

# **BRANCH AND BOUND ALGORITHMS FOR MINIMUM VERTEX COVER PROBLEM**

Optimization Project

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- 1 Problem
- 2 Solution Properties
- 3 Branch and Bound for Vertex Cover

# Problem

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Given a graph  $G = (V, E)$ , a **vertex cover** is a subset of vertices  $S \subseteq V$  such that for every edge  $\{u, v\} \in E$ , at least one of  $u$  or  $v$  is in  $S$ .

The **minimum vertex cover** problem seeks to find a vertex cover of the smallest possible size.

Let  $w : V \rightarrow \mathbb{R}^+$  be a weight function assigning a positive weight to each vertex. The **weighted minimum vertex cover** problem aims to find a vertex cover  $S$  that minimizes the total weight

$$w(S) := \sum_{v \in S} w(v).$$

## Integer Programming Formulation

The weighted minimum vertex cover problem can be formulated as the following integer programming problem:

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V} w(v)x_v \\ \text{subject to} & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\ & x_v \in \{0, 1\}. \end{array} \quad (\text{IP})$$

The vertex cover corresponding to a solution  $x$  is given by  $S = \{v \in V : x_v = 1\}$ .

But solving (IP) is NP-hard in general.

Algorithms make use of the LP-relaxation

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V} w(v)x_v \\ \text{subject to} & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\ & x_v \geq 0. \end{array} \quad (\text{LP})$$

**Note:** Every optimal solution always has  $x_v \leq 1$ , since if  $x_v > 1$  for some vertex  $v$ , we can set  $x_v = 1$  without violating any constraints and get a better solution.

# **Solution Properties**

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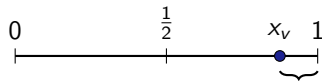
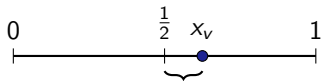
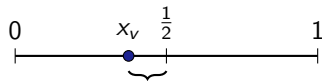
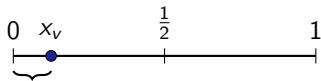
### Theorem 1 (Nemhauser-Trotter)

Let  $x$  be an extreme point of the polytope defined by the constraints of (LP) we have  $x_v \in \{0, \frac{1}{2}, 1\}$  for every  $v \in V$ .



## Optimal Solutions to (LP)

*Proof.* Let  $x$  be an extreme point. Let  $U \subset V$  be the set of vertices such that  $x_v \notin \{0, \frac{1}{2}, 1\}$  for every  $v \in U$ . Suppose for contradiction that  $U$  is non-empty.



## Optimal Solutions to (LP)

Take the minimum distance  $\epsilon$  from  $x_v$  to the closest of  $\{0, \frac{1}{2}, 1\}$  for every  $v \in U$  i.e.

$$\epsilon = \min_{v \in U} \min \left\{ |x_v - 0|, \left| x_v - \frac{1}{2} \right|, |x_v - 1| \right\}.$$

Perturb  $x$  at each  $v \in U$  by  $\epsilon$  by two different ways to get two new solutions  $x^+$  and  $x^-$  defined as follows:

$$x^+(v) = \begin{cases} x_v + \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v - \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise} \end{cases}, \quad x^-(v) = \begin{cases} x_v - \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v + \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise.} \end{cases}$$

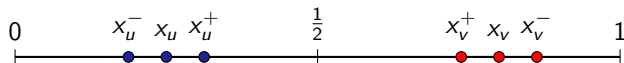
We have  $x = \frac{1}{2}(x^+ + x^-)$ .

## Optimal Solutions to (LP)

To see that both  $x^+$  and  $x^-$  are feasible, there are two cases

(i) If an edge  $uv \in E$  has  $x_u < \frac{1}{2}$  and  $x_v > \frac{1}{2}$ , then

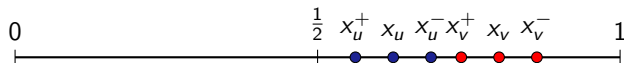
$$x_u^+ + x_v^+ = x_u^- + x_v^- = x_u + x_v \geq 1.$$



The constraint  $x_u + x_v \geq 1$  is satisfied for both  $x^+$  and  $x^-$  since the sum remains at least 1.

(ii) If an edge  $uv \in E$  has both  $x_u, x_v > \frac{1}{2}$ . Then  $x_u^+, x_u^-, x_v^+, x_v^- \geq \frac{1}{2}$ . Hence

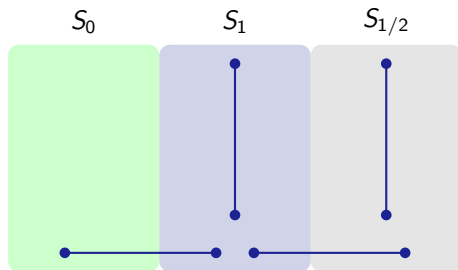
$$x_u^+ + x_v^+ \geq 1 \text{ and } x_u^- + x_v^- \geq 1.$$



## Optimal Solutions to (LP)

Since there is an optimal solution to (LP) that is also an extreme point, we conclude there exists an optimal solution  $x^*$  such that  $x_v^* \in \{0, \frac{1}{2}, 1\}$  for every  $v \in V$ .

Define the set  $S_1$  of vertices with value 1 in  $x^*$  and similarly the sets  $S_0$  and  $S_{1/2}$ .



Possible cases of edges in  $E$  are shown above.

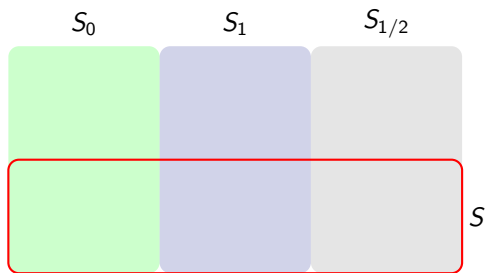
### Theorem 2 (Nemhauser-Trotter)

Let  $x^*$  be an optimal solution to (LP). Then there exists an optimal solution of (IP) that generates a vertex cover  $S \subset V$  such that  $S_1 \subset S \subset S_1 \cup S_{1/2}$ .

## Vertex Cover from (LP)

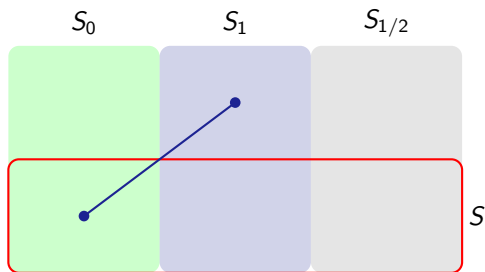
*Proof.* Firstly, we show that  $S \subset S_1 \cup S_{1/2}$ . Suppose not i.e.  $C_0 = S \cap S_0$  is not empty.

For every vertex  $v \in C_0$ , there can only be edges between  $v$  and vertices in  $S_1$ .



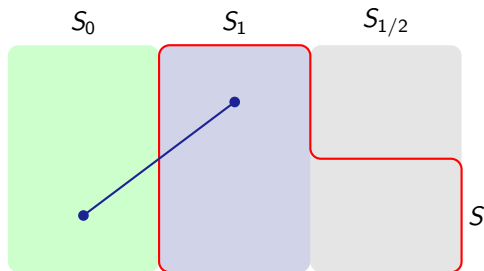
## Vertex Cover from (LP)

Edges from  $C_0$  to  $\overline{C}_1 = S_1 \setminus S$  are covered once by  $S$ .



## Vertex Cover from (LP)

If  $w(\overline{C}_1) \leq w(C_0)$ , we can get a better solution by choosing  $\overline{C}_1$  instead of  $C_0$ .

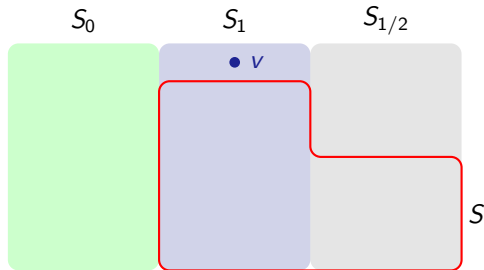


If  $w(C_0) < w(\overline{C}_1)$ , we use the perturbation technique again: add a small  $\epsilon$  to every vertex in  $C_0$  and subtract  $\epsilon$  from every vertex in  $\overline{C}_1$  to get a better feasible solution to (LP), contradicting the optimality of  $x^*$ .



## Vertex Cover from (LP)

Now we prove that  $S_1 \subset S$ . Suppose not, i.e.  $\overline{C}_1$  is nonempty. Let  $v \in \overline{C}_1$ .



If  $w(v) = 0$ , include  $v$  in  $S$  anyway (the value of (IP) does not change).

Consider the case  $w(v) > 0$ . Note that  $v$  cannot have neighbors in  $S_0$ . Hence, we can decrease  $x_v$  from 1 to  $\frac{1}{2}$  and get a better feasible solution to (LP), contradicting the optimality of  $x^*$ .

## **Branch and Bound for Vertex Cover**

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- ① Maintain the current cost and the current best solution.
- ② Extract  $S_0$ ,  $S_1$  and  $S_{1/2}$  from an extreme solution of (LP).
- ③ If adding  $S_1$  exceeds the current best solution, stop (prune this branch).
- ④ If  $S_{1/2}$  is empty, update the current best solution if necessary and stop.
- ⑤ Choose a vertex  $v \in S_{1/2}$ .
- ⑥ Return to step 2 two following graphs in some order:
  - Graph with  $v$  included in the vertex cover (remove  $v$  and its incident edges).
  - Graph with  $v$  excluded from the vertex cover (remove  $v$ 's neighbors and their incident edges, add  $v$  to the current cost).

Different strategies for steps 5 and 6 lead to different algorithms.

We consider three strategies

- Choosing the vertex with the highest degree in  $S_{1/2}$  and include it first.
- Choosing the vertex with the lowest degree in  $S_{1/2}$  and exclude it first.
- Fully-strong branching <sup>1</sup>: choose the vertex and the order whose resulting two LP relaxations have the highest total value.

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<sup>1</sup>Bénichou, Michel, et al. "Experiments in mixed-integer linear programming." Mathematical programming 1.1 (1971): 76-94.

## Results - Strategy 1: Highest Degree Include First

Instance	V	E	Opt VC	BnB Nodes	LP Calls	Time (s)
johnson8-2-4	28	210	21	31	31	0.22
MANN-a9	45	918	42	79	57	0.64
hamming6-2	64	1824	62	265	189	1.92
johnson8-4-4	70	1855	65	323	276	2.64
johnson16-2-4	120	5460	105	183	183	8.87
C125-9	125	6963	121	487	364	11.31
eco-foodweb-baywet	128	2075	86	91	89	2.24
keller4	171	9435	156	2849	2835	47.41
brock200-1	200	14834	194	6723	6701	92.82
bn-macaque	242	3054	150	1851	1851	48.13
inf-USAir97	332	2126	149	791	775	6.37
delaunay_n10	1024	3056	717*	96158	88318	600.04
econ-mahindas	1258	7619	730*	8270	8219	600.12
inf-euroroad	1174	1417	571*	66220	64256	600.01
mk11-b2	6930	20779	909	1	1	1.21

\* indicates timeout (10 min limit)

**Thank you for listening !**