BRANCH AND BOUND ALGORITHM FOR VERTEX COVER PROBLEM

Optimization Project

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Outline

Problem

Original Vertex Cover

Given a graph $G = (V, E)$, a **vertex cover** is a subset of vertices $S \subseteq V$ such that for every edge $\{u, v\} \in E$, at least one of u or v is in S.

The **minimum vertex cover** problem seeks to find a vertex cover of the smallest possible size.

Let $w : V \to \mathbb{R}^+$ be a weight function assigning a positive weight to each vertex. The **weighted minimum vertex cover** problem aims to find a vertex cover S that minimizes the total weight

$$w(S) := \sum_{v \in S} w(v).$$

Integer Programming Formulation

The weighted minimum vertex cover problem can be formulated as the following integer programming problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} w(v) x_v \\ \text{subject to} \quad & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\ & x_v \in \{0, 1\}. \end{aligned} \tag{IP}$$

The vertex cover corresponding to a solution x is given by $S = \{v \in V : x_v = 1\}$.

But solving (IP) is NP-hard in general.

Linear Programing Relaxation

Algorithms make use of the LP-relaxation

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} w(v) x_v \\ \text{subject to} \quad & x_u + x_v \geq 1, \quad \forall \{u, v\} \in E \\ & x_v \geq 0. \end{aligned} \tag{LP}$$

Note: Every optimal solution always has $x_v \leq 1$, since if $x_v > 1$ for some vertex v, we can set $x_v = 1$ without violating any constraints and get a better solution.

Branch-and-Bound for Vertex Cover

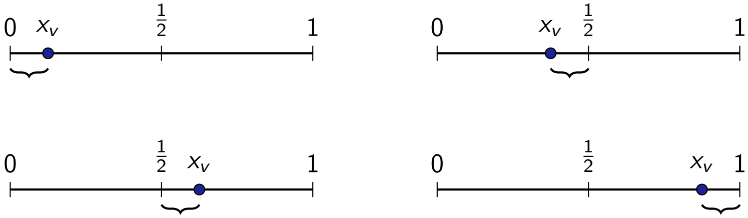
Optimal Solutions of (LP)

Theorem 1 (Nemhauser-Trotter)

Let x be an extreme point of the polytope defined by the constrains of (LP) we have $x_v \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Optimal Solutions of (LP)

Proof. Let x be an extreme point. Let $U \subset V$ be the set of vertices such that $x_v \notin \{0, \frac{1}{2}, 1\}$ for every $v \in U$. Suppose for contradiction that U is non-empty.



Optimal Solutions of (LP)

Take the minimum distance ϵ from x_v to the closest of $\{0, \frac{1}{2}, 1\}$ for every $v \in U$ i.e.

$$\epsilon = \min_{v \in U} \min \left\{ |x_v - 0|, \left| x_v - \frac{1}{2} \right|, |x_v - 1| \right\}.$$

Perturb x at each $v \in U$ by ϵ by two different ways to get two new solutions x^+ and x^- defined as follows:

$$x^+(v) = \begin{cases} x_v + \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v - \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise} \end{cases} \qquad x^-(v) = \begin{cases} x_v - \epsilon & \text{if } v \in U \text{ and } x_v < \frac{1}{2}, \\ x_v + \epsilon & \text{if } v \in U \text{ and } x_v > \frac{1}{2}, \\ x_v & \text{otherwise.} \end{cases}$$

We have $x = \dfrac{1}{2}(x^+ + x^-)$.

Optimal Solutions of (LP)

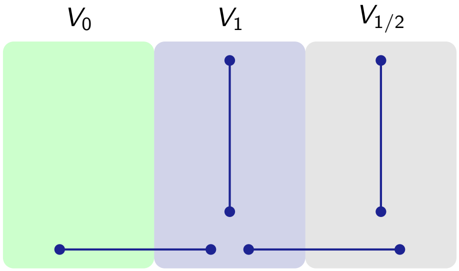
Two see that both x^+ and x^- are feasible, there are two cases

(i) If an

Optimal Solutions of (LP)

Since there is an optimal solution to (LP) that is also an extreme point, we conclude there exists an optimal solution x^* such that $x_v^* \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Define the set V_1 of vertices with value 1 in x^* and similarly the sets V_0 and $V_{1/2}$.



Possible cases of edges in E are shown above.

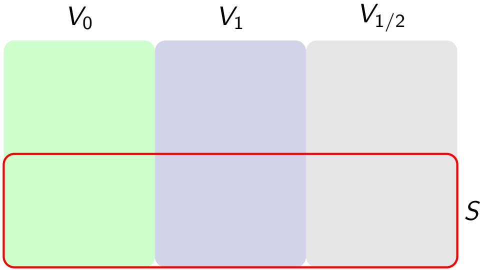
Theorem 2

Let x^* be an optimal solution to (LP). Then there exists an optimal solution of (IP) that generates a vertex cover $S \subset V$ such that $V_1 \subset S \subset V_1 \cup V_{1/2}$.

Vertex Cover from (LP)

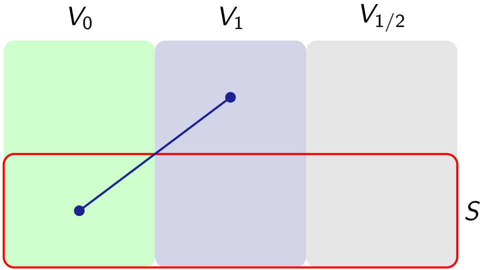
Proof. Firstly, we show that $S \subset V_1 \cup V_{1/2}$. Suppose not i.e. $C_0 = S \cap V_0$ is not empty.

For every vertex $v \in C_0$, there can only be edges between v and vertices in V_1.



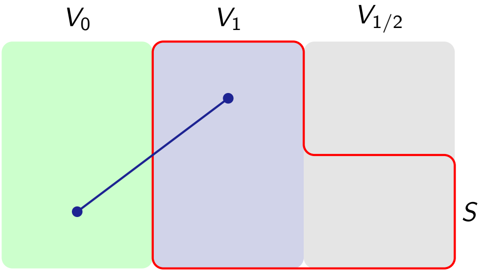
Vertex Cover from (LP)

Edges from C_0 to $\overline{C}_1 = V_1 \setminus S$ are covered once by S.



Vertex Cover from (LP)

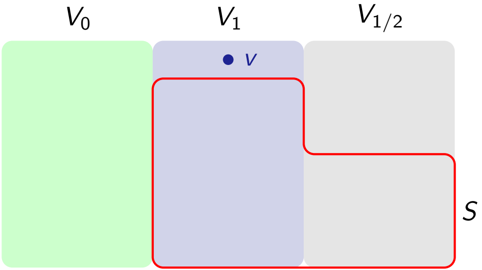
If $w(\overline{C}_1) \leq w(C_0)$, we can get a better solution by choosing $\overline{C}_1$ instead of C_0.



If $w(C_0) < w(\overline{C}_1)$, we use the perturbation technique again: add a small ϵ to every vertex in C_0 and subtract ϵ from every vertex in $\overline{C}_1$ to get a better feasible solution to (LP), contradicting the optimality of x^*.

Vertex Cover from (LP)

Now we prove that $V_1 \subset S$. Suppose not, i.e. $\overline{C}_1$ is nonempty. Let $v \in \overline{C}_1$.



If $w(v) = 0$, include v in S anyway (the value of (IP) does not change).

Consider the case $w(v) > 0$. Note that v cannot have neighbors in V_0. Hence, we can decrease x_v from 1 to $\frac{1}{2}$ and get a better feasible solution to (LP), contradicting the optimality of x^*.

Thank you for listening !