1

1.1

$$\begin{aligned} \text{minimize } & \sum_{u \in V} x_u + \rho \sum_{e \in E} y_e \\ \text{subject to } & x_u + x_v + y_e \geq 1 | \forall e := (u,v) \in E \\ & x_u \in \{0,1\} | \forall u \in V \\ & y_e \in \{0,1\} | \forall e \in E \end{aligned} \tag{IP}$$

1.2

Let's denote the constraint matrix for IP as A', number of nodes n and number of edges m. A' is a $m \times (n+m)$ matrix where rows are edges and columns are nodes and the edges itself. We could rewrite A' as $[A\ I_m]$ where A is a bipartite incidence matrix. During the lecture, we have proved that A is bicoloring equitable therefore totally unimodular. We also proved that $[A\ I_m]$ is totally unimodular if A is totally unimodular. Thus $A' = [A\ I_m]$ is totally modular.

1.3

$$\begin{aligned} \text{minimize } & \sum_{u \in V} x_u + \rho \sum_{e \in E} y_e \\ \text{subject to } & k_e^+ \geq x_u + x_v + y_e \geq k_e^- | \forall e := (u,v) \in E \\ & x_u \in \{0,1\} | \forall u \in V \\ & y_e \in \{0,1\} | \forall e \in E \\ & k_e^+, k_e^- \in \{0,1,2\} | \forall e \in E \\ & k_e^+ \geq k_e^- | \forall e \in E \end{aligned} \tag{GIP}$$

1.4

We can construct the new constraint matrix as $[A'\ I_m\ I_m]$. Now we apply the rule that $[A'\ I_m]$ is totally unimodular if A' is totally unimodular twice. So A'' is totally unimodular.

2

2.1

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in[n]} x_i q_i \\ \text{subject to} \quad & \sum_{i\in[n]} x_i \le k \\ & \sum_{j\in B_i} x_j \le b_B(i) | \forall i \in [m1] \\ & \sum_{j\in C_i} x_j \le b_C(i) | \forall i \in [m2] \\ & x_i \in \{0,1\} | \forall i \in [n] \end{aligned} \tag{ACME}$$

2.2

A counter example:
people: $\{1,2,3\}, k = 2, B_1 = \{1,2\}, C_1 = \{2\}, B_2 = \{3\}, C_2 = \{1,3\}$
$b_B(1) = 2, b_B(2) = 1, b_C(1) = 1, b_C(2) = 1$
Both $\{3\}$ and $\{1,2\}$ are feasible. However neither $\{1,3\}$ nor $\{2,3\}$ is feasible.

2.3

We can think of this problem as a variant of bipartite matching. Each business is a node in a partition p_B and each country as a node in the other partition p_C. Each person is an edge connecting one node in p_B and a node in p_C .Therefore we have a constraint matrix of n rows for each person, m_1 red colored columns and m_2 blue colored columns. The coloring is equitable thus the matrix is totally unimodular.

2.4

Suppose there are two feasible sets of conference members X and Y where $|X| < |Y|$.
$D := Y \setminus X$
We prove it's matroid by making contradictions.
Assume that the problem is not a matroid.$\Rightarrow \nexists d \in D : X + d$ is feasible $\Rightarrow \forall d \in D :$ $X + d$ is infeasible.
However, $|X| < |Y| \Rightarrow D \neq \emptyset \Leftrightarrow \exists d \in D : d \in Y \Rightarrow d \in B_{i^*}$ and $d \in C_{j^*}$ where $B_{i^*} \subset C_{j^*}$ for some i^* and j^*.
According to the assumption, $X + d$ is infeasible. In other words, either (1) $\exists i \in [m_1] : \sum_{k\in B_i} > b_B(i)$ or (2) $\exists j \in [m_2] : \sum_{k\in C_j} > b_C(j)$ and k doesn't matter as $|X| < |Y|$.

- $\forall k \in X : k \notin B_{i^*} \Rightarrow \sum_{k\in B_{i^*}} > b_B(i^*) \Rightarrow |Y|$ is not feasible, contradicts to the premise.
- $\exists k \in X : k \in B_{i^*} \subset C_{j^*} \Rightarrow \sum_{k\in C_{j^*}} > b_C(j^*) \Rightarrow |Y|$ is not feasible, contradicts to the premise.

Therefore, $X + d$ is always feasible. Thus the problem is a matroid.

3

3.1

A counter example:
$p_1 \cap p_2 \neq \emptyset, p_1 \cap p_3 \neq \emptyset, p_2 \cap p_3 = \emptyset$.
$x_1 = 1, x_{i \neq 1} = 0$ is feasible. So is $x_2 = x_3 = 1, x_{i \neq 2, i \neq 3} = 0$. But $x_1 = x_2 = 1$ and $x_1 = x_3 = 1$ are infeasible.

3.2

A counter example:
$v_1 = v_2 = 1, v_{\text{otherwise}} = 0, p_1 \cap p_2 \neq \emptyset$. $x_1 = 0.2, x_2 = 0.8$ is a basic feasible solution which is not integral.

3.3

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in [n]} v_i x_i \\ \text{subject to} \quad & \sum_{k \in [n] : p_k \cap p_i \neq \emptyset} x_k \leq 1 | \forall p_i, i \in [n] \\ & x_i \in \{0, 1\} | \forall i \in [n] \end{aligned} \tag{TIP}$$

3.4

TIP is valid.
$\cap_{k \in K} p_k \neq \emptyset \Rightarrow \forall i, j \in K : p_i \cap p_j \neq \emptyset$
TIP is strictly stronger.
It is clear that $x_i + x_j \leq \sum_{k \text{ includes } i \text{ and } j} x_k$ because x_is are non-negative. If a solution is feasible for TIP, $x_i + x_j \leq \sum_{k \text{ includes } i \text{ and } j} x_k \leq 1$ still holds. Thus it is a feasible solution for the original IP. However, let's say $\mathbf{v}$ is uniform with 1s. $p_1 \cap p_2 \cap p_3 \neq \emptyset . x_1 = x_2 = x_3 = 0.5$ is always feasible and optimal for IP but not feasible for TIP.

4

4.1

Say, we have a basis B in the current iteration. (The initial B_0 is simply $|E|$ spanning trees of each edge in the graph). Calculate $\mathbf{1} A_B^{-1} A_T$ for each T where $A_T|_e = 1$ if $e \in T$. The T has largest $\mathbf{1} A_B^{-1} A_T$ has the minimized reduced cost.

4.2

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} y_e \\ \text{subject to} \quad & \sum_{e \in T} y_e \leq 1 | \forall T \in \mathcal{T} \\ & y_e \text{ free} | \forall e \in E \end{aligned} \tag{Dual}$$

4.3

The objective is to find a spanning tree T^* so that $\sum_{e \in T^*} y_e > 1$.
oracle algorithm:
Given a tree, simply iterate all edges in this tree through DFS/BFS and sum all weights. The running time is of $O(|V| + |E|)$ thus polynomial.