This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TSP.2016.2557304, IEEE Transactions on Signal Processing

Recursive Weighted Myriad Based Filters and their Optimizations

Juan Marcos Ramirez, Jose Luis Paredes

***Abstract*—This paper proposes two new recursive filtering structures based on the nonlinear myriad operator. First, we develop the general class of *recursive weighted myriad* (RWMy) filters as a robust filtering structure against impulsive noise that includes, as particular cases, a normalized version of the linear infinite impulse response (IIR) filter, the recursive mode-type filter, and the non-recursive weighted myriad filter. Secondly, considering the fact that the additive noise that contaminates the previous filter's outputs is no longer impulsive, we introduce a novel class of *recursive hybrid myriad* (RHMy) filters whose structure gathers the advantages of both the weighted myriad and the weighted mean in a single cost function to be minimized. Some properties of the RHMy filter are derived and a fast algorithm to compute the RHMy filter output is proposed. Furthermore, adaptive algorithms for designing the proposed recursive structures, under the equation error formulation framework, are developed. Finally, extensive numerical simulations are shown to evaluate both the iterative update of the adaptive algorithms and the performance of the proposed recursive filters against impulsive noise.**

***Index Terms*—Adaptive filtering, equation error formulation, nonlinear filtering, recursive hybrid myriad, recursive weighted myriad.**

I. Introduction

INFINITE impulse response (IIR) filters are recursive structures that provide a notably better performance than the non-recursive counterpart (finite impulse response, FIR, filters) for the same number of filter coefficients. Indeed, a desired filtering operation can be more accurately obtained using a recursive IIR structure with much fewer filter coefficients compared to the corresponding FIR filter designed for the same filtering task, reducing thus, the computational complexity in terms of number of operations to be performed [1]. Furthermore, the IIR filters are optimal structures under the assumption that the underlying additive noise that contaminates the target signal follows a Gaussian distribution, and, it is well-known that these filters degrade severely their performance in presence of impulsive noise.

Filtering structures based on the median operator are probably the most used nonlinear filtering approach for removing impulsive noise. Under the maximum likelihood (ML) criterion, these filters are optimal when the underlying contamination follows a Laplacian distribution. A recursive structure based on the median operator has been considered to enhance signal features [2], [3], where this recursive filter has been tested mainly in image processing applications [4], [5]. Furthermore, the median filter has been extended to the general class of weighted median filters admitting negative-valued weights, whose inherent structure is able to perform frequency-selective filtering operations [6]. These filters have been used in various signal and image processing applications [7]–[9]. Additionally, a recursive structure based on the weighted median operator has been reported in [10], where its adaptive algorithms are developed under the threshold decomposition framework and the equation error formulation approach. However, median based filters lose efficiency with respect to the linear filters when the underlying contamination approaches to the Gaussian noise [11]. Recently, the class of iterative truncated arithmetic mean (ITM) filters has been developed [12]–[15], that iteratively truncates the outliers, approaching to the median output as the number of iterations increases.

On the other hand, weighted myriad filters are nonlinear filtering structures based on the myriad operator [16], [17], whose output reduces to the ML estimate of location parameter of a set of samples that follow a Cauchy distribution. In general, due to its capability to perform frequency selective filtering operations and its potential in the rejection of a wide variety of additive noise, the class of weighted myriad filters that admits real-valued weights [18], encloses a general family of nonlinear filtering structures successfully used in a wide range of applications [19]–[23].

In this paper, weighted myriad filters are extended to the recursive weighted myriad (RWMy) filters. Interestingly, this class of nonlinear filtering structures includes the linear IIR filter, the outlier-resistant mode-type filter and the non-recursive weighted myriad filter as special cases, since the proposed recursive structures are equipped with tuning parameters that allow to select both the rejection level to impulsive noise and the filtering structure (recursive or non-recursive). Furthermore, considering the fact that the myriad estimate asymptotically follows a normal distribution [24]–[27], we also present the recursive hybrid myriad (RHMy) filters that take advantage of both the rejection capability to impulsive contamination of the weighted myriad and the suitable behavior to Gaussian noise of the linear weighted mean, in a unified objective cost function to be minimized. Furthermore, an algorithm based on fixed-point search is proposed for obtaining the output of this recursive hybrid structure.

Additionally, adaptive algorithms for determining the optimal filter parameters of the proposed recursive filtering structures are developed. These optimization algorithms find the filter parameters minimizing the mean absolute error (MAE) between the filter output and a desired signal. Furthermore, in order to avoid the inherent feedback in the proposed recursive structures, the equation error formulation approach [1], [10] is

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This research is supported by the Council of Scientific, Humanistic, Technological, and Artistic Development of the Universidad de Los Andes (CDCHTA-ULA) under the project I-1336-12-02-B

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used as a recursive decoupling strategy during the filter design stage. The performance of each proposed recursive filter is evaluated by extensive computer simulations. The proposed recursive structures outperform the non-recursive weighted myriad filter with the same number of filter coefficients. It is also observed that the filter output in the bandpass follows closely the shape of the desired signal whereas the cut-off band is highly attenuated. Furthermore, the performance of each proposed filter are tested in the presence of severe impulsive noise, where the outputs of the recursive structures exhibit superior performance compared to others nonlinear filters [10], [12], [14], [15], [18], [28], [29].

II. Preliminary background

Weighted myriad filters are derived from the *sample myriad* operator that is defined as the ML estimate of location parameter for a set of observation samples that follows a Cauchy distribution. Therefore, under the context of robust filtering, the *myriad filter* is specified as a sliding window filter whose output is the sample myriad of the input signal samples inside the observation window.

However, in order to exploit both the statistical relationships and the temporal correlations among observation samples, a more general filtering structure, that includes weights, has been developed in [20]. An extension of this weighted structure, suitable for frequency selective filtering applications, is the general class of *weighted myriad filters (WMy)* admitting real-valued weights proposed by Kalluri and Arce in [18]. More precisely, given a set of N observation samples $\{x_i|_{i=1}^{N}\}$ and a set of weights $\{w_i|_{i=1}^{N}\}$, the output of the weighted myriad filter $\hat{\boldsymbol{\beta}}_K$ is defined as

$$\hat{\boldsymbol{\beta}}_K \triangleq \text{myriad}\left(|w_i| \circ \text{sgn}(w_i)x_i\Big|_{i=1}^{N}; K\right) = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{N} \log\left[K^2 + |w_i|(\text{sgn}(w_i)x_i - \boldsymbol{\beta})^2\right], \quad (1)$$

where $\circ$ denotes the weighting operation; K is a tunable parameter, referred to as *linearity parameter*, that defines the rejection capability to impulsive noise of the WMy operator; and $\text{sgn}(w_i)$ is the sign function defined as $+1$ if $w_i \geq 0$, -1 otherwise. Interestingly, the WMy filter includes the linear FIR filter, when $K \to \infty$, and the outlier-resistant mode-type filter, when $K \to 0$, as particular cases [18]. An exact calculation of the WMy filter output, obtained by direct minimization of (1), is a nontrivial and prohibitively expensive computational task [30]. To overcome this drawback, fast algorithms have been developed for obtaining an estimate of the WMy filter output with high degree of accuracy, where fixed-point search based algorithms [30], the branch and bound search algorithm [31], and the sequential based algorithm [32] are just a few representative examples of this class of fast computation algorithms. This filtering structure has been successfully applied in a wide range of applications, including signal and image processing [20], [21], [23] and communications [19].

III. Recursive weighted myriad based filters

A. Recursive Weighted Myriad Filters

Given a subset of observation samples $\{x_i|_{i=-N_2}^{N_1}\} = \{x[n-i]|_{i=-N_2}^{N_1}\}$ taken from an input sequence $\{x[n]\}$ and a subset of previously computed outputs $\{y_j|_{j=1}^{M}\} = \{y[n-j]|_{j=1}^{M}\}$, whose entries are gathered together to define a recursive sliding window. Furthermore, consider a set of weights $\{g_i|_{i=-N_2}^{N_1}\}$ that specifies the varying levels of reliability of the input samples, and a set of weights $\{h_j|_{j=1}^{M}\}$ that controls the influence of the previous outputs in the current filter output. The output of the *recursive weighted myriad (RWMy) filter* is defined as

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{K_1,K_2} &= \arg\min_{\boldsymbol{\beta}} \mathcal{G}(\boldsymbol{\beta}) \\ &= \arg\min_{\boldsymbol{\beta}} \left\{ \prod_{i=-N_2}^{N_1} \left[K_1^2 + |g_i|(\text{sgn}(g_i)x_i - \boldsymbol{\beta})^2\right] \right. \\ &\cdots \left. \times \prod_{j=1}^{M} \left[K_2^2 + |h_j|(\text{sgn}(h_j)y_j - \boldsymbol{\beta})^2\right] \right\}, \quad (2)\end{aligned}$$

where $(N_1 + N_2 + 1)$ and M are, respectively, the number of input signal samples and the number of previous outputs inside the recursive sliding window; and K_1 and K_2 are tunable parameters that denote the impulsive noise rejection capability over the set of input samples and the set of previous outputs, respectively. Furthermore, as will be detailed shortly, the RWMy filter includes the normalized linear IIR filter as a particular limiting case when $K_1, K_2 \to \infty$, thus, these parameters, in the context of RWMy filter, are also referred to as *linearity parameters*.

Since the logarithm operator is a non-decreasing function, Eq. (2) can be rewritten as

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{K_1,K_2} &= \arg\min_{\boldsymbol{\beta}} \log(\mathcal{G}(\boldsymbol{\beta})) = \arg\min_{\boldsymbol{\beta}} \mathcal{H}(\boldsymbol{\beta}) \\ &= \arg\min_{\boldsymbol{\beta}} \left\{ \sum_{i=-N_2}^{N_1} \log\left[K_1^2 + |g_i|(\text{sgn}(g_i)x_i - \boldsymbol{\beta})^2\right] \right. \\ &\cdots \left. + \sum_{j=1}^{M} \log\left[K_2^2 + |h_j|(\text{sgn}(h_j)y_j - \boldsymbol{\beta})^2\right] \right\}. \quad (3)\end{aligned}$$

A non-causal structure of the proposed RWMy filter is schematically depicted in Fig. 1. As can be seen in this block diagram, the output of the RWMy filter depends on the input signal samples as well as on the feedback of the previously computed outputs. Interestingly, much like linear FIR filters can be generalized to the class of recursive linear filters, namely IIR, the class of nonlinear filters based on the weighted myriad operator can also be extended to a richer structure that includes the previously computed outputs as is highlighted in Fig. 1. A detailed study of the RWMy filter output, is developed next by applying a closer look over the objective cost function $\mathcal{H}(\boldsymbol{\beta})$:

$$\begin{aligned}\mathcal{H}(\boldsymbol{\beta}) &= \sum_{i=-N_2}^{N_1} \log\left[K_1^2 + |g_i|\,(\text{sgn}(g_i)x_i - \boldsymbol{\beta})^2\right] \\ &\cdots + \sum_{j=1}^{M} \log\left[K_2^2 + |h_j|\,(\text{sgn}(h_j)y_j - \boldsymbol{\beta})^2\right]. \quad (4)\end{aligned}$$

Note first that, the proposed recursive nonlinear filters admit real-valued weights, where the signs of the filter weights are uncoupled from their magnitudes and passed to the corresponding samples inside the recursive observation window [18]. In other words, the output of the RWMy filter, at time

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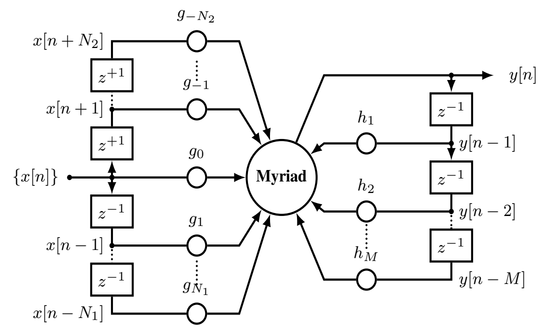

Fig. 1. Structure of the recursive weighted myriad filter.

n, is obtained using both the subset of signed input samples $\{\mathrm{sgn}(g_i)x_i|_{i=-N2}^{N_1}\}$ weighted by the magnitudes of the weights $\{|g_i||_{i=-N_2}^{N_1}\}$ and the subset of signed previous outputs $\{\mathrm{sgn}(h_j)y_j|_{j=1}^{M}\}$ weighted by the magnitudes of the weights $\{|h_j||_{j=1}^{M}\}$. This weighting structure empowers the proposed recursive filters to be used in applications that require robust frequency-selective filtering operations with sharp transition bands. For the sake of conciseness, the extended set of signed samples, that unifies into a single set both the subset of signed input samples and the subset of signed previous outputs, is defined as $\{s_m|_{m=1}^{L}\} = \{\mathrm{sgn}(g_{-N_2})x_{-N_2}, \ldots, \mathrm{sgn}(g_0)x_0, \ldots, \mathrm{sgn}(g_{N_1})x_{N_1}, \mathrm{sgn}(h_1)y_1, \ldots, \mathrm{sgn}(h_M)y_M\}$, with $L = N_1 + N_2 + M + 1$. Furthermore, let denote $\{s_{(m)}|_{m=1}^{L}\}$ the order statistic set of the extended set of signed samples such that $s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(L)}$, where the smallest and largest order statistics are given, respectively, by $s_{(1)} = \min\{s_m|_{m=1}^{L}\}$ and $s_{(L)} = \max\{s_m|_{m=1}^{L}\}$.

From this simplification, some properties of the RWMy filters can easily be derived. First, from (4), it can observed that the RWMy filter has L independent parameters that correspond to a normalized version of the filter weights, i.e. $\left\{\frac{g_{-N_2}}{K_1^2}, \ldots, \frac{g_0}{K_1^2}, \ldots \frac{g_{N_1}}{K_1^2}, \frac{h_1}{K_2^2}, \ldots, \frac{h_M}{K_2^2}\right\}$. Secondly, note in Eq. (3) that $\mathcal{H}(\boldsymbol{\beta}) = \log(\mathcal{G}(\boldsymbol{\beta}))$, where $\mathcal{G}(\boldsymbol{\beta})$ is a polynomial function of $\boldsymbol{\beta}$ with degree $2L$, which has well-defined derivatives in all orders. Thus, the derivative $\mathcal{G}'(\boldsymbol{\beta})$ is also a polynomial function with no more than $(2L-1)$ real roots and local extrema. Furthermore, the derivative of the cost function $\mathcal{H}(\boldsymbol{\beta})$ is given by

$$\begin{aligned}\mathcal{H}'(\boldsymbol{\beta}) &= \mathcal{G}'(\boldsymbol{\beta})/\mathcal{G}(\boldsymbol{\beta}),\\ &= 2\sum_{i=-N2}^{N_1}\frac{|g_i|(\boldsymbol{\beta}-\mathrm{sgn}(g_i)x_i)}{K_1^2+|g_i|(\mathrm{sgn}(g_i)x_i-\boldsymbol{\beta})^2}\\ &\cdots +2\sum_{j=1}^{M}\frac{|h_j|(\boldsymbol{\beta}-\mathrm{sgn}(h_j)y_j)}{K_2^2+|h_j|(\mathrm{sgn}(h_j)y_j-\boldsymbol{\beta})^2}. \end{aligned} \tag{5}$$

Upon a closer look of Eq. (2), it can be observed that $\mathcal{G}(\boldsymbol{\beta}) \neq 0$ for any $\boldsymbol{\beta}$, therefore, the roots of the derivative of the cost function $\mathcal{H}'(\boldsymbol{\beta})$ and the roots of $\mathcal{G}'(\boldsymbol{\beta})$ are identical. Hence, $\mathcal{H}'(\boldsymbol{\beta})$ has at most $(2L-1)$ roots and local extrema. Furthermore, as can be seen in Eq. (4), $\mathcal{H}(\pm\infty) = +\infty$, therefore, the output of the RWMy filter —i.e. the global minimum of the objective cost function— must correspond to one of the local minima of $\mathcal{H}(\boldsymbol{\beta})$, with $\mathcal{H}'(\boldsymbol{\beta}) = 0$.

Furthermore, from Eq. (5), it is straightforward to see that $\mathcal{H}'(\boldsymbol{\beta}) \neq 0$ for $\boldsymbol{\beta} < s_{(1)}$ and $\boldsymbol{\beta} > s_{(L)}$. In consequence, $\mathcal{H}'(\boldsymbol{\beta}) = 0$ is reached for real values of $\boldsymbol{\beta}$ within the range $[s_{(1)}, s_{(L)}]$. Therefore, the output of the RWMy filter is always within the range of the magnitudes of the extended set of signed samples: $s_{(1)} \leq \hat{\boldsymbol{\beta}}_{K_1,K_2} \leq s_{(L)}$.

Interestingly, the class of RWMy filters includes a wide range of filtering operations, where a specific filtering behavior can be derived by applying a suitable tuning on the linearity parameters (K_1, K_2). A particular limiting case of this new class of recursive filters, when both K_1 and K_2 take large values, is developed in the following property.

Property 1. ***(Linear Property)*** *Given a set of input samples $\{x_i|_{i=-N_2}^{N_1}\}$ and a set of previously computed outputs $\{y_j|_{j=1}^{M}\}$; and given the weights of a RWMy filter as $\{g_i|_{i=-N_2}^{N_1}\}$ and $\{h_j|_{j=1}^{M}\}$. The output of the RWMy filter converges to a normalized version of the linear recursive IIR filter as $K_1, K_2 \to \infty$, this is*

$$\hat{\boldsymbol{\beta}}_{\infty,\infty} = \lim_{K_1,K_2\to\infty}\hat{\boldsymbol{\beta}}_{K_1,K_2} = \frac{\sum_{i=-N_1}^{N_2} g_i x_i + \sum_{j=1}^{M} h_j y_j}{\sum_{i=-N_2}^{N_1}|g_i| + \sum_{j=1}^{M}|h_j|}. \tag{6}$$

Proof. The proof follows a similar approach to the linear property of the myriad operator developed in [16]. □

Therefore, the RWMy filter reduces to a linear IIR filter for infinite values of K_1 and K_2. Furthermore, as the linearity parameters decrease, the RWMy filter becomes more robust against impulsive noise. Further, as K_1 and K_2 tend to zero, the proposed filter considers the extended set of signed samples inside the recursive filter window as outliers, giving more reliability to the most repeated sample value. Thus, as specified in the following property, the sample mode of the extended set of signed samples is a particular limiting case of the RWMy filter when $K_1, K_2 \to 0$.

Property 2. ***(Sample Mode Property)*** *Given a set of input samples $\{x_i|_{i=-N_2}^{N_1}\}$ and a set of previously filter outputs $\{y_j|_{j=1}^{M}\}$; and given the filter weights of a RWMy filter as $\{g_i|_{i=-N_2}^{N_1}\}$ and $\{h_j|_{j=1}^{M}\}$. Furthermore, consider the extended set of signed samples defined as $\{s_m|_{m=1}^{L}\} = \{\mathrm{sgn}(g_{-N_2})x_{-N_2}, \ldots, \mathrm{sgn}(g_0)x_0, \ldots, \mathrm{sgn}(g_{N_1})x_{N_1}, \mathrm{sgn}(h_1)y_1, \ldots, \mathrm{sgn}(h_M)y_M\}$, with $L = (N_1+N_2+M+1)$. The output of the RWMy filter is the most repeated value in the extended set of signed samples as $K_1, K_2 \to 0$, this is*

$$\hat{\boldsymbol{\beta}}_{0,0} = \lim_{K_1,K_2\to 0}\hat{\boldsymbol{\beta}}_{K_1,K_2} = \arg\min_{s_0\in\mathcal{M}} \prod_{\substack{m=1,\\ s_m\neq s_0}}^{L} |s_m - s_0|, \tag{7}$$

where $\mathcal{M}$ is the set of the most repeated sample values.

The proof of this property follows a similar outline to that used in property 19 reported by Gonzalez in [16].

Furthermore, for a fixed value of K_1, the recursive component of the objective cost function (4) decreases its influence on the estimation of the filter output as K_2 increases. Thus, as described in the following property, the non-recursive WMy filter introduced in [18] is also a particular limiting case when $K_1 < \infty$ and $K_2 \to \infty$.

Property 3. ***(Non-recursive Weighted Myriad Property)*** *Given a set of input signal components $\{x_i|_{i=-N_2}^{N_1}\}$ and a set of previous output $\{y_j|_{j=1}^{M}\}$; and given the weights of the*

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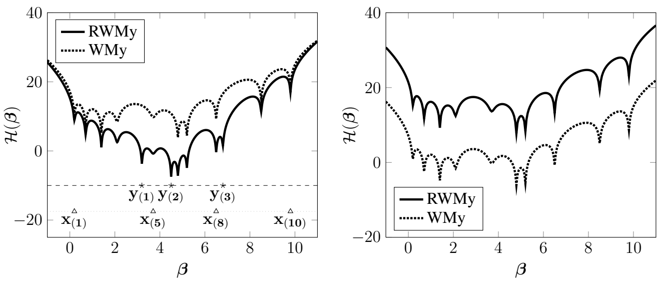

Fig. 2. (left) Objective cost function of the RWMy filter, with $K_1 = K_2 = 0.01$ (solid line), and a shifted version of the cost function of the non-recursive WMy filter, with $K = 0.01$ (dotted line), where just a few input samples are labeled on axis x; (right) Objective cost function of the RWMy filter, with $K_1 = 0.01$ and $K_2 = 10$ (solid line), and the cost function of the non-recursive WMy filter, with $K = 0.01$ (dotted line).

RWMy filter as $\{g_i|_{i=-N_2}^{N_1}\}$ *and* $\{h_j|_{j=1}^{M}\}$. *For a bounded value of* K_1, *the output of the RWMy filter converge to the output of the (non-recursive) WMy filter as* $K_2 \to \infty$, *i.e.*

$$\lim_{K_2\to\infty} \hat{\boldsymbol{\beta}}_{K_1,K_2} = \text{myriad}\left(|g_i| \circ \text{sgn}(g_i)x_i|_{i=-N_2}^{N_1}; K_1\right). \quad (8)$$

Proof. Note that in the cost function given by Eq. (4), each weighted deviation term in the recursive part of $\mathcal{H}(\boldsymbol{\beta})$ decreases its contribution as K_2 becomes larger, i.e. $\log[K_2^2 + |h_j|\left(\text{sgn}(h_j)y_j - \boldsymbol{\beta}\right)^2] \approx \log[K_2^2] = C/M$, where C is a constant. Thus, the output of the RWMy filter can be rewritten as

$$\hat{\boldsymbol{\beta}}_{K_1,K_2} = \arg\min_{\boldsymbol{\beta}} \left\{ \sum_{i=-N_2}^{N_1} \log\left[K_1^2 + |g_i|\left(\text{sgn}(g_i)x_i - \boldsymbol{\beta}\right)^2\right] + C \right\}, \quad (9)$$

where the function $\arg\min$ is not affected by adding the constant C. Therefore, the output of the RWMy filter, as $K_1 < \infty$ and $K_2 \to \infty$, reduces to the output of the non-recursive WMy filter, as specified in (8). □

In order to observe the behavior of the RWMy filters, the cost function $\mathcal{H}(\boldsymbol{\beta})$ for a particular set of observation samples is shown in Fig. 2(left). More precisely, let's consider that, at time n, the sliding observation window captures the following inout samples $\mathbf{x} = [4.8,\ 9.8,\ 3.7,\ 2.1,\ 0.7,\ 6.5,\ 5.2,\ 1.4,\ 0.2,\ 8.5]^T$ and previous outputs $\mathbf{y} = [3.2,\ 4.5,\ 6.8]^T$ and let's assume that the weights of the RWMy filter are given by $\mathbf{g} = [0.74,\ 0.60,\ 0.01,\ 0.04,\ 0.41,\ 0.68,\ 0.72,\ 1.00,\ 0.24,\ 0.34]^T$ and $\mathbf{h} = [0.75,\ 1.25,\ 0.40]^T$. The linearity parameters are set to $K_1 = K_2 = 0.01$. In this work, the output of the RWMy filter is computed using a fixed-point search method, following a similar approach to that developed in [30], but with a recursive window. For comparison purposes, a shifted version of the cost function of the non-recursive WMy filter (dotted line), using $K = 0.01$ and the same set of input samples weighted by the set of coefficients $\mathbf{g}$, is also shown in Fig. 2(left). As can be observed as $\boldsymbol{\beta} \to \pm\infty$, $\mathcal{H}(\boldsymbol{\beta}) \to +\infty$. Furthermore, the cost function of the RWMy filter has local minima at the values of the previous outputs, and these local minima induce a pulled down effect on the global minimum of the cost function around the values of the previous outputs. In consequence, the the global minimum of $\mathcal{H}(\boldsymbol{\beta})$, and hence, the filter output, is highly influenced by the set of recursive samples.

On the other hand, Fig. 2(right) depicts the cost function of the RWMy filter for $K_1 = 0.01$ and $K_2 = 10$ (solid line) and the cost function of the non-recursive WMy filter (dotted line). Note that $K_2/K_1 = 10^3$. As can be seen in this figure, the behavior of the cost function for the RWMy filter, when $K_2 \gg K_1$, looks like a shifted version of the cost function of the non-recursive WMy filter. Therefore, since a shifting of the cost function does not affect the function $\arg\min$, the global minimum in both cost functions is reached at the same value, and, thus, both filtering operations are equivalent. Thus, the RWMy filter includes the class of non-recursive WMy filters as a particular case for large values of K_2/K_1.

B. Scaled Recursive Weighted Myriad Filters

An apparent limitation of the RWMy filters is their incapability to amplify the dynamic range of the signal samples inside the recursive observation window, since the magnitude of the filter output is constrained to the interval $[s_{(1)}, s_{(L)}]$. Furthermore, considering the fact that the proposed filter takes into account previously computed outputs, these recursive samples may be clustered together pulling down the filter output around these clustered samples, this could cause further attenuation on the upcoming outputs as the sliding observation window progresses along the input signal. Further, from Property 1, the RWMy filter, at a particular limiting case when $K_1, K_2 \to \infty$, can be though of as a normalized version of the linear IIR filter, whose normalization factor is the sum of magnitudes of the filter weights, i.e. $\tau = \left\{\sum_{i=-N_2}^{N_1} |g_i| + \sum_{j=1}^{M} |h_j|\right\}$. Indeed, if the normalization factor is removed, the expression of the unconstrained linear IIR filter follows naturally. This scaling operation can also be extended to the RWMy filters for a finite values of K_1 and K_2, obtaining thus the output of the so-called *scaled recursive weighted myriad (SRWMy) filter*, that is

$$\tilde{\boldsymbol{\beta}}_{K_1,K_2} = \left\{ \sum_{i=-N_2}^{N_1} |g_i| + \sum_{j=1}^{M} |h_j| \right\} \hat{\boldsymbol{\beta}}_{K_1,K_2}, \quad (10)$$

where $\hat{\boldsymbol{\beta}}_{K_1,K_2}$ is given by (3). Thus, the amplification of the RWMy filter output can be achieved by simply increasing the magnitude of the filter weights. Once the filter output is amplified, it is fedback into the recursive weighted myriad operator. Note that, this amplification process is applied proportionally among all effective frequency range of the discrete-time signal, including the rejecting bands. Furthermore, for finite values of both the scaling factor τ and the samples inside the recursive observation window, the dynamic range of the output for this scaled version is now extended to $[\tau s_{(1)}, \tau s_{(L)}]$. Therefore, it is straightforward to proof that the SRWMy filter is a stable system under the bounded-input bounded-output (BIBO) criterion as long as $\left\{\sum_{i=-N_2}^{N_1} |g_i| + \sum_{j=1}^{M} |h_j|\right\} < \infty$. Additionally, as it can be observed in (10), the scaling term is the sum of magnitudes of the "unnormalized" values of the filter weights, i.e. $[g_{-N_2}, \ldots, g_0, \ldots, g_{N_1}, h_1, \ldots, h_M]$, therefore, the output of the SRWMy filter depends on $L + 2$ parameters, which correspond to the L weights of the recursive filter window and the linearity parameters K_1 and K_2.

IV. Recursive Hybrid Myriad Filters

Under the perspective of robust filtering based on M-estimates, the output of the RWMy filter can be thought of as the ML estimate of a common location parameter of a

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set of signed inputs $\{\mathrm{sgn}(g_i)x_i|_{i=-N_2}^{N_1}\}$ whose samples are contaminated with additive noise that follows a Cauchy distribution and a set of signed previous outputs $\{\mathrm{sgn}(h_i)y_j|_{j=1}^{M}\}$ that are also corrupted with additive noise obeying a Cauchy distribution. However, this later assumption can be relaxed considering the fact that the myriad estimate asymptotically follow a normal distribution [24]–[27], and, therefore, a Gaussian distribution is a more suitable model to characterize the additive errors that affect the previous filter outputs. This section presents a novel class of hybrid filters that merges the advantages of both the linear and the nonlinear structures —this is, the rejecting capability of impulsive noise of the weighted myriad and the suitable behavior of the weighted mean in Gaussian environments— in a single objective cost function to be minimized.

Given both a subset of observations $\{x_i|_{i=-N_2}^{N_1}\}$ and a subset of previous filter outputs $\{y_j|_{j=1}^{M}\}$, whose samples are gathered inside a recursive sliding window. Furthermore, it is assumed that the influence of the underlying impulsive noise over the previous outputs has been attenuated at a certain level. Also, consider a set of coefficients $\{g_i|_{i=-N_2}^{N_1}\}$ that weights the set of input samples, and a set of weights $\{h_j|_{j=1}^{M}\}$ that controls the influence of previously computed outputs on the current filter response. The output of the *recursive hybrid myriad (RHMy) filter* $\hat{\boldsymbol{\theta}}_K$ is defined as

$$\begin{aligned}\hat{\boldsymbol{\theta}}_K &= \arg\max_{\boldsymbol{\theta}}\{\mathcal{B}(\boldsymbol{\theta})\}\\ &= \arg\max_{\boldsymbol{\theta}}\left\{\prod_{i=-N_2}^{N_1}\frac{K}{\pi}\frac{1}{[K^2+|g_i|(\mathrm{sgn}(g_i)x_i-\boldsymbol{\theta})^2]}\right.\\ &\cdots\left.\times\prod_{j=1}^{M}\sqrt{\frac{|h_j|}{2\pi}}\exp\left(-|h_j|(\mathrm{sgn}(h_j)y_j-\boldsymbol{\theta})^2\right)\right\},\end{aligned}\tag{11}$$

where K is a free tunable parameter that controls the impulsive noise rejection of the non-recursive term in (11). Since the logarithm function is a non-decreasing function, the solution of (11) is equivalent to minimize

$$\begin{aligned}\hat{\boldsymbol{\theta}}_K &= \arg\min_{\boldsymbol{\theta}}\mathcal{F}(\boldsymbol{\theta})\\ &= \arg\min_{\boldsymbol{\theta}}\left\{\sum_{i=-N_2}^{N_1}\log\left[K^2+|g_i|\left(\mathrm{sgn}(g_i)x_i-\boldsymbol{\theta}\right)^2\right]\right.\\ &\cdots\left.+\sum_{j=1}^{M}|h_j|\left(\mathrm{sgn}(h_j)y_j-\boldsymbol{\theta}\right)^2\right\}.\end{aligned}\tag{12}$$

Under the context of ML estimation framework, the output of this recursive hybrid filter can be interpreted as the value that maximizes the likelihood function $\mathcal{B}(\mathbf{x},\mathbf{y};\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a common location parameter of a set of signed input samples that are corrupted by additive noise obeying a Cauchy distribution, where each noise entry is a realization of a Cauchy distribution (with a particular scale parameter given by $S_i = K/\sqrt{|g_i|}$), and a set of signed previous outputs whose additive noise components are characterized using the Gaussian statistical model (where the j-th noise component is related to an individual variance $\sigma_j^2 = 1/|h_j|$). Therefore, as can be seen in (11) and (12), the RHMy filter includes the intrinsic features of both the weighted myriad and the weighted mean in a single cost function to be minimized. To illustrate some properties of the RHMy filter, the derivative of the objective cost function $\mathcal{F}(\boldsymbol{\theta})$ is obtained as

$$\begin{aligned}\mathcal{F}'(\boldsymbol{\theta}) &= 2\sum_{i=-N_2}^{N_1}\frac{|g_i|(\boldsymbol{\theta}-\mathrm{sgn}(g_i)x_i)}{K^2+|g_i|(\mathrm{sgn}(g_i)x_i-\boldsymbol{\theta})^2}\\ &\cdots\ +2\sum_{j=1}^{M}|h_j|(\boldsymbol{\theta}-\mathrm{sgn}(h_j)y_j).\end{aligned}\tag{13}$$

From (13), note that $\mathcal{F}'(\boldsymbol{\theta}) \neq 0$ for $\boldsymbol{\theta} < s_{(1)}$ and $\boldsymbol{\theta} > s_{(L)}$, where $\{s_{(m)}|_{m=1}^{L}\}$ is the order statistics of the extended set of signed samples inside the observation window of the recursive filter. Therefore, each local minimum of the cost function, including the output of the RHMy filter, where $\mathcal{F}'(\boldsymbol{\theta}) = 0$, is reached within the range limited by the extended set of signed samples, i.e $s_{(1)} \leq \boldsymbol{\theta} \leq s_{(L)}$. Furthermore, after some algebraic manipulations, $\mathcal{F}'(\boldsymbol{\theta}) = 0$ can be rewritten as

$$\boldsymbol{\theta} = \frac{\sum_{i=-N_2}^{N_1}\frac{|g_i|\mathrm{sgn}(g_i)x_i}{K^2+|g_i|(\mathrm{sgn}(g_i)x_i-\boldsymbol{\theta})^2}+\sum_{j=1}^{M}|h_j|\mathrm{sgn}(h_j)y_j}{\sum_{i=-N_2}^{N_1}\frac{|g_i|}{K^2+|g_i|(\mathrm{sgn}(g_i)x_i-\boldsymbol{\theta})^2}+\sum_{j=1}^{M}|h_j|}.\tag{14}$$

Upon a closer look of Eq. (14), each local minimum of $\mathcal{F}(\boldsymbol{\theta})$ can be obtained as the weighted mean of the extended set of signed samples $\{s_m|_{m=1}^{L}\}$, where the weights —in this weighted mean operator— have non-negative values. This restricts the estimate of the local extrema to the interval $\{s_{(1)}, s_{(L)}\}$ [30]. Under this framework, and taking into account the fact that each local minimum has an implicit dependence on itself, each local extremum can be thought of as a *fixed point*, i.e. $\boldsymbol{\theta}^* = \mathcal{L}(\boldsymbol{\theta}^*)$, where these fixed points can be computed using a *fixed-point iteration* algorithm, in other words,

$$\mathcal{L}(\boldsymbol{\theta}_q) = \frac{\sum_{i=-N_2}^{N_1}\frac{|g_i|\mathrm{sgn}(g_i)x_i}{K^2+|g_i|(\mathrm{sgn}(g_i)x_i-\boldsymbol{\theta}_q)^2}+\sum_{j=1}^{M}|h_j|\mathrm{sgn}(h_j)y_j}{\sum_{i=-N_2}^{N_1}\frac{|g_i|}{K^2+|g_i|(\mathrm{sgn}(g_i)x_i-\boldsymbol{\theta}_q)^2}+\sum_{j=1}^{M}|h_j|},\tag{15}$$

with q as the iteration index. Among these fixed points, we follows the iterative approach reported in [30] to proposed the following fast algorithm for computing an estimate of the RHMy filter output.

A. Algorithm for computing the RHMy filter output

Step I) Determine the entry of the extended set of signed samples $\{s_m|_{m=1}^{L}\}$ that yields the minimum value of the objective cost function $\mathcal{B}(\boldsymbol{\theta})$, i.e. $\boldsymbol{\theta}_w = \arg\min_{s_m}\mathcal{B}(s_m)$.

Step II) Using $\boldsymbol{\theta}_w$ as initial value, perform P iterations of the fixed-point search algorithm (15). The output of the RHMy filter is the resulting fixed-point on $\boldsymbol{\theta}_w$, i.e. $\hat{\boldsymbol{\theta}} = \mathcal{L}^{(P)}(\boldsymbol{\theta}_w)$

Since the output of the RHMy filter is also restricted to the dynamic range of the extended set of signed samples, a scaled approach can be derived by applying a suitable amplification of the normalized version much like the RWMy filters. In other words, the output of the *scaled recursive hybrid myriad (SRHMy) filter* is defined as $\tilde{\boldsymbol{\theta}} = \left\{\sum_{i=-N_2}^{N_1}|g_i| + \sum_{j=1}^{M}|h_j|\right\}\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is given by (12). This scaled version is stable under BIBO criterion, and its response depends on $(L+1)$ parameters that correspond to the L filter weights and the linearity parameter K of the non-recursive part.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TSP.2016.2557304, IEEE Transactions on Signal Processing

V. Adaptive Algorithms

In this section, adaptive algorithms for obtaining the optimal filter weights of the proposed recursive filtering structures based on the myriad operator are developed. These adaptive algorithms aim at finding the best filter coefficients of the recursive filters based on the myriad operator such that a performance criterion is minimized. To this end, consider a recursive filter based on the myriad operator, whose coefficients, denoted by $\{g_i|_{i=-N_2}^{N_1}\}$ and $\{h_j|_{j=1}^{M}\}$, weigh, respectively, the input samples and the previous filter outputs. The main goal of the filter design is to find the best filter coefficients such that an error function between the filter output $y[n]$ and a desired signal $d[n]$ is minimized. A criterion widely used in designing nonlinear filters [6], [10] is the mean absolute error (MAE). Therefore, under the MAE criterion, the goal is to find the optimal filter parameters such that $J(\mathbf{g}, \mathbf{h}) = E\{|y[n] - d[n]|\}$ is minimized, where $E\{\cdot\}$ denotes the statistical expectation, $\mathbf{g} = [g_{-N_2}, \ldots, g_0, \ldots, g_{N_1}]$, $\mathbf{h} = [h_1, \ldots, h_M]$, and $y[n]$ is the output of the recursive filter based on the myriad operator.

Since direct minimization of $J(\mathbf{g}, \mathbf{h})$ does not lead to a closed-form solution, we resort to the *steepest descent* algorithm, which iteratively updates the filter coefficients in an attempt to converge to the global minimum of the cost function $J(\mathbf{g}, \mathbf{h})$. In essence, the filter weights are iteratively updated according to

$$\begin{aligned} g_i[n+1] &= g_i[n] + \mu\left[-\frac{\partial}{\partial g_i}J(\mathbf{g},\mathbf{h})\right],\ i = -N_2, \ldots, N_1, \\ h_j[n+1] &= h_j[n] + \mu\left[-\frac{\partial}{\partial h_j}J(\mathbf{g},\mathbf{h})\right],\ j = 1, \ldots, M, \quad (16) \end{aligned}$$

where $g_i[n+1]$ and $h_j[n+1]$ are, respectively, the update of the i-th non-recursive weight and the j-th recursive coefficient, at iteration $n+1$; μ is the update step-size, and the gradient of the cost function is given by

$$\begin{aligned} \frac{\partial}{\partial g_i}J(\mathbf{g},\mathbf{h}) &= E\left\{\operatorname{sgn}(e[n])\frac{\partial y}{\partial g_i}\right\},\ i = -N_2, \ldots, N_1, \\ \frac{\partial}{\partial h_j}J(\mathbf{g},\mathbf{h}) &= E\left\{\operatorname{sgn}(e[n])\frac{\partial y}{\partial h_j}\right\},\ j = 1, \ldots, M. \quad (17) \end{aligned}$$

In practical applications, however, the statistics of the filter output is unknown, and, therefore, the statistical expectation defined in (17) cannot be evaluated. We resort to instantaneous estimates for obtaining the gradient of the cost function, as it is common practice in the design of adaptive filters [1], [10], [18], [33]. Thus, the filter weights are updated as

$$\begin{aligned} g_i[n+1] &= g_i[n] - \mu\left[\operatorname{sgn}(e[n])\frac{\partial y}{\partial g_i}[n]\right],\ i = -N_2, \ldots, N_1, \\ h_j[n+1] &= h_j[n] - \mu\left[\operatorname{sgn}(e[n])\frac{\partial y}{\partial h_j}[n]\right],\ j = 1, \ldots, M. (18) \end{aligned}$$

Note that, the gradient of the cost function has to be iteratively computed to update the filter weights, where the computation of derivatives $\left(\frac{\partial y}{\partial g_i}[n],\ \frac{\partial y}{\partial h_j}[n]\right)$, at each iteration, becomes an intractable issue because the explicit feedback in the proposed recursive filters. To overcome this apparent limitation, we follow an optimization framework referred to as *equation error formulation* [1]. This approach has been successfully applied in designing recursive filters based on the median operator [10], and it is based on the fact that is expected that the previous filter outputs $\{y[n-j]|_{j=1}^{M}\}$ be close to the previous the desired signal $\{d[n-j]|_{j=1}^{M}\}$. Thus, during the optimization stage, previous filter outputs are replaced by previous components of the desired signal, i.e. $y[n-j] = d[n-j]$ for $j = 1, \ldots, M$, leading to a two-input single output non-recursive structure, whose derivatives can be determined since the filter no longer introduces feedback in its structure. Figure 3 illustrates schematically the structure of the adaptive algorithms, under the equation error formulation approach. Note that, the adaptive algorithm for obtaining the optimal weights of the proposed recursive structures follows directly once the expressions for computing $\frac{\partial y}{\partial g_i}[n]$ and $\frac{\partial y}{\partial h_j}[n]$, under the equation error formulation, are derived. Hence, the expression for computing of these instantaneous derivatives, and thus, the adaptive algorithms for estimating the optimal filter coefficients, are obtained next for each proposed recursive filter based on the myriad operator.

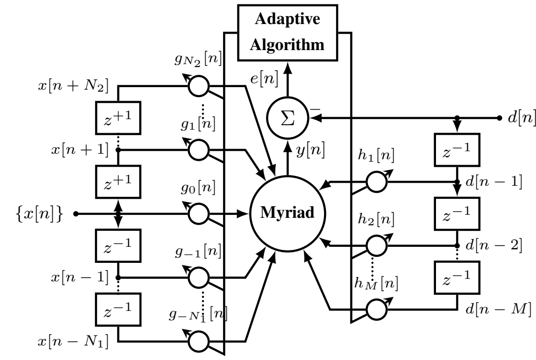

Fig. 3. Structure of the adaptive algorithms for computing the optimal weights of the recursive filters based on the myriad operator under the equation error formulation approach.

Finally, in order to extend the adaptive algorithm for designing structures that perform lowpass type filtering operations, with non-negative real-valued weights, a projector operator $P(u)$ (defined as u if $u \geq 0$, 0 otherwise) is applied over the update of the filter weights. This approach has been previously used for determining the filter weights of nonlinear smoothers [10], [18].

A. Adaptive Recursive Weighted Myriad Filter

As mentioned above, the adaptive algorithms for estimating the optimal filter weights require the evaluation of $\frac{\partial y}{\partial g_i}[n]$ and $\frac{\partial y}{\partial h_j}[n]$. To this end, the output of the recursive filter, under the equation error formulation approach, should be first redefined. Thus, by replacing the previous outputs $\{y_j|_{j=1}^{M}\}$ by the corresponding previous components of the desired signal $\{d_j|_{j=1}^{M}\}$, the output of the RWMy filter, at time n, becomes

$$\begin{aligned} \tilde{y}[n] = \hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} \Bigg\{ &\sum_{i=-N_2}^{N_1} \log\left[K_1^2 + |g_i|\left(\operatorname{sgn}(g_i)x_i - \boldsymbol{\beta}\right)^2\right] \\ &\cdots + \sum_{j=1}^{M} \log\left[K_2^2 + |h_j|\left(\operatorname{sgn}(h_j)d_j - \boldsymbol{\beta}\right)^2\right] \Bigg\}, \quad (19) \end{aligned}$$

and, from (19), we have $\frac{\partial \tilde{y}}{\partial g_i}[n] = \frac{\partial}{\partial g_i}\hat{\boldsymbol{\beta}}$, for $i = -N_2, \ldots, N_1$ and $\frac{\partial \tilde{y}}{\partial h_j}[n] = \frac{\partial}{\partial h_j}\hat{\boldsymbol{\beta}}$ for $j = 1, \ldots, M$; where the expressions for computing the derivatives $(\partial/\partial g_i)\hat{\boldsymbol{\beta}}$ and $(\partial/\partial h_j)\hat{\boldsymbol{\beta}}$ are shown in (20) and (21), respectively. Appendix A presents

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a detailed description of the analytical procedures to evaluate $(\partial/\partial g_i)\hat{\boldsymbol{\beta}}$ and $(\partial/\partial h_j)\hat{\boldsymbol{\beta}}$.

As specified in Section III-A, the output of the RWMy filter depends on the normalized version of the filter weights, i.e. $\left\{\frac{g_{-N_2}}{K_1^2},\ldots,\frac{g_0}{K_1^2},\ldots\frac{g_{N_1}}{K_1^2},\frac{h_1}{K_2^2},\ldots,\frac{h_M}{K_2^2}\right\}$. In other words, this class of filters does not depend on the linearity parameters (K_1, K_2), therefore, the adaptive algorithm for obtaining the optimal weights of the RWMy filter first defines values for the linearity parameters, and then, the optimal coefficients are computed for these previously fixed values. In [34] and [27], authors propose two approaches to compute the linearity parameters based on the impulsiveness of the underlying noise. Furthermore, since the RWMy filters are stable systems under BIBO criterion, the estimated filter coefficients need not to satisfy any additional design constraints unlike linear IIR filters, where the recursive filter coefficients are restricted to the fact that the poles of the filter transfer function must be inside the unit circle.

B. Adaptive Scaled Recursive Weighted Myriad Filter

Under the equation error formulation approach, the output of the SRWMy filter is defined as $\tilde{y}[n] = \tau\hat{\boldsymbol{\beta}} = \left\{\sum_{i=-N_2}^{N_1}|g_i| + \sum_{j=1}^{M}|h_j|\right\}\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is given by (19). Differentiating the instantaneous filter output $\tilde{y}[n]$ with respect to the filter weights $\{g_i|_{i=-N_2}^{N_1}\}$ and $\{h_j|_{j=1}^{M}\}$, we arrive, respectively, to

$$\frac{\partial\tilde{y}}{\partial g_i}[n] = \operatorname{sgn}(g_i)\hat{\boldsymbol{\beta}} + \tau\frac{\partial}{\partial g_i}\hat{\boldsymbol{\beta}}, i = -N_2,\ldots,N_1, \tag{22}$$

$$\frac{\partial\tilde{y}}{\partial h_j}[n] = \operatorname{sgn}(h_j)\hat{\boldsymbol{\beta}} + \tau\frac{\partial}{\partial h_j}\hat{\boldsymbol{\beta}}, j = 1,\ldots,M, \tag{23}$$

where the expressions for computing the derivatives $(\partial/\partial g_i)\hat{\boldsymbol{\beta}}$ and $(\partial/\partial h_j)\hat{\boldsymbol{\beta}}$ are already known from (20) and (21), respectively. Furthermore, as described in Section III-B, the scaled version of the RWMy filter has $(L+2)$ independent parameters, i.e. the L weights of the recursive filter and the linearity parameters (K_1, K_2). Therefore, the adaptive algorithm for computing the optimal filter parameters of the SRWMy filter requires —in addition to update the filter weights— the estimation of the best values of K_1 and K_2. However, in order to determine the optimal values of the linearity parameters some considerations must be previously set. First, since the linearity parameters are given by their squared values, the adaptive algorithm is designed to obtain these squared values. To do so, we define $\mathcal{K}_1 = K_1^2$ and $\mathcal{K}_2 = K_2^2$. Secondly, by the nature of the square function, $\mathcal{K}_1$ and $\mathcal{K}_2$ are restricted to non-negative real values, therefore, the projector operator $P(u)$ is applied to the update of the linearity parameters, following a similar approach to that used in [18] for determining the optimal K of the non-recursive WMy filters. In other words, $\mathcal{K}_1$ and $\mathcal{K}_2$ are continually updated according to

$$\mathcal{K}_\ell[n+1] = P\left(\mathcal{K}_\ell[n] - \mu\operatorname{sgn}(e[n])\left[\tau\frac{\partial}{\partial\mathcal{K}_\ell}\hat{\boldsymbol{\beta}}\right]\right), \quad \ell = 1,2. \tag{24}$$

where $\tau = \left\{\sum_{i=-N_2}^{N_1}|g_i| + \sum_{j=1}^{M}|h_j|\right\}$. The expressions for computing the derivatives $(\partial/\partial\mathcal{K}_1)\hat{\boldsymbol{\beta}}$ and $(\partial/\partial\mathcal{K}_2)\hat{\boldsymbol{\beta}}$ are given by (25) and (26), respectively. Appendix C details the procedures to obtain theses expressions. After obtaining $\mathcal{K}_1$ and $\mathcal{K}_2$, the linearity parameters are derived as $K_1 = +\sqrt{\mathcal{K}_1}$ and $K_2 = +\sqrt{\mathcal{K}_2}$.

C. Adaptive Recursive Hybrid Myriad Filter

In this case, we determine the expressions of $\frac{\partial\tilde{y}}{\partial g_i}[n]$, $\frac{\partial\tilde{y}}{\partial h_j}[n]$ and $\frac{\partial\tilde{y}}{\partial\mathcal{K}}[n]$, where $\tilde{y}[n]$ denotes the output of the SRHMy filter, under equation error formulation approach, this is

$$\begin{aligned}\tilde{y}[n] &= \tau\left\{\hat{\boldsymbol{\theta}}(\mathbf{g},\mathbf{h})\right\}\\ &= \tau\left\{\arg\min_{\boldsymbol{\theta}}\left\{\sum_{i=-N_2}^{N_1}\log\left[K^2 + |g_i|\left(\operatorname{sgn}(g_i)x_i - \boldsymbol{\theta}\right)^2\right]\right.\right.\\ &\cdots \left.\left. + \sum_{j=1}^{M}|h_j|\left(\operatorname{sgn}(h_j)d_j - \boldsymbol{\theta}\right)^2\right\}\right\}\end{aligned} \tag{27}$$

where $\tau = \left\{\sum_{i=-N_2}^{N_1}|g_i| + \sum_{j=1}^{M}|h_j|\right\}$ is the scaling factor; and $\hat{\boldsymbol{\theta}}(\mathbf{g},\mathbf{h})$ is the normalized version of the recursive hybrid myriad filter defined in (12). The instantaneous derivatives of $(\partial\tilde{y}/\partial g_i)$, $(\partial\tilde{y}/\partial h_j)$ and $(\partial\tilde{y}/\partial\mathcal{K})$, that are used in the steepest decent algorithm to update the filter coefficients and the linearity parameter, are obtained as follows

$$\frac{\partial\tilde{y}}{\partial g_i}[n] = \operatorname{sgn}(g_i)\hat{\boldsymbol{\theta}}(\mathbf{g},\mathbf{h}) + \tau\frac{\partial}{\partial g_i}\hat{\boldsymbol{\theta}}(\mathbf{g},\mathbf{h}), i = -N_2,\ldots,N_1 \tag{28}$$

$$\frac{\partial\tilde{y}}{\partial h_j}[n] = \operatorname{sgn}(h_j)\hat{\boldsymbol{\theta}}(\mathbf{g},\mathbf{h}) + \tau\frac{\partial}{\partial h_j}\hat{\boldsymbol{\theta}}(\mathbf{g},\mathbf{h}), j = 1,\ldots,M, \tag{29}$$

$$\frac{\partial\tilde{y}}{\partial\mathcal{K}}[n] = \tau\frac{\partial}{\partial\mathcal{K}}\hat{\boldsymbol{\theta}}(\mathbf{g},\mathbf{h}), \tag{30}$$

where $\mathcal{K} = K^2$; and the quantities $(\partial/\partial g_i)\hat{\boldsymbol{\theta}}$, $(\partial/\partial h_j)\hat{\boldsymbol{\theta}}$, and $(\partial/\partial\mathcal{K})\hat{\boldsymbol{\theta}}$ are, respectively, given by

$$\frac{\partial}{\partial g_i}\hat{\boldsymbol{\theta}} = \frac{-\left[\frac{K^2\operatorname{sgn}(g_i)(\bar{\boldsymbol{\theta}}-\operatorname{sgn}(g_i)x_i)}{[K^2+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\theta}})^2]^2}\right]}{\sum_{i=-N_2}^{N_1}|g_i|\frac{K^2-|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\theta}})^2}{[K^2+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\theta}})^2]^2} + \sum_{j=1}^{M}|h_j|}, \tag{31}$$

$$\frac{\partial}{\partial h_j}\hat{\boldsymbol{\theta}} = \frac{-(\operatorname{sgn}(h_j)\bar{\boldsymbol{\theta}} - d_j)}{\sum_{i=-N_2}^{N_1}|g_i|\frac{K^2-|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\theta}})^2}{[K^2+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\theta}})^2]^2} + \sum_{j=1}^{M}|h_j|}, \tag{32}$$

$$\frac{\partial}{\partial\mathcal{K}}\hat{\boldsymbol{\theta}} = \frac{\sum_{i=-N_2}^{N_1}\frac{|g_i|(\bar{\boldsymbol{\beta}}-\operatorname{sgn}(g_i)x_i)}{\left[\mathcal{K}_1+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\beta}})^2\right]^2}}{\sum_{i=-N_2}^{N_1}|g_i|\frac{\mathcal{K}-|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\theta}})^2}{[\mathcal{K}+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\theta}})^2]^2} + \sum_{j=1}^{M}|h_j|}. \tag{33}$$

It should be pointed out that the derivations of the expressions $(\partial/\partial g_i)\hat{\boldsymbol{\theta}}$, $(\partial/\partial h_j)\hat{\boldsymbol{\theta}}$, and $(\partial/\partial\mathcal{K})\hat{\boldsymbol{\theta}}$ follow similar

$$\frac{\partial}{\partial g_i}\hat{\boldsymbol{\beta}} = -\frac{\frac{K_1^2\operatorname{sgn}(g_i)(\bar{\boldsymbol{\beta}}-\operatorname{sgn}(g_i)x_i)}{\left[K_1^2+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\beta}})^2\right]^2}}{\sum_{i=-N_2}^{N_1}|g_i|\frac{K_1^2-|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\beta}})^2}{\left[K_1^2+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\beta}})^2\right]^2} + \sum_{j=1}^{M}|h_j|\frac{K_2^2-|h_j|(\operatorname{sgn}(h_j)d_j-\bar{\boldsymbol{\beta}})^2}{\left[K_2^2+|h_j|(\operatorname{sgn}(h_j)d_j-\bar{\boldsymbol{\beta}})^2\right]^2}} \tag{20}$$

$$\frac{\partial}{\partial h_j}\hat{\boldsymbol{\beta}} = -\frac{\frac{K_2^2\operatorname{sgn}(h_j)(\bar{\boldsymbol{\beta}}-\operatorname{sgn}(h_j)d_j)}{\left[K_2^2+|h_j|(\operatorname{sgn}(h_j)d_j-\bar{\boldsymbol{\beta}})^2\right]^2}}{\sum_{i=-N_2}^{N_1}|g_i|\frac{K_1^2-|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\beta}})^2}{\left[K_1^2+|g_i|(\operatorname{sgn}(g_i)x_i-\bar{\boldsymbol{\beta}})^2\right]^2} + \sum_{j=1}^{M}|h_j|\frac{K_2^2-|h_j|(\operatorname{sgn}(h_j)d_j-\bar{\boldsymbol{\beta}})^2}{\left[K_2^2+|h_j|(\operatorname{sgn}(h_j)d_j-\bar{\boldsymbol{\beta}})^2\right]^2}} \tag{21}$$

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derivation procedures that those used to obtain, respectively, $(\partial/\partial g_i)\hat{\boldsymbol{\beta}}$, $(\partial/\partial h_j)\hat{\boldsymbol{\beta}}$, and $(\partial/\partial \mathcal{K}_\ell)\hat{\boldsymbol{\beta}}$, highlighted in Appendices A and B, and for the sake of conciseness, the evaluation of these derivatives is not shown. Upon a closer look of the properties of the SRHMy filter, it is straightforward to proof that this recursive structure is stable under BIBO criterion. Therefore, much like the adaptive SRWMy, the adaptive algorithm for obtaining the optimal weights of the SRHMy filter does not require to apply additional constraints on the updated filter weights. Furthermore, the adaptive algorithm for the normalized version of the recursive hybrid myriad filter can be derived from the scaled version, where —for a fixed-valued of K— the instantaneous derivatives $\frac{\partial \tilde{y}}{\partial g_i}[n]$ and $\frac{\partial \tilde{y}}{\partial h_j}[n]$ can be determined using (31) and (32), respectively. Finally, since the adaptive algorithms for obtaining the optimal weights of the proposed recursive filters based on the myriad operator, are nonlinear estimation processes, a convergence analysis becomes intractable, therefore, the suitable bounds of the step-size parameter are not available. For evaluating the performance of the proposed adaptive algorithms, we use the approach of variable step-size $\mu[n]$, where $\mu[n]$ decreases its value as the training stage progresses [10].

VI. Simulation results

A. Performance of the adaptive algorithms

In order to evaluate the performance of the proposed adaptive algorithms, we design recursive filters based on the myriad operator for a bandpass filtering application. More precisely, each recursive filter design consists of a 96-tap bandpass structure, where 64 filter coefficients weigh the non-recursive part of the observation window and the 32 remaining ones weigh the previous outputs. The cut-off frequencies of the desired bandpass filter is $(\omega_1, \omega_2) = (0.075, 0.125)$, with the normalized sampling frequency set to one. At the designing stage, a Bernoulli random sequence, i.e. $x[n] = \pm 1$ with probability $p = 0.5$, is used as the training input signal [33], and the desired signal is obtained by passing the training input signal through a 96-tap linear FIR filter, whose coefficients are computed using the MATLAB's `fir1` function using the same band of interest. For designing the proposed recursive filters, the initial weights are set to identical and normalized values, i.e. $g_i[0] = h_j[0] = 1/96$ for $i = -31, \ldots, 32$ and $j = 1, \ldots, 32$. As mentioned above, a variable step-size parameter $\mu[n]$ is used, where $\mu[n]$ decreases according to $\mu_0 \exp(-n/1000)$, with $\mu_0 = 0.001$ [10]. Furthermore, for designing the normalized versions of the proposed recursive filters, the linearity parameters are fixed to $\mathcal{K}_\ell = 1.00$ for $\ell = 1, 2$, while the initial values of the linearity parameters,

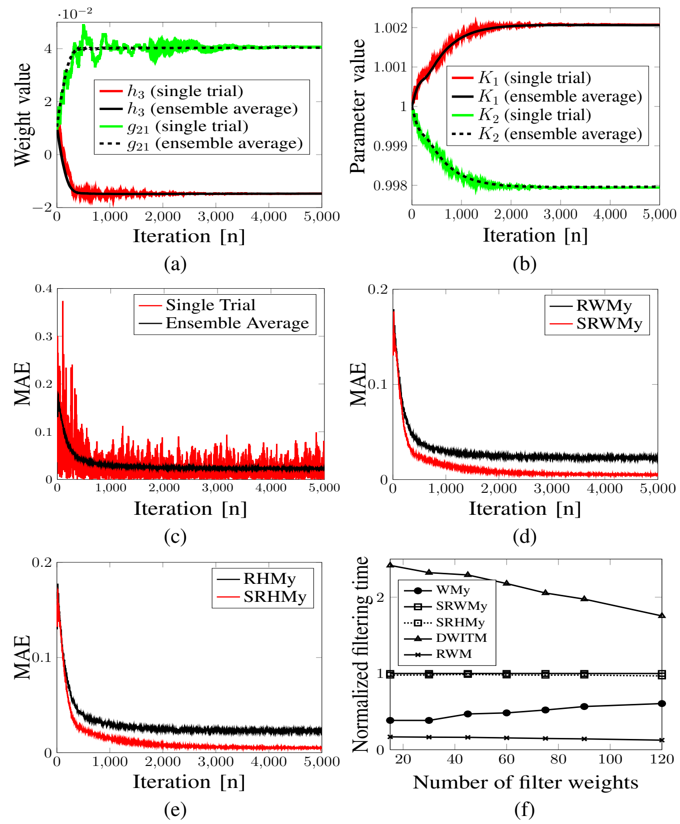

Fig. 4. Performance of adaptive algorithms as they progress, (a) tap-weight value of two coefficients (h_3 and g_{21}) of the SRWMy filter, (b) parameter values (K_1 and K_2) of the SRWMy filter, (c) MAE of the RWMy filter, (d) ensemble average of MAE for the RWMy filter (black line) and the SRWMy filter (red line), (e) ensemble average of MAE for the RHMy filter (black line) and the SRHMy filter (red line). (f) Normalized filtering time versus the number of filter weights. Filtering times are normalized with respect to the values yielded by the SRWMy filter (see Section VI-B).

for designing the corresponding scaled versions, are also set to one, i.e. $\mathcal{K}_\ell[0] = 1.00$ for $\ell = 1, 2$.

To illustrate the behavior of the adaptive SRWMy filter in designing the bandpass filtering structure, the values of some filter parameters are depicted in Fig. 4(a) and 4(b) as the adaptive algorithm progresses. More specifically, Fig. 4(a) shows the transient behavior of two representative filter weights (h_3 and g_{21}), whose indexes are selected for illustrative purposes only; and Fig. 4(b) depicts the transient behavior of the linearity parameters (K_1, K_2). Note that, using a variable step-size, each filter parameter —for a single realization of the adaptive algorithm— follows a behavior similar to a noisy exponential function. Furthermore, the curves of the same filter parameters —by averaging 1000 independent trials— are also shown in Figures 4(a) and 4(b), where at each trial, a random realization of the training input signal is generated. As can

$$\frac{\partial}{\partial \mathcal{K}_1}\hat{\boldsymbol{\beta}} = \frac{\sum_{i=-N_2}^{N_1} \frac{|g_i|(\bar{\boldsymbol{\beta}} - \operatorname{sgn}(g_i)x_i)}{\left[\mathcal{K}_1 + |g_i|(\operatorname{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]^2}}{\sum_{i=-N_2}^{N_1} |g_i| \frac{\mathcal{K}_1 - |g_i|(\operatorname{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2}{\left[\mathcal{K}_1 + |g_i|(\operatorname{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]^2} + \sum_{j=1}^{M} |h_j| \frac{\mathcal{K}_2 - |h_j|(\operatorname{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2}{\left[\mathcal{K}_2 + |h_j|(\operatorname{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2\right]^2}} \tag{25}$$

$$\frac{\partial}{\partial \mathcal{K}_2}\hat{\boldsymbol{\beta}} = \frac{\sum_{j=1}^{M} \frac{|h_j|(\bar{\boldsymbol{\beta}} - \operatorname{sgn}(h_j)d_j)}{\left[\mathcal{K}_2 + |h_j|(\operatorname{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2\right]^2}}{\sum_{i=-N_2}^{N_1} |g_i| \frac{\mathcal{K}_1 - |g_i|(\operatorname{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2}{\left[\mathcal{K}_1 + |g_i|(\operatorname{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]^2} + \sum_{j=1}^{M} |h_j| \frac{\mathcal{K}_2 - |h_j|(\operatorname{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2}{\left[\mathcal{K}_2 + |h_j|(\operatorname{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2\right]^2}} \tag{26}$$

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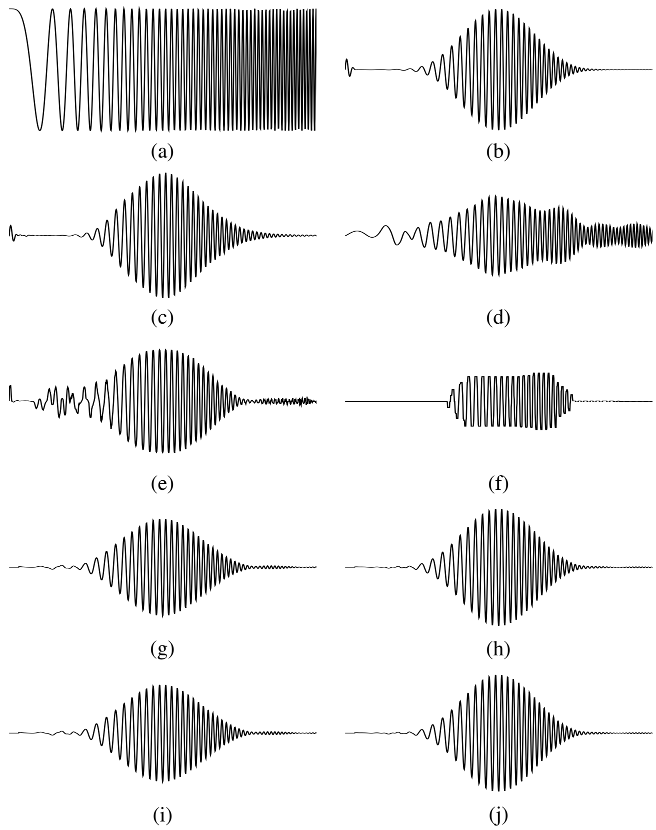

Fig. 5. Bandpass type filtering operations: (a) test input signal, (b) desired signal outputted by a linear FIR filter, (c) linear IIR filter output, (d) non-recursive weighted myriad output with MAE $= 0.0361$, (e) DWITM filter output with MAE $= 0.0040$, (f) RWM filter output with MAE $= 0.0318$, (g) RWMy filter output with MAE $= 0.0029$, (h) SRWMy filter output with MAE $= 0.0005$, (i) RHMy filter output with MAE $= 0.0029$, and (j) SRHMy filter output with MAE $= 0.0005$.

be seen in these figures, the ensemble average curves have a smoother form, where the noise effect over the exponential trend has been notably reduced.

Since the proposed adaptive algorithms obtain the optimal weights by minimizing the MAE, Fig. 4(c) shows the MAE curves as the adaptive RWMy filter algorithm progresses. As can be seen, the learning curve of a single realization (red line) follows a very noisy pattern. Furthermore, the learning curve of the ensemble average using 1000 independent realizations (black line) has a smoother behavior, where the MAE decreases as the adaptive algorithm progresses. This behavior is typical in adaptive filter theory [33]. Finally, Figures 4(d) and 4(e) depict the ensemble average of MAE curves using 1000 independent trials for the proposed adaptive algorithms. More precisely, 4(d) shows the results of the MAE for the RWMy filter and the SRWMy filter, and Fig. 4(e) depicts the MAE curves for the RHMy filter and SRHMy filter. Since the scaled version of the recursive filtering structures suitably amplifies their outputs to minimize the instantaneous error between the filter output and the desired signal, the outputs of these scaled filters exhibit lower error values in MAE curves.

B. Testing the designed bandpass filters

In order to evaluate the performance of the designed bandpass filters, we use a chirp signal as a test input signal. Fig. 5(a) shows the input signal whose instantaneous frequency changes from 0Hz to 400Hz in one second, with a sampling rate of 2 kHz. For comparison purposes, Fig. 5(b) shows the output of the linear FIR filter used to yield the desired signal at the training stage. Furthermore, the output of a 96-tap linear IIR filter is depicted in Fig. 5(c), where the coefficients of this recursive linear filter are obtained using the MATLAB's `yulewalk` function, with the same cut-off frequencies of the desired filter.

Furthermore, the responses of various robust filtering structures are also depicted in Fig. 5, where each nonlinear filter consits of a 96-tap filtering structure. More precisely, the output of a non-recursive WMy filter is shown in Fig. 5(d), where the filter weights of this non-recursive structure are obtained using the optimization algorithm reported in [18]. As can be observed in Fig. 5(d), the non-recursive WMy filter tracks the input signal at the frequency interval of interest but fails to appropriately attenuate the signal at the rejecting bands, specially at the high frequency band. The response of a dual weighted truncated mean (DWITM) filter [14] is shown in Fig 5(e), whose filter coefficients are the same than those obtained for the linear FIR filter. This output signal exhibits a closer tracking of the desired signal compared to that obtained with the WMy filter, but it is not suitably attenuated at the low frequency band. Further, Fig. 5(f) shows the response of the recursive weighted median (RWM) filter, where the filter weights of this recursive structure are obtained using the fast adaptive algorithm reported in [10]. As can be seen in Fig. 5(f), the RWM filter highly attenuate the signal at the cutoff bands, but the output signal does not properly track the desired signal at the pass frequency range.

Furthermore, the outputs of the recursive weighted myriad filter, the recursive hybrid myriad filter, the scaled recursive weighted myriad filter and the scaled recursive hybrid myriad filter are shown, respectively, in Figures 5(e), 5(f), 5(g), and 5(h). Note that, the outputs of the proposed recursive filters follows reliably the shape of the desired signal, where the rejecting bands are highly attenuated. Furthermore, since the optimization algorithms for the scaled versions update the filter parameters such that a suitable scaling is applied to the filter output, the responses of these scaled filters are much closer to the desired signal compared to those yielded by the normalized filters, achieving the smallest values for the mean absolute error (MAE) between filter outputs and the desired signal as shown in the caption of Fig. 5.

The filtering times of the proposed recursive filters are compared to those yielded by the various robust filtering structures specified above; using a desktop with Intel Core i3 CPU 3.1 GHz, RAM 4 GB, and Scientific Fedora 21 operating system. The normalized filtering time (with respect to the filtering times yielded by the SRWMy filter) versus the number of filter weights is shown in Fig. 4(f). Each computing time is obtained by averaging the time results of 1000 realizations, where, at each trial, all filters act on the same clean chirp signal with 2000 samples. As can be observed in Fig. 4(f), the filtering times of the SRHMy filter are very close to those yielded by the RWMy filter. Furthermore, the proposed recursive filters are faster than the DWITM filter, but they are still slower than both the non-recursive WMy filter and the

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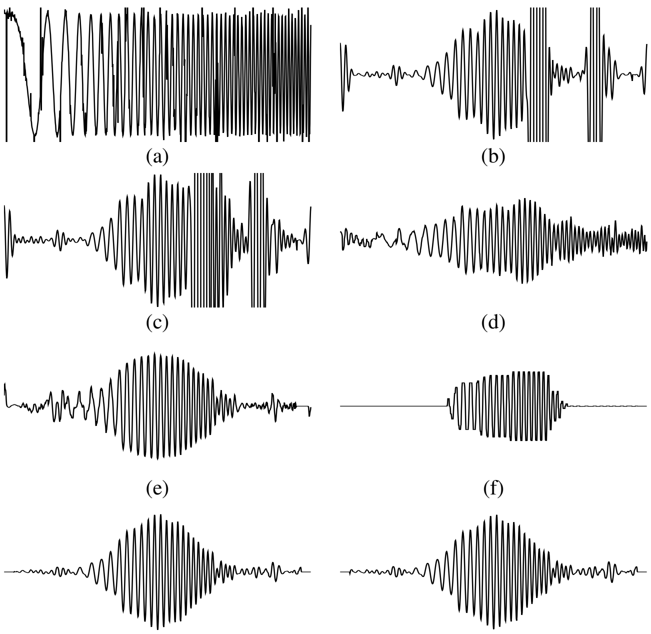

Fig. 6. Performance of the proposed recursive filters acting on impulsive noise: (a) noisy chirp, (b) FIR filter output, (c) IIR filter output, (d) non-recursive weighted myriad filter output, (e) DWITM filter output, (f) RWM filter output, (g) SRWMy filter output, and (f) SRHMy filter output.

TABLE I
MAE OF THE FILTERED CHIRP SIGNAL IN THE PRESENCE OF ADDITIVE NOISE.

Filter	$\alpha = 0.75$	$\alpha = 1.00$	$\alpha = 1.50$	$\alpha = 2.00$	Laplacian
FIR	12.8131	0.6165	0.1244	**0.0684**	**0.0683**
IIR	13.7769	0.6497	0.1405	0.0861	0.0862
DWITM	0.0862	0.0847	0.0833	0.0844	0.0826
RWM	0.1277	0.1265	0.1195	0.1137	0.1120
WMy	0.1899	0.2024	0.1845	0.1681	0.1793
RWMy	0.0785	0.0803	0.0799	0.0814	0.0804
SRWMy	<u>**0.0603**</u>	<u>**0.0605**</u>	<u>**0.0645**</u>	<u>**0.0653**</u>	<u>**0.0616**</u>
RHMy	**0.0772**	**0.0792**	0.0767	0.0768	0.0769
SRWMy	0.0780	0.0798	**0.0763**	0.0753	0.0735

RWM filter.

We also evaluate the performance of the proposed filters against impulsive noise, where the test signal is contaminated with additive symmetric α-stable (SαS) noise ($\alpha = 0.75$, $\gamma = 0.02$) [35], [36], as shown in Fig. 6(a), where this figure is truncated such that all plots have the same scale in the vertical axis. The output of the linear IIR filter is depicted in Fig. 6(b). As can be observed, the linear IIR filter highly degrades its performance in presence of this kind of impulsive noise, yielding the largest MAE. Figures 6(d), 6(e), and 6(f) show the responses of the non-recursive WMy filter, the DWITM filter, and the RWM filter, respectively. Note that, the effects of the impulsive noise are much less severe compared to that observed in the IIR filter. Figures 6(g) and 6(h) show the outputs of the SRWMy filter and SRHMy filter, respectively. As can be seen in these figures, the proposed recursive filters are robust in presence of impulsive noise, outputting responses much closer to the desired signal.

To further evaluate the performance of the various filtering approaches, Table I shows the MAE of the bandpass filter outputs against additive SαS noise over the chirp signal, with

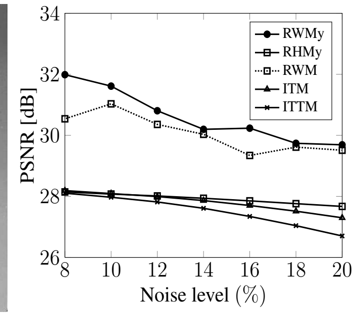

Fig. 7. (left) "Girl" image of size 512×512 pixels, (right) PSNR against the contamination level of the salt and pepper noise.

TABLE II
MAE, MSE AND PSNR OF THE VARIOUS FILTERING TECHNIQUES APPLIED ON THE NOISY "GIRL" IMAGE OF SIZE 512×512 PIXELS. NOISE LEVEL $\epsilon = 20\%$. WINDOW SIZE: $w_s = 5 \times 5$

Filter	MAE	MSE	PSNR
Mean	15.9249	501.6457	21.1270
ITM	4.7940	70.5795	29.6444
ITTM	4.9969	80.0474	29.0979
DWITM	5.3181	156.0369	26.1994
WM	**2.4668**	63.9245	30.0773
RWM	<u>**2.4456**</u>	**50.8626**	**31.0728**
RWMy	2.7207	<u>**30.6572**</u>	<u>**33.2664**</u>
RHMy	4.6666	63.2160	30.1228

dispersion $\gamma = 0.1$ and different values of α. Note that the Gaussian noise is a particular case of the SαS noise, when $\alpha = 2$, where the variance is given by $\sigma^2 = 2\gamma$ [28]. The MAE of the filter outputs against Laplacian noise with dispersion $\sigma_L = 2\gamma$ is also shown in Table I. Each value in Table I is obtained by averaging the MAE values of 1000 experiments, where, for each trial, a different realization of the additive noise is generated. For each noise level, underlined and bold font values represents the smallest MAE; whereas the second smallest value is in font bold. As can be seen in Table I, the SRWMy filter outperforms others filtering structures for all noise levels, and the SRHMy filter has a competitive performance compared to the remaining filter structures.

C. Image Denoising

The performance of each proposed recursive myriad based filter is evaluated under image denoising context. More precisely, the "Girl" image of size 512×512 (see Fig. 7(left)) is corrupted with salt and pepper noise, with contamination level up to $\epsilon = 20\%$. The window size of each applied filter is 5×5. The weights of the non-recursive weighted median (WM) filter and the DWITM filter are obtained using the fast adaptive algorithm developed in [6]. Furthermore, the fast LMA algorithm, reported in [10], is used for obtaining the weights of RWM filter. A image region of size 60×60 in the bottom right part of both the original image and the noisy image are used, respectively, as desired image and input image for the various adaptive algorithms [10], [14].

Table II shows the values of MAE, MSE and PSNR yielded by the various filtering approaches, where each value is obtained as the ensemble average of 10 experiments, and for each trial, a different realization of the salt and pepper noise is generated. The proposed RWMy filter outperforms others filters under the MSE and PSNR, and the weighted

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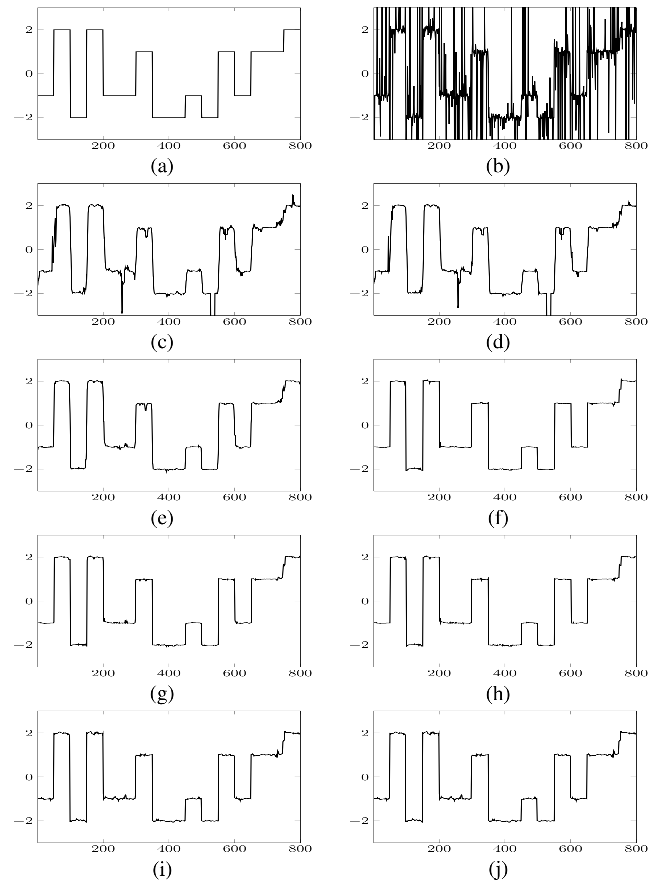

Fig. 8. Performance of various robust filters in the filtering of a powerline communication signal: (a) transmitted signal, (b) noisy signal corrupted with α-stable noise ($\alpha = 0.5$). Filtering responses with (c) ITM filter, (d) ITTM filter, (e) Median filter, (f) Myriad filter, (g) Meridian filter, (h) M-GC filter, (i) SRWMy filter, (j) SRHMy filter.

median based filters yielded the smallest error values under the MAE criterion. Additionally, this experiment is extended for obtaining the PSNR of the filtered images using several percentages of salt and pepper noise (see Fig. 7(right)). For this extension, the test images “Lena”, “Barbara” and “Boat” are also included for obtaining the PSNR values. As can be observed in Fig. 7(right), the RWMy filter outperforms others filtering structures.

D. Powerline Communications

Under the framework of powerline communications, the performance of the proposed recursive filters are evaluated. To this end, let consider the following set of voltage levels $\mathcal{V} = [-2, -1, 1, 2]$. The signal to be transmitted through a power line has amplitudes selected from these voltage levels [28], [29], where each amplitude represents a pair of transmitted bits. The transmitted signal is contaminated with additive SαS noise using different values of α and a scale parameter fixed in $\eta = 0.05$ [28], where the scale parameter and dispersion of a SαS distribution are related as $\eta = \gamma^{1/\alpha}$ [28]. The transmitted signal of a single realization is shown in Fig. 8(a) and the corresponding noisy signal is depicted in Fig. 8(b), where the additive noise follows a SαS distribution with $\alpha = 0.50$. The filter outputs by ITM [12], ITTM [15], Median, Myriad, Meridian [28], and M-GC [29] operators,

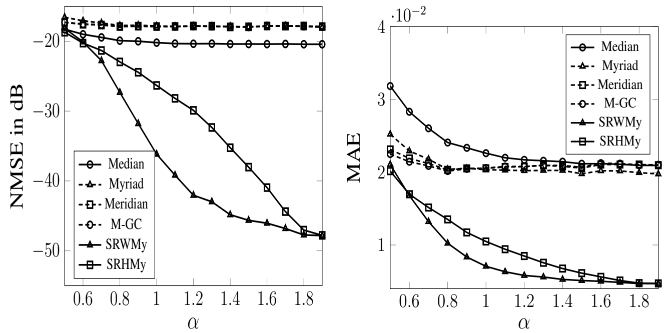

Fig. 9. Performance of various robust filters in the framework of powerline communication enhancement: (left) NMSE versus α, (right) MAE versus α.

with a window size $N = 12$, are shown in Figures 8(c)-(h), respectively. Furthermore, the responses of the SRWMy filter and SRHMy filter, with $N = 8$ and $M = 4$, are shown in Figures 8(i) and (j), respectively. The linearity parameter of the myriad filter, the linearity parameter of the non-recursive part for the SRWMy filter (K_1), and the linearity parameter of the SRHMy filter are obtained using the $\alpha - k$ relationship reported in [34]; and the linearity parameter of the recursive part for the SRWMy is set to $K_2 = 25K_1$. Furthermore, the linearity parameter of the meridian filter is set to one [28], and the parameters used for the M-GC filter are $p = 0.756$ and $\sigma = 0.896$ [29].

As can be observed in Fig. 8, the ITM filter and ITTM filter fail in the impulsive noise supression, and the proposed recursive filters, for this experiment, provide competitive outputs compared to the Median filter, Myriad filter, Meridian filter and M-GC filter, where the impulsive noise is suitably attenuated and the signal structure is well-preserved. Furthermore, the NMSE in dB versus α and the MAE versus α are shown in Fig. 9, with $\mathrm{NMSE[dB]} = 10\log_{10} |\|y - d\|_2^2 / \|d\|_2^2|$, where y is the filter output, d is the noiseless signal, and $\|\cdot\|_2$ denotes the ℓ_2-norm. Each point is obtained by averaging 1000 realizations, and, for each trial, all filtering structures act on a different realization of the transmitted signal which is contaminated with an independent realization of the additive SαS noise. As can be seen in Fig. 9, the proposed recursive filters outperform others robust filtering structures, exhibiting the lower error values in the depicted noise interval. Furthermore, under both NMSE and MAE metrics, the proposed recursive filters improve significantly their performance as the value of α increases (the impulsive level decreases).

VII. CONCLUSIONS

In this paper, we introduce two novel robust recursive filtering structures based on the weighted myriad operator. First, we present the class of *recursive weighted myriad* (RWMy) filters that consider a subset of previously computed outputs as well as a subset of input samples for obtaining the current filter output. Interestingly, under specific conditions of the linearity parameters (K_1, K_2), the class of RWMy filters includes, as particular cases, a normalized version of the linear IIR filter, the recursive sample-mode filter and the non-recursive weighted myriad filter. Furthermore, we present the class of scaled recursive weighted myriad (SRWMy) filters,

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where the output of this kind of filters is obtained by applying an appropriate scaling factor— that depends on the magnitudes of the filter coefficients— over the RWMy output. Further, the class of SRWMy filters includes (but it is a more robust generalization than) the unconstrained linear IIR filters.

Secondly, considering the fact that the output of a myriad based filter is no longer impulsive but it can be characterized by the Gaussian model, we propose the class of *recursive hybrid myriad* (RHMy) filters. In essence, this new nonlinear operator gathers the advantages of the outlier-resistant myriad operator and the suitable behavior of the weighted mean in a single cost function to be jointly minimized. From a closer look of the objective cost function, some properties of the proposed hybrid filter are derived. Furthermore, a fast iterative fixed-point search algorithm for computing the RHMy filter output is presented. The adaptive algorithms to obtain the optimal parameters of the proposed recursive filters based on the myriad operator are developed under the equation error formulation approach. The performance of the proposed recursive filters and their corresponding adaptive algorithms are evaluated using extensive computer simulation examples. Much like the linear IIR filters exhibit advantages over the FIR filters, the proposed recursive weighted myriad filters provide better performance than the (non-recursive) weighted myriad filters, for the same number of filter coefficients. Furthermore, the recursive filters based on the myriad operator outperform significantly the linear IIR filter when the input signal is corrupted with impulsive noise, and yield a comparable performance on noiseless input signals. Finally, the performance of the proposed filters are evaluated and compared with respect to the state-of-the-art nonlinear filters for signal and image processing applications as well as an application in communications.

Appendix A
Evaluation of $(\partial/\partial g_i)\hat{\boldsymbol{\beta}}$ and $(\partial/\partial h_j)\hat{\boldsymbol{\beta}}$

Under the context of equation error formulation, the cost function to be minimized becomes

$$\begin{aligned}\mathcal{Q}(\boldsymbol{\beta}) &= \sum_{i=-N_2}^{N_1} \log\left[K_1^2 + |g_i|(\mathrm{sgn}(g_i)x_i - \boldsymbol{\beta})^2\right] \\ &\cdots + \sum_{j=1}^{M} \log\left[K_2^2 + |h_j|(\mathrm{sgn}(h_j)d_j - \boldsymbol{\beta})^2\right]. \quad (34)\end{aligned}$$

Since the procedures to obtain $(\partial/\partial g_i)\hat{\boldsymbol{\beta}}$ and $(\partial/\partial h_j)\hat{\boldsymbol{\beta}}$ are similar, and for the sake of conciseness, we evaluate only the derivative of $\hat{\boldsymbol{\beta}}$ with respect to g_i, keeping fixed the remaining parameters. To this end, let $\hat{\boldsymbol{\beta}} \equiv \bar{\boldsymbol{\beta}}$ be one of the local minima of $\mathcal{Q}(\boldsymbol{\beta})$, therefore, $\mathcal{Q}'(\bar{\boldsymbol{\beta}}) = 0$. By differentiating the cost function (34) with respect to $\boldsymbol{\beta}$, we have

$$\begin{aligned}\mathcal{P}(\bar{\boldsymbol{\beta}}, g_i, h_j) &= \mathcal{Q}'(\bar{\boldsymbol{\beta}}) = 2\sum_{i=-N_2}^{N_1} \frac{|g_i|(\bar{\boldsymbol{\beta}} - \mathrm{sgn}(g_i)x_i)}{\left[K_1^2 + |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]} \\ &\cdots + 2\sum_{j=1}^{M} \frac{|h_j|(\bar{\boldsymbol{\beta}} - \mathrm{sgn}(h_j)d_j)}{\left[K_2^2 + |h_j|(\mathrm{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2\right]} = 0, \quad (35)\end{aligned}$$

where the function $\mathcal{P}(\bar{\boldsymbol{\beta}}, g_i, h_j)$ clearly shows the dependency of the filter output $\bar{\boldsymbol{\beta}}$ with respect the filter weights $\{g_i|_{i=-N_2}^{N_1}\}$ and $\{h_j|_{j=1}^{M}\}$. By applying *implicit differentiation* on (35), it follows that

$$\left(\frac{\partial \mathcal{P}}{\partial \bar{\boldsymbol{\beta}}}\right) \cdot \left(\frac{\partial \bar{\boldsymbol{\beta}}}{\partial g_i}\right) + \left(\frac{\partial \mathcal{P}}{\partial g_i}\right) = 0. \quad (36)$$

Therefore, the expression for $\frac{\partial \bar{\boldsymbol{\beta}}}{\partial g_i}$ can be obtained from the evaluation of $\frac{\partial \mathcal{P}}{\partial \bar{\boldsymbol{\beta}}}$ and $\frac{\partial \mathcal{P}}{\partial g_i}$. From (35), we can obtain the following expression

$$\begin{aligned}\frac{\partial \mathcal{P}}{\partial \bar{\boldsymbol{\beta}}} &= 2\sum_{i=-N_2}^{N_1} |g_i| \frac{K_1^2 - |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2}{\left[K_1^2 + |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]^2} \\ &\cdots + 2\sum_{j=1}^{M} |h_j| \frac{K_2^2 - |h_j|(\mathrm{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2}{\left[K_2^2 + |h_j|(\mathrm{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2\right]^2}. \quad (37)\end{aligned}$$

For obtaining $\frac{\partial \mathcal{P}}{\partial g_i}$, we take advantage of the fact that $g_i = \mathrm{sgn}(g_i)|g_i|$ in order to avoid the evaluation of $(\partial/\partial g_i)\mathrm{sgn}(g_i)$ which is not differentiable everywhere. Thus, Eq. (35) is rearranged as

$$\begin{aligned}\mathcal{P}(\bar{\boldsymbol{\beta}}, g_i) &= 2\sum_{i=-N_2}^{N_1} \frac{|g_i|\bar{\boldsymbol{\beta}} - g_i x_i}{K_1^2 + |g_i|(\bar{\boldsymbol{\beta}}^2 + x_j^2) - 2g_i\bar{\boldsymbol{\beta}}x_i} \\ &\cdots + 2\sum_{j=1}^{M} \frac{|h_j|\bar{\boldsymbol{\beta}} - h_j d_j}{K_2^2 + |h_j|(\bar{\boldsymbol{\beta}}^2 + d_j^2) - 2h_j\bar{\boldsymbol{\beta}}d_j}. \quad (38)\end{aligned}$$

Differentiating (38) with respect to g_i, and after some algebraic manipulations, we obtain

$$\frac{\partial \mathcal{P}}{\partial g_i} = \frac{2K_1^2\mathrm{sgn}(g_i)(\bar{\boldsymbol{\beta}} - \mathrm{sgn}(g_i)x_i)}{\left[K_1^2 + |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]^2} \quad (39)$$

Finally, by substituting (37) and (39) in (36), we derive the expression of the instantaneous estimate of $(\partial/\partial g_i)\hat{\boldsymbol{\beta}}$, given in (20).

Appendix B
Evaluation of $(\partial/\partial \mathcal{K}_1)\hat{\boldsymbol{\beta}}$ and $(\partial/\partial \mathcal{K}_2)\hat{\boldsymbol{\beta}}$

For the sake of conciseness, we only obtain the derivative of $\hat{\boldsymbol{\beta}}$ with respect to $\mathcal{K}_1$, keeping constants the remaining variables. The evaluation of $(\partial/\partial \mathcal{K}_2)\hat{\boldsymbol{\beta}}$ follows a similar approach to that highlighted. Note that if $\hat{\boldsymbol{\beta}} \equiv \bar{\boldsymbol{\beta}}$ is one of the local minima $\mathcal{Q}(\boldsymbol{\beta})$, $\mathcal{Q}'(\bar{\boldsymbol{\beta}}) = 0$, where $\mathcal{Q}(\boldsymbol{\beta})$ is the cost function under the equation error formulation approach, given by (34). Thus, $\bar{\boldsymbol{\beta}}$ satisfies

$$\begin{aligned}\mathcal{R}(\bar{\boldsymbol{\beta}}, \mathcal{K}_1) &= \mathcal{Q}'(\boldsymbol{\beta}) = 2\sum_{i=-N_2}^{N_1} \frac{|g_i|(\bar{\boldsymbol{\beta}} - \mathrm{sgn}(g_i)x_i)}{\left[\mathcal{K}_1 + |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]} \\ &\cdots + 2\sum_{j=1}^{M} \frac{|h_j|(\bar{\boldsymbol{\beta}} - \mathrm{sgn}(h_j)d_j)}{\left[\mathcal{K}_2 + |h_j|(\mathrm{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2\right]} = 0, \quad (40)\end{aligned}$$

where the function $\mathcal{R}(\bar{\boldsymbol{\beta}}, \mathcal{K}_1)$ indicates the dependence of the output $\bar{\boldsymbol{\beta}}$ with respect to $\mathcal{K}_1$. By applying implicit differentiation on (40) with respect to $\mathcal{K}_1$, we have

$$\left(\frac{\partial \mathcal{R}}{\partial \bar{\boldsymbol{\beta}}}\right) \cdot \left(\frac{\partial \bar{\boldsymbol{\beta}}}{\partial \mathcal{K}_1}\right) + \left(\frac{\partial \mathcal{R}}{\partial \mathcal{K}_1}\right) = 0. \quad (41)$$

From (40), we derive the following expressions

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TSP.2016.2557304, IEEE Transactions on Signal Processing

$$\frac{\partial \mathcal{R}}{\partial \bar{\boldsymbol{\beta}}} = 2\sum_{i=-N_2}^{N_1} |g_i| \frac{\mathcal{K}_1 - |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2}{\left[\mathcal{K}_1 + |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]^2}$$
$$\cdots \quad +2\sum_{j=1}^{M} |h_j| \frac{\mathcal{K}_2 - |h_j|(\mathrm{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2}{\left[\mathcal{K}_2 + |h_j|(\mathrm{sgn}(h_j)d_j - \bar{\boldsymbol{\beta}})^2\right]^2}, \quad (42)$$
$$\frac{\partial \mathcal{R}}{\partial \mathcal{K}_1} = -2\sum_{i=-N_2}^{N_1} \frac{|g_i|(\bar{\boldsymbol{\beta}} - \mathrm{sgn}(g_i)x_i)}{\left[\mathcal{K}_1 + |g_i|(\mathrm{sgn}(g_i)x_i - \bar{\boldsymbol{\beta}})^2\right]^2}. \quad (43)$$

By substituting (42) and (43) in (41), we obtain the evaluation of $(\partial/\partial\mathcal{K}_1)\hat{\boldsymbol{\beta}}$ depicted in (25).

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