.

If we introduce an external force F_{ext}, then the ODE becomes:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$$

We know the solution to the homogeneous equation. Also, if $b = 0$, we can define the **natural frequency** as:

$$w_m = \sqrt{\frac{k}{m}}$$

such that the solution is:

$$x(t) = C_1 \cos(w_m t) + C_2 \sin(w_m t) \quad \text{from the characteristic equation solved via the quadratic formula}$$

And if $b \neq 0$, we can define the **damped frequency** as:

$$w_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{from the characteristic equation solved via the quadratic formula}$$

such that the solution is:

$$x(t) = e^{-\frac{b}{2m}t}(C_1 \cos(w_d t) + C_2 \sin(w_d t))$$

We standardize the homogeneous solution as $x_h(t)$:

$$\ddot{x} + \frac{b}{m}\dot{x} + w_m^2 x = 0$$

where w_m is the natural frequency.

Theorem 3.3.1

If $x_p(t)$ is a particular solution to the non-homogeneous equation $m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$, then the general solution is given by:

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ is the general solution to the corresponding homogeneous equation.

Proof: Let $x_h(t)$ be the general solution to the homogeneous equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

Then, we can write the general solution to the non-homogeneous equation as:

$$x(t) = x_h(t) + x_p(t)$$

Substituting into the non-homogeneous equation gives:

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= m(\ddot{x}_h + \ddot{x}_p) + b(\dot{x}_h + \dot{x}_p) + k(x_h + x_p) \\ &= (m\ddot{x}_h + b\dot{x}_h + kx_h) + (m\ddot{x}_p + b\dot{x}_p + kx_p) \\ &= 0 + F_{\text{ext}} = F_{\text{ext}} \end{aligned}$$

Thus, $x(t) = x_h(t) + x_p(t)$ is indeed a solution to the non-homogeneous equation. Since $x_h(t)$ represents the general solution to the homogeneous equation, this form captures all possible solutions to the non-homogeneous equation. ☺

3.3.1 Steps to solve a non-homogeneous linear ODE

1. Find the homogeneous solution $x_h(t)$ by solving the characteristic equation.
2. Find a particular solution $x_p(t)$ using an appropriate method (e.g., undetermined coefficients, variation of parameters).
3. Combine the homogeneous and particular solutions to get the general solution:

$$x(t) = x_h(t) + x_p(t)$$

4. (Optional) Apply initial conditions to find the specific solution if needed.

3.3.2 Method of undetermined coefficients

Suppose that we have a non-homogeneous ODE of the form:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}(t)$$

We suppose that $F_{\text{ext}}(t) = ky(t)$ where $y(t)$ is a known function. We also consider the case with no damping first ($b = 0$):

$$m\ddot{x} + kx = ky(t)$$

Rearranging gives:

$$m\ddot{x} + k(x - y) = 0$$

We think of $x-y$ as the deviation from the equilibrium position y. From a physical perspective, the system will oscillate around the equilibrium position y, which sets some natural frequency of oscillation.

$$y(t) = A\cos(\omega t)$$

Then inputting wih $\omega_m = \sqrt{\frac{k}{m}}$ gives:

$$\ddot{x} + \omega_m^2 x = \omega_m^2 A\cos(\omega t)$$

Seeing the cos term, we guess a particular solution of the form:

$$x_p(t) = B\cos(\omega t)$$

Plugging this into the ODE gives:

$$\begin{aligned} x_p(t) &= B\cos(\omega t) \\ \ddot{x}_p &= -B\omega^2\cos(\omega t) \\ -B\omega^2\cos(\omega t) + \omega_m^2 B\cos(\omega t) &= \omega_m^2 A\cos(\omega t) \end{aligned}$$

This is true if:

$$\boxed{B = \frac{\omega_m^2}{\omega_m^2 - \omega^2} A}$$

where ω_m is the natural frequency of the system and ω is the frequency of the external force. We can see that if ω is close to ω_m, then B becomes very large, which corresponds to the phenomenon of **resonance**. In reality, with $\omega_m = \omega$, we would need to modify our guess. We will later see how to deal with resonance.

In general, if $F_{\text{ext}}(t) = A\cos(\omega t) + B\sin(\omega t)$ appears to be a sinusoidal function, we can guess:

$$x_p(t) = C\cos(\omega t) + D\sin(\omega t)$$

Then, we can plug this into the ODE and solve for C and D.

3.3.3 Exponential inputs

Similarly, we can have $F_{\text{ext}}(t) = Ae^{rt}$, which gives:

$$m\ddot{x} + b\dot{x} + kx = Ae^{rt}$$

Then, we can guess a particular solution of the form:

$$x_p(t) = Be^{rt}$$

Plugging this into the ODE gives:

$$\begin{aligned} x_p(t) &= Be^{rt} \\ \dot{x}_p &= rBe^{rt} \\ \ddot{x}_p &= r^2Be^{rt} \\ mr^2Be^{rt} + brBe^{rt} + kBe^{rt} &= Ae^{rt} \end{aligned}$$

So we can see that to solve for B, we can divide both sides by e^{rt} (which is never zero) to get:

$$B = \frac{A}{mr^2 + br + k}$$

Note:-

In general, when we have an m^{th} order ODE, we can use a similar approach to find particular solutions for exponential inputs. We have:

$$a_m x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1\dot{x} + a_0 x = Ae^{rt}$$

Then, we apply the function:

$$P(r) = a_m r^m + a_{m-1}r^{m-1} + \cdots + a_1 r + a_0$$

We have thus obtained the **exponential response formula (ERF)**:

$$\boxed{x_p(t) = \frac{Ae^{rt}}{p(r)} \quad \text{where } p(r) = a_m r^m + a_{m-1}r^{m-1} + \cdots + a_1 r + a_0 \quad p(r) \neq 0}$$

Example 3.3.1

Find the particular solution of:

$$\ddot{x} + \dot{x} + 2x = 4e^{3t}$$

Then find the general solution.

Solution: We can identify the homogeneous part of the ODE as:

$$\ddot{x} + \dot{x} + 2x = 0$$

The characteristic equation is:

$$r^2 + r + 2 = 0$$

Solving for the roots gives:

$$r = \frac{-1 \pm \sqrt{1-8}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$$

Thus, the homogeneous solution is:

$$x_h(t) = e^{-\frac{1}{2}t}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

For the particular solution, we can apply the exponential response formula:

$$p(r) = r^2 + r + 2$$

Then, evaluating at $r = 3$ gives:

$$p(3) = 3^2 + 3 + 2 = 14$$

Thus, the particular solution is:

$$x_p(t) = \frac{4e^{3t}}{14} = \frac{2}{7}e^{3t}$$

Finally, the general solution is:

$$\boxed{x(t) = e^{-\frac{1}{2}t}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}t\right)\right) + \frac{2}{7}e^{3t}}$$

Example 3.3.2

Find the particular solution of:

$$\ddot{x} + \dot{x} + 2x = \cos t$$

Solution: We first complexify the equation as:

$$\ddot{z} + \dot{z} + 2z = e^{it}$$

It is possible to do this because the real and imaginary parts of the solution will also be solutions to the original equation, so we can extract the real part at the end to get the particular solution.

$$p(s) = s^2 + s + 2$$

Then evaluating at $s = i$ gives:

$$p(i) = i^2 + i + 2 = 1 + i$$

Applying ERF gives:

$$z_p(t) = \frac{e^{it}}{1+i} = \frac{1-i}{2}e^{it} = \frac{1}{2}(1-i)e^{it}$$

Thus the particular solution to this complexified equation is:

$$\begin{aligned} z_p(t) &= \frac{1}{2}e^{it} - \frac{i}{2}e^{it} \\ &= \frac{1}{2}(\cos t + i\sin t) - \frac{i}{2}(\cos t + i\sin t) \\ &= \frac{1}{2}\cos t + \frac{i}{2}\sin t - \frac{i}{2}\cos t + \frac{1}{2}\sin t \\ &= \frac{1}{2}(\cos t + \sin t) + i\left(\frac{1}{2}(\sin t - \cos t)\right) \end{aligned}$$

We can then extract the real part to get the particular solution to the original equation:

$$\begin{aligned} x_p(t) &= \mathrm{Re}(z_p(t)) \\ &= \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ &= \frac{\sqrt{2}}{2}\cos\left(t - \frac{\pi}{4}\right) \end{aligned}$$

3.3.4 Resonance

An operator takes one function as input and produces another function as output. For example, the operator D takes a function $f(t)$ and produces its derivative $\dot{f}(t)$. We can also have higher-order operators like D^2 which takes a function and produces its second derivative.

The **identity operator** I is:

$$Ix = x$$

We can take linear combinations of operators. For example:

$$(D^2 + 2D + 2I)x = \ddot{x} + 2\dot{x} + 2x$$

The characteristic polynomial associated with this operator is:

$$p(s) = s^2 + 2s + 2$$

Meanwhile, $P(D)$ represents the linear combination of operators:

$$p(D) = D^2 + 2D + 2I$$

Then, we can write the ODE as:

$$p(D)x$$

Now consider an input function $q(t)$, which gives us the non-homogeneous ODE:

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1\dot{x} + a_0 x = q(t)$$

We extract the characteristic polynomial:

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Then, we can write the ODE as:

$$p(D)x = q(t)$$

So the operator $p(D)$ represents the system with $x(t)$ as the response, and the input is represented by $q(t)$. Let $Lx = q$ for some operator L. Our goal is to find the inverse operator L^{-1} such that:

$$x = L^{-1}q$$

Recall the ERF which solves $p(D)x = Ae^{rt}$ for two conditions:

1. $p(r) \neq 0$.
2. A constant.

We note that if on the LHS we found the characteristic polynomial $p(s)$ with roots $r_1, r_2, \ldots, r_n$, but on the RHS we have $q(t) = Ae^{rt}$ where r is one of the roots of $p(s)$, then we have a problem because $p(r) = 0$, which means that the ERF does not apply. This corresponds to the phenomenon of **resonance**.

So we have that $p(D)x = Ae^{rt}$ where $p(r) = 0$. We'd like to make the function $p(D)$ more explicit to find some way to solve for a particular solution.

We repeatedly apply D to e^{rt}:

$$\begin{aligned}
Ie^{rt} &= e^{rt} \\
De^{rt} &= re^{rt} \\
D^2e^{rt} &= r^2e^{rt} \\
&\vdots \\
D^n e^{rt} &= r^n e^{rt} \\
p(D)e^{rt} &= p(r)e^{rt} = 0
\end{aligned}$$

By the ERF, $xp(t) = \frac{1}{p(r)}e^{rt}$ is a particular solution. But $P(D)e^{rt} = p(r)e^{rt}$ is true even if $p(r) = 0$. We start from the key identity:

$$p(D)e^{rt} = p(r)e^{rt}$$

Then we differentiate both sides with respect to r:

$$\frac{\partial}{\partial r}\left[p(D)e^{rt}\right] = \frac{\partial}{\partial r}\left[p(r)e^{rt}\right]$$

The operator $p(D)$ does not depend on r, so we can pull it out of the derivative:

$$p(D)\frac{\partial}{\partial r}e^{rt} = p'(r)e^{rt} + p(r)te^{rt}$$

Since $p(r) = 0$, this simplifies to:

$$p(D)\frac{\partial}{\partial r}e^{rt} = p'(r)e^{rt}$$

Now we get the partial derivative of e^{rt} with respect to r:

$$\frac{\partial}{\partial r}e^{rt} = te^{rt}$$

Thus, we have:

$$p(D)(te^{rt}) = p'(r)e^{rt}$$

Comparing this to the original equation $p(D)x = Ae^{rt}$, we can see that if we let $x = \frac{A}{p'(r)}te^{rt}$, then we have:

$$p(D)\left(\frac{A}{p'(r)}te^{rt}\right) = Ae^{rt}$$

Thus, we now have a formula for the particular solution in the case of resonance:

$$\boxed{x_p(t) = \frac{A}{p'(r)}te^{rt} \quad \text{where } p(r) = 0}$$

3.3.5 Generalization of resonance to higher-order ODEs

We might have a case where $p(r) = 0$, but also $p'(r) = 0$. In this case, we can differentiate the key identity again:

$$\frac{\partial^2}{\partial r^2}\left[p(D)e^{rt}\right] = \frac{\partial^2}{\partial r^2}\left[p(r)e^{rt}\right]$$

This gives us:

$$p(D)\frac{\partial^2}{\partial r^2}e^{rt} = p''(r)e^{rt} + 2p'(r)te^{rt} + p(r)t^2e^{rt}$$

Since $p(r) = 0$ and $p'(r) = 0$, this simplifies to:

$$p(D)\frac{\partial^2}{\partial r^2}e^{rt} = p''(r)e^{rt}$$

Now we get the second partial derivative of e^{rt} with respect to r:

$$\frac{\partial^2}{\partial r^2}e^{rt} = t^2e^{rt}$$

Thus, we have:

$$p(D)(t^2e^{rt}) = p''(r)e^{rt}$$

In general, the **generalized exponential response formula** for the case of resonance is:

$$\boxed{x_p(t) = \frac{A}{p^{(m+1)}(r)}t^{m+1}e^{rt} \quad \text{where } p(r) = p'(r) = \cdots = p^{(m)}(r) = 0}$$

Example 3.3.3

Find the particular solution of:

$$\ddot{x} - 4x = e^{-2t}$$

Solution: We first find the homogeneous solution by solving the characteristic equation:

$$r^2 - 4 = 0$$

which gives us the roots $r = 2$ and $r = -2$. Thus, the homogeneous solution is:

$$x_h(t) = C_1 e^{2t} + C_2 e^{-2t}$$

We see that -2 is a root of the characteristic polynomial, so we have resonance. We can apply the generalized exponential response formula to find the particular solution:

$$p(r) = r^2 - 4$$

Then we compute $p'(-2)$:

$$p'(r) = 2r \quad \Rightarrow \quad p'(-2) = -4$$

Thus, the particular solution is:

$$x_p(t) = \frac{1}{p'(-2)} t e^{-2t} = -\frac{1}{4} t e^{-2t}$$

Note:-

From a physical perspective, although we are at resonance, the particular solution does decay as t increases, which is a consequence of the fact that the input function e^{-2t} also decays as t increases. If we had an input function that did not decay, then we would have a particular solution that grows without bound as t increases, which is the classical notion of resonance.

Example 3.3.4

Find the particular solution of:

$$\ddot{x} + 4x = \cos(2t)$$

Solution: We first find the homogeneous solution by solving the characteristic equation:

$$r^2 + 4 = 0$$

which gives us the roots $r = 2i$ and $r = -2i$. Thus, the homogeneous solution is:

$$x_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

Complexifying the equation gives us:

$$\ddot{z} + 4z = e^{2it}$$

And hence we see that $2i$ is a root of the characteristic polynomial, so we have resonance. We can apply the generalized exponential response formula to find the particular solution:

$$p(r) = r^2 + 4$$

Then we compute $p'(2i)$:

$$p'(r) = 2r \quad \Rightarrow \quad p'(2i) = 4i$$

Thus, the particular solution to the complexified equation is:

$$\begin{aligned} z_p(t) &= \frac{te^{2it}}{4i} \\ &= \frac{t}{4i}(\cos(2t) + i\sin(2t)) \\ &= \frac{t}{4}\sin(2t) - \frac{t}{4} i\cos(2t) \end{aligned}$$

Then we can extract the real part to get the particular solution to the original equation:

$$x_p(t) = \frac{t}{4}\sin(2t)$$

This solution grows without bound as t increases, which is the classical notion of resonance.

3.3.6 Steps to using complexification to solve for particular solutions

1. Complexify the equation by replacing the real input function with a complex exponential function. For example, if the input is $\cos(\omega t)$, we can replace it with $e^{i\omega t}$.
2. Apply the exponential response formula to find a particular solution to the complexified equation.
3. Extract the real part of the particular solution to get a particular solution to the original equation (if the original input was a cosine function) or extract the imaginary part to get a particular solution to the original equation (if the original input was a sine function).

3.3.7 Polynomial inputs

If we have a polynomial input, we can guess a particular solution that is also a polynomial of the same degree. For example, if we have:

$$\ddot{x} + 4x = t^2$$

Then we can guess a particular solution of the form:

$$x_p(t) = At^2 + Bt + C$$

Then we can compute $\dot{x}_p$ and $\ddot{x}_p$:

$$\begin{aligned} \dot{x}_p &= 2At + B \\ \ddot{x}_p &= 2A \end{aligned}$$

Then we can plug this into the ODE to get:

$$\begin{aligned} 2A + 4(At^2 + Bt + C) &= t^2 \\ (4A)t^2 + (4B)t + (2A + 4C) &= t^2 \end{aligned}$$

Thus, we have the system of equations:

$$\begin{aligned} 4A &= 1 \\ 4B &= 0 \\ 2A + 4C &= 0 \end{aligned}$$

which gives us $A = \frac{1}{4}$, $B = 0$, and $C = -\frac{1}{8}$. Thus, the particular solution is:

$$\boxed{x_p(t) = \frac{1}{4}t^2 - \frac{1}{8}}$$

3.3.8 Reduction of order

Second-order ODEs may also have the form:

$$p(t)\ddot{x} + q(t)\dot{x} + r(t)x = 0$$

Example 3.3.5

Find the general solution to the ODE:

$$\ddot{x} + \dot{x} = t$$

Solution: We first find the homogeneous solution by solving the characteristic equation:

$$\begin{aligned} p(s) = s^2 + s &= 0 \\ s(s+1) &= 0 \end{aligned}$$

which gives us the roots $r = 0$ and $r = -1$. Thus, the homogeneous solution is:

$$x_h(t) = C_1 + C_2e^{-t}$$

We can use a substitution $u = \dot{x}$ to reduce the order of the ODE. Then we have $\dot{u} = \ddot{x}$, so we can rewrite the ODE as:

$$\dot{u} + u = t$$

Then if we guess $u = At + B$, and $\dot{u} = A$, we can plug this into the ODE to get:

$$\begin{aligned} A + At + B &= t \\ A &= 1 \\ B &= -1 \end{aligned}$$

Thus, we have $u(t) = t - 1$, and so $x(t) = \int u(t)dt = \int (t-1)dt = \frac{t^2}{2} - t + C_3$. We just let $C_3 = 0$ since it is already captured by the homogeneous solution. Thus, the particular solution is:

$$x_p(t) = \frac{t^2}{2} - t$$

Finally, the general solution is:

$$\boxed{x(t) = C_1 + C_2e^{-t} + \frac{t^2}{2} - t}$$

3.3.9 Variation of parameters

We might obtain an equation like:

$$3\ddot{x} + 8\dot{x} + 6x = (t^2 + 1)e^{-t}$$

Let us suppose some function $u(t)$ exists such that $x_p(t) = u(t)e^{-t}$ is a particular solution to the above equation. Then we can compute $\dot{x}_p$ and $\ddot{x}_p$:

$$\begin{aligned} \dot{x}_p &= \dot{u}e^{-t} - ue^{-t} \\ \ddot{x}_p &= \ddot{u}e^{-t} - 2\dot{u}e^{-t} + ue^{-t} \end{aligned}$$

Then we can plug this into the ODE to get:

$$\begin{aligned} 3(\ddot{u}e^{-t} - 2\dot{u}e^{-t} + ue^{-t}) + 8(\dot{u}e^{-t} - ue^{-t}) + 6(ue^{-t}) &= (t^2 + 1)e^{-t} \\ 3\ddot{u}e^{-t} + 2\dot{u}e^{-t} + ue^{-t} &= (t^2 + 1)e^{-t} \end{aligned}$$

Then we have:

$$3\ddot{u} + 2\dot{u} + u = t^2 + 1$$

To solve this, we guess $u = At^2 + Bt + C$, and then we can compute $\dot{u}$ and $\ddot{u}$:

$$\begin{aligned} \dot{u} &= 2At + B \\ \ddot{u} &= 2A \end{aligned}$$

Then we can plug this into the ODE to get:

$$\begin{aligned} 3(2A) + 2(2At + B) + (At^2 + Bt + C) &= t^2 + 1 \\ 6A + 4At + 2B + At^2 + Bt + C &= t^2 + 1 \end{aligned}$$

Grouping by powers of t gives:

$$At^2 + (4A + B)t + (6A + 2B + C) = t^2 + 1$$

Thus we have the system of equations:

$$\begin{aligned} A &= 1 \\ 4A + B &= 0 \\ 6A + 2B + C &= 1 \end{aligned}$$

We get that $A = 1$, $B = -4$, and $C = 3$.
So we have $u(t) = t^2 - 4t + 3$, and thus the particular solution is:

$$x_p(t) = u(t)e^{-t} = (t^2 - 4t + 3)e^{-t}$$

$$\boxed{x_p(t) = (t^2 - 4t + 3)e^{-t}}$$

Note:-

We often end up with a polynomial multiplied by an exponential. To evaluate an integral involving this expression more easily, you can apply the operator trick.

$$\boxed{\int P(t)e^{\alpha t}dt = e^{\alpha t}\left(\frac{P(t)}{\alpha} - \frac{P'(t)}{\alpha^2} + \frac{P''(t)}{\alpha^3} - \cdots\right)}$$

This works because when we do integration by parts, we continue until we get a derivative equal to zero, which just returns a constant. We can also apply this operator trick to solve ODEs. For example, if we have:

$$(D - \alpha I)x = P(t)e^{\alpha t}$$

Then we can apply the operator trick to get:

$$x = \frac{P(t)}{\alpha}e^{\alpha t} - \frac{P'(t)}{\alpha^2}e^{\alpha t} + \frac{P''(t)}{\alpha^3}e^{\alpha t} - \cdots$$

Note that this is similar to how we got the generalized exponential response formula for the case of resonance, where we had to take derivatives of the characteristic polynomial until we got a non-zero value.

Chapter 4

Fourier series

Fourier series are a way to represent a periodic function as an infinite sum of sine and cosine functions. A function f is said to be **periodic** with period $2L$ if it satisfies the following condition:

$$f(t+2L) = f(t), \quad \forall t \in \mathbb{R}$$

The timepoints at which a value reoccurs $4L, 8L \ldots$ are also called **harmonics**.

Any window of width $2L$ will define the function for all t. We build up to this interval from the cosine and sine functions, which are defined as their behaviour over the window $[-\pi, \pi]$.

Note:-

$\cos(mt)$ has a period of $\frac{2\pi}{m}$, and $\sin(mt)$ has a period of $\frac{2\pi}{m}$. Furthermore, if $f(t)$ and $g(t)$ are periodic functions with period $2L$, then the function $h(t) = af(t) + bg(t)$ is also periodic with period $2L$ for any constants a and b. This is because:

$$\begin{aligned} h(t+2L) &= af(t+2L) + bg(t+2L) \\ &= af(t) + bg(t) \\ &= h(t) \end{aligned}$$

4.1 Fourier coefficients

A Fourier series has the form:

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(mt) + b_m \sin(mt)\right)}$$

The coefficients a_0, a_m, b_m are called the **Fourier coefficients** of the function f.

Intuitively, for any given point $f(t*)$, we can think of the Fourier series as a sum of infinitely many sine and cosine waves that add up to $f(t*)$. The Fourier coefficients determine the amplitude of each sine and cosine wave in the series. The more terms we include in the series, the closer the approximation will be to the original function $f(t)$.

We wish to define the Fourier coefficients so that we can reconstruct the original function.

We begin by considering a_0. We define the function:

$$\text{Average}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

Effectively, this is the average value of the function f over one period. We can also write this as:

$$\text{Average}(f) = \frac{1}{2L} \int_{-L}^{L} f(t)dt$$

The average value of $\cos(mt)$ and $\sin(mt)$ over one period is zero, so we have:

$$\text{Average}(f) = \frac{a_0}{2}$$

Thus, we can define a_0 as:

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt}$$

4.1.1 Properties of sin and cos integrals

The following can be obtained by using trig identities, converting to complex exponentials, or integration by parts.

$$\int_{-\pi}^{\pi} \cos(mt)\sin(mt)dt = 0$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

In general, we can compute the integrals by also considering odd and even functions. For example, $\cos(mt)\sin(nt)$ is an odd function, so its integral over $[-\pi, \pi]$ is zero.

4.1.2 Deriving the Fourier coefficients

Intuitively, a_m and b_m tell us "how much" of that term is present in the function.

To derive a_m, we use the "average" function again, and integrate over the period:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)dt$$

The multiplication by $\cos(mt)$ allows us to "pick out" the a_m term from the Fourier series, since the integrals of $\cos(mt)\cos(nt)$ and $\sin(mt)\cos(nt)$ are zero for $m \neq n$. This reflects the orthogonality of the sine and cosine functions.

$$\int_{-\pi}^{\pi} a_m\cos(mt)\cos(nt)dt = \begin{cases} 0 & m \neq n \\ \pi a_m & m = n \neq 0 \\ 2\pi a_0 & m = n = 0 \end{cases}$$

Similarly, we can derive b_m as:

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)dt$$

$$\int_{\pi}^{\pi} b_m\sin(mt)\sin(nt)dt = \begin{cases} 0 & m \neq n \\ \pi b_m & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

Then the **only** nonzero terms occur when $m = n \neq 0$.

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)dt, \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)dt}$$

By change of variables (on t), we can also write the Fourier coefficients as:

$$\boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi t}{L}\right)dt, \quad b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)dt}$$

This allows us to have the function defined on any interval of length $2L$, not just $[-\pi, \pi]$.

4.1.3 Functions defined as Fourier series

Let us consider the **square wave** function defined as:

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$

Notice that this function is odd.
We can compute the Fourier coefficients, starting with the constant a_0 term:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = 0$$

Then, since f is odd, we have $a_m = 0$ for all m. We can compute b_m as follows:

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)dt \\ &= 2\int_{0}^{\pi} f(t)\sin(mt)dt \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(mt)dt \\ &= \frac{2}{\pi}\left[-\frac{\cos(mt)}{m}\right]_0^{\pi} \\ &= \frac{2}{m\pi}\left[-\cos(m\pi)+1\right] \\ &= \frac{2}{m\pi}\left[1-\cos(m\pi)\right] \end{aligned}$$

As a table:

m	$\cos(m\pi)$	$1-\cos(m\pi)$
1	-1	2
2	1	0
3	-1	2
4	1	0
5	-1	2
$\vdots$	$\vdots$	$\vdots$

Then we have that:

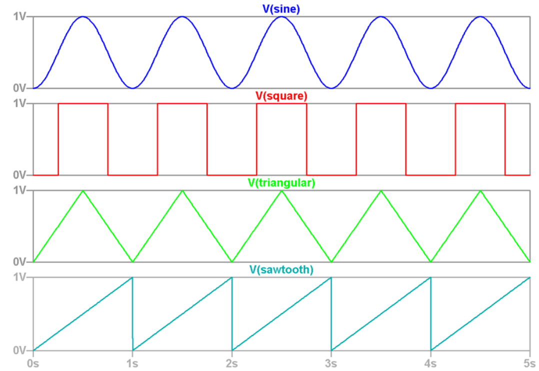
$$b_m = \begin{cases} \frac{4}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

Thus, the Fourier series for the square wave function is:

$$f(t) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right)$$

Continuity, differentiability, and rate of decay of terms

Consider different waves: a square wave, triangle wave, and a wave consisting of parabolas.



The square wave is discontinuous, the triangle wave is continuous but not differentiable, and the parabola wave is continuous and differentiable.

The Fourier coefficients of the square wave decay as $\frac{1}{m}$, the Fourier coefficients of the triangle wave decay as $\frac{1}{m^2}$, and the Fourier coefficients of the parabola wave decay as $\frac{1}{m^3}$. This occurs because the Fourier coefficients are related to the smoothness of the function. The smoother the function, the faster the Fourier coefficients decay. In general, if a function is k times differentiable, then its Fourier coefficients decay at least as fast as $\frac{1}{m^{k+1}}$.

Note:-

Piecewise continuous functions are functions that are continuous on each piece of their domain, but may have a finite number of jump discontinuities. For example, the square wave function is piecewise continuous, as it is continuous on the intervals $(-\pi, 0)$ and $(0, \pi)$, but has a jump discontinuity at $t = 0$.
We can define the left and right limits of a function f at a point a as follows:

$$f(a^-) = \lim_{t \to a^-} f(t), \quad f(a^+) = \lim_{t \to a^+} f(t)$$

A **nice function** is defined if we have $f(a) = \frac{1}{2}\left(f(a^-) + f(a^+)\right)$ for all a. In other words, the function is defined at the point of discontinuity as the average of the left and right limits. If a function is nice, then its Fourier series converges to the function at all points.

4.1.4 Manipulating Fourier series

$$\boxed{\sin(\theta + \frac{\pi}{2}) = \cos(\theta), \quad \cos(\theta + \frac{\pi}{2}) = -\sin(\theta)}$$

- We can take linear combinations of Fourier series.
- We can shift a Fourier series by a constant.
- We can stretch a Fourier series by a constant.

Example 4.1.1 (Linear combinations of Fourier series)

We can take linear combinations of Fourier series by taking linear combinations of the Fourier coefficients. For example, if we have two functions f and g with Fourier coefficients a_m^f, b_m^f and a_m^g, b_m^g respectively, then the Fourier coefficients of the function $h = af + bg$ are given by:

$$\begin{aligned} f(t) &= 1 + 2\mathrm{sq}(t) \\ &= 1 + \frac{8}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right) \\ &= 1 + \frac{8}{\pi}\left(\sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3t + \frac{\pi}{2}\right) + \frac{1}{5}\sin\left(5t + \frac{\pi}{2}\right) + \cdots\right) \\ &= 1 + \frac{8}{\pi}\left(\cos(t) + \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) + \cdots\right) \end{aligned}$$

Example 4.1.2 (Shifting a Fourier series)

We can shift a Fourier series by a constant by using the angle addition formulas for sine and cosine. For example, if we have a function $f(t)$ with Fourier coefficients a_m and b_m, then the Fourier coefficients of the function $g(t) = f(t - \frac{\pi}{2})$ are given by:

$$\begin{aligned} f(t) &= \mathrm{sq}(t - \frac{\pi}{2}) \\ &= \frac{4}{\pi}\left(\sin\left(t - \frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3t - \frac{3\pi}{2}\right) + \cdots\right) \\ &= \frac{4}{\pi}\left(-\cos(t) + \frac{1}{3}\cos(3t) + \cdots\right) \end{aligned}$$

Example 4.1.3 (Stretching a Fourier series)

We can stretch a Fourier series by a constant by using the angle addition formulas for sine and cosine. For example, if we have a function $f(t)$ with Fourier coefficients a_m and b_m, then the Fourier coefficients of the function $g(t) = f(2t)$ are given by:

$$\begin{aligned} f(t) &= \mathrm{sq}(2t) \\ &= \frac{4}{\pi}\left(\sin(2t) + \frac{1}{3}\sin(6t) + \cdots\right) \end{aligned}$$

Then this function has period π, since the period of $\sin(2t)$ is π.

4.1.5 Differentiating and integrating Fourier series

We can differentiate a Fourier series term by term, since the Fourier series converges uniformly. For example, if we have a function $f(t)$ with Fourier coefficients a_m and b_m, then the Fourier coefficients of the function $g(t) = f'(t)$ are given by:

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m\cos(mt) + b_m\sin(mt)) \\ f'(t) &= \sum_{m=1}^{\infty}(-ma_m\sin(mt) + mb_m\cos(mt)) \end{aligned}$$

Similarly, we can integrate a Fourier series term by term. For example, if we have a function $f(t)$ with Fourier coefficients a_m and b_m, then the Fourier coefficients of the function $g(t) = \int f(t)dt$ are given by:

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m\cos(mt) + b_m\sin(mt)) \\ \int f(t)dt &= \frac{a_0}{2}t + \sum_{m=1}^{\infty}\left(\frac{a_m}{m}\sin(mt) - \frac{b_m}{m}\cos(mt)\right) \end{aligned}$$

Example 4.1.4 (Differentiating the square wave function)

We can differentiate the square wave function term by term to get:

$$\begin{aligned} f(t) &= \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \cdots\right) \\ f'(t) &= \frac{4}{\pi}(\cos(t) + \cos(3t) + \cdots) \end{aligned}$$

Then we have that $f'(t)$ is a sum of cosine functions, which is a periodic function with period 2π.