Error bound of a Fourier–Galerkin method for periodic solution modulation

Isabella Li

Abstract

While highly effective for periodic problems, Fourier–Galerkin methods are more challenging to apply to autonomous ordinary differential equations due to the phase invariance inherent in such systems; the phases are arbitrary, leading to a non-uniqueness in Fourier representations. Established results in the literature [1] demonstrate that the error of this numerical method converges at a polynomial rate as the truncation order increases. Drawing inspiration from a convergence analysis of Fourier–Galerkin methods for non-autonomous ODEs, we present a novel and significantly simpler proof for this known polynomial convergence rate. Our approach leverages a similar analytical framework, offering a more direct and accessible derivation of the error bounds.

Keywords: Fourier–Galerkin method, Autonomous ODEs, Periodic solutions, Convergence analysis

Contents

1 Introduction

Periodic solutions are prevalent in various scientific and engineering fields, ranging from celestial mechanics [2] to biological rhythms [3] and fluid dynamics [4]. Crucial for understanding the behavior of dynamical systems, numerical methods serve as indispensable tools for the approximation of solutions for nonlinear systems. Among the many numerical approaches, spectral methods, particularly the Fourier-Galerkin method, are highly attractive due to their congruence with periodic problems and potential for high-order accuracy and efficient implementation, particularly when the solutions exhibit sufficient smoothness.

Inspired by the analytical framework developed in [1], we applied a similar analytical machinery to the autonomous setting, offering a novel and considerably simpler proof for the convergence of the Fourier-Galerkin method when applied to periodic solutions of autonomous ODEs. We give novel and considerably simpler proof for the known polynomial convergence in this setting. This new derivation clarifies the underlying mechanisms of convergence and offers a powerful framework for future analyses of related problems, complementing existing proofs by offering a more direct and less technical pathway to the same result.

In practice, we can directly target stable or even unstable periodic regimes without long transient simulations—a decisive benefit in large-scale models of structures and flows [5–7]. In mechanical and structural dynamics, Fourier–Galerkin computations predict backbone curves and response amplitudes of nonlinear oscillators and large finite-element models [6, 8–10]. In fluids and aeroelasticity, nonlinear frequency-domain solvers compute periodic flows and limit-cycle oscillations more efficiently than long time-marching and can converge to unstable cycles that time integration cannot capture [5, 7, 11].

In Section 2, we introduce the problem of finding periodic solutions in the form of solving a boundary value problem and detail the Fourier-Galerkin formulation. Then, we present the main theorem and its proof in Section 3 and Section 4, respectively. In Section 5, we use numerical examples to illustrate the predicted convergence trend for the error bound of our approximation. In Section 6, we discuss opportunities for multi-frequency analysis and computer-assisted implementations.

2 Preliminary work

2.1 Boundary value problem

Consider a dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n, \tag{1}$$

where $\boldsymbol{f}$ is sufficiently smooth with respect to $\boldsymbol{y}$ in the region K, which is a compact set of $\mathbb{R}^n$.

Definition 1. *Let T be a positive number. A solution $\boldsymbol{x}(t)$ is said to be a T**-periodic solution** of system (1) if it satisfies*

$$\boldsymbol{x}(0) = \boldsymbol{x}(T).$$

Now, we will focus on isolated periodic solutions, which is defined as follows.

Definition 2. *Let $\boldsymbol{x}(t)$ be a periodic solution of system (1). It is said to be **isolated (hyperbolic)** if it has exactly one Floquet multiplier equal to one while the moduli of other Floquet multipliers are all different from one.*

An intuitive interpretation of an isolated periodic solution is that there exists a neighborhood U of its orbit such that it is the unique periodic solution of system (1) that lies in U.

We consider another dynamical system highly correlated to system (1):

$$\dot{\boldsymbol{x}} = \frac{T}{2\pi}\boldsymbol{f}(\boldsymbol{x}). \tag{2}$$

Next, we construct the following boundary value problem (BVP) of $(T, \boldsymbol{x}(\theta)) \in \mathbb{R}_+ \times C(\mathbb{T})$ where $\mathbb{T} = [0, 2\pi]/_{(0\sim 2\pi)}$:

$$\mathbf{0} \equiv \boldsymbol{G}(\theta; T, \boldsymbol{x}(\cdot)) := \dot{\boldsymbol{x}}(\theta) - \frac{T}{2\pi}\boldsymbol{f}(\boldsymbol{x}(\theta)), \quad T \in \mathbb{R}_+,\ \boldsymbol{x} \in C(\mathbb{T}). \tag{3}$$

It is equivalent to system (1) in a way that for any solution $(T, \tilde{\boldsymbol{x}}(\theta))$ of BVP (3), $\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(2\pi t/T)$ is a T-periodic solution of system (1), and any periodic solution of system (1) also induces a solution of BVP (3).

For a solution $(T, \tilde{\boldsymbol{x}}(\theta))$ of BVP (3), we say it is **isolated** if $\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(2\pi t/T)$ is an isolated periodic solution of system (1).

Next, we briefly review the Fourier–Galerkin projection method, a widely used numerical method to solve for periodic solutions of a differential equation.

2.2 Fourier–Galerkin projection method

Consider a trigonometric polynomial

$$\hat{\boldsymbol{x}}(\theta) = \boldsymbol{c}_0 + \sum_{j=1}^{M} (\boldsymbol{c}_j \cos j\theta + \boldsymbol{d}_j \sin j\theta),$$

where $\boldsymbol{c}_j, \boldsymbol{d}_j \in \mathbb{R}^n$. It follows that $\boldsymbol{G}(\theta; T, \hat{\boldsymbol{x}}(\cdot))$ is a periodic function and can thus be approximated by its truncated Fourier series

$$\begin{aligned} P_M \boldsymbol{G}(\theta; T, \hat{\boldsymbol{x}}(\cdot)) = \boldsymbol{v}_0(T, \hat{\boldsymbol{x}}(\cdot)) &+ \sum_{j=1}^{M} \boldsymbol{v}_j(T, \hat{\boldsymbol{x}}(\cdot)) \cos j\theta \\ &+ \sum_{j=1}^{M} \boldsymbol{w}_j(T, \hat{\boldsymbol{x}}(\cdot)) \sin j\theta, \end{aligned} \tag{4}$$

where $\boldsymbol{v}_j(T, \hat{\boldsymbol{x}}(\cdot))$ and $\boldsymbol{w}_j(T, \hat{\boldsymbol{x}}(\cdot))$ are the Fourier coefficients of $\boldsymbol{G}(\theta; T, \hat{\boldsymbol{x}}(\cdot))$. Note that for any 2π-periodic function f, we denote its truncated Fourier series of order M by $P_M f$. For brevity, define

$$\boldsymbol{\alpha} = [T, \boldsymbol{c}_0, \boldsymbol{c}_1, ..., \boldsymbol{c}_M, \boldsymbol{d}_1, ..., \boldsymbol{d}_M] \in \mathbb{R}_+ \times \mathbb{R}^{(2M+1)n}.$$

Since any trigonometric polynomial is completely determined by its coefficients, it follows that $P_M \boldsymbol{G}(\theta; T, \hat{\boldsymbol{x}}(\cdot))$ is determined by the Fourier coefficients of $\boldsymbol{G}$ up to order M and $(T, \hat{\boldsymbol{x}}(\cdot))$ is determined by $\boldsymbol{\alpha}$. As a result, we define the following function of $\boldsymbol{\alpha}$ as an equivalent version of Eq. (4):

$$\hat{\boldsymbol{G}}(\boldsymbol{\alpha}) := [\boldsymbol{v}_0(\boldsymbol{\alpha}), \boldsymbol{v}_1(\boldsymbol{\alpha}), ..., \boldsymbol{v}_M(\boldsymbol{\alpha}), \boldsymbol{w}_1(\boldsymbol{\alpha}), ..., \boldsymbol{w}_M(\boldsymbol{\alpha})], \tag{5}$$

where we consider $\boldsymbol{\alpha}$ and $(T, \hat{\boldsymbol{x}}(\cdot))$ as equivalent, a harmless abuse of notation.

Definition 3. *$\boldsymbol{\alpha}$ is a* ***Fourier–Galerkin approximation*** *of BVP (3) if*

$$\hat{\boldsymbol{G}}(\boldsymbol{\alpha}) = \mathbf{0}. \tag{6}$$

In addition, M is the ***truncation order*** *and Eq. (6) is the* ***Fourier–Galerkin equation****.*

2.3 Problem setting

Assume $(T^*, \boldsymbol{x}^*(\theta)) \in \mathbb{R}_+ \times C(\mathbb{T})$ is an isolated solution of BVP (3). We concern about the following fundamental problems:

1. The existence of Fourier–Galerkin approximation;
2. The error estimation of Fourier–Galerkin approximation. Specifically, any Fourier–Galerkin approximation $\boldsymbol{\alpha}$ induces a positive number T and a trigonometric polynomial $\hat{\boldsymbol{x}}(\theta)$. The error is defined as

   $$\mathrm{Err} = \left(\|\boldsymbol{x}^* - \hat{\boldsymbol{x}}\|_2^2 + |T^* - T|^2\right)^{1/2}.$$

   We concern about the error as the truncation order M tends to infinity.

Note that the solution of Eq. (6) is not unique due to the inherent property of autonomous ordinary differential equations that any solution of an autonomous ordinary differential equation remains a solution under time translation. However, this non-uniqueness is inconsequential, since any two solutions differing by a time translation represent the same physical (or dynamical) behavior. Thus, for simplicity, we eliminate this non-uniqueness by adding one more restriction equation to Eq. (6):

We first perform a coordinate transformation. Let $\{\boldsymbol{e}_1, ...\boldsymbol{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Since the vector field does not vanish along a periodic orbit, we can complete $\frac{\boldsymbol{f}(\boldsymbol{y}_0^*)}{|\boldsymbol{f}(\boldsymbol{y}_0^*)|}$ with $n-1$ vectors $\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_{n-1}\}$ such that $\left\{\frac{\boldsymbol{f}(y_0^*)}{|\boldsymbol{f}(y_0^*)|}, \boldsymbol{n}_1, \ldots, \boldsymbol{n}_{n-1}\right\}$ forms an orthonormal basis of $\mathbb{R}^n$ where $\boldsymbol{y}_0^* = \boldsymbol{x}^*(0)$ is the initial value. For $p \in \mathbb{R}$ and $\boldsymbol{q} \in \mathbb{R}^{n-1}$, we use $\boldsymbol{y'} = (p, \boldsymbol{q})$ to denote the coordinate under the new basis:

$$\boldsymbol{y} = \sum_{j=1}^{n} y_j \boldsymbol{e}_j = p\,\frac{\boldsymbol{f}(\boldsymbol{y}_0)}{|\boldsymbol{f}(\boldsymbol{y}_0)|} + \sum_{j=1}^{n-1} q_j \boldsymbol{n}_j.$$

In particular, we have $(\boldsymbol{f}(\boldsymbol{y}_0))' = (|\boldsymbol{f}(\boldsymbol{y}_0)|, \mathbf{0})$. Let $\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2]$ denote the coordinate transition matrix:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y} = \mathbf{Q}\boldsymbol{y}' = \begin{bmatrix} \mathbf{Q}_1, \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} p \\ \boldsymbol{q} \end{bmatrix}.$$

To eliminate the degree of freedom in the tangent direction, we add one more restriction equation to the Fourier–Galerkin equation Eq. (6):

$$0 = H(\boldsymbol{\alpha}) := \langle \hat{\boldsymbol{x}}(0) - \boldsymbol{y}_0^*, \boldsymbol{b} \rangle = \sum_{j=0}^{M} \langle \boldsymbol{c}_j, \boldsymbol{b} \rangle - \langle \boldsymbol{y}_0^*, \boldsymbol{b} \rangle, \tag{7}$$

where $\boldsymbol{b} \in \mathbb{R}^n$ is a constant vector such that

$$\langle \boldsymbol{f}\left(\boldsymbol{y}_0^*\right), \boldsymbol{b} \rangle > 0.$$

Define

$$\boldsymbol{F}(\boldsymbol{\alpha}) := [\hat{\boldsymbol{G}}(\boldsymbol{\alpha}), H(\boldsymbol{\alpha})]. \tag{8}$$

We will prove the existence of the solution of

$$\boldsymbol{F}(\boldsymbol{\alpha}) = \mathbf{0}. \tag{9}$$

Then, by Theorem 1, we show that the error of this solution is bounded by a function of M which decreases monotonically to zero.

3 Main theorem

Theorem 1. *Consider the BVP (3). Suppose $\boldsymbol{f}$ is C^2 in some compact region $K \subset \mathbb{R}^n$. If there is an isolated solution $(T^*, \boldsymbol{x}^*(\theta))$ where $\boldsymbol{x}^*(\theta)$ lies inside K, then there exists a positive number L such that there exists a solution $\bar{\boldsymbol{\alpha}}$ of Eq. (9) that is locally unique for any truncation order $M > L$.*

Moreover, we have error estimation as follows:

$$\left(\|\boldsymbol{x}^* - \hat{\boldsymbol{x}}\|_2^2 + |T^* - T|^2\right)^{1/2} \lesssim M^{-1}, \quad \forall\ M > L. \tag{10}$$

4 Proof of main theorem

We present Proposition 2 in [1] here for completeness.

Lemma 1. *Let*

$$\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0} \tag{11}$$

be a given real system of equations where $\boldsymbol{\alpha}$ and $\boldsymbol{F}(\boldsymbol{\alpha})$ are vectors of $\mathbb{R}^N$ and $\boldsymbol{F}(\boldsymbol{\alpha})$ is a continuously differentiable function of $\boldsymbol{\alpha}$ defined in some region $\Omega \subset \mathbb{R}^N$. Assume that Eq. (11) has an approximate solution $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$ for which the Jacobian matrix $\mathbf{J}(\boldsymbol{\alpha})$ of $\boldsymbol{F}(\boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$ is invertible at $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$.

If there is a positive constant δ and a nonnegative constant $\kappa < 1$ such that

(i) $\Omega_\delta := \{\boldsymbol{\alpha} : |\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}| \leqslant \delta\} \subset \Omega$,

(ii) $|\mathbf{J}(\boldsymbol{\alpha}) - \mathbf{J}(\hat{\boldsymbol{\alpha}})| \leqslant \kappa/M', \qquad \forall \boldsymbol{\alpha} \in \Omega_\delta$,

(iii) $\dfrac{M'r}{1-\kappa} \leqslant \delta$,

where r and $M' > 0$ are numbers such that

$$|\boldsymbol{F}(\hat{\boldsymbol{\alpha}})| \leqslant r \quad \textit{and} \quad |\mathbf{J}^{-1}(\hat{\boldsymbol{\alpha}})| \leqslant M',$$

then the system Eq. (11) has one and only one solution $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$ in Ω_δ and

$$\|\bar{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}\| \leqslant \frac{M'r}{1-\kappa}.$$

□

Now, consider the truncated Fourier series of $\boldsymbol{x}^*(\theta)$,

$$P_M \boldsymbol{x}^*(\theta) = \boldsymbol{c}_0^* + \sum_{j=1}^{M} (\boldsymbol{c}_j^* \cos j\theta + \boldsymbol{d}_j^* \sin j\theta).$$

And define

$$\boldsymbol{\alpha}^* = [T^*, \boldsymbol{c}_0^*, \boldsymbol{c}_1^*, ..., \boldsymbol{c}_M^*, \boldsymbol{d}_1^*, ..., \boldsymbol{d}_M^*].$$

We will verify that the conditions in Lemma 1 are all satisfied by Eq. (9) with $\boldsymbol{\alpha}^*$ as the approximate solution. Specifically, we prove in Section 4.1 the Jacobian matrix of $\boldsymbol{F}$ in Eq. (9) is invertible and the norm of its inverse is uniformly bounded for sufficiently large M and prove in Section 4.2 the Lipschitz property satisfied by the

Jacobian matrix. Then in Section 4.3 we use the lemma to finish the proof of the existence of solution of Eq. (9).

Because $\boldsymbol{f}$ is smooth in the compact region K, there exists a positive constant C such that

$$\max_{\boldsymbol{y}\in K}\left(\sum_{|\beta|\leqslant 2}|D^{\beta}\boldsymbol{f}(\boldsymbol{y})|^2\right)^{1/2}\leqslant C,$$

where $\boldsymbol{\beta}=[\beta_1,...,\beta_n]\in\mathbb{N}^n$ is the partial derivative index that

$$|\boldsymbol{\beta}|=\sum_{i=1}^{n}\beta_i,\qquad D^{\boldsymbol{\beta}}\boldsymbol{f}(\boldsymbol{y})=\frac{\partial^{|\boldsymbol{\beta}|}\boldsymbol{f}}{\partial^{\beta_1}y_1...\partial^{\beta_n}y_n}(\boldsymbol{y}).$$

We adopt from [1] the following notations

$$\sigma(M)=\sqrt{2}\left(\sum_{k=1}^{\infty}\frac{1}{(M+k)^2}\right)^{1/2},\qquad \sigma_1(M)=\frac{1}{M+1}.$$

4.1 Inverse of the Jacobian matrix

Given the perturbation $\Delta\boldsymbol{y}\in\mathbb{R}^n$ and $\Delta T\in\mathbb{R}$, consider the linearized system of system (2) at $\boldsymbol{x}^*(\theta)$,

$$\begin{cases}\dot{\boldsymbol{u}}(\theta) &= \dfrac{T^*}{2\pi}D_{\boldsymbol{y}}\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)\boldsymbol{u}(\theta)+\dfrac{\Delta T}{2\pi}\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)+\boldsymbol{g}(\theta),\\ \boldsymbol{u}(0) &= \Delta\boldsymbol{y}\end{cases}. \tag{12}$$

where $\boldsymbol{g}(\cdot)$ is a 2π-periodic function. Let $\mathbf{E}(\theta)$ denote the fundamental matrix of the homogeneous variational problem with $\mathbf{E}(0)=\mathbf{I}_n$ and $\mathbf{E}'(\theta)=\mathbf{Q}^{-1}\mathbf{E}(\theta)\mathbf{Q}$ being the fundamental matrix in $\boldsymbol{y}'$-coordinate.

Denote by $\boldsymbol{\varphi}(\theta;\boldsymbol{y},T)$ the flow map of system (2). By Lemma 5, we have

$$\mathbf{E}(\theta)\Delta\boldsymbol{y}=D_{\boldsymbol{y}}\boldsymbol{\varphi}(\theta;\boldsymbol{y}_0^*,T^*)\Delta\boldsymbol{y} \tag{13}$$

which implies

$$\mathbf{E}(\theta)=D_{\boldsymbol{y}}\boldsymbol{\varphi}(\theta;\boldsymbol{y}_0^*,T^*).$$

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) = D_{\boldsymbol{y}}\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)\frac{T^*}{2\pi}\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)$$

which implies that $\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) = \mathbf{E}(\theta)\boldsymbol{f}(\boldsymbol{y}_0^*)$. In particular, for $\theta = 2\pi$, it follows that

$$\boldsymbol{f}(\boldsymbol{y}_0^*) = \mathbf{E}(2\pi)\boldsymbol{f}(\boldsymbol{y}_0^*).$$

Therefore, $\boldsymbol{f}(\boldsymbol{y}_0^*)$ is the eigenvector of $\mathbf{E}(2\pi)$ and in the $\boldsymbol{y}'$-coordinate we have

$$\mathbf{E}'(2\pi) = \begin{bmatrix} 1 & \mathbf{E}_1 \\ \mathbf{0} & \mathbf{E}_2 \end{bmatrix},$$

where $\mathbf{E}_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ is the Jacobian matrix of the Poincaré map defined on the hypersurface $\boldsymbol{y}_0^* + \mathrm{span}\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_{n-1}\}$.

We show by the next lemma that Eq. (12) induces a bounded operator

$$\begin{aligned}\mathcal{H} : \mathbb{R}\times C(\mathbb{T}) &\to \mathbb{R}\times C(\mathbb{T}),\\ (\Delta p, \boldsymbol{g}(\theta)) &\mapsto (\Delta T, \boldsymbol{u}(\theta)),\end{aligned}$$

such that $\boldsymbol{u}(\cdot)$ is the periodic solution of Eq. (12) with initial value $\Delta\boldsymbol{y}' = (\Delta p, \Delta\boldsymbol{q})$ and parameter ΔT.

Lemma 2. *Suppose $(T^*, \boldsymbol{x}^*(\theta))$ is isolated, then for any $\Delta p \in \mathbb{R}$ and continuous 2π-periodic function $\boldsymbol{g}(\theta)$, there exists a unique pair $(\Delta T, \boldsymbol{u}(\theta)) \in \mathbb{R}\times C(\mathbb{T})$ such that $u(\cdot)$ solves Eq. (12) with initial value $\Delta\boldsymbol{y}$ that satisfies $\Delta\boldsymbol{y}' = (\Delta p, \Delta\boldsymbol{q})$ and parameter ΔT. Moreover, there exists a positive number C_H such that*

$$\left(|\Delta T|^2 + |\Delta\boldsymbol{q}|^2\right)^{1/2} \leqslant C_H\|\boldsymbol{g}\|_2, \tag{14}$$

$$\left(|\Delta T|^2 + \|\boldsymbol{u}\|_2^2\right)^{1/2} \leqslant C_H\left(\|\boldsymbol{g}\|_2^2 + |\Delta p|^2\right)^{1/2}. \tag{15}$$

Proof. By Lemma 6,

$$\boldsymbol{u}(\theta) = \mathbf{E}(\theta)\Delta\boldsymbol{y} + \mathbf{E}(\theta)\int_0^\theta \mathbf{E}^{-1}(\nu)\left[\frac{\Delta T}{2\pi}\boldsymbol{f}\left(\boldsymbol{x}^*(\nu)\right) + \boldsymbol{g}(\nu)\right]\mathrm{d}\nu.$$

In the $\boldsymbol{y}'$-coordinate, the last equation becomes

$$\boldsymbol{u}'(\theta) = \mathbf{E}'(\theta)\Delta\boldsymbol{y}' + \mathbf{E}'(\theta)\int_0^\theta \mathbf{E}'(\nu)^{-1}\left[\frac{\Delta T}{2\pi}\boldsymbol{f}'\left(\boldsymbol{x}^*(\nu)\right) + \boldsymbol{g}'(\nu)\right]\mathrm{d}\nu. \tag{16}$$

Let $\Delta\boldsymbol{y}' = (\Delta p, \Delta\boldsymbol{q})$. Since we are seeking a periodic solution $\boldsymbol{u}(\theta)$, it follows that

$$\begin{aligned}\mathbf{0} = &\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2 - \mathbf{I}_{n-1}\end{bmatrix}\Delta\boldsymbol{q} + \frac{\Delta T}{2\pi}\mathbf{E}'(2\pi)\int_0^{2\pi}\mathbf{E}'(\nu)^{-1}\boldsymbol{f}'\left(\boldsymbol{x}^*(\nu)\right)\mathrm{d}\nu\\ &+ \mathbf{E}'(2\pi)\int_0^{2\pi}\mathbf{E}'(\nu)^{-1}\boldsymbol{g}(\nu)\mathrm{d}\nu.\end{aligned} \tag{17}$$

Note that $\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) = \mathbf{E}(\theta)\boldsymbol{f}(\boldsymbol{y}_0^*)$, thus similarly in the $\boldsymbol{y}'$-coordinate we have

$$\frac{\Delta T}{2\pi}\mathbf{E}'(\theta)\int_0^\theta \mathbf{E}'(\nu)^{-1}\boldsymbol{f}'\left(\boldsymbol{x}^*(\nu)\right)\mathrm{d}\nu = \frac{\theta\Delta T}{2\pi}\mathbf{E}'(\theta)\boldsymbol{f}'(\boldsymbol{y}_0^*) = \frac{\theta\Delta T}{2\pi}\boldsymbol{f}'\left(\boldsymbol{x}^*(\theta)\right). \tag{18}$$

Then we can simplify Eq. (17) into

$$\begin{aligned}0 &= \begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2 - \mathbf{I}_{n-1}\end{bmatrix}\Delta\boldsymbol{q} + \boldsymbol{f}'(\boldsymbol{y}_0^*)\Delta T + \mathbf{E}'(2\pi)\int_0^{2\pi}\mathbf{E}'(\nu)^{-1}\boldsymbol{g}(\nu)\mathrm{d}\nu\\ &= \begin{bmatrix}|\boldsymbol{f}(\boldsymbol{y}_0^*)| & \mathbf{E}_1\\ \mathbf{0} & \mathbf{E}_2 - \mathbf{I}_{n-1}\end{bmatrix}\begin{bmatrix}\Delta T\\ \Delta\boldsymbol{q}\end{bmatrix} + \mathbf{E}'(2\pi)\int_0^{2\pi}\mathbf{E}'(\nu)^{-1}\boldsymbol{g}(\nu)\mathrm{d}\nu.\end{aligned} \tag{19}$$

Now by assumption that $(T^*, \boldsymbol{x}^*(\theta))$ is isolated, the matrix $(\mathbf{E}_2 - \mathbf{I}_{n-1})$ is non-singular and thus the linear equation above has a unique solution $(\Delta T, \Delta\boldsymbol{q})$ which determines a periodic function $\boldsymbol{u}'(\cdot)$ by Eq. (16).

To verify that $\mathcal{H}$ is bounded, notice that there exists a positive number C_1 such that

$$\left|\begin{bmatrix}|\boldsymbol{f}(\boldsymbol{y}_0^*)| & \mathbf{E}_1\\ \mathbf{0} & \mathbf{E}_2 - \mathbf{I}_{n-1}\end{bmatrix}^{-1}\right| \leqslant C_1,$$

and a positive number C_E such that

$$\max_{\theta\in\mathbb{T}}|\mathbf{E}'(\theta)| \leqslant C_E, \qquad \max_{\theta\in\mathbb{T}}|\mathbf{E}'(\theta)^{-1}| \leqslant C_E.$$

Therefore, we have

$$(\Delta T^2 + |\Delta\boldsymbol{q}|^2)^{1/2} \leqslant 2\pi C_1 C_E^2\|\boldsymbol{g}\|_2. \tag{20}$$

Now, by Eq. (18), we can write

$$\boldsymbol{u}'(\theta) = \mathbf{E}'(\theta)\Delta\boldsymbol{y}' + \frac{\theta\Delta T}{2\pi}\boldsymbol{f}'\left(\boldsymbol{x}^*(\theta)\right) + \mathbf{E}'(\theta)\int_0^\theta \mathbf{E}'(\nu)^{-1}\boldsymbol{g}'(\nu)\mathrm{d}\nu.$$

We can derive the bound of $\|\boldsymbol{u}\|_2$ as

$$\|u\|_2 \leqslant C_E(|\Delta\boldsymbol{q}| + |\Delta p|) + C\Delta T + 2\pi C_E^2\|\boldsymbol{g}\|_2. \tag{21}$$

Because $|\Delta\boldsymbol{q}|$ and $|\Delta T|$ are bounded by $\|\boldsymbol{g}\|$, and $|\Delta p|$ according to Eq. (20), there exists a positive number $C_H > 2\pi C_1 C_E^2$ such that

$$\left(|\Delta T|^2 + \|\boldsymbol{u}\|_2^2\right)^{1/2} \leqslant C_H\left(\|\boldsymbol{g}\|_2^2 + |\Delta p|^2\right)^{1/2},$$

so the conclusion follows. □

Denote by $\mathbf{J}(\alpha)$ the Jacobian matrix of $\boldsymbol{F}$ at $\boldsymbol{\alpha}$. In order to prove that $\mathbf{J}(\boldsymbol{\alpha}^*)$ is invertible and to derive the norm of its inverse, consider the linear system

$$J(\boldsymbol{\alpha}^*)\boldsymbol{\xi} + \boldsymbol{\gamma} = \mathbf{0}, \tag{22}$$

where

$$\boldsymbol{\xi} = [\Delta T, \Delta\boldsymbol{c}_0, \Delta\boldsymbol{c}_1, ..., \Delta\boldsymbol{c}_M, \Delta\boldsymbol{d}_1, ..., \Delta\boldsymbol{d}_M],$$
$$\boldsymbol{\gamma} = [\Delta\boldsymbol{v}_0, \Delta\boldsymbol{v}_1, ..., \Delta\boldsymbol{v}_M, \Delta\boldsymbol{w}_1, ..., \Delta\boldsymbol{w}_M, \Delta H].$$

Define

$$\begin{aligned}\boldsymbol{z}_{\boldsymbol{\xi}}(\theta) &= \Delta\boldsymbol{c}_0 + \sqrt{2}\sum_{j=1}^{M}(\Delta\boldsymbol{c}_j\cos j\theta + \Delta\boldsymbol{d}_j\sin j\theta),\\ \boldsymbol{z}_{\boldsymbol{\gamma}}(\theta) &= \Delta\boldsymbol{v}_0 + \sqrt{2}\sum_{j=1}^{M}(\Delta\boldsymbol{v}_j\cos j\theta + \Delta\boldsymbol{w}_j\sin j\theta).\end{aligned} \tag{23}$$

Then by Eq. (22), we have

$$\begin{cases}\dot{\boldsymbol{z}}_\xi(\theta) & = P_M\left\{\dfrac{T^*}{2\pi}D_{\boldsymbol{y}}\boldsymbol{f}\left(P_M\boldsymbol{x}^*(\theta)\right)\boldsymbol{z}_{\boldsymbol{\xi}}(\theta) + \dfrac{\Delta T}{2\pi}\boldsymbol{f}\left(P_M\boldsymbol{x}^*(\theta)\right)\right\} + \boldsymbol{z}_{\boldsymbol{\gamma}}(\theta)\\ 0 & = \langle\boldsymbol{z}_{\boldsymbol{\xi}}(0), \boldsymbol{b}\rangle + \Delta H\end{cases}. \tag{24}$$

We have the following lemma:

Lemma 3. *If* $(T^*, \boldsymbol{x}^*(\theta))$ *is isolated then there exists a positive number* C_I *and* M_1 *such that for any solution* $(\Delta T, \boldsymbol{z}_{\boldsymbol{\xi}}(\theta))$ *(if any exists) and* $M > M_1$*, we have*

$$\left(|\Delta T|^2 + \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2^2\right)^{1/2} \leqslant C_I \left(|\Delta H|^2 + \|\boldsymbol{z}_{\boldsymbol{\gamma}}\|_2^2\right)^{1/2}. \tag{25}$$

Proof. For any periodic solution $\boldsymbol{z}_{\boldsymbol{\xi}}(\theta)$ of Eq. (24) we have

$$\dot{\boldsymbol{z}}_{\xi}(\theta) = \frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) + \frac{\Delta T}{2\pi} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) + \boldsymbol{z}_{\boldsymbol{\gamma}}(\theta) + \boldsymbol{\eta}_1(\theta) + \boldsymbol{\eta}_2(\theta),$$

where

$$\begin{aligned}
\boldsymbol{\eta}_1(\theta) &= -(I - P_M)\left\{\frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) + \frac{\Delta T}{2\pi} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)\right\}, \\
\boldsymbol{\eta}_2(\theta) &= P_M \left\{\frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) - \frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta)\right\} \\
&\quad + P_M \left\{\frac{\Delta T}{2\pi} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) - \frac{\Delta T}{2\pi} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)\right\}.
\end{aligned}$$

Let

$$\begin{aligned}
\boldsymbol{u}_1(\theta) &= \frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) + \frac{\Delta T}{2\pi} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right), \\
\boldsymbol{u}_2(\theta) &= \frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) - \frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) \\
&\quad + \frac{\Delta T}{2\pi} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) - \frac{\Delta T}{2\pi} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right).
\end{aligned}$$

Then we have

$$\|\boldsymbol{\eta}_1\|_2 \leqslant \sigma_1(M) \|\dot{\boldsymbol{u}}\|_2.$$

Differetiating $\boldsymbol{u}_1(\theta)$ with respect to θ,

$$\begin{aligned}
\dot{\boldsymbol{u}}_1(\theta) &= \frac{T^*}{2\pi} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)\right] \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) + \frac{\Delta T}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \frac{T^*}{2\pi} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \\
&\quad + \frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) \dot{\boldsymbol{z}}_{\boldsymbol{\xi}}(\theta).
\end{aligned}$$

Substituting $\dot{\boldsymbol{z}}_{\boldsymbol{\xi}}(\theta)$ by Eq. (24) and using Bessel's inequality,

$$\|\boldsymbol{u}_1\|_2 \leqslant \frac{(T^*)^2 C^2}{2\pi^2} \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2 + \frac{T^* C^2}{2\pi^2} |\Delta T| + \frac{T^* C}{2\pi} \|\boldsymbol{z}_{\boldsymbol{\gamma}}\|_2. \tag{26}$$

Therefore,

$$\|\boldsymbol{\eta}_1\|_2 \leqslant \sigma_1(M) \left(\frac{(T^*)^2 C^2}{2\pi^2} \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2 + \frac{T^* C^2}{2\pi^2} |\Delta T| + \frac{T^* C}{2\pi} \|\boldsymbol{z}_{\boldsymbol{\gamma}}\|_2 \right). \tag{27}$$

To derive the bound of $\|\boldsymbol{\eta}_2\|_2$, note that

$$\begin{aligned} |D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) - D_{\boldsymbol{y}} \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)| &\leqslant C |(I - P_M)\boldsymbol{x}^*(\theta)|, \\ |\boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) - \boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)| &\leqslant C |(I - P_M)\boldsymbol{x}^*(\theta)|. \end{aligned}$$

Therefore,

$$|\boldsymbol{u}_2(\theta)| \leqslant \left(\frac{T^* C}{2\pi} |\boldsymbol{z}_{\boldsymbol{\xi}}(\theta)| + \frac{C \Delta T}{2\pi} \right) |(I - P_M)\boldsymbol{x}^*(\theta)|.$$

By Cauchy-Schwartz inequality,

$$\|\boldsymbol{\eta}_2\|_2 \leqslant \sigma_1(M) \left(\frac{(T^*)^2 C^2}{4\pi^2} \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2 + \frac{T^* C^2 \Delta T}{4\pi^2} \right). \tag{28}$$

Suppose $\boldsymbol{z}'_{\boldsymbol{\xi}}(0) = (\Delta p, \Delta \boldsymbol{q})$ and $\boldsymbol{b}' = (b_p, \boldsymbol{b}_{\boldsymbol{q}})$, then

$$b_p \Delta p + \langle \boldsymbol{b}_{\boldsymbol{q}}, \Delta \boldsymbol{q} \rangle + \Delta H = 0. \tag{29}$$

Because $b_p > 0$ by assumption, we can bound Δp by terms of $\Delta \boldsymbol{q}$ and ΔH,

$$|\Delta p| \leqslant \frac{|\boldsymbol{b}||\Delta \boldsymbol{q}| + |\Delta H|}{b_p}. \tag{30}$$

Now, by the second inequality (15) in Lemma 2,

$$\begin{aligned} \left(|\Delta T|^2 + \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2^2\right)^{1/2} &\leqslant C_H \left(|\Delta p|^2 + \|\boldsymbol{z}_{\boldsymbol{\gamma}} + \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2\|_2^2\right)^{1/2} \\ &\leqslant C_H (|\Delta p| + \|\boldsymbol{z}_{\boldsymbol{\gamma}}\|_2 + \|\boldsymbol{\eta}_1\|_2 + \|\boldsymbol{\eta}_2\|_2). \end{aligned} \tag{31}$$

Using the Eq. (30) and the first inequality (14) in Lemma 2,

$$\left(|\Delta T|^2 + \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2^2\right)^{1/2} \leqslant C_H \left(\frac{|\Delta H|}{b_p} + \left(1 + \frac{C_H |\boldsymbol{b}|}{b_p}\right) (\|\boldsymbol{z}_{\boldsymbol{\gamma}}\|_2 + \|\boldsymbol{\eta}_1\|_2 + \|\boldsymbol{\eta}_2\|_2) \right). \tag{32}$$

By Eqs. (26) and (28), we have:

$$
\begin{aligned}
&\left(|\Delta T|^2 + \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2^2\right)^{1/2} \\
&\quad \leqslant \sigma_1(M)\left(1+\frac{C_H|\boldsymbol{b}|}{b_p}\right)\left(\frac{(T^*)^2C^2}{2\pi^2}\|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2 + \frac{T^*C^2}{2\pi^2}|\Delta T| + \frac{T^*C}{2\pi}\|\boldsymbol{z}_{\boldsymbol{\gamma}}\|_2\right) \\
&\qquad + \sigma_1(M)\left(1+\frac{C_H|\boldsymbol{b}|}{b_p}\right)\left(\frac{(T^*)^2C^2}{4\pi^2}\|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2 + \frac{T^*C^2\Delta T}{4\pi^2}\right) \\
&\qquad\quad + \frac{C_H|\Delta H|}{b_p} + \left(1+\frac{C_H|\boldsymbol{b}|}{b_p}\right)\|\boldsymbol{z}_{\gamma}\|_2.
\end{aligned}
\tag{33}
$$

Note that all coefficients of $\|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2$ and $|\Delta T|$ contains term $\sigma_1(M)$ which converges to zero as M tends to ∞. Thus, there exists a sufficiently large number M_1 and a positive number C_I such that for all $M > M_1$:

$$
\left(|\Delta T|^2 + \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2^2\right)^{1/2} \leqslant C_I\left(|\Delta H|^2 + \|\boldsymbol{z}_{\boldsymbol{\gamma}}\|_2^2\right)^{1/2}. \tag{34}
$$

□

The lemma above yields the following corollary.

Corollary 1. *If $(T^*, \boldsymbol{x}^*(\theta))$ is isolated, then there exists nonnegative constants M_1 and C_I such that for any $M > M_1$, $\mathbf{J}(\boldsymbol{\alpha}^*)$ is invertible and*

$$
|\mathbf{J}^{-1}(\boldsymbol{\alpha}^*)| \leqslant C_I, \tag{35}
$$

where the constant C_I is independent of M.

Proof. Suppose $\boldsymbol{\xi}$ and $\boldsymbol{\gamma}$ satisfy Eq. (22) and let $\boldsymbol{z}_{\boldsymbol{\xi}}(\theta)$ and $\boldsymbol{z}_{\boldsymbol{\gamma}}(\theta)$ be the trigonometric polynomials defined as Eq. (23). Now set $\boldsymbol{\gamma} = \mathbf{0}$. Then $\boldsymbol{z}_{\boldsymbol{\gamma}}(\theta) \equiv \mathbf{0}$ and $\Delta H = 0$. Then, $\Delta T = 0$ and $\|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2 = 0$ by Lemma 3. Thus, by Parseval's identity, $|\boldsymbol{\xi}| = 0$ and $\mathbf{J}(\boldsymbol{\alpha}^*)$ is invertible. Eq. (35) follows directly from the inequality in Lemma 3. □

4.2 Lipschitz property of the Jacobian matrix

Suppose $\boldsymbol{\alpha} = [T, \boldsymbol{c}_0, \boldsymbol{c}_1, ..., \boldsymbol{c}_M, \boldsymbol{d}_1, ..., \boldsymbol{d}_M]$ is another vector in $\mathbb{R}_+ \times \mathbb{R}^{(2M+1)n}$. Let $\hat{\boldsymbol{x}}(\theta)$ be the trigonometric polynomial determined by $\boldsymbol{\alpha}$,

$$\hat{\boldsymbol{x}}(\theta) = \boldsymbol{c}_0 + \sqrt{2}\sum_{j=1}^{M} (\boldsymbol{c}_j \cos j\theta + \boldsymbol{d}_j \sin j\theta).$$

We have the following lemma:

Lemma 4. *Suppose $\boldsymbol{\alpha}$ satisfies the* ***convexity condition*** *that for any $0 \leqslant \lambda \leqslant 1$,*

$$\lambda \boldsymbol{x}^*(\theta) + (1-\lambda)\hat{\boldsymbol{x}}(\theta) \in K, \qquad \forall\ \theta \in [0, 2\pi],$$

Then for sufficiently large M, we have

$$|(\mathbf{J}(\boldsymbol{\alpha}) - \mathbf{J}(\boldsymbol{\alpha}^*))| \leqslant \frac{C\left(1+\sqrt{(T^*)^2+1}\right)}{2\pi}|\boldsymbol{\alpha} - \boldsymbol{\alpha}^*|.$$

Proof. Take an arbitrary $\boldsymbol{\xi} \in \mathbb{R}^{1+(2M+1)n}$. Let $\boldsymbol{\gamma}, \boldsymbol{\gamma}^* \in \mathbb{R}^{1+(2M+1)n}$ such that

$$\mathbf{J}\left(\boldsymbol{\alpha}^{(*)}\right)\boldsymbol{\xi} + \boldsymbol{\gamma}^{(*)} = 0,$$

where superscript $(*)$ means for either case of having a superscript $*$ or not. We write $\boldsymbol{\xi}$ and $\boldsymbol{\gamma}^{(*)}$ as

$$\boldsymbol{\xi} = [\Delta\boldsymbol{\mu}, \Delta\boldsymbol{c}_0, \Delta\boldsymbol{c}_1, ..., \Delta\boldsymbol{c}_M, \Delta\boldsymbol{d}_1, ..., \Delta\boldsymbol{d}_M, \Delta\theta_{\xi}^{\max}, \Delta\theta_{\xi}^{\min}],$$
$$\boldsymbol{\gamma}^{(*)} = \left[\Delta\boldsymbol{v}_0^{(*)}, \Delta\boldsymbol{v}_1^{(*)}, ..., \Delta\boldsymbol{v}_M^{(*)}, \Delta\boldsymbol{w}_1^{(*)}, ..., \Delta\boldsymbol{w}_M^{(*)}, \Delta H^{(*)}\right].$$

Define

$$\boldsymbol{z}_{\boldsymbol{\xi}}(\theta) = \Delta\boldsymbol{c}_0 + \sqrt{2}\sum_{j=1}^{M}(\Delta\boldsymbol{c}_j \cos j\theta + \Delta\boldsymbol{d}_j \sin j\theta),$$
$$\boldsymbol{z}_{\boldsymbol{\gamma}^{(*)}}(\theta) = \Delta\boldsymbol{v}_0^{(*)} + \sqrt{2}\sum_{j=1}^{M}(\Delta\boldsymbol{v}_j^{(*)} \cos j\theta + \Delta\boldsymbol{w}_j^{(*)} \sin j\theta).$$

Then, $\boldsymbol{z}_{\boldsymbol{\xi}}(\theta)$ satisfies the following equations

$$\begin{cases} \dot{\boldsymbol{z}}_{\xi}(\theta) & = P_M \left\{ \dfrac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) + \dfrac{\Delta T}{2\pi} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) \right\} + \boldsymbol{z}_{\boldsymbol{\gamma}^*}(\theta) \\ \dot{\boldsymbol{z}}_{\xi}(\theta) & = P_M \left\{ \dfrac{T}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\hat{\boldsymbol{x}}(\theta)\right) \boldsymbol{z}_{\boldsymbol{\xi}}(\theta) + \dfrac{\Delta T}{2\pi} \boldsymbol{f}\left(\hat{\boldsymbol{x}}(\theta)\right) \right\} + \boldsymbol{z}_{\boldsymbol{\gamma}}(\theta) \end{cases}.$$

Subtracting the first equation by the second equation above, and by Bessel's inequality we have

$$\begin{aligned} \|\boldsymbol{z}_{\boldsymbol{\gamma}^*} - \boldsymbol{z}_{\boldsymbol{\gamma}}\|_2 \leqslant & \left\| \left(\frac{T^*}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*\right) - \frac{T}{2\pi} D_{\boldsymbol{y}} \boldsymbol{f}\left(\hat{\boldsymbol{x}}\right) \right) \boldsymbol{z}_{\boldsymbol{\xi}} \right\|_2 \\ & + \frac{\Delta T}{2\pi} \left\| \left(D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*\right) - D_{\boldsymbol{y}} \boldsymbol{f}\left(\hat{\boldsymbol{x}}\right)\right) \right\|_2 . \end{aligned} \tag{36}$$

Note that for any $\boldsymbol{y} \in \mathbb{R}^n$,

$$\begin{aligned} & \left(D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) - D_{\boldsymbol{y}} \boldsymbol{f}\left(\hat{\boldsymbol{x}}(\theta)\right)\right) \boldsymbol{y} \\ & \quad = \int_0^1 \sum_{i=1}^n \left(\frac{\partial D_{\boldsymbol{y}} \boldsymbol{f}}{\partial y_i} \left(\lambda P_M \boldsymbol{x}^*(\theta) + (1-\lambda) \hat{\boldsymbol{x}}(\theta)\right) \boldsymbol{y} \cdot \left(P_M x_i^*(\theta) - \hat{x}_i(\theta)\right) \right) \mathrm{d}\lambda. \end{aligned}$$

With the assumption that $\boldsymbol{\alpha}$ satisfies the convexity condition, it follows that for sufficiently large M,

$$\left| D_{\boldsymbol{y}} \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) - D_{\boldsymbol{y}} \boldsymbol{f}\left(\hat{\boldsymbol{x}}(\theta)\right) \right| \leqslant C \left| P_M \boldsymbol{x}^*(\theta) - \hat{\boldsymbol{x}}(\theta) \right|.$$

Substituting the last inequality into Eq. (36), we have

$$\begin{aligned} \|\boldsymbol{z}_{\boldsymbol{\gamma}^*} - \boldsymbol{z}_{\boldsymbol{\gamma}}\|_2 & \leqslant \left(\frac{C(T^* - T)}{2\pi} + \frac{T^* C}{2\pi} \left\| P_M \boldsymbol{x}^* - \hat{\boldsymbol{x}} \right\|_2 \right) \|\boldsymbol{z}_{\boldsymbol{\xi}}\|_2 \\ & \quad + \frac{C \Delta T}{2\pi} \left\| P_M \boldsymbol{x}^* - \hat{\boldsymbol{x}} \right\|_2 \\ & \leqslant \frac{C \left(1 + \sqrt{(T^*)^2 + 1}\right)}{2\pi} |\boldsymbol{\alpha}^* - \boldsymbol{\alpha}| \cdot |\boldsymbol{\xi}|. \end{aligned} \tag{37}$$

On the other hand,

$$\Delta H = -\langle \boldsymbol{z}_{\boldsymbol{\xi}}(0), \boldsymbol{b} \rangle = \Delta H^*.$$

Therefore,

$$|\boldsymbol{\gamma} - \boldsymbol{\gamma}^*| = \|\boldsymbol{z}_{\boldsymbol{\gamma}^*} - \boldsymbol{z}_{\boldsymbol{\gamma}}\|_2.$$

Now, since $\boldsymbol{\xi}$ is selected arbitrarily, it follows from Eq. (37) that

$$|(\mathbf{J}(\boldsymbol{\alpha}) - \mathbf{J}(\boldsymbol{\alpha}^*))| \leqslant \frac{C\left(1 + \sqrt{(T^*)^2+1}\right)}{2\pi} |\boldsymbol{\alpha} - \boldsymbol{\alpha}^*|.$$

□

4.3 Proof of Theorem 1

Proof. By assumption, there exists a small positive number δ_0 such that

$$U = \{\boldsymbol{y} : |\boldsymbol{y} - \boldsymbol{x}^*(\theta)| \leqslant \delta_0, \text{ for some } \theta \in \mathbb{T}\} \subset K.$$

Then, there exists a positive number M_0 such that $P_M \boldsymbol{x}^*(\theta) \in U$ for any $\theta \in \mathbb{T}$ and $M > M_0$. For $M > M_0$, define the region

$$V_M = \{\boldsymbol{y} : |\boldsymbol{y} - P_M \boldsymbol{x}^*(\theta)| \leqslant \delta_0 - C\sigma(M), \text{ for some } \theta \in \mathbb{T}\}.$$

Then

$$V_M \subset U \subset K.$$

Next, define

$$\Omega_M = \left\{\boldsymbol{\alpha} : |\boldsymbol{\alpha} - \boldsymbol{\alpha}^*| \leqslant \frac{\delta_0 - C\sigma(M)}{\sqrt{2M+1}}\right\}.$$

For any $\boldsymbol{\alpha} \in \Omega_M$, let $\hat{\boldsymbol{x}}(\theta)$ be the trigonometric polynomial determined by $\boldsymbol{\alpha}$ which lies inside K.

For any $M > M_0$, we have

$$\frac{\mathrm{d}P_M \boldsymbol{x}^*(\theta)}{\mathrm{d}\theta} = \frac{T^*}{2\pi} P_M \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right) + \boldsymbol{r}(\theta),$$

where

$$\boldsymbol{r}(\theta) = \frac{T^*}{2\pi} P_M \left\{\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right) - \boldsymbol{f}\left(P_M \boldsymbol{x}^*(\theta)\right)\right\}.$$

By Bessel's inequality,

$$\begin{aligned}
\|\boldsymbol{r}\|_2 &\leqslant \frac{T^*}{2\pi}\left\|\int_0^1 D_{\boldsymbol{y}}\boldsymbol{f}\left(\lambda\boldsymbol{x}^* + (1-\lambda)P_M\boldsymbol{x}^*\right)\cdot((I-P_M)\boldsymbol{x}^*)\,\mathrm{d}\lambda\right\|_2 \\
&\leqslant \frac{T^*C}{2\pi}\|(I-P_M)\boldsymbol{x}^*\|_2 \\
&\leqslant \frac{(T^*)^2C^2}{4\pi^2(M+1)}.
\end{aligned}$$

Therefore, by Parseval's identity,

$$\left|\hat{\boldsymbol{G}}(\boldsymbol{\alpha}^*)\right| = \|\boldsymbol{r}\|_2 \leqslant \frac{(T^*)^2C^2}{4\pi^2(M+1)}.$$

Moreover,

$$\begin{aligned}
|H(\boldsymbol{\alpha}^*)| &= |\langle P_M\boldsymbol{x}^*(0) - \boldsymbol{x}^*(0), \boldsymbol{b}\rangle| \leqslant |\boldsymbol{b}|\|(I-P_M)\boldsymbol{x}^*\|_\infty \\
&\leqslant \sqrt{2M+1}|\boldsymbol{b}|\|(I-P_M)\boldsymbol{x}^*\|_2 \\
&\leqslant \frac{\sqrt{2M+1}|\boldsymbol{b}|}{(M+1)^2}\left\|\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\boldsymbol{x}^*(\theta)\right\|_2 \\
&\leqslant \frac{T^*|\boldsymbol{b}|\sqrt{2M+1}}{2\pi(M+1)^2}\left\|\frac{\mathrm{d}}{\mathrm{d}\theta}\boldsymbol{f}\left(\boldsymbol{x}^*(\theta)\right)\right\|_2 \\
&\leqslant \frac{(T^*)^2C^2|\boldsymbol{b}|\sqrt{2M+1}}{4\pi^2(M+1)^2}.
\end{aligned}$$

Therefore, for sufficiently large M, we have

$$|F(\boldsymbol{\alpha}^*)| \leqslant \frac{(T^*)^2C^2\sqrt{|\boldsymbol{b}|^2+1}}{4\pi^2(M+1)}. \tag{38}$$

By Corollary 1, there exists positive numbers C_I and M_1 such that $\mathbf{J}(\boldsymbol{\alpha}^*)$ is invertible and

$$|\mathbf{J}^{-1}(\boldsymbol{\alpha}^*)| \leqslant C_I,$$

for any $M > M_1$. By Lemma 4,

$$|(\mathbf{J}(\boldsymbol{\alpha}) - \mathbf{J}(\boldsymbol{\alpha}^*))| \leqslant \frac{C\left(1+\sqrt{(T^*)^2+1}\right)}{2\pi}|\boldsymbol{\alpha}-\boldsymbol{\alpha}^*|,$$

for any $\boldsymbol{\alpha} \in \Omega_M$ provided $M > M_0$.

Take an arbitrary $\kappa \in (0,1)$, then there exists a positive number M_2 such that

$$\frac{C_I}{1-\kappa}\frac{(T^*)^2C^2\sqrt{|\boldsymbol{b}|^2+1}}{4\pi^2(M+1)} \leqslant \frac{\delta_0 - C\sigma(M)}{\sqrt{2M+1}} \leqslant \frac{2\pi\kappa}{C_I C\left(1+\sqrt{(T^*)^2+1}\right)}, \tag{39}$$

for any $M > M_2$.

Now, for any $M > M_2$, we can take a positive number δ_M such that

$$\frac{C_I}{1-\kappa}\frac{(T^*)^2C^2}{4\pi^2(M+1)} \leqslant \delta_M \leqslant \frac{\delta_0 - C\sigma(M)}{\sqrt{2M+1}}. \tag{40}$$

Consider the set

$$\Omega_{\delta_M} = \{\boldsymbol{\alpha} : |\boldsymbol{\alpha} - \boldsymbol{\alpha}^*| \leqslant \delta_M\}. \tag{41}$$

Note that $\Omega_{\delta_M} \subset \Omega_M$. For any $\boldsymbol{\alpha} \in \Omega_{\delta_M}$ we have

$$|\mathbf{J}(\boldsymbol{\alpha}) - \mathbf{J}(\boldsymbol{\alpha}^*)| \leqslant \frac{C\left(1+\sqrt{(T^*)^2+1}\right)}{2\pi}|\boldsymbol{\alpha} - \boldsymbol{\alpha}^*| \leqslant \frac{\kappa}{C_I}. \tag{42}$$

Further, from Eq. (40),

$$\frac{C_I|F(\boldsymbol{\alpha}^*)|}{1-\kappa} \leqslant \delta_M. \tag{43}$$

Eqs. (41)(42)(43) show that the conditions of Lemma 1 are all satisfied. Thus, by the lemma we conclude that Eq. (9) has a unique solution $\bar{\boldsymbol{\alpha}}$ in Ω_{δ_M}. □

5 Numerical examples

We complement the single-example demonstration with three autonomous systems: the Van der Pol oscillator, the Rayleigh oscillator and the Brusselator model. For each system, we integrate each system to show that the error decays polynomially. We estimate the period T^*, gradually increase the Fourier truncation order M and measure L^2 error as defined in Eq. (10) over one period to confirm the polynomial decay predicted by Theorem 1 and provide practical guidance for choosing M.

5.1 Van der Pol oscillator

The Van der Pol oscillator is a standard example of an autonomous dynamical system, with periodic orbits for $\mu > 0$. The dynamics are $x_1' = x_2$, $x_2' = \mu(1 - x_1^2)x_2 - x_1$ with $\mu = 1$. The table shows the L^2 error over one period for increasing M, and the estimated period T^*.

Table 1 Error vs. truncation order in Van der Pol

System	M	L2 error	T* (s)
Van der Pol ($\mu = 1$)	4	1.722e-01	6.66400
Van der Pol ($\mu = 1$)	8	1.660e-02	6.66400
Van der Pol ($\mu = 1$)	16	1.601e-04	6.66400
Van der Pol ($\mu = 1$)	32	8.666e-05	6.66400
Van der Pol ($\mu = 1$)	64	6.141e-05	6.66400

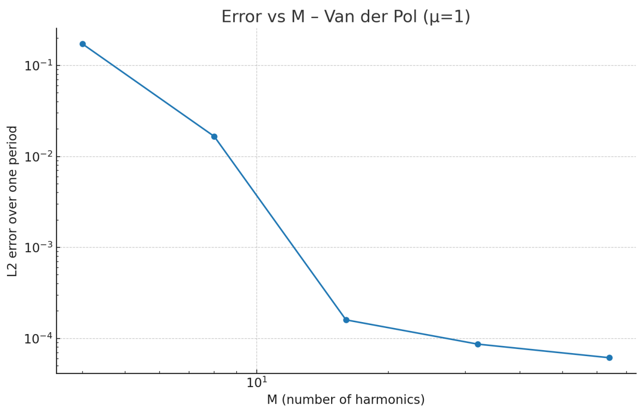

Fig. 1 L^2 error vs. M in log–log scale in Van der Pol

5.2 Rayleigh oscillator

Next, we explore the Raleigh oscillator to show that the error decay vs. the truncation order M is not specific for a particular nonlinear structure. It is a standard Liénard-type oscillator where the nonlinearity acts on x_2. The dynamics are $x_1' = x_2$, $x_2' = \varepsilon(1 - x_2^2)x_2 - x_1$ with $\varepsilon = 0.5$.

Table 2 Error vs. truncation order M in Raleigh

System	M	L2 error	T* (s)
Rayleigh ($\epsilon = 0.5$)	4	5.343e-03	6.38000
Rayleigh ($\epsilon = 0.5$)	8	9.990e-05	6.38000
Rayleigh ($\epsilon = 0.5$)	16	4.327e-05	6.38000
Rayleigh ($\epsilon = 0.5$)	32	3.080e-05	6.38000
Rayleigh ($\epsilon = 0.5$)	64	2.185e-05	6.38000

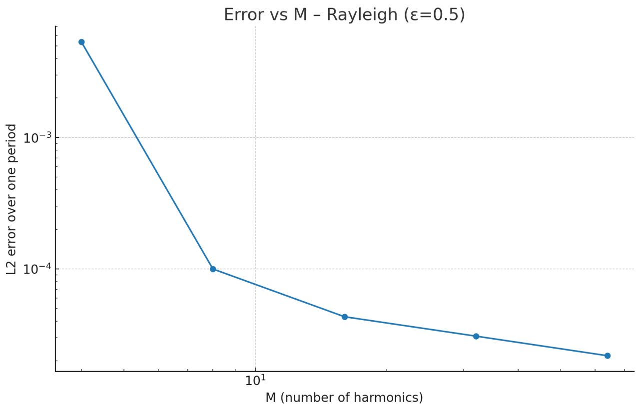

Fig. 2 L^2 error vs. M in log–log scale in Raleigh

5.3 Brusselator

Finally, we consider the Brusselator, a non-Liénard dynamical system and a classical model of chemical oscillations with a non-sinusoidal limit cycle, distinct from the previous two. The dynamics are $x' = A - (B+1)x + x^2y$, $y' = Bx - x^2y$ with $(A, B) = (1, 2.5)$.

Table 3 Error vs. truncation order M in Brusselator

System	M	L2 error	T* (s)
Brusselator (A=1, B=2.5)	4	7.654e-02	6.57800
Brusselator (A=1, B=2.5)	8	1.179e-02	6.57800
Brusselator (A=1, B=2.5)	16	4.148e-04	6.57800
Brusselator (A=1, B=2.5)	32	4.294e-05	6.57800
Brusselator (A=1, B=2.5)	64	3.039e-05	6.57800

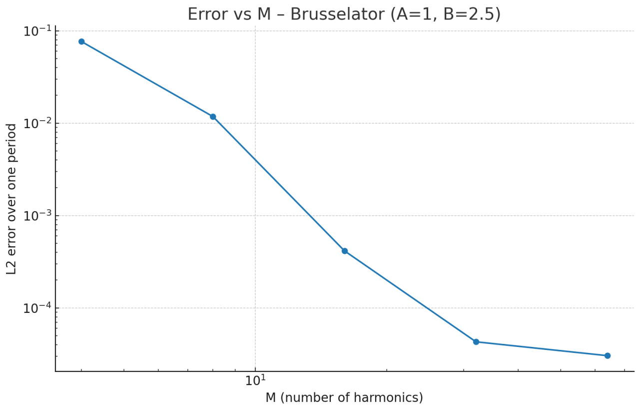

Fig. 3 L^2 error vs. M in log–log scale in Brusselator

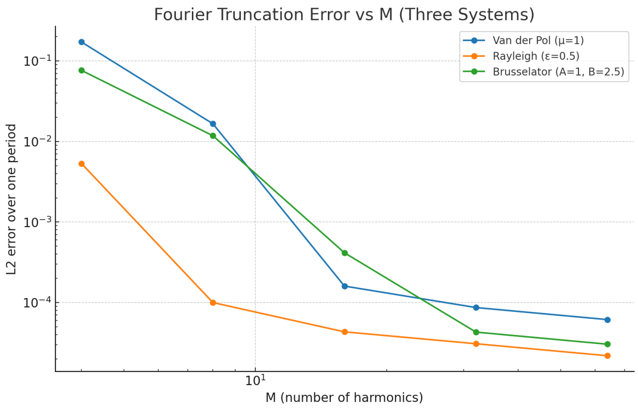

Fig. 4 L^2 error vs. M in log–log scale comparison across 3 systems

6 Future Work

We provided a simpler proof of the convergence of the Fourier–Galerkin method for periodic solutions of autonomous ODEs, but there are opportunities for further theoretical breakthroughs. We could establish explicit convergence constants and sharper error bounds that quantify the dependence on system smoothness or nonlinearity. Moreover, this paper analyzes the Fourier–Galerkin method for periodic solutions, which we could extend to multi-frequency systems. Furthermore, we can utilize computer-assisted proofs to establish specific error bounds for high-dimensional or engineering-scale systems.

7 Acknowledgements

I am deeply grateful for Prof. Zigan Wang's generous guidance on my project, especially his advice to explore the Fourier–Galerkin methods. I would like to thank him for volunteering his time to give me his invaluable support and his suggestions throughout the writing of this paper. I express my gratitude to all my teachers and friends in mathematics for giving me inspiration and support. I owe special thanks to my parents for their encouragement that kept me motivated.

Appendix

Lemmas

Lemma 5. *Suppose $\boldsymbol{u}(\cdot)$ is the solution to Eq. (12). Then the following equality holds*

$$D_{\boldsymbol{y}}\boldsymbol{\varphi}(\theta;\boldsymbol{y},T)\Delta\boldsymbol{y} + D_T\boldsymbol{\varphi}(\theta;\boldsymbol{y},T)\Delta T = \boldsymbol{u}(\theta). \tag{44}$$

Lemma 6. *Let $\mathbf{E}(\theta)$ denote the fundamental matrix of the linear differential equation,*

$$\dot{\boldsymbol{u}}(\theta) = \mathbf{A}(\theta)\boldsymbol{u}(\theta) + \boldsymbol{w}(\theta), \tag{45}$$

with $\mathbf{E}(0) = \mathbf{I}_n$. Suppose $\boldsymbol{v}(\cdot)$ is a solution to Eq. (45). Then

$$\boldsymbol{v}(\theta) = \mathbf{E}(\theta)\boldsymbol{v}(0) + \mathbf{E}(\theta)\int_0^\theta \mathbf{E}^{-1}(\nu)\boldsymbol{w}(\nu)\mathrm{d}\nu. \tag{46}$$

References

[1] Urabe, M.: Galerkin's procedure for nonlinear periodic systems. Archive for Rational Mechanics and Analysis **20**(1), 1–12 (1965)

[2] Naoz, S. (2016). The eccentric Kozai–Lidov effect and its applications. *Annual Review of Astronomy and Astrophysics*, *54*, 441–489. https://doi.org/10.1146/annurev-astro-081915-023315

[3] Goldbeter, A. (1995). A model for circadian oscillations in the *Drosophila* period protein (PER). *Proceedings of the Royal Society of London. Series B: Biological Sciences*, *261*(1362), 319–324. https://doi.org/10.1098/rspb.1995.0153

[4] Rockwell, D. (1978). Self-sustained oscillations of impinging free shear layers. *Annual Review of Fluid Mechanics*, *11*(1), 67–94. https://doi.org/10.1146/annurev.fl.11.010179.000435

[5] Hall, K.C., Thomas, J.P., Clark, W.S.: Computation of unsteady nonlinear flows in cascades using a harmonic balance technique. AIAA Journal **40**(5), 879–886 (2002)

[6] Cardona, A., al.: A multiharmonic method for nonlinear vibration of finite element models. International Journal for Numerical Methods in Engineering **37**(9), 1593–1608 (1994)

[7] Sierra-Ausín, J., *et al.*: Efficient computation of time-periodic compressible flows with analytical jacobians. Journal of Computational Physics **467**, 111398 (2022)

[8] Lau, S.T., Zhang, W.: Nonlinear vibrations of piecewise-linear systems by incremental harmonic balance. Journal of Applied Mechanics **59**(1), 153–160 (1992)

[9] Wang, S., Hua, L., Yang, C., *et al.*: Nonlinear vibrations of a piecewise-linear quarter-car truck model by incremental harmonic balance. Nonlinear Dynamics **92**(4), 1719–1737 (2018)

[10] Krack, M., Gross, J.: Harmonic balance for nonlinear vibration problems. Applied Mechanics Reviews **71**(2), 020802 (2019)

[11] Rigas, G., Oxley, O., Morgans, A., *et al.*: Nonlinear input/output analysis: Application to boundary-layer transition. Journal of Fluid Mechanics **909**, 9 (2021)