

BRANCH AND BOUND ALGORITHM FOR VERTEX COVER PROBLEM

Optimization Project

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Problem

Given a graph $G = (V, E)$, a **vertex cover** is a subset of vertices $S \subseteq V$ such that for every edge $\{u, v\} \in E$, at least one of u or v is in S .

The **minimum vertex cover** problem seeks to find a vertex cover of the smallest possible size.

Let $w : V \rightarrow \mathbb{R}^+$ be a weight function assigning a positive weight to each vertex. The **weighted minimum vertex cover** problem aims to find a vertex cover S that minimizes the total weight

$$w(S) := \sum_{v \in S} w(v).$$

The weighted minimum vertex cover problem can be formulated as the following integer programming problem:

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V} w(v)x_v \\ \text{subject to} & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\ & x_v \in \{0, 1\}. \end{array} \quad (\text{IP})$$

The vertex cover corresponding to a solution x is given by $S = \{v \in V : x_v = 1\}$.

But solving (IP) is NP-hard in general.

Algorithms make use of the LP-relaxation

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V} w(v)x_v \\ \text{subject to} & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\ & x_v \geq 0. \end{array} \quad (\text{LP})$$

Note: Every optimal solution always has $x_v \leq 1$, since if $x_v > 1$ for some vertex v , we can set $x_v = 1$ without violating any constraints and get a better solution.

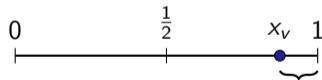
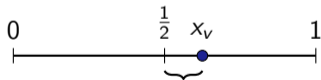
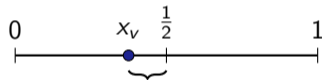
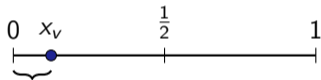
Solution Properties

Theorem 1 (Nemhauser-Trotter)

Let x be an extreme point of the polytope defined by the constraints of (LP) we have $x_v \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Optimal Solutions of (LP)

Proof. Let x be an extreme point. Let $U \subset V$ be the set of vertices such that $x_v \notin \{0, \frac{1}{2}, 1\}$ for every $v \in U$. Suppose for contradiction that U is non-empty.



Optimal Solutions of (LP)

Take the minimum distance ϵ from x_v to the closest of $\{0, \frac{1}{2}, 1\}$ for every $v \in U$ i.e.

$$\epsilon = \min_{v \in U} \min \left\{ |x_v - 0|, \left| x_v - \frac{1}{2} \right|, |x_v - 1| \right\}.$$

Perturb x at each $v \in U$ by ϵ by two different ways to get two new solutions x^+ and x^- defined as follows:

$$x^+(v) = \begin{cases} x_v + \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v - \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise} \end{cases}, \quad x^-(v) = \begin{cases} x_v - \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v + \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise.} \end{cases}$$

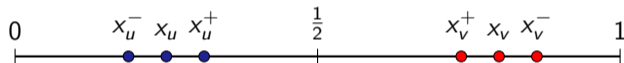
We have $x = \frac{1}{2}(x^+ + x^-)$.

Optimal Solutions of (LP)

To see that both x^+ and x^- are feasible, there are two cases

- (i) If an edge $uv \in E$ has $x_u < \frac{1}{2}$ and $x_v > \frac{1}{2}$, then

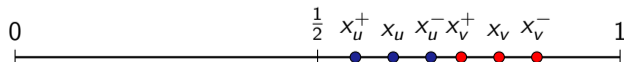
$$x_u^+ + x_v^+ = x_u^- + x_v^- = x_u + x_v \geq 1.$$



The constraint $x_u + x_v \geq 1$ is satisfied for both x^+ and x^- since the sum remains at least 1.

- (ii) If an edge $uv \in E$ has both $x_u, x_v > \frac{1}{2}$. Then $x_u^+, x_u^-, x_v^+, x_v^- \geq \frac{1}{2}$. Hence

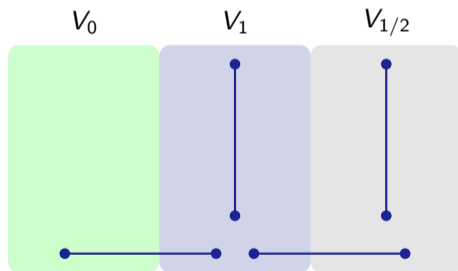
$$x_u^+ + x_v^+ \geq 1 \text{ and } x_u^- + x_v^- \geq 1.$$



Optimal Solutions of (LP)

Since there is an optimal solution to (LP) that is also an extreme point, we conclude there exists an optimal solution x^* such that $x_v^* \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Define the set V_1 of vertices with value 1 in x^* and similarly the sets V_0 and $V_{1/2}$.



Possible cases of edges in E are shown above.

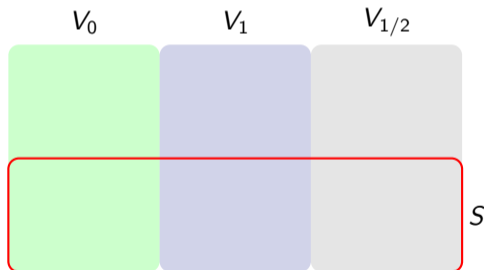
Theorem 2

Let x^* be an optimal solution to (LP). Then there exists an optimal solution of (IP) that generates a vertex cover $S \subset V$ such that $V_1 \subset S \subset V_1 \cup V_{1/2}$.

Vertex Cover from (LP)

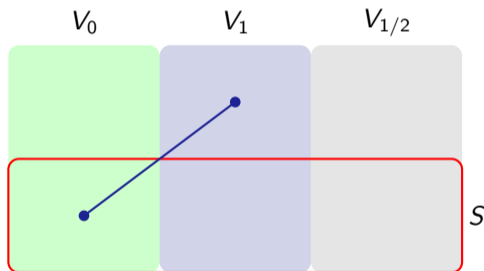
Proof. Firstly, we show that $S \subset V_1 \cup V_{1/2}$. Suppose not i.e. $C_0 = S \cap V_0$ is not empty.

For every vertex $v \in C_0$, there can only be edges between v and vertices in V_1 .



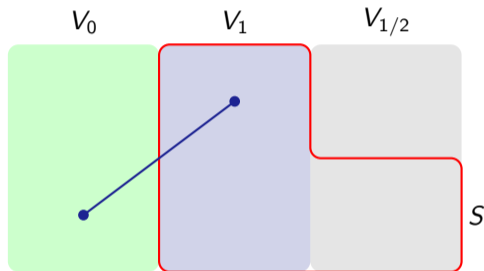
Vertex Cover from (LP)

Edges from C_0 to $\overline{C}_1 = V_1 \setminus S$ are covered once by S .



Vertex Cover from (LP)

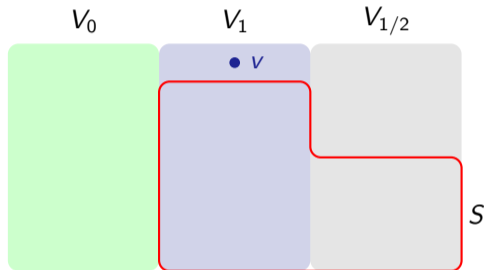
If $w(\overline{C}_1) \leq w(C_0)$, we can get a better solution by choosing \overline{C}_1 instead of C_0 .



If $w(C_0) < w(\overline{C}_1)$, we use the perturbation technique again: add a small ϵ to every vertex in C_0 and subtract ϵ from every vertex in \overline{C}_1 to get a better feasible solution to (LP), contradicting the optimality of x^* .

Vertex Cover from (LP)

Now we prove that $V_1 \subset S$. Suppose not, i.e. \overline{C}_1 is nonempty. Let $v \in \overline{C}_1$.



If $w(v) = 0$, include v in S anyway (the value of (IP) does not change).

Consider the case $w(v) > 0$. Note that v cannot have neighbors in V_0 . Hence, we can decrease x_v from 1 to $\frac{1}{2}$ and get a better feasible solution to (LP), contradicting the optimality of x^* .

Thank you for listening !