BRANCH AND BOUND ALGORITHM FOR VERTEX COVER PROBLEM

Optimization Project

Students: CHAU Dang Minh
LAM Nhat Quan

Outline

Problem

Original Vertex Cover

Given a graph $G = (V, E)$, a **vertex cover** is a subset of vertices $S \subseteq V$ such that for every edge $\{u, v\} \in E$, at least one of u or v is in S.

The **minimum vertex cover** problem seeks to find a vertex cover of the smallest possible size.

Let $w : V \to \mathbb{R}^+$ be a weight function assigning a positive weight to each vertex. The **weighted minimum vertex cover** problem aims to find a vertex cover S that minimizes the total weight

$$w(S) := \sum_{v \in S} w(v).$$

Integer Programming Formulation

The weighted minimum vertex cover problem can be formulated as the following integer programming problem:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} w(v) x_v \\
\text{subject to} \quad & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\
& x_v \in \{0, 1\}.
\end{aligned}
\tag{IP}
$$

The vertex cover corresponding to a solution x is given by $S = \{v \in V : x_v = 1\}$.

But solving (IP) is NP-hard in general.

Linear Programing Relaxation

Algorithms make use of the LP-relaxation

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} w(v) x_v \\
\text{subject to} \quad & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\
& x_v \geq 0.
\end{aligned}
\tag{LP}
$$

Note: Every optimal solution always has $x_v \leq 1$, since if $x_v > 1$ for some vertex v, we can set $x_v = 1$ without violating any constraints and get a better solution.

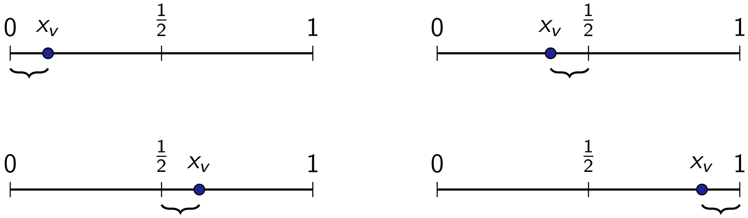
Solution Properties

Theorem 1 (Nemhauser-Trotter)

Let x be an extreme point of the polytope defined by the constrains of (LP) we have $x_v \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Optimal Solutions to (LP)

Proof. Let x be an extreme point. Let $U \subset V$ be the set of vertices such that $x_v \notin \{0, \frac{1}{2}, 1\}$ for every $v \in U$. Suppose for contradiction that U is non-empty.



Optimal Solutions to (LP)

Take the minimum distance ϵ from x_v to the closest of $\{0, \frac{1}{2}, 1\}$ for every $v \in U$ i.e.

$$\epsilon = \min_{v \in U} \min \left\{ |x_v - 0|, \left| x_v - \frac{1}{2} \right|, |x_v - 1| \right\}.$$

Perturb x at each $v \in U$ by ϵ by two different ways to get two new solutions x^+ and x^- defined as follows:

$$x^+(v) = \begin{cases} x_v + \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v - \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise} \end{cases} \qquad x^-(v) = \begin{cases} x_v - \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v + \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise.} \end{cases}$$

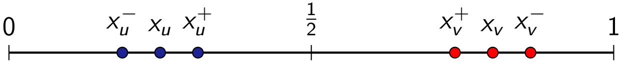
We have $x = \dfrac{1}{2}(x^+ + x^-)$.

Optimal Solutions to (LP)

Two see that both x^+ and x^- are feasible, there are two cases

(i) If an edge $uv \in E$ has $x_u < \frac{1}{2}$ and $x_v > \frac{1}{2}$, then

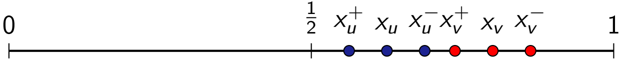
$$x_u^+ + x_v^+ = x_u^- + x_v^- = x_u + x_v \geq 1.$$



The constraint $x_u + x_v \geq 1$ is satisfied for both x^+ and x^- since the sum remains at least 1.

(ii) If an edge $uv \in E$ has both $x_u, x_v > \frac{1}{2}$. Then $x_u^+, x_u^-, x_v^+, x_v^- \geq \frac{1}{2}$. Hence

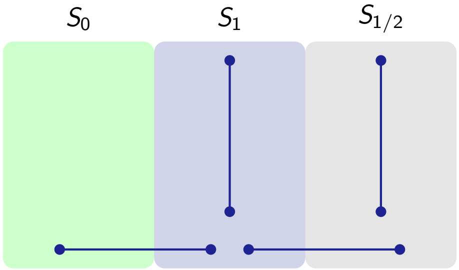
$$x_u^+ + x_v^+ \geq 1 \text{ and } x_u^- + x_v^- \geq 1.$$



Optimal Solutions to (LP)

Since there is an optimal solution to (LP) that is also an extreme point, we conclude there exists an optimal solution x^* such that $x_v^* \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Define the set S_1 of vertices with value 1 in x^* and similarly the sets S_0 and $S_{1/2}$.



Possible cases of edges in E are shown above.

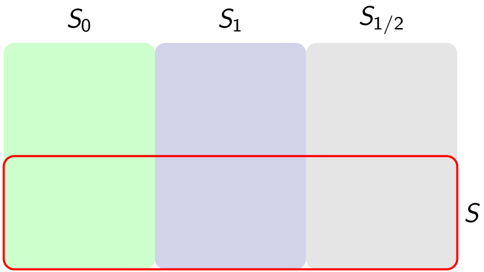
Theorem 2

Let x^* be an optimal solution to (LP). Then there exists an optimal solution of (IP) that generates a vertex cover $S \subset V$ such that $S_1 \subset S \subset S_1 \cup S_{1/2}$.

Vertex Cover from (LP)

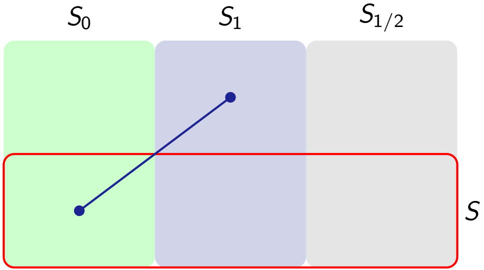
Proof. Firstly, we show that $S \subset S_1 \cup S_{1/2}$. Suppose not i.e. $C_0 = S \cap S_0$ is not empty.

For every vertex $v \in C_0$, there can only be edges between v and vertices in S_1.



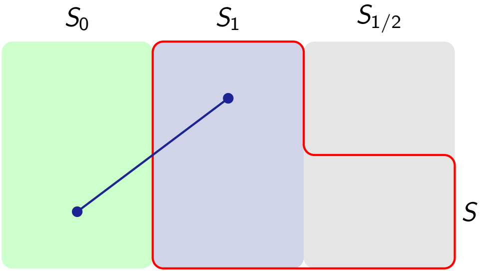
Vertex Cover from (LP)

Edges from C_0 to $\overline{C}_1 = S_1 \setminus S$ are covered once by S.



Vertex Cover from (LP)

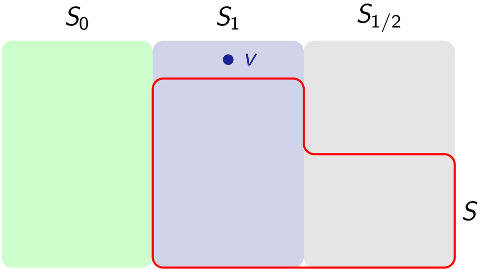
If $w(\overline{C}_1) \leq w(C_0)$, we can get a better solution by choosing $\overline{C}_1$ instead of C_0.



If $w(C_0) < w(\overline{C}_1)$, we use the perturbation technique again: add a small ϵ to every vertex in C_0 and subtract ϵ from every vertex in $\overline{C}_1$ to get a better feasible solution to (LP), contradicting the optimality of x^*.

Vertex Cover from (LP)

Now we prove that $S_1 \subset S$. Suppose not, i.e. $\overline{C}_1$ is nonempty. Let $v \in \overline{C}_1$.



If $w(v) = 0$, include v in S anyway (the value of (IP) does not change).

Consider the case $w(v) > 0$. Note that v cannot have neighbors in S_0. Hence, we can decrease x_v from 1 to $\frac{1}{2}$ and get a better feasible solution to (LP), contradicting the optimality of x^*.

Branch and Bound for Vertex Cover

General Algorithm

1. Maintain the current cost and the current best solution.
2. Extract S_0, S_1 and $S_{1/2}$ from an extreme solution of (LP).
3. If adding S_1 exceeds the current best solution, stop (prune this branch).
4. If $S_{1/2}$ is empty, update the current best solution if necessary and stop.
5. Choose a vertex $v \in S_{1/2}$.
6. Return to step 2 two following graphs in some order:
   - Graph with v included in the vertex cover (remove v and its incident edges).
   - Graph with v excluded from the vertex cover (remove v's neighbors and their incident edges, add v to the current cost).

Different strategies for steps 5 and 6 lead to different algorithms.

Experiments

We consider three strategies

- Choosing the vertex with the highest degree in $S_{1/2}$ and include it first.
- Choosing the vertex with the lowest degree in $S_{1/2}$ and exclude it first.
- Fully-strong branching [1]: choose the vertex and the order whose resulting two LP relaxations have the highest total value.

[1] Bénichou, Michel, et al. "Experiments in mixed-integer linear programming." Mathematical programming 1.1 (1971): 76-94.

Thank you for listening !