

This assignment is **due on Sep 15** and should be submitted on Gradescope. All submitted work must be *done individually* without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies. You may like to refer to the attached "Advice on how to do the assignment".

Problem 1. (10 Points)

- a) Let $A \in \mathbb{R}^{m \times n}$ be a matrix and consider the polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \geq 0, x \geq 0\}$. Suppose for every $x \in \mathcal{P}$ we have $x_1 = 0$. Prove that there exists $y \in \mathbb{R}^m$ such that $A^T y \leq 0$, $y \geq 0$ and $y \cdot A_1 < 0$.
- b) Consider the following linear program in standard form

$$\begin{aligned} & \text{minimize} && c \cdot x \\ & \text{subject to} && Ax = b \\ & && x \geq 0 \end{aligned}$$

Let x^* be an **optimal** basic feasible solution to this problem induced by basis B . Assume that B is non-degenerate. Prove that the dual of this linear program has a unique optimal solution.

- c) Let $\mathcal{P} = \{p \in \mathbb{R}^n : Ax \leq b\}$ and $\mathcal{Q} = \{q \in \mathbb{R}^n : \hat{A}x \leq \hat{b}\}$ such that $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Use LP duality to prove that there exists $c \in \mathbb{R}^n$ such that $(p - q) \cdot c > 0$ for every $p \in \mathcal{P}$ and $q \in \mathcal{Q}$.

Problem 2. (10 Points)

- a) Suppose we are playing a two-player zero-sum game, such that no two outcomes in the payoff matrix P have the same value. I claim the number of pure equilibria to this game is always either 0 or 1. Am I correct? Provide a short proof or a counterexample.
- b) Suppose we are playing a simplified version of the board game Battleship. You must choose two *adjacent* squares on a 2×3 grid on which to hide your battleship. I then choose a single square to target with my attack. If I hit your battleship it sinks and I win the game. If I miss you win the game. What is the mixed equilibrium for this game and (assuming we both follow our equilibrium strategies) what is the probability that I win?
- c) Suppose we are playing a two-player zero-sum game with payoff matrix $P \in \mathbb{R}^{m \times n}$ such that for a particular $k \in [n]$ we have $p_{i,j} \geq p_{i,k}$ for all $i \in [m]$, $j \in [n]$. I claim that column k can be removed from P without changing the value of the game. Am I correct? Provide a short proof or a counterexample.

Problem 3. (10 Points) I have n final-year students $\{s_1, s_2, \dots, s_n\}$ in need of capstone projects and m available projects $\{p_1, p_2, \dots, p_m\}$. Every student picks k projects they are interested in and ranks them from 1 to k . If a student is assigned to their j th ranked project we say the student has *unhappiness* j . Every student must be assigned to exactly one project. We may assign multiple students to the same project, but no project may have more than 3 students assigned to it. Students may not be assigned to projects that are not in their k preferred projects. **Your task** is to assign students to projects in a way that minimizes **average unhappiness**.

- Write down an integer linear program that models this problem.
- Write down the linear relaxation of that integer program.
- Show that the linear relaxation will always have an integral optimal solution. Show this **without** applying unimodularity.
- Hence describe how this problem can be solved using linear programming in polynomial time.

Problem 4. (10 Points) We are interested in the Square Traveling Salesman Problem. In this problem we are given a set $P \subset \mathbb{R}^2$ of cities. The cost of traveling between cities p and $q \in P$ is the square of the euclidean between them, i.e. $\|p - q\|_2^2$. The Square Traveling Salesman Problem is to find a tour that visits every city in P and returns to where it started, while minimizing the total cost of traveling between the cities.

Let X_n be a set of n points chosen independently and uniformly at random from the unit square $[0, 1]^2$. Let $f(X_n)$ be the cost of an optimal Square Traveling Salesman Tour on X_n . We are interested in understanding the asymptotic (in n) behaviour of $\mathbb{E}(f(X_n))$. This seems difficult to understand theoretically, so we will investigate it empirically instead.

- Write down the integer linear programs for the Traveling Salesman Problem and the Square Traveling Salesman Problem. Does it matter that $\|p - q\|_2^2$ is not a linear function?
- Sample a random set of points X_n for values of $n = \{2, 4, 8, 16, \dots\}$ and compute $f(X_n)$ using gurobi. Continue increasing n by a powers of 2 until the solver takes more than a few seconds to return.
- For each value of n used in the previous step sample 10 additional point sets X_n , compute $f(X_n)$ for each point set and calculate the average value of $f(X_n)$ for each n .
- Plot this estimation of $\mathbb{E}(f(X_n))$ against n (Include the plots in your assignment).
- Therefore propose a hypothesis function $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ such that $\mathbb{E}(f(X_n)) = \Theta(h(n))$ based on your experimental data.

Advice on how to do the assignment

- Assignments should be typed and submitted as pdf (no pdf containing text as images, no handwriting).
- Start by typing your student ID at the top of the first page of your submission. Do not type your name.
- Submit only your answers to the questions. Do not copy the questions.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for “your worst answer”, as this indicates how well you understood the question.
- You can use the material presented in the lecture slides or lecture notes without proving it. You do not need to write more than necessary.
- When giving answers to questions, always prove/explain/motivate your answers.
- When giving an algorithm as an answer, the algorithm does not have to be given as (pseudo-)code.
- If you do give (pseudo-)code, then you still have to explain your code and your ideas in plain English.
- Unless otherwise stated, we always ask about worst-case analysis, worst case running times, etc.
- If you use further resources (books, scientific papers, the internet, ...) to formulate your answers, then add references to your sources and explain it in your own words. Only citing a source doesn't show your understanding and will thus get you very few (if any) marks. Copying from any source without reference is plagiarism.
- Finally, to make marking run more smoothly, please specify in with your Gradescope submission, which pages in your submission cover which problem.