Master 2 Mathematics and Computer Science Symbolic Dynamics. Lecture 3

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One-sided shift spaces

Overview

- One-sided shift spaces
- Decidability of conjugacy of one-sided shifts of finite type

One-sided shift spaces

A *one-sided shift space* is a closed subset X of $A^{\mathbb{N}}$ such that $S(X) \subseteq X$.

One-sided shift spaces are usually defined as closed subsets such that $S(X) = X$, but we do not require this stronger condition here.

The set $A^{\mathbb{N}}$ itself is a one-sided shift space, called the *one-sided full shift*.

For a two-sided sequence $x \in A^{\mathbb{Z}}$, we define $x^+ = x_0x_1\cdots$.
If X is a two-sided shift space, then the set $X^+ = \{x^+ \mid x \in X\}$ is a one-sided shift space.

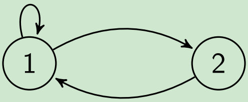
One-sided shift spaces of finite type

A one-sided shift space is *of finite type* if it is the set X_F of one-sided sequences over A avoiding all words of some finite set $F \subseteq A^*$.

A *one-sided edge shift* is the set X_G of right-infinite paths in a finite directed graph G. Note that the paths may start at any state.

Example

The one-sided edge shift X_G represented by the directed graph G:



is also defined by the adjacency matrix of G, that is, by the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

One-sided conjugacy

The *one-sided (sliding) block code* defined by f is the map $\varphi : X \to B^{\mathbb{N}}$ defined by $\varphi(x) = y$ if for every $i \in \mathbb{N}$, $y_i = f(x_{[i,i+n]})$, where $f : \mathcal{B}_{n+1}(X) \to B$.

n is the anticipation, and no memory is allowed.

$$\begin{array}{l} x_0 \cdots x_{i-1}\,\boxed{x_i}\boxed{\;\cdots\;}\boxed{x_{i+n}}\,x_{i+n+1} \quad \cdots \\ \qquad\qquad\quad f \downarrow \\ y_0 \cdots y_{i-1}\,\boxed{y_i}\,y_{i+1} \quad \cdots \end{array}$$

A *one-sided conjugacy* $\varphi\colon X \to Y$ is a bijective one-sided block code. Its inverse is also a one-sided block code.

One-sided edge shifts

Proposition

Any one-sided shift of finite type is conjugate to a one-sided edge shift.

Proof.

Almost the same proof as for two-sided edge shifts.

Let $X = \mathsf{X}_F$ with F finite, and let n be the maximal size of words in F. We may assume that all words in F have size n.
Let $\mathcal{A} = (Q, E)$, where Q is the set of words of length $n-1$ with edges $a_0 a_1 \ldots a_{n-2} \xrightarrow{a_0} a_1 \ldots a_{n-2} a$, where $a_0 a_1 \ldots a_{n-2} a \notin F$. We keep only the trim part of this automaton, that is, only the states having at least one outgoing edge.

Then all paths of $\mathcal{A}$ labeled by a word w of length $n-1$ start at the same state q_w. $\square$

proof continued.

Let Y be the set of right-infinite paths of $\mathcal{A}$.
Let $\phi\colon Y \to X$ defined by the 1-block map $f\colon E \to A$ with $f(e = (p, a, q)) = a$.
Then the one-sided sliding block code $\varphi\colon X \to Y$ with memory 0 and anticipation $n-1$ defined by the n-block map $g\colon A^n \to E$ with $g(a_0 a_1 \ldots a_{n-1}) = (p, a_0, q)$, where $p = a_0 a_1 \ldots a_{n-2}$ and $q = a_1 a_2 \ldots a_{n-1}$.
The map φ is the inverse of ϕ.

$\square$

Out-splitting (reminder, see Lecture 2)

Let X_G be a one-sided edge shift defined by a directed graph $G = (V, E)$. We may assume that the graph is *trim*, that is, that each vertex has at least one outgoing edge.

An *out-splitting* of G is a transformation of G into a graph $G' = (V', E')$ obtained by selecting a vertex s and partitioning the set of edges going out of s into two non-empty sets E_1 and E_2.

- $V' = V \setminus \{s\} \cup \{s_1, s_2\}$,
- E' contains all edges of E neither starting at or ending in s,
- E' contains the edge (s_1, a, t) for each edge $(s, a, t) \in E_1$, and the edge (s_2, a, t) for each edge $(s, a, t) \in E_2$, so long as $t \neq s$,
- E' contains the edges (t, a, s_1) and (t, a, s_2) if (t, a, s) in E, when $t \neq s$,
- E' contains the edges (s_1, a, s_1) and (s_1, a, s_2) if (s, a, s) in E_1, and the edges (s_2, a, s_1) and (s_2, a, s_2) if $(s, a, s) \in E_2$.

Out-splitting (reminder, see Lecture 2)

Example

The graph G' in the right part of the figure is an out-split of the graph G in the left part of the figure. Here, $s = 1$, and the partition of the outgoing edges of 1 is $\{E_1, E_2\}$, where E_1 contains the loop around 1, and E_2 contains the two edges going from 1 to 2.

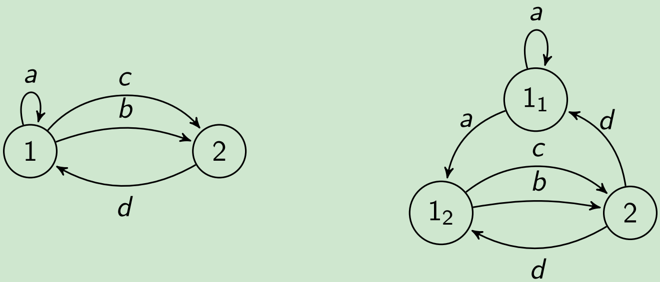


Figure: An out-splitting.

Out-merging

The inverse operation of an out-splitting is referred to as an *out-merging*. An out-merging of a directed graph $G' = (V', E')$ can be performed if there are two vertices s_1, s_2 of G' such that the adjacency matrix M' satisfies:

- the column of index s_1 is equal to the column of index s_2 of M'.

The adjacency matrix of G is thus the matrix M obtained by adding the rows of index s_2 to the row of index s_1 of M' and then removing the column of index s_2 afterward.

The graph G is called an *elementary amalgamation* of G'. Notice that even if M' has 0-1 entries, M may not have 0-1 entries.

Let M' be the adjacency matrix of a directed graph G', and $(V_1, V_2, \ldots, V_k)$ be a partition of V' into classes such that if s, t belong to the same class, then the columns of indices s and t of M' are identical.

When at least one set of the partition has a size greater than 1, we can perform a *general merging*. We define a graph K of adjacency matrix N obtained by merging all states of each $V_i = \{s_{i,1}, \ldots s_{i,k_i}\}$ into a single state $s_{i,1}$.

The row in N corresponding to $s_{i,1}$ is obtained by summing the rows of the states of V_i in M' and removing the columns $s_{i,2}, \cdots, s_{i,k_i}$.

The graph K is called a *general amalgamation* of G'.

Decomposition theorem

Proposition (R. Williams 1973)

Let X (resp. Y) be a one-sided edge shift defined by an irreducible directed graph G (resp. H), Then X and Y are conjugate if and only if there is a sequence of out-splittings and out-mergings from G to H.

Proof.

The same proof as the proof for two-sided edge shifts. Here we use only out-splittings and out-mergings. $\square$

Two out-merging transformations commute

Proposition (R. Williams 1973)

If G and H are amalgamations of a common directed graph L, then they have a common amalgamation K.

Proposition (R. Williams 1973)

Let G and H be irreducible directed graphs that define one-sided edge shifts X_G and X_H. Then X_G and X_H are conjugate if and only if G and H have the same total amalgamation.

It also holds for one-sided edge shifts defined by trim directed graphs.

Decidability of conjugacy of one-sided shifts of finite type

Corollary (R. Williams 1973)

It is decidable whether two one-sided shifts of finite are conjugate.

One-sided conjugacy

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ G' \arrow[dr] & & G'' \arrow[dl] \\ & H & \end{tikzcd}$$

Two out-merging transformations commute.

There is a unique graph, up to a renaming of the vertices, obtained by performing elementary out-mergings until we cannot perform anymore. This graph is called the *total amalgamation* of G.

Two out-merging transformations commute

Proposition (R. Williams 1973)

If G and H are amalgamations of a common directed graph L, then they have a common amalgamation K.

Proof.

Let us assume that there is an out-merging of G with adjacency matrix M into G' with adjacency matrix M', obtained by merging s_1 and s_2 into s_1, and an out-merging of G into G'' with adjacency matrix M'', obtained by merging s_3 and s_4 into s_3.
We may assume that the set $\{s_3, s_4\}$ is distinct from the set $\{s_1, s_2\}$.

Let us show that there is a graph H that is an out-merging of both G' and G''.

□

Two out-merging transformations commute

Proof.

Thus, by hypothesis,

- the columns of index s_1 and s_2 of M are equal.
- the columns of index s_3 and s_4 of M are equal.

The matrix M' obtained by adding the rows of index s_2 to the row of index s_1 of M and then removing the column of index s_2 afterward.

The matrix M'' obtained by adding the rows of index s_4 to the row of index s_3 of M and then removing the column of index s_3 afterward. □

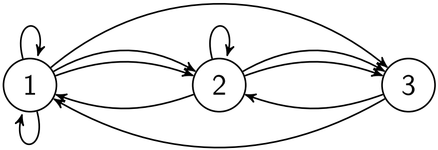
Two out-merging transformations commute

Proof.

- If both s_3, s_4 are distinct from s_1 and s_2, we define H as the out-merging of G' obtained by merging s_3 and s_4 into s_3. It is trivial that H is also the out-merging of G'' obtained by merging s_1 and s_2 into s_1.
- If $s_3 = s_1$, then we define H as the out-merging of G' obtained by merging s_1 and s_4 into s_1.
  Indeed, the columns of index s_1, s_2, s_3 and s_4 of M are equal.
  Hence, the columns of index s_1 and s_4 of M' are equal.
  It is clear that H is also the out-merging of G'' obtained by merging s_1 and s_2 into s_1.

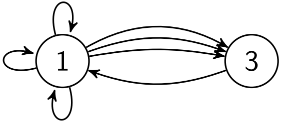
□

If G is the following graph:



$$M = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

Its total amalgamation is H:



$$N = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$$