This assignment is **due on Aug 24** and should be submitted on Gradescope. All submitted work must be *done individually* without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies. You may like to refer to the attached "Advice on how to do the assignment".

Problem 1. (9 Marks) Consider the following linear program.

$$\begin{aligned} \text{maximize} \quad & 5x - 2y \\ \text{subject to} \quad & ax + y \leq t - 5 \\ & x, y \geq 0 \end{aligned}$$

Where $a, t \in \mathbb{R}$ are fixed constants. Describe *all* values of a and t for which:

a) The given linear program is infeasible.

b) The given linear program is feasible and has bounded objective.

c) The given linear program is feasible and has unbounded objective.

In each case give a short explanation why your answer is correct.

Solution 1.

a) The linear program is infeasible when $t < 5$ and $a \geq 0$. This is infeasible because it would mean $ax + y \geq 0$ but $t - 5 < 0$ meaning the first inequality cannot be satisfied.

b) The linear program is feasible and bounded if $t \geq 5$ and $a > 0$. Because $5x - 2y \leq \frac{5}{a}(ax + y) \leq \frac{5}{a}(t - 5))$.

c) The linear program is feasible and unbounded if $t \geq 5$ and $a = 0$, then $y = 0$, $x \to \infty$ is feasible). Or if $a < 0$, in which case $y = 0$ is as $x \to \infty$ is also feasible for large x.

These alternatives account for all possible values of t and a, so they completely describe the possible scenarios.

Problem 2. (9 Marks) Consider the following linear program in standard form, where $A \in \mathbb{R}^{m \times n}$ with $m \geq 2$ and $n \geq 3$.

$$\begin{aligned} \text{minimize} \quad & c \cdot x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Let Q be the set of all optimal solutions to this linear program. You may assume that Q is non-empty.

a) Prove that Q is a polyhedron.

b) Using a standard LP solver, how could you find the optimal solution $q \in Q$ that has the largest first coordinate, i.e. $q_1 \geq p_1$ for all $p \in Q$.

c) Is Q always bounded? If it is, provide a short proof. If it is not, provide a counterexample.

Solution 2.

a) We know the linear program must be feasible and bounded because it has a non-empty set of optimal solutions. Let the optimal value of the linear program be v. Then $Q = \{x \in \mathbb{R}^n : Ax = b, x \geq 0, c \cdot x = v\}$. Clearly this last additional constraint we have introduced is also linear, so Q is an intersection of (finitely-many) linear constraints and hence a polyhedron.

b) First solve the given linear program to find its optimal value v, and then solve the following linear program

$$\begin{aligned} \text{maximize} \quad & x_1 \\ \text{subject to} \quad & Ax = b \\ & c \cdot x = v \\ & x \geq 0 \end{aligned}$$

Return the optimal solution given by the solver for this linear program. The constraints enforce that the solution must be in Q, while the optimal ensures that x_1 is maximized among the elements of Q.

c) No, Q is not always bounded. Consider the following linear program.

$$\begin{aligned} \text{maximize} \quad & x - y \\ \text{subject to} \quad & z - \tfrac{1}{2}x - \tfrac{1}{2}y = 0 \\ & x - y = 0 \\ & x, y, z \geq 0 \end{aligned}$$

Any point along the (unbounded) ray (t, t, t) for $t \geq 0$ is both feasible and optimal.

Problem 3. (12 Marks) Consider the following linear program in standard form. In each case $A \in \mathbb{R}^{m \times n}$ with $m \geq 2$ and $n \geq 3$.

$$\begin{aligned} \text{minimize} \quad & c \cdot x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Determine whether each of the following statements is true or false. If a statement is true, provide a short proof. If a statement is false, provide a counterexample.

a) If B is a feasible basis, it is not possible for every reduced cost $c_j - c_B^T A_B^{-1} A_j$ to be negative, i.e. it is not possible that $c^T - c_B^T A_B^{-1} A < 0$.

b) It is not possible for *every* basic feasible solution to be degenerate, i.e. it is not possible that $x_B \not> 0$ for every feasible basis B where $x_B = A_B^{-1} b$.

c) For every $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ there always exists some $c \in \mathbb{R}^n$, $c \neq 0$ such that the given LP is *bounded*.

d) For every $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ there always exists some $c \in \mathbb{R}^n$, $c \neq 0$ such that the given LP is *unbounded*.

Solution 3.

a) This statement is true. The reduced costs of indices already in the basis will always be zero.
To see this consider that $A_B^{-1} A_B = I_m$ by definition, so if $j \in B$, $A_B^{-1} A_j$ must be the vector of all zeros with a 1 in the jth position. Hence $c_B^T A_B^{-1} A_j = c_j$ and the reduced cost of j is 0.

b) This statement is false. Consider the following linear program in standard form.

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 + x_3 \\ \text{subject to} \quad & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ & x \geq 0 \end{aligned}$$

This program has a single basis, $\{1, 2, 3\}$, which is degenerate.

c) This statement is true. Let $c = a_1$ where a_1 is the first row of A. Then $c \cdot x = b_1$ for all feasible x.

d) This statement is false. Consider the following LP, where I_n is the $n \times n$ identity matrix.

$$\begin{aligned} \text{minimize} \quad & c \cdot x \\ \text{subject to} \quad & I_n x = \vec{0} \\ & x \text{ free} \end{aligned}$$

The only feasible solution to this LP is $x = \vec{0}$, so the only possible optimal solution to this program is $c \cdot \vec{0} = 0$, so this LP is bounded regardless of c.

Problem 4. (10 Marks) You are put in charge of production planning at the newly-opened Worcestershire Widget Factory (WWF). Here is some information about the WWF:

- The WWF produces widgets; widgets may be sold for \$5 per widget.

- However each year i there is a limited total demand for widgets, d_i; so no more than d_i widgets can be sold that year.
- Widgets produced in one year will no longer be fashionable in future years, so widgets must be discarded if they are not sold in the year of their production.
- An apprentice widget-maker working in the WWF can produce 300 widgets per year, while an expert widget-maker working in the WWF can produce 700 widgets per year.
- Each apprentice has an annual salary of $1000 and each expert has an annual salary of $1500. Salaries are paid in a single, lump sum at the end of the year.
- All salaries must be paid using money earned that year or earlier years. You cannot go into debt to pay salaries.
- You are the first employee of the WWF; the WWF is about to start its first year of operation.
- At the start of each year you may hire as many apprentices as you would like, but you cannot hire any experts.
- At the end of each year each apprentice working for the WWF finishes their training and becomes an expert; this happens after salaries are paid so they are paid the apprentice rate for their first year.
- At the end of each year you can fire as many apprentices and as many experts as you like. This happens after salaries are paid. Any staff who are not fired continue working for the WWF.

You have been hired by the WWF on a three year contract, so your goal is to maximize the total net profit (widget sales minus salaries) added together over the next three years. The WWF's analysts have predicted demand of d_1, d_2 and d_3 for the next three years respectively, and they are never wrong. **Your task** is to formulate this problem as an integer linear program; it does not need to be in standard form.

a) Define your variables and their intended meaning.
b) Define your constraints and how they enforce the above meaning.
c) Define the objective function.

Solution 4.

a) We start with the decision variables:

- a_i be the number of apprentices we hire in year i.
- Let f_i and g_i be the number of apprentices and experts respectively we fire at the end of year i.
- Let s_i be the number of widgets sold in year i, we need this because we may not be able to sell everything we produce.

We will also use some convenience variables to help us.

- Let e_i be the number of experts working for us at the start of year i.

$a_i, f_i, g_i, s_i, e_i \in \mathbb{Z}_{\geq 0}$ because we are only dealing with whole widgets and whole employees, with no negatives allowed.

b) We have the following constraints

- $e_1 = 0$ because we do not start with any experts.
- $e_i = e_{i-1} + a_{i-1} - f_{i-1} - g_{i-1}$ for $i \in \{2, 3\}$ because our number of experts is how many we had the previous year, plus newly-upgraded apprentices, minus the people we fired.
- $s_i \leq d_i$ for $i \in \{1, 2, 3\}$ because we cannot sell more widgets than there is demand for widgets.
- $s_i \leq 300a_i + 700e_i$ for $i \in \{1, 2, 3\}$ because we cannot sell more widgets than we produce.
- $5 \sum_{j=1}^{i} s_j \geq 1000 \sum_{j=1}^{i} a_j + 1500 \sum_{j=1}^{i} e_j$ for $i \in \{1, 2, 3\}$ because we have to be able to afford to pay salaries at the end of the year. This constraint is necessary because there might not be enough demand for workers to meet their own salary in a given year.

c) Our objective function is $5(s_1 + s_2 + s_3) - 1000(a_1 + a_2 + a_3) - 1500(e_1 + e_2 + e_3)$. This is the total number of widgets sold multiplied by their price, minus the salary paid to apprentices and experts each year.

Advice on how to do the assignment

- Assignments should be typed and submitted as pdf (no pdf containing text as images, no handwriting).
- Start by typing your student ID at the top of the first page of your submission. Do not type your name.
- Submit only your answers to the questions. Do not copy the questions.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for "your worst answer", as this indicates how well you understood the question.
- You can use the material presented in the lecture slides or lecture notes without proving it. You do not need to write more than necessary.
- When giving answers to questions, always prove/explain/motivate your answers.
- When giving an algorithm as an answer, the algorithm does not have to be given as (pseudo-)code.
- If you do give (pseudo-)code, then you still have to explain your code and your ideas in plain English.
- Unless otherwise stated, we always ask about worst-case analysis, worst case running times, etc.
- If you use further resources (books, scientific papers, the internet, ...) to formulate your answers, then add references to your sources and explain it in your own words. Only citing a source doesn't show your understanding and will thus get you very few (if any) marks. Copying from any source without reference is plagiarism.
- Finally, to make marking run more smoothly, please specify in with your Gradescope submission, which pages in your submission cover which problem.