GLM fundamentals
A gentle theory of the three pillars of response, link and linear predictor

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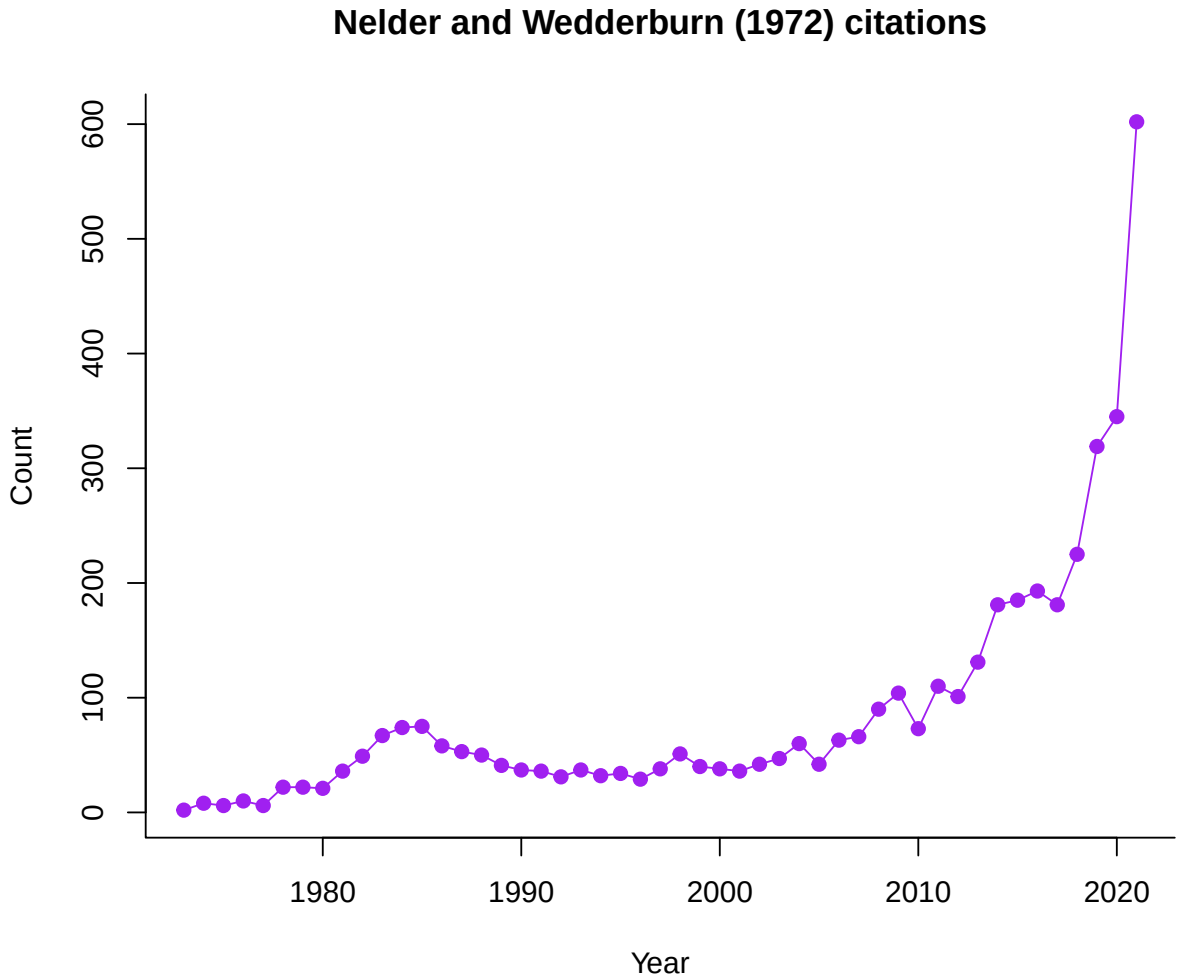
Outline

A little bit about GLMs

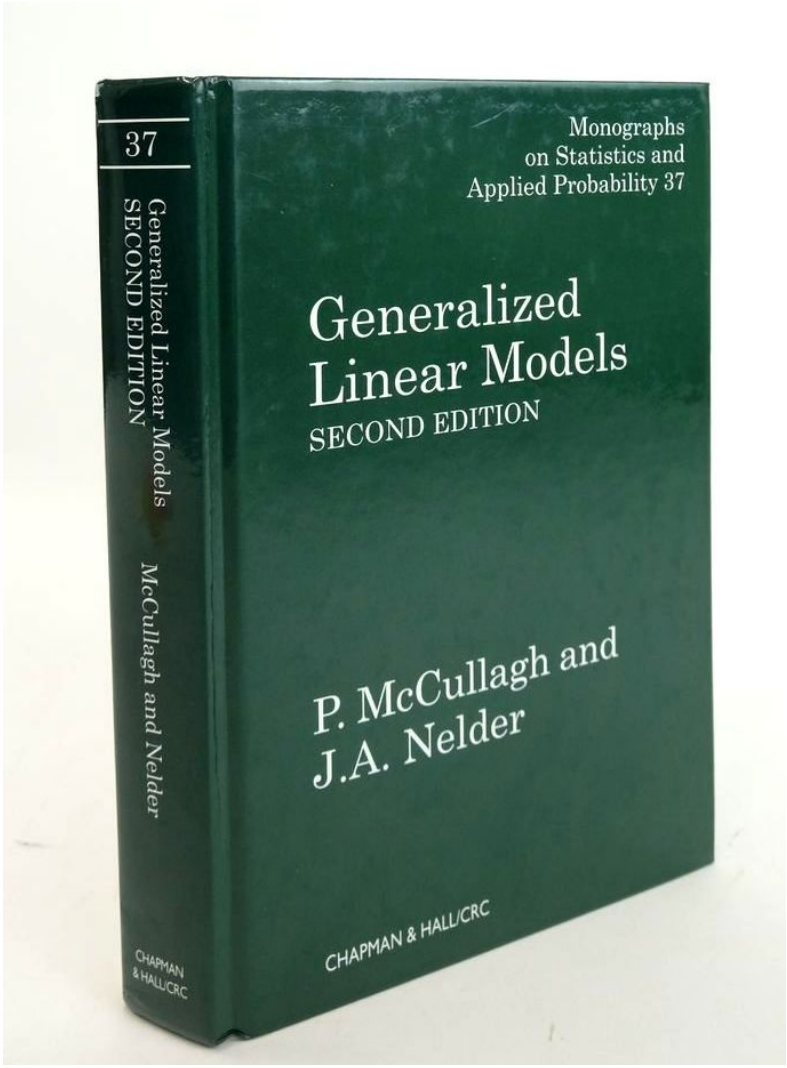
Developed by Nelder and Wedderburn (1972)[1]



[1] Nelder, J. and Wedderburn, R. (1972). Generalized Linear Models. *Journal of the Royal Statistical Society. Series A (General)*, 135 (3), 370–384.

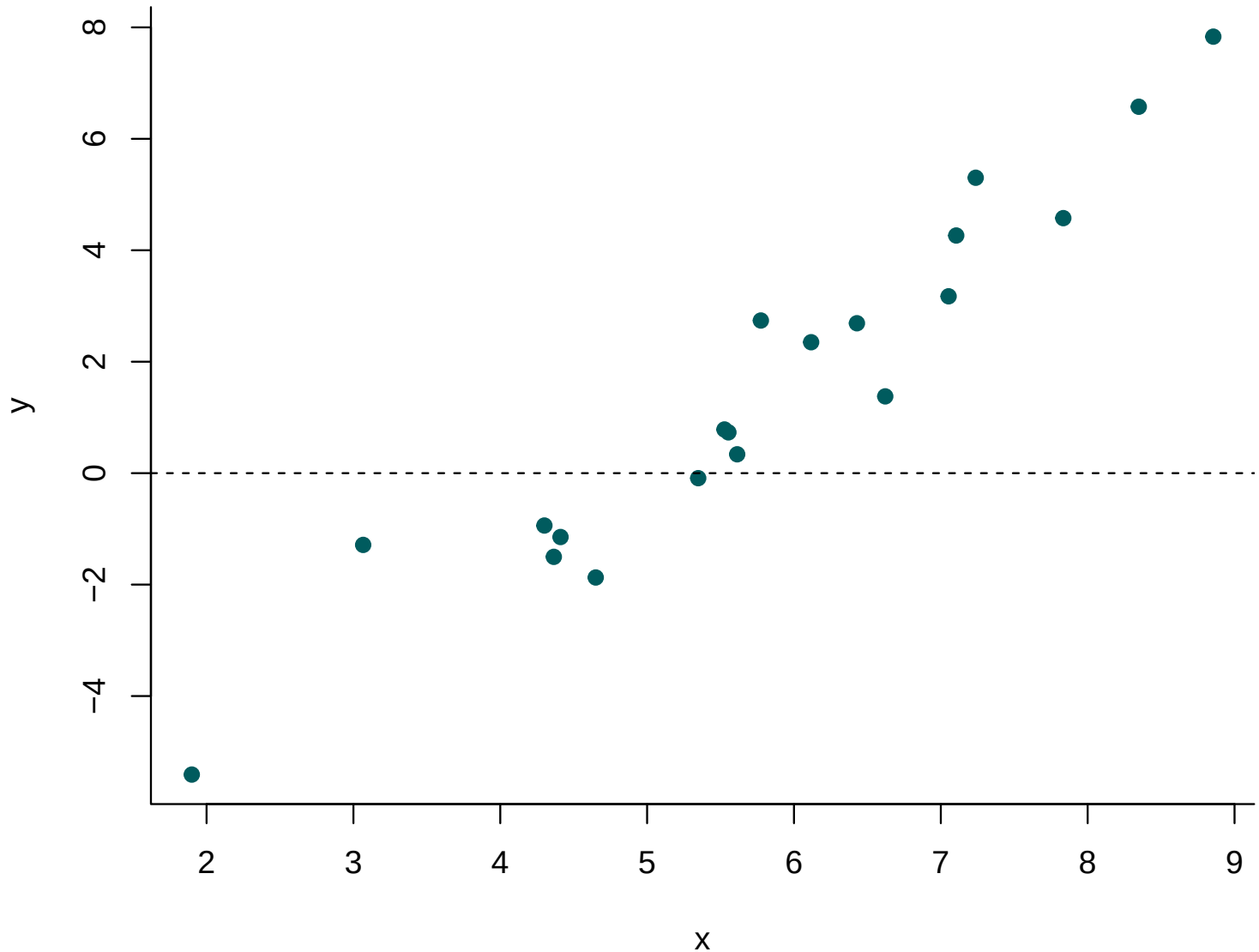
A little bit about GLMs

Famous book - McCullagh and Nelder (cited >43,000 times)

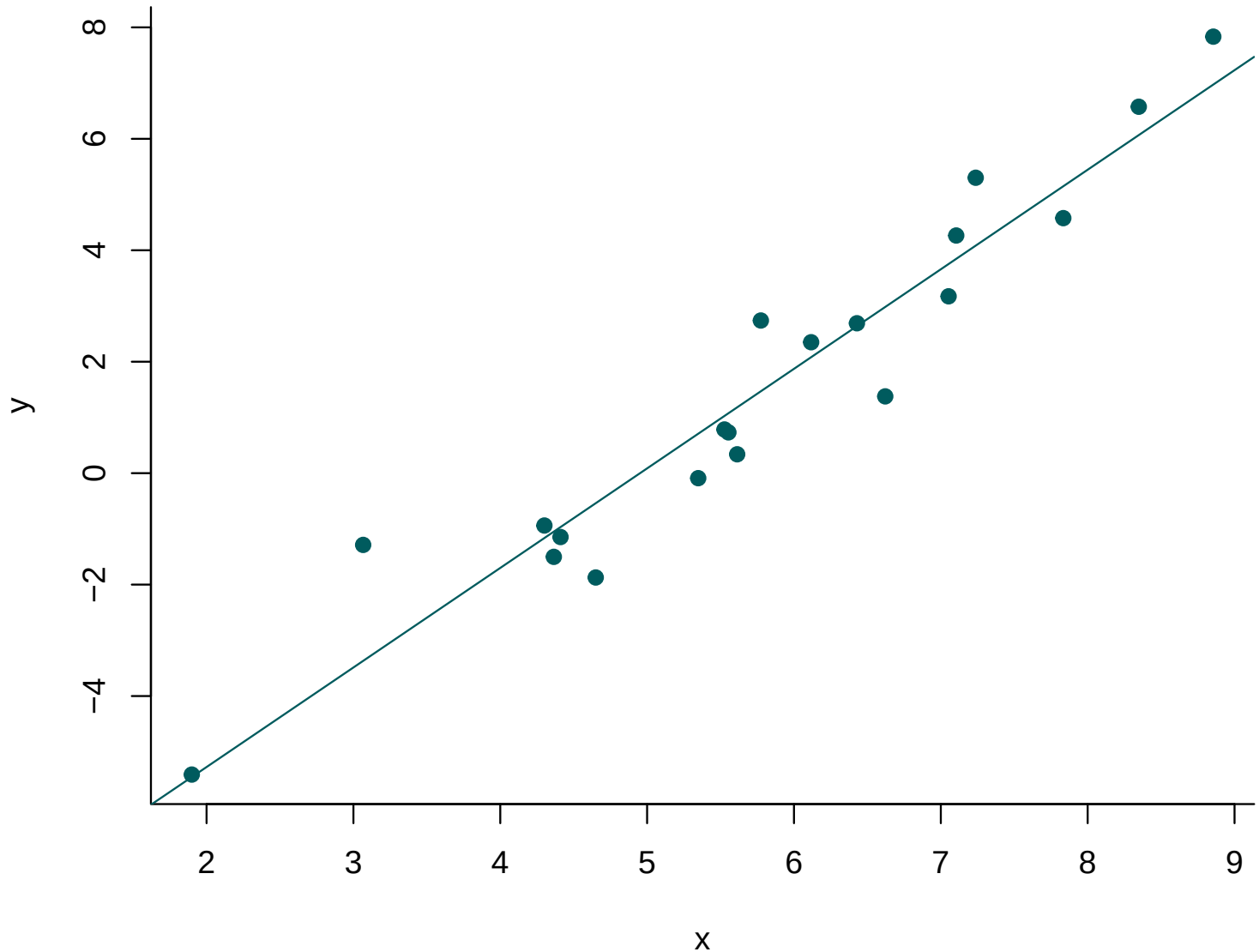


Outline

Linear regression - normal errors



Linear regression - normal errors



Linear regression - normal errors

For the linear regression

$$y \sim \mathrm{N}(\mu, \sigma^2)$$

where

$$\mu = a + bx$$

what are a and b?

Nonnormal errors

Often errors are non-normally distributed:

- May exhibit a lot of skew
- Exhibit heavy tails
- Discrete variable (e.g. count, especially low counts)
- May be bounded (e.g. binary data)

In these cases, the use of normal distribution is typically not appropriate

Nonnormal errors

Traditional approach has been to transform the data prior to analysis to achieve:

- Constancy of the variance (normality)
- Additivity of effects $bx + cz$

Transformations sometimes work spectacularly well (lognormal data), where log transforming stabilizes the variance and results in additivity but for other data (counts) most transforms cannot achieve both goals

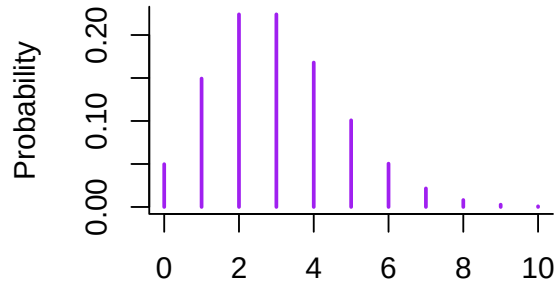
Generalized Linear Models (GLMs)

GLMs allow one to fit normal and non-normal error models from the exponential family, including:

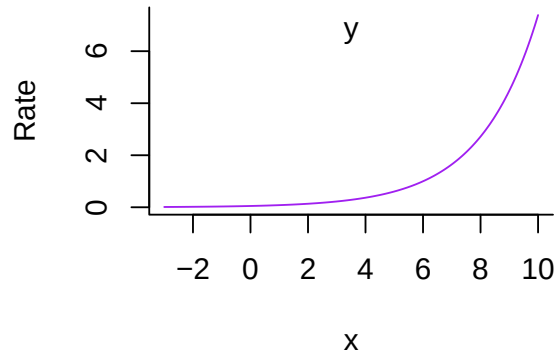
- Binomial
- Poisson
- Gamma
- Normal
- ...

Three components of a GLM

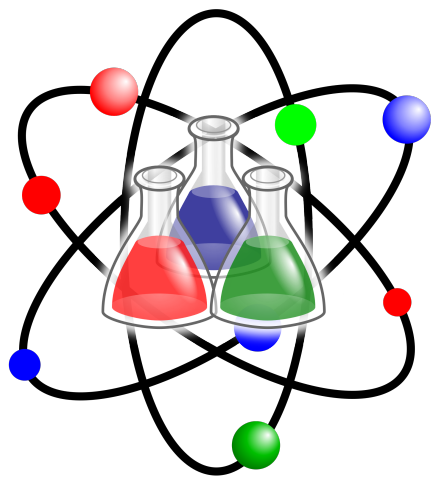
Distribution



Link function



Linear predictor



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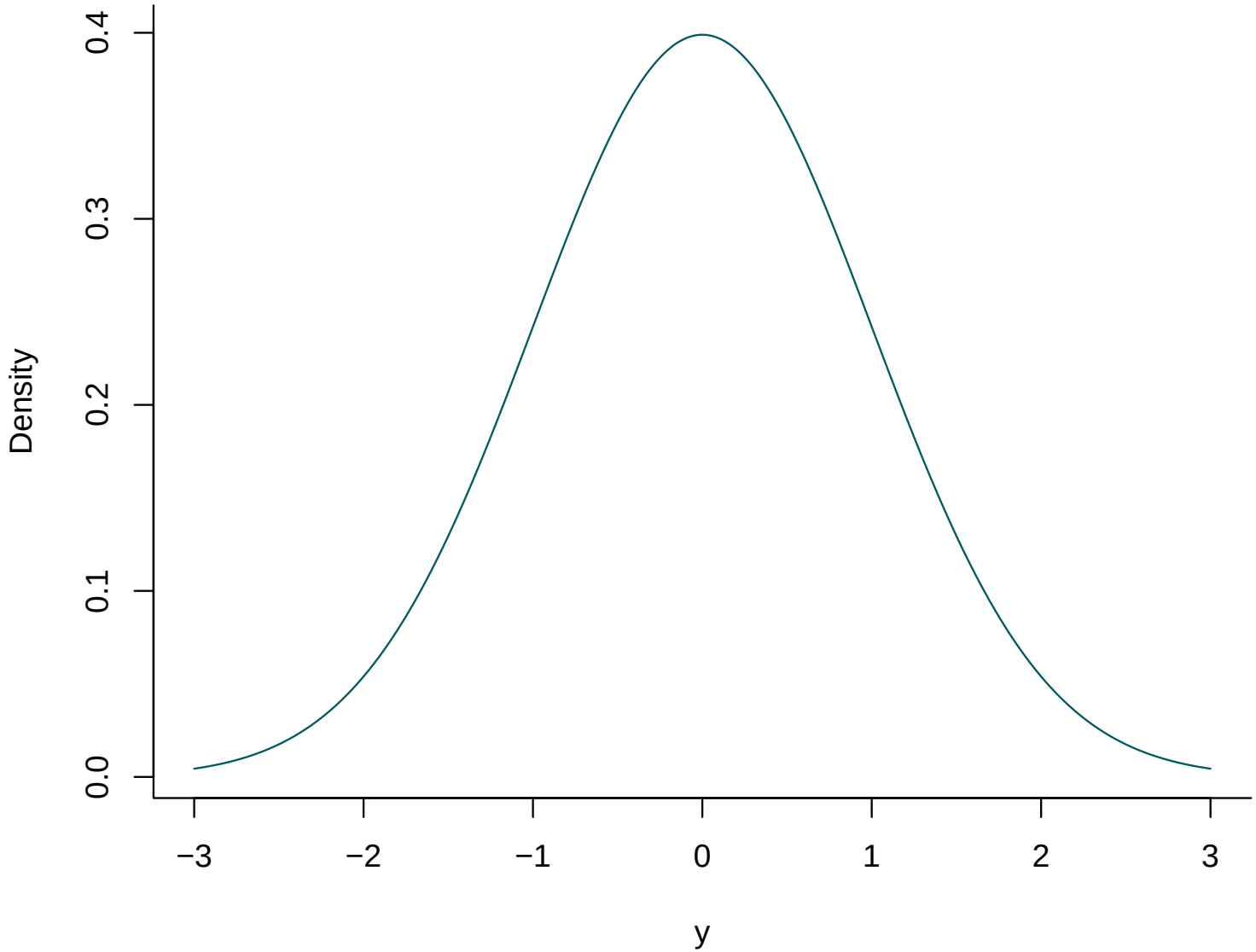
Distribution



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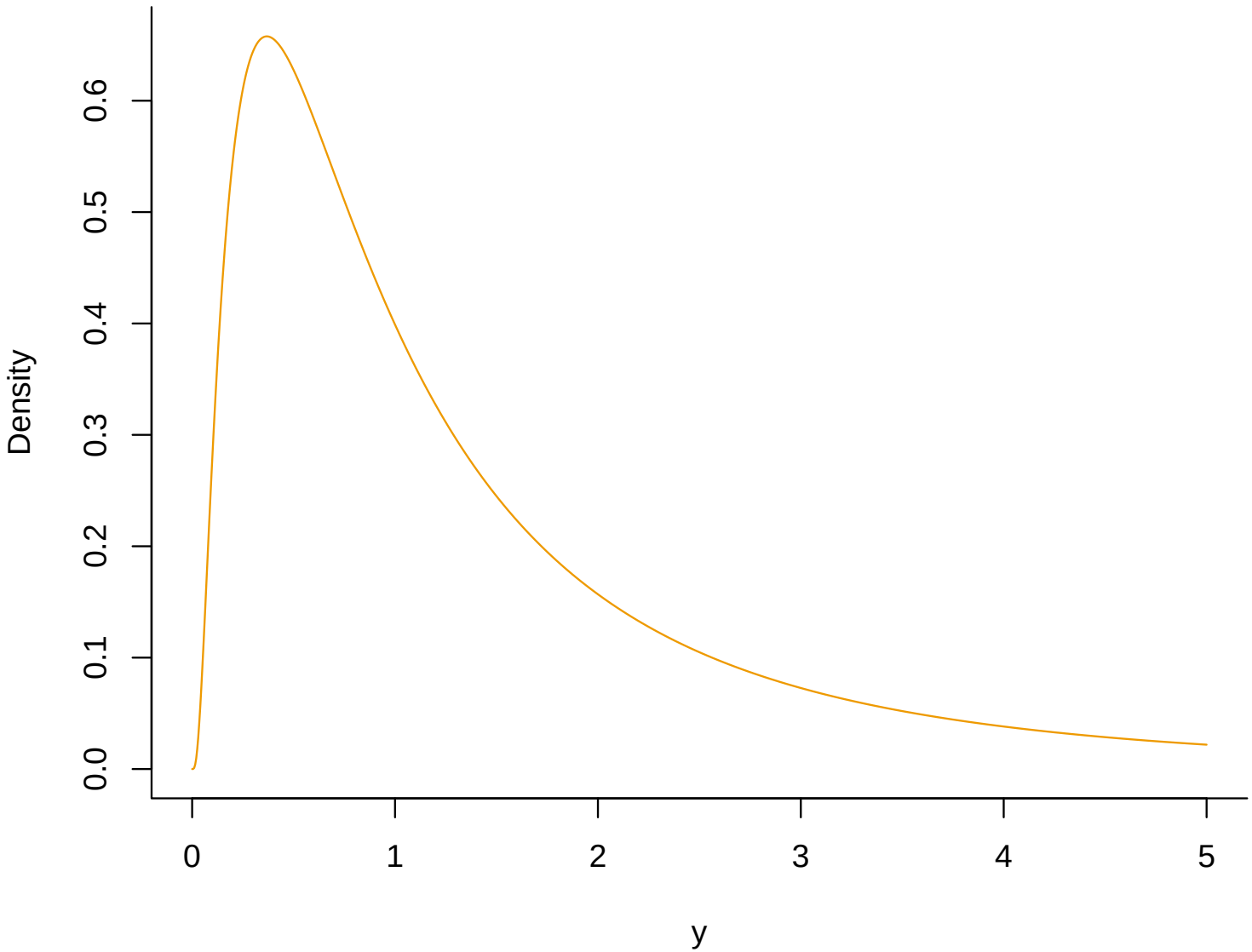

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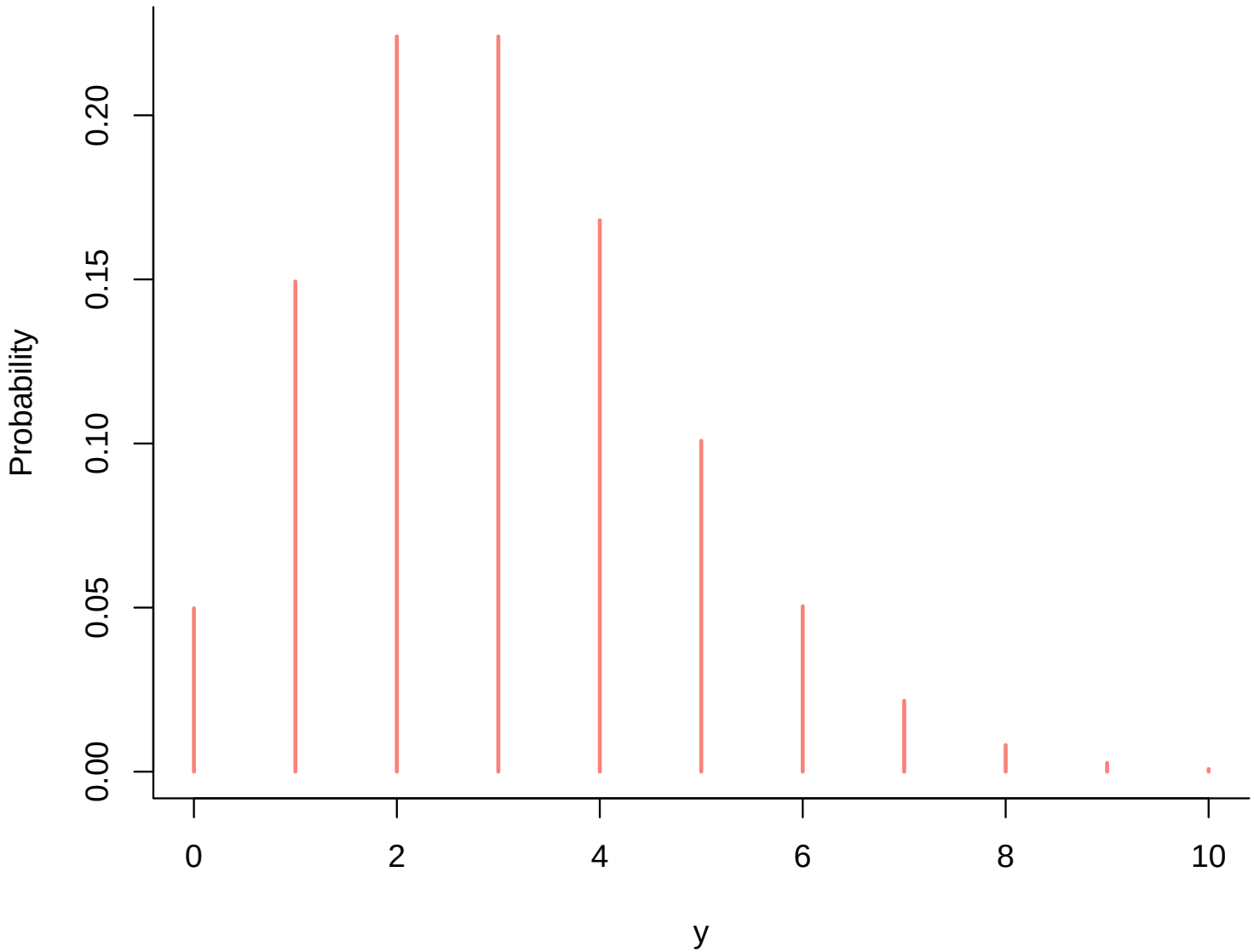

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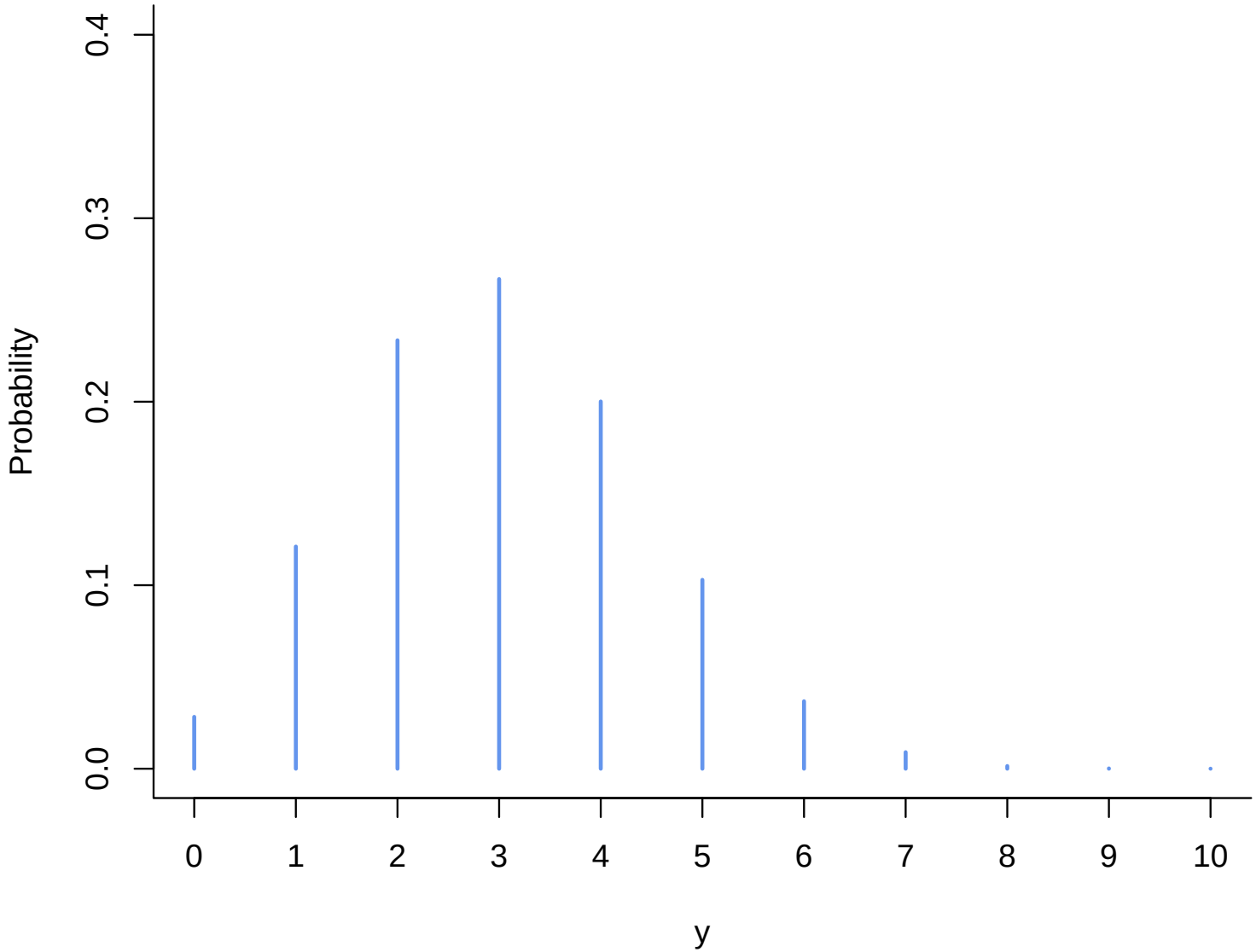
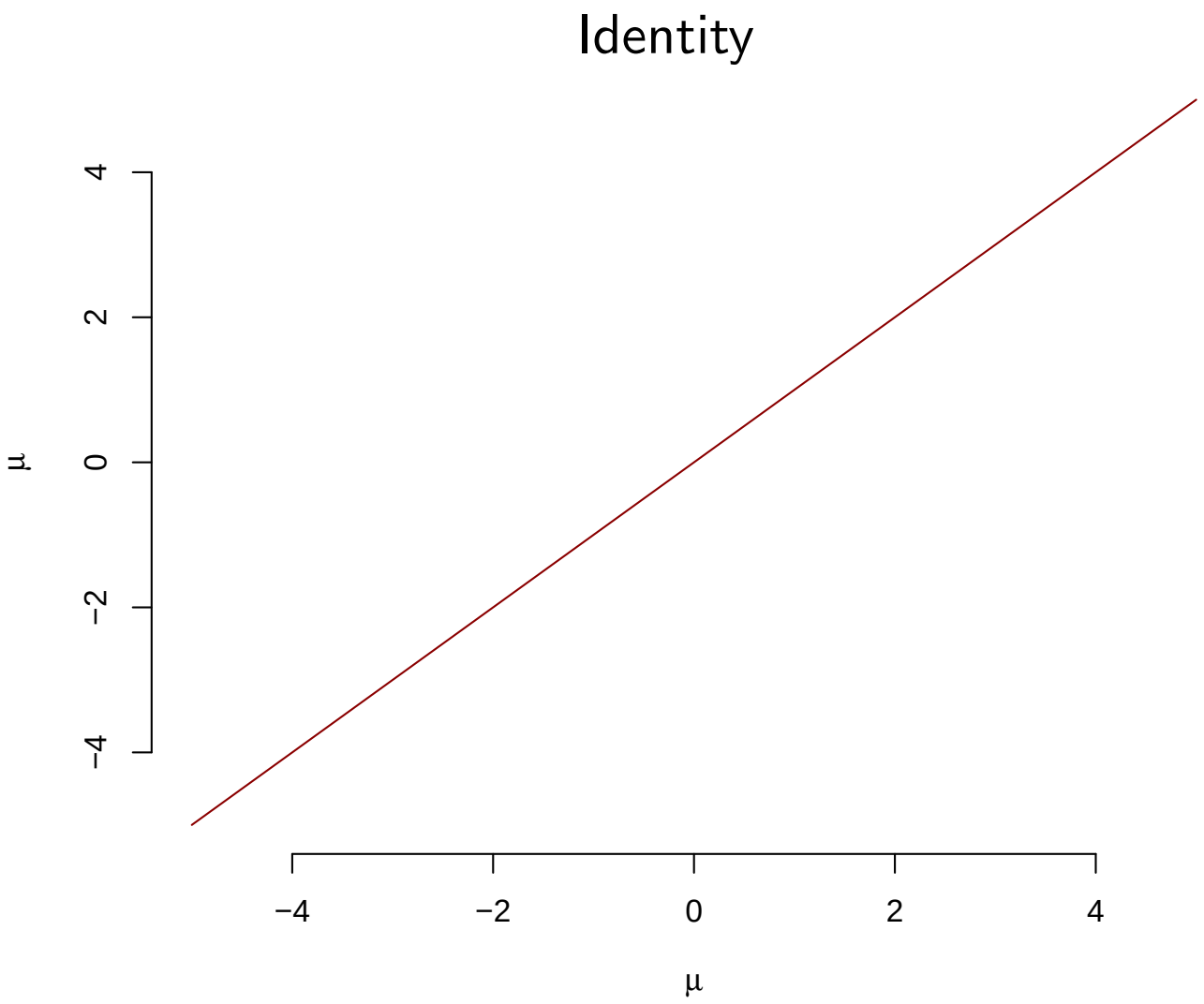


Link function

Link function transforms parameter from its natural scale (positive, between zero and one, ...) to the real line.
Think of it as a way to make sure the parameter is bounded correctly

Link function examples



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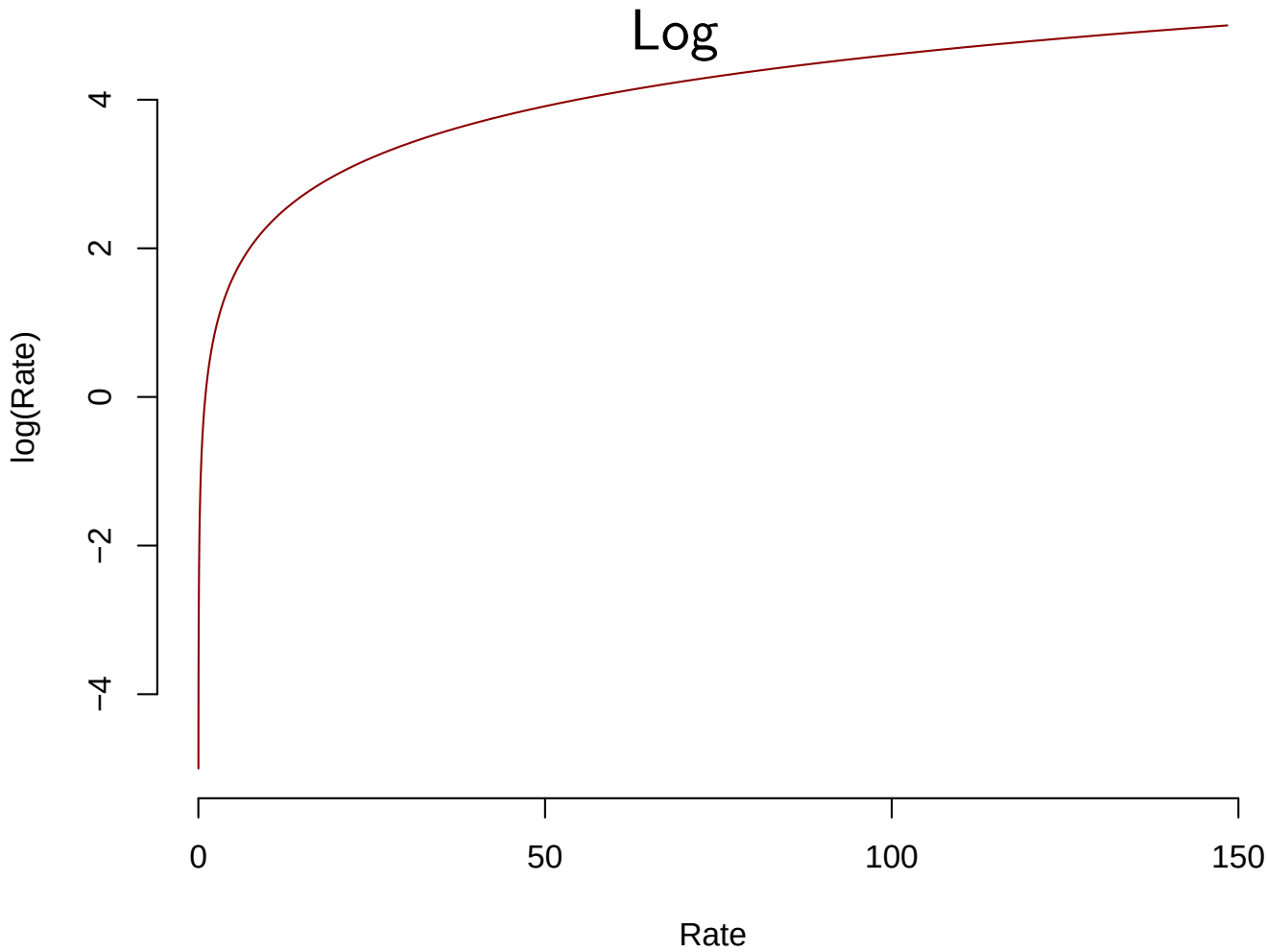
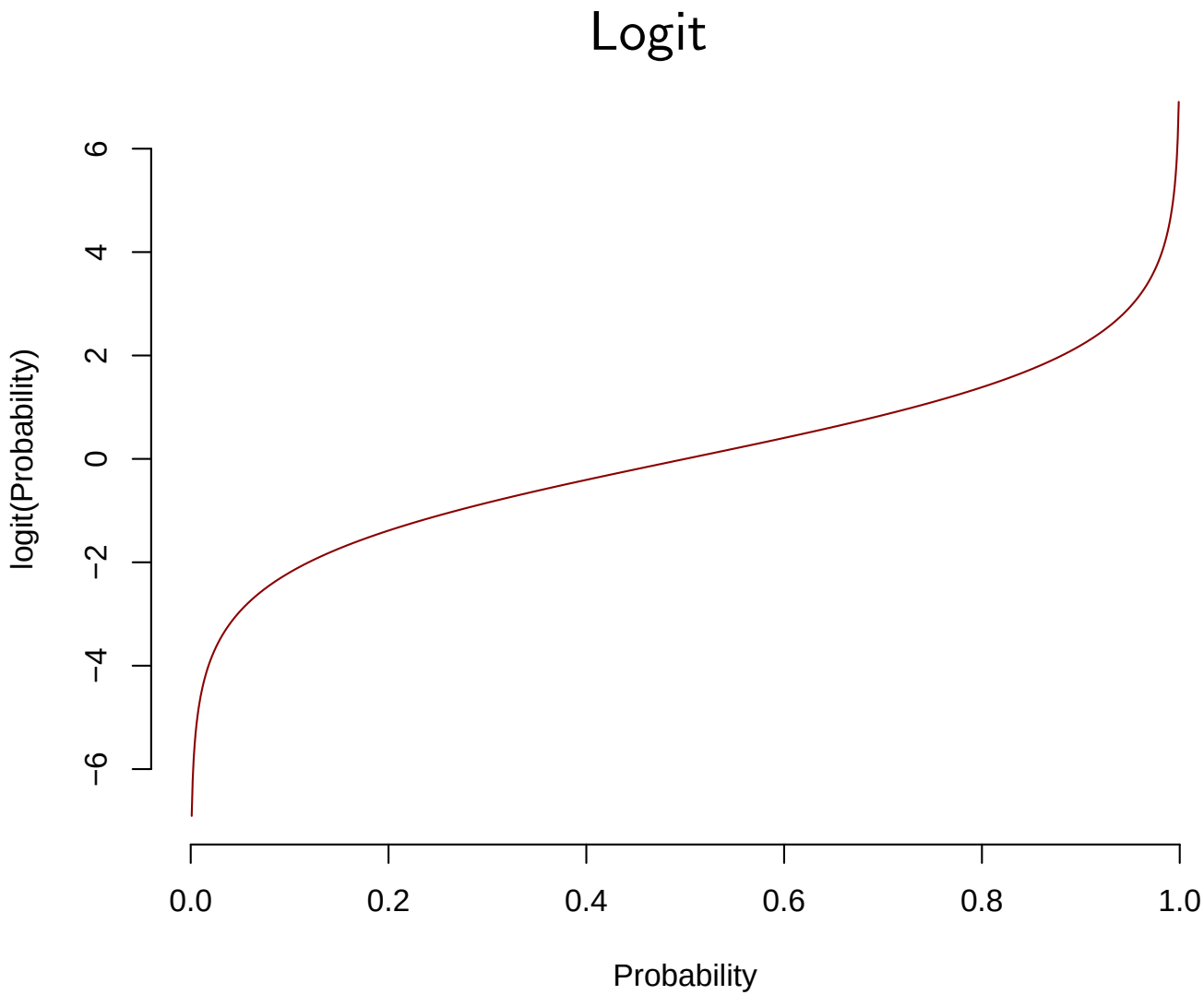
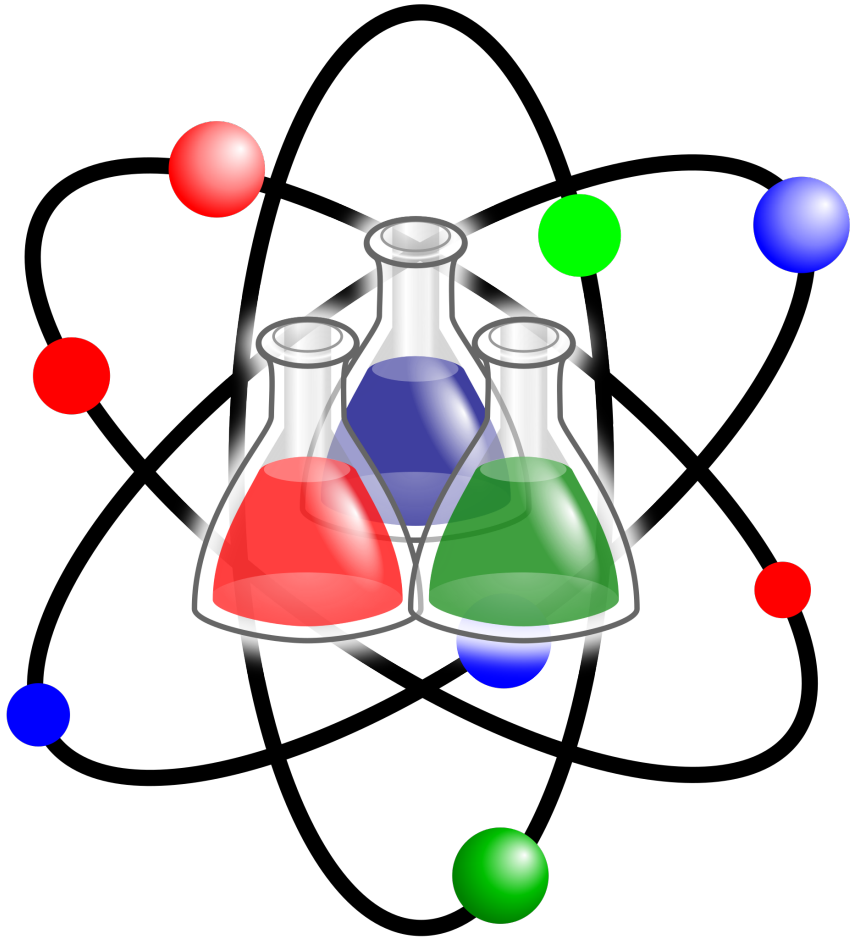


Link function examples



Linear predictor

Where the science (aka hypotheses) enters!



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Linear predictor

Example - for a whale survey we might be interested in whether the sighting rate depends on water surface temperature

$$\log(\lambda) = a + b * SST$$

our inference concerns whether $b = 0$ or not

Poisson GLM

Distribution: $y \sim \text{Pois}(\lambda)$

Link function: $\log(\lambda) = \eta$

Linear predictor: $\eta = a + b * SST$

NB: a and b are what `glm` in R will estimate and report

Link function: binomial

The default link function relating the mean and the linear predictor for Binomial is the logit

$$\text{logit}(p) = \eta$$

where η is the linear predictor (where your covariates enter)

$$\text{logit}(p) = \ln\left(\frac{p}{1-p}\right)$$

Linear predictor: binomial

Example - interested in how maturity changes over length of fish

$$\text{logit}(p) = a + b * length$$

Binomial GLM

Distribution: $y \sim \mathrm{Bin}(n, p)$

Link function: $\mathrm{logit}(p) = \eta$

Linear predictor: $\eta = a + b * length$

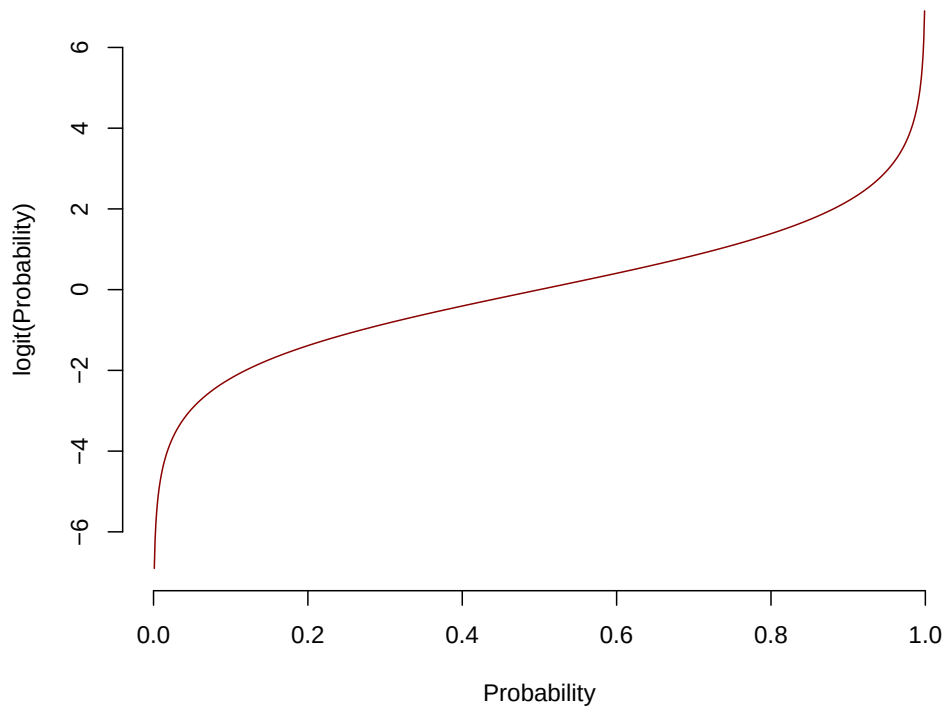
NB: a and b are what `glm` in R will estimate and report

Outline

Inverse link function

Distribution	Link function	Inverse link
Poisson	$\log$	$\exp$
Binomial	logit	logit^{-1} (`plogis` in R)

Use the inverse link to get predictions on the scale of the data (will see more on this in the practicals)



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- GLMs allow for natural distributions matching the data type
- Three components:
  - Distribution
  - Link function
  - Linear predictor

Questions?