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A theoretical analysis of blade coating for third-grade fluid

M Sajid, M Mughees, N Ali and H Shahzad

Abstract

In this article, coating process for a third-grade fluid using a simple fixed blade on a moving substrate is discussed. The analysis is carried out for both exponential (flexible blade) and plane (stiff blade) coaters. Coupled partial differential equations are simplified by applying the lubrication approximation theory and assuming that the coating layer thickness is much less than the blade length. The obtained system is normalized using suitable scales. A numerical solution of the governing boundary value problem is developed for various third-grade parameters. In addition, a perturbation solution is also obtained for small third-grade parameters. Interesting quantities such as pressure, pressure gradient, velocity, and load are computed and shown graphically and in tables. How the involved parameters influence the results is examined. The load on the blade is the most important physical quantity in the present work as it controls the coating thickness and quality. We found that the blade load increases as the third-grade material parameter β increases.

Keywords

Blade coating, third-grade fluid, regular perturbation, lubrication approximation theory, numerical solution

Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan

Corresponding author:
M Mughees, International Islamic University, H 10, Islamabad 44000, Pakistan.
Email: solangi123@yahoo.com

Introduction

Coating is a process in which the surface of an object (called substrate) is covered with some material. Coating is applied to decorate or to protect an object. Coating may completely or partially cover the substrate. Nowadays, coating processes are very important in many ways such as coating in food, coating in industry, coating in paints, newspapers, catalogs, fibers, photographic film, metal coating, and many more. A simple example is to spread butter over toast with a knife. This simple example was used by Booth[1] in which the relationship between the blade angle and coat weight was described. In fact, in the blade coating process, the fluid flows in a narrow gap between the blade and the moving substrate. The moving substrate induces the shear stress which causes the flow. Coatings improve the quality, efficiency, and life of the substrate. Many industrially important fluids have rheological properties like shear thinning, shear thickening, and viscoelasticity. These important industrial fluids are categorized as third-grade fluids.

Trist[2] was the first one to attribute the blade coating by a coating machine in a continuous manner in which a bread wrapping paper was coated by an oil phase emulsion. This technique was improved in 1950s and as a result of which the system of blade coating started growing. Several applications and different geometries related to blade coating have been introduced since then. Nowadays, blade coating systems run at more than 20 m/s. Mechanically, the blade coating system can be divided into stiff (beveled) blade systems and flexible (bent) blade systems. To understand the blade coating phenomenon, the book by Middleman[3] and the article presented by Ruschak[4] are very helpful. Kistler and Schweizer[5] comprehensively presented in detail the coating flows for Newtonian fluids. However, the fluids used in blade coating industry mostly exhibit non-Newtonian behavior. Greener and Middleman[6] presented a theoretical roll coating study using viscoelastic and power-law fluids. Later, Savage[7] corrected the error made by Greener and Middleman[6] in treating the power-law fluid and added the case when the moving roller speed is slightly different. Rose et al.[8] discussed a brief investigation on the analysis of power-law fluid using both exponential as well as plane coater. Sullivan[9] used a microscope to observe the contact between the moving substrate surface and the needle. Sullivan and Middleman[10] used a rigid blade over a coating roll to examine the experimental and theoretical behavior of viscoelastic and viscous liquids. Sinha and Singh[11] studied roll coating model of power-law fluid using cavitation effects. Hsu et al.[12] discussed the numerical and experimental results of the separating forces using a power-law fluid in blade coating. Saita[13] and Saita and Seriven[14] developed the elastohydrodynamic model for the first time and presented the analysis of an undeformable incompressible substrate along with a thin and flexible blade for narrow channel flow beneath the blade. Corvalan and Saita[15] used finite element method to analyze the deformation of compressible substrate and a thin blade in which blade coating involved a nonlinear elastohydrodynamic problem. Later, Elkouh and Yang[16] enhanced the work presented by Flumerfelt et al.[17] in Rayleigh step bearing using the same power-law fluid. Tichy[18] discussed the limiting case of weak viscoelasticity using Maxwell fluid model in a simple blade coater. Cameron[19] calculated the analytical solution for Newtonian fluids using perturbation technique and obtained zeroth-order solution.

Aidun[20] pointed out some complexities in blade coating like web permeability and compressibility, blade flexibility, liquid–gas interface at the both ends of device, changes in shear rate that liquid experiences during the flow under the blade, and pressure gradient regions enclosed by the regions where pressure is nearly constant. Chen and Seriven[21] analyzed the blade coating using a compressible and porous web with a fixed blade. Eklund[22] presented that for blade coating, the coating weight decreases to a minimum value for small loads and attains an intermediate load and after that phenomenon reverses and an increase in coating weight starts with an increase in blade loadings. Pranckh and Seriven[23] analyzed the effect of inertial force in the blade coating process for two-dimensional fluid flow using a flexible and thin blade. Earlier, Tayler[24] and Coyle[25] employed the same model discussed by Pranckh and Seriven,[23] assuming that both the layers may be compressed only toward transverse direction and also that extent of local deformation supposed to be proportional to local normal load which is applied on substrate surface. Hwang[26] proposed the approximate solution using a power-law fluid flow in a simple blade coating process. Dien and Elrod[27] discussed the plane flow for a power-law fluid using the simple blade coater in which exact equations were solved numerically. Recently, Siddiqui et al.[28] developed mathematical model using Williamson fluid and adopted Adomian decomposition technique to find pressure gradient, flow rate, and velocity, analytically. Rana et al.[29] used a Powell-Eyring fluid model to explain the blade coating process. Most recently, Sajid et al.[30] presented mathematical analysis for the effects of partial slip and applied magnetic field for the blade coating of a viscous fluid over a moving substrate. Most of the coating processes, performed in industry, are based on the theoretical results obtained from Newtonian fluids whereas most fluids used in industrial coatings have rheologically complex properties. In this study, we attempt to explain the effects of these rheological properties which may be helpful to improve the coating process. We present a blade coating analysis for a third-grade fluid. The detail mathematical model and numerical and analytic solutions are presented.

Mathematical model

Consider a two-dimensional blade coating model in a Cartesian coordinate system (Figure 1) where we have:

- An incompressible third-grade fluid flows in the narrow gap between the moving substrate and the blade.
- A substrate moving with velocity U is placed at $y = 0$.
- The substrate motion is along x direction.
- L is the blade length with heights:
  - H_1 at $x = 0$
  - H_0 at $x = L$.

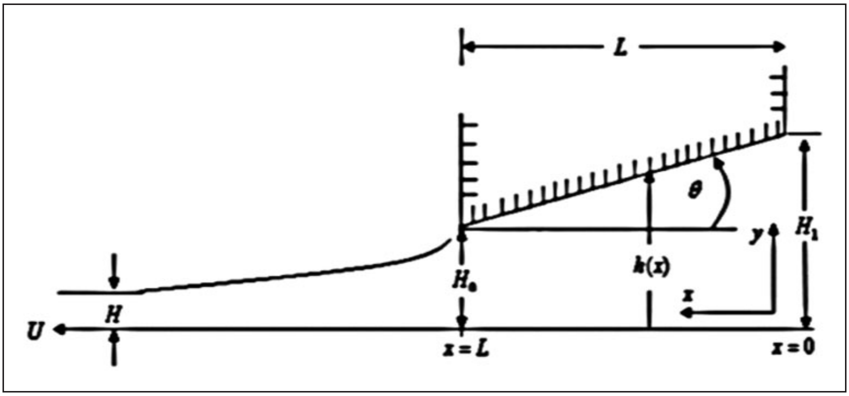

Figure 1. The problem geometry.

- The blade is fixed to maintain the uniform thickness layer H on the moving substrate.

For a third order fluid, the extra stress tensor $\mathbf{S}$ is given by Dunn and Rajagopal[31]

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1 \tag{1}$$

where μ is the dyanamic viscosity; α_1, α_2, β_1, β_2, and β_3 are the material constants; $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are Rivlin–Ericksen tensors

$$\mathbf{A}_1 = (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T \tag{2}$$

$$\mathbf{A}_n = \left(\frac{\partial}{\partial t} + (\nabla.\mathbf{V})\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_{n-1} \tag{3}$$

Thermodynamic constraints suggest[32]

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3} \tag{4}$$

For an incompressible flow, in the absence of body forces, the basic governing equations are

$$\nabla.\mathbf{V} = 0 \tag{5}$$

$$\rho\frac{D\mathbf{V}}{Dt} = -\nabla\bar{p} + \mathrm{div}\mathbf{S} \tag{6}$$

where $\mathbf{V}$, ρ, $\bar{p}$, t, and $\mathbf{S}$ represent the velocity, the fluid density, the hydrodynamic pressure, the time, and the extra stress tensor, respectively, for a third-grade fluid.

The blade coating process is represented by two-dimensional flow having velocity

$$\boldsymbol{V} = [\bar{u}(\bar{x}, \bar{y}), \bar{v}(\bar{x}, \bar{y}), 0] \tag{7}$$

The steady two-dimensional equations for a third-grade fluid are thus given by

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{8}$$

$$\begin{aligned}
&\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} \\
&= -\frac{1}{\rho}\ \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\mu}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) + \frac{\alpha_1}{\rho}\left\{\frac{\partial \bar{u}}{\partial \bar{x}}\left(13\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) + \bar{u}\left(\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} + \frac{\partial^3 \bar{u}}{\partial \bar{y}^2 \partial \bar{x}}\right)\right. \\
&\quad \left. + \bar{v}\left(\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + \frac{\partial^3 \bar{u}}{\partial \bar{x}^2 \partial \bar{y}}\right) + 2\frac{\partial \bar{v}}{\partial \bar{x}}\left(2\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}}\right) + 3\frac{\partial \bar{u}}{\partial \bar{y}}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}}\right)\right\} \\
&\quad + 2\frac{\alpha_2}{\rho}\left\{4\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial \bar{u}}{\partial \bar{y}}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}}\right) + \frac{\partial \bar{v}}{\partial \bar{x}}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}}\right)\right\} \\
&\quad + \frac{\beta_3}{\rho}\left\{40\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + 24\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} + 24\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{v}}{\partial \bar{x}}\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} + 12\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{v}}{\partial \bar{x}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right. \\
&\quad + 8\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + 8\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{v}}{\partial \bar{x}}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + 8\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + 6\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + 6\left(\frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \\
&\quad \left. - 4\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{v}}{\partial \bar{x}}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - 2\left(\frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - 2\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right\}
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} \\
&= -\frac{1}{\rho}\ \frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\mu}{\rho}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right) + \frac{\alpha_1}{\rho}\left\{\frac{\partial \bar{v}}{\partial \bar{y}}\left(13\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{x}^2}\right) + \bar{u}\left(\frac{\partial^3 \bar{v}}{\partial \bar{x}^3} + \frac{\partial^3 \bar{v}}{\partial \bar{y}^2 \partial \bar{x}}\right)\right. \\
&\quad \left. + \bar{v}\left(\frac{\partial^3 \bar{v}}{\partial \bar{y}^3} + \frac{\partial^3 \bar{v}}{\partial \bar{x}^2 \partial \bar{y}}\right) + 2\frac{\partial \bar{u}}{\partial \bar{y}}\left(2\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{x}\partial \bar{y}}\right) + 3\frac{\partial \bar{v}}{\partial \bar{x}}\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{x}\partial \bar{y}}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&+2\frac{\alpha_2}{\rho}\left\{4\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial^2\bar{v}}{\partial\bar{y}^2}+\frac{\partial\bar{u}}{\partial\bar{y}}\left(\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{y}}\right)+\frac{\partial\bar{v}}{\partial\bar{x}}\left(\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{y}}\right)\right\}\\
&+\frac{\beta_3}{\rho}\left\{40\left(\frac{\partial\bar{v}}{\partial\bar{y}}\right)^2\frac{\partial^2\bar{v}}{\partial\bar{y}^2}+24\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{y}}+24\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{y}}+12\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\right.\\
&+8\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+8\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{y}^2}+8\left(\frac{\partial\bar{v}}{\partial\bar{y}}\right)^2\frac{\partial^2\bar{v}}{\partial\bar{x}^2}+6\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\frac{\partial^2\bar{v}}{\partial\bar{x}^2}+6\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)^2\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\\
&\left.-4\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{y}^2}-2\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)^2\frac{\partial^2\bar{v}}{\partial\bar{y}^2}-2\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\frac{\partial^2\bar{v}}{\partial\bar{y}^2}\right\}
\end{aligned}
\tag{10}
$$

Invoking dimensionless variables

$$
u=\frac{\bar{u}}{U},\quad y=\frac{\bar{y}}{H_0},\quad x=\frac{\bar{x}}{L},\quad p=\frac{\bar{p}H_0{}^2}{\mu UL} \tag{11}
$$

and assuming that the coating thickness H is much smaller than the blade length L. So, as a result the value $H/L \ll 1$. Also for convenience, a nearly parallel flow is supposed in the narrow gap between the moving substrate and the blade so that the lubrication approximation theory can be applied in such a way that $\partial/\partial x \ll \partial/\partial y$ and $v \ll u$, where the velocity components in the x and y direction are represented by u and v, respectively. In the light of above approximations, the governing equations, in the present case, reduce substantially and take the form

$$
\frac{\partial^2 u}{\partial y^2}+3\beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}=\frac{dp}{dx} \tag{12}
$$

$$
\frac{\partial p}{\partial y}=0 \tag{13}
$$

where $\beta=\frac{2\beta_3 U^2}{\mu H_0{}^2}$.
here are the dimensions of

$$U : LT^{-1}$$

$$H_0 : L$$

$$\mu : ML^{-1}T^{-1}$$

$$\beta_3 : ML^{-1}T$$

The appropriate boundary conditions for the considered flow situation are

$$u(0) = 1 \quad \text{and}\, u(h) = 0 \tag{14}$$

and conditions on pressure at the end points are given by

$$p(0) = 0 \quad \text{and}\, p(1) = 0 \tag{15}$$

The flow rate is given by

$$H = \int_0^h u \mathrm{d}y \tag{16}$$

Blade load can be expressed as

$$L = \int_0^h p(x) \mathrm{d}x \tag{17}$$

where h for plane (stiff blade) and exponential (flexible blade) coater[7] is given by

$$\left.\begin{aligned} &\textit{for plane coater}, \quad h(x) = k - (k-1)x \\ &\textit{for exponential coater}, \quad h(x) = k^{1-x} \end{aligned}\right\} \tag{18}$$

Perturbation solution for small β

The governing equation (12) involves a nonlinear term with β. Assuming small β we can expand velocity, pressure, coating thickness, and load in the following way

$$\left.\begin{aligned} u &= u^0 + \beta u^1 + \dots, \\ p &= p^0 + \beta p^1 + \dots, \\ H &= H^0 + \beta H^1 + \dots, \\ L &= L^0 + \beta L^1 + \dots \end{aligned}\right\} \tag{19}$$

Zeroth-order solution

Substituting series (19) in equations (12) and (14) to (17), the boundary value problem at zeroth-order takes the form

$$\frac{\partial p^0}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial y}\right) \tag{20}$$

$$u^0(0) = 1, \quad u^0(h) = 0 \tag{21}$$

$$p^0(0) = 0, \quad p^0(1) = 0 \tag{22}$$

$$H^0 = \int_0^h u^0 \mathrm{d}y \tag{23}$$

$$L^0 = \int_0^1 p^0(x) \mathrm{d}x \tag{24}$$

(a) For plane coater (stiff blade)

The zeroth-order solution of equation (20) together with conditions given by equation (21) for the plane coater is

$$u^0 = -\frac{\big(k(-1+x) - x + y\big)\Big((1+k)(k+x-kx)^2 + 3(-1+k)(k(-1+x)+x)y\Big)}{(1+k)(k+x-kx)^3} \tag{25}$$

The pressure gradient is presented in equation (26) and pressure in equation (27) by using conditions given in equation (22)

$$\frac{\partial p^0}{\partial x} = \frac{6(-1+k)(k(-1+x)+x)}{(1+k)(k+x-kx)^3} \tag{26}$$

$$p^0 = \frac{6(-1+k)(-1+x)x}{(1+k)(k+x-kx)^2} \tag{27}$$

substituting equation (25) in equation (23), one can obtain flow rate given by

$$H^0 = \frac{k}{k+1} \tag{28}$$

and load can be represented as

$$L^0 = \frac{6(2-2k+(1+k)\mathrm{Log}[k])}{(-1+k)^2(1+k)} \tag{29}$$

(b) For exponential coater (flexible blade)

In case of exponential coater, the solution of equation (20) along with equation (21) is given by

$$u^0 = -\frac{(k - k^x y)\left(-2 + 3k^{-1+x}\left(2 - \frac{3k^x(1+k)}{1+k+k^2}\right)y\right)}{2k} \tag{30}$$

Pressure gradient and pressure with conditions (22) are presented in equation (31) and in equation (32), respectively

$$\frac{\partial p^0}{\partial x} = 3k^{-2+2x}\left(2 - \frac{3k^x(1+k)}{1+k+k^2}\right) \tag{31}$$

$$p^0 = \frac{3(k - k^x)(-1 + k^x)(k + k^x(1+k))}{k^2(1+k+k^2)\mathrm{Log}[k]} \tag{32}$$

Flow rate and load are given by equations (33) and (34), respectively, for the case of exponential coater

$$H^0 = \frac{3k(1+k)}{4(1+k+k^2)} \tag{33}$$

$$L^0 = \frac{-1 + \frac{1}{k^2} + \frac{6\mathrm{Log}[k]}{1+k+k^2}}{2\mathrm{Log}[k]^2} \tag{34}$$

First-order solution

The first-order boundary value problem in the present case is given below

$$\frac{\partial p^1}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u^1}{\partial y}\right) + 3\left(\frac{\partial u^0}{\partial y}\right)^2 \frac{\partial}{\partial y}\left(\frac{\partial u^0}{\partial y}\right) \tag{35}$$

$$u^1(0) = 0, \quad u^1(h) = 0 \tag{36}$$

$$p^1(0) = 0, \quad p^1(1) = 0 \tag{37}$$

$$H^1 = \int_0^h u^1 \mathrm{d}y \tag{38}$$

$$L^1 = \int_0^1 p^1(x)\mathrm{d}x \tag{39}$$

(a) For plane coater (stiff blade)

Introducing equation (36) in equation (35), the following relation can be obtained

$$\begin{aligned} u^1 = {} & \frac{1}{25k(1+k)^3(k+x-kx)^9} 6(-1+k)^2 y(k(-1+x)-x+y)(26k^8(-1+x)^6 \\ & + 26x^6 - 2k^7(-1+x)^5(-61+79x) + k^6(-1+x)^4(-4-64x+428x^2+375y) \\ & - k^3(-1+x)x(2x^2(380+x(-715+353x)) + 150x(-2+5x)y - 675y^2) \\ & + kx^3(36x^2-158x^3+375xy-225y^2) - k^5(-1+x)^3(2x(18+x(9+353x)) \\ & - 75(1+5x)y - 225y^2) + k^2x^2(4x^2(90+x(-198+107x)) + 75x(-6+5x)y \\ & + 225(3-2x)y^2) + 5k^4(-1+x)^2(2x^2(45+82(-1+x)x) \\ & + 30(3-5x)xy + 45(1+2x)y^2)) \end{aligned} \tag{40}$$

The pressure gradient is presented by the relation

$$\frac{\partial p^1}{\partial x} = -\frac{12\left[\left\{\begin{array}{c} \dfrac{540k^3}{(1+k)^3} + \dfrac{810k^2(k(-1+x)-x)}{(1+k)^2} + \dfrac{480k(k+x-kx)^2}{1+k} \\ -105(k+x-kx)^3 - \dfrac{2(-1+k)^2(13+k(-1+13k))(k+x-kx)^4}{k(1+k)^3} \end{array}\right\}\right]}{25(k+x-kx)^7} \tag{41}$$

In this case, one can obtain the pressure from equation (41) after introducing conditions given in equation (37)

$$p^1 = \frac{\left\{\begin{array}{l} 12(-1+k)x(k^3(79+k(-32+k(77+26k))) \\ \quad -3k^2(-21+k(29+k(-56+k(59+39k))))x \\ \quad +(-1+k)k(-87+k(3+k(43+k(233+208k))))x^2 \\ \quad -(-1+k)^2(-13+k(27+k(167+7k(21+26k))))x^3 \\ \quad +6(-1+k)^3k(13+k(8+13k))x^4 - (-1+k)^4(13+k(8+13k))x^5) \end{array}\right\}}{25k(1+k)^3(k+x-kx)^6} \tag{42}$$

The corresponding flow rate and load are shown, respectively, by the relations given in equations (43) and (44)

$$H^1 = -\frac{2(-1+k)^2(13+k(-1+13k))}{25k(1+k)^3} \tag{43}$$

$$L^1 = \frac{12(-1+k)(13+k(-1+13k))}{25k^2(1+k)^3} \tag{44}$$

(b) For exponential coater (flexible blade)

For the case of exponential coater, the first-order velocity is given by equation (45) after solving equation (35) along with conditions (36) we obtain

$$\begin{aligned} u^1 =& \frac{1}{8(1+k+k^2)^3} k^{-6-x} y \Big\{ 432k^{3+4x}(1+k+k^2)^3 \\ & - \frac{1}{175(1+k+k^2)} 9k^{3+3x}(-8505k^{4x}(1+k)^3(1+k+k^2) \\ & + 17010k^{3x}(1+k)^2(1+k+k^2)^2 - 13440k^{2x}(1+k)(1+k+k^2)^3 \\ & + 3920k^x(1+k+k^2)^4 + 3(-1+k)^2(1+k)(88+k(440+k(745+k(604 \\ & + k(745+88k(5+k))))))) + \frac{1}{175(1+k+k^2)} 9k^{2+4x}(-8505k^{4x}(1+k)^3 \\ & \times (1+k+k^2) + 17010k^{3x}(1+k)^2(1+k+k^2)^2 - 13440k^{2x}(1+k)(1+k+k^2)^3 \\ & + 3920k^x(1+k+k^2)^4 + 3(-1+k)^2(1+k)(88+k(440+k(745+k(604 \\ & + k(745+88k(5+k)))))))y + 1458k^{10x}(1+k)^3y^3 + 81k^{8x}(1+k)y(27k^2(1+k)^2 \\ & + 80k(1+k)(1+k+k^2)y + 24(1+k+k^2)^2y^2) \\ & + 18k^{1+6x}(1+k+k^2)(99k^2(1+k)^2 + 240k(1+k)(1+k+k^2)y \\ & + 64(1+k+k^2)^2y^2) - 27k^{7x}(27k^3(1+k)^3 + 198k^2(1+k)^2(1+k+k^2)y \\ & + 176k(1+k)(1+k+k^2)^2y^2 + 16(1+k+k^2)^3y^3) \\ & - 2916k^{9x}(1+k)^2y^2(y+k(1+k)(1+y)) - 72k^{2+5x}(1+k+k^2)^2 \\ & \times (16y+k(1+k)(21+16y)) \Big\} \end{aligned} \tag{45}$$

The following relation gives the pressure gradient

$$\begin{aligned} \frac{\partial p^1}{\partial x} =& \frac{1}{700(1+k+k^2)^4} 9k^{-4+3x} \Big\{ -8505k^{4x}(1+k)^3(1+k+k^2) \\ & + 17010k^{3x}(1+k)^2(1+k+k^2)^2 - 13440k^{2x}(1+k)(1+k+k^2)^3 \end{aligned}$$

$$+ 3920k^x(1+k+k^2)^4 + 3(-1+k)^2(1+k)(88+k(440+k(745 +k(604+k(745+88k(5+k))))))\Big\} \tag{46}$$

Pressure, in present case, takes the form

$$p^1 = \frac{1}{700k^4(1+k+k^2)^4\mathrm{Log}[k]}9(-1215k^{7x}(^{1+k)3}(1+k+k^2) + 2835k^{6x}(1+k)^2 \times (1+k+k^2)^2 - 2688k^{5x}(1+k)(1+k+k^2)^3 + 980k^{4x}(1+k+k^2)^4 + (-1+k)^2k^{3x}(1+k)(88+k(440+k(745+k(604+k(745+88k(5+k)))))) 0 - k^3(88+k(1332+k(1449+k(1390+k(1449+4k(333+22k)))))) \tag{47}$$

and flow rate in this case is given by

$$H^1 = -\frac{9(-1+k)^2(1+k)(88+k(440+k(745+k(604+k(745+88k(5+k))))))}{2800k(1+k+k^2)^4} \tag{48}$$

For exponential coater the following expression represents the blade load

$$L^1 = \frac{1}{49000k^4(1+k+k^2)^4\mathrm{Log}[k]^2}3((-1+k)(1+k)(1+k+k^2)(7489+k(22467 + k(29809+k(29662+k(70626+k(2+k)(14831+7489k(1+k))))))) - 210k^3(88+k(1332+k(1449+k(1390+k(1449+4k(333+22k))))))\mathrm{Log}[k]) \tag{49}$$

Numerical solution for all β

For obtaining the numerical solution, we replace the velocity with stream function ψ by assuming

$$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x} \tag{50}$$

Hence coating thickness H becomes

$$H = \int_0^h \frac{\partial\psi}{\partial y}\,\mathrm{d}y \tag{51}$$

which suggests the additional boundary conditions

$$\psi(h) = H \quad \text{and}\, \psi(0) = 0 \tag{52}$$

In terms of stream function ψ, the boundary conditions (14) take the form

$$\frac{\partial \psi}{\partial y}(0) = 1 \quad \text{and} \; \frac{\partial \psi}{\partial y}(h) = 0 \tag{53}$$

By introducing equation (50) in equation (12), we get

$$\frac{\partial^3 \psi}{\partial y^3} + 3\beta \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 \frac{\partial^3 \psi}{\partial y^3} = \frac{dp}{dx} \tag{54}$$

Eliminating pressure

$$\frac{\partial^4 \psi}{\partial y^4} + 6\beta \frac{\partial^2 \psi}{\partial y^2}\left(\frac{\partial^3 \psi}{\partial y^3}\right)^2 + 3\beta \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{55}$$

Equation (55) together with conditions given in equations (52) and (53) gives the stream function ψ

$$\text{Load is given by the relation } L = \int_0^1 p(x)\mathrm{d}x \tag{56}$$

For the numerical solution, we choose a value of H and solve equation (55) subject to the boundary conditions (52) and (53) to get the stream function ψ. Substituting the obtained value of ψ into equation (54), one can obtain the pressure gradient dp/dx which further gives the pressure upon integration. Finally, load is calculated through equation (56). A root finding algorithm is implemented to modify H so that $p(1) = 0$.

Results and discussions

The blade coating process for a third-grade fluid both for the plane as well as the exponential coater is discussed here. How the parameters k and β affect the pressure, pressure gradient, velocity, and load are observed by using the numerical technique known as the shooting method and also by the perturbation method for small β. However, the tabulated data are presented only for numerical values of blade load for both the parameters k and β. How the above-mentioned parameters

Table 1. Exponential coater load.

β	$k=2$	$k=3$	$k=4$	$k=6$	$k=8$	$k=10$
0.000	0.162232	0.158389	0.140993	0.112744	0.094371	0.082058
0.001	0.162504	0.158260	0.141166	0.112916	0.094409	0.082101
0.01	0.163211	0.159175	0.141596	0.113220	0.094725	0.082347
0.05	0.167009	0.162036	0.144118	0.114913	0.096046	0.083476
0.09	0.170852	0.165085	0.146326	0.116512	0.097394	0.084577
0.1	0.171725	0.165784	0.146868	0.116848	0.097644	0.084859
0.5	0.206665	0.191931	0.167146	0.130828	0.108607	0.094029
0.9	0.239271	0.215753	0.185214	0.143121	0.118064	0.101916

Table 2. Plane coater load.

β	$k=2$	$k=3$	$k=4$	$k=6$	$k=8$	$k=10$
0.000	0.158938	0.148693	0.125468	0.089769	0.067660	0.053182
0.001	0.159163	0.148617	0.125539	0.089799	0.067676	0.053194
0.01	0.160201	0.149304	0.125910	0.090225	0.067927	0.053400
0.05	0.163985	0.151951	0.128192	0.091594	0.068973	0.054116
0.09	0.167598	0.154893	0.130179	0.092983	0.069891	0.054719
0.1	0.168616	0.155429	0.130799	0.093293	0.070062	0.054863
0.5	0.201986	0.178974	0.147654	0.103363	0.076643	0.059377
0.9	0.233234	0.199670	0.162087	0.111409	0.081732	0.062790

affect the pressure, pressure gradient, and velocity are shown in graphical form. Tables 1 and 2 give numerical data for different loads with different the third-grade parameters β and k for both exponential and plane coater. It is observed from Table 1 (exponential coater) that for a fixed k, the blade load increases gradually as the third-grade parameter β increases whereas blade load decreases as the thickness parameter k increases. Table 2 (plane coater) shows the load corresponding to the parameters β and k. It is obvious from this table that blade load increases as β increases and the blade load decreases as the parameter k increases. One can observe from both tables that load for viscous (Newtonian) fluid ($\beta = 0$) is less than that for third-grade fluid. Since the blade load ensures the coating thickness and quality, therefore blade load must be adjusted according to the coating material and the coating substrate requirements.

In all figures, panel (a) is the plane coater and panel (b) is the exponential coater. Figures 2 to 7 show the results of numerical solution for plane coater as well as for exponential coater. In Figure 2(a), the pressure is plotted against different thickness parameter k keeping the third-grade fluid parameter β as constant for the plane coater. We see that the pressure distribution is sharper near the blade tip for large k. Moreover, the maximum pressure shifts toward the blade tip.

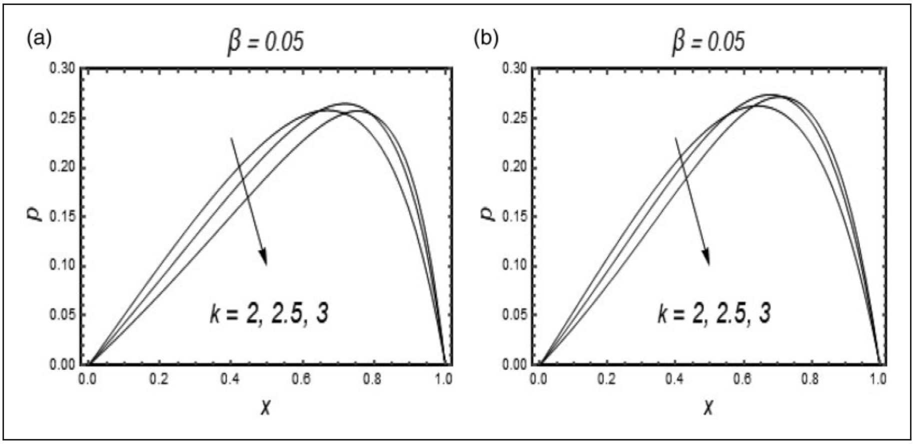

Figure 2. (a) Influence of k on the pressure distribution (plane coater) and (b) influence of k on the pressure distribution (exponential coater).

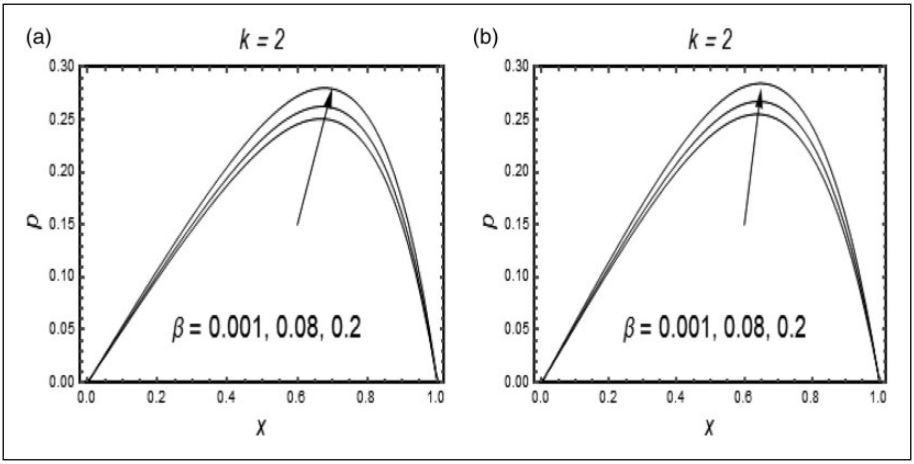

Figure 3. (a) Influence of β on the pressure distribution (plane coater) and (b) influence of β on the pressure distribution (exponential coater).

A similar phenomenon is noted for the exponential coater and can be seen in Figure 2(b). The effects of third-grade fluid parameter β on the pressure distribution, keeping k as constant, can be seen in Figure 3(a) and (b). It is worth mentioning here that the effects of β on pressure distribution are quiet opposite to that for k. For both the cases (exponential coater as well as plane coater), the pressure increases with an increase in β. Figure 4(a) and (b) show how k affects the pressure gradient. For both cases, an increase in k reduces the pressure gradient. Figure 5(a) and (b) illustrates how β affects the pressure gradient for both plane and

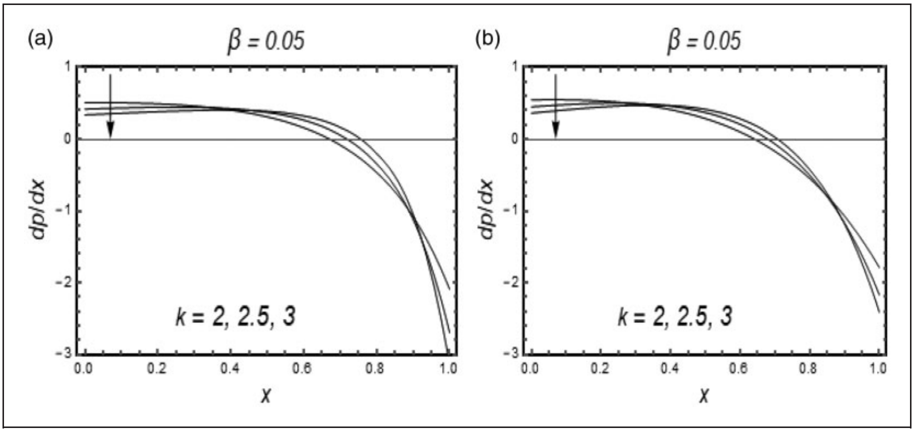

Figure 4. (a) Influence of k on pressure gradient (plane coater) and (b) influence of k on pressure gradient (exponential coater).

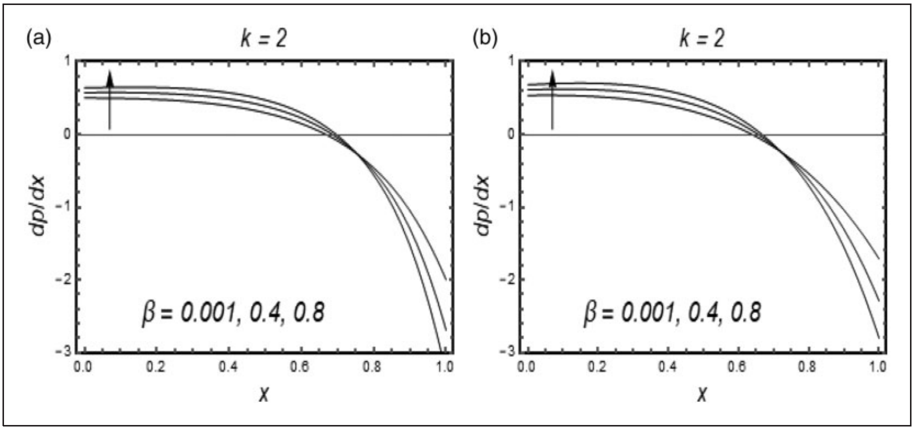

Figure 5. (a) Influence of β on pressure gradient (plane coater) and (b) influence of β on pressure gradient (exponential coater).

exponential coater. The pressure gradient increases when β increases. Figures 6 and 7 depict how k and β affect the velocity profile, for the plane and exponential coater, respectively. Here the observed phenomenon predicts that the fluid velocity increases for the corresponding increase in both the parameters (k and β). Figures 8 and 9 compare both the perturbed values as well as the numerical values of pressure for different k and β separately. The perturbed values are with dashed lines whereas the numerical values are with solid lines. In Figure 8, p is plotted versus k to see the how β affects the pressure. One sees in Figure 8(a) and (b) that for both plane coater and exponential coater, pressure

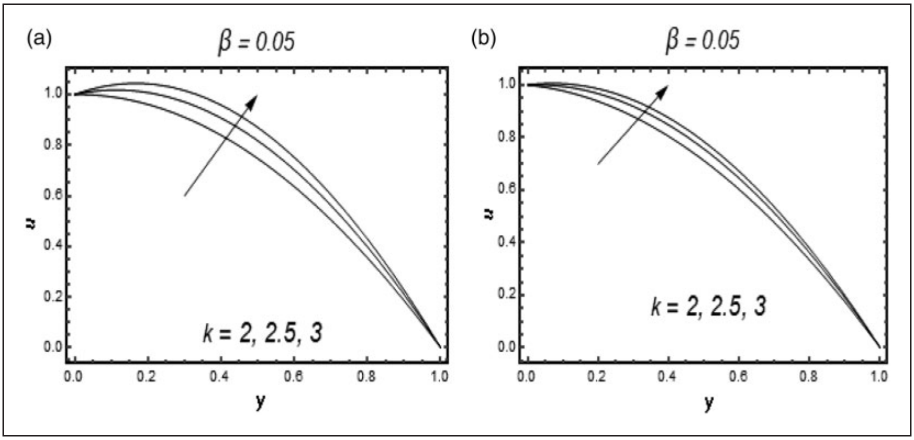

Figure 6. (a) Velocity profile for different k versus y (plane coater) and (b) velocity profile for different k versus y (exponential coater).

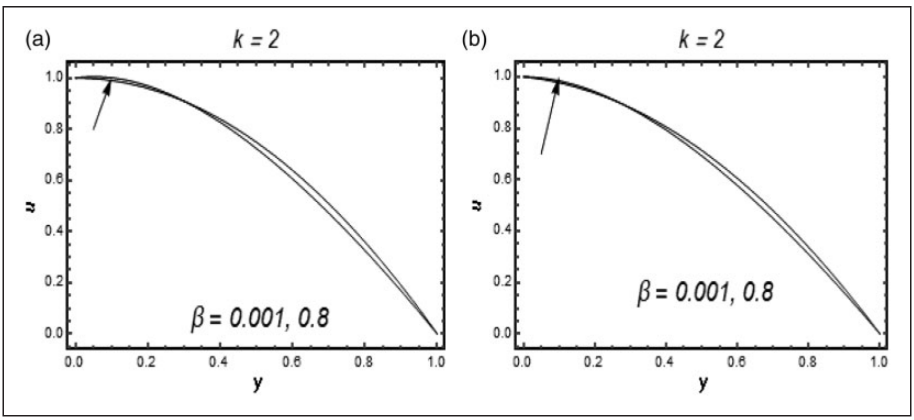

Figure 7. (a) Velocity profile for different β versus y (plane coater) and (b) velocity profile for different β versus y (exponential coater).

increases with an increasing β and one observes from the same figure that the influence of β is reduced for large k. It is also notable that the perturbed values match with numerical value for very small β. In Figure 9(a) and (b), p is plotted versus β to see the impact of thickness parameter k. It is evident from Figure 9 that the pressure reduces for increasing k. Also, from Figure 9, one sees that the perturbed solution matches with the numerical solution for small β but variation increases as β increases. We observe that the fluid apparent viscosity increases as the third-grade parameter β increases which results in a decreased velocity. While a

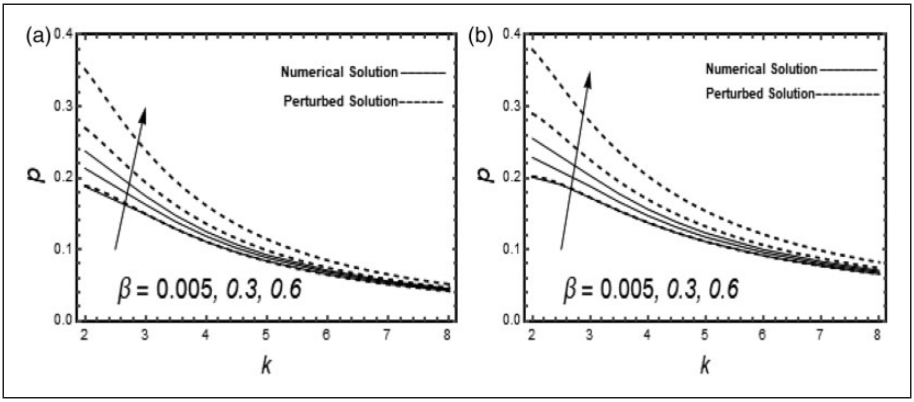

Figure 8. (a) p for different β versus k (plane coater) and (b) p for different β versus k (exponential coater).

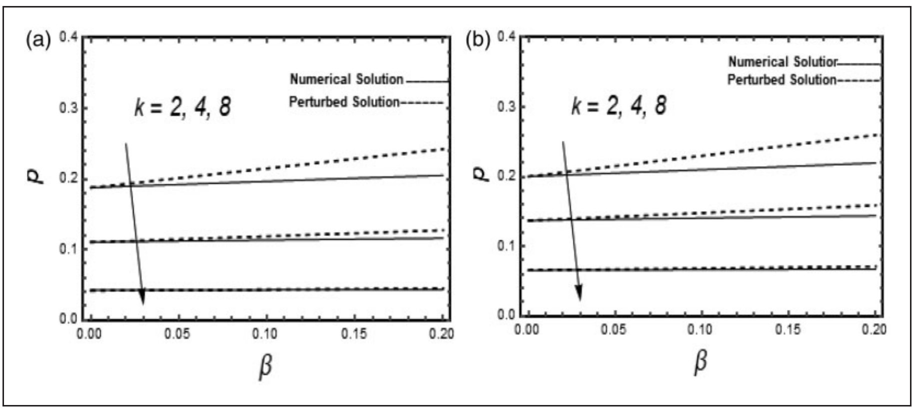

Figure 9. (a) p for different k versus β (plane coater) and (b) p for different k versus β (exponential coater).

decrease in β causes a relevant decrease in the apparent viscosity and enhances the fluid velocity.

Concluding remarks

In this paper the process of coating a solid substrate for a simple fixed blade coater is examined using third-grade fluid by applying lubrication approximation theory. The perturbed solution for pressure, pressure gradient, and velocity is obtained up to the first order. Numerical solution is also obtained for the governing equation using the stream function by a well-known technique termed as the shooting method. Both numerical and perturbed results are compared, and we observed that numerical and perturbation results match very well for small third-grade fluid parameter β. One concludes from the tabulated values that load increases gradually as the third-grade parameter β increases and a reverse phenomenon is observed for the parameter k for both the cases (plane coater and exponential coater).

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ORCID iD

M Mughees http://orcid.org/0000-0002-6274-0819

References

1. Booth GL. *Coating equipment and processes*. New York: Lockwood Publishing Co. Inc., 1970.
2. Trist AR. *Machine for the coating of webs of paper and like absorbent material*. Patent 2,368,176, 2,593,074 and 2,796,846, USA, 1945.
3. Middleman S. *Fundamentals of polymer processing*. New York: McGraw-Hill College, 1977.
4. Ruschak KJ. Coating flows. *Annu Rev Fluid Mech* 1985; 17: 65–89.
5. Kistler SF and Schweizer PM. *Liquid film coating*. London: Chapman & Hall, 1997.
6. Greener Y and Middleman S. Blade coating of a viscoelastic fluid. *Polym Eng Sci* 1974; 14: 791–796.
7. Savage MD. Variable speed coating with purely viscous non-Newtonian fluids. *Z Angew Math Phys* 1983; 34: 358.
8. Ross AB, Wilson SK and Duffy BR. Blade coating of a power law fluid. *Phys Fluids* 1999; 11: 958–970.
9. Sullivan T. *Fluid dynamics in a blade-over-roll coater*. PhD Thesis, University of California, USA, 1986.
10. Sullivan TM and Middleman S. Film thickness in blade coating of viscous and viscoelastic fluids. *J Non-Newtonian Fluid Mech* 1986; 21: 13–38.
11. Sinha P and Singh C. Lubrication of a cylinder on a plane with a non-Newtonian fluid considering cavitation. *Trans ASME J Lubr Technol* 1982; 104: 168.
12. Hsu TC, Malone M, Laurence RL, et al. Separating forces in blade coating of viscous and viscoelastic liquids. *J Non-Newtonian Fluid Mech* 1985; 18: 273.

13. Saita FA. *Elastohydrodynamics and flexibleblade coating*. PhD Thesis, University of Minnesota, USA, 1984.
14. Saita FA and Seriven LE. Coating flow analysis and the physics of flexible blade coating, In: *Proceedings of Tappi J coating conference*, Atlanta: TAPPI Press, 1985, pp.13–21.
15. Corvalan CM and Saita FA. Blade coating on a compressible substrate. *Chem Eng Sci* 1995; 50: 1769–1783.
16. Elkouh AF and Yang DF. Flow of a power-law fluid in a Rayleigh step. *Trans ASME J Tribol* 1991; 113: 428.
17. Flumerfelt RW, Pierick MW, Cooper SL, et al. Generalized plane Couette flow of a non-Newtonian fluid. *Ind Eng Chem Fund* 1969; 8: 354.
18. Tichy JA. Non-Newtonian lubrication with the convected Maxwell model. *Trans ASME J Tribol* 1996; 118: 344.
19. Cameron A. *Principles of lubrication*. London: Longmans, 1966.
20. Aidun CK. Principles of hydrodynamic instability: application in coating systems. *Tappi J* 1991; 74: 213–220.
21. Chen KS and Seriven LE. Liquid penetration into a deformable porous substrate. *Tappi J* 1990; 73: 151–161.
22. Eklund DE. Influence of blade geometry and blade pressure on appearance of a coated surface. *Tappi J* 1984; 67: 66–70.
23. Pranckh FR and Seriven LE. The physics of blade coating of a deformable substrate. *Tappi J* 1990; 73: 163–173.
24. Tayler AB. Fluid flow between a roller and absorbent compressible paper. *Q J Mech Appl Math* 1978; 31: 481–495.
25. Coyle DJ. Forward roll coating with deformable rolls: a simple one-dimensional elastohydrodynamics model. *Chem Engng Sci* 1988; 43: 2673–2684.
26. Hwang SS. Non-Newtonian liquid blade coating process. *J Fluids Eng* 1982; 104: 469–475.
27. Dien IK and Elrod HG. A generalized steady-state Reynolds equation for non-Newtonian fluids, with application to journal bearings. *Trans ASME J Lubr Technol* 1983; 105: 385.
28. Siddiqui AM, Bhatti S, Rana MA, et al. Blade coating analysis of a Williamson fluid. *Results Phys* 2017; 7: 2845–2850.
29. Rana MA, Siddiqui AM, Bhatti S, et al. The study of the blade coating process lubricated with Powell-Eyring Fluid. *J Nanofluids* 2018; 7: 1–10.
30. Sajid M, Shahzad H, Mughees M, et al. Mathematical modeling of slip and magnetohydrodynamics effects in blade coating. *J Plast Film Sheet* 2018; 35: 9–21.
31. Dunn JE and Rajagopal KR. Fluids of differential type: critical review and thermodynamic analysis. *Int J Engng Sci* 1995; 33: 689–729.
32. Fosdick RL and Rajagopal KR. Thermodynamics and stability of fluids of third grade. *Proc Roy Soc Lond A* 1980; 339: 351–377.

Biographies

M Sajid is a professor in Department of Mathematics and Statistics, IIU. His research interest includes fluid mechanics, applied mathematics, and computational fluid dynamics.

M Mughees is assistant professor of Mathematics in Punjab Higher Education Department and a PhD scholar in IIU.

N Ali is an associate professor in Department of Mathematics and Statistics, IIU. His research interest includes fluid mechanics, peristalsis, and blood flows.

H Shahzad is a PhD scholar at College of Mechanical Engineering and Applied Electronics Technology, Beijing University of Technology, China.