Problem 1

a

$t < 5 \wedge a \geq 0 \Rightarrow$ the given linear program is infeasible.
Such t and a simply make the feasible region empty as the intersection of $ax + y \leq t - 5$, $x \geq 0$ and $y \geq 0$ is $\emptyset$.

b

$t \geq 5 \wedge a > 0 \Rightarrow$ the given linear program is feasible and bounded.
The optimal(max) objective occurs at $x = \frac{t-5}{a}, y = 0$. The other corners, $x = 0, y = 0$ and $x = 0, y = t - 5$, trivially have a smaller objective value.

c

$a < 0 \vee (t \geq 5 \wedge a = 0) \Rightarrow$ the given linear program is feasible and unbounded.

- $a < 0$: x^+ is not bounded, $y = 0$ is feasible. Therefore $\max(5x - 2y) = \max(5x)$ is unbounded.
- $t \geq 5 \wedge a = 0$: similarly

Problem 2

a

Since Q is non-empty, we could arbitrarily pick a solution x^* from Q. Let's denote $C := c \cdot x^*$ for now. We assume that Q is the set of optimal solutions. Therefore, we could rewrite Q as $\{x \in \mathbb{R}^n : c \cdot x = C, Ax = b, x \geq 0\}$, which is trivially a polyhedron. (We could further reconstruct the set into a structure alike the definition of polyhedra in the lecture notes by stacking LHS constraints row by row to a new matrix A' and RHS constraints to a new column b'.) Thus, Q is, by definition, an instance of a polyhedron.

b

1. We use the LP solver to solve the given $LP1$ to get the set Q.
2. Similar to what I have done in section 2.a ($C := c \cdot x^*$ where $x^* \in Q$), now we could define a new feasible region of a new $LP2$, where $LP2$ is subject to $c \cdot x = C, Ax = b, x \geq 0$.
3. To get the largest first coordinate, we define an objective mapping $c' \in \mathbb{Z}^n$ in such a way:

$$c'_1 = -1$$
$$\forall i \in [2, n] \subset \mathbb{Z} : c'_i = 0$$

In another form, $c' = [-1, 0, ..., 0]^T$. $LP2$ minimizes $c' \cdot x$, which is simply $-x_1$.

4. Now let the LP solver handle $LP2$. The optimal solution for $LP2$ is the answer for the question.

c

Q is not always bounded.
I prefer to abuse $c = 0$ to make life easy. Here is a counterexample,

$$c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The optimal solution set Q is not bounded. For example,

$$x = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, k \in \mathbb{R}^+$$

$x \in Q$ while k is unbounded.

Problem 3

a

The statement is true.
Assume that B is a feasible basis. In other words,

$$\begin{aligned} &\exists x_B : x_B = A_B^{-1} b \\ &\Rightarrow \exists i \in B : c_i - c_B^T A_B^{-1} A_i \\ &= c_i - c_B^T v \\ &\text{where } v\text{'s } i\text{th element is 1 while others are 0 since } A_B^{-1} A_B = I_m \\ &= c_i - c_i = 0 \end{aligned}$$

We have found such a column index i where its reduced cost is 0. So it is not possible that every reduced cost is negative.

b

The statement is false.
A counter example:

$$c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For this instance, there is one and only one feasible basis with a solution $[0, 0, 0]^T$, which is degenerate. So it is possible that every(the only one in the instance) basic feasible solution is degenerate.

c

The statement is true.
We denote a_i as ith row of A and b_i as ith element of b. Therefore, $\forall A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m : \exists c := a_i : cx = a_i x = b_i$. In other words, the given LP objective is always bounded at $[b_i, b_i]$ for such c.

d

The statement is false.
A classic counter example again:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The only feasible solution is $x = [0, 0, 0]^T$. And $\forall c : cx = 0$. In other words, the objective is bounded no matter what c is. Therefore, we have found such A and b where there does not exist a c so that the LP objective is unbounded.

Problem 4

a

- w_i: number of widgets manufactured at year i
- s_i: number of widgets sold at year i
- a_i^+: number of apprentices starting working at the beginning of year i
- a_i^-: number of apprentices fired at the end of year i
- e_i^+: number of experts starting working at the beginning of year i
- e_i^-: number of experts fired at the end of year i
- r_i: profits occurring at year i.

b

- $i \in \{1, 2, 3\}$
  only plan for next 3 years
- $w_i = 300a_i^+ + 700e_i^+$
  An apprentice widget-maker working in the WWF can produce 300 widgets per year, while an expert widget-maker working in the WWF can produce 700 widgets per year.
- $0 \le s_i \le d_i$
  However each year i there is a limited total demand for widgets, d_i; so no more than d_i widgets can be sold that year.

- $w_i \geq s_i$
  Widgets produced in one year will no longer be fashionable in future years, so widgets must be discarded if they are not sold in the year of their production.

- $r_i = 5s_i - 1000a_i^+ - 1500e_i^+$
  widgets may be sold for \$5 per widget.
  Each apprentice has an annual salary of \$1000 and each expert has an annual salary of \$1500. Salaries are paid in a single, lump sum at the end of the year.
  At the end of each year each apprentice working for the WWF finishes their training and becomes an expert; this happens after salaries are paid so they are paid the apprentice rate for their first year.

- $\forall i : \sum_{j=1}^{i} r_j \geq 0$
  All salaries must be paid using money earned that year or earlier years. You cannot go into debt to pay salaries.

- $i \in \{1,2\} : e_{i+1}^+ = a_i^+ - a_i^- + e_i^+ - e_i^-$
  $e_1^+ = 0$
  $i \in \{1,2,3\} : 0 \leq e_i^- \leq e_i^+$ and $0 \leq a_i^- \leq a_i^+$ and $a_i^+ \geq 0$
  At the end of each year each apprentice working for the WWF finishes their training and becomes an expert; this happens after salaries are paid so they are paid the apprentice rate for their first year.
  At the start of each year you may hire as many apprentices as you would like, but you cannot hire any experts.
  At the end of each year you can fire as many apprentices and as many experts as you like.

c

objective: maximize $r_1 + r_2 + r_3$