

Probability - Exercise Sheet 2 - Random Variables and Expectation

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Exercise 1. (Number of Hamiltonian paths in a tournament.) A tournament T_n is a orientation of the edges of K_n . We say that T_n admits a Hamiltonian path if there exists $\sigma \in \mathcal{S}_n$ such that $(\sigma(1), \sigma(2)), \dots, (\sigma(n-1), \sigma(n)) \in T_n$.

We want to show that for every integer n there exists one tournament with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

Let us choose a random tournament T_n . For any $\sigma \in \mathcal{S}_n$, let A_σ be the event that $(\sigma(1), \sigma(2)), \dots, (\sigma(n-1), \sigma(n)) \in T_n$. Let X be the number of Hamiltonian paths.

1. Give an expression of X in terms of the events A_σ .
2. Prove that for every $\sigma \in \mathcal{S}_n$, $\mathbb{P}(A_\sigma) = \frac{1}{2^{n-1}}$.
3. Compute $\mathbb{E}(X)$.
4. Prove the expected result.

Solution. To be precise, the sample space Ω is the set of all tournaments, the σ -algebra \mathcal{A} is the power set of Ω , and the probability measure \mathbb{P} is defined as $\mathbb{P}(\{T_n\}) = \frac{1}{2^{\binom{n}{2}}}$, for every tournament T_n . Also, $X : \Omega \rightarrow \mathbb{N}$.

1. We have $X = \sum_{\sigma \in \mathcal{S}_n} \mathbf{1}_{A_\sigma}$.
2. For every T_n such that $(i, j) \in T_n$, there exists exactly one T'_n such that $(j, i) \in T'_n$, and vice versa. Hence, there are as many tournaments containing the edge (i, j) as those containing the edge (j, i) . Half of the tournaments contain the edge (i, j) and the other half contain the edge (j, i) . Therefore, for every $i, j \in [n]$, we have

$$\mathbb{P}((i, j) \in T_n) = \mathbb{P}((j, i) \in T_n) = \frac{1}{2}.$$

Furthermore, for every $\sigma \in \mathcal{S}_n$, the edges $(\sigma(1), \sigma(2)), \dots, (\sigma(n-1), \sigma(n))$ are distinct and the orientations of the edges are independent. Therefore,

$$\mathbb{P}(A_\sigma) = \mathbb{P}((\sigma(1), \sigma(2)) \in T_n, \dots, (\sigma(n-1), \sigma(n)) \in T_n) = \prod_{i=1}^{n-1} \mathbb{P}((\sigma(i), \sigma(i+1)) \in T_n) = \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}}.$$

3. We have $\mathbb{E}(X) = \sum_{\sigma \in \mathcal{S}_n} \mathbb{E}(\mathbf{1}_{A_\sigma}) = \sum_{\sigma \in \mathcal{S}_n} \mathbb{P}(A_\sigma) = n! \frac{1}{2^{n-1}} = \frac{n!}{2^{n-1}}$.
4. Since $\mathbb{E}(X) = \frac{n!}{2^{n-1}} \leq \max_{T_n \in \Omega} X(T_n)$, there exists at least one tournament whose the number of Hamiltonian paths is greater than $\frac{n!}{2^{n-1}}$.

Exercise 3. Show that the variance of the number of fixed points of a random permutation is 1.

Solution. Let σ be a random permutation in \mathcal{S}_n . Let X be the number of fixed points of σ . For every $i \in [n]$, let $X_i = \mathbf{1}_{\{\sigma(i)=i\}}$. We have $X = \sum_{i=1}^n X_i$. We have $\mathbb{E}(X_i) = \mathbb{P}(\sigma(i) = i) = \frac{1}{n}$. Therefore,

On the other hand,

$$\mathbb{E}(X) = \sum_{i=1}^n \mathbb{E}(X_i) = n \cdot \frac{1}{n} = 1.$$

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E} \left(\left(\sum_{i=1}^n X_i \right)^2 \right) \\
&= \mathbb{E} \left(\sum_{i=1}^n X_i^2 + 2 \sum_{1 \leq i < j \leq n} X_i X_j \right) \\
&= \sum_{i=1}^n \mathbb{E}(X_i^2) + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}(X_i X_j)
\end{aligned}$$

Since X_i is an indicator variable, $X_i^2 = X_i$, so $\mathbb{E}(X_i^2) = \mathbb{E}(X_i) = \frac{1}{n}$. For $i \neq j$, we have

$$\mathbb{E}(X_i X_j) = \mathbb{P}(\sigma(i) = i \text{ and } \sigma(j) = j).$$

The remaining $n - 2$ elements can be permuted arbitrarily, so

$$\mathbb{E}(X_i X_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(X^2) &= n \cdot \frac{1}{n} + 2 \cdot \binom{n}{2} \cdot \frac{1}{n(n-1)} \\
&= 1 + 2 \cdot \frac{n(n-1)}{2} \cdot \frac{1}{n(n-1)} \\
&= 1 + 1 = 2
\end{aligned}$$

Thus, $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2 - 1^2 = 1$.