4 Mostly optimization

Group 1
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I hereby declare that all solutions are entirely my own work, without having taken part of other solutions.
The number of hours spent: 20hours (Min Wu)
The number of hours has been present in supervision for this module: 3h
(keeping track)
(WHAT IS THE REVENUE?)
Consider this data for sales and pricing of chocolate bars:

year	quantity	unit price
2014	10300	15 kr
2015	8100	17 kr
2016	7400	18 kr

a) Assuming that this data gives an indication of the price sensitivity of chocolate bar sales, how can you estimate a) for which price revenue is maximized, b) for which price profit is maximized, if we have a production cost of 3 kr per bar (ignoring fixed costs). Perform any calculations as far as possible.
a)The price vs quantity is plotted, see the following.
The relationship between price and quantity is likely linked to a linear function y=ax+b. The equation can be solved by the two known data (15,10300) and (18, 7400).The function is thus quantity = -986* price + 25029 with R^2=0.99.

The income will be the quantity * price: income=quantity*price
2014 income: 10,300*15 kr =154,500 kr
2015 income: 8,100*17 kr= 137,700kr
2016 income: 7,400*18 kr= 133,200kr

Income = quantity*price, which quantity= -986*price +25,029.
The relationship between income and price then become:
Income =-986*price^2+25,029*price.
The maximum can be solved when derivative of income is equal to 0.
the derivative income:
income'= -1,972*price+25,029
then income'=0:
-1,972*price +25,029=0
price=12.7 kr
Conclusion: when the price is set to 12.7kr, the revenue is maximized and it is 158,836kr

b) Profit = quantity * (price - production cost)
= quantity * (price-3)
The relationship between profit and quantity, the profit become:
Profit = (-986*price+25,029)*(price-3)
= -986 * price ^2 + 27,987* price - 75,087
The maximum profit can be obtained when the derivative of profit equals 0

Derivative of profit:
Profit'=-1,972 * price + 27,987
Profit'=0
-1,972*price+27,987=0
price = 14.2 kr
Conclusion: when the price is set to 14.2 kr the profit is maximized and it is 123, 511.4kr

b) Even though you do not know much about the specific situation, hypothesize other possible reasons for the decline in chocolate bar sales.

The price of candy is decreased, so more people tended to buy candy instead of the chocolate bars.

(investigating the abstract)
(LEAST SQUARES METHOD)
Fitting a curve graphically can work fine, but there is a limitation in that you then do not have a precise criterion for what is a good fit. The most common criterion in mathematics is the least squares method (look at any other websites if you like), which minimizes the sum of the squares of the errors between the points and the fitted function. Note how the curve fitting then becomes a well defined optimization problem!

Mathematica uses the least squares criterion in the function Fit, which finds the best linear combination (=weighted sum) of a set of base functions to minimize the quadratic error. For the (EMPIRICAL CURVE FITTING) from the last module, try this out by automatically fitting a quadratic polynomial to the points:

```
data = {{88.0, 57.9}, {224.7, 108.2}, {365.3,
149.6}, {687.0,
    228.07}, {4332, 778.434}, {10760, 1428.74},
{30684,
    2839.08}, {60188, 4490.8}, {90467, 5879.13}};
f = Fit[data, {1, x, x^2}, x]
```

For the quadratic polynomial we use the base functions 1,x and x^2, and the function will try to fit a linear combination (weighted average) of these functions. So this means that we ask to find a function of the form ax^2+bx+c. Then you can plot with:

```
p1 = ListPlot[data];
p2 = Plot[f, {x, 0, 150000}, PlotStyle -> Red];
Show[p1, p2, PlotRange -> {{0, 150000}, {0,
8000}}]
```

a) As the answer to the question, show the function and the plot.

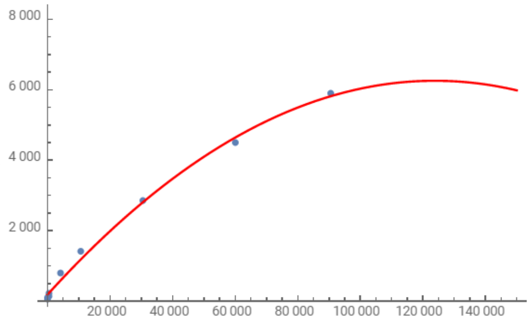
The function is 185.701+0.0980238 x-3.95858*10^-7 *x^2

The plot is

```
In[]:= data = {{88.0, 57.9}, {224.7, 108.2}, {365.3, 149.6}, {687.0, 228.07}, {4332, 778.434}, {10760,
         1428.74}, {30684, 2839.08}, {60188, 4490.8}, {90467, 5879.13}};
       f = Fit[data, {1, x, x^2}, x]
Out[]= 185.701 + 0.0980238 x - 3.95858×10^-7 x^2

In[]:= p1 = ListPlot[data];
       p2 = Plot[f, {x, 0, 150000}, PlotStyle → Red]; Show[p1, p2, PlotRange → {{0, 150000}, {0, 8000}}]
```

Out[]=



b) Why do you think it may be good to minimize the sum of square errors. Why not simply the sum of the errors for each point?

The error of each point will be positive or negative. The sum of the errors will be zero which indicates no error which is not true.

For example:

error1: 40, error2:-20, error3: -10 and error4: -10

the sum of those errors:

error1+error2+error3+error4=0

But the errors is not zero. By squaring the errors this problem is overcome. The variance is obtained by taking the average of the sum of the squares errors.

(SIMPLE ASSIGNMENT PROBLEM)

Take a careful look at how we modelled this problem in the lecture. Try Mathematica for this problem, formulated as a linear programming problem (no integer constraints). Make sure that you understand how the problem is formulated in Mathematica. See what solution you get. Is it fractional or did you get an integer solution?

```
NMinimize[
 {1 x11 + 3 x12 + 5 x13 + 1 x14 + 4 x21 + 5 x22 +
3 x23 + 2 x24 +
   7 x31 + 4 x32 + 6 x33 + 9 x34 + 8 x41 + 4 x42
+ 7 x43 + 3 x44,
  x11 + x12 + x13 + x14 == 1,
  x21 + x22 + x23 + x24 == 1,
  x31 + x32 + x33 + x34 == 1,
  x41 + x42 + x43 + x44 == 1,
  x11 + x21 + x31 + x41 == 1,
  x12 + x22 + x32 + x42 == 1,
  x13 + x23 + x33 + x43 == 1,
  x14 + x24 + x34 + x44 == 1,
  0 <= x11 <= 1, 0 <= x12 <= 1, 0 <= x13 <= 1, 0
<= x14 <= 1,
  0 <= x21 <= 1, 0 <= x22 <= 1, 0 <= x23 <= 1, 0
<= x24 <= 1,
  0 <= x31 <= 1, 0 <= x32 <= 1, 0 <= x33 <= 1, 0
<= x34 <= 1,


   0 <= x41 <= 1, 0 <= x42 <= 1, 0 <= x43 <= 1, 0
 <= x44 <= 1},
  {x11, x12, x13, x14, x21, x22, x23, x24, x31,
 x32, x33, x34, x41,
   x42, x43, x44}
  ]
```

To find the minimum

x11+3x12+5x13+x14+4x21+5x22+3x23+2x24+7x31+4x32+6x33+9x34+8x41+4x42+7x43+3x44

varibles:
x11,x12,x13,x14,x21,x22,x23,x24,x31,x32,x33,x34,x41,x42,x43,x44

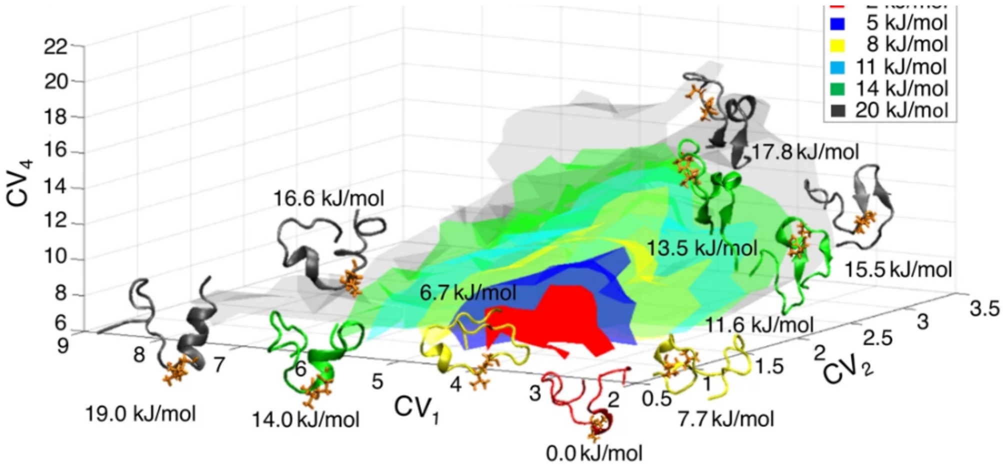
Constraints:
x11+x12+x13+x14=1
x21+x22+x23+x24=1
x31+x32+x33+x34=1
x41+x42+x43+x44=1
x11+x21+x31+x41=1
x12+x22+x32 +x42=1
x13+x23+x33 +x43=1
x14+x24+x34 +x44=1
0<=x11,x12,x13,x14,x21,x22,x23,x24,x31,x32,x33,x34,x41,x42,x43,x44<=1

Result:
The minimum number is 11
when
(x11,1), (x12,0), (x13,0), (x14,0), (x21, 0), (x22, 0), (x23, 1), (x24,0), (x31, 0), (x32, 1), (x33, 0), (x34, 0), (x41, 0), (x42,0),(x43,0) and (x44, 1)

x11, x23, x32 and x44 equal 1
the reast variables are 0.

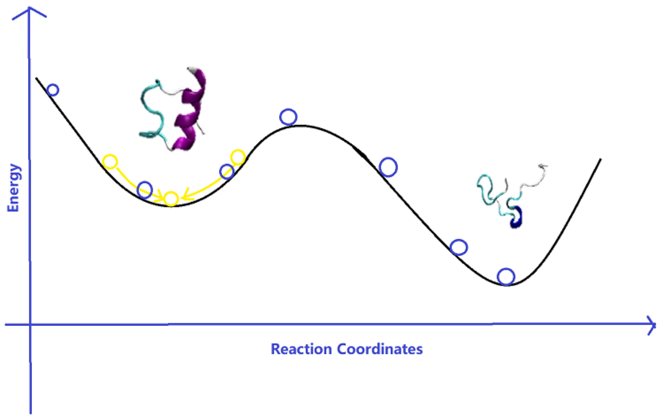
(WHEN IS AN OPTIMAL SOLUTION GUARANTEED?)
Numerical optimization algorithms often work with some variant of the following iterative approach:
- **Make sure to be in a feasible point.**
- **Take a small step in a direction that will improve the objective function while still staying within the feasible set.**
- **Repeat until no improvement.**

A natural and important question is then if an algorithm of this kind is guaranteed to find the globally optimal solution, or if it is at risk to get stuck in a local optimum. a) Give examples of optimization problems in one and two variables, where this approach will always work, and examples of problems where it may fail.
HINT The two-dimensional problem can be seen as finding a highest or a lowest point in a landscape, within a limited geographical area.



The figure above is the 3D-dimensional representation of the free energy landscape of an intrinsically disordered peptide as a function of three collective variables (C1, CV2 and CV4) CV4 is for the number of contacts between the side-chain heavy atoms of different residues. CV1 and CV2 count the number of fragments of 6 residues belonging to α-helix or β-sheet structures respectively.

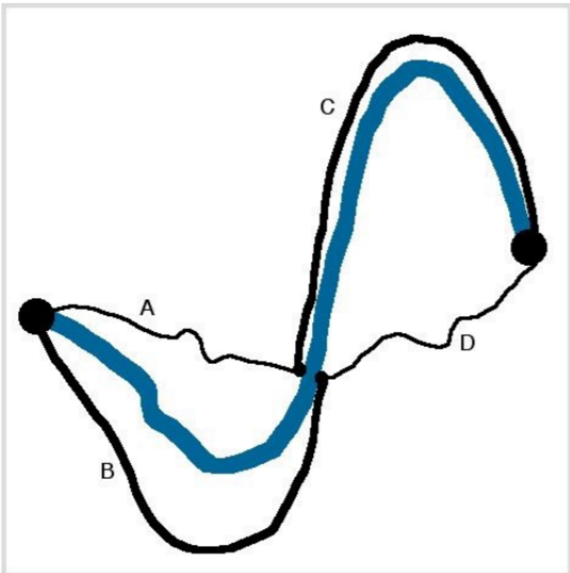
For an intrinsically disordered protein, for example, an alpha-Synuclein protein which has been identified to play critical roles in Parkinson's Disease. For an intrinsically disordered protein which forms intrinsically disordered structures at the global minimum. However, as the energy increased it forms some relatively stable structures (more beta-sheet or alpha-helix formed structure) at some local minimums which interact with other functional proteins leading to signal transformations or diseases, for example, seeing the figure below:



b) (Voluntary) What is the most general case you can think of, where finding the global optimum can be guaranteed? Consider both the character of the objective function and the shape of the feasible set.

By different starting points which have no relationship between each other and all reactive pathways are calculated together with the constraints, the global optimum is likely to be guaranteed at the lowest point in a landscape.

(investigating the world)
(BRIDGE PROBLEM)'
Consider the road network below. The figure illustrates the roads between two larger cities along a river. Roads B and C are large roads and have a fixed travel time of about 30 minutes independently of the traffic load. The roads A and D are mountain roads and the travel time is estimated to about 10+x minutes where x is the traffic intensity in cars per minute (in one direction). During rush hours the total traffic between the cities in either direction s is about 20 cars per minute.
a) Assuming that every individual tries to optimize his/her own travel time, what will be the travel time during rush hours? Motivate your answer. Hint: consider if all individual decisions of drivers could eventually lead to some equilibrium state with constant traffic flows? (note that this is not a single optimization problem, but a more complex situation with many agents where each agent optimizes for itself)



Conditions:
B and C roads, time 30min independent of the traffic load.
A and D, 10+x min, in rush time traffic loading is cars/min
In the rush time, the total traffic between the cities in either direction s is about 20 cars per minute.

Variables:
The number of cars goes through B and D: x1
The number of cars goes through A and C: x2

Object function
For each car, the time spend to go trough B and D
30+10+x1
go through A and C
30+10+x2

The sum of the total number will be
(30+10+x1)x1+(30+10+x2)x2
The problem is to find the minimum of the function (30+10+x1)x1+(30+10+x2)x2

Constraints
In a rush time: x1+x2=20
x1 and x2 >=0
x1 and x2 are integers.

Results

```
In[7]:= NMinimize[{(30 + 10 + x1) x1 + (30 + 10 + x2) x2, x1 + x2 == 20 , x1 ≥ 0, x2 ≥ 0, x1 ∈ Integers, x2 ∈
        Integers}, {x1, x2}]
Out[7]= {1 000., {x1 → 10, x2 → 10}}
```

The minimum total sum of the time is 1000 minutes and with 10 cars go through A and D and 10 cars go through B and C. Eventually, the equilibrium state is the number of person goes A and C equals to the number of person goes B and D.

b) In order to improve traffic flow it is decided to build a bridge over the river between the two small communities. The travel time over the bridge is about 1 minute independently of the traffic volumes we are considering. Again, assuming that every individual tries to optimize his/her own travel time, what will be the new travel time between the large cities? Motivate your answer, discuss the result and draw qualitative conclusions.

Conditions:
B and C roads, time 30min independent of the traffic load.
A and D, time 10+x min, x is the cars per minute
Go through the bridge 1min independent of the traffic load.
In the rush time, the total traffic between the cities in either direction s is about 20 cars per minute.

Variables:
The number of cars goes through B and D: x1
The number of cars goes through B and C: x2
The number of cars goes through A and C: x3
The number of cars goes through A and D: x4

Object function
For each car the time spend to go trough B and D
30+10+(x1+x4)
go through A and C
30+10+(x3+x4)
go through B and C
30 + 1 + 30
go through A and D

10+ (x3+x4)+ 1 + 10 + (x1+x4)
The sum of the total number will be
(30+10+(x1+x4))x1+ (30+10+(x3+x4))x3+(61)x2+(10+(x3+x4)+1+10+(x1+x4))x4

The problem is to find the minimum of the function:
(40+(x1+x4))x1+(40+(x3+x4))x3+61x2+(21+(x3+2x4+x1))x4

Constraints
In a rush time: x1+x2+x3+x4=20
x1, x2, x3, x4 >=0
x1,x2,x3 and x4 are integers

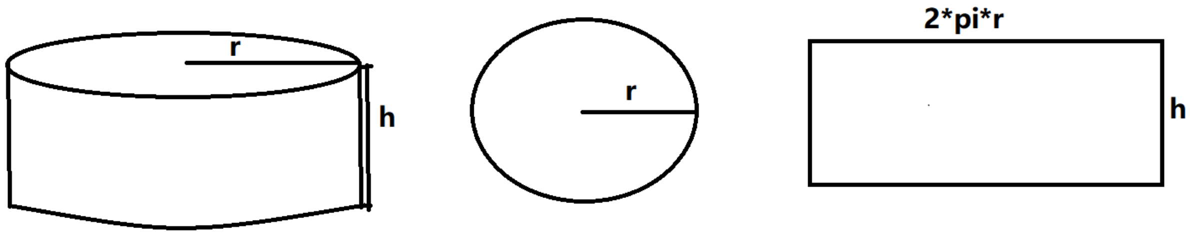
Results
The minimal sum of the time is 1000 minutes with x1 and x3 equal to 10, but x2, x4 equals 0.The bridge seems useless in this case.

```
In[10]:= NMinimize[{(40 + (x1 + x4)) x1 + (40 + (x3 + x4)) x3 + 61 x2 + (31 + (x3 + 2 x4 + x1)) x4, x1 + x2 + x3 + x4 ==
    20 , x1 ≥ 0, x2 ≥ 0, x3 ≥ 0, x4 ≥ 0, x1 ∈ Integers, x2 ∈ Integers, x3 ∈ Integers, x4 ∈
    Integers}, {x1, x2, x3, x4}]
Out[10]= {1 000., {x1 → 10, x2 → 0, x3 → 10, x4 → 0}}
```

OPTIMAL SHAPE OF CAN

Take a careful look at how we modelled this problem in the lecture. Try Mathematica for this problem and find out what the best shape is! (Note that this is a non-linear optimization problem, which are usually harder to solve, but this is a very small one)

```
FindMinimum[{2 Pi r h + 2 Pi r^2 , Pi r^2 h == 1, r
>= 0, h >= 0}, {r, h}]
```



The shape of a can contains a circle and a rectangle.
and the volume of the cylinder is pi*r^2*h.
The surface of a can equals 2*pi*r*h+pi*r^2
The problem is to find the minimum of the can surface.
Object function:
2*pi*r*h+pi*r^2
Variable:
r and h
Constraints:

The volume of the cylinder is fixed, assume to be 1 pi*r^2*h=1
r and h >=0
Results:

```
In[9]:= FindMinimum[{2 Pi r h + 2 Pi r^2 , Pi r^2 h == 1, r ≥ 0, h ≥ 0}, {r, h}]

Out[9]= {5.53581, {r → 0.541926, h → 1.08385}}
```

The minimal surface of a can is 5.54 with r 0.54 and h 1.08.

(EMERGENCY CARE PROBLEM)

The following problem is a so called facility location problem. A city wishes to make a long term study to decide where to best locate emergency care. The city has been partitioned into regions, and it has been decided that an emergency care site can acceptably service regions of the city which are within a driving distance of 8 minutes. The goal is to choose a set of stations at minimum cost. There are seven regions to cover, and six potential sites have been identified. Distances in minutes between regions and potential sites:

Site #	1	2	3	4	5	6
Region 1	15	3	12	5	17	20
Region 2	12	9	13	16	3	4
Region 3	13	16	9	4	7	11
Region 4	3	7	6	22	5	18
Region 5	4	22	12	5	16	14
Region 6	8	10	5	16	13	5
Region 7	13	10	5	6	13	21

The cost for locating emergency care on the respective sites:

	Cost
Site 1	710 000
Site 2	610 000
Site 3	650 000
Site 4	910 000
Site 5	720 000
Site 6	570 000

a) Model this problem mathematically by defining variables, constraints and an objective function. To get started, you can simply begin to define some variables, write some equations and see what you get along the way. (It is best if you can make the constraints and the objective function linear, since then the problem becomes

easier to solve mathematically. For links about linear programming see below. Note that in this step you are not solving the problem, just defining it. Hint: think about what I said in the lecture about how to define the variables.
Site 1 can connect to region 4, 5 and 6 because the distances are smaller than 8 minuter. So as to the rest of sites see bellowing
S1 (4,5,6)
S2 (1,4)
S3 (4,6,7)
S4 (1,3,5,7)
S5 (2,3,4)
S6 (2,6)
Assume the numbers of site1, site 2.. site 6 equal to x1, x2...x6.
The problem is to find the minimun of object function:
710000x1+610000x2+650000x3+910000x4+720000x5+570000x6
Variables:
site1, x1; site2, x2; site3, x3; site4, x4; site5, x5; site6,x6
Constraints
x1,x2,x3,x4,x5,x6 >=0
The maximun sites' building must be smaller than 6:
0<x1+x2+x3+x4+x5+x6<=6
The region1 only accessible from site2 and site4 so that
1<=x2+x4<=2
same to the rest regions
region2:
1<=x5+x6<=2
region3:
1<=x4+x5<=2
region4:
1<=x1+x2+x3+x5<=4
region5
1<=x1+x4<=2
region6:
1<=x1+x3+x6<=3
All x1,x2,x3,x4,x5,x6 are intergers.

b) Now try to solve your model by using the Mathematica function NMinimize. Try to solve it as a plain linear programming problem with continuous variables (without using any special options of the NMinimize function or constraints to say that the variables are integer or binary - as we discussed in the introductory lecture). Describe any difficulties you run into. What conclusions can you draw from the solution you obtain? Then solve the original integer problem eg. by solving variations of the LP multiple times, or by using some integer option in Mathematica, and give the answer.

Without any special options of the NMinimin function or constraints for variables of integer or binary , I could get the minimal cost 1.67E6 units but with the numbers of sites in float type. It is not realistic to have 0.333 site located.

```
In[9]:= NMinimize[ {710000 x1 + 610000 x2 + 650000 x3 + 910000 x4 + 720000 x5 + 570000 x6 , 0 < x1 + x2 + x3 + x4 + x5 + x6 ≤ 6, 1 ≤
        x2 + x4 ≤ 2, 1 ≤ x5 + x6 ≤ 2, 1 ≤ x4 + x5 ≤ 2, 1 ≤ x1 + x2 + x3 + x5 ≤ 4, 1 ≤ x1 + x4 ≤ 2 , 1 ≤ x1 + x3 + x6 ≤ 3, x1 ≥ 0,
        x2 ≥ 0, x3 ≥ 0, x4 ≥ 0, x5 ≥ 0, x6 ≥ 0  }, {x1, x2, x3, x4, x5, x6} ]
Out[9]= {1.66667×10^6, {x1 → 0.333333, x2 → 0.333333, x3 → 0., x4 → 0.666667, x5 → 0.333333, x6 → 0.666667}}
```

By using the integer option in Mathematica, the minimal cost of locating sites is 2.04E6 units with x1 (site1),x2 (site2) and x5 (site5) equal to 1 and x3, x4, x6 equal to 0.
The cost is minimized by locating site1, site2 and site5.

```
In[10]:= NMinimize[ {710000 x1 + 610000 x2 + 650000 x3 + 910000 x4 + 720000 x5 + 570000 x6 , 0 < x1 + x2 + x3 + x4 + x5 + x6 ≤ 6, 1 ≤
         x2 + x4 ≤ 2, 1 ≤ x5 + x6 ≤ 2, 1 ≤ x4 + x5 ≤ 2, 1 ≤ x1 + x2 + x3 + x5 ≤ 4, 1 ≤ x1 + x4 ≤ 2 , 1 ≤ x1 + x3 + x6 ≤ 3, x1 ≥ 0,
         x2 ≥ 0, x3 ≥ 0, x4 ≥ 0, x5 ≥ 0, x6 ≥ 0 , x1 ∈ Integers, x2 ∈ Integers, x3 ∈ Integers, x4 ∈ Integers, x5 ∈
         Integers, x6 ∈ Integers  }, {x1, x2, x3, x4, x5, x6} ]
Out[10]= {2.04×10^6, {x1 → 1, x2 → 1, x3 → 0, x4 → 0, x5 → 1, x6 → 0}}
```

c) Ask yourself if this is the only way to handle this problem? For example, in this problem we assumed that we should have a maximum number of minutes from each region of the city. Is this the only way to think about this? If you can, elaborate on any idea you might have. Note that while this problem only has a small number of variables and therefore can be solved by brute force combinatorial search, this would be useless for larger problems. For larger problems the more mathematical approach is much more powerful.

The shortest distance from each region or the traffic loading between each region, the maximum number of person is able to be treated.

(REASONING - PSYCHOLOGICAL TEST)
For each of the four cards below there is a letter on one side and a digit on the other.

F 8 U 3

For these cards it is also claimed that if there is a vowel on one side, there is an odd number on the other side. What is the least number of cards you need to turn to verify this, and which cards do you need to turn?
Problem:
To verify there is a vowel on one side and the other side number is an odd
Constraints
a vowel -> an odd (Does not mean an odd -> a vowel), it is a sufficient but not necessary condition
Solution:
I need two times to verify this. The cards I need to turn is card U and card 8.
Reason:
Card F is not a vowel, it doesn't matter if the other side is an odd number or not, it is not included in constraint.

Card 3 is an odd number, it doesn’t matter if the other side is a vowel or not, it is not included the constraints.
Card U is a vowel, the other side must be check, if it is not odd, it will break the constraint.
Card 8 is an even number, the other side must be check, if the other side is a vowel, it will break the constraint.

(finally…)
(MID-COURSE FEEDBACK)
The purpose of this question is to ensure a common understanding of what we are trying to do in this course, and to identify any possible problems. Your answers may influence our feedback and other actions during the rest of the course.
a) Please write what you think is the purpose of this course. If you like, you may also write a personal answer in terms of how you see the purpose of the course for you.
The course for me is to study how to think or plan my daily work mathematically. I am a problem-solution oriented person. I want to apply those mathematical methods which I studied in my daily life which help me be more logical and systematic.
b) Does the course work well for you? If not, please explain!
Yes, it works very well for me.
c) Any other comments?
Some mathematical knowledge is lacked when solving the problems in modules. It might become better if I can be given some hints during the lecture on Monday.
d) What is your math background? The main distinction for us is “high school” or “university”, but you are welcome to provide additional information if you wish. Please also give your main field of study in your BSc.
My math background is at the university level and I passed the mathematic 4 from Sweden before I took the course. I have 6 year- experience about data analysis by doing research at university. My background is theoretical chemistry. I am interested in being a data scientists now.

(finally…)
(SELF-CHECK)
Self-check passed!