Université Gustave Eiffel
M2 – Semester 1 – 2025/2026

Analysis - Exercises

CHAU Dang Minh

Exercise 10. Show that the function $f : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ such that $f(M) = M^{-1}$ is differentiable and compute its differential.

Solution. Let $\|\cdot\|$ be a norm on $\mathrm{GL}_n(\mathbb{R})$. Firstly, note that for any $A, B \in \mathrm{GL}_n(\mathbb{R})$, we have

$$A^{-1} - B^{-1} = A^{-1}BB^{-1} - A^{-1}AB^{-1} = A^{-1}(BB^{-1} - AB^{-1}) = A^{-1}(B - A)B^{-1}.$$

Let $A \in \mathrm{GL}_n(\mathbb{R})$, we want to show that f is differentiable at A. For any $H \in \mathrm{GL}_n(\mathbb{R})$ such that $A+H \in \mathrm{GL}_n(\mathbb{R})$, we have

$$f(A+H) - f(A) = (A+H)^{-1} - A^{-1} = (A+H)^{-1}(A - (A+H))A^{-1} = -(A+H)^{-1}HA^{-1}.$$

Choose $L : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ such that $L(H) = -A^{-1}HA^{-1}$. For any $H, K \in \mathrm{GL}_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$, we have

$$L(\alpha H + K) = -A^{-1}(\alpha H + K)A^{-1} = -\alpha A^{-1}HA^{-1} - A^{-1}KA^{-1} = \alpha L(H) + L(K).$$

Hence, L is linear. To show that L is continuous on $\mathrm{GL}_n(\mathbb{R})$, we only have to show that L is continuous at 0. Indeed, for any $H \in \mathrm{GL}_n(\mathbb{R})$, we have

$$\|L(H)\| = \|-A^{-1}HA^{-1}\| \leq \|A^{-1}\|\|H\|\|A^{-1}\| = \|A^{-1}\|^2\|H\|.$$

Thus, for any $\varepsilon > 0$, there exists $\delta = \dfrac{\varepsilon}{\|A^{-1}\|^2}$ such that if $\|H\| < \delta$, then $\|L(H)\| < \varepsilon$. This shows that L is continuous at 0, hence continuous on $\mathrm{GL}_n(\mathbb{R})$. Now, we have

$$\begin{aligned}
\|f(A+H) - f(A) - L(H)\| &= \|-(A+H)^{-1}HA^{-1} + A^{-1}HA^{-1}\| \\
&= \|(A^{-1} - (A+H)^{-1})HA^{-1}\| \\
&= \|A^{-1}H(A+H)^{-1}HA^{-1}\|. \\
&\leq \|A^{-1}\|\|H\|\|(A+H)^{-1}\|\|H\|\|A^{-1}\| \\
&= \|A^{-1}\|^2\|(A+H)^{-1}\|\|H\|^2.
\end{aligned}$$

Hence, let $\theta(A, H) = \dfrac{1}{\|H\|}(f(A+H) - f(A) - L(H))$, we have

$$\|\theta(A, H)\| \leq \|A^{-1}\|^2\|(A+H)^{-1}\|\|H\| \to 0, \text{ as } H \to 0,$$

or $\theta(A, H) \to 0$ as $H \to 0$. Therefore, we have found L and θ such that

$$\|f(A+H) - f(A) - L(H)\| = \|H\| \cdot \theta(A, H)$$

such that L is linear and continuous and $\theta(A, H) \to 0$ as $H \to 0$. This shows that f is differentiable at A. Since A is chosen arbitrarily, we conclude that f is differentiable on $\mathrm{GL}_n(\mathbb{R})$. The differential of f at A is given by

$$df(A)(H) = L(H) = -A^{-1}HA^{-1}.$$

Exercise 11. Consider n points (x_i, y_i) in the plan $\mathbb{R}^2$ with at least two distinct x_i's. Show that there exist two real numbers λ, μ which minimize the sum

$$S(\lambda,\mu) = \sum_{i=1}^{n} (\lambda x_i + \mu - y_i)^2.$$

Solution. We have

$$\frac{\partial S}{\partial \lambda} = 2\sum_{i=1}^{n} (\lambda x_i + \mu - y_i)x_i, \quad \frac{\partial S}{\partial \mu} = 2\sum_{i=1}^{n} (\lambda x_i + \mu - y_i).$$

Let us solve for (λ_0, μ_0) such that $\dfrac{\partial S}{\partial \lambda}(\lambda_0, \mu_0) = 0$ and $\dfrac{\partial S}{\partial \mu}(\lambda_0, \mu_0) = 0$. Equivalently,

$$\begin{cases} \lambda_0 \sum_{i=1}^n x_i^2 + \mu_0 \sum_{i=1}^n x_i = \sum_{i=1}^n y_i x_i, \\ \lambda_0 \sum_{i=1}^n x_i + n\mu_0 = \sum_{i=1}^n y_i. \end{cases} .$$

Multiplying the first equation by n and the second equation by $\sum_{i=1}^n x_i$, then subtract the two equations, we get

$$\lambda_0 \left(n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 \right) = n\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i.$$

Since there are at least two distinct x_i's, we have $n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2 = n\sum_{i=1}^n (x_i - \bar{x})^2 > 0$, where $\bar{x} = \dfrac{1}{n}\sum_{i=1}^n x_i$. Thus,

$$\lambda_0 = \frac{n\sum_{i=1}^n y_i x_i - \sum_{i=1}^n y_i \sum_{i=1}^n x_i}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}$$

Substituting λ_0 into the second equation, we get

$$\mu_0 = \frac{\sum_{i=1}^n y_i - \lambda_0 \sum_{i=1}^n x_i}{n}.$$

$$\frac{\partial^2 S}{\partial \lambda^2} = 2\sum_{i=1}^{n} x_i^2, \quad \frac{\partial^2 S}{\partial \mu^2} = 2n, \quad \frac{\partial^2 S}{\partial \lambda \partial \mu} = \frac{\partial^2 S}{\partial \mu \partial \lambda} = 2\sum_{i=1}^{n} x_i.$$

We will show that the critical point (λ_0, μ_0) minimizes S. For any $(\lambda, \mu) \in \mathbb{R}^2$, we have

$$\left(\frac{\partial^2 S}{\partial \lambda^2}(\lambda,\mu)\right)\left(\frac{\partial^2 S}{\partial \mu^2}(\lambda,\mu)\right) - \left(\frac{\partial^2 S}{\partial \lambda \partial \mu}(\lambda,\mu)\right)^2 = 4n\sum_{i=1}^{n} x_i^2 - 4\left(\sum_{i=1}^{n} x_i\right)^2 = 4n\sum_{i=1}^{n}(x_i - \bar{x})^2 > 0,$$

$$\frac{\partial^2 S}{\partial \lambda^2}(\lambda,\mu) = 2\sum_{i=1}^{n} x_i^2 > 0.$$

Therefore, (λ_0, μ_0) is a local minimum of S. Also from the calculation, the Hessian matrix of S is always positive definite. Thus, S is strictly convex, and (λ_0, μ_0) is the unique global minimum of S.