Master 2 Mathematics and Computer Science Symbolic Dynamics. Lecture 1

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Topology and shift transformation

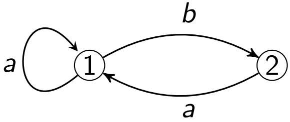
The set $A^{\mathbb{Z}}$ of two-sided infinite sequences of elements of A is a metric space for the distance defined for $x \neq y$ by $d(x,y) = 2^{-r(x,y)}$ where

$$r(x,y) = \inf\{|n| \mid n \in \mathbb{Z}, x_n \neq y_n\}. \tag{1}$$

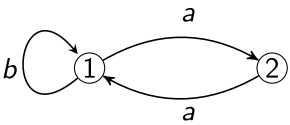
The topology induced by this metric coincides with the product topology on $A^{\mathbb{Z}}$, using the discrete topology on A.
Since a product of compact spaces is compact, $A^{\mathbb{Z}}$ is a compact metric space.

Let S denote the *shift transformation*, defined for $x \in A^{\mathbb{Z}}$ by $S(x) = y$ if $y_n = x_{n+1}$ for $n \in \mathbb{Z}$. It is continuous and one-to-one from $A^{\mathbb{Z}}$ to itself.

Example of a shift of finite type: the golden mean shift



Example of a sofic shift: the even shift



A shift of finite type is sofic: another solution

Let $X = \mathsf{X}_F$ with $F \subseteq A^*$ a finite set.
Let n be the maximal size of words in F.
Let $\mathcal{A}$ be the graph whose states are the words of length $n-1$ that do not contain any word of F, and with edges

$$a_0 a_1 \dots a_{n-2} \xrightarrow{a_0} a_1 \dots a_{n-2} a,$$

where $a_0 a_1 \dots a_{n-2} a$ does not contain any word of F. The set of labels of two-sided infinite paths in $\mathcal{A}$ is equal to $X = \mathsf{X}_F$.

Example with $F = \{bb\}$ on the board.

Language of a shift space

If X is a shift space, the set of blocks of sequences in X is denoted by $\mathcal{B}(X)$. The set of blocks of length n of sequences in X is denoted by $\mathcal{B}_n(X)$.
A language L is called *factorial* if it contains the words occurring as blocks in its elements, that is, if $uvw \in L$, then $v \in L$.
It is *extendable* if every $u \in L$ is *extendable*, that is, there are letters $a, b \in A$ such that $aub \in L$.

Proposition

The language of a shift space is factorial and extendable. Conversely, for every factorial and extendable language L, there is a unique shift space X such that $\mathcal{B}(X) = L$. It is the set $\mathrm{X}(L)$ of sequences $x \in A^{\mathbb{Z}}$ with all their blocks in L. For every factorial and extendable language L and every shift space X, the following equalities hold: $\mathcal{B}(\mathrm{X}(L)) = L$, and $\mathrm{X}(\mathcal{B}(X)) = X$.

Let X be a shift space. For two words u, v such that $uv \in \mathcal{B}(X)$, the set

$$[u \cdot v]_X = \{x \in X \mid x_{[-|u|,|v|)} = uv\}$$

is nonempty. It is called the *cylinder* with basis (u, v). For $v \in \mathcal{B}(X)$, we also define

$$[v]_X = \{x \in X \mid x_{[0,|v|)} = v\}$$

in such a way that $[v]_X = [\varepsilon \cdot v]_X$. The set $[v]_X$ is called the *right cylinder* with basis v.

The open sets contained in X are the unions of cylinders and the clopen sets are the finite unions of cylinders (Exercises).

Irreducible shift

A nonempty shift space X is *irreducible* if, for every $u, v \in \mathcal{B}(X)$, there is a word w such that $uwv \in \mathcal{B}(X)$.

Example

The golden mean shift X is irreducible. Indeed, if $u, v \in \mathcal{B}(X)$, then $uav \in \mathcal{B}(X)$.

Uniformly recurrent shift

A nonempty shift space X is *uniformly recurrent* if for every $w \in \mathcal{B}(X)$ there is an integer $n \geq 1$ such that w occurs in every word of $\mathcal{B}_n(X)$.

As an equivalent definition, a shift space X is uniformly recurrent if for every $n \geq 1$ there is an integer $N = R_X(n)$ such that every word of $\mathcal{B}_n(X)$ occurs in every word of $\mathcal{B}_N(X)$. The function R_X is called the *recurrence function* of X.

Example

Example

The golden mean shift X is not uniformly recurrent since b is in $\mathcal{B}(X)$ although b does not occur in any $a^n \in \mathcal{B}(X)$.

An automaton $\mathcal{A} = (Q, E)$ is a finite directed (multi)graph with edges labeled on A. The set of edges is included in $Q \times A \times Q$.

It is *trim* if each state has at least one outgoing edge and at least one incoming edge.

It is (uncomplete) *deterministic* if, for each state $p \in Q$ and each letter $a \in A$, there is at most one edge labeled by a going out of p.

It is *irreducible* if its graph is strongly connected.

It is a *presentation* of a sofic shift X if X is the set of labels of bi-infinite paths of $\mathcal{A}$.

Minimal automaton in symbolic dynamics

Proposition

Every sofic shift has a trim deterministic presentation.

Proposition

Every irreducible sofic shift has a unique minimal deterministic presentation (irreducible deterministic and with the fewest number of states among these presentations).

Local automaton

A deterministic automaton $\mathcal{A} = (Q, E)$ is *local* if there is an integer n such that, for each word w of length n, all paths labeled by w end in the same state q_w.

Proposition

An irreducible shift X is of finite type if and only if its minimal deterministic automaton is local.

Proof.

Exercise. $\square$

Local automaton

Proposition

An irreducible deterministic automaton is local if and only if it has at most one cycle with a given label.

Proof.

Exercise. $\square$

cycle : path $p = p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} p_2 \ldots \xrightarrow{a_{m-1}} p_m = p$.
m is the length of the cycle.