Master 2 Mathematics and Computer Science Symbolic Dynamics. Lecture 2

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Overview

- Conjugacy and state-splitting
- Perron-Frobenius theorem
- Conjugacy invariants.

Conjugacy and state-splitting

Block maps

Let X be a shift space on the alphabet A, and let B be another alphabet. Given integers m, a with $m + a \geq 0$, a *block map* is a map $f : \mathcal{B}_{m+a+1}(X) \to B$. The *sliding block code* defined by f and (m, a) is the map $\varphi : X \to B^{\mathbb{Z}}$ defined by $\varphi(x) = y$ if for every $i \in \mathbb{Z}$,

$$y_i = f(x_{[i-m,i+a]}).$$

Thus y is computed from x by sliding a window of length $m + a + 1$ on x. The integer m is the *memory* of φ and a is its *anticipation*.

$$\begin{array}{c}
\cdots \quad x_{i-m-1} \boxed{x_{i-m} \mid \cdots \mid x_i \mid \cdots \mid x_{i+a}} x_{i+a+1} \quad \cdots \\
\downarrow f \\
\cdots \quad y_{i-1} \boxed{y_i} y_{i+1} \quad \cdots
\end{array}$$

Figure: The sliding block code.

Sliding block code

Proposition

Let X be a shift space on the alphabet A and let $\varphi\colon X \to B^{\mathbb{Z}}$ be a sliding block code. Then $\varphi(X)$ is a shift space.

Proof.

The map φ is clearly continuous and commutes with the shift, that is, $\varphi \circ S = S \circ \varphi$. Thus $Y = \varphi(X)$ is closed (because the continuous image of a compact space is compact and thus closed). It is shift-invariant because, if $y = \varphi(x)$ with $x \in X$, then $S(y) = S \circ \varphi(x) = \varphi(S(x))$ and thus $S(y) \in Y$. □

Curtis-Hedlund-Lyndon theorem

Let X, Y be shift spaces. A map $\varphi\colon X \to Y$ is a *morphism* if φ is continuous and commutes with the shift map.

Theorem (Curtis, Hedlund, Lyndon)

Let X, Y be shift spaces. A map $\varphi\colon X \to Y$ is a morphism systems if and only if it is a sliding block code from X into Y.

Proof.

A sliding block code is clearly continuous and commutes with the shift.
Conversely, let $\varphi\colon X \to Y$ be a morphism . For every letter b from the alphabet B of Y, the set $[b]_Y$ is clopen and thus $\varphi^{-1}([b]_Y)$ is also clopen. Since a clopen set is a finite union of cylinders, there is an integer n such that $\varphi(x)_0$ depends only on $x_{[-n,n]}$. Set $f(x_{[-n,n]}) = \varphi(x)_0$. Then φ is the sliding block code associated with the block map f. □

Conjugacy

Let X, Y be shift spaces. A map $\varphi \colon X \to Y$ is a *conjugacy* if φ is a bijective morphism.

Its inverse is also a morphism.

It follows from the Curtis-Hedlund-Lyndon theorem that a conjugacy between shift spaces X and Y is the same as a sliding block code from X to Y which is invertible (the inverse is also a sliding block code since the inverse of a conjugacy is a morphism).

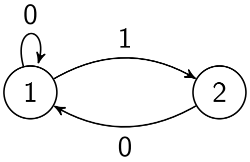
Example

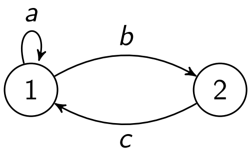
We say that $f\colon A^{m+a+1} \to B$ is a *k-block map*, where $k = m + a + 1$, and that the corresponding sliding block code is a *k-block code*.

The simplest form of a sliding block code, called a 1-*block code*, occurs when $m = a = 0$. In this case, we use the same symbol to denote the 1-block map $\phi\colon A \to B$ and the 1-block code $\phi\colon A^{\mathbb{Z}} \to B^{\mathbb{Z}}$. In this way, we have for every $x \in X$ and $n \in \mathbb{Z}$, $\phi(x)_n = \phi(x_n)$.

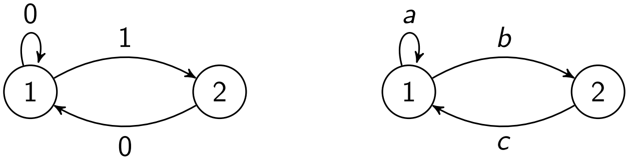
Example

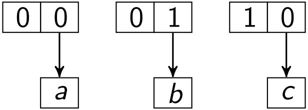
X_F with $F = \{11\}$ is a shift of finite (the golden mean shift).
X_G with $G = \{ac, ba, bb, cc\}$ is a shift of finite.



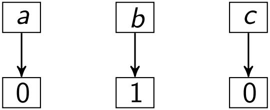


Conjugacy: example





$$\varphi \colon X_F \to X_G .$$



$$\varphi^{-1} \colon X_G \to X_F .$$

Edge shifts

An *edge shift* is the set of bi-infinite paths of a directed (multi)graph.

Proposition

Every shift of finite type is conjugate to an edge shift.

Proof.

Let $X = \mathsf{X}_F$ with F finite, and let n be the maximal size of words in F. We may assume that all words in F have size n.
Let $\mathcal{A} = (Q, E)$, where Q is the set of words of length $n-1$ with edges $a_0a_1 \dots a_{n-2} \xrightarrow{a} a_1 \dots a_{n-2}a$, where $a_0a_1 \dots a_{n-2}a \notin F$. We keep only the trim part of this automaton.

Then $\mathcal{A}$ is deterministic and local (all paths labeled by a word w of length $n-1$ end in the same state q_w). $\square$

proof continued.

Let Y be the set of bi-infinite paths of $\mathcal{A}$.
Let $\phi\colon Y \to X$ defined by the 1-block map $f\colon E \to A$ with $f(e = (p, a, q)) = a$.
Then the sliding block code $\varphi\colon X \to Y$ with anticipation 0 and memory $n-1$ defined by the n-block map $g\colon A^n \to E$ with $g(a_0 a_1 \dots a_{n-1}) = (p, a_{n-1}, q)$, where $p = a_0 a_1 \dots a_{n-2}$ and $q = a_1 a_2 \dots a_{n-1}$.
The map φ is the inverse of ϕ.

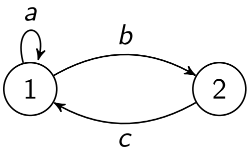
$\square$

Edge shifts

If we take a minimal deterministic presentation of an irreducible shift of finite type, then, up to a conjugacy, one may assume that all labels are distinct and the shift can be defined by the transition matrix of the graph G of the presentation.

Transition matrix of an automaton

$M = (m_{pq})_{p,q \in Q}$, where m_{pq} is the number of edges from p to q.



$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

If G is an automaton or a graph, the edge shift defined by G is denoted X_G.

State splitting of an automaton

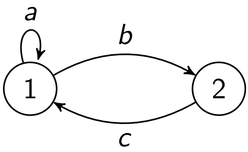
An *out-splitting* of an automaton $\mathcal{A} = (Q, E)$ is a local transformation of $\mathcal{A}$ into an automaton $\mathcal{B} = (Q', E')$ obtained by selecting a state s and partitioning the set of edges going out of s into two non-empty sets E_1 and E_2.

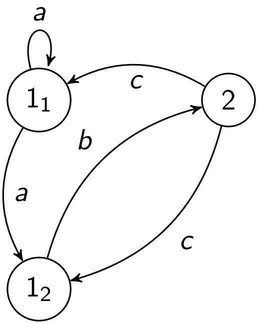
- $Q' = Q \setminus \{s\} \cup \{s_1, s_2\}$,
- E' contains all edges of E neither starting at or ending in s,
- E' contains the edge (s_1, a, t) for each edge $(s, a, t) \in E_1$, and the edge (s_2, a, t) for each edge $(s, a, t) \in E_2$, if $t \neq s$.
- E' contains the edges (t, a, s_1) and (t, a, s_2) if (t, a, s) in E, when $t \neq s$,
- E' contains the edges (s_1, a, s_1) and (s_1, a, s_2) if (s, a, s) in E_1, and the edges (s_2, a, s_1) and (s_2, a, s_2) if $(s, a, s) \in E_2$.

An *input state splitting* is defined similarly.

The inverse operation is called an *output merging*, possible whenever s_1 and s_2 have the *same input edges*.

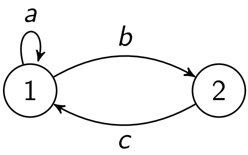
Output state splitting of an automaton

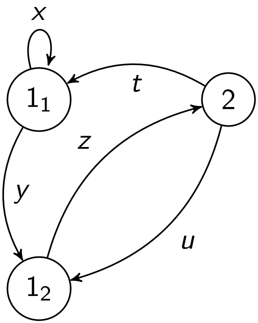




The state 1 is split into two states 1_1 and 1_2 with $E_1 = \{(1, a, 1)\}$ and $E_2 = \{(1, b, 2)\}$.

Output state splitting of a graph





Proposition

Let G be a graph and H a split graph of G. Then X_G and X_H are conjugate.

Proof.

Let $G = (Q, E)$ (all labels are distinct).
Let $H = (Q', E')$ be an outsplit of G, obtained after splitting the state s into s_1, s_2 according to the partition E_1, E_2 of edges going out of s.
Let X_G be the edge shift defined by G and X_H be the edge shift defined by H.
Then X_G and X_H are conjugate. $\square$

Proof.

Indeed, let $\varphi_{GH}\colon\ \colon E^{\mathbb{Z}} \to E'^{\mathbb{Z}}$ be the $(0,1)$-sliding block code defined by the 2-block map $f\colon \mathcal{B}_2(X_G) \to E'$, where

$$
\begin{aligned}
f((t,a,u)(u,b,v)) &= (t,a,u) && \text{if } t,u \neq s,\\
f((t,a,s)(s,b,v)) &= (t,a,s_1) && \text{if } t \neq s \text{ and } (s,b,v) \in E_1,\\
f((t,a,s)(s,b,v)) &= (t,a,s_2) && \text{if } t \neq s \text{ and } (s,b,v) \in E_2,\\
f((s,a,t)(t,b,u)) &= (s_1,a,t) && \text{if } t \neq s \text{ and } (s,a,t) \in E_1,\\
f((s,a,t)(t,b,u)) &= (s_2,a,t) && \text{if } t \neq s \text{ and } (s,a,t) \in E_2,\\
f((s,a,s)(s,b,t)) &= (s_1,a,s_1) && \text{if } (s,b,t) \in E_1,\\
f((s,a,s)(s,b,t)) &= (s_1,a,s_2) && \text{if } (s,b,t) \in E_2.
\end{aligned}
$$

defines a conjugacy from X_G onto X_H. Its inverse is the sliding block code defined by the 1-block map $g\colon E' \to E$, where $g(t,a,u) = (\pi(t),\pi(a),\pi(u))$, with $\pi(t) = t$ if $t \neq s_1, s_2$, $\pi(s_1) = \pi(s_2) = s$, $\pi(a) = a$.

Complete out-splitting

Let $G = (Q, E)$ be a graph. All edges have distinct labels; we may omit them.

The *complete out-splitting* of G is the graph $H = (Q', E')$ where

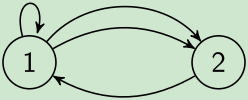
- $Q' = \{s_e \mid s \in Q, e = (s, t) \in E\}$,
- $E' = \{(s_e, t_f) \mid e = (s, t)\}$

Proposition

Let G be a graph and H the complete out-splitting of G. Then X_G and X_H are conjugate. There is a sequence of out-splittings from G to H.

Example

Compute the complete out-splitting of



Decomposition Theorem

An *trim graph* is a graph such that each state has at least one incoming edge and at least one outgoing edge.

Theorem (Decomposition Theorem, R. Williams 1973)

Two edge shifts X_G and X_H defined by trim graphs G and H are conjugate if and only if there is a sequence of (input and output) state splittings and (input and output) state mergings from G to H.

Higher block shift

Let X be a shift space on the alphabet A and let $k \geq 1$ be an integer.

The map $\gamma_k : X \to \mathcal{B}_k(X)^{\mathbb{Z}}$ defined for $x \in X$ by $y = \gamma_k(x)$ if, for every $n \in \mathbb{Z}$,

$$y_n = \langle x_n \cdots x_{n+k-1} \rangle, \tag{1}$$

is the k-th *higher block code*.
One also says that γ_k is a *coding by overlapping blocks* of length k.

The set $X^{(k)} = \gamma_k(X)$ is a shift space on $\mathcal{B}_k(X)$, called the k-th *higher block shift* of X.

Higher block code

The higher block code is an isomorphism of shift spaces, and the inverse of γ_k is the map $\pi_k \colon y \mapsto x$ such that, for all n, x_n is the first letter of the word u such that $y_n = \langle u \rangle$.

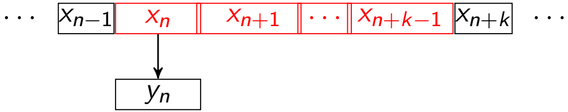


Figure: The k-th higher block code.

Thus, $X^{(k)}$ is conjugate to X.

Reduction to a 1-block code

Lemma

Let $\varphi\colon X \to Y$ be a sliding block code. Then there is a higher block shift $X^{(k)}$ of X, a conjugacy $\gamma\colon X \to X^{(k)}$ and a 1-block code $\phi\colon X^{(k)} \to Y$ such that $\phi \circ \gamma = \varphi$.

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[dr, "\varphi"'] & X^{(k)} \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$

Proof.

Let $\varphi\colon X \to Y$ be a (m, a)-sliding block code.
Set $k = m + a + 1$ and $\gamma = \gamma_k \circ S^{-m}$.
Thus, $\gamma(x)_i = \langle x_{i-m} \cdots x_{i+a} \rangle$.
Put $\phi = \varphi \circ \gamma^{-1}$.
ϕ is a 1-block code. $\square$

Lemma

Let $\varphi\colon X_G \to X_H$ be a $(0, a)$-sliding block code with $a > 0$. Then there is an out-splitting G' of G such $X_{G'} = {X_G}^{(2)}$ and a conjugacy $\gamma\colon X_G \to {X_G}^{(2)}$ and a $(0, a-1)$-sliding block code $\varphi_2\colon {X_G}^{(2)} \to X_H$ such that $\varphi_2 \circ \gamma = \varphi$.

$$\begin{tikzcd}
X_G \arrow[r, "\gamma"] \arrow[dr, "\varphi"'] & {X_G}^{(2)} \arrow[d, "\varphi_2"] \\
 & X_H
\end{tikzcd}$$

Proof.

Let $G' = (Q', E')$ be the complete out-splitting of G. Let $H = (R, F)$. Assume that φ is defined by the block map $f: E^{a+1} \to F$.

- $Q' = \{s_e \mid s \in Q, e = (s,t) \in E\}$,
- $E' = \{(s_e, t_f) \mid e = (s,t)\}$

We define $\varphi_2 : E' \to F$ by the $(0, a-1)$-block map $g: {E'}^a \to F$ with $g((s_{e_0}, s_{e_1})(s_{e_1}, s_{e_2}) \ldots (s_{e_{a-1}}, s_{e_a})) = f(e_0 e_1 \ldots e_a)$.

We get $\varphi_2 \circ \gamma = \varphi$.

We check that $X_{G'}$ is equal to ${X_G}^{(2)}$, up to a renaming of the states.
Let $\phi: E' \to \mathcal{B}_2(X_G)$ be the 1-block code defined by $\phi((s_e, t_f)) = (e, f)$. The inverse $\phi^{-1}: \mathcal{B}_2(X_G) \to E'$ is the 1-block code defined by $\phi^{-1}((e,f)) = (s_e, t_f)$, where $e = (s,t)$.

Reduction of the anticipation of ϕ^{-1}

Lemma

Let $\phi\colon X_G \to X_H$ be a 1-block conjugacy whose inverse has memory m' and anticipation $a' \geq 1$.
Then, there are out-splitting G' of G and H' of H and a 1-block conjugacy $\phi'\colon X_{G'} \to X_{H'}$ such that

$$\phi = \varphi_{H'H} \circ \phi' \circ \varphi_{GG'}.$$

$$\begin{array}{ccc} X_G & \xrightarrow{\varphi_{GG'}} & X_{G'} \\ \phi \downarrow & & \downarrow \phi' \\ X_H & \xleftarrow{\varphi_{H'H}} & X_{H'} \end{array}$$

and whose inverse has memory m' and anticipation $a' - 1$.

Proof.

Let $\phi \colon X_G \to X_H$ be a 1-block conjugacy whose inverse ϕ^{-1} has memory m' and anticipation $a' \geq 1$.

Let $H' = (Q_{H'}, E_{H'})$ be the complete out-splitting of H. The edges of H' are (s_e, t_f) where $e = (s, t)$, $f = (t, u)$ are edges of H.

Let $G' = (Q_{G'}, E_{G'})$ be the out-splitting of G obtained by splitting each state v into states v_e, where e is an edge of H.
The edges going out of each state v of G are partitioned into the sets $\Delta(v)_e = \{(v, w) \text{ edge of } G \mid \phi((v, w)) = e\}$. The edges of G' are (v_e, w_f) where $\phi((v, w)) = e$.

We define $\phi' \colon E_{G'} \to E_{H'}$ by $\phi'((v_e, w_f)) = (s_e, t_f)$, where $e = (s, t)$. Then ϕ' is a 1-block conjugacy.

□

Reduction of the anticipation of ϕ^{-1}

Proof.

By hypothesis, ϕ^{-1} has memory m' and anticipation a'.
With $e_i = (s_i, s_{i+1})$, $f_i = (v_i, v_{i+1})$, we have

$$\begin{array}{ccc}
\dots f_{-1} \cdot f_0 f_1 \dots & \xrightarrow{\varphi_{GG'}} & \dots ((v_{-1})_{e_{-1}}, (v_0)_{e_0}) \cdot ((v_0)_{e_0}, (v_1)_{e_1})((v_1)_{e_1}, (v_2)_{e_2}) \dots \\
\phi \downarrow & & \downarrow \phi' \\
\dots e_{-1} \cdot e_0 e_1 \dots & \xleftarrow{\varphi_{H'H}} & \dots ((s_{-1})_{e_{-1}}, (s_0)_{e_0}) \cdot ((s_0)_{e_0}, (s_1)_{e_1})((s_1)_{e_1}, (s_2)_{e_2}) \dots
\end{array}$$

Hence ϕ'^{-1} has memory m' and anticipation $a' - 1$. $\square$

Proof of the decomposition theorem.

Let us assume that there is a conjugacy $\varphi\colon X_G \to X_H$. We may assume that φ is a $(0, m+a)$-block code (with a composition with S^{-m}).
We reduce the anticipation of ${\phi'_i}^{-1}$ with out-splittings and the memory of ${\phi'_j}^{-1}$ with in-splittings, and get

$$\begin{tikzcd}
X_G \arrow[r, "\gamma"] \arrow[dr, "\varphi"'] & (X_G)^{(m+a)} \arrow[r, "\varphi_{GG'_1}"] \arrow[d, "\phi"'] & X_{G'_1} \arrow[r, "\varphi_{G'_1G'_2}"] \arrow[d, "\phi'_1"] & X_{G'_2} \arrow[r] \arrow[d, "\phi'_2"] & \dots \arrow[r] & X_{G'_{m'+a'}} \arrow[d, "\phi'_{m'+a'}"] \\
 & X_H & X_{H'_1} \arrow[l, "\varphi_{H'_1H}"'] & X_{H'_2} \arrow[l, "\varphi_{H'_2H'_1}"'] & \dots \arrow[l] & X_{H'_{m'+a'}} \arrow[l]
\end{tikzcd}$$

Thus $\phi'_{m'+a'}$ is a 1-block conjugacy whose inverse is a 1-block conjugacy, implying that the graphs $G'_{m'+a'}$ and $H'_{m'+a'}$ are equal since they are trim.
Conversely, if there is a sequence of splittings and mergings from G to H, then X_G and X_H are conjugate. □

Let G be the graph with transition matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We can perform an out-merging of the states 2 and 3 of G (since the columns 2 and 3 are identical) and get G_1 with transition matrix

$$M_1 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

We can perform an in-merging of the states 2 and 3 of G (since the rows 2 and 3 are identical) and get G_2 with transition matrix

$$M_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

And no mergings are possible from G_1 or G_2.

Open problem

Open problem

It is not known whether conjugacy between shifts of finite type is decidable, even for irreducible shifts of finite type.

Strong shift equivalence

Two nonnegative integer matrices M, N are *elementary equivalent* if there are, possibly nonsquare, matrices R, S such that

$$M = RS, N = SR.$$

Two nonnegative integer matrices M, N are *strong shift equivalent* if there is a sequence of elementary equivalences from M to N:

$$\begin{aligned}
M &= R_0 S_0, S_0 R_0 = M_1, \\
M_1 &= R_1 S_1, S_1 R_1 = M_2, \\
&\vdots \\
M_\ell &= R_\ell S_\ell, S_\ell R_\ell = N.
\end{aligned}$$

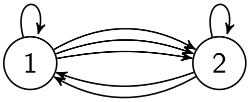
Theorem (Classification Theorem, R. Williams 1973)

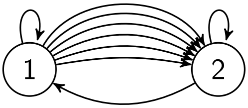
Two edge shifts defined by matrices M and N are conjugate if and only if M and N are strong shift equivalent.

Conjugacy: examples

Let X and Y be the edge shifts defined by the graphs given by the matrices M and N respectively:

$$M = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 6 \\ 1 & 1 \end{bmatrix}$$





We may see these graphs as automata where all edge labels are distinct.

The shifts X and Y are conjugate (K. Baker, using computer research to prove strong shift equivalence).

It is not known whether X and Y defined by M_k and N_k are conjugate for $k \geq 4$.

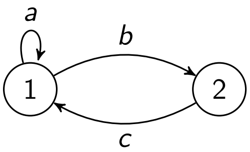
$$M_k = \begin{bmatrix} 1 & k \\ k-1 & 1 \end{bmatrix}, \quad N_k = \begin{bmatrix} 1 & k(k-1) \\ 1 & 1 \end{bmatrix}$$

Perron-Frobenius theorem

Transition matrix of a graph

Let $G = (Q, E)$ be a graph. Its transition matrix is a nonnegative integer matrix M where

$$M = (m_{pq})_{p,q \in Q}, \text{ where } m_{pq} \text{ is the number of edges from } p \text{ to } q.$$



$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Irreducible and primitive matrices

A nonnegative square matrix (with real coefficients) M is *irreducible* if for every pair s, t of indices, there is an integer $n \geq 1$ such that $M^n_{s,t} > 0$. Otherwise, M is *reducible*.
A matrix M is reducible if and only if, up to a permutation of the indices, it can be written

$$M = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$$

for some matrices U, V, W with U, W being square matrices of dimension ≥ 1.

A nonnegative square matrix M is *primitive*, if there is some integer $n \geq 1$ such that all entries of M^n are positive.
The least such n is called the *exponent* of M, denoted $\exp(M)$.
A primitive matrix is irreducible but the converse is not necessarily true.

Irreducible matrix

Lemma

If M is a nonnegative $Q \times Q$ irreducible matrix, then $(I + M)^{n-1} > 0$, where $n = \text{Card } Q$.

Proof.

Let G be the graph whose adjacency matrix is $I + M$.
Thus, $s \to t$ is an edge if and only if $(I + M)_{st} > 0$.
Since M is irreducible, there is a path of length at most $n - 1$ from s to t in G.
Since the state s has a self-loop, there is a path of length $n - 1$ from s to t in G.
Hence, $(I + M)^{n-1}_{st} > 0$ for all states $s, t \in Q$. $\square$

Periods

The *period* of an irreducible nonnegative square matrix $M \neq 0$ is the greatest common divisor of the integers n such that M^n has a positive diagonal coefficient. By convention, the period of $M = 0$ is 1. If M has period p, then M and M^p have, up to a permutation of indices, the forms:

$$M = \begin{bmatrix} 0 & M_1 & 0 & \dots & 0 \\ 0 & 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_{p-1} \\ M_p & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad M^p = \begin{bmatrix} D_1 & 0 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & D_{p-1} & 0 \\ 0 & 0 & \dots & 0 & D_p \end{bmatrix}$$

Thus M^p is block diagonal, with each diagonal block D_i primitive. An irreducible matrix is primitive if and only if it has period 1.

Example

The three matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

are nonnegative. The first one is reducible. The second one is irreducible but not primitive since it has period 2. The third one is primitive with exponent 2.

The Perron-Frobenius theorem

Theorem

Let M be a nonnegative real $Q \times Q$-matrix. Then

1. *M has an eigenvalue λ_M such that $|\mu| \leq \lambda_M$ for every eigenvalue μ of M.*
2. *There corresponds to λ_M a nonnegative eigenvector v, and a positive one if M is irreducible. If M is irreducible, λ_M is the only eigenvalue with a nonnegative eigenvector.*
3. *If M is primitive, the sequence (M^n/λ_M^n) converges to the matrix yx where x, y are positive left and right eigenvectors relative to λ_M with $\sum_{s \in Q} y_s = 1$ and $\sum_{s \in Q} x_s y_s = 1$.*

If M is irreducible, then λ_M is simple. The matrix M is primitive if and only if $|\mu| < \lambda_M$ for every other eigenvalue μ of M.

Spectral radius

An *eigenvector* of a square real matrix M for the eigenvalue λ (a real or complex number) is a **non null** vector v (with real or complex coefficients) such that $Mv = \lambda v$.

The *spectral radius* of a square real matrix is the real number

$$\rho(M) = \max\{|\lambda| \mid \lambda \text{ eigenvalue of } M\}.$$

The theorem states in particular that if a matrix M is irreducible, $\rho(M)$ is an eigenvalue of M that is algebraically simple. Furthermore, if M is primitive, any eigenvalue of M other than $\rho(M)$ has modulus less than $\rho(M)$.

Proof of Perron-Frobenius Points 1 and 2

Proposition

Any nonnegative matrix M has a real eigenvalue λ_M such that $|\lambda| \leq \lambda_M$ for any eigenvalue λ of M, and there corresponds to λ_M a nonnegative eigenvector v.
If M is irreducible, there corresponds to λ_M a positive eigenvector v, and λ_M is the only eigenvalue with a nonnegative eigenvector.

Proof of Perron-Frobenius Points 1 and 2

Proof.

We first assume that M **is irreducible**.
For any nonnegative real vector $v \neq 0$, let

$$r_M(v) = \min\{(Mv)_s / v_s \mid v_s \neq 0\}.$$

Thus $r_M(v)$ is the largest real number r such that $Mv \geq rv$.
One has $r_M(\lambda v) = r_M(v)$ for any nonnegative nonzero real number λ.
Moreover, the mapping $v \to r_M(v)$ is continuous on the set of nonnegative nonzero vectors.
The set X of nonnegative vectors v such that $\|v\| = 1$ is compact.
Define $\lambda_M = \max\{r_M(v) \mid v \in X\}$.
Since a continuous function on a compact set reaches its maximum on this set, there is an $x \in X$ such that $r_M(x) = \lambda_M$.
Since $r_M(v) = r_M(\lambda v)$ for $\lambda \geq 0, \lambda \neq 0$, we have
$\lambda_M = \max\{r_M(v) \mid v \geq 0, v \neq 0\}$. $\square$

Proof.

We show that $Mx = \lambda_M x$.
By the definition of the function r_M, we have $Mx \geq \lambda_M x$.
Set $y = Mx - \lambda_M x$. Then $y \geq 0$.
Assume $Mx \neq \lambda_M x$. Then $y \neq 0$.
Since $(I + M)^n > 0$ for some $n \geq 1$, this implies that $(I + M)^n y > 0$.
But $(I + M)^n y = (I + M)^n (Mx - \lambda_M x) = M(I + M)^n x - \lambda_M (I + M)^n x = Mz - \lambda_M z$ with $z = (I + M)^n x$.
This shows that $Mz > \lambda_M z$, which implies that $r_M(z) > \lambda_M$, a contradiction.
Thus, λ_M is an eigenvalue with a nonnegative eigenvector x.
Since $(I + M)^n x = (1 + \lambda_M)^n x$ is positive, we get $x > 0$. $\square$

Proof of Perron-Frobenius Points 1 and 2

Proof.

Let us now show that $\lambda_M \geq |\lambda|$ for each real or complex eigenvalue λ of M.
Indeed, let v be an eigenvector corresponding to λ.
Then $Mv = \lambda v$.
Let $|v|$ be the nonnegative vector with coordinates $|v_s|$. Then $M|v| \geq |\lambda||v|$ by the triangular inequality.
By the definition of the function r_M, this implies $r_M(|v|) \geq |\lambda|$ and consequently $\lambda_M \geq |\lambda|$. $\square$

Proof.

We show that if M is irreducible, λ_M is a simple eigenvalue.

Let v be an eigenvector of M for λ_M (with complex coefficients). We have seen that $w = |v|$ is a nonnegative eigenvector of M for λ_M.
We have $w > 0$. Indeed, since $(I + M)^n > 0$ and $w \neq 0$, we have $(I + M)^n w > 0$, And $(I + M)^n w = (1 + \lambda_M)^n w$, implying $w > 0$. Hence, v has no null coefficient and $|v|$ is a positive eigenvector of M for λ_M.

Now let w, w' be two eigenvectors of M for λ_M.
Let $z = w'_s w - w_s w'$. Then $Mz = \lambda_M z$ and $z_s = 0$, implying $z = 0$ and w is colinear to w'.

□

Proof of Perron-Frobenius Points 1 and 2

Proof.

We show that if M is irreducible, λ_M is the only eigenvalue with a nonnegative eigenvector.

Let $Mv = \lambda v$ with $v \geq 0$, $v \neq 0$. Hence λ is a nonnegative real number.
Then $(I + M)^n v = (1 + \lambda)^n v > 0$ implying $v > 0$.

Let D be the diagonal matrix with coefficients v_s and
$N = D^{-1}MD$.
We have $n_{st} = m_{st} v_t / v_s$ and $\sum_t n_{st} = \sum_t m_{st} v_t / v_s = \lambda$.
If w is a positive eigenvector of M for λ_M, then $D^{-1}w$ be a positive eigenvector of N for λ_M.
Let w be a positive eigenvector of N for λ_M, normalized in such a way that $w_s \leq 1$ for all s and $w_{s_0} = 1$ for some s_0.
Then $\lambda_M = \lambda_M w_{s_0} = \sum_t n_{s_0 t} w_t \leq \sum_t n_{s_0 t} = \lambda$.
Thus $\lambda = \lambda_M$.

Proof of Perron-Frobenius Points 1 and 2

Proof.

If M **is reducible**, we may consider a triangular decomposition:

$$M = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}.$$

Applying by induction the theorem to U and W, we obtain nonnegative eigenvectors u and w for the eigenvalues λ_U and λ_W of U and W. We prove that $\max(\lambda_U, \lambda_W)$ is an eigenvalue of M with some nonnegative eigenvector.

If $\lambda_U \geq \lambda_W$, then λ_U is an eigenvalue of M with nonnegative eigenvector $\begin{bmatrix} u \\ 0 \end{bmatrix}$.

□

Proof of Perron-Frobenius Points 1 and 2

Proof.

If $\lambda_U < \lambda_W$, then λ_W is an eigenvalue of M with nonnegative eigenvector $\begin{bmatrix} u' \\ w \end{bmatrix}$, where $u' = \sum_{n \geq 0} (U/\lambda_W)^n {\lambda_W}^{-1} Vw$.
Indeed, the spectral radius of U/λ_W is < 1, implying the convergence of $\sum_{n \geq 0} (U/\lambda_W)^n$ (Berstel Lemma).

Conversely, if λ is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} u \\ w \end{bmatrix}$, then λ is an eigenvalue of W if $w \neq 0$, and is an eigenvalue of U if $w = 0$.

We set $\lambda_M = \max(\lambda_U, \lambda_W)$.
Hence, $\max\{|\lambda| \mid \lambda \text{ eigenvalue of } M\} = \lambda_M$. □

Example

The following matrix M is primitive (hence irreducible).

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues are the roots φ, φ' of $\det(zI - M) = z^2 - z - 1$, where

$$\varphi = \frac{1+\sqrt{5}}{2}, \quad \varphi' = \frac{1-\sqrt{5}}{2}.$$

The vector $\begin{bmatrix} \varphi \\ 1 \end{bmatrix}$ is a (right) positive eigenvector for φ.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi \\ 1 \end{bmatrix} = \begin{bmatrix} \varphi + 1 \\ \varphi \end{bmatrix} = \begin{bmatrix} \varphi^2 \\ \varphi \end{bmatrix} = \varphi \begin{bmatrix} \varphi \\ 1 \end{bmatrix}.$$

Conjugacy invariants

Entropy

The (topological) entropy of a shift space X is

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_n(X)).$$

The limit exists.

Lemma (Fekete Lemma)

Let $a_1, a_2, \ldots$ be a sequence of nonnegative numbers such that $a_{m+n} \leq a_m + a_n$ for $m, n \geq 1$, then $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_{n\geq 1} a_n/n$.

Proof.

Let $\alpha = \inf_{n\geq 1} a_n/n$. We have $\inf_{n\geq 1} a_n/n \geq \alpha$ for all $n \geq 1$.
Fix $\varepsilon > 0$. There is $k \geq 1$ such that $a_k/k < \alpha + \varepsilon/2$.
Then for $0 \leq j < k$ and $m \geq 1$,

$$\frac{a_{mk+j}}{mk+j} \leq \frac{a_{mk}}{mk+j} + \frac{a_j}{mk+j} \leq \frac{a_{mk}}{mk} + \frac{a_j}{mk}$$
$$\leq \frac{ma_k}{mk} + \frac{ja_1}{mk} \leq \frac{a_k}{k} + \frac{a_1}{m} < \alpha + 1/2\varepsilon + \frac{a_1}{m}.$$

For $n = mk + j$ large enough, $a_1/m < \varepsilon/2$ and $a_n/n < \alpha + \varepsilon$. $\square$

Lemma

If X is a shift space,

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_n(X)).$$

exists and is equal to

$$\inf_{n\geq 1} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_n(X)).$$

Proof.

Let $m, n \geq 1$,

$$\mathrm{Card}(\mathcal{B}_{m+n}(X)) \leq \mathrm{Card}(\mathcal{B}_m(X)) \times \mathrm{Card}(\mathcal{B}_n(X)).$$

$\square$

Proposition

If there is a sliding block map $\varphi\colon X \to Y$ which is onto, then $h(Y) \leq h(X)$.

Proof.

If φ is a (m, a)-sliding block code, every block in $\mathcal{B}_n(Y)$ is the image of a block in $\mathcal{B}_{n+m+a}(X)$.
Hence, $\mathrm{Card}(\mathcal{B}_n(Y)) \leq \mathrm{Card}(\mathcal{B}_{n+m+a}(X))$.
Thus,

$$\begin{aligned}
h(Y) &= \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_n(Y)) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_{n+m+a}(X)) \\
&= \lim_{n\to\infty} \left(\frac{n+m+a}{n}\right) \frac{1}{n+m+a} \log \mathrm{Card}(\mathcal{B}_{n+m+a}(X)) \\
&= h(X).
\end{aligned}$$

Corollary

If X and Y are conjugate, then $h(X) = h(Y)$.

Example

If X is the full shift $A^{\mathbb{Z}}$, then $h(X) = \log \mathrm{Card}(A)$.

Hence, the full shift on two letters is not conjugate to the full shift on three letters.

Computation of the entropy of a sofic shift

Proposition

Let $\mathcal{A} = (Q, E)$ be a trim deterministic automaton presenting a sofic shift X and $G = (Q, F)$ be the graph of $\mathcal{A}$. Then $h(X) = h(X_G)$.

Proof.

There is a 1-block sliding block code from X_G onto X (with no memory and no anticipation). Thus $h(X) \leq h(X_G)$.
If $\mathcal{A}$ has k states, and since $\mathcal{A}$ is deterministic, $\mathrm{Card}(\mathcal{B}_n(X_G)) \leq k\,\mathrm{Card}(\mathcal{B}_n(X))$. Thus $h(X) \geq h(X_G)$. $\square$

Let $G = (Q, E)$ be a graph with adjacency matrix M, we have

$$\mathcal{B}_n(X_G) = \sum_{s,t \in Q} (M^n)_{st}.$$

If G is strongly connected, i.e. if M is irreducible, there is a positive eigenvector v for λ_M.
Let $c = \min v_q$, $d = \max v_q$.
Since $\sum_{t \in Q} (M^n)_{st} v_t = \lambda_M^n v_s$,

$$\sum_{s,t \in Q} (M^n)_{st} v_t = \sum_{s \in Q} \lambda_M^n v_s.$$

$$c \sum_{s,t \in Q} (M^n)_{st} \leq \sum_{s,t \in Q} (M^n)_{st} v_t \leq d \operatorname{Card}(Q) \lambda_M^n.$$

$$\sum_{s,t \in Q} (M^n)_{st} \leq (d/c) \operatorname{Card}(Q) \lambda_M^n.$$

Computation of the entropy of a sofic shift

Similarly

$$(c/d)\lambda_M^n \le \sum_{s,t \in Q} (M^n)_{st}.$$

Proposition

Let $\mathcal{A} = (Q, E)$ be an irreducible deterministic automaton presenting an irreducible sofic shift X and M its adjacency matrix. Then $h(X) = \log \lambda_M$.

The result holds for a trim deterministic automaton presenting a sofic shift X with a reduction to the irreducible components of M (exercise).

Computation of the entropy of a sofic shift: example

Example

For the even shift,

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$h(X) = \log \varphi, \text{ where } \varphi = \frac{1+\sqrt{5}}{2}.$$

The even shift and the golden mean shift have the same entropy but are not conjugate.

Periodic points in a shift space

A point x of a shift space X is *periodic* if $S^n(x) = x$ for some $n \geq 1$ and we say that x has *period* n.

If x is periodic, the smallest positive integer n for which $S^n(x) = x$, called the least period of x, divides all periods of x.

Let

$$p_n(X) = \mathrm{Card}\{x \in X \mid S^n(x) = x\}.$$

Proposition

Let $\varphi\colon X \to Y$ be a sliding block map. If x is a periodic point of X and has period n, then $\varphi(x)$ is periodic and has period n and the least period of $\varphi(x)$ divides the least period of x. If X and Y are conjugate, then $p_n(X) = p_n(Y)$ for each $n \geq 1$.

Proposition

Let G be a graph of transition matrix M, the number of cycles of length n in G is $\mathrm{tr}(M^n)$ and this equals the number of points in X_G with period n.

Zeta function

The zeta function of a shift space X is the formal series

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X)}{n} z^n\right).$$

Proposition

If X and Y are conjugate, then $\zeta_X = \zeta_Y$.

Zeta function

Example

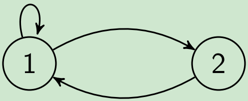
Let X be the full shift on a 2-letter alphabet.
Then, $p_n(X) = 2^n$ for each $n \geq 1$.

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{2^n}{n} z^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(2z)^n}{n}\right)$$
$$= \exp(-\log(1-2z)) = \frac{1}{1-2z}.$$

Zeta function

Example

Let G be the graph



Its transition matrix is

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We have $p_n(X_G) = \mathrm{tr}(M^n) = \varphi^n + \varphi'^n$, where

$$\varphi = \frac{1+\sqrt{5}}{2}, \quad \varphi' = \frac{1-\sqrt{5}}{2}.$$

are the roots of $\det(zI - M) = z^2 - z - 1$.

Zeta function

Example

$$\zeta_{X_G}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\varphi^n + \varphi'^n}{n} z^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\varphi^n}{n} + \frac{\varphi'^n}{n} z^n\right)$$

$$= \exp(-\log(1-\varphi z) - \log(1-\varphi' z)) = \frac{1}{(1-\varphi z)(1-\varphi' z)}$$

$$= \frac{1}{1-z-z^2} = \frac{1}{\det(I - Mz)}.$$

Zeta function of a shift of finite type

Theorem

Let G be a graph with adjacency matrix M. Then

$$\zeta_{X_G}(z) = \frac{1}{\det(I - Mz)}.$$

Proof.

We have

$$p_n(X_G) = \mathrm{tr}(M^n) = \lambda_1^n + \lambda_2^n + \cdots + \lambda_{|Q|}^n,$$

where $\lambda_1, \lambda_2, \ldots \lambda_{|Q|}$ are the roots of the characteristic polynomial of M listed with multiplicities.

$$\begin{aligned}
\zeta_{X_G}(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{\lambda_1^n + \lambda_2^n + \cdots + \lambda_{|Q|}^n}{n} z^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(\lambda_1 z)^n}{n} + \cdots + \sum_{n=1}^{\infty} \frac{(\lambda_{|Q|} z)^n}{n}\right) \\
&= \prod_{k=1}^{|Q|} \frac{1}{1 - \lambda_k z} = \frac{1}{\det(I - Mz)}.
\end{aligned}$$

Observe that the zero eigenvalues "do not count". □

The zeta function is an invariant stronger than entropy

If two edge shifts have the same zeta function, then they have the same entropy.
We have

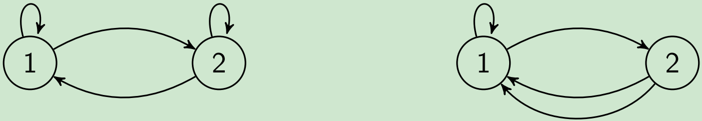
$$\zeta_{X_G}(z) = \prod_{k=1}^{|Q|} \frac{1}{1 - \lambda_k z} = \frac{1}{\det(I - Mz)}.$$

So the zeta function determines the list of nonzero eigenvalues of the M (with repeated eigenvalues listed according to their multiplicity), called the *nonzero spectrum of* M.

Thus, it determines the largest eigenvalue of M, and thus the entropy.
If G is strongly connected $h(X_G) = \log \lambda_M$. Otherwise $h(X_G) = \max \log \lambda_{M_i}$, where M_i is the adjacency matrix of all strongly connected components of G.

Example

Let G and H be the graphs



Show that the two edge shifts X_G and X_H are not conjugate although they have the same entropy.

Let M be the adjacency matrix of G and N the adjacency matrix of N.

$$\zeta_{X_G}(z) = \frac{1}{\det(I - Mz)}, \zeta_{X_H}(z) = \frac{1}{\det(I - Nz)}.$$

We have

$$\det(I - Mz) = \det \begin{bmatrix} 1 - z & -z \\ -z & 1 - z \end{bmatrix} = 1 - 2z.$$

$$\det(I - Nz) = \det \begin{bmatrix} 1 - z & -z \\ -2z & 1 \end{bmatrix} = 1 - z - 2z^2.$$

Hence X_G and X_H have not the same zeta function, and thus are not conjugate.
We can also see that $p_2(X_G) = 4$ and $p_2(X_H) = 5$.

We have

$$\det(zI - M) = \det \begin{bmatrix} z-1 & -1 \\ -1 & z-1 \end{bmatrix} = z^2 - 2z = z(z-2).$$

Also obtained directly from $\det(I - Mz)$ by changing z into $1/\lambda$ and mutiplying by λ^2: $\lambda^2(1 - 2/\lambda) = \lambda(\lambda - 2)$.

$$\det(zI - N) = \det \begin{bmatrix} z-1 & -1 \\ -2 & z \end{bmatrix} = z^2 - z - 2 = (z-2)(z+1).$$

Hence, M and N have the same largest eigenvalue. Thus, X_G and X_H have the same entropy.