Université Gustave Eiffel
M2 – Semester 1 – 2025/2026

Analysis - Exercises

CHAU Dang Minh

Exercise 8. Let (u_n) be a sequence defined by $u_0 \in \mathbb{R}$ and $u_{n+1} = u_n + e^{-u_n}$. Show that $u_n \to +\infty$ and

$$u_n = \ln n + \frac{\ln n}{2n} + o\left(\frac{\ln n}{n}\right).$$

Solution. We first show that $u_n \to +\infty$. For every $n \in \mathbb{N}$, we have $u_{n+1} - u_n = e^{-u_n} > 0$, or $u_{n+1} > u_n$. Hence (u_n) is a strictly increasing sequence. Suppose that (u_n) is bounded above by some $M \in \mathbb{R}$. Then, it converges to some limit $L \in \mathbb{R}$. Taking the limit on both sides of the recurrence relation, we get

$$L = L + e^{-L} \iff e^{-L} = 0.$$

This is a contradiction, since $e^{-L} > 0$ for every $L \in \mathbb{R}$. Hence (u_n) is not bounded above, and since it is increasing, we must have $u_n \to +\infty$.

Let $a_n = e^{u_n}$, then $a_n \to \infty$. Also, $u_n = \ln(a_n)$. We have

$$a_{n+1} = e^{u_{n+1}} = e^{u_n + e^{-u_n}} = e^{u_n} e^{e^{-u_n}} = a_n e^{\frac{1}{a_n}} \geq a_n \left(1 + \frac{1}{a_n}\right) = a_n + 1.$$

Therefore, $a_n \geq n + a_0 = n + e^{u_0} > n$ for all $n \in \mathbb{N}$. Since $\dfrac{1}{a_n} \to 0$, we can use the Taylor expansion of e^x at 0 to get

$$a_{n+1} = a_n \left(1 + \frac{1}{a_n} + \frac{1}{2a_n^2} + O\left(\frac{1}{a_n^3}\right)\right) = a_n + 1 + \frac{1}{2a_n} + O\left(\frac{1}{a_n^2}\right).$$

We also have

$$\begin{aligned}
a_n - a_0 &= \sum_{k=0}^{n-1} (a_{k+1} - a_k) = \sum_{k=0}^{n-1} \left(1 + \frac{1}{2a_k} + O\left(\frac{1}{a_k^2}\right)\right) \\
&= n + \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{a_k} + O\left(\sum_{k=0}^{n-1} \frac{1}{a_k^2}\right) \\
&\leq n + \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{a_0} + O\left(\sum_{k=0}^{n-1} \frac{1}{n^2}\right) \\
&\leq n + \ln n + O(1),
\end{aligned}$$

or $a_n \leq n + \ln n + O(1)$. Now let $b_n = a_n - n - \dfrac{1}{2} \ln n$. Then we have

$$b_{n+1} - b_n = a_{n+1} - a_n - 1 - \frac{1}{2}(\ln(n+1) - \ln n) = \frac{1}{2a_n} + O\left(\frac{1}{a_n^2}\right) - \frac{1}{2}(\ln(n+1) - \ln n).$$

Since $\ln(n+1) - \ln n = \ln\left(1 + \dfrac{1}{n}\right) = \dfrac{1}{n} - \dfrac{1}{2n^2} + O\left(\dfrac{1}{n^3}\right)$, we obtain

$$\begin{aligned}
b_{n+1}-b_n &= \frac{1}{2a_n}+O\left(\frac{1}{a_n^2}\right)-\frac{1}{2}\left(\frac{1}{n}-\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)\right)\\
&= \frac{1}{2}\left(\frac{1}{a_n}-\frac{1}{n}\right)+\frac{1}{4n^2}+O\left(\frac{1}{n^3}+\frac{1}{a_n^2}\right).
\end{aligned}$$

From $a_n \le n+\ln n+O(1)$, for $n\ge 1$, we have

$$\frac{1}{a_n}\ge \frac{1}{n+\ln n+O(1)}=\frac{1}{n}\cdot\frac{1}{1+\frac{\ln n}{n}+O\left(\frac{1}{n}\right)}=\frac{1}{n}-\frac{\ln n}{n^2}+O\left(\frac{\ln^2 n}{n^3}\right).$$

Therefore, $0>\frac{1}{a_n}-\frac{1}{n}>-\frac{\ln n}{n^2}+O\left(\frac{\ln^2 n}{n^3}\right)$ or $0<\left|\frac{1}{a_n}-\frac{1}{n}\right|<\frac{\ln n}{n^2}+O\left(\frac{\ln^2 n}{n^3}\right)$. For some $N_1\in\mathbb{N}$, we have $\frac{\ln n}{n^2}<\frac{1}{n^{3/2}}$ for every $n>N_1$. Using the fact that $\sum_{k=1}^{n}\frac{1}{k^p}<\infty$, for every $p>1$, we get

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^2}\le\sum_{n=1}^{N_1}\frac{\ln n}{n^2}+\sum_{n=N_1+1}^{\infty}\frac{\sqrt{n}}{n^2}=\sum_{n=1}^{N_1}\frac{\ln n}{n^2}+\sum_{n=N_1+1}^{\infty}\frac{1}{n^{3/2}}<\infty.$$

Using the same argument, we also have $\sum_{n=1}^{\infty}O\left(\frac{\ln^2 n}{n^3}\right)<\infty$. Hence, $\sum_{n=1}^{\infty}\left|\frac{1}{a_n}-\frac{1}{n}\right|<\infty$.

From the calculation of $b_{n+1}-b_n$, there exist $C>0$ and $N_2\in\mathbb{N}$ such that for every $n>N_2$, we have

$$|b_{n+1}-b_n|\le\frac{1}{2}\left|\frac{1}{a_n}-\frac{1}{n}\right|+\frac{1}{4n^2}+C\left(\frac{1}{n^3}+\frac{1}{a_n^2}\right)<\frac{1}{2}\left|\frac{1}{a_n}-\frac{1}{n}\right|+\frac{1}{4n^2}+C\left(\frac{1}{n^3}+\frac{1}{n^2}\right).$$

Therefore, $\sum_{n=1}^{\infty}|b_{n+1}-b_n|<\infty$, implying that $\sum_{n=1}^{\infty}(b_{n+1}-b_n)$ converges. Hence, $b_n=b_0+\sum_{k=1}^{n}(b_k-b_{k-1})$ converges to some limit $b\in\mathbb{R}$. Therefore, $a_n=n+\frac{1}{2}\ln n+b+o(1)$, implying that

$$u_n=\ln n+\ln\left(1+\frac{\frac{1}{2}\ln n+b+o(1)}{n}\right).$$

Since $\frac{\frac{1}{2}\ln n+b+o(1)}{n}\to 0$, we can use the Taylor expansion of $\ln(1+x)$ at 0 to get

$$u_n=\ln n+\frac{\frac{1}{2}\ln n+b+o(1)}{n}+o\left(\frac{\ln n}{2n}+\frac{b}{n}+\frac{o(1)}{n}\right)=\ln n+\frac{\ln n}{2n}+o\left(\frac{\ln n}{n}\right).$$