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Film flow on a rotating disk

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Unsteady liquid film flow on a rotating disk is analyzed by asymptotic methods for low and high Reynolds numbers. The analysis elucidates how a film of uniform thickness thins when the disk is set in steady rotation. In the low Reynolds number analysis two time scales for the thinning film are identified. The long-time-scale analysis ignores the initial acceleration of the fluid layer and hence is singular at the onset of rotation. The singularity is removed by matching the long-time-scale expansion for the transient film thickness with a short-time-scale expansion that accounts for fluid acceleration during spinup. The leading order term in the long-time-scale solution for the transient film thickness is shown to be a lower bound for film thickness for all time. A short-time analysis that accounts for boundary layer growth at the disk surface is also presented for arbitrary Reynolds number. The analysis becomes invalid either when the boundary layer has a thickness comparable to that of the thinning film, or when nonlinear effects become important.

I. INTRODUCTION

In this paper we present an asymptotic solution to the unsteady Navier–Stokes equations for film flow on a rotating disk. We restrict attention to film flow with a planar interface and assume that the disk radius is much larger than the film thickness so that edge effects are confined to a small region of the total film area and can therefore be neglected. The work of Emslie *et al.*,[1] as discussed below, suggests that the assumption of a planar interface is not unreasonable. A principal feature of the solution is that the radial and azimuthal components of velocity assume a linear dependence on radial distance that allows the r dependence to be factored out of the governing equations and boundary conditions. The pressure field, independent of radial position, can then be expressed in terms of the axial component of velocity induced by the thinning film. The principal objective of this paper is to derive an asymptotic solution to the resulting nonlinear, time-dependent boundary value problem in the limit of vanishingly small Reynolds number. However, a short-time analysis for arbitrary Reynolds number is also undertaken to clarify the effects of viscosity on how the film thins after the disk is impulsively started from rest.

This work is motivated in part by our interest in studying centrifugal spinning of liquid films as a convenient method for coating a discrete planar substrate, such as a disk, with a uniform film. In the coating literature this technique is referred to as spin coating, and is widely used in the manufacture of integrated circuits,[2] magnetic and optical disks for data storage,[3] color television screens,[1] and optical mirrors.[4] In many applications the coating liquid is initially applied to the disk as a thick layer, which is then thinned by rapidly spinning the disk. The final film thickness and its uniformity are known to depend on several process parameters, most notably the rotational speed, liquid viscosity, and, when solvents are present, the rate of solvent evaporation.

The first detailed hydrodynamic analysis of spin coating of a Newtonian liquid was given by Emslie *et al.*[1] They assumed that the local centrifugal force per unit volume (given by $\rho r\Omega^2$, where Ω is the angular velocity of the disk) is uniform across the thinning film and is balanced solely by viscous shear across the film due to the radial liquid flux. The pressure is taken to be everywhere uniform and inertial forces other than centrifugal are assumed to be unimportant. As noted by Acrivos *et al.*,[5] these assumptions are expected to be valid when the Reynolds number $\mathrm{Re} = h_0^2\Omega/\nu$ is much less than unity (h_0 being a characteristic length scale for the thinning film, and ν the kinematic viscosity). Emslie *et al.* were able to show that initial irregularities in the film profile would tend to smooth out with spinning, and a fluid layer that was initially uniform would maintain its uniformity with spinning. These predictions are exclusively for Newtonian liquids and are not necessarily valid for non-Newtonian liquids. For instance, Acrivos *et al.*[5] have shown that a uniform film cannot, in general, be obtained by spinning a power-law liquid.

In recent years several mathematical models for spin coating of photoresists have been proposed in the literature.[6–8] Although additional effects, such as solvent evaporation and non-Newtonian behavior, have been accounted for in the analyses, the hydrodynamic approximations used are typically those employed by Emslie *et al.* To the best of our knowledge there have been no attempts in the literature to develop a rigorous theory for the hydrodynamics of spin coating that goes beyond the analysis of Emslie *et al.*

In Sec. II we state the mathematical problem, and introduce the form for the velocity field that allows the r dependence to be factored out of the Navier–Stokes equations and boundary conditions. We show that it is possible then to derive a general expression for the pressure field in terms of the velocity component normal to the disk surface.

Section III is devoted to the asymptotic analysis in the limit of vanishingly small Reynolds number. We develop an analytic solution based on a long-time-scale expansion. This solution does not satisfy the initial conditions and must be matched, in an asymptotic sense, to a short-time-scale solution. The mathematical problem discussed in Sec. III is similar in several aspects to the low Reynolds number solution for the time-dependent draining of a thin fluid layer between

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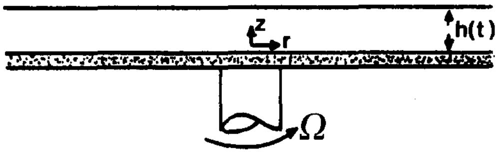

FIG. 1. Schematic of film flow on a rotating disk. The transient film thickness is denoted by $h(t)$.

parallel plates under a constant applied force recently discussed by Weinbaum *et al.*[9] In our problem the driving force is centrifugal acceleration. The other significant difference is the additional independent equation required for the azimuthal component of velocity.

In Sec. IV a short-time analysis is given for the boundary layer flow that is formed after the disk is impulsively started from rest. This analysis, valid for all Re, draws heavily on the work of Lawrence *et al.*[10] In a follow-up paper to Ref. 9 these authors consider the boundary layer formation that occurs when a thin fluid layer is suddenly drained between parallel plates under a constant applied force. We show that the governing equations that describe boundary layer growth for our problem are to leading order identical to those given by Benton.[11] He analyzed flow due to a rotating disk started impulsively from rest in an infinite expanse of fluid. By matching the axial velocity in the boundary layer with its counterpart in the overlying inviscid layer, we are able to derive an expression for how the film thins with time. The asymptotic solutions are discussed in Sec. V and their relation to the work of Emslie *et al.* displayed. There are some similarities between our work at arbitrary Re and the time-dependent spinup of a rotating fluid analyzed by Greenspan and Howard.[12] For a discussion of these similarities readers are referred to Benton's article.

II. MATHEMATICAL FORMULATION

We consider a uniform liquid film on a horizontal plane disk of arbitrary large diameter. The disk is made to rotate in its own plane with a steady angular velocity Ω. We restrict our attention to flows that maintain a film thickness that is independent of the radial coordinate r, and azimuthal coordinate θ. The coordinate axes as well as the general dimensions of the flow are indicated in Fig. 1. The initial film thickness is h_0.

We take the fluid velocity in the thinning film to be independent of θ:

$$\mathbf{v} = u(r,z,t)\mathbf{e}_r + v(r,z,t)\mathbf{e}_\theta + w(r,z,t)\mathbf{e}_z, \tag{1}$$

where $\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z$ are the unit vectors along the coordinate axes. The equations of continuity and motion for an incompressible Newtonian fluid satisfying (2.1) are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{2a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right], \tag{2b}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right], \tag{2c}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right), \tag{2d}$$

where ν is the kinematic viscosity of the liquid, ρ the density, and p the pressure. The boundary conditions that (2) must satisfy are as follows. At the surface of the rotating disk the adherence and impenetrability conditions stipulate that

$$u(r,0,t) = 0, \quad v(r,0,t) = \Omega r, \quad w(r,0,t) = 0. \tag{3}$$

At the free surface $z = h(t)$ the normal and tangential components of the traction vector must vanish:

$$-p + 2\mu\frac{\partial w}{\partial z} = 0, \tag{4a}$$

$$\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0. \tag{4b}$$

In addition, the kinematic condition dictates that the instantaneous velocity of the free surface and the z component of the fluid velocity be equivalent:

$$\frac{dh}{dt} = w(r,h,t). \tag{5}$$

The initial conditions for motion in the liquid film are

$$\mathbf{v} = \mathbf{0}, \quad h = h_0, \quad \frac{dh}{dt} = 0, \quad \text{at } t = 0. \tag{6}$$

Equations (1)–(6) define a time-dependent, two-dimensional free boundary value problem.

We will seek a solution to (1)–(6) of the form (cf. von Karman[13])

$$u = rf(z,t), \quad v = rg(z,t), \quad w = w(z,t). \tag{7}$$

Substituting (7) into the continuity equation (2a) provides a relation between f and w:

$$2f + w_z = 0. \tag{8}$$

(In what follows we will use a subscript to denote partial differentiation, i.e., $w_z \equiv \partial w/\partial z$.) The r component of momentum in terms of (7) becomes

$$f_t + f^2 - g^2 + wf_z - \nu f_{zz} = -(1/\rho)(1/r)p_r. \tag{9}$$

It is evident that the r dependence can be factored out of (9) by defining the pressure to be

$$p(r,z,t) = -\rho(r^2/2)A(z,t) + B(z,t). \tag{10}$$

The quantities $A(z,t)$ and $B(z,t)$ are unknown functions of integration. The w component of momentum in terms of (7) and (10) becomes

$$\rho(w_t + ww_z - \nu w_{zz}) = \rho(r^2/2)A_z - B_z. \tag{11}$$

Thus in order for the right-hand side of (11) to be independent of r, we must have $A_z = 0$. Moreover, since the viscous term in the normal stress boundary condition (4a) is independent of r, this implies that $A(z,t) = 0$. Integrating (11) with respect to z from z to $z = h(t)$ gives an expression for $B(z,t)$:

$$B(z,t) = B(h,t) + \rho \int_z^h w_t \, dz + \rho \left(\frac{w^2}{2} - \nu w_z\right)_z^h. \quad (12)$$

The boundary term $B(h,t)$ can be evaluated with (4a). Hence, the pressure in the thinning film is given by

$$p(z,t) = \rho \left(\nu w_z + \frac{w^2}{2}\right)_{z=h(t)} - \rho \left(\frac{w^2}{2} - \nu w_z - \int_z^{h(t)} w_t \, dz\right). \quad (13)$$

Finally, the r dependence can be factored out of the θ component of momentum to give

$$g_t + 2fg + wg_z = \nu g_{zz}. \quad (14)$$

The boundary conditions for f and g follow directly from (3) and (4):

$$f(t,0) = 0, \quad g(t,0) = \Omega, \quad w(t,0) = 0, \quad (15)$$

$$f_z(t,h) = 0, \quad g_z(t,h) = 0, \quad (16a)$$

$$h_t = w(t,h). \quad (16b)$$

The initial conditions for f and g become

$$f(0,z) = 0, \quad g(0,z) = 0, \quad w(0,z) = 0, \quad (17)$$

and

$$h = h_0, \quad h_t = 0, \quad \text{at } t = 0. \quad (18)$$

The Navier–Stokes equations (2) have been reduced to a coupled set of three partial differential equations for f, g, and w, viz. (8), (9), and (14), subject to the boundary and initial conditions given by (15)–(18). The kinematic condition (16b) provides the additional equation for determining the film thickness. Interestingly, the pressure field, a consequence of normal viscous stresses and inertia of the axial component of velocity induced by the thinning film, can be determined from (13) once w and h are known, and thus need not be solved for simultaneously with the other dependent variables.

Although the analysis presented so far is strictly correct for a flat interface, the more complicated case of a nonuniform interface $h = h(r,t)$ can be analyzed in a similar manner in the limit of zero capillary number (infinite surface tension) by the technique of domain perturbations (Joseph[14]). In the limit of zero capillary number the selection of a uniform interface as a base state leads to a sequence of problems involving cylindrical coordinates in which the r dependence can be factored out of the perturbation equations. Of course, higher-order corrections to the pressure field will now depend on r as well as z. The case of a nonuniform interface will be considered in future work.

III. ASYMPTOTIC ANALYSIS FOR SMALL Re

When the Reynolds number is much less than unity we anticipate that the centrifugal force [g^2 in (9)] will be of comparable magnitude to viscous shear across the film induced by the radial liquid flux (cf. Emslie *et al.*[1]). A balance between viscous shear and centrifugal force may be used to determine a characteristic time scale t_c for the flow. To proceed with the analysis we introduce the following dimensionless variables:

$$\tau = \frac{t}{t_c}, \quad \xi = \frac{z}{h_0}, \quad H = \frac{h}{h_0}, \quad F = ft_c, \quad W = \frac{wt_c}{h_0}, \quad G = \frac{g}{\Omega}. \quad (19)$$

On substituting these variables into (8), (9), and (14), and then requiring in the limit of vanishingly small Reynolds number ($\mathrm{Re} \equiv h_0^2 \Omega/\nu$) that the centrifugal force be balanced by viscous shear across the film, we obtain the appropriate expression for the time scale t_c:

$$t_c = \nu / h_0^2 \Omega^2. \quad (20)$$

The meaning of this time scale becomes clear when one recalls that the time scale for vorticity generated at the disk boundary to spread through the film on spinup is h_0^2/ν, whereas the time scale for inertial spinning is $1/\Omega$. Thus t_c may be interpreted as the time required after spinup for the film to thin in the absence of all inertial forces other than centrifugal.

In terms of the dimensionless variables (19) and (20) the boundary value problem for F, G, W, and H becomes

$$2F + W_\xi = 0, \quad (21a)$$

$$\mathrm{Re}^2(F_\tau + F^2 + WF_\xi) - G^2 = F_{\xi\xi}, \quad (21b)$$

$$\mathrm{Re}^2(G_\tau + 2FG + WG_\xi) = G_{\xi\xi}, \quad (21c)$$

with boundary conditions

$$F(\tau,0) = 0, \quad G(\tau,0) = 1, \quad W(\tau,0) = 0, \quad F_\xi(\tau,H) = 0, \quad G_\xi(\tau,H) = 0, \quad H_\tau = W(\tau,H), \quad (22)$$

and initial conditions

$$F(0,\xi) = 0, \quad G(0,\xi) = 0, \quad W(0,\xi) = 0, \quad H = 1, \quad H_\tau = 0. \quad (23)$$

To construct an asymptotic solution for $\mathrm{Re} \ll 1$, we assume the following forms for the expansions of the dependent variables:

$$\begin{aligned} F(\xi,\tau;\mathrm{Re}) &\sim F_0(\xi,H) + \mathrm{Re}^2 F_1(\xi,H,H_\tau) + O(\mathrm{Re}^4), \\ G(\xi,\tau;\mathrm{Re}) &\sim G_0(\xi,H) + \mathrm{Re}^2 G_1(\xi,H,H_\tau) + O(\mathrm{Re}^4), \\ W(\xi,\tau;\mathrm{Re}) &\sim W_0(\xi,H) + \mathrm{Re}^2 W_1(\xi,H,H_\tau) + O(\mathrm{Re}^4), \\ H(\tau;\mathrm{Re}) &\sim H_0(\tau) + \mathrm{Re}^2 H_1(\tau) + O(\mathrm{Re}^4). \end{aligned} \quad (24)$$

As we will show shortly these expansions are not valid for all time. In particular they are singular at $\tau = 0$, and thus represent the long-time-scale solution. A short-time-scale solution will be required to satisfy the initial conditions.

In the limit $\mathrm{Re} \to 0$ with τ and ξ fixed, the zero-order problem becomes

$$2F_0 + W_{0\xi} = 0, \quad (25a)$$

$$F_{0\xi\xi} + G_0^2 = 0, \quad (25b)$$

$$G_{0\xi\xi} = 0, \quad (25c)$$

with boundary conditions

$$F_0(\tau,0) = 0, \quad G_0(\tau,0) = 1, \quad W_0(\tau,0) = 0, \quad F_{0\xi}(\tau,H) = 0, \quad G_{0\xi}(\xi,H) = 0. \quad (26)$$

Since we have lost the local inertial terms in the zero-order problem, the initial conditions on the velocity field must be discarded at this level of approximation. The solution to (25) satisfying (26) is

$$F_0(\xi,H) = -\xi^2/2 + H\xi, \quad G_0(\xi,H) = 1,$$
$$W_0(\xi,H) = \xi^3/3 - H\xi^2. \tag{27}$$

The first-order problem at $O(\mathrm{Re}^2)$ is

$$2F_1 + W_{1\xi} = 0, \tag{28a}$$

$$F_{1\xi\xi} = F_{0\tau} + F_0^2 + W_0 F_{0\xi} - 2G_0 G_1, \tag{28b}$$

$$G_{1\xi\xi} = G_{0\tau} + 2F_0 G_0 + W_0 G_{0\xi}, \tag{28c}$$

with boundary conditions

$$G_1(\tau,0) = 0, \quad F_1(\tau,0) = 0, \quad W_1(\tau,0) = 0,$$
$$G_{1\xi}(\tau,H) = 0, \quad F_{1\xi}(\tau,H) = 0. \tag{29}$$

Because the right-hand side of (28c) is known from the zero-order problem, (28c) can be integrated to find

$$G_1 = -\frac{\xi^4}{12} + \frac{H\xi^3}{3} - \frac{2H^3\xi}{3}. \tag{30}$$

The expression for G_1 together with (27) can be substituted into (28b) to give

$$F_{1\xi\xi} = H_\tau\,\xi + \frac{\xi^4}{12} - \frac{H\xi^3}{3} + \frac{4}{3}H^3\xi. \tag{31}$$

Integrating twice and applying (29) we obtain

$$F_1 = H_\tau\left(\frac{\xi^3}{6} - \frac{H^2\xi}{2}\right) + \frac{\xi^6}{360} - \frac{H\xi^5}{60} + \tfrac{2}{9}H^3\xi^3 - \tfrac{3}{5}H^5\xi. \tag{32}$$

Finally, from the continuity equation (28a) and boundary condition (29) we obtain the expression for W_1:

$$W_1 = H_\tau\left(\frac{H^2\xi^2}{2} - \frac{\xi^4}{12}\right) - \frac{\xi^7}{1260} + \frac{H\xi^6}{180} - \tfrac{1}{9}H^3\xi^4 + \tfrac{3}{5}H^5\xi^2. \tag{33}$$

The kinematic condition at (22) provides the additional relation needed to determine the film thickness as a function of time:

$$H_\tau \sim W_0(H,\tau) + \mathrm{Re}^2 W_1(H,\tau)$$
$$= -\frac{2}{3}H^3 + \mathrm{Re}^2\left(\frac{5}{12}H_\tau H^4 + \frac{311}{630}H^7\right). \tag{34}$$

Substituting the asymptotic expansion for H [see (24)] into (34) and separating powers of Re yields the following differential equations for H_0 and H_1:

$$H_{0\tau} = -\tfrac{2}{3}H_0^3, \tag{35a}$$

$$H_{1\tau} = -2H_0 H_1^2 + \tfrac{5}{12}H_{0\tau}H_0^4 + \tfrac{311}{630}H_0^7. \tag{35b}$$

These equations are readily integrated to give

$$H_0 = \left(\frac{3}{4\tau + C_0}\right)^{1/2},$$
$$H_1 = C_1\left(\frac{3}{4\tau + C_0}\right)^{3/2} - \frac{17}{105}\left(\frac{3}{4\tau + C_0}\right)^{5/2}. \tag{36}$$

The constants of integration C_0 and C_1 are to be determined from the initial conditions (23). Because the velocity components given by the expansions for F, G, and W do not satisfy the initial conditions, an "inner" expansion valid for short times is required, which must then be matched to the long-time-scale solution (36) to determine C_0 and C_1.

Short-time-scale analysis: At short time scales the local inertial terms in (21) are of the same order of magnitude as the viscous and centrifugal terms included in the long-time-scale analysis. In order to reflect this, we must introduce a new time scale that characterizes the time required for vorticity to diffuse through the film, i.e., a time scale of order h_0^2/ν. The appropriate dimensionless variables for the short-time-scale analysis are

$$T = \frac{t\Omega}{\mathrm{Re}} = \frac{\tau}{\mathrm{Re}^2}, \ \overline{F} = \frac{f}{\Omega\mathrm{Re}} = F, \ \overline{W} = \frac{w}{h_0\Omega\mathrm{Re}} = W,$$
$$\overline{H} = h/h_0 = H, \quad \eta = z/h_0 = \xi. \tag{37}$$

Note in the short-time-scale analysis only the time variable is stretched. Substituting these variables into (21) and (22) gives

$$2\overline{F} + \overline{W}_\eta = 0, \tag{38a}$$

$$\overline{F}_T + \mathrm{Re}^2(\overline{F}^2 + \overline{W}\overline{F}_\eta) - \overline{G}^2 = \overline{F}_{\eta\eta}, \tag{38b}$$

$$\overline{G}_T + \mathrm{Re}^2(2\overline{F}\overline{G} + \overline{W}\overline{G}_\eta) = \overline{G}_{\eta\eta}, \tag{38c}$$

with boundary conditions

$$\overline{F}(T,0) = 0, \quad \overline{G}(T,0) = 1, \quad \overline{W}(T,0) = 0,$$
$$\overline{F}_\eta(T,\overline{H}) = 0, \quad \overline{G}_\eta(T,\overline{H}) = 0, \quad \overline{H}_T = \mathrm{Re}^2\overline{W}(T,\overline{H}). \tag{39}$$

The initial conditions in terms of the short-time-scale variables are

$$\overline{F}(0,\eta) = 0, \quad \overline{G}(0,\eta) = 0, \quad \overline{W}(0,\eta) = 0,$$
$$\overline{H} = 1, \quad \overline{H}_T = 0. \tag{40}$$

The zero-order problem for the short-time-scale solution in the limit $\mathrm{Re}\to 0$ with T and η held fixed follows by setting all terms multiplied by Re in (38)–(40) to zero. This yields a linear set of partial differential equations. An important consequence of the limiting process is that at zero order the film thickness can be determined directly from the kinematic condition at (39). The result is

$$\overline{H}_{0T} = 0 \quad \text{or}\ \overline{H}_0 = 1. \tag{41a}$$

(Here, as before, a subscript zero or one on a dependent variable denotes the order of the approximation.) Thus at the zero-order approximation the axial coordinate η is specified in the range $0 \leqslant \eta \leqslant 1$.

The governing equations for G_0 follow directly from (38)–(40):

$$\overline{G}_{0T} = \overline{G}_{0\eta\eta}, \quad 0 \leqslant \eta \leqslant 1, \quad T > 0,$$
$$\overline{G}_0(T,0) = 1, \quad \overline{G}_{0\eta}(T,1) = 0, \quad \overline{G}_0(0,\eta) = 0. \tag{41b}$$

Equations (41) admit a separable solution given by

$$G_0 = 1 - 2\sum_{n>0} B_n(\eta,T;\lambda_n), \tag{42}$$

where

$$B_n \equiv \frac{\sin(\lambda_n \eta)}{\lambda_n} e^{-\lambda_n^2 T} \quad \text{and} \quad \lambda_n = \frac{n\pi}{2}, \quad n = 1,3,5. \tag{43}$$

The zero-order problem for $\overline{F}_0$ is

$$\overline{F}_{0T} - \overline{F}_{0\eta\eta} = \overline{G}_0^2, \quad 0 \leqslant \eta \leqslant 1, \quad T > 0,$$
$$\overline{F}_0(T,0) = 0, \quad \overline{F}_{0\eta}(T,1) = 0, \quad \overline{F}_0(0,\eta) = 0. \tag{44}$$

Equations (44) are conveniently solved by finite Fourier transforms. If we define the sine transforms for $\overline{F}_0$ and $\overline{G}_0^2$,

$$\widehat{F} \equiv \int_0^1 \overline{F}_0(T,\eta)\sin(\lambda_m \eta)d\eta, \tag{45a}$$

$$\widehat{G} \equiv \int_0^1 \overline{G}_0^2(T,\eta)\sin(\lambda_m \eta)d\eta, \tag{45b}$$

with $\lambda_m = m\pi/2$, and m odd, then (44) becomes

$$\frac{d\widehat{F}}{dT} + \lambda_m^2 \widehat{F} = \widehat{G}, \quad \widehat{F}(0) = 0. \tag{46}$$

The general solution to (46) is

$$\widehat{F} = e^{-\lambda_m^2 T} \int_0^T \widehat{G} e^{\lambda_m^2 t} dT, \tag{47}$$

and after inverting the transform for $\widehat{F}$, we obtain

$$\overline{F}_0(T,\eta) = 2 \sum_{m>0} \sin(\lambda_m \eta) e^{-\lambda_m^2 T} \int_0^T \widehat{G} e^{\lambda_m^2 t} dt. \tag{48}$$

To proceed further it is necessary to evaluate $\widehat{G}$. From (43) we can write the transform for $\overline{G}_0^2$ as

$$\widehat{G} = \int_0^1 \left\{ \left[1 - 4 \sum_{n>0} B_n + 4\left(\sum_{n>0} B_n \right)^2 \right] \sin(\lambda_m \eta) \right\} d\eta. \tag{49}$$

The infinite series in the integrand of (49) can be integrated term by term to yield

$$\widehat{G} = \frac{1}{\lambda_m}(1 - 2e^{-\lambda_m^2 T}) - 8 \sum_{n>0} A_{nm}^0 e^{-2\lambda_n^2 T}$$
$$- 16 \sum_{p>0} \sum_{n>p} A_{nm}^p e^{-(\lambda_p^2 + \lambda_n^2)T}, \tag{50}$$

where

$$A_{nm}^0 \equiv \frac{1}{\lambda_m(\lambda_m^2 - 4\lambda_n^2)},$$
$$A_{nm}^p \equiv \frac{\lambda_p}{[(\lambda_n^2 - \lambda_m^2)^2 + \lambda_p^4 - 2\lambda_p^2(\lambda_n^2 + \lambda_m^2)]}. \tag{51}$$

The subscripts n, m, and p are summed over all odd positive integers. Substituting the above expression for $\widehat{G}$ into (48) and integrating with respect to time yields the required expression for $\overline{F}_0(T,\eta)$:

$$\overline{F}_0(T,\eta) = \eta - \frac{\eta^2}{2} - 2 \sum_{m>0} \frac{\sin(\lambda_m \eta)}{\lambda_m^3} e^{-\lambda_m^2 T} - 4 \sum_{m>0} \frac{\sin(\lambda_m \eta)}{\lambda_m} T e^{-\lambda_m^2 T} + 16 \sum_{m>0} \sin(\lambda_m \eta) \sum_{n>0} \frac{A_{nm}^0}{(2\lambda_n^2 - \lambda_m^2)}$$
$$\times (e^{-2\lambda_n^2 T} - e^{-\lambda_m^2 T}) + 32 \sum_{m>0} \sin(\lambda_m \eta) \sum_{p>0} \sum_{n>p} \frac{A_{nm}^p}{(\lambda_p^2 + \lambda_n^2 - \lambda_m^2)} (e^{-(\lambda_p^2 + \lambda_n^2)T} - e^{-\lambda_m^2 T}). \tag{52}$$

Note, in the derivation of (52) we have used the result that

$$\eta - \frac{\eta^2}{2} = 2 \sum_{m>0} \frac{\sin(\lambda_m \eta)}{\lambda_m^3}. \tag{53}$$

Once $\overline{F}_0$ is found, the continuity equation (38a) can be integrated to find $\overline{W}_0$:

$$\overline{W}_0(T,\eta) = -2 \int_0^\eta \overline{F}_0 \, d\eta = \frac{\eta^3}{3} - \eta^2 - 4 \sum_{m>0} \frac{1}{\lambda_m^4} [\cos(\lambda_m \eta) - 1] e^{-\lambda_m^2 T} - 8 \sum_{m>0} \frac{1}{\lambda_m^2} [\cos(\lambda_m \eta) - 1] T e^{-\lambda_m^2 T}$$
$$+ 32 \sum_{m>0} \frac{1}{\lambda_m} [\cos(\lambda_m \eta) - 1] \sum_{n>0} \frac{A_{nm}^0}{(2\lambda_n^2 - \lambda_m^2)} (e^{-2\lambda_n^2 T} - e^{-\lambda_m^2 T}) + 64 \sum_{m>0} \frac{1}{\lambda_m}$$
$$\times [\cos(\lambda_m \eta) - 1] \sum_{p>0} \sum_{n>p} \frac{A_{nm}^p}{(\lambda_p^2 + \lambda_n^2 - \lambda_m^2)} (e^{-(\lambda_p^2 + \lambda_n^2)T} - e^{-\lambda_m^2 T}). \tag{54}$$

This completes the short-time-scale solution at zero order. However, to determine the constants C_0 and C_1 in the long-time-scale solution (36), it is necessary to compute the film thickness for the short-time-scale solution to order Re^2. The kinematic condition to order Re^2 yields

$$\overline{H}_{1T} = \overline{W}_0(T,1). \tag{55}$$

Substituting (54) evaluated at $\eta = 1$ into (55) and integrating with respect to time gives the required order Re^2 correction for the film thickness:

$$\overline{H}_1 = -\frac{2}{3} T - 4 \sum_{m>0} \frac{1}{\lambda_m^6} (e^{-\lambda_m^2 T} - 1) - 8 \sum_{m>0} \frac{1}{\lambda_m^4} \left[e^{-\lambda_m^2 T} \left(T + \frac{1}{\lambda_m^2} \right) - \frac{1}{\lambda_m^2} \right] + 32 \sum_{m>0} \frac{1}{\lambda_m}$$
$$\times \sum_{n>0} \frac{A_{nm}^0}{(2\lambda_n^2 - \lambda_m^2)} \left(\frac{1}{2\lambda_n^2} (e^{-2\lambda_n^2 T} - 1) - \frac{1}{\lambda_m^2} (e^{-\lambda_m^2 T} - 1) \right) + 64 \sum_{m>0} \frac{1}{\lambda_m}$$
$$\times \sum_{p>0} \sum_{n>p} \frac{A_{nm}^p}{(\lambda_p^2 + \lambda_n^2 - \lambda_m^2)} \left(\frac{1}{(\lambda_p^2 + \lambda_n^2)} (e^{-(\lambda_p^2 + \lambda_n^2)T} - 1) - \frac{1}{\lambda_m^2} (e^{-\lambda_m^2 T} - 1) \right). \tag{56}$$

To determine the constants C_0 and C_1 in (36) we introduce an intermediate time scale $S = \mathrm{Re}^{2\alpha} T$, with $0<\alpha<1$, and then match up the expansions for H and $\overline{H}$ in their region of overlap (Nayfeh[15]). Requiring that the coefficients for each power of Re be equal yields $C_0 = 3$, and

$$C_1 = \frac{17}{105} + 12 \sum_{m>0} \frac{1}{\lambda_m^6} + 32 \sum_{m>0} \frac{1}{\lambda_m} \sum_{n>0} \frac{A_{mn}^0}{2\lambda_n^2 \lambda_m^2} + 64 \sum_{m>0} \frac{1}{\lambda_m} \sum_{p>0} \sum_{n>p} \frac{A_{nm}^p}{\lambda_m^2(\lambda_p^2 + \lambda_n^2)}. \qquad (57)$$

The infinite sequence on the right side of (57) converges rapidly to a value of 0.66087, and hence $C_1 = 0.8227$.

Finally, we can determine a single composite uniform expansion for the transient film thickness by adding the long-time-scale solution (36) to the short-time-scale solution (56) and subtracting from the result their common part:

$$H^c(\tau;\mathrm{Re}) = H_0(\tau) + \tfrac{4}{3}\tau + \mathrm{Re}^2 \times [\overline{H}_1(\tau/\mathrm{Re}^2) + H_1(\tau) - 0.6609]. \qquad (58)$$

In arriving at (58) we have expressed the short-time-scale contribution $\overline{H}_1(T)$ to the film thickness in terms of τ.

IV. SHORT-TIME ANALYSIS FOR ARBITRARY Re

The underlying assumption in Sec. III in developing the short-time solution is that the effects of viscosity are felt throughout the film on spinup. At low Re this is a reasonable assumption. However, at large Re we would anticipate a boundary layer to form at the disk surface in which the effects of viscosity are felt. The boundary layer thus provides a pathway for liquid to leave the disk during spinup, with liquid being supplied to it from an overlying inviscid layer. It is instructive to determine how the fluid layer thins under these circumstances. The problem is similar to the short-time solution for arbitrary Re of Lawrence *et al.*,[10] except that there is an additional equation for the azimuthal component of velocity.

To analyze the boundary layer growth at the disk surface, we follow the procedure detailed in Ref. 10. We suppose that the characteristic time scale t_c for the flow is ϵ/Ω, where ϵ is a small constant. Introducing $t_c = \epsilon/\Omega$ into (19), the boundary value problem for F, G, W, and H becomes

$$2F + W_\xi = 0, \qquad (59a)$$

$$\mathrm{Re}(F_\tau + F^2 + WF_\xi) - \epsilon^2 \mathrm{Re} G^2 = \epsilon F_{\xi\xi}, \qquad (59b)$$

$$\mathrm{Re}(G_\tau + 2FG + WG_\xi) = \epsilon G_{\xi\xi}, \qquad (59c)$$

with boundary conditions

$$F(\tau,0) = 0, \quad G(\tau,0) = 1, \quad W(\tau,0) = 0,$$
$$F_\xi(\tau,H) = 0, \quad G_\xi(\tau,H) = 0, \quad H_\tau = W(\tau,H), \qquad (60)$$

and initial conditions

$$F(0,\xi) = 0, \quad G(0,\xi) = 0, \quad W(0,\xi) = 0, \quad H(0) = 1. \qquad (61)$$

To keep the notation simple, we have used the same symbols for the dependent and independent variable as in (21).

Since the effects of viscosity are confined to the boundary layer, there is no azimuthal or radial component of velocity in the overlying inviscid layer. The required solution to (59) as $\epsilon \to 0$, holding ξ, Re fixed, is thus

$$W(\tau,\xi;\epsilon) \sim \Delta_1(\epsilon) a_1(\tau) + \Delta_2(\epsilon) a_2(\tau) + \cdots. \qquad (62)$$

The functions $a_1(\tau)$ and $a_2(\tau)$ and the scalings $\Delta_1(\epsilon)$ and $\Delta_2(\epsilon)$ are to be determined by matching (62) with an inner solution that satisfies the boundary conditions at the disk surface. To reflect the presence of a boundary layer, ξ is replaced by a stretched coordinate $\zeta = \xi/\epsilon^{1/2}$, and new scalings for F and W are introduced:

$$\widetilde{F} = F/\epsilon^2, \quad \widetilde{W} = W/\epsilon^{5/2}. \qquad (63)$$

It is not necessary to scale G as it is of order unity in the boundary layer; thus $\widetilde{G} = G$. In terms of the inner variables, (59) becomes

$$\begin{aligned} &2\widetilde{F} + \widetilde{W}_\zeta = 0, \\ &\mathrm{Re}(\widetilde{F}_\tau + \epsilon^2 \widetilde{F}^2 + \epsilon^2 \widetilde{W}\widetilde{F}_\zeta) - \mathrm{Re}\widetilde{G}^2 = \widetilde{F}_{\zeta\zeta}, \\ &\mathrm{Re}(\widetilde{G}_\tau + \epsilon^2 \widetilde{F}\widetilde{G} + \epsilon^2 \widetilde{W}\widetilde{G}_\zeta) = \widetilde{G}_{\zeta\zeta}. \end{aligned} \qquad (64)$$

In the limit $\epsilon \to 0$, the leading-order terms satisfy

$$2\widetilde{F} + \widetilde{W}_\zeta = 0, \quad \mathrm{Re}\widetilde{G}_\tau = \widetilde{G}_{\zeta\zeta}, \quad \mathrm{Re}(\widetilde{F}_\tau - \widetilde{G}^2) = \widetilde{F}_{\zeta\zeta}. \qquad (65)$$

The boundary conditions at the disk surface are

$$\widetilde{F}(0,\tau) = 0, \quad \widetilde{W}(0,\tau) = 0, \quad \widetilde{G}(0,\tau) = 1. \qquad (66)$$

Since $\widetilde{F}$ and $\widetilde{G}$ must match with their respective components in the inviscid layer, we have

$$\widetilde{F}(\zeta,\tau) \to 0, \quad \widetilde{G}(\zeta,\tau) \to 0, \quad \text{as } \zeta \to \infty. \qquad (67)$$

To solve (65)–(67) we introduce the similarity variable $\sigma = \zeta(\mathrm{Re}/4\tau)^{1/2}$ and let $\widetilde{G} = \tilde{g}(\sigma)$, $\widetilde{F} = \tau \tilde{f}(\sigma)$, $\widetilde{W} = 4\tau^{3/2}\tilde{w}(\sigma)/\mathrm{Re}^{1/2}$. Equations (65) become

$$\tilde{g}'' + 2\sigma\tilde{g}' = 0, \quad \tilde{f}'' + 2\sigma\tilde{f}' - 4\tilde{f} = -4\tilde{g}^2, \quad \tilde{w}' = -\tilde{f}. \qquad (68)$$

The boundafy conditions for the above set are

$$\begin{aligned} &\tilde{f}(0) = 0, \quad \tilde{f}(\infty) = 0, \quad \tilde{g}(0) = 1, \\ &\tilde{g}(\infty) = 0, \quad \tilde{w}(0) = 0. \end{aligned} \qquad (69)$$

The prime denotes differentiation with respect to σ. These equations are identical to Benton's[11] leading-order set that he derived to describe flow due to a rotating disk started impulsively from rest, the time-dependent analog of von Karman's problem. Closed form solutions for $\tilde{f}$, $\tilde{g}$, and $\tilde{w}$ are given by Nigam[16] (see also Benton[11]). The expressions for $\widetilde{G}$, $\widetilde{F}$, and $\widetilde{W}$ in terms of the similarity variable σ are

$$\widetilde{G}(\zeta,\tau) = \operatorname{erfc}\sigma,$$

$$\widetilde{F}(\zeta,\tau) = (2\tau/\pi)[(1+2\sigma^2)\operatorname{erfc}\sigma - 2\pi^{-1/2}\sigma e^{-\sigma^2}] - 2\tau(\sigma\operatorname{erfc}\sigma - \pi^{-1/2}e^{-\sigma^2})^2,$$

$$\begin{aligned} \widetilde{W}(\zeta,\tau) = &-\frac{8\tau^{3/2}}{3\pi\mathrm{Re}^{1/2}}[(3\sigma+2\sigma^3)\operatorname{erfc}\sigma - 2\pi^{-1/2}(1+\sigma^2)e^{-\sigma^2}] + \frac{8\tau^{3/2}}{3\mathrm{Re}^{1/2}}\sigma(\sigma\operatorname{erfc}\sigma - \pi^{-1/2}e^{-\sigma^2})^2 \\ &+ \frac{8\tau^{3/2}}{3(\pi\mathrm{Re})^{1/2}}e^{-\sigma^2}\operatorname{erfc}\sigma - \frac{8\tau^{3/2}}{3}\left(\frac{2}{\pi\mathrm{Re}}\right)^{1/2}\operatorname{erfc}(2^{1/2}\sigma) - \frac{8\tau^{3/2}}{3(\pi\mathrm{Re})^{1/2}}(2\pi^{-1} - 2^{1/2} + 1), \end{aligned} \qquad (70)$$

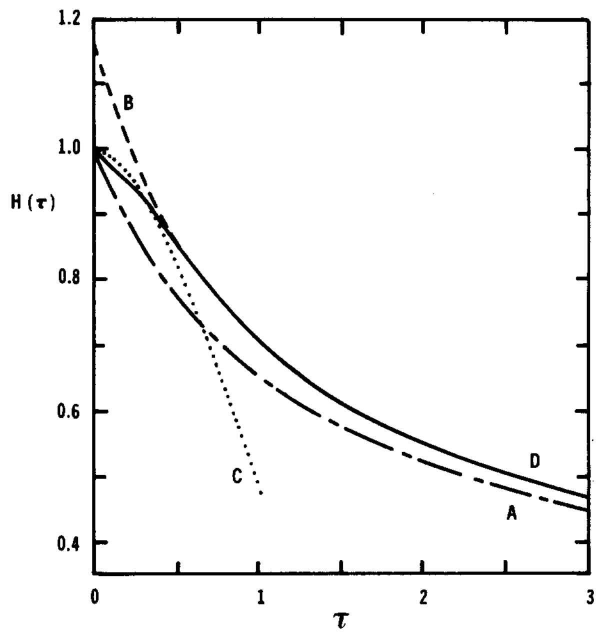

FIG. 2. Transient film thickness $H(\tau)$ for low Reynolds number flow ($\mathrm{Re} = 0.5$): A: · — · — · one-term long-time-scale expansion equation (36); B: — — — — two-term long-time-scale expansion equation (36); C: ------ two-term short-time-scale expansion equation (56); D: ——— composite uniform expansion equation (58).

where erfc is the complementary error function. Matching $w(\zeta,\tau)$ with the axial velocity (62) in the inviscid layer gives

$$\Delta_1(\epsilon) = \epsilon^{5/2}, \quad a_1(\tau) = -\frac{8\tau^{3/2}}{3(\pi \mathrm{Re})^{1/2}}(2\pi^{-1} - 2^{1/2} + 1). \tag{71}$$

In order to determine $\Delta_2(\epsilon)$ and $a_2(\tau)$ in (62), it is necessary to compute $\widetilde{F}$, $\widetilde{G}$, and $\widetilde{W}$ to order ϵ^2. Because the governing equations at this order are complicated by the presence of inhomogeneous terms, it appears that matching with the outer solution is best accomplished numerically. We have not explored this possibility.

To determine the shape of the free surface we let

$$H(\tau,\epsilon) \sim 1 + \epsilon^{5/2} H_1(\tau) \tag{72}$$

and from the kinematic condition we find

$$H_1(\tau) = -[16\tau^{5/2}/15(\pi \mathrm{Re})^{1/2}](2\pi^{-1} - 2^{1/2} + 1). \tag{73}$$

Thus the free surface thins according to

$$H(\tau) \sim 1 - [16/15(\pi \mathrm{Re})^{1/2}](2\pi^{-1} - 2^{1/2} + 1)\tau^{5/2}. \tag{74}$$

The artificial parameter ϵ has been removed in (7) by taking $t_c = 1/\Omega$. The result given in (74) holds for arbitrary Reynolds number but obviously becomes invalid when nonlinear effects become important, or when the boundary layer thickness becomes comparable to that of the film such that viscous effects dominate throughout the film.

Benton[11] showed by calculating higher-order corrections to (70) that the first-order approximation becomes inaccurate when $\tau \sim 0.5$. Homsy and Hudson[17] analyzed Benton's problem numerically and presented results for the entire history of the transient. From their results we can estimate that the thickness of the viscous region at $\tau = 0.5$ is $\xi \sim 2.5\mathrm{Re}^{-1/2}$. Setting $H(0.5) = 2.5\mathrm{Re}^{-1/2}$ in (74), it then follows that the viscous region is of the order of the film thickness when $\mathrm{Re} \sim 6.4$. This calculation shows that at high Re the short-time solution for the film thickness (74) becomes invalid long before the boundary layer has reached the free surface.

V. DISCUSSION

The asymptotic expressions for the transient film thickness derived in Sec. III are displayed in Fig. 2. For the purpose of comparison we have chosen the long-time-scale τ as the independent variable for all the expansions, and have taken $\mathrm{Re} = 0.5$ to be a representative value for low Reynolds number flows.

The zero-order term $H_0(\tau)$ is shown in Fig. 2 as the dashed curve A. This plot is identical to what Emslie *et al.*[1] found for the film thickness when they took the initial height of the fluid layer to be independent of the coordinate r. The two-term long-time-scale solution is shown as the upper dashed curve B. As expected for short times ($\tau < 0.2$ for the case $\mathrm{Re} = 0.5$) this expansion becomes singular and predicts a film thickness greater than unity. As noted previously the singular nature of the long-time-scale expansion occurs because the velocity field for the long-time-scale solution does not satisfy the initial conditions. Interestingly, the second-order correction $H_1(\tau)$ is always positive, hence $H_0(\tau)$ represents a lower bound for the transient film thickness. It should be noted that the choice of τ as the independent variable in Fig. 2 suppresses the Reynolds number dependence of H_0. When this dependence is made explicit, one obtains the expected result that with decreasing Reynolds number, the relative resistance of the film to thinning increases, and a longer spinning time is required to achieve a given reduction in the initial film thickness. As expected the difference between curve A, the lower bound for $H(\tau)$, and curve B diminishes with increasing time, and with decreasing Reynolds number. For example, at $\mathrm{Re} = 0.5$ and $\tau = 0.5$, curve B predicts a film thickness about 11% higher than that predicted by H_0. At $\tau = 3$ the difference in predicted film thicknesses is about 3%.

The composite uniform expansion given by (58) (curve D in Fig. 2) merges with the two-term long-time-scale expansion around $\tau = 0.5$, and becomes essentially indistinguishable from it for $\tau > 0.6$. For $\tau < 0.5$ transient effects due to the initial acceleration of the disk become important, and these effects are accounted for in the composite expansion through the short-time-scale solution, curve C in Fig. 2.

The composite expansion predicts a film of zero thickness when the spinning time becomes indefinitely large, a result that is obviously physically unreasonable. Not accounted for in this analysis are interfacial effects that eventually dominate the hydrodynamics at the rim of the disk when the film becomes sufficiently thin. In particular, contact line forces will prevent any liquid from being sloughed off the disk, thus enabling the liquid film to reach gyrostatic equilibrium. When solvent is present evaporation effects must be taken into account to determine the final film thickness.[8]

The pressure distribution in the thinning film follows directly from (13). Substituting the long-time-scale vari-

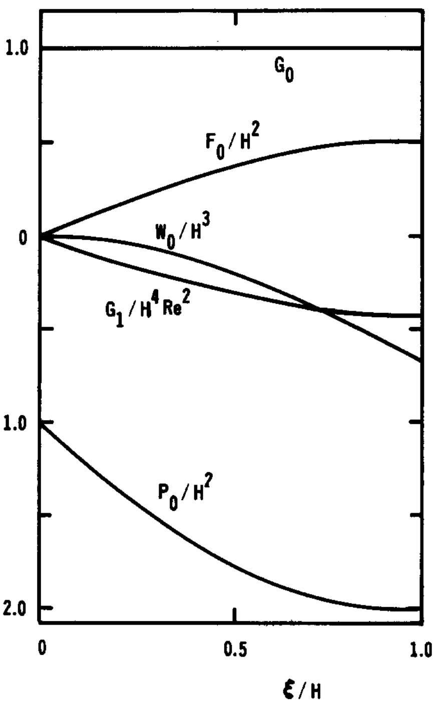

FIG. 3. Velocity and pressure distributions for transient film flow on a rotating disk.

ables and making the pressure dimensionless with $\rho h_0^2\Omega^2$ results in

$$P(\tau,\xi;\mathrm{Re}) = W_\xi + (W_\xi)_H - \mathrm{Re}^2\left[\frac{W^2}{2} - \left(\frac{W^2}{2}\right)_H + \int_\xi^H W_\tau \, d\xi\right]. \quad (75)$$

The zero-order term for the pressure distribution is thus

$$P_0(\tau,\xi) = W_{0\xi} + (W_{0\xi})_H = \xi^2 - 2H_0\xi - H_0^2. \quad (76)$$

In Fig. 3 we have plotted the pressure distribution, scaled with H^2, against ξ/H. Similar plots are also shown for G_0, F_0, W_0, and G_1. (Each variable has been scaled with an appropriate power of H to make it independent of H when plotted against ξ/H.)

It is evident from (76) that the pressure distribution at zero order is determined solely by normal viscous stresses induced by the thinning film, and as Fig. 3 shows, the pressure in the film increases monotonically with distance from the interface, attaining its maximum value at the disk surface. In order to satisfy the normal stress boundary condition, the pressure at the interface of the thinning film must be negative with respect to ambient.

A view of some of the hydrodynamic aspects of centrifugal spinning of liquids may be gained by inspection of the velocity distributions plotted in Fig. 3. It is seen that the velocity field in the thinning film is dominated by the azimuthal component G_0, which has a magnitude of unity throughout the fluid layer. In comparison the radial component of velocity, F_0, has a magnitude of order H^2 and is parabolic in ξ, the latter feature being typical of lubrication flows. The axial component W_0 is even smaller still, having a magnitude of order H^3. Because the first-order correction G_1 is negative for all values of ξ, the azimuthal surface velocity of the film lags the angular velocity of disk by $5\mathrm{Re}^2H^4/12$.

VI. CONCLUDING REMARKS

In this paper we have derived an asymptotic solution that describes the thinning of a fluid layer on a rotating disk when the Reynolds number for the flow is small. The solution results from matching a long-time-scale expansion that ignores the initial acceleration of the fluid layer with a short-time-scale expansion that accounts for fluid inertia. The leading term in the long-time-scale solution is identical to the result obtained by Emslie *et al.*,[1] and is shown to be a lower bound for the transient film thickness for all $\tau > 0$. Although Emslie *et al.* assumed a uniform pressure throughout the thinning film, the rigorous asymptotic analysis shows that there is an axial pressure distribution at zero order, which is induced by normal viscous stresses.

Finally, it should be pointed out that the theory is based on the assumption that the interface remains flat during spinning. Consistent with this assumption is the property that the radial and azimuthal velocities depend linearly on the radial coordinate, which allows the problem to be formulated as an exact solution to the time-dependent Navier–Stokes equations, in a manner quite akin to von Karman's formulation of the problem of a disk rotating in a large quiescent body of fluid. Although Emslie *et al.*[1] have shown that a flat interface is the preferred shape during spinning at low Re, the approximation may well be inappropriate at high Re. Circumferential waves and helical waves are known to occur,[18] though a detailed analysis of their dynamics has not been undertaken. Since the asymptotic analysis present in Sec. IV for arbitrary Re quickly becomes invalid at high Re due to nonlinear effects, it seems fruitful to solve the governing equations presented in Sec. II numerically to gain further insight into film dynamics at high Re. This possibility is presently under investigation.

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