Université Gustave Eiffel
M2 – Semester 1 – 2025/2026

Probability - Exercise Sheet 2 - Random Variables and Expectation

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Exercise 1. (Number of Hamiltonian paths in a tournament.) A tournament T_n is a orientation of the edges of K_n. We say that T_n admits a Hamiltonian path if there exists $\sigma \in \mathcal{S}_n$ such that $(\sigma(1), \sigma(2)), \ldots, (\sigma(n-1), \sigma(n)) \in T_n$.

We want to show that for every integer n there exists one tournament with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

Let us choose a random tournament T_n. For any $\sigma \in \mathcal{S}_n$, let A_σ be the event that $(\sigma(1), \sigma(2)), \ldots, (\sigma(n-1), \sigma(n)) \in T_n$. Let X be the number of Hamiltonian paths.

1. Give an expression of X in terms of the events A_σ.
2. Prove that for every $\sigma \in \mathcal{S}_n$, $\mathbb{P}(A_\sigma) = \frac{1}{2^{n-1}}$.
3. Compute $\mathbb{E}(X)$.
4. Prove the expected result.

Solution. To be precise, the sample space Ω is the set of all tournaments, the σ-algebra $\mathcal{A}$ is the power set of Ω, and the probability measure $\mathbb{P}$ is defined as $\mathbb{P}(\{T_n\}) = \frac{1}{2^{\binom{n}{2}}}$, for every tournament T_n. Also, $X : \Omega \to \mathbb{N}$.

1. We have $X = \sum_{\sigma \in \mathcal{S}_n} \mathbf{1}_{A_\sigma}$.
2. For every T_n such that $(i, j) \in T_n$, there exists exactly one T'_n such that $(j, i) \in T'_n$, and vice versa. Hence, there are as many tournaments containing the edge (i, j) as those containing the edge (j, i). Half of the tournaments contain the edge (i, j) and the other half contain the edge (j, i). Therefore, for every $i, j \in [n]$, we have

   $$\mathbb{P}((i, j) \in T_n) = \mathbb{P}((j, i) \in T_n) = \frac{1}{2}.$$

   Furthermore, for every $\sigma \in \mathcal{S}_n$, the edges $(\sigma(1), \sigma(2)), \ldots, (\sigma(n-1), \sigma(n))$ are distinct and the orientations of the edges are independent. Therefore,

   $$\mathbb{P}(A_\sigma) = \mathbb{P}((\sigma(1), \sigma(2)) \in T_n, \ldots, (\sigma(n-1), \sigma(n)) \in T_n) = \prod_{i=1}^{n-1} \mathbb{P}((\sigma(i), \sigma(i+1)) \in T_n) = \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}}.$$

3. We have $\mathbb{E}(X) = \sum_{\sigma \in \mathcal{S}_n} \mathbb{E}(\mathbf{1}_{A_\sigma}) = \sum_{\sigma \in \mathcal{S}_n} \mathbb{P}(A_\sigma) = n! \frac{1}{2^{n-1}} = \frac{n!}{2^{n-1}}$.
4. Since $\mathbb{E}(X) = \frac{n!}{2^{n-1}} \leq \max_{T_n \in \Omega} X(T_n)$, there exists at least one tournament whose the number of Hamiltonian paths is greater than $\frac{n!}{2^{n-1}}$.