Master 2 Mathematics and Computer Science
Symbolic Dynamics. Lecture 1

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Topology and shift transformation

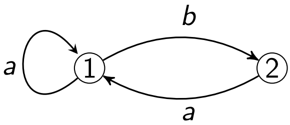
The set $A^{\mathbb{Z}}$ of two-sided infinite sequences of elements of A is a metric space for the distance defined for $x \neq y$ by $d(x, y) = 2^{-r(x,y)}$ where

$$r(x, y) = \inf\{|n| \mid n \in \mathbb{Z}, x_n \neq y_n\}. \tag{1}$$

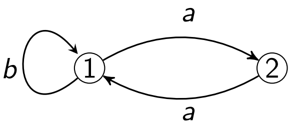
The topology induced by this metric coincides with the product topology on $A^{\mathbb{Z}}$, using the discrete topology on A.
Since a product of compact spaces is compact, $A^{\mathbb{Z}}$ is a compact metric space.

Let S denote the *shift transformation*, defined for $x \in A^{\mathbb{Z}}$ by $S(x) = y$ if $y_n = x_{n+1}$ for $n \in \mathbb{Z}$. It is continuous and one-to-one from $A^{\mathbb{Z}}$ to itself.

Example of a shift of finite type: the golden mean shift



Example of a sofic shift: the even shift



A shift of finite type is sofic: another solution

Let $X = \mathsf{X}_F$ with $F \subseteq A^*$ a finite set.
Let n be the maximal size of words in F.
Let $\mathcal{A}$ be the graph whose states are the words of length $n-1$ that do not contain any word of F, and with edges

$$a_0 a_1 \dots a_{n-2} \xrightarrow{a_0} a_1 \dots a_{n-2} a,$$

where $a_0 a_1 \dots a_{n-2} a$ does not contain any word of F. The set of labels of two-sided infinite paths in $\mathcal{A}$ is equal to $X = \mathsf{X}_F$.

Example with $F = \{bb\}$ on the board.

Language of a shift space

If X is a shift space, the set of blocks of sequences in X is denoted by $\mathcal{B}(X)$. The set of blocks of length n of sequences in X is denoted by $\mathcal{B}_n(X)$.
A language L is called *factorial* if it contains the words occurring as blocks in its elements, that is, if $uvw \in L$, then $v \in L$.
It is *extendable* if every $u \in L$ is *extendable*, that is, there are letters $a, b \in A$ such that $aub \in L$.

Proposition

The language of a shift space is factorial and extendable. Conversely, for every factorial and extendable language L, there is a unique shift space X such that $\mathcal{B}(X) = L$. It is the set $\mathrm{X}(L)$ of sequences $x \in A^{\mathbb{Z}}$ with all their blocks in L. For every factorial and extendable language L and every shift space X, the following equalities hold: $\mathcal{B}(\mathrm{X}(L)) = L$, and $\mathrm{X}(\mathcal{B}(X)) = X$.

Cylinders

Let X be a shift space. For two words u, v such that $uv \in \mathcal{B}(X)$, the set

$$[u \cdot v]_X = \{x \in X \mid x_{[-|u|,|v|)} = uv\}$$

is nonempty. It is called the *cylinder* with basis (u, v). For $v \in \mathcal{B}(X)$, we also define

$$[v]_X = \{x \in X \mid x_{[0,|v|)} = v\}$$

in such a way that $[v]_X = [\varepsilon \cdot v]_X$. The set $[v]_X$ is called the *right cylinder* with basis v.

The open sets contained in X are the unions of cylinders and the clopen sets are the finite unions of cylinders (Exercises).

Irreducible shift

A nonempty shift space X is *irreducible* if, for every $u, v \in \mathcal{B}(X)$, there is a word w such that $uwv \in \mathcal{B}(X)$.

Example

The golden mean shift X is irreducible. Indeed, if $u, v \in \mathcal{B}(X)$, then $uav \in \mathcal{B}(X)$.

Uniformly recurrent shift

A nonempty shift space X is *uniformly recurrent* if for every $w \in \mathcal{B}(X)$ there is an integer $n \geq 1$ such that w occurs in every word of $\mathcal{B}_n(X)$.

As an equivalent definition, a shift space X is uniformly recurrent if for every $n \geq 1$ there is an integer $N = R_X(n)$ such that every word of $\mathcal{B}_n(X)$ occurs in every word of $\mathcal{B}_N(X)$. The function R_X is called the *recurrence function* of X.

Example

Example

The golden mean shift X is not uniformly recurrent since b is in $\mathcal{B}(X)$ although b does not occur in any $a^n \in \mathcal{B}(X)$.

Deterministic automaton in symbolic dynamics

An automaton $\mathcal{A} = (Q, E)$ is a finite directed (multi)graph with edges labeled on A. The set of edges is included in $Q \times A \times Q$.

It is *trim* if each state has at least one outgoing edge and at least one incoming edge.

It is (uncomplete) *deterministic* if, for each state $p \in Q$ and each letter $a \in A$, there is at most one edge labeled by a going out of p.

It is *irreducible* if its graph is strongly connected.

It is a *presentation* of a sofic shift X if X is the set of labels of bi-infinite paths of $\mathcal{A}$.

Minimal automaton in symbolic dynamics

Proposition

Every sofic shift has a trim deterministic presentation.

Proposition

Every irreducible sofic shift has a unique minimal deterministic presentation (irreducible deterministic and with the fewest number of states among these presentations).

Local automaton

A deterministic automaton $\mathcal{A} = (Q, E)$ is *local* if there is an integer n such that, for each word w of length n, all paths labeled by w end in the same state q_w.

Proposition

An irreducible shift X is of finite type if and only if its minimal deterministic automaton is local.

Proof.

Exercise. $\square$

Local automaton

Proposition

An irreducible deterministic automaton is local if and only if it has at most one cycle with a given label.

Proof.

Exercise. $\square$

cycle : path $p = p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} p_2 \ldots \xrightarrow{a_{m-1}} p_m = p$.
m is the length of the cycle.

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Curtis-Hedlund-Lyndon theorem

Let X, Y be shift spaces. A map $\varphi\colon X \to Y$ is a *morphism* if φ is continuous and commutes with the shift map.

Theorem (Curtis, Hedlund, Lyndon)

Let X, Y be shift spaces. A map $\varphi\colon X \to Y$ is a morphism systems if and only if it is a sliding block code from X into Y.

Proof.

A sliding block code is clearly continuous and commutes with the shift.
Conversely, let $\varphi\colon X \to Y$ be a morphism . For every letter b from the alphabet B of Y, the set $[b]_Y$ is clopen and thus $\varphi^{-1}([b]_Y)$ is also clopen. Since a clopen set is a finite union of cylinders, there is an integer n such that $\varphi(x)_0$ depends only on $x_{[-n,n]}$. Set $f(x_{[-n,n]}) = \varphi(x)_0$. Then φ is the sliding block code associated with the block map f. $\square$

Edge shifts

An *edge shift* is the set of bi-infinite paths of a directed (multi)graph.

Proposition

Every shift of finite type is conjugate to an edge shift.

Proof.

Let $X = \mathsf{X}_F$ with F finite, and let n be the maximal size of words in F. We may assume that all words in F have size n.
Let $\mathcal{A} = (Q, E)$, where Q is the set of words of length $n-1$ with edges $a_0a_1 \ldots a_{n-2} \xrightarrow{a} a_1 \ldots a_{n-2}a$, where $a_0a_1 \ldots a_{n-2}a \notin F$. We keep only the trim part of this automaton.

Then $\mathcal{A}$ is deterministic and local (all paths labeled by a word w of length $n-1$ end in the same state q_w). $\square$

State splitting of an automaton

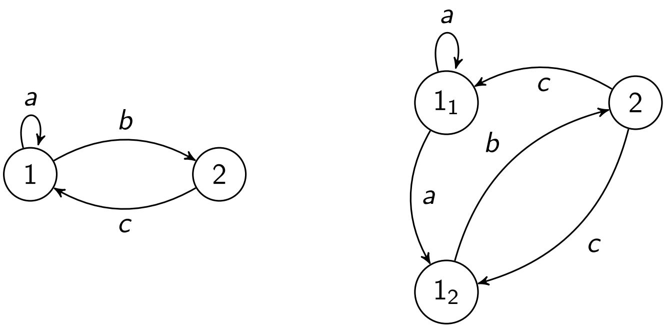
An *out-splitting* of an automaton $\mathcal{A} = (Q, E)$ is a local transformation of $\mathcal{A}$ into an automaton $\mathcal{B} = (Q', E')$ obtained by selecting a state s and partitioning the set of edges going out of s into two non-empty sets E_1 and E_2.

- $Q' = Q \setminus \{s\} \cup \{s_1, s_2\}$,
- E' contains all edges of E neither starting at or ending in s,
- E' contains the edge (s_1, a, t) for each edge $(s, a, t) \in E_1$, and the edge (s_2, a, t) for each edge $(s, a, t) \in E_2$, if $t \neq s$.
- E' contains the edges (t, a, s_1) and (t, a, s_2) if (t, a, s) in E, when $t \neq s$,
- E' contains the edges (s_1, a, s_1) and (s_1, a, s_2) if (s, a, s) in E_1, and the edges (s_2, a, s_1) and (s_2, a, s_2) if $(s, a, s) \in E_2$.

An *input state splitting* is defined similarly.

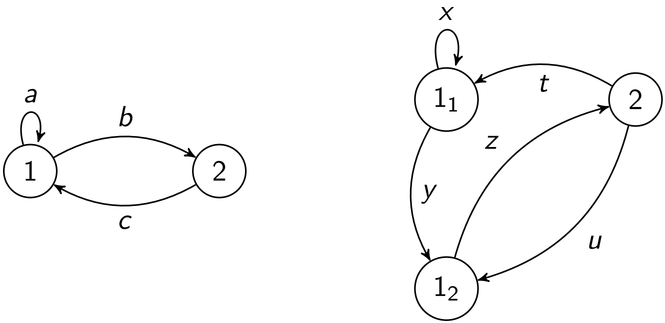
The inverse operation is called an *output merging*, possible whenever s_1 and s_2 have the *same input edges*.

Output state splitting of an automaton



The state 1 is split into two states 1_1 and 1_2 with $E_1 = \{(1, a, 1)\}$ and $E_2 = \{(1, b, 2)\}$.

Output state splitting of a graph



Proposition

Let G be a graph and H a split graph of G. Then X_G and X_H are conjugate.

Proof.

Let $G = (Q, E)$ (all labels are distinct).
Let $H = (Q', E')$ be an outsplit of G, obtained after splitting the state s into s_1, s_2 according to the partition E_1, E_2 of edges going out of s.
Let X_G be the edge shift defined by G and X_H be the edge shift defined by H.
Then X_G and X_H are conjugate. □

Strong shift equivalence

Two nonnegative integer matrices M, N are *elementary equivalent* if there are, possibly nonsquare, matrices R, S such that

$$M = RS, N = SR.$$

Two nonnegative integer matrices M, N are *strong shift equivalent* if there is a sequence of elementary equivalences from M to N:

$$\begin{aligned} M &= R_0 S_0, S_0 R_0 = M_1, \\ M_1 &= R_1 S_1, S_1 R_1 = M_2, \\ &\vdots \\ M_\ell &= R_\ell S_\ell, S_\ell R_\ell = N. \end{aligned}$$

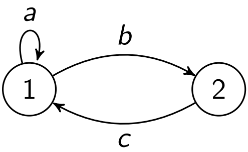
Theorem (Classification Theorem, R. Williams 1973)

Two edge shifts defined by matrices M and N are conjugate if and only if M and N are strong shift equivalent.

Transition matrix of a graph

Let $G = (Q, E)$ be a graph. Its transition matrix is a nonnegative integer matrix M where

$$M = (m_{pq})_{p,q \in Q}, \text{ where } m_{pq} \text{ is the number of edges from } p \text{ to } q.$$



$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Irreducible and primitive matrices

A nonnegative square matrix (with real coefficients) M is *irreducible* if for every pair s, t of indices, there is an integer $n \geq 1$ such that $M^n_{s,t} > 0$. Otherwise, M is *reducible*.
A matrix M is reducible if and only if, up to a permutation of the indices, it can be written

$$M = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$$

for some matrices U, V, W with U, W being square matrices of dimension ≥ 1.

A nonnegative square matrix M is *primitive*, if there is some integer $n \geq 1$ such that all entries of M^n are positive.
The least such n is called the *exponent* of M, denoted $\exp(M)$.
A primitive matrix is irreducible but the converse is not necessarily true.

Lemma

If M is a nonnegative $Q \times Q$ irreducible matrix, then $(I + M)^{n-1} > 0$, where $n = \mathrm{Card}\, Q$.

Proof.

Let G be the graph whose adjacency matrix is $I + M$.
Thus, $s \to t$ is an edge if and only if $(I + M)_{st} > 0$.
Since M is irreducible, there is a path of length at most $n - 1$ from s to t in G.
Since the state s has a self-loop, there is a path of length $n - 1$ from s to t in G.
Hence, $(I + M)^{n-1}_{st} > 0$ for all states $s, t \in Q$. $\square$

Periods

The *period* of an irreducible nonnegative square matrix $M \neq 0$ is the greatest common divisor of the integers n such that M^n has a positive diagonal coefficient. By convention, the period of $M = 0$ is 1. If M has period p, then M and M^p have, up to a permutation of indices, the forms:

$$M = \begin{bmatrix} 0 & M_1 & 0 & \dots & 0 \\ 0 & 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_{p-1} \\ M_p & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad M^p = \begin{bmatrix} D_1 & 0 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & D_{p-1} & 0 \\ 0 & 0 & \dots & 0 & D_p \end{bmatrix}$$

Thus M^p is block diagonal, with each diagonal block D_i primitive. An irreducible matrix is primitive if and only if it has period 1.

The Perron-Frobenius theorem

Theorem

Let M be a nonnegative real $Q \times Q$-matrix. Then

1. *M has an eigenvalue λ_M such that $|\mu| \leq \lambda_M$ for every eigenvalue μ of M.*
2. *There corresponds to λ_M a nonnegative eigenvector v, and a positive one if M is irreducible. If M is irreducible, λ_M is the only eigenvalue with a nonnegative eigenvector.*
3. *If M is primitive, the sequence (M^n/λ_M^n) converges to the matrix yx where x, y are positive left and right eigenvectors relative to λ_M with $\sum_{s \in Q} y_s = 1$ and $\sum_{s \in Q} x_s y_s = 1$.*

If M is irreducible, then λ_M is simple. The matrix M is primitive if and only if $|\mu| < \lambda_M$ for every other eigenvalue μ of M.

Spectral radius

An *eigenvector* of a square real matrix M for the eigenvalue λ (a real or complex number) is a **non null** vector v (with real or complex coefficients) such that $Mv = \lambda v$.

The *spectral radius* of a square real matrix is the real number

$$\rho(M) = \max\{|\lambda| \mid \lambda \text{ eigenvalue of } M\}.$$

The theorem states in particular that if a matrix M is irreducible, $\rho(M)$ is an eigenvalue of M that is algebraically simple. Furthermore, if M is primitive, any eigenvalue of M other than $\rho(M)$ has modulus less than $\rho(M)$.

Proof of Perron-Frobenius Points 1 and 2

Proposition

Any nonnegative matrix M has a real eigenvalue λ_M such that $|\lambda| \leq \lambda_M$ for any eigenvalue λ of M, and there corresponds to λ_M a nonnegative eigenvector v.
If M is irreducible, there corresponds to λ_M a positive eigenvector v, and λ_M is the only eigenvalue with a nonnegative eigenvector.

Entropy

The (topological) entropy of a shift space X is

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_n(X)).$$

The limit exists.

Similarly

$$(c/d)\lambda_M^n \leq \sum_{s,t \in Q} (M^n)_{st}.$$

Proposition

Let $\mathcal{A} = (Q, E)$ be an irreducible deterministic automaton presenting an irreducible sofic shift X and M its adjacency matrix. Then $h(X) = \log \lambda_M$.

The result holds for a trim deterministic automaton presenting a sofic shift X with a reduction to the irreducible components of M (exercise).

Periodic points in a shift space

A point x of a shift space X is *periodic* if $S^n(x) = x$ for some $n \geq 1$ and we say that x has *period* n.
If x is periodic, the smallest positive integer n for which $S^n(x) = x$, called the least period of x, divides all periods of x.
Let

$$p_n(X) = \mathrm{Card}\{x \in X \mid S^n(x) = x\}.$$

Proposition

Let $\varphi \colon X \to Y$ be a sliding block map. If x is a periodic point of X and has period n, then $\varphi(x)$ is periodic and has period n and the least period of $\varphi(x)$ divides the least period of x. If X and Y are conjugate, then $p_n(X) = p_n(Y)$ for each $n \geq 1$.

Proposition

Let G be a graph of transition matrix M, the number of cycles of length n in G is $\mathrm{tr}(M^n)$ and this equals the number of points in X_G with period n.

Zeta function

The zeta function of a shift space X is the formal series

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X)}{n} z^n\right).$$

Proposition

If X and Y are conjugate, then $\zeta_X = \zeta_Y$.

Zeta function of a shift of finite type

Theorem

Let G be a graph with adjacency matrix M. Then

$$\zeta_{X_G}(z) = \frac{1}{\det(I - Mz)}.$$

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One-sided shift spaces

A *one-sided shift space* is a closed subset X of $A^{\mathbb{N}}$ such that $S(X) \subseteq X$.

One-sided shift spaces are usually defined as closed subsets such that $S(X) = X$, but we do not require this stronger condition here.

The set $A^{\mathbb{N}}$ itself is a one-sided shift space, called the *one-sided full shift*.

For a two-sided sequence $x \in A^{\mathbb{Z}}$, we define $x^+ = x_0 x_1 \cdots$.
If X is a two-sided shift space, then the set $X^+ = \{x^+ \mid x \in X\}$ is a one-sided shift space.

Out-merging

The inverse operation of an out-splitting is referred to as an *out-merging*. An out-merging of a directed graph $G' = (V', E')$ can be performed if there are two vertices s_1, s_2 of G' such that the adjacency matrix M' satisfies:

- the column of index s_1 is equal to the column of index s_2 of M'.

The adjacency matrix of G is thus the matrix M obtained by adding the rows of index s_2 to the row of index s_1 of M' and then removing the column of index s_2 afterward.

The graph G is called an *elementary amalgamation* of G'. Notice that even if M' has 0-1 entries, M may not have 0-1 entries.

General amalgamation

Let M' be the adjacency matrix of a directed graph G', and $(V_1, V_2, \ldots, V_k)$ be a partition of V' into classes such that if s, t belong to the same class, then the columns of indices s and t of M' are identical.

When at least one set of the partition has a size greater than 1, we can perform a *general merging*. We define a graph K of adjacency matrix N obtained by merging all states of each $V_i = \{s_{i,1}, \ldots s_{i,k_i}\}$ into a single state $s_{i,1}$.

The row in N corresponding to $s_{i,1}$ is obtained by summing the rows of the states of V_i in M' and removing the columns $s_{i,2}, \cdots, s_{i,k_i}$.

The graph K is called a *general amalgamation* of G'.

Two out-merging transformations commute

Proposition (R. Williams 1973)

If G and H are amalgamations of a common directed graph L, then they have a common amalgamation K.

Proposition (R. Williams 1973)

Let G and H be irreducible directed graphs that define one-sided edge shifts X_G and X_H. Then X_G and X_H are conjugate if and only if G and H have the same total amalgamation.

It also holds for one-sided edge shifts defined by trim directed graphs.

Two out-merging transformations commute

Proposition (R. Williams 1973)

If G and H are amalgamations of a common directed graph L, then they have a common amalgamation K.

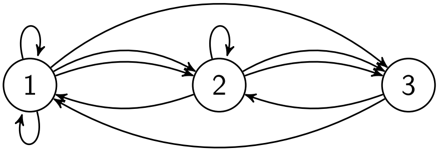
Proof.

Let us assume that there is an out-merging of G with adjacency matrix M into G' with adjacency matrix M', obtained by merging s_1 and s_2 into s_1, and an out-merging of G into G'' with adjacency matrix M'', obtained by merging s_3 and s_4 into s_3.
We may assume that the set $\{s_3, s_4\}$ is distinct from the set $\{s_1, s_2\}$.

Let us show that there is a graph H that is an out-merging of both G' and G''.

□

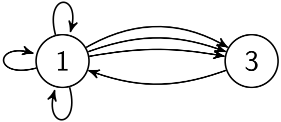
If G is the following graph:



$$M = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

Total amalgamation

Its total amalgamation is H:



$$N = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$$

Master 2 Mathematics and Computer Science
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Lettre coding

A substitution $\sigma\colon A^* \to B^*$ is a *letter coding* if it is of constant length 1. Letter codings, also called *letter-to-letter* substitutions, play an important role in the definition of morphic sequences (see later).
They are the substitutions preserving length, meaning that $|\sigma(w)| = |w|$ for every $w \in A^*$. They also correspond to 1-block sliding block codes.

For a substitution $\sigma\colon A^* \to B^*$, we define

$$|\sigma| = \max_{a \in A} |\sigma(a)|, \quad \text{and} \quad \langle \sigma \rangle = \min_{a \in A} |\sigma(a)| \tag{2}$$

Composition matrix

Let $\sigma\colon A^* \to B^*$ be a substitution. The *composition matrix* of σ is the $(B \times A)$-matrix $M = M(\sigma)$ defined by

$$M_{b,a} = |\sigma(a)|_b,$$

where $|\sigma(a)|_b$ is the number of occurrences of the letter b in the word $\sigma(a)$. Thus, the composition vector of each $\sigma(a)$ is the column of index a of the matrix $M(\sigma)$.
If $\sigma\colon B^* \to C^*$ and $\tau\colon A^* \to B^*$ are substitutions, we have

$$M(\sigma \circ \tau) = M(\sigma)M(\tau).$$

Indeed, for every $a \in A$ and $c \in C$, we have

$$M(\sigma \circ \tau)_{c,a} = |\sigma \circ \tau(a)|_c = \sum_{b \in B} |\sigma(b)|_c |\tau(a)|_b = (M(\sigma)M(\tau))_{c,a}.$$

The transpose of $M(\sigma)$ is called the *adjacency matrix*.

Composition matrix

For a word $w \in A^*$, we denote by $\ell(w)$ the column vector $(|w|_a)_{a \in A}$, called the *composition vector* of w.
The composition matrix satisfies, for every $w \in A^*$, the equation

$$\ell(\sigma(w)) = M(\sigma)\ell(w). \tag{3}$$

Example

The composition matrix of $\sigma : a \mapsto ab, b \mapsto aa$ is

$$M(\sigma) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

Iteration of a substitution

A substitution $\sigma \colon A^* \to A^*$ from A^* into itself is an endomorphism of the monoid A^*. It can be iterated, that is, its powers σ^n for $n \geq 1$ are also substitutions.

Let $\sigma \colon A^* \to A^*$ be an iterable substitution. The *language* of σ, denoted by $\mathcal{L}(\sigma)$ is the set of words occurring as blocks in the words $\sigma^n(a)$ for some $n \geq 0$ and some $a \in A$.
It follows from the definition that

$$\sigma(\mathcal{L}(\sigma)) \subseteq \mathcal{L}(\sigma). \tag{4}$$

The language $\mathcal{L}(\sigma)$ is decidable (exercise).

Substitution shift

Let $\sigma \colon A^* \to A^*$ be an iterable substitution.
The *substitution shift* defined by σ is the shift space $\mathsf{X}(\sigma)$ consisting of all $x \in A^{\mathbb{Z}}$ whose finite blocks belong to $\mathcal{L}(\sigma)$.

Show that it is a shift space.

Since $\sigma(\mathcal{L}(\sigma)) \subseteq \mathcal{L}(\sigma)$ by (4), we have also

$$\sigma(\mathsf{X}(\sigma)) \subseteq \mathsf{X}(\sigma). \tag{5}$$

Note that $\mathcal{B}(\mathsf{X}(\sigma)) \subseteq \mathcal{L}(\sigma)$, but the converse inclusion may not hold, as shown in the example below.

Example

Consider the substitution $\sigma\colon a \mapsto ab, b \mapsto b$. We have $\mathcal{L}(\sigma) = ab^* \cup b^*$ but $\mathsf{X}(\sigma) = b^\infty$, and thus $\mathcal{B}(\mathsf{X}(\sigma)) = b^*$.

Erasable and growing letters

Let $\sigma\colon A^* \to A^*$ be an iterable substitution. A letter $a \in A$ is *erasable* if $\sigma^n(a) = \varepsilon$ for some $n \geq 1$.
A word is *erasable* if it is formed of erasable letters.

A word $w \in A^*$ is *growing* for σ if the sequence $(|\sigma^n(w)|)_n$ is unbounded.
A word is growing if and only if at least one of its letters is growing.
The substitution σ itself is said to be *growing* if all letters are growing.
We have the following property of growing letters.

Proposition

If $a \in A$ is growing for σ, then for every $r \geq 0$, $\sigma^{r\,\mathrm{Card}(A)}(a)$ contains at least $r+1$ non-erasable letters. In particular, $\lim_{n\to+\infty} |\sigma^n(a)| = +\infty$.

Primitive substitutions

An iterable substitution $\sigma\colon A^* \to A^*$ is *primitive* if there is an integer $n \geq 1$ such that for every $a, b \in A$ one has $|\sigma^n(a)|_b \geq 1$.

For a primitive substitution σ, except the trivial case $A = \{a\}$ and $\sigma(a) = a$, every letter is growing and $\mathcal{L}(\sigma) = \mathcal{B}(\mathrm{X}(\sigma))$ (exercise).

A substitution shift $X = \mathrm{X}(\sigma)$ is *primitive* if σ is primitive, and not the identity on a one-letter alphabet.

Exercise

Show that $\mathcal{L}(\sigma) = \mathcal{B}(\mathsf{X}(\sigma))$ if and only if $\mathcal{L}(\sigma)$ is extendable, *i.e.* if for each $u \in \mathcal{L}(\sigma)$, there are letters a, b such that $aub \in \mathcal{L}(\sigma)$.

Minimal shift spaces

A shift space X is *minimal* if it is nonempty and if, for every subshift $Y \subseteq X$, one has $Y = \emptyset$ or $Y = X$.

Equivalently, X is minimal if and only if the closure of the orbit $\mathcal{O}(x) = \{S^n(x) \mid n \in \mathbb{Z}\}$ of x is equal to X, for every $x \in X$.

A shift space is minimal if and only if the closure $\mathcal{O}^+(x) = \{S^n(x) \mid n \in \mathbb{N}\}$ of x is equal to X, for every $x \in X$.

Indeed, if X is minimal and Y equal to the closure of $\mathcal{O}^+(x)$, then $Z = \cap_{n \geq 0} S^n(Y)$ is nonempty shift contained in X, thus equal to X. (It is nonempty by compacity as a decreasing sequence of nonempty compact sets).

Return words

Let X be a shift space. Given a word $u \in \mathcal{B}(X)$, a *return word* to u in X is a nonempty word w such that $wu \in \mathcal{B}(X)$ and wu has exactly two occurrences of u: one as a prefix and one as a suffix.

By convention, a return word to the empty word is a letter. The set of return words to u in X is denoted by $\mathcal{R}_X(u)$.

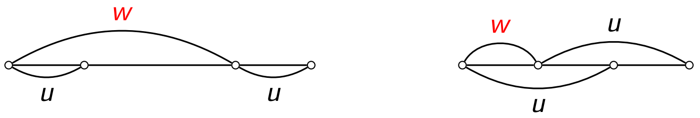


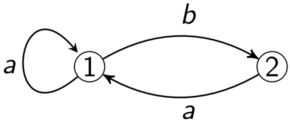
Figure: Return word to u.

The set of return words to u is a *suffix code*, that is, a set S of nonempty words such that no element of S is a proper suffix of another one.

Example

Example

The set of return words to b in the golden mean shift X is $\mathcal{R}_X(b) = ba^+$.



Uniform recurrence

A nonempty shift space X is *recurrent* if it is irreducible, that is, for every $u, v \in \mathcal{B}(X)$ there is a block $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$.

A nonempty shift space X is *uniformly recurrent* if for every $w \in \mathcal{B}(X)$ there is an integer $n \geq 1$ such that w occurs in every word of $\mathcal{B}_n(X)$.

As an equivalent definition, a shift space X is uniformly recurrent if for every $n \geq 1$ there is an integer $N = R_X(n)$ such that every word of $\mathcal{B}_n(X)$ occurs in every word of $\mathcal{B}_N(X)$. The function R_X is called the *recurrence function* of X.

Remark: Uniform recurrence implies recurrence

Uniform recurrence implies recurrence.

Indeed, let $u, v \in \mathcal{B}(X)$ and $n \geq 1$ such that u and v occur in every word of $\mathcal{B}_n(X)$.
Then every word w in $\mathcal{B}_{2n}(X)$ contains a block uzv for some block z, since u appears in the first half of w and v in the second half.

Minimality and uniform recurrence

Proposition

A shift space is minimal if and only if it is uniformly recurrent.

Proof.

Assume first that X is a minimal shift space and consider $u \in \mathcal{B}(X)$. Since X is minimal, the forward orbit $\mathcal{O}^+(x) = \{S^n(x) \mid n \geq 0\}$ of every $x \in X$ is dense, and thus the integer $n(x) = \min\{n > 0 \mid S^n x \in [u]_X\}$ exists.
The map $x \mapsto n(x)$ is continuous since the set of x such that $n(x) = n$ is the open set $S^{-n}([u]_X) \setminus \cup_{i=1}^{n-1} S^{-i}([u]_X)$. Since the map $x \mapsto n(x)$ is continuous on a compact space, the integers $n(x)$ are bounded. Then u occurs in every word $w \in \mathcal{B}(X)$ of length $|u| + \max n(x)$. Thus, X is uniformly recurrent.
Conversely, if X is uniformly recurrent, the orbit of every $x \in X$ is dense, and thus X is minimal. $\square$

Primitive substitution shifts are minimal

Proposition

Let $\sigma \colon A^* \to A^*$ be a substitution distinct from the identity on a one-letter alphabet. If σ is primitive, then it is growing, and $\mathsf{X}(\sigma)$ is minimal. The converse is true if, additionally, every letter is in $\mathcal{B}(X)$.

Proof.

Let $\sigma \colon A^* \to A^*$ be primitive. Since the trivial case $A = \{a\}$ and $\sigma(a) = a$ is excluded, we have $\mathcal{B}(\mathsf{X}(\sigma)) = \mathcal{L}(\sigma)$.
Let $n \geq 1$ be such that every $b \in A$ occurs in every $\sigma^n(a)$ for $a \in A$. $\square$

Examples

Example

The Fibonacci substitution $\sigma : a \mapsto ab, b \mapsto a$ is primitive. According to the proposition, the Fibonacci shift $\mathsf{X}(\sigma)$ is minimal.

Example

The Thue-Morse substitution $\sigma : a \mapsto ab, b \mapsto ba$, is primitive. Accordingly to the proposition, the Thue-Morse shift $\mathsf{X}(\sigma)$ is minimal.

Prolongable

A substitution $\sigma\colon A^* \to A^*$ is *prolongable* (or *right prolongable*) on $u \in A^+$ if $\sigma(u)$ begins with u and u is growing.

In this case, there is a unique right-infinite sequence, denoted $\sigma^\omega(u)$ such that each $\sigma^n(u)$ is a prefix of $\sigma^\omega(u)$.

One has, of course $\sigma^\omega(u) = \lim_{n\to\infty} \sigma^n(u)$.

Note also that $\sigma^\omega(u)$ is a right-infinite fixed point of σ.

Return words

Proposition

A shift space X is uniformly recurrent if and only if it is irreducible, and for every $u \in \mathcal{B}(X)$ the set of return words to u is finite.

Proof.

Assume first that X is uniformly recurrent. Let $u \in \mathcal{B}_n(X)$ and let $v \in \mathcal{B}(X)$ be of length $R_X(n) - n + 1$ with $vu \in \mathcal{B}(X)$. Then vu has length $R_X(n) + 1$ and thus u has a second occurrence in vu. This shows that v has a suffix in $\mathcal{R}_X(u)$. Thus $\max\{|w| + n - 1 \mid w \in \mathcal{R}_X(u), u \in \mathcal{B}_n(X)\} \leq R_X(n)$ and $\mathcal{R}_X(u)$ is finite.

$\square$

Computation of $\mathcal{R}_X(u)$ when $X = \mathsf{X}(\sigma)$ is minimal, u is a **prefix** of a fixed point x of σ and $w \in \mathcal{R}_X(u)$.
The word w can be an arbitrary element of $\mathcal{R}_X(u)$, for instance the prefix of x in $\mathcal{R}_X(u)$.

Computation of the return words of prefixes of a fixed point

ReturnWords(u, w)

▷ u is a prefix of $x = \sigma^\omega(a)$ and $w \in \mathcal{R}_X(u)$
▷ Returns in R the set $\mathcal{R}_X(u)$
$R \leftarrow \emptyset$
$S \leftarrow \{w\}$
▷ S is the set of return words to be processed
while $S \neq \emptyset$ **do**
    $r \leftarrow$ an element of S
    $S \leftarrow S \setminus \{r\}$
    $R \leftarrow R \cup \{r\}$
    $r(1), \cdots, r(k) \leftarrow \sigma(r)$
    ▷ The words $r(i)$ are the decomposition of $\sigma(r)$ in return words to u
    for $i \leftarrow 1$ **to** k **do**
        if $r(i) \notin R \cup S$ **then**
            $S \leftarrow S \cup r(i)$
return R

Example

Let $\sigma\colon a \mapsto ab, b \mapsto ba$ be the Thue-Morse substitution.
$\sigma^\omega(a) = abbabaabbaababba\ldots$.
$u = ab$.
$w = abb$. $S = \{abb\}$.

1. $r = abb$. $S = \emptyset$. $R = \{abb\}$. $\sigma(abb) = abb\,aba$. $S = \{aba\}$
2. $r = aba$. $S = \emptyset$. $R = \{abb, aba\}$. $\sigma(aba) = abba\,ab$.
   $S = \{abba, ab\}$
3. $r = ab$. $S = \{abba\}$. $R = \{abb, aba, abba, ab\}$.
   $\sigma(ab) = abba$. $S = \{abba\}$
4. $r = abba$. $S = \emptyset$. $R = \{abb, aba, abba, ab\}$.
   $\sigma(abba) = abb\,aba\,ab$. $S = \emptyset$

Thus, $\mathcal{R}_X(ab) = \{ab, aba, abb, abba\}$.

Block complexity

The *block complexity*, or just *complexity*, of a shift space X is the sequence $(p_X(n))_{n\geq 0}$ with $p_X(n) = \mathrm{Card}(\mathcal{B}_n(X))$.

We also write $p_x(n) = \mathrm{Card}(\mathcal{B}_n(x))$ for an individual sequence x.

Theorem (Morse, Hedlund)

Let x be a two-sided sequence. The following conditions are equivalent.

(i) *For some $n \geq 1$, one has $p_x(n) \leq n$.*

(ii) *For some $n \geq 1$, one has $p_x(n) = p_x(n+1)$.*

(iii) *x is periodic.*

Moreover, in this case, the least period of x is $\max p_x(n)$.

Linear recurrence

A shift space is *linearly recurrent* if it is minimal and if there is an integer $n \geq 1$ and a real number $K \geq 0$ such that, for every $u \in \mathcal{B}_{\geq n}(X)$, the length of every return word to u in X is bounded by $K|u|$.
We say that X is (K, n)-linearly recurrent.

We say that X is linearly recurrent with constant K. We say that X is linearly recurrent if it is K-linearly recurrent for some $K \geq 1$.

The lower bound of the numbers K such that X is K-linearly recurrent is called the *minimal constant* of linear recurrence.

Primitive substitution shifts are linearly recurrent

Proposition

A primitive substitution shift $\mathsf{X}(\sigma)$ is linearly recurrent.

Proposition

A primitive substitution shift $\mathsf{X}(\sigma)$ is linearly recurrent with minimal constant $K(\sigma) \leq kR|\sigma|$, where k is such that $|\sigma^n| \leq k\langle\sigma^n\rangle$ for all $n \geq 1$ and R is the maximal length of a return word to a word of $\mathcal{B}_2(\mathsf{X}(\sigma))$.

Block complexity of primitive substitution shifts

Proposition

If $\sigma : A^* \to A^*$ is a primitive substitution that is not the identity on a one-letter alphabet and such that $X = \mathrm{X}(\sigma)$ is not periodic, then $p_X(n) = \Theta(n)$.

Proof.

Since X is not periodic, we have $p_X(n) \geq n+1$ for every $n \geq 1$ by the Morse-Hedlund theorem. Thus $p_X(n) = \Omega(n)$. □

Block complexity of linearly recurrent shift

Proposition

Every linearly recurrent shift has at most linear complexity. More precisely, a shift X is (K, n_0)-linearly recurrent if and only if, for $n \geq n_0$, every word of $\mathcal{B}_n(X)$ occurs in every word of $\mathcal{B}_m(X)$ when $m > (K+1)n - 2$. In this case, $p_X(n) \leq Kn$ for every $n \geq n_0$.

Master 2 Mathematics and Computer Science
Symbolic Dynamics. Lecture 6

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Let $\sigma\colon A^* \to B^*$ be a substitution. A *σ-representation* of $y \in B^{\mathbb{Z}}$ is a pair (x, k) of a sequence $x \in A^{\mathbb{Z}}$ and an integer k such that

$$y = S^k(\sigma(x)). \tag{1}$$

The σ-representation (x, k) is *centered* if $0 \le k < |\sigma(x_0)|$.

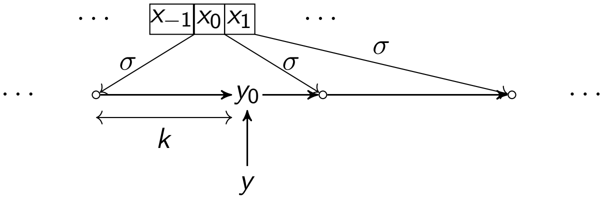

Figure: A centered σ-representation (x, k) of y.

Note, in particular, that a centered σ-representation (x, k) is such that $\sigma(x_0) \neq \varepsilon$.

σ-representation

Note that if y has a (not necessarily centered) σ-representation (x, ℓ), then it has also a centered σ-representation (x', k), where x' a shift of x.

Indeed, assume $\ell \geq 0$ (the case $\ell < 0$ is symmetric). Let $i \geq 0$ be such that $|\sigma(x_0 \cdots x_{i-1})| \leq \ell < |\sigma(x_0 \cdots x_i)|$. Set $k = \ell - |\sigma(x_0 \cdots x_{i-1})|$ and $x' = S^i x$. Then $S^k \sigma(x') = S^{k+|\sigma(x_0 \cdots x_{i-1})|}\sigma(x) = S^\ell \sigma(x) = y$ and $0 \leq k < |\sigma(x'_0)|$. Thus, (x', k) is a centered σ-representation of y.

For a shift space X on A, the set of points in $B^{\mathbb{Z}}$ having a σ-representation (x, k) with $x \in X$ is a shift space on B, which is the closure under the shift of $\sigma(X)$.
Indeed, if (x, k) is a σ-representation of y, then $S(y)$ has the σ-representation (x', k') with

$$(x', k') = \begin{cases} (x, k+1) & \text{if } k+1 < |\sigma(x_0)| \\ (S(x), 0) & \text{otherwise.} \end{cases}$$

Recognizability

Let X be a shift space on A.
The substitution $\sigma\colon A^* \to B^*$ is *recognizable* in X if every $y \in B^{\mathbb{Z}}$ has **at most one** centered σ-representation (x, k) such that $x \in X$.

Thus, in informal terms, for a sequence y on B, there is at most one way to desubstitute y to obtain a sequence in X.

Example

The substitution $\sigma\colon a \mapsto a, b \mapsto ab, c \mapsto abb$ is recognizable in the full shift $X = \{a, b, c\}^{\mathbb{Z}}$.
Indeed, let Y be the closure under the shift of $\sigma(X)$.
Any two consecutive occurrences of a are separated by a block of zero, one or two b, which determines the rule of σ to be used for desubstitution. Formally, we have

$$\begin{aligned}
\sigma([a]_X) &= [aa]_Y, \\
\sigma([b]_X) &= [aba]_Y, \quad S\sigma([b]_X) = [a \cdot ba]_Y \\
\sigma([c]_X) &= [abba]_Y, \quad S\sigma([c]_X) = [a \cdot bba]_Y, \quad S^2\sigma([c]_X) = [ab \cdot ba]_Y
\end{aligned}$$

and these sets form a partition of Y.

A *coding substitution* for a set U of nonempty words on A is a substitution $\phi\colon B^* \to A^*$ such that its restriction to B is a bijection onto U. The set U is called a *code* if ϕ is injective and a *circular code* if ϕ is circular.

Proposition

Let X be a minimal shift space on A and let $u \in \mathcal{B}(X)$. Any coding substitution $\phi\colon B^* \to A^*$ for the set $\mathcal{R}_X(u)$ of return words to u is circular.

Proof.

Since wu contains exactly two occurrences of u for each $w \in \mathcal{R}_X(u)$, for each $y \in X$, there is a unique sequence $z = \cdots w_{-1} \cdot w_0 w_1 \cdots$ with $w_i \in \mathcal{R}_X(u)$, and a unique integer k such that $y = S^k(z)$ with $0 \leq k < |w_0|$. Since ϕ is a coding substitution, for each $w_i \in \mathcal{R}_X(u)$, there is a unique $b_i \in B$ such that $\phi(b_i) = w_i$. Hence, there is a unique $x \in B^{\mathbb{Z}}$ and k with $0 \leq k < |\phi(x_0)|$ such that $y = S^k\phi(x)$. $\square$

Proposition

Let $\sigma\colon A^* \to A^*$ be a substitution. Every point y in $\mathsf{X}(\sigma)$ has a σ-representation $y = S^i(\sigma(x))$ for some $i \geq 0$, and x in $\mathsf{X}(\sigma)$.

Elementary substitution

A substitution $\sigma\colon A^* \to C^*$ is *elementary* if for every alphabet B and every pair of substitutions $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} C^*$ such that $\sigma = \alpha \circ \beta$, one has $\mathrm{Card}(B) \geq \mathrm{Card}(A)$.

In this case, one has in particular $\mathrm{Card}(C) \geq \mathrm{Card}(A)$.

Moreover, σ is non-erasing (Exercise).

Note that the property of being elementary is decidable.
Indeed, if $\sigma \colon A^* \to C^*$ is a substitution consider the finite family $\mathcal{F}$ of sets $U \subset C^*$ such that $\sigma(A) \subset U^* \subset C^*$ with every $u \in U$ occurring in some $\sigma(a)$ for $a \in A$.
Then σ is elementary if and only if $\mathrm{Card}(U) \geq \mathrm{Card}(A)$ for every $U \in \mathcal{F}$.

Elementary substitution

Proposition

Let $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} C^*$ be substitutions. If $\alpha \circ \beta$ is elementary, then β is elementary.

Proof.

Let $A^* \xrightarrow{\gamma} D^* \xrightarrow{\delta} B^*$ be such that $\beta = \delta \circ \gamma$. Then $\alpha \circ \beta = \alpha \circ (\delta \circ \gamma) = (\alpha \circ \delta) \circ \gamma$. This implies $\mathrm{Card}(D) \geq \mathrm{Card}(A)$. Thus β is elementary. $\square$

Elementary substitution

A sufficient condition for a substitution to be elementary can be formulated in terms of its composition matrix.

Proposition

If the rank of $M(\sigma)$ is equal to $\mathrm{Card}(A)$, then σ is elementary.

Proof.

Indeed, if $\sigma = \alpha \circ \beta$ with $\beta \colon A^* \to B^*$ and $\alpha \colon B^* \to C^*$, then $M(\sigma) = M(\alpha)M(\beta)$. If $\mathrm{rank}(M(\sigma)) = \mathrm{Card}(A)$, then

$$\mathrm{Card}(A) = \mathrm{rank}(M(\sigma)) \leq \mathrm{rank}(M(\alpha)) \leq \mathrm{Card}(B).$$

Thus σ is elementary. □

This condition is not necessary. For example, the Thue-Morse substitution $\sigma \colon a \mapsto ab, b \mapsto ba$ is elementary, but its composition matrix has rank one.

Elementary substitution

If $\sigma\colon A^* \to C^*$ is a substitution, we define

$$\ell(\sigma) = \sum_{a \in A} (|\sigma(a)| - 1). \tag{2}$$

We say that a decomposition $\sigma = \alpha \circ \beta$ with $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ is *trim* if

(i) α is non-erasing,

(ii) for each $b \in B$ there is an $a \in A$ such that $\beta(a)$ contains b.

Proposition

Let $\sigma = \alpha \circ \beta$ with $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ be a trim decomposition of σ. Then

$$\ell(\alpha \circ \beta) \geq \ell(\alpha) + \ell(\beta). \tag{3}$$

By a symmetric version, an elementary substitution $\sigma\colon A^* \to C^*$ is injective on $A^{-\mathbb{N}}$. Since a substitution which is injective on $A^{\mathbb{N}}$ and on $A^{-\mathbb{N}}$ is injective on $A^{\mathbb{Z}}$, we obtain the following corollary of Proposition 6.

Proposition

An elementary substitution $\sigma\colon A^* \to C^*$ is injective on $A^{\mathbb{Z}}$.

A substitution $\sigma\colon A^* \to B^*$ is *recognizable in X for aperiodic points* if **every aperiodic point** $y \in B^{\mathbb{Z}}$ has at most one centered representation **in** X.

We say that σ is *fully recognizable for aperiodic points* if it is recognizable in the full shift for aperiodic points.

Aperiodic substitution

A substitution σ is *aperiodic* if $X(\sigma)$ contains no periodic point.

Theorem (B. Mossé 1992, B. Mossé 1996)

Any aperiodic substitution is recognizable in $X(\sigma)$.

Recognizability for aperiodic points

Theorem (J. Karhumäki, J. Maňuch, W. Plandowski 2003)

An elementary substitution is fully recognizable for aperiodic points.

Recognizability for aperiodic points

Theorem (Berthé et al. 2018 for non-erasing substitutions, B. et al. 2022)

Any morphism $\sigma \colon A^* \to A^*$ *is recognizable for aperiodic points in* $X(\sigma)$.

Lemma

Let $\sigma \colon A^* \xrightarrow{\sigma} A^*$ *be a substitution and* $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} A^*$ *such that* $\sigma = \alpha \circ \beta$. *If* σ *is not recognizable in* $X(\sigma)$, *then* $\sigma \circ \alpha$ *is not fully recognizable. The same statement holds for the recognizability for aperiodic points.*