Université Gustave Eiffel
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Probability - Exercise Sheet 2 - Random Variables and Expectation

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Exercise 1. (Number of Hamiltonian paths in a tournament.) A tournament T_n is a orientation of the edges of K_n. We say that T_n admits a Hamiltonian path if there exists $\sigma \in \mathcal{S}_n$ such that $(\sigma(1), \sigma(2)), \ldots, (\sigma(n-1), \sigma(n)) \in T_n$.

We want to show that for every integer n there exists one tournament with at least $\dfrac{n!}{2^{n-1}}$ Hamiltonian paths.

Let us choose a random tournament T_n. For any $\sigma \in \mathcal{S}_n$, let A_σ be the event that $(\sigma(1), \sigma(2)), \ldots, (\sigma(n-1), \sigma(n)) \in T_n$. Let X be the number of Hamiltonian paths.

1. Give an expression of X in terms of the events A_σ.
2. Prove that for every $\sigma \in \mathcal{S}_n$, $\mathbb{P}(A_\sigma) = \dfrac{1}{2^{n-1}}$.
3. Compute $\mathbb{E}(X)$.
4. Prove the expected result.

Solution. To be precise, the sample space Ω is the set of all tournaments, the σ-algebra $\mathcal{A}$ is the power set of Ω, and the probability measure $\mathbb{P}$ is defined as $\mathbb{P}(\{T_n\}) = \dfrac{1}{2^{\binom{n}{2}}}$, for every tournament T_n. Also, $X : \Omega \to \mathbb{N}$.

1. We have $X = \sum_{\sigma \in \mathcal{S}_n} \mathbf{1}_{A_\sigma}$.
2. For every T_n such that $(i, j) \in T_n$, there exists exactly one T_n' such that $(j, i) \in T_n'$, and vice versa. Hence, there are as many tournaments containing the edge (i, j) as those containing the edge (j, i). Half of the tournaments contain the edge (i, j) and the other half contain the edge (j, i). Therefore, for every $i, j \in [n]$, we have
$$\mathbb{P}((i, j) \in T_n) = \mathbb{P}((j, i) \in T_n) = \frac{1}{2}.$$
Furthermore, for every $\sigma \in \mathcal{S}_n$, the edges $(\sigma(1), \sigma(2)), \ldots, (\sigma(n-1), \sigma(n))$ are distinct and the orientations of the edges are independent. Therefore,
$$\mathbb{P}(A_\sigma) = \mathbb{P}((\sigma(1), \sigma(2)) \in T_n, \ldots, (\sigma(n-1), \sigma(n)) \in T_n) = \prod_{i=1}^{n-1} \mathbb{P}((\sigma(i), \sigma(i+1)) \in T_n) = \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}}.$$
3. We have $\mathbb{E}(X) = \sum_{\sigma \in \mathcal{S}_n} \mathbb{E}(\mathbf{1}_{A_\sigma}) = \sum_{\sigma \in \mathcal{S}_n} \mathbb{P}(A_\sigma) = n! \dfrac{1}{2^{n-1}} = \dfrac{n!}{2^{n-1}}$.
4. Since $\mathbb{E}(X) = \dfrac{n!}{2^{n-1}} \leq \max\limits_{T_n \in \Omega} X(T_n)$, there exists at least one tournament whose the number of Hamiltonian paths is greater than $\dfrac{n!}{2^{n-1}}$.

Exercise 2. (Balancing vectors) Let $v_1, \ldots, v_n \in \mathbb{R}^n$ such that $|v_i| = 1$ for every $i \in [n]$, where $|\cdot|$ is the euclidean norm.

1. We want to prove that there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that
$$|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n| \leq \sqrt{n},$$
and also that there exist $\varepsilon_1', \ldots, \varepsilon_n' \in \{-1, 1\}$ such that
$$|\varepsilon_1' v_1 + \ldots + \varepsilon_n' v_n| \geq \sqrt{n}.$$

Let $(\chi_1, \ldots, \chi_n)$ be a vector of symmetric Rademacher random variables, i.e. $\mathbb{P}(\chi_i = 1) = \mathbb{P}(\chi_i = -1) = \dfrac{1}{2}$ for every $i \in [n]$.

(a) Give the expression of the second moment of the random variable $X = |\chi_1 v_1 + \ldots + \chi_n v_n|$.

(b) Conclude.

2. We assume now $|v_i| \leq 1$ for every $i \in [n]$. Let $(p_i)_{i \in [n]} \in [0,1]^n$. By considering Bernoulli random variables χ_i with parameters p_i, show that there exists $(\eta_1, \ldots, \eta_n) \in [0,1]^n$ such that

$$|(p_1 v_1 + \ldots + p_n v_n) - (\eta_1 v_1 + \ldots + \eta_n v_n)| \leq \frac{\sqrt{n}}{2}.$$

Solution.

1. We have

$$\begin{aligned}
\mathbb{E}[X^2] &= \mathbb{E}[|\chi_1 v_1 + \ldots + \chi_n v_n|^2] \\
&= \mathbb{E}\left(\sum_{i=1}^n \chi_i^2 |v_i|^2 + 2 \sum_{1 \leq i < j \leq n} \chi_i \chi_j \langle v_i, v_j \rangle\right) \\
&= \mathbb{E}\left(\sum_{i=1}^n \chi_i^2 |v_i|^2\right) + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}(\chi_i \chi_j \langle v_i, v_j \rangle) \\
&= \mathbb{E}\left(\sum_{i=1}^n 1\right) + 2 \sum_{1 \leq i < j \leq n} \langle v_i, v_j \rangle \mathbb{E}(\chi_i \chi_j) \\
&= n + 2 \sum_{1 \leq i < j \leq n} \langle v_i, v_j \rangle \cdot 0 \\
&= n
\end{aligned}$$

If for every $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ we have $|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n| > \sqrt{n}$, then $X > \sqrt{n}$ almost surely, or $X^2 > \sqrt{n}$ almost surely. Hence $\mathbb{E}[X^2] > n$, which is a contradiction. Therefore, there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that $|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n| \leq \sqrt{n}$. The proof of the second part is similar.

2. Let χ_i be a Bernoulli random variable with parameter p

Exercise 3. Show that the variance of the number of fixed points of a random permutation is 1.

Solution. Let σ be a random permutation in $\mathcal{S}_n$. Let X be the number of fixed points of σ. For every $i \in [n]$, let $X_i = \mathbf{1}_{\{\sigma(i)=i\}}$. We have $X = \sum_{i=1}^n X_i$. We have $\mathbb{E}(X_i) = \mathbb{P}(\sigma(i) = i) = \dfrac{1}{n}$. Therefore,

On the other hand,

$$\mathbb{E}(X) = \sum_{i=1}^n \mathbb{E}(X_i) = n \cdot \frac{1}{n} = 1.$$

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}\left(\left(\sum_{i=1}^n X_i\right)^2\right) \\
&= \mathbb{E}\left(\sum_{i=1}^n X_i^2 + 2 \sum_{1 \leq i < j \leq n} X_i X_j\right) \\
&= \sum_{i=1}^n \mathbb{E}(X_i^2) + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}(X_i X_j)
\end{aligned}$$

Since X_i is an indicator variable, $X_i^2 = X_i$, so $\mathbb{E}(X_i^2) = \mathbb{E}(X_i) = \frac{1}{n}$. For $i \neq j$, we have

$$\mathbb{E}(X_i X_j) = \mathbb{P}(\sigma(i) = i \text{ and } \sigma(j) = j).$$

The remaining $n-2$ elements can be permuted arbitrarily, so

$$\mathbb{E}(X_i X_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(X^2) &= n \cdot \frac{1}{n} + 2 \cdot \binom{n}{2} \cdot \frac{1}{n(n-1)} \\
&= 1 + 2 \cdot \frac{n(n-1)}{2} \cdot \frac{1}{n(n-1)} \\
&= 1 + 1 = 2
\end{aligned}$$

Thus, $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2 - 1^2 = 1$.