Master 2 Mathematics and Computer Science
Symbolic Dynamics. Lecture 6

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Let $\sigma \colon A^* \to B^*$ be a substitution. A *σ-representation* of $y \in B^{\mathbb{Z}}$ is a pair (x, k) of a sequence $x \in A^{\mathbb{Z}}$ and an integer k such that

$$y = S^k(\sigma(x)). \tag{1}$$

The σ-representation (x, k) is *centered* if $0 \leq k < |\sigma(x_0)|$.

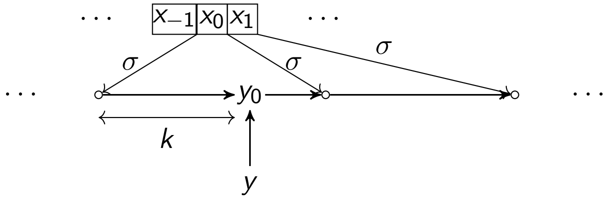

Figure: A centered σ-representation (x, k) of y.

Note, in particular, that a centered σ-representation (x, k) is such that $\sigma(x_0) \neq \varepsilon$.

σ-representation

Note that if y has a (not necessarily centered) σ-representation (x, ℓ), then it has also a centered σ-representation (x', k), where x' a shift of x.

Indeed, assume $\ell \geq 0$ (the case $\ell < 0$ is symmetric). Let $i \geq 0$ be such that $|\sigma(x_0 \cdots x_{i-1})| \leq \ell < |\sigma(x_0 \cdots x_i)|$. Set $k = \ell - |\sigma(x_0 \cdots x_{i-1})|$ and $x' = S^i x$. Then $S^k\sigma(x') = S^{k+|\sigma(x_0 \cdots x_{i-1})|}\sigma(x) = S^\ell\sigma(x) = y$ and $0 \leq k < |\sigma(x'_0)|$. Thus, (x', k) is a centered σ-representation of y.

For a shift space X on A, the set of points in $B^{\mathbb{Z}}$ having a σ-representation (x, k) with $x \in X$ is a shift space on B, which is the closure under the shift of $\sigma(X)$.
Indeed, if (x, k) is a σ-representation of y, then $S(y)$ has the σ-representation (x', k') with

$$(x', k') = \begin{cases} (x, k+1) & \text{if } k+1 < |\sigma(x_0)| \\ (S(x), 0) & \text{otherwise.} \end{cases}$$

Recognizability

Let X be a shift space on A.
The substitution $\sigma\colon A^* \to B^*$ is *recognizable* in X if every $y \in B^{\mathbb{Z}}$ has **at most one** centered σ-representation (x, k) such that $x \in X$.

Thus, in informal terms, for a sequence y on B, there is at most one way to desubstitute y to obtain a sequence in X.

Example

Example

The substitution $\sigma\colon a \mapsto a, b \mapsto ab, c \mapsto abb$ is recognizable in the full shift $X = \{a, b, c\}^{\mathbb{Z}}$.
Indeed, let Y be the closure under the shift of $\sigma(X)$.
Any two consecutive occurrences of a are separated by a block of zero, one or two b, which determines the rule of σ to be used for desubstitution. Formally, we have

$$\begin{aligned}
\sigma([a]_X) &= [aa]_Y, \\
\sigma([b]_X) &= [aba]_Y, \quad S\sigma([b]_X) = [a \cdot ba]_Y \\
\sigma([c]_X) &= [abba]_Y, \quad S\sigma([c]_X) = [a \cdot bba]_Y, \quad S^2\sigma([c]_X) = [ab \cdot ba]_Y
\end{aligned}$$

and these sets form a partition of Y.

A *coding substitution* for a set U of nonempty words on A is a substitution $\phi \colon B^* \to A^*$ such that its restriction to B is a bijection onto U. The set U is called a *code* if ϕ is injective and a *circular code* if ϕ is circular.

Proposition

Let X be a minimal shift space on A and let $u \in \mathcal{B}(X)$. Any coding substitution $\phi \colon B^* \to A^*$ for the set $\mathcal{R}_X(u)$ of return words to u is circular.

Proof.

Since wu contains exactly two occurrences of u for each $w \in \mathcal{R}_X(u)$, for each $y \in X$, there is a unique sequence $z = \cdots w_{-1} \cdot w_0 w_1 \cdots$ with $w_i \in \mathcal{R}_X(u)$, and a unique integer k such that $y = S^k(z)$ with $0 \leq k < |w_0|$. Since ϕ is a coding substitution, for each $w_i \in \mathcal{R}_X(u)$, there is a unique $b_i \in B$ such that $\phi(b_i) = w_i$. Hence, there is a unique $x \in B^{\mathbb{Z}}$ and k with $0 \leq k < |\phi(x_0)|$ such that $y = S^k \phi(x)$. $\square$

Existence of a representation

Proposition

Let $\sigma \colon A^* \to A^*$ be a substitution. Every point y in $\mathsf{X}(\sigma)$ has a σ-representation $y = S^i(\sigma(x))$ for some $i \geq 0$, and x in $\mathsf{X}(\sigma)$.

Elementary substitution

A substitution $\sigma\colon A^* \to C^*$ is *elementary* if for every alphabet B and every pair of substitutions $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} C^*$ such that $\sigma = \alpha \circ \beta$, one has $\mathrm{Card}(B) \geq \mathrm{Card}(A)$.

In this case, one has in particular $\mathrm{Card}(C) \geq \mathrm{Card}(A)$.

Moreover, σ is non-erasing (Exercise).

Elementary substitution

Note that the property of being elementary is decidable.
Indeed, if $\sigma\colon A^* \to C^*$ is a substitution consider the finite family $\mathcal{F}$ of sets $U \subset C^*$ such that $\sigma(A) \subset U^* \subset C^*$ with every $u \in U$ occurring in some $\sigma(a)$ for $a \in A$.
Then σ is elementary if and only if $\mathrm{Card}(U) \geq \mathrm{Card}(A)$ for every $U \in \mathcal{F}$.

Elementary substitution

Proposition

Let $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} C^*$ be substitutions. If $\alpha \circ \beta$ is elementary, then β is elementary.

Proof.

Let $A^* \xrightarrow{\gamma} D^* \xrightarrow{\delta} B^*$ be such that $\beta = \delta \circ \gamma$. Then $\alpha \circ \beta = \alpha \circ (\delta \circ \gamma) = (\alpha \circ \delta) \circ \gamma$. This implies $\mathrm{Card}(D) \geq \mathrm{Card}(A)$. Thus β is elementary. $\square$

Elementary substitution

A sufficient condition for a substitution to be elementary can be formulated in terms of its composition matrix.

Proposition

If the rank of $M(\sigma)$ is equal to $\mathrm{Card}(A)$, then σ is elementary.

Proof.

Indeed, if $\sigma = \alpha \circ \beta$ with $\beta \colon A^* \to B^*$ and $\alpha \colon B^* \to C^*$, then $M(\sigma) = M(\alpha)M(\beta)$. If $\mathrm{rank}(M(\sigma)) = \mathrm{Card}(A)$, then

$$\mathrm{Card}(A) = \mathrm{rank}(M(\sigma)) \leq \mathrm{rank}(M(\alpha)) \leq \mathrm{Card}(B).$$

Thus σ is elementary. $\square$

This condition is not necessary. For example, the Thue-Morse substitution $\sigma \colon a \mapsto ab, b \mapsto ba$ is elementary, but its composition matrix has rank one.

If $\sigma\colon A^* \to C^*$ is a substitution, we define

$$\ell(\sigma) = \sum_{a \in A} (|\sigma(a)| - 1). \tag{2}$$

We say that a decomposition $\sigma = \alpha \circ \beta$ with $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ is *trim* if

(i) α is non-erasing,

(ii) for each $b \in B$ there is an $a \in A$ such that $\beta(a)$ contains b.

Proposition

Let $\sigma = \alpha \circ \beta$ with $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ be a trim decomposition of σ. Then

$$\ell(\alpha \circ \beta) \geq \ell(\alpha) + \ell(\beta). \tag{3}$$

Elementary substitution

By a symmetric version, an elementary substitution $\sigma \colon A^* \to C^*$ is injective on $A^{-\mathbb{N}}$. Since a substitution which is injective on $A^{\mathbb{N}}$ and on $A^{-\mathbb{N}}$ is injective on $A^{\mathbb{Z}}$, we obtain the following corollary of Proposition 6.

Proposition

An elementary substitution $\sigma \colon A^* \to C^*$ is injective on $A^{\mathbb{Z}}$.

Recognizability for aperiodic points

A substitution $\sigma\colon A^* \to B^*$ is *recognizable in X for aperiodic points* if **every aperiodic point** $y \in B^{\mathbb{Z}}$ has at most one centered representation **in** X.

We say that σ is *fully recognizable for aperiodic points* if it is recognizable in the full shift for aperiodic points.

Aperiodic substitution

A substitution σ is *aperiodic* if $X(\sigma)$ contains no periodic point.

Theorem (B. Mossé 1992, B. Mossé 1996)

Any aperiodic substitution is recognizable in $X(\sigma)$.

Theorem (J. Karhumäki, J. Maňuch, W. Plandowski 2003)

An elementary substitution is fully recognizable for aperiodic points.

Recognizability for aperiodic points

Theorem (Berthé et al. 2018 for non-erasing substitutions, B. et al. 2022)

Any morphism $\sigma\colon A^* \to A^*$ is recognizable for aperiodic points in $X(\sigma)$.

Lemma

Let $\sigma : A^* \xrightarrow{\sigma} A^*$ *be a substitution and* $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} A^*$ *such that* $\sigma = \alpha \circ \beta$. *If* σ *is not recognizable in* $X(\sigma)$, *then* $\sigma \circ \alpha$ *is not fully recognizable. The same statement holds for the recognizability for aperiodic points.*