Algebra - Exercises

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Exercise 2. Let $A \in \mathcal{M}_{p,q}$ and $B \in \mathcal{M}_{q,p}$. Show that $\det(I_p + AB) = \det(I_q + BA)$.

Solution. Let $\operatorname{rank} A = r$. Then there exist invertible matrices $P \in \mathcal{M}_p$ and $Q \in \mathcal{M}_q$, and a matrix $D \in \mathcal{M}_{p,q}$ such that $d_{ii} = 1$ for $i \in [r]$ and $d_{ij} = 0$ otherwise, such that $A = PDQ$. We have

$$\det(I_p + AB) = \det(P^{-1}(I_p + AB)P) = \det(P^{-1}(I_p + PDQB)P) = \det(I_p + D(QBP)).$$

Let $C = QBP \in \mathcal{M}_{q,p}$, we rewrite $\det(I_p + AB) = \det(I_p + DC)$. Similarly, $\det(I_q + BA) = \det(I_q + CD)$. We have to show that $\det(I_p + DC) = \det(I_q + CD)$. From the form of D, the matrices DC and CD are triangular and agree on the first r diagonal entries, while the other diagonal entries are zeros. Therefore, $I_p + DC$ and $I_q + CD$ are triangular and agree on the on the first r diagonal entries, while the other diagonal entries are ones. Since the determinant of a triangular matrix is the product of its diagonal entries, we have $\det(I_p + DC) = \det(I_q + CD)$.

For $u, v \in \mathbb{R}^n$, we have $\det(I_n + uv^\top) = 1 + v^\top u = 1 + \sum\limits_{i=1}^{n} u_i v_i$.

Exercise 3. (Kernel Lemma) Let V be a $\mathbb{K}$-vector space and $f \in \operatorname{End}(V)$. Let $P = P_1 \dots P_r \in \mathbb{K}[X]$ with $P_1, \dots, P_r$ in $\mathbb{K}[X]$ and pairwise coprime. Then

$$\ker(P(f)) = \bigoplus_{i=1}^{r} \ker(P_i(f)).$$

Proof. Firstly we have $f^m \circ f^n = f^n \circ f^m = f^{m+n}$ for any nonnegative integers m and n. Hence for $P, Q \in \mathbb{K}[X]$, we have $P(f) \circ Q(f) = PQ(f) = QP(f) = Q(f) \circ P(f)$.

We prove the lemma by induction.

- For $r = 2$, we have $P = P_1P_2$ with $P_1, P_2 \in \mathbb{K}[X]$ coprime. Since P_1 and P_2 are coprime, there exist $U, V \in \mathbb{K}[X]$ such that
$$UP_1 + VP_2 = 1.$$
Let $x \in \ker(P(f))$. Then, $P(f)(x) = P_1P_2(f)(x) = 0$. On the other hand,
$$UP_1(f)(x) + VP_2(f)(x) = x.$$
Let $x_1 = UP_1(f)(x)$ and $x_2 = VP_2(f)(x)$. We have $x = x_1 + x_2$. Furthermore,
$$P_1(f)(x_1) = P_1VP_2(f)(x) = V(f) \circ (P_1P_2)(f)(x) = 0,$$
implying that $x_1 \in \ker(P_1(f))$. Similarly, $x_2 \in \ker(P_2(f))$. Therefore,
$$\ker(P(f)) \subseteq \ker(P_1(f)) + \ker(P_2(f)).$$
Conversely, let $x_1 \in \ker(P_1(f))$. We have $P(f)(x_1) = P_2(f) \circ P_1(f)(x_1) = 0$, or $x_1 \in \ker(P_1(f))$. That means $\ker(P_1(f)) \subset \ker(P(f))$. Similarly, $\ker(P_2(f)) \subset \ker(P(f))$. Hence,
$$\ker(P_1(f)) + \ker(P_2(f)) \subseteq \ker(P(f)).$$
Thus, $\ker(P(f)) = \ker(P_1(f)) + \ker(P_2(f))$. To show that the sum is direct, it is sufficient to show that $\ker(P_1(f)) \cap \ker(P_2(f)) = \{0\}$. Indeed, let $x \in \ker(P_1(f)) \cap \ker(P_2(f))$, then
$$x = UP_1(f)(x) + VP_2(f)(x) = U(f) \circ P_1(f)(x) + V(f) \circ P_2(f)(x) = 0.$$
Therefore, $\ker(P(f)) = \ker(P_1(f)) \oplus \ker(P_2(f))$.

- Suppose that the lemma is true for some $r \geq 2$. Let $P = P_1 \dots P_{r+1}$ with $P_1, \dots, P_{r+1} \in \mathbb{K}[X]$ pairwise coprime. Let $Q = P_1 \dots P_r$. Since P_{r+1} is coprime to each of $P_1, \dots, P_r$, it is also coprime to Q. By the case $r = 2$, we have
$$\ker(P(f)) = \ker(Q(f)) \oplus \ker(P_{r+1}(f)).$$
By the induction hypothesis, we have
$$\ker(Q(f)) = \bigoplus_{i=1}^{r} \ker(P_i(f)).$$
Therefore,
$$\ker(P(f)) = \left(\bigoplus_{i=1}^{r} \ker(P_i(f)) \right) \oplus \ker(P_{r+1}(f)) = \bigoplus_{i=1}^{r+1} \ker(P_i(f)).$$