Université Gustave Eiffel
M2 – Semester 1 – 2025/2026

Analysis - Exercises

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Exercise 26. Let H be a subgroup of the additive group $(\mathbb{R}, +)$. Assume H is not reduced to $\{0\}$. Denote $H_+ = \{s \in H \mid s > 0\}$.

(a) Show that H_+ admits an infimum α in $\mathbb{R}_+$.

(b) Show that whenever $\alpha > 0$ then $\alpha \in H_+$.

(c) Deduce that whenever $\alpha > 0$ then $H = \alpha\mathbb{Z}$.

(d) Show that whenever $\alpha = 0$ then H is dense in $\mathbb{R}$.

(e) **Application:** Prove that $B = \{\cos(n) \mid n \in \mathbb{Z}\}$ is dense in $[-1, 1]$.

Solution.

(a) Since H is not reduced to $\{0\}$, there is an element $h \in H$ such that $h \neq 0$. If $h > 0$, then $h \in H_+$. If $h < 0$, then $-h \in H$ and $-h > 0$, which means that $-h \in H_+$. In both cases, H_+ is non-empty. Moreover, H_+ is bounded below by 0. Since $H_+ \subset \mathbb{R}_+$, by the completeness property of $\mathbb{R}$, H_+ admits an infimum α in $\mathbb{R}_+$.

(b) Suppose that $\alpha \notin H_+$. Then, for every $h \in H_+$, we have $h > \alpha$ or $h - \alpha > 0$. Since H is a subgroup of $(\mathbb{R}, +)$, we have $h - \alpha \in H$. Thus, $h - \alpha \in H_+$. This means that for every $h \in H_+$, there exists an element $h - \alpha \in H_+$ such that $h - \alpha < h$. This contradicts the fact that α is a lower bound of H_+. From this fact, we build a sequence (h_n) in H_+ such that $h = h_0$ is an arbitrary element of H_+ and $h_{n+1} = h_n - \alpha$. This sequence is decreasing and bounded below by α. Thus, it converges to a limit $l \geq \alpha$. However, taking the limit on both sides of the recurrence relation, we get $l = l - \alpha$, which implies that $\alpha = 0$. This contradicts the assumption that $\alpha > 0$. Therefore, $\alpha \in H_+$.

(c) Let $x \in H$. Consider the case $x \geq 0$. Take $n = \lfloor x/\alpha \rfloor \in \mathbb{Z}_{\geq 0}$ and set $r = x - n\alpha$. Then $0 \leq r < \alpha$ and $r \in H$ (since H is a subgroup). If $r > 0$, then $r \in H+$ contradicts the minimality of α in H_+. Hence $r = 0$ and $x = n\alpha$. For $x < 0$, we use the same argument to have $-x = n\alpha$ for some $n \in \mathbb{Z}_+$. Therefore, $H \subset \alpha\mathbb{Z}$. The reverse inclusion is obvious since $\alpha \in H$. Thus, $H = \alpha\mathbb{Z}$.

(d) We will show that for every $x, y \in \mathbb{R}$ such that $0 \leq x < y$, there exists an element $h \in H_+$ such that $x < h < y$. Let $d = y - x \in H$. Since $\alpha = 0 = \inf H_+$, there exists an element $h' \in H$ such that $0 < h' < d$. Choose $n = \left\lfloor \frac{x}{h} \right\rfloor + 1 \in \mathbb{Z}$. Then,

$$x < h'n \leq x + h' < x + d = y.$$

Thus, $h = h'n \in H_+$ satisfies $x < h < y$.

Now we show that every $x \in \mathbb{R}$ is a limit of a sequence of elements of H. Consider the case $x \geq 0$. Using the previous argument, for every $n \in \mathbb{N}^*$, there exists h_n in H such that $x < h_n < x + \frac{1}{n}$. Thus, the sequence (h_n) converges to x. This shows that H is dense in $\mathbb{R}$. For $x < 0$, we use the same argument to show that there exists a sequence (h_n) in H which converges to $-x$. Then, the sequence $(-h_n)$ converges to x. Thus, H is dense in $\mathbb{R}$.

(e) Let $G = \mathbb{Z} + 2\pi\mathbb{Z} = \{m + 2\pi n \mid m, n \in \mathbb{Z}\}$ be a subgroup of $(\mathbb{R}, +)$. If $G = \alpha\mathbb{Z}$ for some $\alpha > 0$, then since $1 \in G$ and $2\pi \in G$, there exist $a, b \in \mathbb{Z}$ such that $1 = a\alpha$ and $2\pi = b\alpha$. Thus, $\pi = \dfrac{b}{2a} \in \mathbb{Q}$, a contradiction. Therefore, from (c), we have $\inf G_+ = 0$ and from (d), G is dense in $\mathbb{R}$. We claim that $G' = \{n \mod 2\pi \mid n \in \mathbb{Z}\}$ is dense in $[0, 2\pi)$. Indeed, for every $x \in [0, 2\pi)$, there exists a sequence (g_n) in G which converges to x. Therefore, the sequence $(g_n \mod 2\pi)$ in G' converges to x. Since the function cos is continuous, we have $\{\cos(n) \mid n \in \mathbb{Z}\} = \{\cos(n) \mid n \in G'\}$ is dense in $[\cos(0), \cos(2\pi)] = [-1, 1]$.