1 Problem and Parameter

Consider a 5DOF Manipulator holding the end-effector (EE) on the given 3D position and pointing (EE's z-axis) at the given direction. D-H parameter is given as follow

i	α_i	a_i	θ_i	d_i
1	$\pi/2$	0	θ_1	0.089159
2	0	-0.425	θ_2	0
3	0	-0.39225	θ_3	0
4	$\pi/2$	0	θ_4	0.10915
5	$-\pi/2$	0	θ_5	0.09465
6	0	0	θ_6	0.0823

Target pose is

$$(p_x, p_y, p_z, \vec{n}),\ \vec{n} = [n_x, n_y, n_z]^T$$

where $\vec{n}$ is a normal vector of the surface of the object.

Since the end-effector should be perpendicular to the surface,

$$^0T_{ee} = \begin{bmatrix} a_x & b_x & n_x & p_x \\ a_y & b_y & n_y & p_y \\ a_z & b_z & n_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

where $a_x, a_y, a_z, b_x, b_y, b_z$ are all variables, and

$$\begin{cases} a_x n_x + a_y n_y + a_z n_z = 0 \\ b_x n_x + b_y n_y + b_z n_z = 0 \\ a_x b_x + a_y b_y + a_z b_z = 0 \\ \| a \|_2 = 1 \\ \| b \|_2 = 1 \end{cases} \tag{2}$$

2 Kinematics

The kinematic function of the manipulator:

$$T = {}^0T_1 {}^1T_2 \cdots {}^4T_5 {}^5T_{ee}$$

where

$$^0T_1 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^1T_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^2T_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$^3T_4 = \begin{bmatrix} c\theta_4 & 0 & s\theta_4 & 0 \\ s\theta_4 & 0 & -c\theta_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^4T_5 = \begin{bmatrix} c\theta_5 & 0 & -s\theta_5 & 0 \\ s\theta_5 & 0 & c\theta_5 & 0 \\ 0 & -1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^5T_{ee} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

To simplify the calculation, we get

$$\begin{aligned} ^1T_3 = {}^1T_2 {}^2T_3 &= \begin{bmatrix} c\theta_2c\theta_3 - s\theta_2s\theta_3 & -c\theta_2s\theta_3 - c\theta_3s\theta_2 & 0 & a_2c\theta_2 + a_3(c\theta_2c\theta_3 - s\theta_2s\theta_3) \\ c\theta_2s\theta_3 + c\theta_3s\theta_2 & c\theta_2c\theta_3 - s\theta_2s\theta_3 & 0 & a_2s\theta_2 + a_3(c\theta_2s\theta_3 + a_3c\theta_3s\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\triangleq \begin{bmatrix} c_{23} & -s_{23} & 0 & a_3c_{23} + a_2c_2 \\ s_{23} & c_{23} & 0 & a_3s_{23} + a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{5}$$

$$^1T_4 = {}^1T_3 {}^3T_4 = \begin{bmatrix} c_{23}c\theta_4 - s_{23}s\theta_4 & 0 & c_{23}s\theta_4 + s_{23}c\theta_4 & a_3c_{23} + a_2c\theta_2 \\ s_{23}c\theta_4 + c_{23}s\theta_4 & 0 & s_{23}s\theta_4 - c_{23}c\theta_4 & a_3s_{23} + a_2s\theta_2 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$
{}^{1}T_5 = {}^{1}T_4{}^{4}T_5 = \begin{bmatrix} (c_{23}c\theta_4 - s_{23}s\theta_4)c\theta_5 & -c_{23}s\theta_4 - s_{23}c\theta_4 & (-c_{23}c\theta_4 + s_{23}s\theta_4)s\theta_5 & d_5(c_{23}s\theta_4 + s_{23}c\theta_4) + a_3c_{23} + a_2c_2 \\ (s_{23}c\theta_4 + c_{23}s\theta_4)c\theta_5 & -s_{23}s\theta_4 + c_{23}c\theta_4 & (-s_{23}c\theta_4 - c_{23}s\theta_4)s\theta_5 & d_5(s_{23}s\theta_4 - c_{23}c\theta_4) + a_3s_{23} + a_2s\theta_2 \\ s\theta_5 & 0 & c\theta_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}
$$

We can precalculate the ${}^{0}T_1$ and ${}^{5}T_{ee}$ to simplify the calculation

$$
({}^{0}T_1)^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ s\theta_1 & -c\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; ({}^{5}T_{ee})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}
$$

So we get

$$
\begin{aligned}
{}^{1}T_5 &= (({}^{0}T_1)^{-1}){}^{0}T_{ee}(({}^{5}T_{ee})^{-1}) \\
&= \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ s\theta_1 & -c\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x & b_x & n_x & p_x \\ a_y & b_y & n_y & p_y \\ a_z & b_z & n_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 a_x + s\theta_1 a_y & c\theta_1 b_x + s\theta_1 b_y & c\theta_1 n_x + s\theta_1 n_y & c\theta_1 p_x + s\theta_1 p_y \\ a_z & b_z & n_z & p_z - d_1 \\ s\theta_1 a_x - c\theta_1 a_y & s\theta_1 b_x - c\theta_1 b_y & s\theta_1 n_x - c\theta_1 n_y & s\theta_1 p_x - c\theta_1 p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 a_x + s\theta_1 a_y & c\theta_1 b_x + s\theta_1 b_y & c\theta_1 n_x + s\theta_1 n_y & -d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y \\ a_z & b_z & n_z & -d_{ee}n_z + p_z - d_1 \\ s\theta_1 a_x - c\theta_1 a_y & s\theta_1 b_x - c\theta_1 b_y & s\theta_1 n_x - c\theta_1 n_y & -d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{9}
$$

2.1 Solve θ_1

From $(3,4)$ in Equ. 7 and Equ.9, we know

$$
d_4 = -d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y
$$

$$
\frac{p_x - d_{ee}n_x}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}} s\theta_1 + \frac{d_{ee}n_y - p_y}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}} c\theta_1 = \frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}}
$$

Set ϕ that

$$
\tan\phi = \frac{p_x - d_{ee}n_x}{d_{ee}n_y - p_y} \Rightarrow \phi = \operatorname{atan2}\left(\frac{p_x - d_{ee}n_x}{d_{ee}n_y - p_y}\right)
$$

$$
\cos(\theta_1 - \phi) = \sin\phi\sin\theta_1 + \cos\phi\cos\theta_1 = \frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}}
$$

$$
\Rightarrow \theta_1 - \phi = \operatorname{acosall}\left(\frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}}\right)
$$

$$
\theta_1 = \operatorname{acosall}\left(\frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}}\right) + \operatorname{atan2}(p_x - d_{ee}n_x, d_{ee}n_y - p_y) \tag{10}
$$

2.2 Solve θ_5

From $(3,3)$ in Equ. 7 and Equ.9, we know

$$
c\theta_5 = s\theta_1 n_x - c\theta_1 n_y
$$

$$
\theta_5 = \operatorname{acosall}(s\theta_1 n_x - c\theta_1 n_y) \tag{11}
$$

2.3 Solve b_x, b_y, b_z

From $(3,2)$ in Equ. 7 and Equ.9 and Equ. 2, we know

$$
\begin{cases} s\theta_1 b_x - c\theta_1 b_y = 0 \\ b_x n_x + b_y n_y + b_z n_z = 0 \\ b_x^2 + b_y^2 + b_z^2 = 1 \end{cases} \Rightarrow \begin{cases} b_y = \dfrac{s\theta_1}{c\theta_1} b_x \\ \dfrac{n_x}{n_z} b_x + \dfrac{n_y}{n_z} b_y = -b_z \end{cases}
$$

$$b_x^2 + \left(\frac{s\theta_1}{c\theta_1} b_x\right)^2 + \left(\frac{n_x}{n_z} b_x + \frac{n_y}{n_z}\frac{s\theta_1}{c\theta_1} b_x\right)^2 = 1$$

$$b_x^2\left(1 + \frac{s\theta_1^2}{c\theta_1^2} + \frac{n_x^2}{n_z^2} + \frac{n_y^2 s\theta_1^2}{n_z^2 c\theta_1^2} + 2\frac{n_x n_y s\theta_1}{n_z^2 c\theta_1}\right) = 1$$

$$\begin{cases} b_x = \pm\sqrt{\dfrac{1}{1 + \frac{s\theta_1^2}{c\theta_1^2} + \frac{n_x^2}{n_z^2} + \frac{n_y^2 s\theta_1^2}{n_z^2 c\theta_1^2} + 2\frac{n_x n_y s\theta_1}{n_z^2 c\theta_1}}} \\ b_y = \dfrac{s\theta_1}{c\theta_1} b_x \\ bz = -\dfrac{n_x}{n_z} b_x - \dfrac{n_y}{n_z} b_y \end{cases} \tag{12}$$

2.4 Solve a_x, a_y, a_z

Since $\vec{a} = \vec{b} \times \vec{n}$,

$$\begin{cases} a_x = b_y n_z - b_z n_y \\ a_y = b_z n_x - b_x n_z \\ a_z = b_x n_y - b_y n_x \end{cases} \tag{13}$$

From $(3,1)$ in the Equ. 9 we must check

$$s\theta_5 = s\theta_1 a_x - c\theta_1 a_y$$

2.5 Balance Equation

$$^1T_2^2T_3^3T_4 = (^0T_1)^{-1}(^0T_{ee})(^5T_{ee})^{-1}(^4T_5)^{-1}$$

$$^4T_5^{-1} = \begin{bmatrix} c_5 & s_5 & 0 & 0 \\ 0 & 0 & -1 & d_5 \\ -s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

$$Left = \begin{bmatrix} c_{234} & 0 & s_{234} & a_3 c_{23} + a_2 c_2 \\ s_{234} & 0 & -c_{234} & a_3 s_{23} + a_2 s_2 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ Right = \begin{bmatrix} Right_{11} & Right_{12} \\ Right_{21} & Right_{22} \end{bmatrix} \tag{15}$$

where

$$Right_{11} = \begin{bmatrix} c\theta_5(c\theta_1 a_x + s\theta_1 a_y) - s\theta_5(c\theta_1 n_x + s\theta_1 n_y) & s\theta_5(c\theta_1 a_x + s\theta_1 a_y) + c\theta_5(c\theta_1 n_x + s\theta_1 n_y) \\ c\theta_5 a_z - s\theta_5 n_z & s\theta_5 a_z + c\theta_5 n_z \end{bmatrix}$$

$$Right_{12} = \begin{bmatrix} -c\theta_1 b_x - s\theta_1 b_y & d_5(c\theta_1 b_x + s\theta_1 b_y) - d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y \\ -b_z & d_5 b_z - d_{ee} n_z + p_z - d_1 \end{bmatrix}$$

$$Right_{21} = \begin{bmatrix} c\theta_5(s\theta_1 a_x - c\theta_1 a_y) - s\theta_5(s\theta_1 n_x - c\theta_1 n_y) & s\theta_5(s\theta_1 a_x - c\theta_1 a_y) + c\theta_5(s\theta_1 n_x - c\theta_1 n_y) \\ 0 & 0 \end{bmatrix}$$

$$Right_{22} = \begin{bmatrix} -s\theta_1 b_x + c\theta_1 b_y & d_5(s\theta_1 b_x - c\theta_1 b_y) - d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y \\ 0 & 1 \end{bmatrix}$$

2.6 Solve θ_3

From $(1,4)$ and $(2,4)$ in Equ. 15 we know (and define m_1, m_2 as)

$$\begin{cases} m_1 \triangleq d_5(c\theta_1 b_x + s\theta_1 b_y) - d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y = a_3 c_{23} + a_2 c_2 \\ m_2 \triangleq d_5 b_z - d_{ee} n_z + p_z - d_1 = a_3 s_{23} + a_2 s_2 \end{cases} \tag{16}$$

$$\begin{cases} m_1^2 = a_3^2 c_{23}^2 + a_2^2 c_2^2 + 2a_2 a_3 c_2 c_{23} \\ m_2^2 = a_3^2 s_{23}^2 + a_2^2 s_2^2 + 2a_2 a_3 s_2 s_{23} \end{cases} \Rightarrow m_1^2 + m_2^2 = a_3^2 + a_2^2 + 2a_2 a_3(c_2 c_{23} + s_2 s_{23})$$

$$\theta_3 = \text{acosall}\left(\frac{m_1^2 + m_2^2 - a_3^2 - a_2^2}{2a_2 a_3}\right) \tag{17}$$

2.7 Solve θ_2

From Equ. 16 we know

$$\begin{cases} a_3(c\theta_2 c\theta_3 - s\theta_2 s\theta_3) + a_2 c\theta_2 = m_1 \\ a_3(s\theta_2 c\theta_3 + c\theta_2 s\theta_3) + a_2 s\theta_2 = m_2 \end{cases} \Rightarrow \begin{cases} \dfrac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} c\theta_2 - s\theta_2 = \dfrac{m_1}{a_3 s\theta_3} \\ \dfrac{a_3 s\theta_3}{a_3 c\theta_3 + a_2} c\theta_2 + s\theta_2 = \dfrac{m_2}{a_3 c\theta_3 + a_2} \end{cases}$$

$$\left(\frac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} + \frac{a_3 s\theta_3}{a_3 c\theta_3 + a_2}\right) c\theta_2 = \frac{m_1}{a_3 s\theta_3} + \frac{m_2}{a_3 c\theta_3 + a_2}$$

$$\theta_2 = \text{acosall} \frac{\frac{m_1}{a_3 s\theta_3} + \frac{m_2}{a_3 c\theta_3 + a_2}}{\frac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} + \frac{a_3 s\theta_3}{a_3 c\theta_3 + a_2}} \tag{18}$$

After solving θ_2 and θ_3, we should check whether θ_2 and θ_3 satisfy the equation 16, because we have used the square function which will lead to multiple solutions.

2.8 Solve θ_4

From $(1,1)$ and $(2,1)$ in Equ. 15 we know

$$\begin{cases} s_{234} = c\theta_5 a_z - s\theta_5 n_z \\ c_{234} = c\theta_5 (c\theta_1 a_x + s\theta_1 a_y) - s\theta_5 (c\theta_1 n_x + s\theta_1 n_y) \end{cases}$$

$$\theta_4 = -\theta_2 - \theta_3 + \text{atan2}(c\theta_5 a_z - s\theta_5 n_z, c\theta_5 (c\theta_1 a_x + s\theta_1 a_y) - s\theta_5 (c\theta_1 n_x + s\theta_1 n_y)) \tag{19}$$

3 Summary

$$\theta_1 = \text{acosall}\left(\frac{d_4}{\sqrt{(p_x - d_{ee} n_x)^2 + (d_{ee} n_y - p_y)^2}}\right) + \text{atan2}\left(p_x - d_{ee} n_x, d_{ee} n_y - p_y\right)$$

$$\theta_5 = \text{acosall}(s\theta_1 n_x - c\theta_1 n_y)$$

$$b_x = \pm \sqrt{\frac{1}{1 + \frac{s\theta_1^2}{c\theta_1^2} + \frac{n_x^2}{n_z^2} + \frac{n_y^2 s\theta_1^2}{n_z^2 c\theta_1^2} + 2\frac{n_x n_y s\theta_1}{n_z^2 c\theta_1}}}$$

$$b_y = \frac{s\theta_1}{c\theta_1} b_x$$

$$bz = -\frac{n_x}{n_z} b_x - \frac{n_y}{n_z} b_y$$

$$\begin{cases} a_x = b_y n_z - b_z n_y \\ a_y = b_z n_x - b_x n_z \\ a_z = b_x n_y - b_y n_x \end{cases}$$

$$m_1 \triangleq d_5(c\theta_1 b_x + s\theta_1 b_y) - d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y$$

$$m_2 \triangleq d_5 b_z - d_{ee} n_z + p_z - d_1$$

$$\theta_3 = \text{acosall}\left(\frac{m_1^2 + m_2^2 - a_3^2 - a_2^2}{2a_2 a_3}\right)$$

$$\theta_2 = \text{acosall} \frac{\frac{m_1}{a_3 s\theta_3} + \frac{m_2}{a_3 c\theta_3 + a_2}}{\frac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} + \frac{a_3 s\theta_3}{a_3 c\theta_3 + a_2}}$$

$$\theta_4 = -\theta_2 - \theta_3 + \text{atan2}(c\theta_5 a_z - s\theta_5 n_z, c\theta_5 (c\theta_1 a_x + s\theta_1 a_y) - s\theta_5 (c\theta_1 n_x + s\theta_1 n_y))$$

4 Exceptional case when $(p_x - d_{ee} n_x)^2 + (d_{ee} n_y - p_y)^2 = 0$

4.1 Solve θ_1

From $(3,4)$ in Equ. 7 and Equ.9, we know

$$d_4 = -d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y$$

But we cannot normalize the equation because $(p_x - d_{ee} n_x)^2 + (d_{ee} n_y - p_y)^2 = 0$.

5 Static Force Calculation

Assume that the manipulator must be able to exert given force on the z-axis of the EE (static force). For a given configuration, calculate the force on each joint.

The angle limit given by UR[1]

i	N/m
1	150
2	150
3	150
4	28
5	28
6	28

Since our requirement is only the force pointing at z-axis, large amount of calculation is omitted.

$$T = {}^0T_1 {}^1T_2 \cdots {}^4T_5 {}^5T_{ee}$$

$$\begin{aligned}
ee_z =& d_1 + a_2 \sin(t_2) \\
&- d_5 (\cos(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3)) - \sin(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2))) \\
&+ a_3 \cos(t_2)\sin(t_3) + a_3 \cos(t_3)\sin(t_2) \\
&- d_{\mathrm{ee}} \sin(t_5)(\cos(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2)) + \sin(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3)))
\end{aligned}$$

$$\frac{dee_z}{dt_1} = 0$$

$$\begin{aligned}
\frac{dee_z}{dt_2} =& a_2 \cos(t_2) \\
&+ d_5 (\cos(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2)) + \sin(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3))) \\
&+ a_3 \cos(t_2)\cos(t_3) - a_3 \sin(t_2)\sin(t_3) \\
&- d_{\mathrm{ee}} \sin(t_5)(\cos(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3)) - \sin(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2))) \\
\frac{dee_z}{dt_3} =& d_5 (\cos(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2)) + \sin(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3))) \\
&+ a_3 \cos(t_2)\cos(t_3) - a_3 \sin(t_2)\sin(t_3) \\
&- d_{\mathrm{ee}} \sin(t_5)(\cos(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3)) - \sin(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2))) \\
\frac{dee_z}{dt_4} =& d_5 (\cos(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2)) + \sin(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3))) \\
&- d_{\mathrm{ee}} \sin(t_5)(\cos(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3)) - \sin(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2))) \\
\frac{dee_z}{dt_5} =& -d_{\mathrm{ee}} \cos(t_5)(\cos(t_4)(\cos(t_2)\sin(t_3) + \cos(t_3)\sin(t_2)) + \sin(t_4)(\cos(t_2)\cos(t_3) - \sin(t_2)\sin(t_3)))
\end{aligned}$$

The Jacobi in velocity convey

$$^0v = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \frac{dee_z}{dt_1} & \frac{dee_z}{dt_2} & \frac{dee_z}{dt_3} & \frac{dee_z}{dt_4} & \frac{dee_z}{dt_5} \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \dot{\Theta} \tag{20}$$

Then the force on each joint is

$$\tau_{5\times1} = J^T F = \begin{bmatrix} * & * & \frac{dee_z}{dt_1} & * & * & * \\ * & * & \frac{dee_z}{dt_2} & * & * & * \\ * & * & \frac{dee_z}{dt_3} & * & * & * \\ * & * & \frac{dee_z}{dt_4} & * & * & * \\ * & * & \frac{dee_z}{dt_5} & * & * & * \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ f \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

6 Robustness on the EE

When the manipulator is near its singularities, it can exert any force because of the singularty in the static force. To get rid of it, we use the

[1]http://www.universal-robots.com/how-tos-and-faqs/faq/ur-faq/max-joint-torques-17260/