Master 2 Mathematics and Computer Science
Symbolic Dynamics. Lecture 2

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Curtis-Hedlund-Lyndon theorem

Let X, Y be shift spaces. A map $\varphi \colon X \to Y$ is a *morphism* if φ is continuous and commutes with the shift map.

Theorem (Curtis, Hedlund, Lyndon)

Let X, Y be shift spaces. A map $\varphi \colon X \to Y$ is a morphism systems if and only if it is a sliding block code from X into Y.

Proof.

A sliding block code is clearly continuous and commutes with the shift.
Conversely, let $\varphi \colon X \to Y$ be a morphism . For every letter b from the alphabet B of Y, the set $[b]_Y$ is clopen and thus $\varphi^{-1}([b]_Y)$ is also clopen. Since a clopen set is a finite union of cylinders, there is an integer n such that $\varphi(x)_0$ depends only on $x_{[-n,n]}$. Set $f(x_{[-n,n]}) = \varphi(x)_0$. Then φ is the sliding block code associated with the block map f. $\square$

Edge shifts

An *edge shift* is the set of bi-infinite paths of a directed (multi)graph.

Proposition

Every shift of finite type is conjugate to an edge shift.

Proof.

Let $X = \mathsf{X}_F$ with F finite, and let n be the maximal size of words in F. We may assume that all words in F have size n.
Let $\mathcal{A} = (Q, E)$, where Q is the set of words of length $n-1$ with edges $a_0a_1 \dots a_{n-2} \xrightarrow{a} a_1 \dots a_{n-2}a$, where $a_0a_1 \dots a_{n-2}a \notin F$. We keep only the trim part of this automaton.

Then $\mathcal{A}$ is deterministic and local (all paths labeled by a word w of length $n-1$ end in the same state q_w). □

State splitting of an automaton

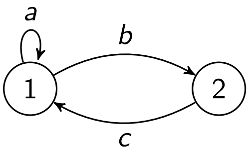
An *out-splitting* of an automaton $\mathcal{A} = (Q, E)$ is a local transformation of $\mathcal{A}$ into an automaton $\mathcal{B} = (Q', E')$ obtained by selecting a state s and partitioning the set of edges going out of s into two non-empty sets E_1 and E_2.

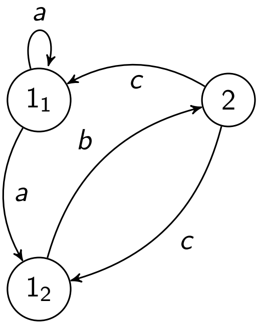
- $Q' = Q \setminus \{s\} \cup \{s_1, s_2\}$,
- E' contains all edges of E neither starting at or ending in s,
- E' contains the edge (s_1, a, t) for each edge $(s, a, t) \in E_1$, and the edge (s_2, a, t) for each edge $(s, a, t) \in E_2$, if $t \neq s$.
- E' contains the edges (t, a, s_1) and (t, a, s_2) if (t, a, s) in E, when $t \neq s$,
- E' contains the edges (s_1, a, s_1) and (s_1, a, s_2) if (s, a, s) in E_1, and the edges (s_2, a, s_1) and (s_2, a, s_2) if $(s, a, s) \in E_2$.

State splitting of an automaton

An *input state splitting* is defined similarly.

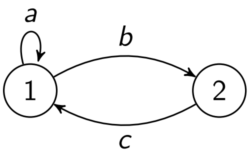
The inverse operation is called an *output merging*, possible whenever s_1 and s_2 have the *same input edges*.

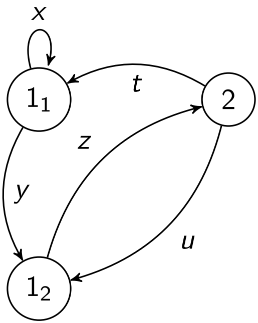




The state 1 is split into two states 1_1 and 1_2 with $E_1 = \{(1, a, 1)\}$ and $E_2 = \{(1, b, 2)\}$.

Output state splitting of a graph





State splitting

Proposition

Let G be a graph and H a split graph of G. Then X_G and X_H are conjugate.

Proof.

Let $G = (Q, E)$ (all labels are distinct).
Let $H = (Q', E')$ be an outsplit of G, obtained after splitting the state s into s_1, s_2 according to the partition E_1, E_2 of edges going out of s.
Let X_G be the edge shift defined by G and X_H be the edge shift defined by H.
Then X_G and X_H are conjugate. □

Strong shift equivalence

Two nonnegative integer matrices M, N are *elementary equivalent* if there are, possibly nonsquare, matrices R, S such that

$$M = RS, N = SR.$$

Two nonnegative integer matrices M, N are *strong shift equivalent* if there is a sequence of elementary equivalences from M to N:

$$\begin{aligned} M &= R_0 S_0, S_0 R_0 = M_1, \\ M_1 &= R_1 S_1, S_1 R_1 = M_2, \\ &\vdots \\ M_\ell &= R_\ell S_\ell, S_\ell R_\ell = N. \end{aligned}$$

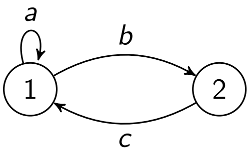
Theorem (Classification Theorem, R. Williams 1973)

Two edge shifts defined by matrices M and N are conjugate if and only if M and N are strong shift equivalent.

Transition matrix of a graph

Let $G = (Q, E)$ be a graph. Its transition matrix is a nonnegative integer matrix M where

$$M = (m_{pq})_{p,q \in Q}, \text{ where } m_{pq} \text{ is the number of edges from } p \text{ to } q.$$



$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Irreducible and primitive matrices

A nonnegative square matrix (with real coefficients) M is *irreducible* if for every pair s, t of indices, there is an integer $n \geq 1$ such that $M^n_{s,t} > 0$. Otherwise, M is *reducible*.
A matrix M is reducible if and only if, up to a permutation of the indices, it can be written

$$M = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$$

for some matrices U, V, W with U, W being square matrices of dimension ≥ 1.

A nonnegative square matrix M is *primitive*, if there is some integer $n \geq 1$ such that all entries of M^n are positive.
The least such n is called the *exponent* of M, denoted $\exp(M)$.
A primitive matrix is irreducible but the converse is not necessarily true.

Irreducible matrix

Lemma

If M is a nonnegative $Q \times Q$ irreducible matrix, then $(I + M)^{n-1} > 0$, where $n = \text{Card } Q$.

Proof.

Let G be the graph whose adjacency matrix is $I + M$.
Thus, $s \to t$ is an edge if and only if $(I + M)_{st} > 0$.
Since M is irreducible, there is a path of length at most $n - 1$ from s to t in G.
Since the state s has a self-loop, there is a path of length $n - 1$ from s to t in G.
Hence, $(I + M)^{n-1}_{st} > 0$ for all states $s, t \in Q$. $\square$

Periods

The *period* of an irreducible nonnegative square matrix $M \neq 0$ is the greatest common divisor of the integers n such that M^n has a positive diagonal coefficient. By convention, the period of $M = 0$ is 1. If M has period p, then M and M^p have, up to a permutation of indices, the forms:

$$M = \begin{bmatrix} 0 & M_1 & 0 & \dots & 0 \\ 0 & 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_{p-1} \\ M_p & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad M^p = \begin{bmatrix} D_1 & 0 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & D_{p-1} & 0 \\ 0 & 0 & \dots & 0 & D_p \end{bmatrix}$$

Thus M^p is block diagonal, with each diagonal block D_i primitive. An irreducible matrix is primitive if and only if it has period 1.

The Perron-Frobenius theorem

Theorem

Let M be a nonnegative real $Q \times Q$-matrix. Then

1. *M has an eigenvalue λ_M such that $|\mu| \leq \lambda_M$ for every eigenvalue μ of M.*
2. *There corresponds to λ_M a nonnegative eigenvector v, and a positive one if M is irreducible. If M is irreducible, λ_M is the only eigenvalue with a nonnegative eigenvector.*
3. *If M is primitive, the sequence (M^n/λ_M^n) converges to the matrix yx where x, y are positive left and right eigenvectors relative to λ_M with $\sum_{s \in Q} y_s = 1$ and $\sum_{s \in Q} x_s y_s = 1$.*

If M is irreducible, then λ_M is simple. The matrix M is primitive if and only if $|\mu| < \lambda_M$ for every other eigenvalue μ of M.

An *eigenvector* of a square real matrix M for the eigenvalue λ (a real or complex number) is a **non null** vector v (with real or complex coefficients) such that $Mv = \lambda v$.

The *spectral radius* of a square real matrix is the real number

$$\rho(M) = \max\{|\lambda| \mid \lambda \text{ eigenvalue of } M\}.$$

The theorem states in particular that if a matrix M is irreducible, $\rho(M)$ is an eigenvalue of M that is algebraically simple. Furthermore, if M is primitive, any eigenvalue of M other than $\rho(M)$ has modulus less than $\rho(M)$.

Proof of Perron-Frobenius Points 1 and 2

Proposition

Any nonnegative matrix M has a real eigenvalue λ_M such that $|\lambda| \leq \lambda_M$ for any eigenvalue λ of M, and there corresponds to λ_M a nonnegative eigenvector v.
If M is irreducible, there corresponds to λ_M a positive eigenvector v, and λ_M is the only eigenvalue with a nonnegative eigenvector.

Entropy

The (topological) entropy of a shift space X is

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_n(X)).$$

The limit exists.

Computation of the entropy of a sofic shift

Similarly

$$(c/d)\lambda_M^n \leq \sum_{s,t \in Q} (M^n)_{st}.$$

Proposition

Let $\mathcal{A} = (Q, E)$ be an irreducible deterministic automaton presenting an irreducible sofic shift X and M its adjacency matrix. Then $h(X) = \log \lambda_M$.

The result holds for a trim deterministic automaton presenting a sofic shift X with a reduction to the irreducible components of M (exercise).

Periodic points in a shift space

A point x of a shift space X is *periodic* if $S^n(x) = x$ for some $n \geq 1$ and we say that x has *period* n.

If x is periodic, the smallest positive integer n for which $S^n(x) = x$, called the least period of x, divides all periods of x.

Let

$$p_n(X) = \mathrm{Card}\{x \in X \mid S^n(x) = x\}.$$

Proposition

Let $\varphi\colon X \to Y$ be a sliding block map. If x is a periodic point of X and has period n, then $\varphi(x)$ is periodic and has period n and the least period of $\varphi(x)$ divides the least period of x. If X and Y are conjugate, then $p_n(X) = p_n(Y)$ for each $n \geq 1$.

Proposition

Let G be a graph of transition matrix M, the number of cycles of length n in G is $\mathrm{tr}(M^n)$ and this equals the number of points in X_G with period n.

The zeta function of a shift space X is the formal series

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X)}{n} z^n\right).$$

Proposition

If X and Y are conjugate, then $\zeta_X = \zeta_Y$.

Zeta function of a shift of finite type

Theorem

Let G be a graph with adjacency matrix M. Then

$$\zeta_{X_G}(z) = \frac{1}{\det(I - Mz)}.$$