Theory of Thermoelectricity.

BY SHIZUWO SANO.

(READ DECEMBER 15, 1906.)

The interesting problem of thermoelectricity was theoretically treated by Clausius,[1] Lord Kelvin[2], C. Neumann[3], Budde[4], Kohlrausch[5], Duhem[6], Lorentz[7], Planck[8], Bucherer[9], Mewes[10], and other eminent authors. But the fact that the discussions on this subject continue to be published until the present time seems to point out that there is certain unsatisfactoriness in the theoretical treatments. In fact some of the results obtained by different authors are not reconcilable to each other.

The most doubtful point in the discussions lies, perhaps, in the applications of the second law of thermodynamics. Whether on the heat product in the thermoelectric circuit which is proportional to the first power of the strength of the electric current, this principle can be applied on the consideration that the process is reversible, the thermal conduction and Joule effect having no direct influence on it,

---

[1] R. Clausius, Pogg. Ann. **90**, p. 513, 1853.

[2] W. Thomson, Trans. Roy. Sc. Edinb. **21**, p. 123, 1854.

[3] C. Neumann, Ber. d. Säch. Ges. d. Wiss., Math-Phys. Kl. **24**, p. 49, 1872.

[4] E. Budde, Pogg. Ann. **153**, p. 343, 1874; Wied. Ann. **21**, p. 277, 1884; **30**, p. 664, 1887.

[5] F. Kohlrausch, Pogg. Ann. **156**, p. 601, 1875; Wied. Ann. **23**, p. 477, 1884; **67**, p. 630, 1899.

[6] P. Duhem, Ann. sc. d. l'Ec. Norm. (3) **2**, p. 405, 1885.

[7] H. A. Lorentz, Arch. Néer. **20**, p. 129, 1885; Wied. Ann. **36**, p. 593, 1889.

[8] M. Planck, Wied. Ann. **36**, p. 624, 1889.

[9] A. H. Bucherer, Wied. Ann. **59**, p. 735, 1896.

[10] R. Mewes, Electrochem. Zeitschr. **3**, p. 263, 1897.

is not easy to decide. So far as I know, Boltzmann[1] was the first to publish a theory based upon the hypothesis that the thermal conduction is related to the Peltier and Thomson effects so intimately that they can not be considered as a mere superposition. The present paper is a humble imitation of Boltzmann's, and is a mere reproduction of it on the principal points.

The following theory is based on several assumptions which are not altogether justified. I assume that the state of each of the metals under consideration can be determined by the co-ordinates used to define the state of a fluid. It is no doubt incomplete not to consider the æolotropy of stress produced by the inequality of temperature and other causes; but this, as well as hysteresis, is too difficult for me to take into account. Probably the general feature of thermoelectric phenomena in solid metals which are in isotropic stress at the state under consideration is quite analogous to the case of fluids. The æolotropies of the Thomson effect and the thermal and electric conductivities are neglected at the same time under the above assumption. Moreover, I have not considered how radiation affects the thermoelectric phenomena. This is another weak point of the present theory.

Little is known as to the mechanism of electric currents in metals, and so far as I know at present there is no phenomenon contradicting decidedly the assumption that the electric conduction in metals is solely due to convection currents. In this paper it is immaterial whether the carriers of electricity are ions as in electrolytic solutions, or electrons. If the latter possess real mass, evidently they can be treated just in the same way as ions; but if, on the contrary, they possess only electromagnetic mass on account of their self-induction, no further change will be required in the following treatment, beyond that by density is meant the sum of the density of ordinary matter and that of electromagnetic mass. If we assume that we are justified to make the above assumption as to the nature of the electric conduction in a portion of a metal of absolute temperature θ, and pass an elementary quantity de of electricity against the electromotive force E in an interval of time, and denote the reversible heat absorbed during the same time by the same portion by H (measured in mechanical units), then we can not apply the thermodynamic principles in the forms that $E\,de+H$ and H/θ are exact differentials, for this

[1] L. Boltzmann, Sitzungsber. d. k. Akad. d. Wissensch. zu Wien 96 (2), p. 1258 1887.

can be done only when there is no flux of matter during the infinitesimal change. It appears to me that in current-bearing conductors the term "flow of heat" has no signification but that of the flow of energy, as there is in this case no standard to distinguish heat from other forms of energy. Thus we shall apply the thermodynamic principles to thermoelectric phenomena in other forms.

Let us suppose that the metal under consideration consists of n chemical components $S_1, S_2, S_3 \ldots S_i, S_{i+1}, \ldots S_j \ldots S_n$ and let ρ_i be the density of the component S_i at a point (x, y, z) expressed in rectangular co-ordinates, and q_i the quantity of electricity carried by the unit mass of the same component. Here $q_1, q_2 \ldots q_n$ are constant and they can not all vanish. Denoting the free energy per unit volume and the electric potential by F and Ψ respectively, it can easily be shown, by the aid of the two laws of thermodynamics, that in equilibrium we have

$$\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial \rho_i}+q_i \Psi\right\}=\frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial \rho_i}+q_i \Psi\right\}=\frac{\partial}{\partial z}\left\{\frac{\partial F}{\partial \rho_i}+q_i \Psi\right\}=0,$$

where $\partial F/\partial \rho_i$ is the differential coefficient of F with respect to ρ_i when the temperature, the densities $\rho_1, \rho_2 \ldots \rho_{i-1}, \rho_{i+1} \ldots \rho_n$, the electric displacement (f, g, h) and the magnetic induction (a, b, c) are constant during the differentiation.[1] In these equations the effect of gravity is neglected, and I shall retain the supposition in the future. Also I obtained from the condition of equilibrium the equation,

$$\begin{aligned}-(\xi_i-\xi_1)=&\left\{a_{1i}+a_{2i}(\alpha^2+\beta^2+\gamma^2)+a_{3i}\alpha^2+\ldots\right\}\frac{\partial\theta}{\partial x}+\left\{a_{4i}\gamma+a_{5i}\alpha\beta+\ldots\right\}\frac{\partial\theta}{\partial y}\\&+\left\{-a_{4i}\beta+a_{5i}\gamma\alpha+\ldots\right\}\frac{\partial\theta}{\partial z}+\left\{b_{1i}^{(1)}+b_{2i}^{(1)}(\alpha^2+\beta^2+\gamma^2)+b_{3i}^{(1)}\alpha^2\right.\\&\left.+\ldots\right\}\mathfrak{X}_1+\ldots+\left\{b_{1i}^{(j)}+b_{2i}^{(j)}(\alpha^2+\beta^2+\gamma^2)+b_{3i}^{(j)}\alpha^2+\ldots\right\}\mathfrak{X}_j\\&+\left\{b_{4i}^{(j)}\gamma+b_{5i}^{(j)}\alpha\beta+\ldots\right\}\mathfrak{Y}_j+\left\{-b_{4i}^{(j)}\beta+b_{5i}^{(j)}\gamma\alpha+\ldots\right\}\mathfrak{Z}_j+\ldots,\end{aligned}$$

where ξ_i is the x-component of the (mean) velocity of the component S_i; α, β, γ the components of the magnetic force; a's, b's functions of $\theta, \rho_1, \rho_2 \ldots \rho_n$, but independent of the magnetic force and the electromotive intensity (X, Y, Z); and $\mathfrak{X}_i$, $\mathfrak{Y}_i$, $\mathfrak{Z}_i$ stand respectively for

[1] S. Sano, Proc. Tôkyô Math.-Phys. Soc. 2, p. 365, 1905; Phys. Zeitschr. 6, p. 566, 1905.

$\frac{\partial}{\partial x}\frac{\partial F}{\partial \rho_i}-q_iX, \frac{\partial}{\partial y}\frac{\partial F}{\partial \rho_i}-q_iY, \frac{\partial}{\partial z}\frac{\partial F}{\partial \rho_i}-q_iZ.$[1] If we neglect the terms multiplied by $a_{2i}, a_3, \ldots b_{2i}^{(j)}, b_{3i}^{(j)}, \ldots,$ the above equation will reduce to

$$-(\xi_i-\xi_1)=a_{1i}\frac{\partial\theta}{\partial x}+b_{1i}^{(1)}\mathfrak{X}_1+\ldots+b_{1i}^{(j)}\mathfrak{X}_j+\ldots+b_{1i}^{(n)}\mathfrak{X}_n. \qquad (1)$$

Of course there is a loss of generality by so doing. But as I shall not consider the Hall effect, the change of electric resistance in magnetic fields and other allied phenomena, I shall adopt the same assumption.

To obtain

$$\mathfrak{X}_i=\mathfrak{Y}_i=\mathfrak{Z}_i=0$$

as the conditions of equilibrium I assumed that the elementary work done on the unit volume during the interval dt when the mass does not displace is

$$Xdf+Ydg+Zdh+\frac{\alpha da+\beta db+\gamma dc}{4\pi}.$$

If we follow the same consideration, and denote the components of the energy flux by Ξ, H, Z, then

$$-\frac{\partial \Xi}{\partial x}-\frac{\partial H}{\partial y}-\frac{\partial Z}{\partial z}=X\frac{\partial f}{\partial t}+Y\frac{\partial g}{\partial t}+Z\frac{\partial h}{\partial t}+\frac{\alpha}{4\pi}\frac{\partial a}{\partial t}+\frac{\beta}{4\pi}\frac{\partial b}{\partial t}+\frac{\gamma}{4\pi}\frac{\partial c}{\partial t},$$

provided there is no flow of heat. By the usual method of transformation we have

$$-\frac{\partial \Xi}{\partial x}-\frac{\partial H}{\partial y}-\frac{\partial Z}{\partial z}=-\frac{1}{4\pi}\frac{\partial(Y\gamma-Z\beta)}{\partial x}-\frac{1}{4\pi}\frac{\partial(Z\alpha-X\gamma)}{\partial y}-\frac{1}{4\pi}\frac{\partial(X\beta-Y\alpha)}{\partial z},$$

whence

$$\Xi=\frac{Y\gamma-Z\beta}{4\pi}, \quad H=\frac{Z\alpha-X\gamma}{4\pi}, \quad Z=\frac{X\beta-Y\alpha}{4\pi}.$$

When the elementary portion we are considering moves as a whole, we must have terms depending upon its velocity (ξ, η, ζ) besides the Poynting flux. Thus we write

$$\Xi=\frac{Y\gamma-Z\beta}{4\pi}+P_x, \quad H=\frac{Z\alpha-X\gamma}{4\pi}+P_y, \quad Z=\frac{X\beta-Y\alpha}{4\pi}+P_z,$$

---

[1] S. Sano, Proc. Tōkyō Math.-Phys. Soc. **2**, p. 465, 1906; Phys. Zeitschr. **7**, p. 318, 1906.

where P_x, P_y, P_z are functions of θ, ρ_1, $\rho_2, \ldots \rho_n$, X, Y, Z, α, β, γ, ξ, η, ζ and vanish when $\xi=\eta=\zeta=0$. Next we must extend the equations to the case in which there are relative displacements of different chemical components. To obtain the above expression of $-(\xi_i-\xi_1)$ I started from the consideration that $\xi-\xi_1$ must vanish in the state of equilibrium, in which

$$\frac{\partial\theta}{\partial x}=0, \quad \mathfrak{X}_1=\mathfrak{X}_2=\ldots=\mathfrak{Y}_1=\ldots=\mathfrak{Z}_1=\ldots=\mathfrak{Z}_n=0.$$

If we follow similar steps of reasoning we shall have

$$\Xi=-a_1\frac{\partial\theta}{\partial x}-b_1^{(1)}\mathfrak{X}_1-\ldots-b_1^{(j)}\mathfrak{X}_j-\ldots-b_1^{(n)}\mathfrak{X}_n+\frac{Y\gamma-Z\beta}{4\pi}+P_x',$$

where a_1, $b_1^{(1)}$, $b_1^{(2)}, \ldots b_1^{(n)}$ are functions of θ, ρ_1, $\rho_2, \ldots \rho_n$, but are independent of the electric and magnetic forces and the velocities of translation of the chemical components, and P_x' is a function formed by introducing the components (ξ_m, η_m, ζ_m) of the mean velocity of all the chemical components, in place of ξ, η, ζ in P_x respectively. In the case we are going to discuss ξ_m, η_m, ζ_m are either zero or extremely small so that we may write

$$\left.\begin{aligned}
\Xi&=-a_1\frac{\partial\theta}{\partial x}-b_1^{(1)}\mathfrak{X}_1-b_1^{(2)}\mathfrak{X}_2-\ldots-b_1^{(j)}\mathfrak{X}_j\ldots-b_1^{(n)}\mathfrak{X}_n+\frac{Y\gamma-Z\beta}{4\pi},\\
H&=-a_1\frac{\partial\theta}{\partial y}-b_1^{(1)}\mathfrak{Y}_1-\ldots\qquad-b_1^{(j)}\mathfrak{Y}_j-\ldots-b_1^{(n)}\mathfrak{Y}_n+\frac{Z\alpha-X\gamma}{4\pi},\\
Z&=-a_1\frac{\partial\theta}{\partial z}-b_1^{(1)}\mathfrak{Z}_1-\ldots\qquad-b_1^{(j)}\mathfrak{Z}_j-\ldots-b_1^{(n)}\mathfrak{Z}_n+\frac{X\beta-Y\alpha}{4\pi}.
\end{aligned}\right\}\quad(2)$$

Let u, v, w be the components of the electric conduction current. Then from (1)

$$\left.\begin{aligned}
u&=\sum_{i=1}^{i=n}q_i\rho_i\xi_i=\xi_1\sum_{i=1}^{i=n}q_i\rho_i-\frac{\partial\theta}{\partial x}\sum_{i=2}^{i=n}q_i\rho_i a_{1i}-\sum_{i=2}^{i=n}\sum_{j=1}^{j=n}q_i\rho_i b_{1i}^{(j)}\mathfrak{X}_j,\\
v&=\sum_{i=1}^{i=n}q_i\rho_i\eta_i=\eta_1\sum_{i=1}^{i=n}q_i\rho_i-\frac{\partial\theta}{\partial y}\sum_{i=2}^{i=n}q_i\rho_i a_{1i}-\sum_{i=2}^{i=n}\sum_{j=1}^{j=n}q_i\rho_i b_{1i}^{(j)}\mathfrak{Y}_j,\\
w&=\sum_{i=1}^{i=n}q_i\rho_i\zeta_i=\zeta_1\sum_{i=1}^{i=n}q_i\rho_i-\frac{\partial\theta}{\partial z}\sum_{i=2}^{i=n}q_i\rho_i a_{1i}-\sum_{i=2}^{i=n}\sum_{j=1}^{j=n}q_i\rho_i b_{1i}^{(j)}\mathfrak{Z}_j.
\end{aligned}\right\}\quad(3)$$

We are going to proceed from (2) and (3).

Except the layer of transition between two metals in contact where the electromotive intensity becomes, perhaps, very great, $\frac{\partial}{\partial \rho_i}\int_0^{f,g,h}(Xdf+Ydh+Zdh)$ which is the term in $\frac{\partial F}{\partial \rho_i}$ dependent on X, Y, Z, is very small so that we may neglect it. Let us assume that the substance is non-magnetic and neglect $\frac{\partial}{\partial \rho_i}\int_0^{a,b,c}\frac{(\alpha da+\beta db+\gamma dc)}{4\pi}$ in $\frac{\partial F}{\partial \rho_i}$. Thus we introduce F_0 in place of F, where F_0 is the value of F when f, g, h, a, b, c vanish, and we write $\frac{\partial}{\partial x}\frac{\partial F_0}{\partial \rho_j}-q_jX$ for $\mathfrak{X}_j$. Since F_0 depends only upon the densities $\rho_1\ \rho_2,\ldots.\rho_n$ and the absolute temperature θ, F_0 can, in pure metals, be determined by the two arguments, θ and the pressure p, so that we write

$$\frac{\partial}{\partial x}\frac{\partial F_0}{\partial \rho_j}=\left(\frac{\partial}{\partial \theta}\frac{\partial F_0}{\partial \rho_j}\right)_p\frac{\partial \theta}{\partial x}.$$

We need not retain the term proportional to dp/dx in the last equation, since we have neglected the effect of gravity and the kinetic energy of the system and have supposed that the substance is non-magnetic. Also we may neglect $\sum_{i=1}^{i=n} q_i\rho_i$ in the interior of pure metals. Hence for such substances (3) and (2) become respectively

$$u=-\omega\frac{\partial \theta}{\partial x}+\sigma X,\quad v=-\omega\frac{\partial \theta}{\partial y}+\sigma Y,\quad w=-\omega\frac{\partial \theta}{\partial z}+\sigma Z; \tag{4}$$

$$\Xi=-\lambda\frac{\partial \theta}{\partial x}+\mu X+\frac{Y\gamma-Z\beta}{4\pi},$$

$$H=-\lambda\frac{\partial \theta}{\partial y}+\mu Y+\frac{Z\alpha-X\gamma}{4\pi},$$

$$Z=-\lambda\frac{\partial \theta}{\partial z}+\mu Z+\frac{X\beta-Y\alpha}{4\pi},$$

where

$$\omega=\sum_{i=1}^{i=n} q_i\rho_i a_{1i}+\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} q_i\rho_i b_{1i}^{(j)}\left(\frac{\partial}{\partial \theta}\frac{\partial F_0}{\partial \rho_j}\right)_p,\quad \sigma=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} q_iq_j\rho_i b_{1i}^{(j)},$$

$$\lambda=a_1+\sum_{j=1}^{j=n} b_1^{(j)}\left(\frac{\partial}{\partial \theta}\frac{\partial F_0}{\partial \rho_j}\right)_p,\quad \mu=\sum_{j=1}^{j=n} q_jb_1^{(j)}.$$

Introducing the relations (4) in the expressions of Ξ, H, Z, we have

$$\left.\begin{aligned}\Xi &= -\varkappa\frac{\partial\theta}{\partial x}+\frac{\mu u}{\sigma}+\frac{Y\gamma-Z\beta}{4\pi},\\ H &= -\varkappa\frac{\partial\theta}{\partial y}+\frac{\mu v}{\sigma}+\frac{Z\alpha-X\gamma}{4\pi},\\ Z &= -\varkappa\frac{\partial\theta}{\partial z}+\frac{\mu w}{\sigma}+\frac{X\beta-Y\alpha}{4\pi},\end{aligned}\right\} \tag{5}$$

where

$$\varkappa=\lambda-\frac{\omega\mu}{\sigma}.$$

It is evident that (4) and (5) can be applied to alloys whose chemical compositions are uniform. In case the chemical compositions of alloys are not uniform, these equations require certain modifications; but as we are going to exclude the latter cases we shall adopt these equations in the following discussions.

They are not altogether new. Equations (4) were frequently employed for the investigation of similar problems[1]; and equations (5) are mere superposition of the Poynting flux on terms analogous to those which enter (4). But as it seemed to me that their derivation is new I have written down fully thus far. These equations are comparatively simple, and can be derived probably in many other ways on different hypotheses. They may be true independently of the hypothesis presented in this paper. Since we are proceeding from the fundamental equations (4) and (5) but not directly from the hypothesis on the constitution of the electric currents, the hypothesis itself does not possess much weight.

From (4) and (5) we have after a slight transformation by dint of Maxwell's equations

$$\begin{aligned}-\frac{\partial\Xi}{\partial x}-\frac{\partial H}{\partial y}-\frac{\partial Z}{\partial z} &= -\frac{\partial}{\partial x}\left(-\varkappa\frac{\partial\theta}{\partial x}+\frac{\mu u}{\sigma}\right)-\frac{\partial}{\partial y}\left(-\varkappa\frac{\partial\theta}{\partial y}+\frac{\mu v}{\sigma}\right)\\ &\quad-\frac{\partial}{\partial z}\left(-\varkappa\frac{\partial\theta}{\partial z}+\frac{\mu w}{\sigma}\right)+\frac{\omega}{\sigma}\left(u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+w\frac{\partial\theta}{\partial z}\right)+\frac{u^2+v^2+w^2}{\sigma}\\ &\quad+X\frac{\partial f}{\partial t}+Y\frac{\partial g}{\partial t}+Z\frac{\partial h}{\partial t}+\frac{\alpha}{4\pi}\frac{\partial a}{\partial t}+\frac{\beta}{4\pi}\frac{\partial b}{\partial t}+\frac{\gamma}{4\pi}\frac{\partial c}{\partial t},\end{aligned}$$

[1]For example, see O. Wiedeburg, Ann. d. Phys. I, p. 758, 1900. He considers the flux of entropy in a medium in which the thermal conduction is going on; but this seems to me to be impossible.

where

$$X\frac{\partial f}{\partial t}+Y\frac{\partial g}{\partial t}+Z\frac{\partial h}{\partial t}+\frac{\alpha}{4\pi}\frac{\partial a}{\partial t}+\frac{\beta}{4\pi}\frac{\partial b}{\partial t}+\frac{\gamma}{4\pi}\frac{\partial c}{\partial t}$$

can be neglected when the changes are show as in ordinary thermoelectric circuits. Since we may put

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{6}$$

except the case in which extremely rapid changes are going on, the expression of the convergence of the energy flux becomes

$$-\frac{\partial \Xi}{\partial x}-\frac{\partial H}{\partial y}-\frac{\partial Z}{\partial z}=-\frac{\partial}{\partial x}\left(-\varkappa\frac{\partial\theta}{\partial x}+\tau u\right)-\frac{\partial}{\partial y}\left(-\varkappa\frac{\partial\theta}{\partial y}+\tau v\right)$$
$$-\frac{\partial}{\partial z}\left(-\varkappa\frac{\partial\theta}{\partial z}+\tau w\right)+\frac{u^2+v^2+w^2}{\sigma}, \tag{7}$$

where τ stands for $\frac{\mu}{\sigma}-\chi$, χ being defined by the equation

$$\left(\frac{\partial\chi}{\partial\theta}\right)_p = \frac{\omega}{\sigma}.$$

Here τ represents the quantity of energy which flows apparently with unit electricity.

It appears that as to the applications of the first law of thermodynamics opinions are not much divided among authors, but it is not so in the case of the applications of the second law. The fundamental equations (4) and (5) have been obtained with the aid of the latter important law. But there is no reason to suppose that further applications of the thermodynamic principles lead to no new result. In fact, nothing has been mentioned before with regard to the signs of $\varkappa$ and σ; but we know that they must be positive as will follow immediately from the second law. Thus further applications in suitable forms are necessary even in my theory in which the same principle has already been applied.

Some writers consider that the absorption of heat proportional to the strength i of the electric current by the thermoelectric circuit from the surrounding medium can be treated as reversible thermal changes with no regard to other irreversible heat effects; that is, if $iQ_1\,ds$ be the term proportional to i in the expression of heat absorbed in unit time by the line-element ds of the circuit in a steady state, then

$$\int \frac{Q_1 ds}{\theta} = 0,$$

the integration being extended throughout the whole circuit. But it appears to me that this consideration is impossible. To prove this I shall consider a special circuit and try to prove that the supposition $\int \frac{Q_1 ds}{\theta} = o$ leads to a result which is contrary to experiments. For this purpose let us consider a very long right circular cylinder of radius a traversed by an electric current and surrounded by a perfect insulator of electricity, and let the system be in a steady state, and suppose that physical conditions are the same in every respect except the directions of directed quantities at any points in the same cross-section equidistant from the axis of the cylinder. Let us use cylindrical coordinates r, θ', z, and let the axis of the cylinder be the axis of z. If u and w are the components of the electric current in the directions of r and z respectively, then (6) becomes

$$\frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial (ru)}{\partial r} = 0, \tag{8}$$

where w and u are given by (4):

$$-w = \omega\frac{\partial \theta}{\partial z} + \sigma\frac{\partial \Psi}{\partial z}, \quad -u = \omega\frac{\partial \theta}{\partial r} + \sigma\frac{\partial \Psi}{\partial r}. \tag{9}$$

In this case (7) may be written

$$\frac{\partial}{\partial z}\left(\varkappa\frac{\partial \theta}{\partial z}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\varkappa\frac{\partial \theta}{\partial r}\right) - \left(w\frac{\partial \tau}{\partial z} + u\frac{\partial \tau}{\partial r}\right) + \frac{w^2+u^2}{\sigma} = 0, \tag{10}$$

by virtue of (8). Equations (8), (9), (10) are too difficult to solve completely. To obtain approximate values let us suppose a to be very small and expand w, u, Ψ, θ in the ascending powers of r and retain only two or three terms for each. Thus we put

$$w = w_0 + w_1 r^2, \quad u = u_1 r + u_2 r^3, \quad \Psi = \Psi_0 + L_1 r^2 + L_2 r^4, \quad \theta = \theta_0 + M_1 r^2 + M_2 r^4,$$

where w_0, w_1, u_1, u_2, Ψ_0, L_1, L_2, θ_0, M_1, M_2 are functions of z but are independent of r. Putting these values of w and u in (8) we have two equations

$$\frac{dw_0}{dz} + 2u_1 = 0, \qquad \frac{dw_1}{dz} + 4u_2 = 0.$$

Writing ω_0 and σ_0 for the values of ω and σ at the axis of the cylinder respectively, we have approximately

$$\omega = \omega_0 + r^2 M_1 \frac{d\omega_0}{d\theta_0}, \qquad \sigma = \sigma_0 + r^2 M_1 \frac{d\sigma_0}{d\theta_0}.$$

Here $\dfrac{d\omega_0}{d\theta_0}$ stands for $\left(\dfrac{\partial \omega_0}{\partial \theta_0}\right)_p$, and $\dfrac{d\sigma_0}{d\theta_0}$ for $\left(\dfrac{\partial \sigma_0}{\partial \theta_0}\right)_p$; we shall use similar notations in the future. Thus by (9)

$$\left.\begin{aligned}
-w_0 &= \omega_0 \frac{d\theta_0}{dz} + \sigma_0 \frac{d\Psi_0}{dz}, \\
-w_1 &= \omega_0 \frac{dM_1}{dz} + \sigma_0 \frac{dL_1}{dz} + M_1 \frac{d\omega_0}{d\theta_0} \frac{d\theta_0}{dz} + M_1 \frac{d\sigma_0}{d\theta_0} \frac{d\Psi_0}{dz}, \\
-u_1 &= 2\omega_0 M_1 + 2\sigma_0 L_1, \\
-u_2 &= 4\omega_0 M_2 + 4\sigma_0 L_2 + 2M_1^2 \frac{d\omega_0}{d\theta_0} + 2L_1 M_1 \frac{d\sigma_0}{d\theta_0}.
\end{aligned}\right\} \tag{11}$$

Since u vanishes at the lateral surface, we obtain

$$u_1 + a^2 u_2 \doteq 0.$$

From those equations we get approximate values of w_0, w_1, u_1, u_2, L_1, L_2, and $\dfrac{d\Psi_0}{dz}$ in terms of θ_0, M_1, M_2, and the first and second differential coefficients of those quantities with respect to z.

If we eliminate u_1 and u_2 from the last equation and those connecting u_1 and u_2 with dw_0/dz and dw_1/dz, we shall have at once

$$\frac{dw_0}{dz} + \frac{a^2}{2}\frac{dw_1}{dz} = 0,$$

which becomes, when integrated,

$$w_0 = \bar{w} - \frac{a^2 w_1}{2},$$

where $\bar{w}$ is independent of z. If i is the strength of the electric current flowing the cylinder in the positive direction of z,

$$i = \int_0^a 2\pi r w dr = \int_0^a 2\pi r \left\{ \bar{w} + w_1 \left(r^2 - \frac{a^2}{2} \right) \right\} dr = \pi a^2 \bar{w},$$

so that $\bar{w}$ denotes the mean electric current in the direction of z. Introducing the value of w_0 in the first equation of (11) we have

$$\frac{d\Psi_0}{dz}=-\frac{\bar{v}}{\sigma_0}-\frac{\omega_0}{\sigma_0^2}\frac{d\theta_0}{dz}+\frac{a^2w_1}{2\sigma_0}.$$

We may now put the value of $\frac{d\Psi_0}{dz}$ thus found in the second equation of (11), and get

$$\frac{dL_1}{dz}=-\frac{w_1}{\sigma_0}+\frac{\bar{v}M_1}{\sigma_0^2}\frac{d\sigma_0}{d\theta_0}-\frac{M_1}{\sigma_0}\frac{d\omega_0}{d\theta_0}\frac{d\theta_0}{dz}+\frac{\omega_0 M_1}{\sigma_0^2}\frac{d\sigma_0}{d\theta_0}\frac{d\theta_0}{dz}-\frac{\omega_0}{\sigma_0}\frac{dM_1}{dz}$$
$$-\frac{a^2w_1M_1}{2\sigma_0^2}\frac{d\sigma_0}{d\theta_0},$$

while the third equation of (11) can easily be transformed into

$$L_1=-\frac{\omega_0M_1}{\sigma_0}-\frac{a^2}{8\sigma_0}\frac{dw_1}{dz}.$$

Eliminating L_1 from the last two equations we have

$$w_1=\frac{\bar{v}M_1}{\sigma_0}\frac{d\sigma_0}{d\theta_0}-\frac{a^2w_1M_1}{2\sigma_0}\frac{d\sigma_0}{d\theta_0}-\frac{a^2}{8\sigma_0}\frac{d\sigma_0}{d\theta_0}\frac{d\theta_0}{dz}\frac{dw_1}{dz}+\frac{a^2}{8}\frac{d^2w_1}{dz^2},$$

of which we may neglect the last three terms in our degree of approximation. Hence

$$w_0=\bar{v}-\frac{a^2w_1}{2}=\bar{v}-\frac{\bar{v}a^2M_1}{2\sigma_0}\frac{d\sigma_0}{d\theta_0},$$
$$u_1=-\frac{1}{2}\frac{dw_0}{dz}=\frac{\bar{v}a^2}{4}\frac{d}{dz}\left(\frac{M_1}{\sigma_0}\frac{d\sigma_0}{d\theta_0}\right),$$
$$u_2=-\frac{1}{4}\frac{dw_1}{dz}=-\frac{\bar{v}}{4}\frac{d}{dz}\left(\frac{M_1}{\sigma_0}\frac{d\sigma_0}{d\theta_0}\right).$$

Thus

$$w=\bar{v}-\frac{\bar{v}a^2M_1}{2\sigma_0}\frac{d\sigma_0}{d\theta_0}+\frac{\bar{v}r^2M_1}{\sigma_0}\frac{d\sigma_0}{d\theta_0},$$
$$u=\frac{\bar{v}(a^2r-r^3)}{4}\frac{d}{dz}\left(\frac{M_1}{\sigma_0}\frac{d\sigma_0}{d\theta_0}\right).$$

We can now determine M_1 and M_2 from (10) by the aid of the expressions of w and u just obtained. We may put

$$\varkappa=\varkappa_0+r^2M_1\frac{d\varkappa_0}{d\theta_0},\qquad \tau=\tau_0+r^2M_1\frac{d\tau_0}{d\theta_0}$$

in (10), and we have

$$\frac{d}{dz}\left(\varkappa_0\frac{d\theta_0}{dz}\right)+4\varkappa_0 M_1-\varpi\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}+\frac{\varpi a^2 M_1}{2\sigma_0}\frac{d\sigma_0}{d\theta_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}$$
$$+\frac{\varpi^2}{\sigma_0}-\frac{\varpi^2 a^2 M_1}{\sigma_0^2}\frac{d\sigma_0}{d\theta_0}=0,$$

$$\frac{d^2}{dz^2}(\varkappa_0 M_1)+8M_1^2\frac{d\varkappa_0}{d\theta_0}+16\varkappa_0 M_2-\varpi\frac{d\tau_0}{d\theta_0}\frac{dM_1}{dz}-\varpi M_1\frac{d^2\tau_0}{dz^2}\frac{d\theta_0}{dz}$$
$$-\frac{\varpi M_1}{\sigma_0}\frac{d\sigma_0}{d\theta_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}+\frac{\varpi^2 M_1}{\sigma_0^2}\frac{d\sigma_0}{d\theta_0}=0,$$

of which the former gives

$$M_1=-\frac{1}{4\varkappa_0}\frac{d}{dz}\left(\varkappa_0\frac{d\theta_0}{dz}\right)+\frac{\varpi}{4\varkappa_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}-\frac{\varpi^2}{4\varkappa_0\sigma_0}$$
$$-\frac{\varpi a^2 M_1}{8\varkappa_0\sigma_0}\frac{d\sigma_0}{d\theta_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}+\frac{\varpi^2 a^2 M_1}{4\varkappa_0\sigma_0^2}\frac{d\sigma_0}{d\theta_0},$$

and the latter gives

$$M_2=-\frac{1}{16\varkappa_0}\frac{d^2}{dz^2}(\varkappa_0 M_1)-\frac{M_1^2}{2\varkappa_0}\frac{d\varkappa_0}{d\theta_0}+\frac{\varpi}{16\varkappa_0}\frac{d\tau_0}{d\theta_0}\frac{dM_1}{dz}$$
$$+\frac{\varpi M_1}{16\varkappa_0}\frac{d^2\tau_0}{d\theta_0^2}\frac{d\theta_0}{dz}+\frac{\varpi M_1}{16\varkappa_0\sigma_0}\frac{d\sigma_0}{d\theta_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}-\frac{\varpi^2 M_1}{16\varkappa_0\sigma_0^2}\frac{d\sigma_0}{d\theta_0}.$$

If we put in the right-hand members of the last equations of M_1 and M_2 the approximate value of M_1

$$-\frac{1}{4\varkappa_0}\frac{d}{dz}\left(\varkappa_0\frac{d\theta_0}{dz}\right)+\frac{\varpi}{4\varkappa_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}-\frac{\varpi^2}{4\varkappa_0\sigma_0},$$

we shall have

$$M_1=-\frac{1}{4\varkappa_0}\frac{d}{dz}\left(\varkappa_0\frac{d\theta_0}{dz}\right)+\frac{\varpi}{4\varkappa_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}-\frac{\varpi^2}{4\varkappa_0\sigma_0}$$
$$+\frac{\varpi a^2}{32\varkappa_0^2\sigma_0}\frac{d\varkappa_0}{d\theta_0}\frac{d\sigma_0}{d\theta_0}\frac{d\tau_0}{d\theta_0}\left(\frac{d\theta_0}{dz}\right)^3+\frac{\varpi a^2}{32\varkappa_0\sigma_0}\frac{d\sigma_0}{d\theta_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}\frac{d^2\theta_0}{dz^2}$$
$$-\frac{\varpi^2 a^2}{32\varkappa_0^2\sigma_0}\frac{d\sigma_0}{d\theta_0}\left(\frac{d\tau_0}{d\theta_0}\right)^2\left(\frac{d\theta_0}{dz}\right)^2-\frac{\varpi^2 a^2}{16\varkappa_0\sigma_0^2}\frac{d\sigma_0}{d\theta_0}\frac{d^2\theta_0}{dz^2}$$
$$-\frac{\varpi^2 a^2}{16\varkappa_0^2\sigma_0^2}\frac{d\varkappa_0}{d\theta_0}\frac{d\sigma_0}{d\theta_0}\left(\frac{d\theta_0}{dz}\right)^2+\frac{3\varpi^3 a^3}{32\varkappa_0^2\sigma_0^2}\frac{d\sigma_0}{d\theta_0}\frac{d\tau_0}{d\theta_0}\frac{d\theta_0}{dz}-\frac{\varpi^4 a^2}{16\varkappa_0^2\sigma_0^3}\frac{d\sigma_0}{d\theta_0},$$

$$M_2=\frac{1}{64\varkappa}\left[\left\{\frac{d^3\varkappa}{d\theta^3}-\frac{2}{\varkappa^2}\left(\frac{d\varkappa}{d\theta}\right)^3\right\}\left(\frac{d\theta}{dz}\right)^4+\left\{6\frac{d^2\varkappa}{d\theta^2}-\frac{4}{\varkappa}\left(\frac{d\varkappa}{d\theta}\right)^2\right\}\left(\frac{d\theta}{dz}\right)^2\frac{d^2\theta}{dz^2}\right.$$
$$\left.+\frac{d\varkappa}{d\theta}\left(\frac{d^2\theta}{dz^2}\right)^2+4\frac{d\varkappa}{d\theta}\frac{d\theta}{dz}\frac{d^3\theta}{dz^3}+\varkappa\frac{d^4\theta}{dz^4}\right]$$

$$+\frac{\varpi}{64\varkappa}\left[\left\{-\frac{d^3\tau}{d\theta^3}+\frac{5}{\varkappa^2}\left(\frac{d\varkappa}{d\theta}\right)^2\frac{d\tau}{d\theta}-\frac{1}{\varkappa}\frac{d^2\varkappa}{d\theta^2}\frac{d\tau}{d\theta}-\frac{1}{\varkappa}\frac{d\varkappa}{d\theta}\frac{d^2\tau}{d\theta^2}\right.\right.$$
$$\left.-\frac{1}{\varkappa\sigma}\frac{d\varkappa}{d\theta}\frac{d\sigma}{d\theta}\frac{d\tau}{d\theta}\right\}\left(\frac{d\theta}{dz}\right)^3+\left\{-4\frac{d^2\tau}{d\theta^2}+\frac{2}{\varkappa}\frac{d\varkappa}{d\theta}\frac{d\tau}{d\theta}\right.$$
$$\left.\left.-\frac{1}{\sigma}\frac{d\sigma}{d\theta}\frac{d\tau}{d\theta}\right\}\frac{d\theta}{dz}\frac{d^2\theta}{dz^2}-2\frac{d\tau}{d\theta}\frac{d^3\theta}{dz^3}\right]$$

$$+\frac{\varpi^2}{64\varkappa}\left[\left\{-\frac{4}{\varkappa^2\sigma}\left(\frac{d\varkappa}{d\theta}\right)^2+\frac{1}{\varkappa\sigma^2}\frac{d\varkappa}{d\theta}\frac{d\sigma}{d\theta}+\frac{2}{\sigma^3}\left(\frac{d\sigma}{d\theta}\right)^2-\frac{1}{\sigma^2}\frac{d^2\sigma}{d\theta^2}\right.\right.$$
$$\left.+\frac{2}{\varkappa}\frac{d\tau}{d\theta}\frac{d^2\tau}{d\theta^2}-\frac{3}{\varkappa^2}\frac{d\varkappa}{d\theta}\left(\frac{d\tau}{d\theta}\right)^2+\frac{1}{\varkappa\sigma}\frac{d\sigma}{d\theta}\left(\frac{d\tau}{d\theta}\right)^2\right\}\left(\frac{d\theta}{dz}\right)^2$$
$$\left.+\left\{-\frac{4}{\varkappa\sigma}\frac{d\varkappa}{d\theta}+\frac{1}{\varkappa}\left(\frac{d\tau}{d\theta}\right)^2\right\}\frac{d^2\theta}{dz^2}\right]$$

$$+\frac{\varpi^3}{64\varkappa}\left\{\frac{5}{\varkappa^2\sigma}\frac{d\varkappa}{d\theta}\frac{d\tau}{d\theta}-\frac{1}{\varkappa\sigma^2}\frac{d\sigma}{d\theta}\frac{d\tau}{d\theta}-\frac{1}{\varkappa\sigma}\frac{d^2\tau}{d\theta^2}\right\}\frac{d\theta}{dz}$$
$$+\frac{\varpi^4}{64\varkappa}\left\{-\frac{2}{\varkappa^2\sigma^2}\frac{d\varkappa}{d\theta}+\frac{1}{\varkappa\sigma^3}\frac{d\sigma}{d\theta}\right\}.$$

We have purposely omitted the suffix o in the right-hand side of the last equation, since in our degree of approximation we need not distinguish different values of the temperature in any cross-section in the expression of M_2. But this is not the case for M_1. If θ, $\varkappa$, σ, τ be the values at the boundary of the cylinder, we may put

$$\theta_0=\theta-a^2M_1', \quad \varkappa_0=\varkappa-a^2M_1'\frac{d\varkappa}{d\theta}, \quad \sigma_0=\sigma-a^2M_1'\frac{d\sigma}{d\theta}, \quad \tau_0=\tau-a^2M_1'\frac{d\tau}{d\theta},$$

where M_1' stands for

$$-\frac{1}{4\varkappa}\frac{d}{dz}\left(\varkappa\frac{d\theta}{dz}\right)+\frac{\varpi}{4\varkappa}\frac{d\tau}{d\theta}\frac{d\theta}{dz}-\frac{\varpi^2}{4\varkappa\sigma}.$$

Putting these values in the above expression of M_1 and neglecting the terms containing a^4 and higher powers of a, we get

$$M_1=-\frac{1}{4\varkappa}\frac{d}{dz}\left(\varkappa\frac{d\theta}{dz}\right)+\frac{\varpi}{4\varkappa}\frac{d\tau}{d\theta}\frac{d\theta}{dz}-\frac{\varpi^2}{4\varkappa\sigma}$$

$$+\frac{a^2}{32\varkappa}\Bigg[\left\{\frac{2}{\varkappa^2}\left(\frac{d\varkappa}{d\theta}\right)^3-2\frac{d^3\varkappa}{d\theta^3}\right\}\left(\frac{d\theta}{dz}\right)^4+\left\{\frac{4}{\varkappa}\left(\frac{d\varkappa}{d\theta}\right)^2-12\frac{d^2\varkappa}{d\theta^2}\right\}\left(\frac{d\theta}{dz}\right)^2\frac{d^2\theta}{dz^2}$$

$$-4\frac{d\varkappa}{d\theta}\left(\frac{d^2\theta}{dz^2}\right)^2-8\frac{d\varkappa}{d\theta}\frac{d\theta}{dz}\frac{d^3\theta}{dz^3}-2\varkappa\frac{d^4\theta}{dz^4}\Bigg]$$

$$+\frac{\bar{w}a^2}{32\varkappa}\Bigg[\left\{2\frac{d^3\tau}{d\theta^3}-\frac{6}{\varkappa^2}\left(\frac{d\varkappa}{d\theta}\right)^2\frac{d\tau}{d\theta}+\frac{2}{\varkappa}\frac{d^2\varkappa}{d\theta^2}\frac{d\tau}{d\theta}+\frac{2}{\varkappa}\frac{d\varkappa}{d\theta}\frac{d^2\tau}{d\theta^2}+\frac{1}{\varkappa\sigma}\frac{d\varkappa}{d\theta}\frac{d\sigma}{d\theta}\frac{d\tau}{d\theta}\right\}\left(\frac{d\theta}{dz}\right)^3$$

$$+\left\{8\frac{d^2\tau}{d\theta^2}+\frac{1}{\sigma}\frac{d\sigma}{d\theta}\frac{d\tau}{d\theta}\right\}\frac{d\theta}{dz}\frac{d^2\theta}{dz^2}+4\frac{d\tau}{d\theta}\frac{d^3\theta}{dz^3}\Bigg]$$

$$\frac{\bar{w}^2a^2}{32\varkappa}\Bigg[\left\{\frac{4}{\varkappa^2\sigma}\left(\frac{d\varkappa}{d\theta}\right)^2-\frac{4}{\sigma^3}\left(\frac{d\sigma}{d\theta}\right)^2+\frac{2}{\sigma^2}\frac{d^2\sigma}{d\theta^2}-\frac{4}{\varkappa}\frac{d\tau}{d\theta}\frac{d^2\tau}{d\theta}+\frac{4}{\varkappa^2}\frac{d\varkappa}{d\theta}\left(\frac{d\tau}{d\theta}\right)^2$$

$$-\frac{1}{\varkappa\sigma}\frac{d\sigma}{d\theta}\left(\frac{d\tau}{d\theta}\right)^2\right\}\left(\frac{d\theta}{dz}\right)^2+\left\{\frac{4}{\varkappa\sigma}\frac{d\varkappa}{d\theta}+\frac{2}{\sigma^2}\frac{d\sigma}{d\theta}-\frac{2}{\varkappa}\left(\frac{d\tau}{d\theta}\right)^2\right\}\frac{d^2\theta}{dz^2}\Bigg]$$

$$+\frac{\bar{w}^3a^2}{32\sigma}\left\{-\frac{6}{\varkappa^2\sigma}\frac{d\varkappa}{d\theta}\frac{d\tau}{d\theta}-\frac{1}{\varkappa\sigma^2}\frac{d\sigma}{d\theta}\frac{d\tau}{d\theta}+\frac{2}{\varkappa\sigma}\frac{d^2\tau}{d\theta^2}\right\}\frac{d\theta}{dz}+\frac{\bar{w}^4a^2}{16\varkappa^3\sigma^2}\frac{d\varkappa}{d\theta}.$$

Now it follows from (5) that the quantity of energy crossing unit area of the bounding surface of the cylinder in unit time is

$$\varXi=-\varkappa\left(\frac{\partial\theta}{\partial r}\right)_{r=a}-\frac{Z\beta}{4\pi},$$

since u vanishes on the surface. Here Z denotes the electromotive intensity in the direction of z, and β the magnetic force in the direction perpendicular to r and z. Since the tangential components of the electromotive intensity and the magnetic force vary continuously across the layer of transition between two substances in contact, the flux of electromagnetic energy in the medium surrounding the cylinder at its neighbourhood is also $-Z\beta/4\pi$, so that the quantity of heat flowing in unit time from the medium to the cylinder across its unit length is

$$Q=2\pi a\varkappa\left(\frac{\partial\theta}{\partial r}\right)_{r=a}=4\pi a^2\varkappa(M_1+2a^2M_2).$$

Introducing the values of M_1 and M_2 already found, the last equation becomes

$$Q=-\pi a^2\frac{d}{dz}\left(\varkappa\frac{d\theta}{dz}\right)-\frac{\pi a^4}{8}\left\{\frac{d^3\varkappa}{d\theta^3}\left(\frac{d\theta}{dz}\right)^4+6\frac{d^2\varkappa}{d\theta^2}\left(\frac{d\theta}{dz}\right)^2\frac{d^2\theta}{dz^2}+3\frac{d\varkappa}{d\theta}\left(\frac{d^2\theta}{dz^2}\right)^2\right.$$

$$\left.+4\frac{d\varkappa}{d\theta}\frac{d\theta}{dz}\frac{d^3\theta}{dz^3}+\varkappa\frac{d^4\theta}{dz^4}\right\}+i\left[\frac{d\tau}{d\theta}\frac{d\theta}{dz}+\frac{a^2}{8}\left\{\left(\frac{d^3\tau}{d\theta^3}-\frac{1}{\varkappa^2}\left(\frac{d\varkappa}{d\theta}\right)^2\frac{d\tau}{d\theta}+\frac{1}{\varkappa}\frac{d^2\varkappa}{d\theta^2}\frac{d\tau}{d\theta}\right.\right.\right.$$

$$+\frac{1}{\varkappa}\frac{d\varkappa}{d\theta}\frac{d^2\tau}{d\theta^2}\Big)\left(\frac{d\theta}{dz}\right)^3+\left(4\frac{d^2\tau}{d\theta^2}+\frac{2}{\varkappa}\frac{d\varkappa}{d\theta}\frac{d\tau}{d\theta}\right)\frac{d\theta}{dz}\frac{d^2\theta}{dz^2}+2\frac{d\tau}{d\theta}\frac{d^3\theta}{dz^3}\Big\}\Big]$$

+terms proportional to i^2, i^3, i^4.

It is not without interest that Q contains terms involving i^3 and i^4. If we retain higher powers of the radius of the cylinder, the expression of Q will contain higher powers of i than i^4.

Now suppose that such a cylinder forms a part of a thermoelectric circuit consisting of any number of metals, and consider in the cylinder two portions AB and BC bounded by the surrounding medium and the three normal sections A, B, and C of which B lies between the other two. Suppose that the portions AB and BC are placed in a bath so regulated that the temperature at the lateral surfaces of the portions is uniform, while the remaining part of the circuit is placed in baths at any temperatures, and that the whole system is in a steady state. Next take another steady state and suppose that the lateral surface of the portion AB is placed in a series of baths at any temperatures, only subject to the condition that the neighbourhoods of the sections A and B retain sensibly the same states as before, and also suppose that the length of the portion BC is so adjusted that the electric current retains the same strength, the temperature of its lateral surface not varying, and that the remaining part of the circuit is in the same state as before. If it were possible to apply the second law of thermodynamics in the form

$$\int\frac{Q_1 ds}{\theta}=0,$$

where the integration is to be extended throughout the entire circuit, then we might apply this equation to the two cases just stated and take the difference between the two corresponding equations. Since the whole circuit except the portions AB and BC is in the same state in the two cases, this part contributes nothing to the final equation. The portion BC has no influence on the same equation as the temperature of its lateral surface is uniform by our supposition. Thus we get

$$\int\frac{Q_1 dz}{\theta}=0,$$

where the integration is to be extended through the cylinder from A to B. Hence by virtue of the expression of Q already obtained Q_1 is equal to

$$\frac{d\tau}{d\theta}\frac{d\theta}{dz}+\frac{a^2}{8}\left\{\left(\frac{d^3\tau}{d\theta^3}+\ldots\right)\left(\frac{d\theta}{dz}\right)^3+\ldots+2\frac{d\tau}{d\theta}\frac{d^3\theta}{dz^3}\right\}.$$

As

$$\int_A^B\frac{1}{\theta}\frac{d\tau}{d\theta}\frac{d\theta}{dz}dz=0,$$

the equation

$$\int_A^B\frac{Q_1}{\theta}dz=0$$

becomes

$$\int_A^B Gdz=0,$$

where

$$G=\frac{1}{8}\left\{\frac{1}{\theta}\frac{d^3\tau}{d\theta^3}-\frac{1}{\theta\varkappa^3}\left(\frac{d\varkappa}{d\theta}\right)^2\frac{d\tau}{d\theta}+\frac{1}{\theta\varkappa}\frac{d^2\varkappa}{d\theta^2}\frac{d\tau}{d\theta}+\frac{1}{\theta\varkappa}\frac{d\varkappa}{d\theta}\frac{d^2\tau}{d\theta^2}\right\}\left(\frac{d\theta}{dz}\right)^3$$
$$+\frac{1}{4}\left(\frac{2}{\theta}\frac{d^2\tau}{d\theta^2}+\frac{1}{\theta\varkappa}\frac{d\varkappa}{d\theta}\frac{d\tau}{d\theta}\right)\frac{d\theta}{dz}\frac{d^2\theta}{dz^2}+\frac{1}{4\theta}\frac{d\tau}{d\theta}\frac{d^3\theta}{dz^3}.$$

After a little transformation we have

$$G=\frac{1}{8}\frac{d}{dz}\left\{\left(\frac{1}{\theta}\frac{d^2\tau}{d\theta^2}+\frac{1}{\theta^2}\frac{d\tau}{d\theta}+\frac{1}{\theta\varkappa}\frac{d\varkappa}{d\theta}\frac{d\tau}{d\theta}\right)\left(\frac{d\theta}{dz}\right)^2+\frac{2}{\theta}\frac{d\tau}{d\theta}\frac{d^2\theta}{dz^2}\right\}$$
$$+\frac{1}{4\theta^3}\frac{d\tau}{d\theta}\left(1+\frac{\theta}{2\varkappa}\frac{d\varkappa}{d\theta}\right)\left(\frac{d\theta}{dz}\right)^3.$$

By our supposition $d\theta/dz$ and $d^2\theta/dz^2$ must vanish at A and B, so that the equation

$$\int^B Gdz=0$$

reduces to

$$\int_A^B\frac{1}{\theta^3}\frac{d\tau}{d\theta}\left(1+\frac{\theta}{2\varkappa}\frac{d\varkappa}{d\theta}\right)\left(\frac{d\theta}{dz}\right)dz=0.$$

In order that the last equation may hold good whatever be the distribution of temperature between A and B, evidently it is necessary that

$$\frac{d\tau}{d\theta}\left(1+\frac{\theta}{2\varkappa}\frac{d\varkappa}{d\theta}\right)=0.$$

As $d\tau/d\theta$ can not vanish in general, this equation will lead to the effect that the thermal conductivity is, under constant pressure, inversely proportional to the square of the absolute temperature, which is contrary to experiments. The cause of such a conclusion as this can be traced back to the equation

$$\int\frac{Q_1 ds}{\theta}=0.$$

Thus it appears to me that this assumption itself is impossible. Since θ which enters the integral is in reality the absolute temperature of the surrounding medium, the above treatment fails entirely, if the temperature does not vary continuously across the layer of transition between the cylinder and the surrounding medium. The same is true if there be a finite flow of energy in the tangential direction in the layer, the flux being dependent upon the temperature.

Next let us consider a thermoelectric circuit consisting of two metallic wires, each of which has uniform chemical composition. We shall suppose that the diameters of the wires are very small and neglect terms containing them as far as possible. Let A be one junction and B the other, and let the suffix $_1$ correspond to the junction A and the suffix $_2$ to the junction B. Denote the quantities referring to the first metal by unaccented letters and those corresponding to the second by accented ones and suppose that an electric current of strength i flows from the first metal to the second across the junction B. Let ds, ds' be the line-elements and A, A' the cross-sections of the wires. We shall suppose that the cross-sections vary slowly and make no abrupt change at the junctions. Lastly we suppose that the layers of transition between the two metals are extremely thin, and assume that the temperatures at both sides of each junction are sensibly the same. We have now from (4)

$$\Psi_1-\Psi_2=i\int_A^B\frac{ds}{\sigma A}+\int_A^B\frac{\omega}{\sigma}\frac{d\theta}{ds}ds,$$

$$\Psi_2'-\Psi_1'=i\int_B^A\frac{ds'}{\sigma' A'}+\int_B^A\frac{\omega'}{\sigma'}\frac{d\theta'}{ds'}ds',$$

supposing that the circuit is not under the influence of an external electromotive force. Hence, by replacing ω/σ by $d\chi/d\theta$,

$$i\left\{\int_A^B \frac{ds}{\sigma A}+\int_B^A \frac{ds'}{\sigma' A'}\right\}=\chi_2'-\chi_2+\chi_1-\chi_1'+\Psi_2'-\Psi_2+\Psi_1-\Psi_1'.$$

Thus the electromotive force of the circuit is

$$E=\chi_2'-\chi_2+\chi_1-\chi_1'+\Psi_2'-\Psi_2+\Psi_1-\Psi_1'. \tag{12}$$

To obtain an expression of the Peltier coefficient let us take a section C in the first metal and another D in the second metal, and let the junction B lie between those sections, and denote by H the heat absorbed from the lateral surface of the portion CBD in unit time when the system is in a steady state. Also let the quantities referring to C and D be represented by adding the suffixes C and D respectively. Now it is evident from (5) that H is equal to the sum of

$$i\left(\frac{\mu_D'}{\sigma_D'}-\frac{\mu_C}{\sigma_C}\right)-\varkappa_D'A_D'\left(\frac{d\theta'}{ds'}\right)_D+\varkappa_C A_C\left(\frac{d\theta}{ds}\right)_C$$

and the surface-integral of the Poynting flux outwards over the whole surface of the portion CBD. But the latter quantity is equal to $i(\Psi_D'-\Psi_C)$. Hence

$$H=i\left(\frac{\mu_D'}{\sigma_D'}-\frac{\mu_C}{\sigma_C}\right)-\varkappa_D'A_D'\left(\frac{d\theta'}{ds'}\right)_D+\varkappa_C A_C\left(\frac{d\theta}{ds}\right)_C+i(\Psi_D'-\Psi_C).$$

As before from (4) we obtain

$$\Psi_C-\Psi_2=\chi_2-\chi_C+iR_{CB}, \quad \Psi_2'-\Psi_D'=\chi_D'-\chi_2'+iR_{BD},$$

where R_{CB} is the electric resistance of the wire CB and R_{BD} that of BD. Hence

$$H=i\left(\frac{\mu_D'}{\sigma_D'}-\frac{\mu_C}{\sigma_C}+\chi_2'-\chi_2+\chi_C-\chi_D'+\Psi_2'-\Psi_2\right)-i^2(R_{CB}+R_{BD})$$
$$-\varkappa_D'A_D'\left(\frac{d\theta'}{ds'}\right)_D+\varkappa_C A_C\left(\frac{d\theta}{ds}\right)_C$$

$$=i(\tau_D'-\tau_C+\chi_2'-\chi_2+\Psi_2'-\Psi_2)-i^2(R_{CB}+R_{BD})$$
$$-\varkappa_D'A_D'\left(\frac{d\theta'}{ds'}\right)_D+\varkappa_C A_C\left(\frac{d\theta}{ds}\right)_C,$$

of which the last two terms vanish provided the temperature-gradients along the axes of the wires vanish at C and D. As $d\theta/ds$ and $d\theta'/ds'$

can not vanish at the same time in the junction, such sections C and D can not both approach indefinitely to B. If $d\theta/ds$ and $d\theta'/ds'$ are zero at C and D and the temperatures at these sections are really the same as that at B, then there are maxima or minima of temperature in the portion CBD. If we suppose now that the conditions

$$\frac{d\theta}{ds}=\frac{d\theta'}{ds'}=0$$

are satisfied at C and D which are sufficiently near B, and that the temperatures at C and D are sensibly the same as that at B, we may neglect $R_{CB}+R_{BD}$ and substitute τ_2 for τ_C and τ_2' for τ_D' in the above expression of H. Hence denoting by H_2 the quantity of heat absorbed by the portion CBD from its lateral surface when unit electricity passes in the direction CD, and dropping the suffix 2, we have

$$H=\tau'-\tau+\chi'-\chi+\Psi'-\Psi. \tag{13}$$

This equation when combined with (12) gives

$$E=H_2-H_1+\tau_2-\tau_2'-\tau_1+\tau_1', \tag{14}$$

as is well known.

If we proceed just in the manner employed in obtaining the above expression of H, we shall have for the amount of heat absorbed in unit time from the unit length of the lateral surface of the wire

$$Q=-\frac{d}{ds}\left(\varkappa A\frac{d\theta}{ds}\right)+i\frac{d}{d\theta}\left(\frac{\mu}{\sigma}\right)\frac{d\theta}{ds}-iX_s,$$

where it is supposed that s increases in the positive direction of i, and X_s stands for the electromotive intensity in the same direction. By virtue of (4) this equation can be easily transformed into

$$Q=-\frac{d}{ds}\left(\varkappa A\frac{d\theta}{ds}\right)+i\frac{d\tau}{d\theta}\frac{d\theta}{ds}-\frac{i^2}{\sigma A}. \tag{15}$$

The absorption of heat in unit time from the parts of the lateral surface corresponding to the layers of transition between the two metals must be negligibly small. If not, the temperature-gradient in the surrounding medium, normal to the lateral surface at the corresponding parts, would be extremely great. Therefore, using the same notation as before,

$$H=\int_C^B Qds+\int_B^D Q'ds',$$

which by means of the equation just obtained becomes

$$H=\varkappa_C A_C\left(\frac{d\theta}{ds}\right)_C-\varkappa_2 A_2\left(\frac{d\theta}{ds}\right)_2+\varkappa_2' A_2'\left(\frac{d\theta'}{ds'}\right)_2-\varkappa_D' A_D'\left(\frac{d\theta'}{ds'}\right)_D$$
$$+i(\tau_2-\tau_C+\tau_D'+\tau_2')-i^2(R_{CB}+R_{BD}).$$

If we make the same assumption as before with respect to the temperatures and their space-variations at C and D the last equation will reduce to

$$H=A_2\left\{\varkappa_2'\left(\frac{d\theta'}{ds'}\right)_2-\varkappa_2\left(\frac{d\theta}{ds}\right)_2\right\},$$

since $A_2'=A_2$ by our supposition. Again omitting the suffix 2,

$$i\Pi=A\left(\varkappa'\frac{d\theta'}{ds'}-\varkappa\frac{d\theta}{ds}\right). \tag{16}$$

I shall now apply on the circuit the second law of thermodynamics in the form

$$\int_A^B\frac{Qds}{\theta}+\int_B^A\frac{Q'ds'}{\theta'}\leqq 0 \tag{17}$$

which by means of (15) becomes

$$\int_A^B\left\{\frac{d}{ds}\left(\varkappa A\frac{d\theta}{ds}\right)-i\frac{d\tau}{d\theta}\frac{d\theta}{ds}+\frac{i^2}{\sigma A}\right\}\frac{ds}{\theta}+\int_B^A\left\{\frac{d}{ds'}\left(\varkappa' A'\frac{d\theta'}{ds'}\right)\right.$$
$$\left.-i\frac{d\tau'}{d\theta'}\frac{d\theta'}{ds'}+\frac{i^2}{\sigma' A'}\right\}\frac{ds'}{\theta'}\geqq 0.$$

This relation may also be written as follows:

$$\frac{A_2}{\theta_2}\left\{\varkappa_2\left(\frac{d\theta}{ds}\right)_2-\varkappa_2'\left(\frac{d\theta'}{ds'}\right)_2\right\}-\frac{A_1}{\theta_1}\left\{\varkappa_1\left(\frac{d\theta}{ds}\right)_1-\varkappa_1'\left(\frac{d\theta'}{ds'}\right)_1\right\}$$
$$+\int_A^B\left\{\frac{\varkappa A}{\theta^2}\left(\frac{d\theta}{ds}\right)^2-\frac{i}{\theta}\frac{d\tau}{ds}\frac{d\theta}{ds}+\frac{i^2}{\theta\sigma A}\right\}ds$$
$$+\int_B^A\left\{\frac{\varkappa' A'}{\theta'^2}\left(\frac{d\theta'}{ds'}\right)^2-\frac{i}{\theta'}\frac{d\tau'}{d\theta'}\frac{d\theta'}{ds'}+\frac{i^2}{\theta'\sigma' A'}\right\}ds'\geqq 0.$$

By making use of equation (16),

$$i^2\left\{\int_A^B \frac{ds}{\theta\sigma A} + \int_B^A \frac{ds'}{\theta'\sigma' A'}\right\} + i\left\{\frac{\Pi_1}{\theta_1} - \frac{\Pi_2}{\theta_2} - \int_A^B \frac{1}{\theta}\frac{d\tau}{d\theta}\frac{d\theta}{ds}\,ds - \int_B^A \frac{1}{\theta'}\frac{d\tau'}{d\theta'}\frac{d\theta'}{ds'}\,ds'\right\} + \int_A^B \frac{\varkappa A}{\theta^2}\left(\frac{d\theta}{ds}\right)^2 ds + \int_B^A \frac{\varkappa' A'}{\theta'^2}\left(\frac{d\theta'}{ds'}\right)^2 ds' \geqq 0,$$

where the co-efficient of i can be written in another form. If we differentiate (14) with respect to θ_2 and drop the suffix 2, we shall have

$$\frac{dE}{d\theta} = \frac{d\Pi}{d\theta} + \frac{d(\tau-\tau')}{d\theta},$$

so that

$$\int_{\theta_1}^{\theta_2}\left(\frac{1}{\theta}\frac{dE}{d\theta} - \frac{\Pi}{\theta^2}\right)d\theta = \frac{\Pi_2}{\theta_2} - \frac{\Pi_1}{\theta_1} + \int_{\theta_1}^{\theta_2}\frac{1}{\theta}\frac{d(\tau-\tau')}{d\theta}\,d\theta.$$

Hence we get

$$i^2\left\{\int_A^B \frac{ds}{\theta\sigma A} + \int_B^A \frac{ds'}{\theta'\sigma' A'}\right\} - i\int_{\theta_1}^{\theta_2}\left(\frac{1}{\theta}\frac{dE}{d\theta} - \frac{\Pi}{\theta^2}\right)d\theta + \int_A^B \frac{\varkappa A}{\theta^2}\left(\frac{d\theta}{ds}\right)^2 ds + \int_B^A \frac{\varkappa' A'}{\theta'^2}\left(\frac{d\theta'}{ds'}\right)^2 ds' \geqq 0.$$

Now produce a new electromotive force by induction. In this case the circuit is not, strictly speaking, in a steady state, but the system may be looked upon in some favourable cases as if it were in a steady state. Since equation (16) must be satisfied at the junctions the distributions of temperature along the circuit at a steady state can not be altogether arbitrary. By assuming different values for i the neighbourhoods of the junction necessarily have different temperature distributions, so that the co-efficient of i^2 and the terms not containing i in the above relation are not constant. Take two sections A' and B' in the first metal, of which the former lies near A and the latter near B. By suitable arrangements we may make the distributions of temperature along the lateral surface of $A'B'$ unchanged, while i varies at will. The part of the co-efficient of i^2 which depends upon the portion AA' is small provided the latter is very short, and we may so arrange that at any time $\int_{A}^{A'} \frac{\varkappa A}{\theta^2}\left(\frac{d\theta}{ds}\right)^2 ds$ assumes an insignificant value. The same thing is true for the portion $B'B$ and for the other wire. Hence we may suppose that the co-efficient of i^2, and the terms not multiplied by i in the above relation are independent of i. Let us assume that $dE/d\theta$ and Π are also independent of i. Then, in order that the above relation may hold good for all values of i it is necessary that

$$\int_{A}^{B} \frac{ds}{\theta\sigma A} + \int_{B}^{A} \frac{ds'}{\theta'\sigma' A'} > 0,$$

$$\left\{ \int_{\theta_1}^{\theta_2} \left(\frac{1}{\theta} \frac{dE}{d\theta} - \frac{\Pi}{\theta^2} \right) d\theta \right\}^2 - 4 \left\{ \int_{A}^{B} \frac{ds}{\theta\sigma A} + \int_{B}^{A} \frac{ds'}{\theta'\sigma' A'} \right\} \left\{ \int_{A}^{B} \frac{\varkappa A}{\theta^2} \left(\frac{d\theta}{ds} \right) ds \right.$$
$$\left. + \int_{B}^{A} \frac{\varkappa' A'}{\theta'^2} \left(\frac{d\theta'}{ds'} \right)^2 ds' \right\} \leqq 0. \tag{18}$$

From the first of these relations we infer that the specific electric conductivity is positive. When combined with the latter, this gives the conclusion that the thermal conductivity is also positive.

The quantity

$$P = \left\{ \int_{A}^{B} \frac{ds}{\theta\sigma A} + \int_{B}^{A} \frac{ds'}{\theta'\sigma' A'} \right\} \left\{ \int_{A}^{B} \frac{\varkappa A}{\theta^2} \left(\frac{d\theta}{ds} \right)^2 ds + \int_{B}^{A} \frac{\varkappa' A'}{\theta'^2} \left(\frac{d\theta'}{ds'} \right)^2 ds' \right\}$$

which enters (18) depends not only upon the temperatures at the junctions but also upon the distribution of temperature along the wires and upon their cross-sections. Its minimum value, when the temperatures of the junctions are given, can be easily found to be

$$\left\{ \int_{\theta_1}^{\theta_2} \left(\sqrt{\frac{\varkappa}{\theta^3\sigma}} + \sqrt{\frac{\varkappa'}{\theta^3\sigma'}} \right) d\theta \right\}^2.$$

We may now suppose that the difference of the temperatures of the junctions is very small. And as we are quite at liberty to assign any lengths and any cross-sections to the wires and to take any distributions of temperature along them, we may put the minimum value of P in (18), and we have at once

$$\left| \theta \frac{dE}{d\theta} - \Pi \right| \leqq 2 \left(\sqrt{\frac{\theta\varkappa}{\sigma}} + \sqrt{\frac{\theta\varkappa'}{\sigma'}} \right). \tag{19}$$

This relation was obtained by Boltzmann.[1]

[1] L. Boltzmann, loc. cit.

Conversely, if (19) be satisfied, (18) and therefore (17) will be satisfied whatever be the distributions of temperature and cross-sections along the thermoelectric circuit. Since the relation (19) does not necessarily require that $dE/d\theta$ and Π vanish at the same temperature, it seems to me that the contrary case is theoretically conceivable. This is coutrary to Lecher's theory.[1] The chief disagreement of his result with that described in this paper originates from the difference in the manner of application of the second law of thermodynamics. He does not take the conduction of heat into account. It seems rather strange that some of those who take the Peltier coefficient into account which has a certain connection to $\varkappa d\theta/ds$ as in (16), ignore the corresponding terms in the expression of Q.

Although Boltzmann's relation (19) is not contradictory to the well-known Thomson-Clausuis's equation

$$\Pi=\theta\frac{dE}{d\theta}, \tag{20}$$

yet it is too indefinite, and it is quite desirous to prove (20) theoretically, if possible. To some of the theoretical derivations of (20) applies the objection already mentioned though the objection itself is not free from defects. The relation (19) has been derived from the second law of thermodynamics. But we must not be so hasty as to conclude from what has been mentioned in this paper that (20) can not be deduced from the same law; for we may consider different thermoelectric arrangements and apply the same principle in other ways than (17). It appears to me doubtful at present, however, that such a relation as (20) can ever be deduced from the principle of energetics only. On the other hand, atomistic theories enter deep into the internal constitution of matter and deduce several relations which are otherwise hard to obtain; and the proper way to reach (20) theoretically, as it seems to me at present, is to proceed parallel to the courses taken by Riecke,[2] Drude,[3] and Lorentz.[4] If the hypothesis described in the beginning of this paper were near the truth, then the quantity which measures the Thomson effect involves those per-

[1] E. Lecher, Physik. Zeitschr. 7, p. 34. 1906; Ann. d. Phys. 20, p. 480, 1906.

[2] E. Riecke, Wied, Ann. 65, p. 353 and p. 545, 1898.

[3] P. Drude, Ann. d. Phys. 1, p. 566, and 3, p. 369, 1900.

[4] H. A. Lorentz, Versl. Akad. v. Wetensch. t. Amsterdam 13, p. 493, 1905; Arch. Néer. (2) 10, p. 336, 1905.

taining irreversible processes, and probably it might not be connected directly with quantities referring to purely reversible ones.

According to Liebenow,[1] the electromotive force of an unequally heated wire has a close relation with its thermal conductivity and its specific electric resistance; but his theory was criticised by Voigt.[2]

Voigt[3] treats the problem of thermoelectricity in a general manner by supposing some parts of the phenomena to be reversible and by making use of a function somewhat resembling the thermodynamic potential, so that he arrives at a more definite conclusion than in this paper. According to his theory, the x-component of the electromotive intensity when the electric current vanishes is $-\dfrac{\partial \Theta}{\partial x}$, while the same quantity is $\dfrac{\omega}{\sigma}\dfrac{\partial \theta}{\partial x}$ if we use (4). Again the flow of energy associated with that of unit electricity is by the same author $\Theta-\theta d\Theta/d\theta$, which is τ in my notation. If we assume that it is possible to combine those results, we shall have immediately

$$\Theta=-\chi+C, \quad \tau+\chi=\frac{\mu}{\sigma}=\frac{\theta\omega}{\sigma}+C,$$

where C is independent of θ. He also obtains

$$\Pi=\theta\left(\frac{d\Theta}{d\theta}-\frac{d\Theta'}{d\theta}\right),$$

where Θ' refers to the second metal. Combining the last equation with (13) and making use of the expression of $\tau+\chi$ written down just now, we get

$$\Psi'-\Psi=C-C'. \tag{21}$$

Here C' is independent of the temperature and depends only upon the pressure and the nature of the second metal. In this case (20) is satisfied.

According to (21) the difference of electric potentials between two metals in contact is independent of the temperature. Szarvassi[4] obtains a more definite result to the effect that the metals are in the same potential. I have written down (21) merely for the sake of compari-

[1] C. Liebenow, Wied. Ann. 68, p. 316, 1899.
[2] W. Voigt, Wied. Ann. 69, p. 706, 1899.
[3] W. Voigt, Gött. Nachr., 1895, Heft 2; Wied. Ann. 67, p. 717, 1899.
[4] A. Szarvassi, Ann, d. Phys. 17, p. 248, 1905.

son; the equation does not seem satisfactory to me. The difference of potentials has been the subject of many discussions and controversies, and it does not appear probable that this state will be put to a termination in the near future.

On a quartz half shadow polarization apparatus.

S. NAKAMURA.

(READ JANUARY 26, 1907).

I described and discussed this apparatus in a paper published in the "Centralblatt für Mineralogie, Geologie und Paläontologie, 1905" p. 267 under the title "Ueber einen Quarz halbschattenapparat." As I have not yet shown this apparatus to you, I wish to show you to-day how it is applied for the saccharimetry and for the measurement of extinction angles in crystals begging you for the full particulars to refer to the above paper.

In the well known Laurent's apparatus, the light first passes through a Nicol and then one half of it passes through a half wave length plate of quartz or mica cut parallel to the optical axis. In the Lippich's apparatus, the half shadow is created by a large polariser Nicol and another small Nicol behind it. The chief defect of the Laurent's instrument is that, it can be used only with that particular color for which the crystal plate is made to give a retardation of one-half wave length. For light of other colors, the ray through the plate is elliptically polarized, and the condition required for a half shadow instrument is not satisfied by it. In the Lippich's apparatus, on the other hand, though undoubtedly it is the best of the kind, for it can be used for any color and the sensibility can be easily altered by turning the larger Nicol, yet we must remember that the intensities of light in the two parts of the field are not exactly equal owing to the reflection and the absorption in the smaller Nicol.

Now in the new type of the half shadow apparatus, the rotatory polarization in quartz is taken advantage of. Macé de Lipinay* first proposed and described it, but he has not given the theory of

* Jour. de phys. (2) 4, 267; 1885. (3) 9, 585; 1900.