Symbolic Dynamics
Exercises

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1 Study of an infinite word defined by a substitution

Let $A = \{a, b\}$ and let $\sigma : A \to A^*$ be the substitution defined by

$$\sigma(a) = abaa \qquad \text{and} \qquad \sigma(b) = babb.$$

1. How many fixed points does the substitution σ admit in $A^{\mathbb{N}}$? In the rest, let u denote the fixed point of σ that begins with a.

2. Show that the sequence u is uniformly recurrent.

3. Show that the set of its factors is invariant under exchanging the letters a and b.

4. List the factors of u of length at most 6. List the right special factors of u of length 5. Draw the Rauzy graphs of u of order 1 to 5. What does the complexity of u seem to be?

5. Is the set of factors of u invariant under reversal (mirror image)?

6. The goal of this question is to prove the following decomposition lemma: every factor W of u of length at least 5 can be written uniquely as

$$W = R\,\sigma(V)\,S$$

where V is a factor of u, $R \in \{\varepsilon, a, aa, baa, b, bb, abb\}$, and $S \in \{\varepsilon, a, ab, aba, b, ba, bab\}$.

   (a) **Existence.** Show that if $u = u_0 u_1 \cdots u_n$, then

   $$\sigma(u_n) = u_{4n} u_{4n+1} u_{4n+2} u_{4n+3}.$$

   Deduce the existence of such a decomposition.

(b) **Uniqueness.** Show that R is determined by a prefix of W of length at most 5; to do this, by a case analysis, show that for each possible value of R, that value determines a prefix of W of length at most 5. Conclude by observing that determining R implies determining S.

7. Show that if W is a right special factor of u of length at least 5, then W decomposes uniquely as

$$W = R\sigma(V),$$

where V is a right special factor of u and $R \in \{\varepsilon, a, aa, baa, b, bb, abb\}$. You may use the fact that the suffix of W of length 5 is, in particular, right special.

8. Show that for every $n \geq 2$, u has the same number of right special factors of length n. Give this number.

9. Deduce the complexity function of u.

2 Complexity functions of the form $p(n) = n+k$, for all n

1. Let $k \geq 1$. Construct, from the Fibonacci word, an example u of a sequence whose complexity function $p_u(n)$ satisfies

$$p_u(n) = n + k, \qquad \text{for all } n.$$

2. Let u be a sequence over the alphabet A such that every letter of A appears in u. Show that if the complexity function of u satisfies

$$\exists n \in \mathbb{N}, \quad p_u(n) < n + \mathrm{Card}(A) - 1,$$

then u is ultimately periodic.

3. Let $k \geq 1$. Throughout this question, $u = (u_n)_{n\in\mathbb{N}}$ is a non-recurrent sequence whose complexity function satisfies

$$p_u(n) = n + k, \qquad \text{for all } n.$$

(a) Let $a = u_0$ be the first letter of u. The goal is to prove by contradiction that the letter a does not extend to the left in u. Assume, on the contrary, that a extends to the left in u. What are the two possible shapes of the order-1 word graph of u? Deduce that the $k+1$ letters of A appear infinitely often in u. Deduce from the non-recurrence of u and Question 2 a contradiction.

(b) What is the complexity of the sequence $(u_n)_{n\geq 1}$?

(c) Deduce that if u is non-recurrent, then there exist $l \geq 1$, a subalphabet A_0 of A, letters $a_1, a_2, \ldots, a_l \in A$ such that $A = A_0 \cup \{a_1, \ldots, a_l\}$ and a recurrent sequence w over the alphabet A_0 such that

$$u = a_1 a_2 \cdots a_l w.$$

What is the complexity of w?

3 Central words

1. Give the Christoffel word of slope 5/4 over the alphabet $\{x < y\}$. Give its Cayley graph. Give its dual Christoffel word.

2. Give the inverses of 5 and 4 modulo 9. Construct all non-constant words of periods 5 and 4 of length 7.

3. Show that if two words U and V satisfy $UV = VU$, then there exist a word W and $i, j \in \mathbb{N}$ such that $U = W^i$ and $V = W^j$. Give an inductive proof and a proof based on the Fine–Wilf theorem.

4 Sturmian words

Show that the set of Sturmian factors is closed under reversal (mirror image).

5 Episturmian words

An infinite word is called *episturmian* if its language is closed under reversal and it has a unique right special factor of each length. Show that frequencies are uniform for an episturmian word.

6 Chacon word

The Chacon word $u = \sigma^\infty(0)$ is defined over $\{0, 1\}$ by the substitution $\sigma : 0 \mapsto 0010,\ 1 \mapsto 1$. Show that the Chacon word is uniformly recurrent. *Hint:* Show that the words b_n defined by

$$b_0 = 1, \qquad b_{n+1} = b_n b_n 1 b_n \quad \text{for all } n$$

are prefixes of u. Show that frequencies are uniform in the Chacon word.

7 Frequencies

Consider the substitution $\sigma : 0 \mapsto 0011,\ 1 \mapsto 0101$. Let $u = \sigma^\infty(0)$. Show that $(O(u), T)$ is minimal and uniquely ergodic. Do factor frequencies exist? Show that the values of frequencies on cylinder sets are not determined by their values on letter-cylinder sets.