This assignment is **due on Oct 27** and should be submitted on Gradescope. All submitted work must be *done individually* without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies. You may like to refer to the attached "Advice on how to do the assignment".

Problem 1. (10 points) We consider a variant of the minimum spanning tree problem. In this problem you are given an undirected graph $G(V, E)$ with edge weights $w : E \to \mathbb{R}_{\geq 0}$ as well as vertex capacities $k : V \to \mathbb{Z}_{\geq 1}$. Your goal is to choose a minimum-weight set of edges F such that $G(V, F)$ is connected and the degree of each vertex $u \in V$ with respect to F is at most k_u.

a) Write down an integer program formulation of this problem.

b) Apply Lagrangian relaxation to the degree constraint on every vertex and write down the relaxed integer program.

c) How can we solve this relaxed program in polynomial time?

Problem 2. (10 points) You are the marketing manager for Dr. Alex B. Honest, who is in the running to be elected President of Fakelandia. Your team has designed n different advertising campaigns promoting Dr. Honest that could be conducted in any of Fakelandia's m states. Market research has shown that running campaign i in state j would generate $v_{i,j}$ new votes for Dr. Honest, but would cost \$ $c_{i,j}$.

Dr. Honest has instructed you to not run any campaign in more than one state, and not run more than one campaign in any state. (NB. There is no concept of a campaign being run multiple times in one state, either it is run or it is not.)

You also have a total budget of \$ b to spend on advertising, but no reward for spending less than the budget.

Your task is to choose which campaigns to run in which states, such that the total number of new votes for Dr. Honest is maximized, while not going over budget and not disobeying Dr. Honest's instructions.

a) Write down an integer program formulation of this problem.

b) Apply Lagrangian relaxation to the constraints that no campaign can run in more than one state, and that no state can have more than one campaign, write down the resulting integer program. What is the conventional name of this problem? Can it be solved in polynomial time?

c) Instead, apply Lagrangian relaxation to the constraint that no more than \$ b can be spent on advertising, write down the resulting integer program. What is the conventional name of this problem? Can it be solved in polynomial time?

Problem 3. (10 points) Inspired by Dr. Honest, you have decided to run for SRC Council. You have identified n voters; in order to be elected you will need all n to vote for you. These voters will always vote for their friends, but have different thresholds for what they consider friendship. For voter i to consider you a friend you must attend the same event at least k_i times. You check the clubs and societies calendar, there are m events coming up before SRC election day. You then check Facebook and see who is attending each event; each event j will be attended by a subset $E_j \subseteq [n]$ of the voters. Luckily no events are at the same time, so there is no restriction on the events you can attend; however, you find campus society events very boring. **Your goal** is to choose a minimum number of events to attend, such that you go to an event with each voter i at least k_i times. **Your task** is to model this problem as a minimum submodular cover problem.

$$\text{minimize}|C| \text{ such that } f(C) = f(\mathcal{U}) \text{ for } C \subseteq \mathcal{U}$$

You decide that you should let $\mathcal{U} = \{E_1, E_2, \ldots, E_m\}$.

a) Write down a function $f(C) : 2^{\mathcal{U}} \to \mathbb{R}$ such that solving the given minimum submodular cover problem solves your event-attending problem.

b) Prove that this function is monotone non-decreasing.

c) Prove that this function is submodular.

Problem 4. (10 points) In order to improve your chances of being elected to SRC Council you decide to spread a negative rumour about your closest rival. Consider the undirected, connected social network graph $G(V, E)$ where V is the set of students and there is an edge $(u, v) \in E$ if student u and student v are friends. Today you can tell your false rumour to up to k students. Tomorrow those people will tell the rumour to all their friends, the next day everyone who has heard the rumour will tell it to all *their* friends, and so on. However, the election is in only 6 days, i.e. the rumour will only spread 5 times. **Your goal** is to maximize the number of people who have heard the rumour by election day. **Your task** is to model this problem as a maximal submodular cover problem.

$$\text{maximize} f(C) \text{ such that } |C| \leq k \text{ and } C \subseteq \mathcal{U}$$

a) What should the set $\mathcal{U}$ be?

b) Write down a function $F(C) : 2^{\mathcal{U}} \to \mathbb{R}$ such that solving the given maximal submodular coverage problem solves your rumour-spreading problem.

c) Prove that this function is monotone non-decreasing.

d) Prove that this function is submodular.

Advice on how to do the assignment

- Assignments should be typed and submitted as pdf (no pdf containing text as images, no handwriting).
- Start by typing your student ID at the top of the first page of your submission. Do not type your name.
- Submit only your answers to the questions. Do not copy the questions.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for "your worst answer", as this indicates how well you understood the question.
- You can use the material presented in the lecture slides or lecture notes without proving it. You do not need to write more than necessary.
- When giving answers to questions, always prove/explain/motivate your answers.
- When giving an algorithm as an answer, the algorithm does not have to be given as (pseudo-)code.
- If you do give (pseudo-)code, then you still have to explain your code and your ideas in plain English.
- Unless otherwise stated, we always ask about worst-case analysis, worst case running times, etc.
- If you use further resources (books, scientific papers, the internet, ...) to formulate your answers, then add references to your sources and explain it in your own words. Only citing a source doesn't show your understanding and will thus get you very few (if any) marks. Copying from any source without reference is plagiarism.
- Finally, to make marking run more smoothly, please specify in with your Gradescope submission, which pages in your submission cover which problem.