Université Gustave Eiffel
M2 – Semester 1 – 2025/2026

Complexity - Exercise Sheet 1

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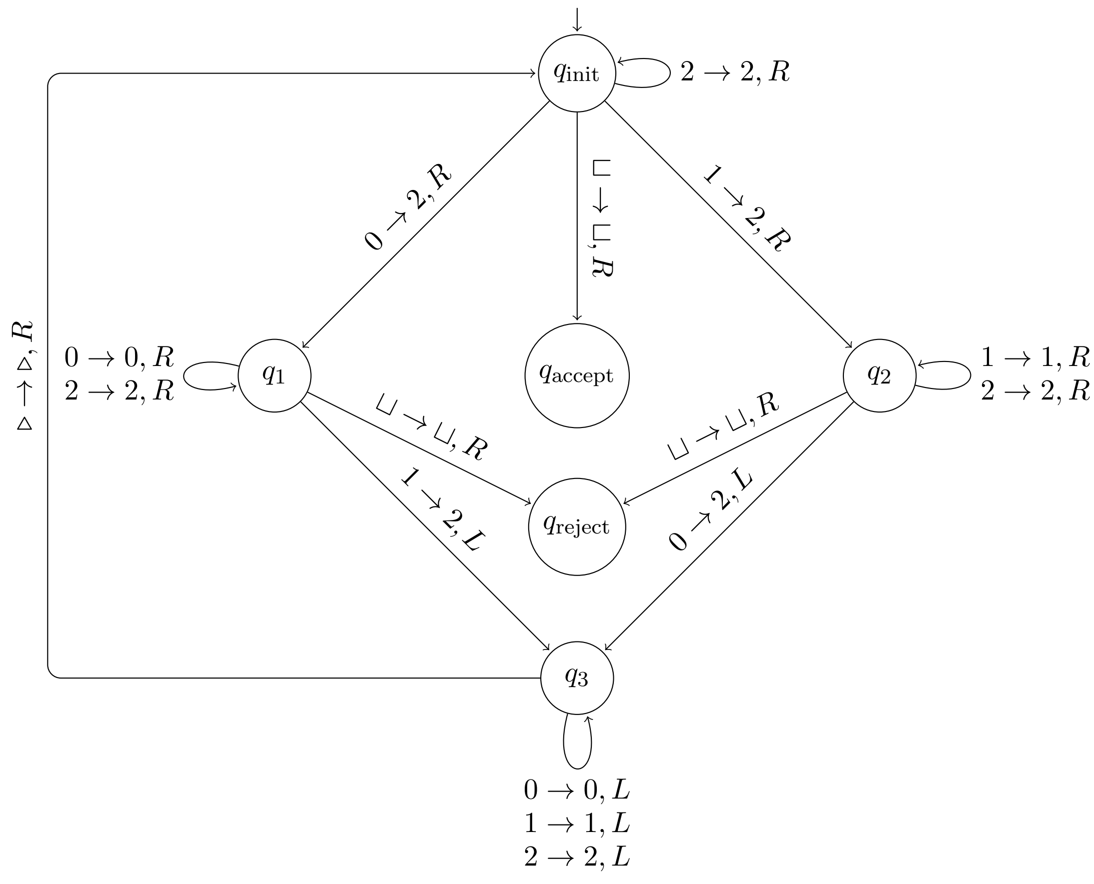
Exercise 1. Describe in detail a Turing machine that decides the language

$$\mathcal{L} = \{w \in \{0,1\}^* \mid w \text{ contains equally 0s and 1s}\}.$$

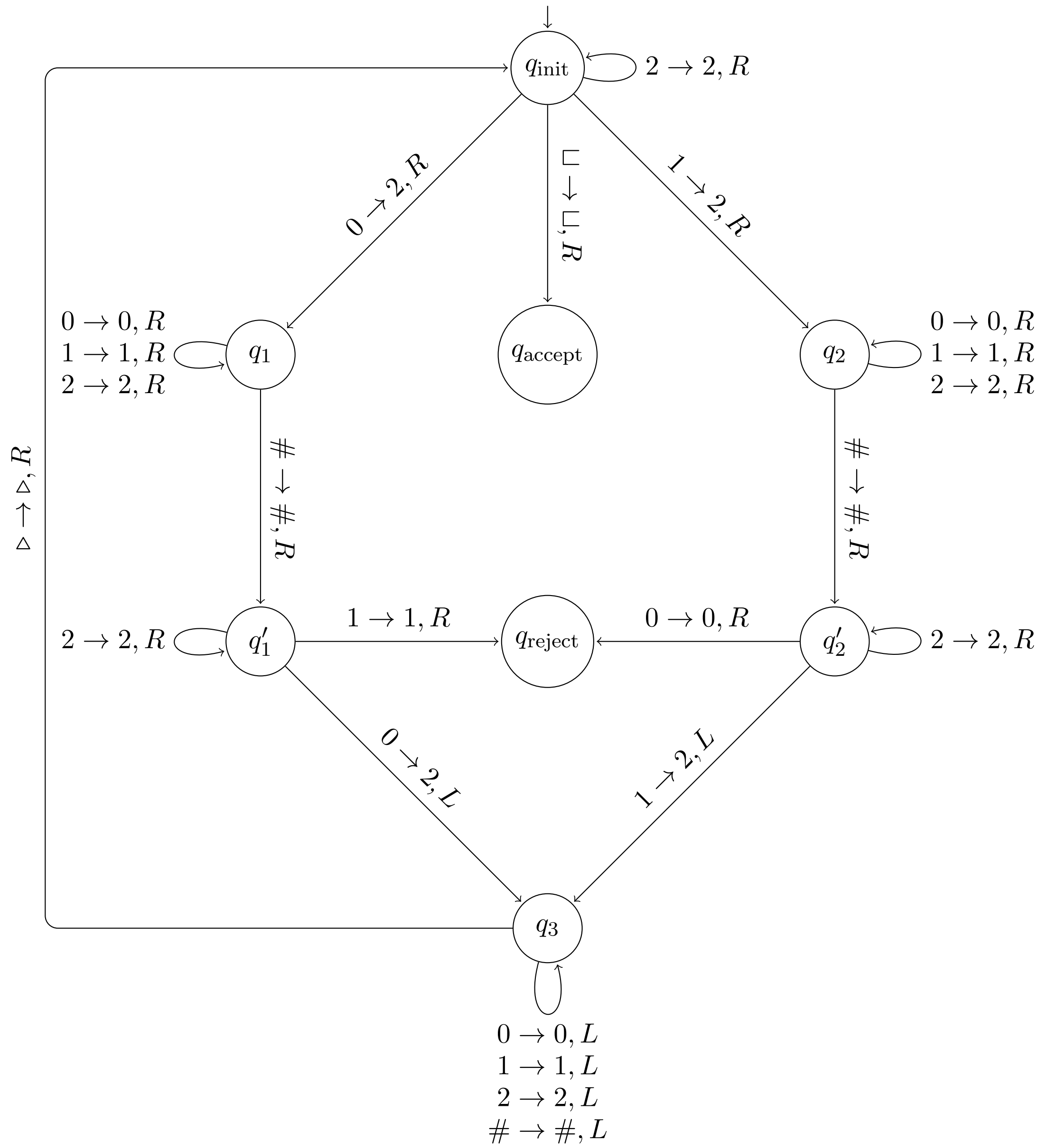
We use the convention that the head points to the symbol right after the start symbol $\triangleright$. We need another symbol 2 in our alphabet. The idea is that when the string is on the tape, we will traverse it multiple times and at each time, we modify a pair of 0 and 1 to 2 and 2. A string is in $\mathcal{L}$ if it can be modified to a string of all 2's.

- At the state q_{init}, if the tape contains all 2's followed by blanks, the original input is accepted. If 0 is read, the machine goes to the state q_1. If 1 is read, the machine goes to the state q_2.
- At the state q_1, we wait for a 1, so we loop around when encountering a 0 or 2. If a blank is encountered before a 1 can be read, the string is rejected. If there is a 1, we move to state q_3. Similarly, q_2 is the state where we intend to wait for a 0.
- At the state q_3, we move to the left until encountering $\triangleright$, by which we move back to q_{init} and start reading the next symbol.



Exercise 2. Describe a TM that decides the language $\mathcal{L} = \{w\#w \mid w \in \{0,1\}^*\}$.

The idea is similar to Exercise 1. The difference is that if 0 is encountered, we *wait for another* 0 *only after* #. The case of encountering 1 is the same.



Exercise 5. Let $\mathcal{L}(M)$ be the language recognized by the TM M i.e. the set of inputs $w \in \{0,1\}^*$ such that M halts with output 1 (on other inputs it may halt with output 0 or not halt). Consider the problem $E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } \mathcal{L}(M) = \varnothing\}$. Show that if E_{TM} were decidable, then the halting problem would also be decidable. Conclude that E_{TM} is undecidable.

For any encoding $\langle M, w \rangle$ of a TM and an input, construct a TM M' such that

- If $\forall x \in \{0,1\}^*, M'(x) = 1$ or $\mathcal{L}(M') = \{0,1\}^*$, then M halts on w;
- If $\forall x \in \{0,1\}^*, M'(x)$ does not halt or $\mathcal{L}(M') = \varnothing$, then M does not halt on w.

If E_{TM} were decidable, then there would be a TM N such that

$$N(\langle M' \rangle) = 1 \Leftrightarrow \mathcal{L}(M') = \varnothing \Leftrightarrow \ M \text{ does not halt on } w,$$

$$N(\langle M' \rangle) = 0 \Leftrightarrow \mathcal{L}(M') \neq \varnothing \Leftrightarrow M \text{ halts on } w.$$

So it would be possible to use N to decide if a TM M halts or not, or the halting problem would be decidable. But in fact, it is not. Thus, E_{TM} is undecidable.