Université Gustave Eiffel
M2 – Semester 1 – 2025/2026

**Probability - Exercise Sheet 1 - Probability and Random Experiments**

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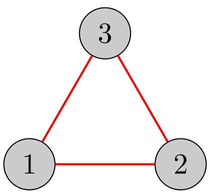
**Exercise 2. (Random coloring of a complete graph. Ramsey numbers.)** Let $K_n, n \geq 2$ be the complete graph of $n$ vertices. Let $R(k,l)$ be the smallest integer $n$ such that any two colorings of the edges of $K_n$ with red and blue contains either a red complete subgraph of $k$ vertices or a blue complete subgraph of $l$ vertices. We want to show that $R(k,k) \geq 2^{k/2}$ for all $k \geq 3$.

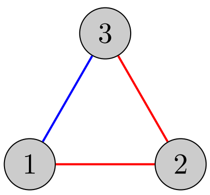
1. Show that $R(2,2) = 2, R(3,2) = 3, R(3,3) = 4$.
2. Fix $n$ and color randomly the edges of $K_n$ with red and blue. For any $R \subset [n]$ of cardinality $k$, let $K_R$ be the complete subgraph of $K_n$ with vertices in $R$.
   (a) Give a probabilistic model for the random choice of colors for the edges of $K_n$.
   (b) Prove that the probability that there exists $R$ such that $K_R$ is monochromatic is bounded by $\binom{n}{k}\frac{2}{2^{\binom{k}{2}}}$.
   (c) Prove that if $\binom{n}{k}\frac{2}{2^{\binom{k}{2}}} < 1$, then $R(k,k) > n$.
   (d) Prove that for all $k \geq 3$, if $n \leq \lfloor 2^{k/2} \rfloor$, then $\binom{n}{k}\frac{2}{2^{\binom{k}{2}}} < 1$.
   (e) Complete the proof.

*Solution.*

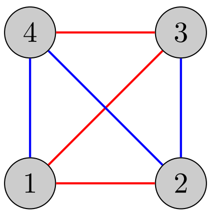
1. We note that $R(k,l) \geq \max\{k,l\}$. Furthermore, if $m > n$ then $K_m$ contains $K_n$ as a complete subgraph. Hence if $K_n$ has a satisfying coloring, then so does $K_m$. Therefore we only need to find the smallest $n$ such that any coloring of $K_n$ contains either a red complete subgraph of $k$ vertices or a blue complete subgraph of $l$ vertices.
   - In any coloring of $K_2$, the only edge is either red (left) or blue (right), hence it contains either a red complete subgraph of 2 vertices or a blue complete subgraph of 2 vertices. Therefore, $R(2,2) = 2$.


   - The coloring where all three edges are in red satisfies the fact that there is a red 3-complete subgraph. In other colorings, at least one edge is in blue, which makes a blue 2-complete subgraph. Therefore, $R(3,2) = 3$.






   - Consider the following coloring of $K_4$. Neither a red 3-complete subgraph nor a blue 3-complete subgraph exists. Therefore, $R(3,3) > 4$.



2.

(a) The sample space contains $2^{\binom{n}{2}}$ possible colorings of the edges of $K_n$ i.e.

$$\Omega = \{f \mid f : [n] \times [n] \to \{\text{red}, \text{blue}\}\}.$$

We have $|\Omega| = \binom{n}{2}$. The $\sigma$-algebra $\mathcal{F}$ is chosen to be the power set of $\Omega$. The probability measure $\mathbb{P}$ is uniform on $\Omega$ i.e. for any event $A \in \mathcal{F}$,

$$\mathbb{P}(A) = \frac{|A|}{2^{\binom{n}{2}}}.$$

(b) The event that there exists $R$ such that $K_R$ is monochromatic is

$$E = \bigcup_{\substack{R \subset [n] \\ |R|=k}} A_R.$$

where $A_R$ is the event that $K_R$ is monochromatic. For each subset $R \subset [n]$ of cardinality $k$, we have

$$|A_R| = 2 \times 2^{\binom{n}{2}-\binom{k}{2}} = 2^{k+1},$$

since the only two ways to color $K_R$ in a monochromatic way are all red or all blue and for each of these two ways, there are $2^{\binom{n}{2}-\binom{k}{2}}$ ways to color the remaining edges.

$$\mathbb{P}(A_R) = \frac{2 \times 2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}}} = \frac{2}{2^{\binom{k}{2}}}, \forall R \subset [n], |R| = k.$$

On the other hand, there are $\binom{n}{k}$ ways to choose $R$. By $\sigma$-additivity of $\mathbb{P}$, we have

$$\mathbb{P}(E) = \mathbb{P}\left(\bigcup_{\substack{R \subset [n] \\ |R|=k}} A_R\right) \leq \sum_{\substack{R \subset [n] \\ |R|=k}} \mathbb{P}(A_R) = \binom{n}{k}\frac{2}{2^{\binom{k}{2}}}.$$

(c) Suppose that $R(k,k) \leq n$. Then for any coloring of $K_n$, there exists a red complete subgraph of $k$ vertices or a blue complete subgraph of $k$ vertices. In other words, the event $E$ always happens, i.e. $\mathbb{P}(E) = 1$. Contrapositively, given that $\mathbb{P}(E) = \binom{n}{k}\frac{2}{2^{\binom{k}{2}}} < 1$, we must have $R(k,k) > n$.

(d) Firstly, notice that for $k \leq n < m$, $\binom{n}{k} < \binom{m}{k}$. It is sufficient to show that $\binom{n}{k} < \binom{n+1}{k}$. Equivalently,

$$\begin{aligned} &\frac{n!}{k!(n-k)!} < \frac{(n+1)!}{k!(n+1-k)!} \\ \iff &(n-k+1)(n-k+2)\dots n < (n-k)(n-k+1)\dots(n+1) \end{aligned}$$

The last inequality is true, since $n-k+i < n-k+i+1$ for all $i = 1, 2, \dots, k$. Therefore, it suffices to show that $\binom{\lfloor 2^{k/2} \rfloor}{k}\frac{2}{2^{\binom{k}{2}}} < 1$.

For $k = 3$, we have $\lfloor 2^{3/2} \rfloor = 2\sqrt{2} = 2$. Hence $\binom{\lfloor 2^{3/2} \rfloor}{3}\frac{2}{2^{\binom{3}{2}}} = \binom{2}{3}\frac{2}{2^3} = 0 < 1$.

For $k = 4$, we have $\lfloor 2^{4/2} \rfloor = 4$. Hence $\binom{\lfloor 2^{4/2} \rfloor}{4}\frac{2}{2^{\binom{4}{2}}} = \binom{4}{4}\frac{2}{2^6} = \frac{1}{32} < 1$.

For $k \geq 5$, we have

$$
\begin{aligned}
\binom{\lfloor 2^{k/2} \rfloor}{k} \frac{2}{2^{\binom{k}{2}}} &= \frac{\left(\lfloor 2^{k/2} \rfloor - k + 1\right) \ldots \lfloor 2^{k/2} \rfloor}{k!} \frac{2}{2^{\binom{k}{2}}} \\
&\leq \frac{\left(2^{k/2} - k + 1\right) \ldots 2^{k/2}}{k!} \frac{2}{2^{\binom{k}{2}}} \\
&\leq \frac{2^{k^2/2}}{k!} \frac{2}{2^{k(k-1)/2}} = \frac{2^{k/2+1}}{k!} = \frac{2^{k/2+1}}{2 \ldots k} \\
&\leq \frac{2^{k/2+1}}{2^{k-1}} = 2^{2-k/2} \\
&\leq 2^{2-5/2} = \frac{1}{\sqrt{2}} \\
&< 1.
\end{aligned}
$$

(e) Take $n = \lfloor 2^{k/2} \rfloor$, from questions (c) and (d), we have $R(k,k) > \lfloor 2^{k/2} \rfloor$, which means

$$R(k,k) \geq \lfloor 2^{k/2} \rfloor + 1 > 2^{k/2}.$$