

Master 2 Mathematics and Computer Science

Symbolic Dynamics. Lecture 3

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One-sided shift spaces

A *one-sided shift space* is a closed subset X of $A^{\mathbb{N}}$ such that $S(X) \subseteq X$.

One-sided shift spaces are usually defined as closed subsets such that $S(X) = X$, but we do not require this stronger condition here.

The set $A^{\mathbb{N}}$ itself is a one-sided shift space, called the *one-sided full shift*.

For a two-sided sequence $x \in A^{\mathbb{Z}}$, we define $x^+ = x_0x_1 \cdots$.

If X is a two-sided shift space, then the set $X^+ = \{x^+ \mid x \in X\}$ is a one-sided shift space.

Out-merging

The inverse operation of an out-splitting is referred to as an *out-merging*. An out-merging of a directed graph $G' = (V', E')$ can be performed if there are two vertices s_1, s_2 of G' such that the adjacency matrix M' satisfies:

- the column of index s_1 is equal to the column of index s_2 of M' .

The adjacency matrix of G is thus the matrix M obtained by adding the rows of index s_2 to the row of index s_1 of M' and then removing the column of index s_2 afterward.

The graph G is called an *elementary amalgamation* of G' . Notice that even if M' has 0-1 entries, M may not have 0-1 entries.

General amalgamation

Let M' be the adjacency matrix of a directed graph G' , and (V_1, V_2, \dots, V_k) be a partition of V' into classes such that if s, t belong to the same class, then the columns of indices s and t of M' are identical.

When at least one set of the partition has a size greater than 1, we can perform a *general merging*. We define a graph K of adjacency matrix N obtained by merging all states of each

$V_i = \{s_{i,1}, \dots, s_{i,k_i}\}$ into a single state $s_{i,1}$.

The row in N corresponding to $s_{i,1}$ is obtained by summing the rows of the states of V_i in M' and removing the columns

$s_{i,2}, \dots, s_{i,k_i}$.

The graph K is called a *general amalgamation* of G' .

Two out-merging transformations commute

Proposition (R. Williams 1973)

If G and H are amalgamations of a common directed graph L , then they have a common amalgamation K .

Proposition (R. Williams 1973)

Let G and H be irreducible directed graphs that define one-sided edge shifts X_G and X_H . Then X_G and X_H are conjugate if and only if G and H have the same total amalgamation.

It also holds for one-sided edge shifts defined by trim directed graphs.

Two out-merging transformations commute

Proposition (R. Williams 1973)

If G and H are amalgamations of a common directed graph L , then they have a common amalgamation K .

Proof.

Let us assume that there is an out-merging of G with adjacency matrix M into G' with adjacency matrix M' , obtained by merging s_1 and s_2 into s_1 , and an out-merging of G into G'' with adjacency matrix M'' , obtained by merging s_3 and s_4 into s_3 .

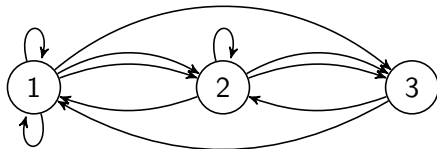
We may assume that the set $\{s_3, s_4\}$ is distinct from the set $\{s_1, s_2\}$.

Let us show that there is a graph H that is an out-merging of both G' and G'' .



Total amalgamation

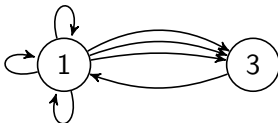
If G is the following graph:



$$M = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

Total amalgamation

Its total amalgamation is H :



$$N = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$$