The Matrix problems

Coding the Matrix, 2015

For auto-graded problems, edit the file The_Matrix_problems.py to include your solution.

Problem 1: Compute the following matrix-vector products (I recommend you not use the computer to compute these):

1. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * [0.5, 0.5]$
2. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} * [1.2, 4.44]$
3. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} * [1, 2, 3]$

Problem 2: What 2×2 matrix M satisfies $M * [x, y] = [y, x]$ for all vectors $[x, y]$?

Problem 3: What 3×3 matrix M satisfies $M * [x, y, z] = [z + x, y, x]$ for all vectors $[x, y, z]$?

Problem 4: What 3×3 matrix M satisfies $M * [x, y, z] = [2x, 4y, 3z]$ for all vectors $[x, y, z]$?

Problem 5: For each of the following problems, answer whether the given matrix-matrix product is valid or not. If it is valid, give the number of rows and the number of columns of the resulting matrix (you need not provide the matrix itself).

1. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$
2. $\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix}^T$
4. $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}^T$

5. $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}^T$

7. $\begin{bmatrix} 2 & 1 & 5 \end{bmatrix}^T \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}$

Problem 6: Compute:

1. $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 6 \\ 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

5. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 1 & -3 \\ 2 & 2 & -2 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ (Remember the superscript T means "transpose".)

Problem 7: Let

$$A = \begin{bmatrix} 2 & 0 & 1 & 5 \\ 1 & -4 & 6 & 2 \\ 3 & 0 & -4 & 2 \\ 3 & 4 & 0 & -2 \end{bmatrix}$$

For each of the following values of the matrix B, compute AB and BA. (I recommend you not use the computer to compute these.)

1. $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ 2. $B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ 3. $B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Problem 8: Let

$$A = \begin{bmatrix} 4 & 2 & 1 & -1 \\ 1 & 5 & -2 & 3 \\ 4 & 4 & 4 & 0 \\ -1 & 6 & 2 & -5 \end{bmatrix}$$

For each of the following values of the matrix B, compute AB and BA without using a computer. (To think about: Which definition of matrix-matrix multiplication is most useful here? What does a nonzero entry at position (i, j) in B contribute to the j^{th} column of AB? What does it contribute to the i^{th} row of BA?)

(a) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Problem 9: Compute the result of the following matrix multiplications.

(a) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 & 2 \\ -2 & 6 & 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(e) $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}^T \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ (Remember the superscript T means "transpose".)

Problem 10: Write the procedure `lin_comb_mat_vec_mult(M,v)`, which multiplies `M` times `v` using the linear-combination definition. For this problem, the only operation on `v` you are allowed is getting the value of an entry using brackets: `v[k]`. The vector returned must be computed as a linear combination.

Problem 11: Write `lin_comb_vec_mat_mult(v,M)`, which multiply `v` times `M` using the linear-combination definition. For this problem, the only operation on `v` you are allowed is getting the value of an entry using brackets: `v[k]`. The vector returned must be computed as a linear combination.

Problem 12: Write `dot_product_mat_vec_mult(M,v)`, which multiplies `M` times `v` using the dot-product definition. For this problem, the only operation on `v` you are allowed is taking the dot-product of `v` and another vector and `v`: `u*v` or `v*u`. The entries of the vector returned must be computed using dot-product.

Problem 13: Write `dot_product_vec_mat_mult(v,M)`, which multiplies `v` times `M` using the dot-product definition. For this problem, the only operation on `v` you are allowed is taking the dot-product of `v` and another vector and `v`: `u*v` or `v*u`. The entries of the vector returned must be computed using dot-product.

Problem 14: `Mv_mat_mat_mult(A,B)`, using the matrix-vector multiplication definition of matrix-matrix multiplication. For this procedure, the only operation you are allowed to do on `A` is matrix-vector multiplication, using the * operator: `A*v`. Do *not* use the named procedure `matrix_vector_mul` or any of the procedures defined in the previous problem.

Problem 15: `vM_mat_mat_mult(A,B)`, using the vector-matrix definition. For this procedure, the only operation you are allowed to do on `A` is vector-matrix-vector multiplication, using the * operator: `v*A`. Do *not* use the named procedure `vector_matrix_mul` or any of the procedures defined in the previous problem.

Ungraded problem: Let A be a matrix whose column labels are countries and whose row labels are votes taken in the United Nations (UN), where $A[i,j]$ is $+1$ or -1 or 0 depending on whether country j votes in favor of or against or neither in vote i.

As in the politics lab, we can compare countries by comparing their voting records. Let $M = A^T A$. Then M's row and column labels are countries, and $M[i,j]$ is the dot-product of country i's voting record with country j's voting record.

The provided file `UN_voting_data.txt` has one line per country. The line consists of the country name, followed by $+1$'s, -1's, and zeroes, separated by spaces. Read in the data and form the matrix A. Then form the matrix $M = A^T A$. (Note: this will take quite a while—from fifteen minutes to an hour, depending on your computer.)

Use M to answer the following questions.

1. Which pair of countries are most opposed? (They have the most negative dot-product.)
2. What are the ten most opposed pairs of countries?
3. Which pair of distinct countries are in the greatest agreement (have the most positive dot-product)?

Hint: the items in $M.f$ are key-value pairs where the value is the dot-product. You can use a comprehension to obtain a list of value-key pairs, and then sort by the value, using the expression `sorted([(value,key) for key,value in M.f.items()])`.

Problem 16: Here is my code for creating a dictionary of $n \times n$ button vectors:

```
def D(n): return {(i,j) for i in range(n) for j in range(n)}

def button_vectors(n):
  return {(i,j):Vec(D(n),dict([((x,j),one) for x in range(max(i-1,0), min(i+2,n))]
                              +[((i,y),one) for y in range(max(j-1,0), min(j+2,n))]))
```

```
                    for (i,j) in D(n)}
```

You will use `solve` to try to solve some starting configurations of 9×9 *Lights Out*. Use `solver` to solve an appropriate matrix-vector equation, and check if the solution it proposes is actually a solution to the equation. (For a vector over $GF(2)$, taking the dot-product of the vector with itself is not a good way to see whether the vector is zero—why?)

1. Consider the initial configuration given by the following vector.

```
b1 = Vec(D(9), {(7, 7):one, (6, 2):one, (3, 7):one,
 (2, 5):one, (8, 5):one, (7, 2):one, (1, 2):one,
(6, 3):one, (5, 0):one, (0, 4):one, (2, 2):one,
 (6, 4):one, (5, 4):one, (0, 0):one, (1, 4):one,
 (8, 7):one, (0, 8):one, (6, 5):one, (2, 7):one,
 (8, 3):one, (7, 0):one, (4, 6):one, (6, 8):one,
 (7, 4):one, (0, 6):one, (1, 8):one, (7, 8):one, (2, 4):one})
```

   What putative solution is obtained by `solver`? Is it a solution to the equation?
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2. Repeat with the following vector.

```
b2=Vec(D(9), {(3,4):one, (6,7):one})
```

Problem 17:

1. Use a formula given in lecture to solve the linear system $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. Use the formula to solve the linear system $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3. Use your solutions to find a 2×2 matrix M such that $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ times M is an identity matrix.

4. Calculate M times $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ and calculate $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ times M, and use the following criterion to decide whether M is the inverse of $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$:

Matrices A and B are inverses of each other if and only if both AB and BA are identity matrices.

Problem 18: For each of the parts below, use

Matrices A and B are inverses of each other if and only if both AB and BA are identity matrices.

to demonstrate that the pair of matrices given are or are not inverse of each other.

1. matrices $\begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -9 & 5 \end{bmatrix}$ over $\mathbb{R}$

2. matrices $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ over $\mathbb{R}$

3. matrices $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & \frac{1}{6} \\ -2 & \frac{1}{2} \end{bmatrix}$ over $\mathbb{R}$

4. matrices $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ over $GF(2)$

number of rows # number of columns >>> no inverse