1 Problem and Parameter

Consider a 5DOF Manipulator holding the end-effector (EE) on the given 3D position and pointing (EE's z-axis) at the given direction. D-H parameter is given as follow

i	α_i	a_i	θ_i	d_i
1	$\pi/2$	0	θ_1	0.089159
2	0	-0.425	θ_2	0
3	0	-0.39225	θ_3	0
4	$\pi/2$	0	θ_4	0.10915
5	$-\pi/2$	0	θ_5	0.09465
6	0	0	θ_6	0.0823

Target pose is

$$(p_x, p_y, p_z, \vec{n}),\ \vec{n} = [n_x, n_y, n_z]^T$$

where $\vec{n}$ is a normal vector of the surface of the object.

Since the end-effector should be perpendicular to the surface,

$$^0T_{ee} = \begin{bmatrix} a_x & b_x & n_x & p_x \\ a_y & b_y & n_y & p_y \\ a_z & b_z & n_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

where $a_x, a_y, a_z, b_x, b_y, b_z$ are all variables, and

$$\begin{cases} a_x n_x + a_y n_y + a_z n_z = 0 \\ b_x n_x + b_y n_y + b_z n_z = 0 \\ a_x b_x + a_y b_y + a_z b_z = 0 \\ \quad\quad\quad\quad \| a \|_2 = 1 \\ \quad\quad\quad\quad \| b \|_2 = 1 \end{cases} \tag{2}$$

2 Kinematics

The kinematic function of the manipulator:

$$T = {}^0T_1 {}^1T_2 \cdots {}^4T_5 {}^5T_{ee}$$

where

$$^0T_1 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^1T_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^2T_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$^3T_4 = \begin{bmatrix} c\theta_4 & 0 & s\theta_4 & 0 \\ s\theta_4 & 0 & -c\theta_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^4T_5 = \begin{bmatrix} c\theta_5 & 0 & -s\theta_5 & 0 \\ s\theta_5 & 0 & c\theta_5 & 0 \\ 0 & -1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ ^5T_{ee} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

To simplify the calculation, we get

$$\begin{aligned} ^1T_3 = {}^1T_2 {}^2T_3 &= \begin{bmatrix} c\theta_2 c\theta_3 - s\theta_2 s\theta_3 & -c\theta_2 s\theta_3 - c\theta_3 s\theta_2 & 0 & a_2 c\theta_2 + a_3(c\theta_2 c\theta_3 - s\theta_2 s\theta_3) \\ c\theta_2 s\theta_3 + c\theta_3 s\theta_2 & c\theta_2 c\theta_3 - s\theta_2 s\theta_3 & 0 & a_2 s\theta_2 + a_3(c\theta_2 s\theta_3 + a_3 c\theta_3 s\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\triangleq \begin{bmatrix} c_{23} & -s_{23} & 0 & a_3 c_{23} + a_2 c_2 \\ s_{23} & c_{23} & 0 & a_3 s_{23} + a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{5}$$

$$^1T_4 = {}^1T_3 {}^3T_4 = \begin{bmatrix} c_{23} c\theta_4 - s_{23} s\theta_4 & 0 & c_{23} s\theta_4 + s_{23} c\theta_4 & a_3 c_{23} + a_2 c\theta_2 \\ s_{23} c\theta_4 + c_{23} s\theta_4 & 0 & s_{23} s\theta_4 - c_{23} c\theta_4 & a_3 s_{23} + a_2 s\theta_2 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$ {}^1T_5 = {}^1T_4 {}^4T_5 = \begin{bmatrix} (c_{23}c\theta_4 - s_{23}s\theta_4)c\theta_5 & -c_{23}s\theta_4 - s_{23}c\theta_4 & (-c_{23}c\theta_4 + s_{23}s\theta_4)s\theta_5 & d_5(c_{23}s\theta_4 + s_{23}c\theta_4) + a_3c_{23} + a_2c_2 \\ (s_{23}c\theta_4 + c_{23}s\theta_4)c\theta_5 & -s_{23}s\theta_4 + c_{23}c\theta_4 & (-s_{23}c\theta_4 - c_{23}s\theta_4)s\theta_5 & d_5(s_{23}s\theta_4 - c_{23}c\theta_4) + a_3s_{23} + a_2s\theta_2 \\ s\theta_5 & 0 & c\theta_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7} $$

We can precalculate the ${}^0T_1$ and ${}^5T_{ee}$ to simplify the calculation

$$ ({}^0T_1)^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ s\theta_1 & -c\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; ({}^5T_{ee})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8} $$

So we get

$$ \begin{aligned} {}^1T_5 &= (({}^0T_1)^{-1}){}^0T_{ee}(({}^5T_{ee})^{-1}) \\ &= \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ s\theta_1 & -c\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x & b_x & n_x & p_x \\ a_y & b_y & n_y & p_y \\ a_z & b_z & n_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c\theta_1 a_x + s\theta_1 a_y & c\theta_1 b_x + s\theta_1 b_y & c\theta_1 n_x + s\theta_1 n_y & c\theta_1 p_x + s\theta_1 p_y \\ a_z & b_z & n_z & p_z - d_1 \\ s\theta_1 a_x - c\theta_1 a_y & s\theta_1 b_x - c\theta_1 b_y & s\theta_1 n_x - c\theta_1 n_y & s\theta_1 p_x - c\theta_1 p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_{ee} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c\theta_1 a_x + s\theta_1 a_y & c\theta_1 b_x + s\theta_1 b_y & c\theta_1 n_x + s\theta_1 n_y & -d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y \\ a_z & b_z & n_z & -d_{ee}n_z + p_z - d_1 \\ s\theta_1 a_x - c\theta_1 a_y & s\theta_1 b_x - c\theta_1 b_y & s\theta_1 n_x - c\theta_1 n_y & -d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{9} $$

2.1 Solve θ_1

From $(3,4)$ in Equ. 7 and Equ.9, we know

$$ d_4 = -d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y $$

$$ \frac{p_x - d_{ee}n_x}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}} s\theta_1 + \frac{d_{ee}n_y - p_y}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}} c\theta_1 = \frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}} $$

Set ϕ that

$$ \tan\phi = \frac{p_x - d_{ee}n_x}{d_{ee}n_y - p_y} \Rightarrow \phi = \operatorname{atan2}\left(\frac{p_x - d_{ee}n_x}{d_{ee}n_y - p_y}\right) $$

$$ \cos(\theta_1 - \phi) = \sin\phi \sin\theta_1 + \cos\phi\cos\theta_1 = \frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}} $$

$$ \Rightarrow \theta_1 - \phi = \operatorname{acosall}\left(\frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}}\right) $$

$$ \theta_1 = \operatorname{acosall}\left(\frac{d_4}{\sqrt{(p_x - d_{ee}n_x)^2 + (d_{ee}n_y - p_y)^2}}\right) + \operatorname{atan2}(p_x - d_{ee}n_x, d_{ee}n_y - p_y) \tag{10} $$

2.2 Solve θ_5

From $(3,3)$ in Equ. 7 and Equ.9, we know

$$ c\theta_5 = s\theta_1 n_x - c\theta_1 n_y $$

$$ \theta_5 = \operatorname{acosall}(s\theta_1 n_x - c\theta_1 n_y) \tag{11} $$

2.3 Solve b_x, b_y, b_z

Compared to the derivation in Tmech, we do following improvements:

2.3.1 If $n_z \neq 0$

From $(3,2)$ in Equ. 7 and Equ.9 and Equ. 2, we know

$$\begin{cases} s\theta_1 b_x - c\theta_1 b_y = 0 \\ b_x n_x + b_y n_y + b_z n_z = 0 \\ b_x^2 + b_y^2 + b_z^2 = 1 \end{cases} \Rightarrow \begin{cases} b_y = \dfrac{s\theta_1}{c\theta_1} b_x \\ \dfrac{n_x}{n_z} b_x + \dfrac{n_y}{n_z} b_y = -b_z \end{cases}$$

$$b_x^2 + \left(\frac{s\theta_1}{c\theta_1} b_x\right)^2 + \left(\frac{n_x}{n_z} b_x + \frac{n_y}{n_z}\frac{s\theta_1}{c\theta_1} b_x\right)^2 = 1$$

$$b_x^2\left(1 + \frac{s\theta_1^2}{c\theta_1^2} + \frac{n_x^2}{n_z^2} + \frac{n_y^2 s\theta_1^2}{n_z^2 c\theta_1^2} + 2\frac{n_x n_y s\theta_1}{n_z^2 c\theta_1}\right) = 1$$

$$\begin{cases} b_x = \pm\sqrt{\dfrac{1}{1 + \frac{s\theta_1^2}{c\theta_1^2} + \frac{n_x^2}{n_z^2} + \frac{n_y^2 s\theta_1^2}{n_z^2 c\theta_1^2} + 2\frac{n_x n_y s\theta_1}{n_z^2 c\theta_1}}} \\ b_y = \dfrac{s\theta_1}{c\theta_1} b_x \\ bz = -\dfrac{n_x}{n_z} b_x - \dfrac{n_y}{n_z} b_y \end{cases} \tag{12}$$

2.3.2 If $n_z = 0$

From $(3,2)$ in Equ. 7 and Equ.9 and Equ. 2, we know

$$\begin{cases} s\theta_1 b_x - c\theta_1 b_y = 0 \\ b_x n_x + b_y n_y + b_z n_z = 0 \\ b_x^2 + b_y^2 + b_z^2 = 1 \end{cases} \Rightarrow \begin{cases} s\theta_1 b_x - c\theta_1 b_y = 0 \\ b_x n_x + b_y n_y = 0 \\ b_x^2 + b_y^2 + b_z^2 = 1 \end{cases} \Rightarrow \begin{cases} b_y = \dfrac{s\theta_1}{c\theta_1} b_x \\ (s\theta_1 = n_x, c\theta_1 = -n_y \text{ or opposite}) \end{cases}$$

Hence

$$\begin{cases} b_x = \pm c\theta_1 \sqrt{1 - b_z^2} \\ b_y = \dfrac{s\theta_1}{c\theta_1} b_x \\ b_z = \forall \in (-1, 1) \end{cases} \text{ or } \begin{cases} b_x = 0 \\ b_y = 0 \\ b_z = -1 \text{ or } 1 \end{cases}$$

2.4 Solve a_x, a_y, a_z

Since $\vec{a} = \vec{b} \times \vec{n}$,

$$\begin{cases} a_x = b_y n_z - b_z n_y \\ a_y = b_z n_x - b_x n_z \\ a_z = b_x n_y - b_y n_x \end{cases} \tag{13}$$

From $(3,1)$ in the Equ. 9 we must check

$$s\theta_5 = s\theta_1 a_x - c\theta_1 a_y$$

2.5 Balance Equation

$$^1T_2 {}^2T_3 {}^3T_4 = ({}^0T_1)^{-1}({}^0T_{ee})({}^5T_{ee})^{-1}({}^4T_5)^{-1}$$

$$^4T_5^{-1} = \begin{bmatrix} c_5 & s_5 & 0 & 0 \\ 0 & 0 & -1 & d_5 \\ -s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

$$Left = \begin{bmatrix} c_{234} & 0 & s_{234} & a_3 c_{23} + a_2 c_2 \\ s_{234} & 0 & -c_{234} & a_3 s_{23} + a_2 s_2 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ Right = \begin{bmatrix} Right_{11} & Right_{12} \\ Right_{21} & Right_{22} \end{bmatrix} \tag{15}$$

where

$$Right_{11} = \begin{bmatrix} c\theta_5(c\theta_1 a_x + s\theta_1 a_y) - s\theta_5(c\theta_1 n_x + s\theta_1 n_y) & s\theta_5(c\theta_1 a_x + s\theta_1 a_y) + c\theta_5(c\theta_1 n_x + s\theta_1 n_y) \\ c\theta_5 a_z - s\theta_5 n_z & s\theta_5 a_z + c\theta_5 n_z \end{bmatrix}$$

$$Right_{12} = \begin{bmatrix} -c\theta_1 b_x - s\theta_1 b_y & d_5(c\theta_1 b_x + s\theta_1 b_y) - d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y \\ -b_z & d_5 b_z - d_{ee} n_z + p_z - d_1 \end{bmatrix}$$

$$Right_{21} = \begin{bmatrix} c\theta_5(s\theta_1 a_x - c\theta_1 a_y) - s\theta_5(s\theta_1 n_x - c\theta_1 n_y) & s\theta_5(s\theta_1 a_x - c\theta_1 a_y) + c\theta_5(s\theta_1 n_x - c\theta_1 n_y) \\ 0 & 0 \end{bmatrix}$$

$$Right_{22} = \begin{bmatrix} -s\theta_1 b_x + c\theta_1 b_y & d_5(s\theta_1 b_x - c\theta_1 b_y) - d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y \\ 0 & 1 \end{bmatrix}$$

2.6 Solve θ_3

From $(1,4)$ and $(2,4)$ in Equ. 15 we know (and define m_1, m_2 as)

$$\begin{cases} m_1 \triangleq d_5(c\theta_1 b_x + s\theta_1 b_y) - d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y = a_3 c_{23} + a_2 c_2 \\ m_2 \triangleq d_5 b_z - d_{ee} n_z + p_z - d_1 = a_3 s_{23} + a_2 s_2 \end{cases} \tag{16}$$

$$\begin{cases} m_1^2 = a_3^2 c_{23}^2 + a_2^2 c_2^2 + 2a_2 a_3 c_2 c_{23} \\ m_2^2 = a_3^2 s_{23}^2 + a_2^2 s_2^2 + 2a_2 a_3 s_2 s_{23} \end{cases} \Rightarrow m_1^2 + m_2^2 = a_3^2 + a_2^2 + 2a_2 a_3 (c_2 c_{23} + s_2 s_{23})$$

$$\theta_3 = \text{acosall}\left(\frac{m_1^2 + m_2^2 - a_3^2 - a_2^2}{2a_2 a_3}\right) \tag{17}$$

2.7 Solve θ_2

From Equ. 16 we know

$$\begin{cases} a_3(c\theta_2 c\theta_3 - s\theta_2 s\theta_3) + a_2 c\theta_2 = m_1 \\ a_3(s\theta_2 c\theta_3 + c\theta_2 s\theta_3) + a_2 s\theta_2 = m_2 \end{cases} \Rightarrow \begin{cases} \dfrac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} c\theta_2 - s\theta_2 = \dfrac{m_1}{a_3 s\theta_3} \\ \dfrac{a_3 s\theta_3}{a_3 c\theta_3 + a_2} c\theta_2 + s\theta_2 = \dfrac{m_2}{a_3 c\theta_3 + a_2} \end{cases}$$

$$\left(\frac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} + \frac{a_3 s\theta_3}{a_3 c\theta_3 + a_2}\right) c\theta_2 = \frac{m_1}{a_3 s\theta_3} + \frac{m_2}{a_3 c\theta_3 + a_2}$$

$$\theta_2 = \text{acosall}\frac{\frac{m_1}{a_3 s\theta_3} + \frac{m_2}{a_3 c\theta_3 + a_2}}{\frac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} + \frac{a_3 s\theta_3}{a_3 c\theta_3 + a_2}} \tag{18}$$

After solving θ_2 and θ_3, we should check whether θ_2 and θ_3 satisfy the equation 16, because we have used the square function which will lead to multiple solutions.

2.8 Solve θ_4

From $(1,1)$ and $(2,1)$ in Equ. 15 we know

$$\begin{cases} s_{234} = c\theta_5 a_z - s\theta_5 n_z \\ c_{234} = c\theta_5(c\theta_1 a_x + s\theta_1 a_y) - s\theta_5(c\theta_1 n_x + s\theta_1 n_y) \end{cases}$$

$$\theta_4 = -\theta_2 - \theta_3 + \text{atan2}(c\theta_5 a_z - s\theta_5 n_z, c\theta_5(c\theta_1 a_x + s\theta_1 a_y) - s\theta_5(c\theta_1 n_x + s\theta_1 n_y)) \tag{19}$$

3 Summary

$$\theta_1 = \text{acosall}\left(\frac{d_4}{\sqrt{(p_x - d_{ee} n_x)^2 + (d_{ee} n_y - p_y)^2}}\right) + \text{atan2}\left(p_x - d_{ee} n_x, d_{ee} n_y - p_y\right)$$

$$\theta_5 = \text{acosall}(s\theta_1 n_x - c\theta_1 n_y)$$

$$b_x = \pm\sqrt{\frac{1}{1 + \frac{s\theta_1^2}{c\theta_1^2} + \frac{n_x^2}{n_z^2} + \frac{n_y^2 s\theta_1^2}{n_z^2 c\theta_1^2} + 2\frac{n_x n_y s\theta_1}{n_z^2 c\theta_1}}}$$

$$b_y = \frac{s\theta_1}{c\theta_1} b_x$$

$$bz = -\frac{n_x}{n_z} b_x - \frac{n_y}{n_z} b_y$$

$$\begin{cases} a_x = b_y n_z - b_z n_y \\ a_y = b_z n_x - b_x n_z \\ a_z = b_x n_y - b_y n_x \end{cases}$$

$$m_1 \triangleq d_5(c\theta_1 b_x + s\theta_1 b_y) - d_{ee}(c\theta_1 n_x + s\theta_1 n_y) + c\theta_1 p_x + s\theta_1 p_y$$

$$m_2 \triangleq d_5 b_z - d_{ee} n_z + p_z - d_1$$

$$\theta_3 = \mathrm{acosall}(\frac{m_1^2 + m_2^2 - a_3^2 - a_2^2}{2a_2 a_3})$$

$$\theta_2 = \mathrm{acosall}\frac{\frac{m_1}{a_3 s\theta_3} + \frac{m_2}{a_3 c\theta_3 + a_2}}{\frac{a_3 c\theta_3 + a_2}{a_3 s\theta_3} + \frac{a_3 s\theta_3}{a_3 c\theta_3 + a_2}}$$

$$\theta_4 = -\theta_2 - \theta_3 + \mathrm{atan2}(c\theta_5 a_z - s\theta_5 n_z, c\theta_5(c\theta_1 a_x + s\theta_1 a_y) - s\theta_5(c\theta_1 n_x + s\theta_1 n_y))$$

4 Exceptional case when $(p_x - d_{ee} n_x)^2 + (d_{ee} n_y - p_y)^2 = 0$

4.1 Solve θ_1

From $(3,4)$ in Equ. 7 and Equ.9, we know

$$d_4 = -d_{ee}(s\theta_1 n_x - c\theta_1 n_y) + s\theta_1 p_x - c\theta_1 p_y$$

But we cannot normalize the equation because $(p_x - d_{ee} n_x)^2 + (d_{ee} n_y - p_y)^2 = 0$.

5 A singular configuration case

Consider following EE pose

$$(p_x, p_y, p_z, n_x, n_y, n_z) = [0.5285, 0.1091, 0.1757, 0, 1, 0], d_{ee} = 0.09 \tag{20}$$

And we go through each equation to check the validity.