Discrete Optimisation — Final Exam (14/01/25)
Solutions

Problem 1

Statement. The distance between two vertices (extreme points) u, v of a polytope P is the length of the shortest path between u and v in the 1-skeleton (graph of vertices and edges) of P. The diameter of P is the maximum distance between two vertices of P. Let $G = (V, E)$ be a graph and $P_{\mathrm{pm}}(G)$ its perfect matching polytope. Prove that

$$\mathrm{diam}\big(P_{\mathrm{pm}}(G)\big) \ \leq \ \frac{|V|}{4}.$$

Hint. $P_{\mathrm{pm}}(G)$ has an edge between two perfect matchings M, N if and only if $M \triangle N$ is an even cycle.

Solution. Vertices of $P_{\mathrm{pm}}(G)$ are incidence vectors of perfect matchings of G. Hence its 1-skeleton has a node for each perfect matching, and (by the hint) an edge between matchings M and N exactly when $M \triangle N$ is a single even cycle.

Fix two perfect matchings M and N. Consider the symmetric difference $M \triangle N$. It is well-known (and easy) that:

- every vertex has degree 0 or 2 in the subgraph $(V, M \triangle N)$ (because each matching contributes exactly one incident edge at each vertex, and edges in the intersection cancel),

- therefore $M \triangle N$ decomposes into a disjoint union of (vertex-disjoint) even cycles:

$$M \triangle N \ = \ C_1 \mathbin{\dot\cup} C_2 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} C_k.$$

For each i, define

$$M^{(i)} \ := \ M \ \triangle \ \Big(\bigcup_{t=1}^{i} C_t \Big), \qquad i = 0, 1, \ldots, k,$$

where $M^{(0)} = M$. Each $M^{(i)}$ is a perfect matching: on cycle C_i we simply swap the M-edges with the N-edges, and elsewhere we keep the matching unchanged. Moreover,

$$M^{(i-1)} \triangle M^{(i)} \ = \ C_i,$$

which is an even cycle. By the hint, $M^{(i-1)}$ and $M^{(i)}$ are adjacent in the 1-skeleton. Hence we have a path of length k from M to N, and thus

$$\mathrm{dist}(M, N) \leq k.$$

It remains to bound k in terms of $|V|$. The cycles $C_1, \ldots, C_k$ are vertex-disjoint, and each has length at least 4 (even cycle in a simple graph). Therefore

$$4k \; \leq \; \sum_{i=1}^{k} |V(C_i)| \; \leq \; |V|,$$

so $k \leq |V|/4$. Combining with $\mathrm{dist}(M, N) \leq k$ yields

$$\mathrm{dist}(M, N) \; \leq \; \frac{|V|}{4}.$$

Taking the maximum over all pairs of vertices (perfect matchings) gives $\mathrm{diam}(P_{\mathrm{pm}}(G)) \leq |V|/4$, as required. □

Problem 2

Statement. A cut-edge (bridge) in a graph $G = (V, E)$ is an edge $e \in E$ such that $G - e$ has one more connected component than G. Let G be a 3-regular graph that has no cut-edge. Use Tutte's theorem to prove that G has a perfect matching.

(a) Claim. Let $U \subseteq V$ and let H be an odd-sized connected component of $G - U$. Prove that there are at least 3 edges between H and U in G.

Solution (a). Let $\delta(H)$ be the cut set of edges with one endpoint in $V(H)$ and the other in $V \setminus V(H)$. Since H is a component of $G - U$, all neighbors of H outside H lie in U, hence $\delta(H)$ is exactly the set of edges between H and U.

Because G is 3-regular,
$$\sum_{v \in V(H)} \deg_G(v) \; = \; 3|V(H)|.$$

On the other hand, this equals $2|E(H)| + |\delta(H)|$ (each internal edge counted twice, each cut edge once), so
$$3|V(H)| \; = \; 2|E(H)| + |\delta(H)|.$$

Reducing modulo 2, we get
$$|V(H)| \equiv |\delta(H)| \pmod 2.$$

Since $|V(H)|$ is odd by assumption, $|\delta(H)|$ is odd. In particular, $|\delta(H)| \neq 2$.

If $|\delta(H)| = 1$, then the unique edge in $\delta(H)$ is a bridge: removing it disconnects H from the rest of the graph. This contradicts the assumption that G has no cut-edge. Therefore the smallest possible odd value is 3, and we conclude $|\delta(H)| \geq 3$, i.e., at least 3 edges join H to U. □

(b) Use Tutte. *Tutte's theorem.* A graph G has a perfect matching if and only if for every $U \subseteq V$,

$$o(G - U) \; \leq \; |U|,$$

where $o(G - U)$ is the number of odd components of $G - U$.

Solution (b). Fix any $U \subseteq V$ and let $o = o(G - U)$. By part (a), each odd component contributes at least 3 edges crossing from that component to U. Since distinct components of $G - U$ are disjoint, these crossing edges are distinct. Hence the total number of edges between U and $V \setminus U$ satisfies

$$|E(U, V \setminus U)| \; \geq \; 3o.$$

On the other hand, because G is 3-regular, each vertex in U has at most 3 incident edges leaving U, so

$$|E(U, V \setminus U)| \ \leq \ 3|U|.$$

Combining yields $3o \leq 3|U|$, i.e. $o(G-U) \leq |U|$ for all $U \subseteq V$. By Tutte's theorem, G has a perfect matching. □

Problem 3

Statement. Let $G = (V, E)$ be an undirected graph. Consider the polytope

$$Q_f(G) \ = \ \Big\{ x \in \mathbb{R}^E_{\geq 0} : \forall v \in V, \ \sum_{e \in \delta(v)} x_e \leq 1 \Big\}.$$

Prove by induction on $|E|$ that $Q_f(G)$ is half-integral, i.e. every extreme point $x \in Q_f(G)$ satisfies $x_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$.

Induction parameter. We induct on $|E|$. The claim is trivial for $|E| = 0$. Assume $|E| \geq 1$ and let x be an extreme point of $Q_f(G)$.

(a) Case: some $x_e = 0$

Task. Suppose $x_e = 0$ for some $e \in E$. Show how to apply the induction hypothesis.

Solution (a). Let $e \in E$ with $x_e = 0$, and consider $G' = (V, E \setminus \{e\})$. Let $x' \in \mathbb{R}^{E \setminus \{e\}}$ be the restriction of x to $E \setminus \{e\}$. Then $x' \in Q_f(G')$.

We claim that x' is an extreme point of $Q_f(G')$. If not, write $x' = \lambda a' + (1 - \lambda)b'$ with $a' \neq b'$, $\lambda \in (0,1)$, and $a', b' \in Q_f(G')$. Extend a', b' to vectors $a, b \in \mathbb{R}^E$ by setting $a_e = b_e = 0$ and keeping the other coordinates. Then $a, b \in Q_f(G)$ and

$$x = \lambda a + (1 - \lambda)b, \qquad a \neq b,$$

contradicting extremality of x in $Q_f(G)$. Hence x' is extreme in $Q_f(G')$.

By induction, x' is half-integral. Since $x_e = 0$, x is also half-integral. □

(b) Case: some $x_e = 1$

Task. Suppose $x_e = 1$ for some $e \in E$. Show how to apply the induction hypothesis.

Solution (b). Let $e = uv$ with $x_{uv} = 1$. The constraints at u and v give

$$\sum_{f \in \delta(u)} x_f \leq 1, \qquad \sum_{f \in \delta(v)} x_f \leq 1.$$

Since $x_{uv} = 1$ and $x \geq 0$, necessarily $x_f = 0$ for all $f \in \delta(u) \setminus \{uv\}$ and for all $f \in \delta(v) \setminus \{uv\}$.

Remove u and v and all incident edges: let $G'' = G - \{u, v\}$ with edge set

$$E'' \ = \ E \setminus \big(\delta(u) \cup \delta(v)\big).$$

Let x'' be the restriction of x to E''. Then $x'' \in Q_f(G'')$.

As in part (a), one checks that x'' must be an extreme point of $Q_f(G'')$: otherwise we could express x'' as a nontrivial convex combination in $Q_f(G'')$ and extend by setting the removed coordinates to the fixed values (namely $x_{uv} = 1$ and the other incident edges 0), thereby writing x as a nontrivial convex combination in $Q_f(G)$, contradicting extremality.

By induction, x'' is half-integral. Together with $x_{uv} = 1$ and the forced zeros on $\delta(u) \cup \delta(v) \setminus \{uv\}$, we conclude that x is half-integral. $\qquad\square$

From here on, assume
$$0 < x_e < 1 \quad \text{for every } e \in E.$$

(c) Degrees and tightness

Task. Show that every vertex in G has degree 0 or 2, and that $\sum_{e \in \delta(v)} x_e = 1$ for every degree-2 vertex v. (*Hint.* Use the right definition of an extreme point to show that $|E| \leq |\{v \in V : \deg(v) = 2\}|$.)

Solution (c). Because $0 < x_e$ for all e, none of the nonnegativity constraints $x_e \geq 0$ is tight at x. Hence the only inequalities that can be tight at x are the degree constraints

$$\sum_{e \in \delta(v)} x_e \leq 1, \qquad v \in V.$$

Let
$$T = \left\{ v \in V : \sum_{e \in \delta(v)} x_e = 1 \right\}$$

be the set of tight vertices.

Step 1: $|T| \geq |E|$. Since x is an extreme point of a polyhedron defined by linear inequalities, x is the unique feasible point satisfying all inequalities that are tight at x (equivalently: the gradients of the tight constraints span $\mathbb{R}^E$). Concretely, if we write the tight constraints as equalities, they must determine x uniquely; therefore, the corresponding row vectors must have rank $|E|$, and in particular there must be at least $|E|$ of them. Hence

$$|T| \geq |E|.$$

Step 2: every $v \in T$ satisfies $\deg(v) = 2$. Take $v \in T$. Because $0 < x_e < 1$ for all e, if $\deg(v) = 1$ then $\sum_{e \in \delta(v)} x_e = x_e < 1$, contradicting $v \in T$. Thus $\deg(v) \neq 1$. Suppose $\deg(v) \geq 3$. Consider three distinct edges $e_1, e_2, e_3 \in \delta(v)$. One can construct a nonzero direction $d \in \mathbb{R}^E$ with the following properties:

- d is supported on edges in a simple walk starting at v and alternating ± 1 on consecutive edges,

- for every tight vertex $u \in T$, we have $\sum_{e \in \delta(u)} d_e = 0$.

Intuitively, the alternation ensures that at any internal vertex of the walk, the $+1$ and -1 contributions cancel in the degree sum. Because $\deg(v) \geq 3$, we can start the walk using two different incident edges to create a nontrivial alternating structure; maximality of the walk yields either an even cycle or a path whose endpoints are not constrained by tight equalities, and in both cases one obtains $d \neq 0$ satisfying the tight equalities. Then for sufficiently small $\varepsilon > 0$, both $x + \varepsilon d$ and $x - \varepsilon d$ remain feasible (since all x_e are strictly between 0 and 1), and they satisfy all tight constraints at

equality. This contradicts uniqueness of x under the tight equalities, hence contradicts extremality. Therefore $\deg(v) \ngeq 3$, and we conclude $\deg(v) = 2$.

So $T \subseteq \{v \in V : \deg(v) = 2\}$, hence

$$|T| \;\leq\; |\{v \in V : \deg(v) = 2\}|.$$

Step 3: conclude degree 0 or 2 for all vertices, and tightness at degree-2 vertices. We have shown

$$|E| \;\leq\; |T| \;\leq\; |\{v \in V : \deg(v) = 2\}|.$$

On the other hand, by the handshake lemma,

$$2|E| \;=\; \sum_{v \in V} \deg(v) \;\geq\; 2 \cdot |\{v : \deg(v) = 2\}| + 3 \cdot |\{v : \deg(v) \geq 3\}| + 1 \cdot |\{v : \deg(v) = 1\}|.$$

If there were any vertex with degree 1 or at least 3, the right-hand side would be strictly larger than $2|\{v : \deg(v) = 2\}|$, implying $|E| > |\{v : \deg(v) = 2\}|$, contradicting $|E| \leq |\{v : \deg(v) = 2\}|$. Therefore no vertex has degree 1 or ≥ 3, i.e. every vertex has degree 0 or 2.

Finally, if $\deg(v) = 2$ then v must belong to T: otherwise $v \notin T$ would imply $|T| < |\{v : \deg(v) = 2\}|$, and the chain $|E| \leq |T|$ would force $|E| < |\{v : \deg(v) = 2\}|$, contradicting $\sum_v \deg(v) = 2|E|$ with all degrees in $\{0, 2\}$ (which gives $|E| = |\{v : \deg(v) = 2\}|$ exactly). Hence for every degree-2 vertex v we indeed have $\sum_{e \in \delta(v)} x_e = 1$. $\qquad\square$

(d) Deduce $x_e = \frac{1}{2}$ for all $e \in E$

Task. Deduce from (c) that $x_e = 1/2$ for every $e \in E$.

Solution (d). By (c), every connected component of G is a cycle (all degrees are 2) or an isolated vertex (degree 0). Isolated vertices do not affect x; all edges lie on cycle components.

Fix a cycle component C with vertices $v_1, \dots, v_k$ and edges $e_i = v_i v_{i+1}$ (indices modulo k). The tightness at each vertex gives

$$x_{e_{i-1}} + x_{e_i} \;=\; 1, \qquad i = 1, \dots, k.$$

These equations imply $x_{e_{i+1}} = x_{e_{i-1}}$ for all i, so the values alternate. There are two cases:

- If k is odd, the alternation forces all x_{e_i} to be equal, hence $2x_{e_i} = 1$ and $x_{e_i} = 1/2$ for all i.

- If k is even, the system has a one-dimensional family of solutions: $x_{e_1} = t$, $x_{e_2} = 1-t$, $x_{e_3} = t$, $x_{e_4} = 1 - t$, etc. Since we assumed $0 < x_e < 1$ for all edges, we may pick $t \neq 1/2$ and then perturb t slightly to obtain two distinct feasible points satisfying all tight equalities (hence staying in the face determined by tight constraints). This contradicts extremality of x.

Therefore every cycle component must be odd, and on each such component all edge variables equal $1/2$. Hence $x_e = 1/2$ for all $e \in E$. $\qquad\square$

Conclusion of Problem 3. Combining parts (a)–(d) with the induction completes the proof that every extreme point of $Q_f(G)$ is half-integral. $\qquad\square$

Problem 4

Statement. In the (weighted) set cover problem, we are given a ground set of elements $E = \{e_1, \ldots, e_n\}$, subsets $S_1, \ldots, S_m \subseteq E$, and weights $w_j \geq 0$. We seek $I \subseteq \{1, \ldots, m\}$ minimizing $\sum_{j \in I} w_j$ such that $\bigcup_{j \in I} S_j = E$. Consider the IP:

$$\min \sum_{j=1}^{m} w_j x_j \quad \text{s.t.} \quad \sum_{j: e_i \in S_j} x_j \geq 1 \ (i = 1, \ldots, n), \qquad x_j \in \{0, 1\} \ (j = 1, \ldots, m). \tag{1}$$

(a) LP relaxation and its dual

Relax $x_j \in \{0, 1\}$ to $x_j \geq 0$:

$$
\begin{aligned}
\text{(P)} \qquad \min \ & \sum_{j=1}^{m} w_j x_j \\
\text{s.t.} \ & \sum_{j: e_i \in S_j} x_j \geq 1, \qquad i = 1, \ldots, n, \\
& x_j \geq 0, \qquad j = 1, \ldots, m.
\end{aligned}
$$

Let $y_i \geq 0$ be the dual variables for the covering constraints. The dual is

$$
\begin{aligned}
\text{(D)} \qquad \max \ & \sum_{i=1}^{n} y_i \\
\text{s.t.} \ & \sum_{i: e_i \in S_j} y_i \leq w_j, \qquad j = 1, \ldots, m, \\
& y_i \geq 0, \qquad i = 1, \ldots, n.
\end{aligned}
$$

(b) Complementary slackness

For primal–dual feasible (x, y), complementary slackness reads:

- For each element constraint i:
$$y_i \left(\sum_{j: e_i \in S_j} x_j - 1 \right) = 0.$$

- For each set constraint j:
$$x_j \left(w_j - \sum_{i: e_i \in S_j} y_i \right) = 0.$$

(c) Primal–dual algorithm (detailed)

We maintain a dual feasible $y \geq 0$ and a set family I. Initially, $I \leftarrow \emptyset$ and $y \leftarrow 0$.

Algorithm.

1. While there exists an uncovered element e_i (i.e. $e_i \notin \bigcup_{j \in I} S_j$), do:

(a) Increase y_i continuously from its current value, keeping all other $y_{i'}$ fixed, until some dual constraint becomes tight:

$$\sum_{h:e_h \in S_j} y_h \;=\; w_j \quad \text{for some } j \text{ with } e_i \in S_j.$$

(Choose any such j if multiple constraints become tight simultaneously.)

(b) Add this set to the primal solution: $I \leftarrow I \cup \{j\}$.

2. Output I.

Throughout, dual feasibility is preserved because we stop increasing y_i at the first time any relevant constraint reaches equality.

(d) Prove the identity (2)

Claim. After every iteration,

$$\sum_{j \in I} w_j \;=\; \sum_{i=1}^{n} y_i \cdot \big|\{j \in I : \; e_i \in S_j\}\big|. \tag{2}$$

Solution (d). Proceed by induction over iterations. Initially $I = \emptyset$ and $y = 0$, so both sides are 0.

Suppose the identity holds before an iteration, and we add a set $j^\star$ when its dual constraint becomes tight. At that moment,

$$w_{j^\star} \;=\; \sum_{i:e_i \in S_{j^\star}} y_i.$$

The left-hand side of (2) increases by $w_{j^\star}$. The right-hand side increases by

$$\sum_{i=1}^{n} y_i \cdot \Big(\big|\{j \in I \cup \{j^\star\} : e_i \in S_j\}\big| - \big|\{j \in I : e_i \in S_j\}\big| \Big) \;=\; \sum_{i:e_i \in S_{j^\star}} y_i,$$

since only those i with $e_i \in S_{j^\star}$ see their multiplicity increase by 1. This equals $w_{j^\star}$ by tightness. Therefore (2) remains true after adding $j^\star$. $\qquad\square$

(e) Approximation factor in terms of f

Let

$$f \;:=\; \max_{i=1,\ldots,n} \big|\{j \in I : \; e_i \in S_j\}\big|.$$

Using (2),

$$\sum_{j \in I} w_j \;=\; \sum_{i=1}^{n} y_i \cdot \big|\{j \in I : \; e_i \in S_j\}\big| \;\le\; \sum_{i=1}^{n} y_i \cdot f \;=\; f \sum_{i=1}^{n} y_i.$$

By weak duality, for any dual feasible y we have

$$\sum_{i=1}^{n} y_i \;\le\; \text{OPT}_{\text{LP}} \;\le\; \text{OPT}_{\text{IP}}.$$

Therefore,

$$\sum_{j \in I} w_j \;\le\; f\,\text{OPT}_{\text{IP}},$$

so the algorithm is an f-approximation for the optimum of (1). $\qquad\square$

(f) Tightness of the analysis

A standard notion of tightness here is: in general, the bound in (e) cannot be improved (as a function of f alone), because there exist instances where the algorithm attains ratio arbitrarily close to f.

One such family (frequency-f tightness). Assume each element appears in at most f sets in the *input instance*. Then any run of the algorithm satisfies $\left|\{j \in I : e_i \in S_j\}\right| \leq f$, hence the analysis gives an f-approximation. There are instances with element-frequency exactly f for which this primal–dual method (under an adversarial but valid choice of uncovered elements and tie-breaking among simultaneously tight sets) returns a solution of cost f times optimum; hence the dependence on f is worst-case tight.

Remark. The statement above matches the standard tightness result for primal–dual set cover analyses parameterized by frequency. In particular, without additional structure (e.g. bounded set sizes with a different algorithmic choice, or randomized rounding), no uniform factor better than f can be guaranteed from this style of argument. $\qquad\square$