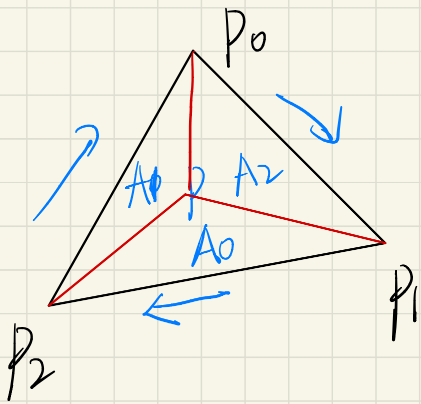
Barycentric Coordinate and Gradient

Mathematical Formulation



屏幕空间重心坐

$$p = b_0 p_0 + b_1 p_1 + b_2 p_2$$

$$\begin{cases} b_0 = \dfrac{A_0}{A_0 + A_1 + A_2} \\ b_1 = \dfrac{A_1}{A_0 + A_1 + A_2} \\ b_2 = \dfrac{A_2}{A_0 + A_1 + A_2} \\ b_0 + b_1 + b_2 = 1 \end{cases}$$

面积计算式

$$A_0 = |\overrightarrow{pp_1} \times \overrightarrow{pp_2}|$$

$$= \begin{vmatrix} x_1 - x & y_1 - y \\ x_2 - x & y_2 - y \end{vmatrix}$$

$$= (x_1 - x)(y_2 - y) - (x_2 - x)(y_1 - y)$$

$$= (y_1 - y_2)x + (x_2 - x_1)y + (x_1 y_2 - x_2 y_1)$$

$$= \alpha_0 x + \beta_0 y + \gamma_0$$

$$A_1 = |\overrightarrow{pp_2} \times \overrightarrow{pp_0}|$$

$$= (y_2 - y_0)x + (x_0 - x_2)y + (x_2 y_0 - x_0 y_2)$$

$$= \alpha_1 x + \beta_1 y + \gamma_1$$

$$A_2 = |\overrightarrow{pp_0} \times \overrightarrow{pp_1}|$$

$$= (y_0 - y_1)x + (x_1 - x_0)y + (x_0 y_1 - x_1 y_0)$$

$$= \alpha_2 x + \beta_2 y + \gamma_2$$

基于平面坐标的重心插值

$$\begin{cases} u = b_0 u_0 + b_1 u_1 + b_2 u_2 \\ v = b_0 v_0 + b_1 v_1 + b_2 v_2 \end{cases}$$

基于透视的重心插值

$$\begin{cases} u = \dfrac{b_0 \frac{u_0}{w_0} + b_1 \frac{u_1}{w_1} + b_2 \frac{u_2}{w_2}}{\frac{b_0}{w_0} + \frac{b_1}{w_1} + \frac{b_2}{w_2}} \end{cases}$$

$$= \frac{w_1 w_2 b_0 u_0 + w_0 w_2 b_1 u_1 + w_0 w_1 b_2 u_2}{w_1 w_2 b_0 + w_0 w_2 b_1 + w_0 w_1 b_2}$$

$$\begin{cases} b_0' = \dfrac{w_1 w_2 A_0}{w_1 w_2 A_0 + w_2 w_0 A_1 + w_0 w_1 A_2} \\[2ex] b_1' = \dfrac{w_2 w_0 A_1}{w_1 w_2 A_0 + w_2 w_0 A_1 + w_0 w_1 A_2} \\[2ex] b_2' = \dfrac{w_0 w_1 A_2}{\boxed{w_1 w_2 A_0 + w_2 w_0 A_1 + w_0 w_1 A_2}} \end{cases}$$

M

$$\begin{cases} u = b_0' u_0 + b_1' u_1 + b_2' u_2 \\ v = b_0' v_0 + b_1' v_1 + b_2' v_2 \end{cases}$$

$$\frac{\partial u}{\partial b_0'} = u_0 \quad \frac{\partial u}{\partial b_1'} = u_1 \quad \frac{\partial u}{\partial b_2'} = u_2$$

$$\frac{\partial v}{\partial b_0'} = v_0 \quad \frac{\partial v}{\partial b_1'} = v_1 \quad \frac{\partial v}{\partial b_2'} = v_2$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial [b_0' \ b_1' \ b_2']} \begin{bmatrix} \frac{\partial b_0'}{\partial x} \\ \frac{\partial b_1'}{\partial x} \\ \frac{\partial b_2'}{\partial x} \end{bmatrix}$$

$$= [u_0 \quad u_1 \quad u_2] \begin{bmatrix} \frac{\partial b_0'}{\partial x} \\ \frac{\partial b_1'}{\partial x} \\ \frac{\partial b_2'}{\partial x} \end{bmatrix}$$

$$\frac{\partial b_0'}{\partial x} = \frac{w_1 w_2 \alpha_0}{M} - \frac{w_1 w_2 A_0}{M^2} \frac{\partial (w_1 w_2 A_0 + w_0 w_2 A_1 + w_0 w_1 A_2)}{\partial x}$$

$$= \frac{w_1 w_2 \alpha_0}{M} - \frac{w_1 w_2 A_0}{M^2} (\underbrace{w_1 w_2 \alpha_0 + w_0 w_2 \alpha_1 + w_0 w_1 \alpha_2}_{\rightarrow N})$$

$$\frac{\partial u}{\partial x} = \begin{bmatrix} u_0 & u_1 & u_2 \end{bmatrix} \begin{bmatrix} \frac{w_1 w_2 \alpha_0}{M} - \frac{w_1 w_2 A_0}{M^2} N \\ \frac{w_0 w_2 \alpha_1}{M} - \frac{w_0 w_2 A_1}{M^2} N \\ \frac{w_0 w_1 \alpha_2}{M} - \frac{w_0 w_1 A_2}{M^2} N \end{bmatrix}$$

$$= \frac{u_0 w_1 w_2 \alpha_0}{M} + \frac{u_1 w_0 w_2 \alpha_1}{M} + \frac{u_2 w_0 w_1 \alpha_2}{M}$$

$$- \frac{w_1 w_2 A_0 u_0 N}{M^2} - \frac{w_0 w_2 A_1 u_1 N}{M^2} - \frac{w_0 w_1 A_2 u_2 N}{M^2}$$

$$= \frac{u_0 w_1 w_2 \alpha_0}{M} + \frac{u_1 w_0 w_2 \alpha_1}{M} + \frac{u_2 w_0 w_1 \alpha_2}{M} - \frac{uN}{M}$$

$$= \frac{(u_0 - u) w_1 w_2 \alpha_0 + (u_1 - u) w_0 w_2 \alpha_1 + (u_2 - u) w_0 w_1 \alpha_2}{M}$$

$$= \frac{w_1 w_2 \alpha_0 (u_0 - u) + w_0 w_2 \alpha_1 (u_1 - u) + w_0 w_1 \alpha_2 (u_2 - u)}{w_1 w_2 A_0 + w_0 w_2 A_1 + w_0 w_1 A_2}$$

$$= \frac{\frac{\alpha_0}{w_0}(u_0 - u) + \frac{\alpha_1}{w_1}(u_1 - u) + \frac{\alpha_2}{w_2}(u_2 - u)}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}$$

$\Rightarrow \quad u_0 = 1 \quad u_1 = 0 \quad u_2 = 0$

$$\therefore \frac{\partial u}{\partial x} = \frac{\frac{\alpha_0}{w_0}(1-u) + \frac{\alpha_1}{w_1}(-u) + \frac{\alpha_2}{w_2}(-u)}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}$$

$$= \frac{\frac{\alpha_0}{w_0} - u\left(\frac{\alpha_0}{w_0} + \frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2}\right)}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}$$

$$u = b_0' u_0 + b_1' u_1 + b_2' u_2$$

$$= b_0' u_0 = \frac{\frac{A_0}{w_0}}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}$$

$$\frac{\partial u}{\partial x} = \left(\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}\right)\frac{\alpha_0}{w_0} - \frac{A_0}{w_0}\left(\frac{\alpha_0}{w_0} + \frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2}\right)$$

$$= \frac{A_0}{w_0}\left(-\frac{\alpha_1}{w_1} - \frac{\alpha_2}{w_2}\right) + \frac{A_1}{w_1}\frac{\alpha_0}{w_0} + \frac{A_2}{w_2}\frac{\alpha_0}{w_0}$$

$$= \frac{\frac{A_0}{w_0}\left(-\frac{\alpha_1}{w_1} - \frac{\alpha_2}{w_2}\right) + \frac{A_1}{w_1}\frac{\alpha_0}{w_0} + \frac{A_2}{w_2}\frac{\alpha_0}{w_0}}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}\left(\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}\right)$$

$$= \left(\frac{\frac{A_0}{w_0}}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}\left(-\frac{\alpha_1}{w_1} - \frac{\alpha_2}{w_2}\right) + \frac{\frac{A_1}{w_1}}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}\frac{\alpha_0}{w_0} + \frac{\frac{A_2}{w_2}\frac{\alpha_0}{w_0}}{\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}}\right)\left(\frac{A_0}{w_0} + \frac{A_1}{w_1} + \frac{A_2}{w_2}\right)$$

$$\frac{\partial u}{\partial x}=\left[b_0'\left(-\frac{\alpha_1}{w_1}-\frac{\alpha_2}{w_2}\right)+b_1'\frac{\alpha_0}{w_0}+b_2'\frac{\alpha_0}{w_0}\right]\left(\frac{A_0}{w_0}+\frac{A_1}{w_1}+\frac{A_2}{w_2}\right)$$

$$W=\frac{\frac{A_0}{w_0}w_0+\frac{A_1}{w_1}w_1+\frac{A_2}{w_2}w_2}{\frac{A_0}{w_0}+\frac{A_1}{w_1}+\frac{A_2}{w_2}}=\frac{A_0+A_1+A_2}{\frac{A_0}{w_0}+\frac{A_1}{w_1}+\frac{A_2}{w_2}}$$

$$\therefore \frac{A_0}{w_0}+\frac{A_1}{w_1}+\frac{A_2}{w_2}=\frac{W}{A_0+A_1+A_2}=\frac{W}{Area}$$

$$\therefore \frac{\partial u}{\partial x}=\frac{W}{Area}\left[b_0'\left(-\frac{\alpha_1}{w_1}-\frac{\alpha_2}{w_2}\right)+b_1'\frac{\alpha_0}{w_0}+b_2'\frac{\alpha_0}{w_0}\right]$$

同理

$$\frac{\partial u}{\partial y}=\frac{W}{Area}\left[b_0'\left(-\frac{\beta_1}{w_1}-\frac{\beta_2}{w_2}\right)+b_1'\frac{\beta_0}{w_0}+b_2'\frac{\beta_0}{w_0}\right]$$

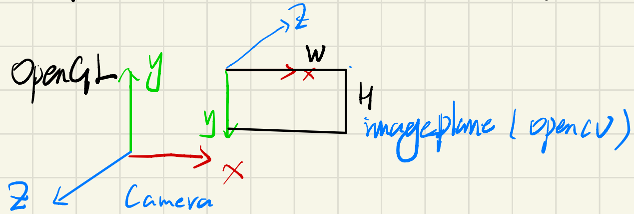
$$\frac{\partial v}{\partial x}=\frac{W}{Area}\left[b_0'\frac{\alpha_1}{w_1}+b_1'\left(-\frac{\alpha_0}{w_0}-\frac{\alpha_2}{w_2}\right)+b_2'\frac{\alpha_1}{w_1}\right]$$

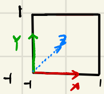
$$\frac{\partial v}{\partial y}=\frac{W}{Area}\left[b_0'\frac{\beta_1}{w_1}+b_1'\left(-\frac{\beta_0}{w_0}-\frac{\beta_2}{w_2}\right)+b_2'\frac{\beta_1}{w_1}\right]$$

$$P_{level0}=2^{level}P_{level}+0.5\left(2^{level}-1\right)$$

$$C_{level+1}=0.5\,C_{level}-0.5$$

2. OpenCV 投影矩阵与OpenGL投影矩阵的转换





imageplane (NDC)

OpenCV
$$\begin{cases} x = f_x \frac{X}{-Z} + c_x \\ y = f_y \frac{-Y}{-Z} + c_y \end{cases}$$

$$x \in [0, W] = \frac{f_x X + c_x(-Z)}{-Z}$$

$$y \in [0, H] = \frac{f_y(-Y) + c_y(-Z)}{-Z}$$

NDC (OpenGL)
$$\begin{bmatrix} a & 0 & c & 0 \\ 0 & b & d & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{cases} x_{clip} = aX + CZ \\ y_{clip} = bY + dZ \\ z_{clip} = AX + BZ \\ w_{clip} = -Z \end{cases} \rightarrow$$

$$x_{NDC} = \frac{x_{clip}}{w_{clip}} = \frac{aX + CZ}{-Z}$$

$$y_{NDC} = \frac{y_{clip}}{w_{clip}} = \frac{bY + dZ}{-Z}$$

$$z_{NDC} = \frac{z_{clip}}{w_{clip}} = \frac{AX + BZ}{-Z}$$

OpenCV → NDC

$$\frac{2x}{W} - 1 \in (-1, 1)$$

$$\frac{aX + CZ}{-Z} = \frac{2(f_x X + c_x(-Z))}{-Z\,W} - 1$$

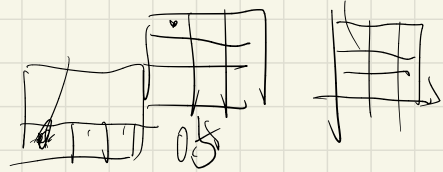
$$\frac{aX}{-Z} - C = \frac{2f_x}{W}\frac{X}{-Z} + \frac{2c_x}{W} - 1$$

$$a = \frac{2f_x}{W} \quad c = 1 - \frac{2c_x}{W}$$

$$b\frac{Y}{-Z} - d = -2\,\frac{f_y\frac{-Y}{-Z} + c_y}{H} + 1$$

$$= \frac{2f_y}{H}\frac{Y}{-Z} + 1 - \frac{2c_y}{H}$$

$$\therefore \begin{cases} b = \frac{2f_y}{H} \\ d = \frac{2c_y}{H} - 1 \end{cases}$$



图像金字塔的坐标

$$P_{level0} = 2^{level} P_{level} + 0.5(2^{level} - 1)$$

$$C_{level+1} = 0.5\,C_{level} - 0.5$$

Rasterize Backword

$$\begin{cases} [u_0, u_1, u_2] = [1, 0, 0] \\ [v_0, v_1, v_2] = [0, 1, 0] \end{cases}$$

$$\begin{cases} u = b_0' u_0 + b_1' u_1 + b_2' u_2 \\ v = b_0' v_0 + b_1' v_1 + b_2' v_2 \end{cases}$$

$\downarrow$

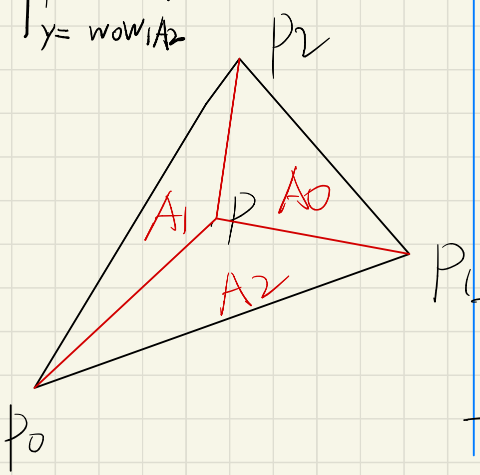
$$\begin{cases} u = b_0' \\ v = b_1' \end{cases}$$

$$b_0' = \frac{w_1 w_2 A_0}{w_1 w_2 A_0 + w_0 w_2 A_1 + w_0 w_1 A_2}$$

$$b_1' = \frac{w_0 w_2 A_1}{w_1 w_2 A_0 + w_0 w_2 A_1 + w_0 w_1 A_2}$$

$$b_2' = \frac{w_0 w_1 A_2}{w_1 w_2 A_0 + w_0 w_2 A_1 + w_0 w_1 A_2}$$

令
$$\begin{cases} \alpha = w_1 w_2 A_0 \\ \beta = w_0 w_2 A_1 \\ \gamma = w_0 w_1 A_2 \end{cases}$$



$$A_0 = \left| \overrightarrow{P_1 P} \times \overrightarrow{P_2 P} \right|$$

$$= \begin{vmatrix} \frac{x_1}{w_1} - x & \frac{y_1}{w_1} - y \\ \frac{x_2}{w_2} - x & \frac{y_2}{w_2} - y \end{vmatrix}$$

$$\alpha = w_1 w_2 A_0$$
$$= (x_1 - w_1 x)(y_2 - w_2 y) - (y_1 - w_1 y)(x_2 - w_2 x)$$

$$\frac{\partial \alpha}{\partial x_1} = y_2 - w_2 y = P_2 y$$

$$\frac{\partial \alpha}{\partial y_1} = -(x_2 - w_2 x) = -P_2 x$$

$$\frac{\partial \alpha}{\partial w_1} = -x(y_2 - w_2 y) + y(x_2 - w_2 x)$$
$$= -f_x P_2 y + f_y P_2 x$$

$$\frac{\partial \alpha}{\partial x_2} = -(y_1 - w_1 y) = -P_1 y$$

$$\frac{\partial \alpha}{\partial y_2} = (x_1 - w_1 x) = P_1 x$$

$$\frac{\partial \alpha}{\partial w_2} = -y(x_1 - w_1 x) + x(y_1 - w_1 y) = -f_y P_1 x + f_x P_1 y$$

$$\frac{\partial \beta}{\partial x_0} = -P_2 y$$

$$\frac{\partial \beta}{\partial y_0} = P_2 x$$

$$\frac{\partial \beta}{\partial w_0} = -f_y P_2 x + f_x P_2 y$$

$$\frac{\partial \beta}{\partial x_2} = P_0 y$$

$$\frac{\partial \beta}{\partial y_2} = -P_0 x$$

$$\frac{\partial \beta}{\partial w_2} = -f_x P_0 y + f_y P_0 x$$

$$\frac{\partial \gamma}{\partial x_0} = P_1 y$$

$$\frac{\partial \gamma}{\partial y_0} = -P_1 x$$

$$\frac{\partial \gamma}{\partial w_0} = -f_x P_1 y + f_y P_1 x$$

$$\frac{\partial y}{\partial x_1} = -p_0 y$$

$$\frac{\partial y}{\partial y_1} = p_0 x$$

$$\frac{\partial y}{\partial w_1} = -f_y p_0 x + f_x p_0 y$$

$$b_0' = \frac{\alpha}{\alpha+\beta+y}$$

$$\frac{\partial b_0'}{\partial \alpha} = \frac{1}{\alpha+\beta+y} - \frac{\alpha}{(\alpha+\beta+y)^2} = \frac{\beta+y}{(\alpha+\beta+y)^2}$$

$$\frac{\partial b_0'}{\partial \beta} = -\frac{\alpha}{(\alpha+\beta+y)^2}$$

$$\frac{\partial b_0'}{\partial y} = -\frac{\alpha}{(\alpha+\beta+y)^2}$$

$$\frac{\partial b_1'}{\partial \alpha} = \frac{-\beta}{(\alpha+\beta+y)^2}$$

$$\frac{\partial b_1'}{\partial \beta} = \frac{1}{\alpha+\beta+y} - \frac{\beta}{(\alpha+\beta+y)^2} = \frac{\alpha+y}{(\alpha+\beta+y)^2}$$

$$\frac{\partial b_1'}{\partial y} = \frac{-\beta}{(\alpha+\beta+y)^2}$$

$$\frac{\partial L}{\partial x_0} = \frac{\partial L}{\partial [u,v]} \frac{\partial [u,v]}{\partial x_0}$$

$$= [dy.x,\ dy.y] \cdot \begin{bmatrix} \frac{\partial b_0'}{\partial x_0} \\ \frac{\partial b_1'}{\partial x_0} \end{bmatrix}$$

$$\frac{\partial b_0'}{\partial x_0} = \frac{\partial b_0'}{\partial \beta} \frac{\partial \beta}{\partial x_0} + \frac{\partial b_0'}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial x_0}$$

$$= -\frac{\alpha}{(\alpha+\beta+\gamma)^2} \frac{\partial \beta}{\partial x_0} + \frac{-\alpha}{(\alpha+\beta+\gamma)^2} \frac{\partial \gamma}{\partial x_0}$$

$$= \frac{-b_0'}{\alpha+\beta+\gamma} \frac{\partial \beta}{\partial x_0} + \frac{-b_0'}{\alpha+\beta+\gamma} \frac{\partial \gamma}{\partial x_0}$$

$$\frac{\partial b_1'}{\partial x_0} = \frac{\partial b_1'}{\partial \beta} \frac{\partial \beta}{\partial x_0} + \frac{\partial b_1'}{\partial \gamma} \frac{\partial \gamma}{\partial x_0}$$

$$= \frac{a+c}{(\alpha+\beta+\gamma)^2} \frac{\partial \beta}{\partial x_0} + \frac{-\beta}{(\alpha+\beta+\gamma)^2} \frac{\partial \gamma}{\partial x_0} = \frac{b_0'+b_2'}{\alpha+\beta+\gamma} \frac{\partial \beta}{\partial x_0} + \frac{-b_1'}{\alpha+\beta+\gamma} \frac{\partial \gamma}{\partial x_0}$$

$$\frac{\partial L}{\partial x_0} = [dy.x,\ dy.y] \begin{bmatrix} \frac{-b_0'}{\alpha+\beta+\gamma} \frac{\partial \beta}{\partial x_0} + \frac{-b_0'}{\alpha+\beta+\gamma} \frac{\partial \gamma}{\partial x_0} \\ \frac{b_0'+b_2'}{\alpha+\beta+\gamma} \frac{\partial \beta}{\partial x_0} + \frac{-b_1'}{\alpha+\beta+\gamma} \frac{\partial \gamma}{\partial x_0} \end{bmatrix}$$

$\parallel$

gP0x

$$= \frac{[dy.x,\ dy.y]}{\alpha+\beta+\gamma} \begin{bmatrix} -b_0' \frac{\partial \beta}{\partial x_0} - b_0' \frac{\partial \gamma}{\partial x_0} \\ (b_0'+b_2') \frac{\partial \beta}{\partial x_0} - b_1' \frac{\partial \gamma}{\partial x_0} \end{bmatrix}$$

$$= [g b_0 \quad g b_1] \begin{bmatrix} -b_0' \frac{\partial \beta}{\partial x_0} - b_0' \frac{\partial \gamma}{\partial x_0} \\ (1-b_1') \frac{\partial \beta}{\partial x_0} - b_1' \frac{\partial \gamma}{\partial x_0} \end{bmatrix}$$

$$= [g_{b_0}\ \ g_{b_1}] \begin{bmatrix} -b_0\left(\frac{\partial \beta}{\partial x_0} + \frac{\partial \gamma}{\partial x_0}\right) \\ -b_1\left(\frac{\partial \beta}{\partial x_0} + \frac{\partial \gamma}{\partial x_0}\right) + \frac{\partial \beta}{\partial x_0} \end{bmatrix}$$

$$= -\underbrace{[g_{b_0} \cdot b_0 + g_{b_1} b_1]}_{g_{bb}} \left(\frac{\partial \beta}{\partial x_0} + \frac{\partial \gamma}{\partial x_0}\right) + g_{b_1} \frac{\partial \beta}{\partial x_0}$$

$$= -g_{bb}\left[\frac{\partial \beta}{\partial x_0} + \frac{\partial \gamma}{\partial x_0}\right] + g_{b_1} \frac{\partial \beta}{\partial x_0}$$

$$= -g_{bb}[-P_{2y} + P_{1y}] + g_{b_1}(-P_{2y})$$

$$= g_{bb}(P_{2y} - P_{1y}) - g_{b_1} P_{2y}$$

同理

$$\frac{\partial L}{\partial y_0} = -g_{bb}\left[\frac{\partial \beta}{\partial y_0} + \frac{\partial \gamma}{\partial y_0}\right] + g_{b_1} \frac{\partial \beta}{\partial y_0}$$

$$\| \quad g_{P_{0y}} = -g_{bb}[P_{2x} - P_{1x}] + g_{b_1} P_{2x}$$

$$= g_{bb}(P_{1x} - P_{2x}) + g_{b_1} P_{2x}$$

$$\frac{\partial L}{\partial w_0} = -g_{bb}\left[\frac{\partial \beta}{\partial w_0} + \frac{\partial \gamma}{\partial w_0}\right] + g_{b_1} \frac{\partial \beta}{\partial w_0}$$

$$g_{P_{0w}} = -g_{bb}[-f_y P_{2x} + f_x P_{2y} - f_x P_{1y} + f_y P_{1x}] + g_{b_1}(-f_y P_{2x} + f_x P_{2y})$$

$$= -g_{bb}[f_y(P_{1x} - P_{2x}) + f_x(P_{2y} - P_{1y})] + g_{b_1}(f_x P_{2y} - f_y P_{2x})$$

$$= g_{bb}[f_y(P_{2x} - P_{1x}) + f_x(P_{1y} - P_{2y})] + g_{b_1}(f_x P_{2y} - f_y P_{2x})$$

$$= f_x[g_{bb}(P_{1y} - P_{2y}) + g_{b_1} P_{2y}] + f_y[g_{bb}(P_{2x} - P_{1x}) - g_{b_1} P_{2x}]$$

$$= -f_x [gb_0 (p_{2y} - p_{1y}) - gb_1 p_{2y}] - f_y [gb_0 (p_{1x} - p_{2x}) + gb_1 p_{2x}]$$

$$= -f_x \, g p_{0x} - f_y \, g p_{0y}$$

$$g p_1 w = -f_x \, g p_{1x} - f_y \, g p_{1y}$$

$$g p_2 w = -f_x \, g p_{2x} - f_y \, g p_{2y}$$

$$g p_{1x} = \frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial [u,v]} \frac{\partial [u,v]}{\partial x_1}$$

$$= [dy.x, \; dy.y] \begin{bmatrix} \frac{\partial b_0'}{\partial \alpha} \frac{\partial \alpha}{\partial x_1} + \frac{\partial b_0'}{\partial \gamma} \frac{\partial \gamma}{\partial x_1} \\ \frac{\partial b_1'}{\partial \alpha} \frac{\partial \alpha}{\partial x_1} + \frac{\partial b_1'}{\partial \gamma} \frac{\partial \gamma}{\partial x_1} \end{bmatrix}$$

$$= [dy.x, \; dy.y] \begin{bmatrix} \frac{\beta+\gamma}{(\alpha+\beta+\gamma)^2} \frac{\partial \alpha}{\partial x_1} + \frac{-\alpha}{(\alpha+\beta+\gamma)^2} \frac{\partial \gamma}{\partial x_1} \\ \frac{-\beta}{(\alpha+\beta+\gamma)^2} \frac{\partial \alpha}{\partial x_1} + \frac{-\beta}{(\alpha+\beta+\gamma)^2} \frac{\partial \gamma}{\partial x_1} \end{bmatrix}$$

$$= [gb_0 \;\; gb_1] \begin{bmatrix} (b_1' + b_2') \frac{\partial \alpha}{\partial x_1} - b_0' \frac{\partial \gamma}{\partial x_1} \\ -b_1' \frac{\partial \alpha}{\partial x_1} - b_1' \frac{\partial \gamma}{\partial x} \end{bmatrix}$$

$$= [gb_0, \; gb_1] \begin{bmatrix} (1 - b_0') \frac{\partial \alpha}{\partial x_1} - b_0' \frac{\partial \gamma}{\partial x_1} \\ -b_1' \frac{\partial \alpha}{\partial x} - b_1' \frac{\partial \gamma}{\partial x} \end{bmatrix}$$

$$= [gb_0, gb_1] \begin{bmatrix} -b_0' \dfrac{\partial \alpha}{\partial x_1} - b_0' \dfrac{\partial \gamma}{\partial x_1} + \dfrac{\partial \alpha}{\partial x_1} \\ -b_1' \dfrac{\partial \alpha}{\partial x_1} - b_1' \dfrac{\partial \gamma}{\partial x_1} \end{bmatrix}$$

$$= -gb_0 b_0' \left(\frac{\partial \alpha}{\partial x_1} + \frac{\partial \gamma}{\partial x_1}\right) + gb_0 \frac{\partial \alpha}{\partial x_1} - gb_1 b_1' \left(\frac{\partial \alpha}{\partial x_1} + \frac{\partial \gamma}{\partial x_1}\right)$$

$$= -(gb_0 b_0' + gb_1 b_1')\left(\frac{\partial \alpha}{\partial x_1} + \frac{\partial \gamma}{\partial x_1}\right) + gb_0 \frac{\partial \alpha}{\partial x_1}$$

$$= -gbb\left(\frac{\partial \alpha}{\partial x_1} + \frac{\partial \gamma}{\partial x_1}\right) + gb_0 \frac{\partial \alpha}{\partial x_1}$$

$$= -gbb(p_2y - p_0y) + gb_0 p_2y$$

$$= gbb(p_0y - p_2y) + gb_0 p_2y$$

$$gp_1y = \frac{\partial L}{\partial y_1} = -gbb\left(\frac{\partial \alpha}{\partial y_1} + \frac{\partial \gamma}{\partial y_1}\right) + gb_0 \frac{\partial \alpha}{\partial x_1}$$

$$= -gbb(-p_2x + p_0x) + gb_0(-p_2x)$$

$$= gbb(p_2x - p_0x) - gb_0 p_2x$$

$$gp_2x = \frac{\partial L}{\partial x_2} = \frac{\partial L}{\partial [u,v]}\frac{\partial [u,v]}{\partial x_2}$$

$$= [dy.x, dy.y]\begin{bmatrix} \frac{\partial b_0'}{\partial \alpha}\frac{\partial \alpha}{\partial x_2} + \frac{\partial b_0'}{\partial \beta}\frac{\partial \beta}{\partial x_2} \\ \frac{\partial b_1'}{\partial \alpha}\frac{\partial \alpha}{\partial x_2} + \frac{\partial b_1'}{\partial \beta}\frac{\partial \beta}{\partial x_2} \end{bmatrix}$$

$$= [dy.x, dy.y]\begin{bmatrix} \frac{\beta+\gamma}{(\alpha+\beta+\gamma)^2}\frac{\partial \alpha}{\partial x_2} + \frac{-\alpha}{(\alpha+\beta+\gamma)^2}\frac{\partial \beta}{\partial x_2} \\ \frac{-\beta}{(\alpha+\beta+\gamma)^2}\frac{\partial \alpha}{\partial x_2} + \frac{\alpha+\gamma}{(\alpha+\beta+\gamma)^2}\frac{\partial \beta}{\partial x_2} \end{bmatrix}$$

$$= [gb_0, gb_1]\begin{bmatrix} (1-b_0')\frac{\partial \alpha}{\partial x_2} - b_0'\frac{\partial \beta}{\partial x_2} \\ -b_1'\frac{\partial \alpha}{\partial x_2} + (1-b_1')\frac{\partial \beta}{\partial x_2} \end{bmatrix}$$

$$= [gb_0, gb_1]\begin{bmatrix} -b_0'\left(\frac{\partial \alpha}{\partial x_2} + \frac{\partial \beta}{\partial x_2}\right) + \frac{\partial \alpha}{\partial x_2} \\ -b_1'\left(\frac{\partial \alpha}{\partial x_2} + \frac{\partial \beta}{\partial x_2}\right) + \frac{\partial \beta}{\partial x_2} \end{bmatrix}$$

$$= -gbb\left(\frac{\partial \alpha}{\partial x_2} + \frac{\partial \beta}{\partial x_2}\right) + gb_0\frac{\partial \alpha}{\partial x_2} + gb_1\frac{\partial \beta}{\partial x_2}$$

$$= -gbb(-p_1y + p_0y) + gb_0(-p_1y) + gb_1 p_0y$$

$$= gbb(p_1y - p_0y) - gb_0 p_1y + gb_1 p_0y$$

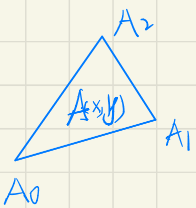
$$g_{\beta_2 y} = \frac{\partial L}{\partial y_2} = \frac{\partial L}{\partial [u,v]}\frac{\partial [u,v]}{\partial y_2}$$

$$= -g_{bb}\left[\frac{\partial \alpha}{\partial y_2} + \frac{\partial \beta}{\partial y_2}\right] + g_{b0}\frac{\partial \alpha}{\partial y_2} + g_{b1}\frac{\partial \beta}{\partial y_2}$$

$$= -g_{bb}[P_{1x} - P_{0x}] + g_{b0}P_{1x} - g_{b1}P_{0x}$$

$$= g_{bb}[P_{0x} - P_{1x}] + g_{b0}P_{1x} - g_{b1}P_{0x}$$

Interpolate 插值函数，可以用于顶点渲染或者计算每个像素的纹理坐标。



考虑像素位置 $p(x,y)$，用A表示某处的属性 A_0, A_1, A_2 表示像素p所在的三角形的3个顶点属性，重心坐标 (u,v) 是Raster在像素位置为 (x,y) 处的结果。由此我们可以得到

$$A = uA_0 + vA_1 + (1-u-v)A_2$$

插值属性对顶点属性的导数

$$\frac{\partial A}{\partial A_0} = u \quad \frac{\partial A}{\partial A_1} = v \quad \frac{\partial A}{\partial A_2} = (1-u-v)$$

插值属性对 $[u,v]$ 的导数

$$\frac{\partial A}{\partial u} = A_0 - A_2 \qquad \frac{\partial A}{\partial v} = A_1 - A_2$$

texture

屏幕坐标为 $[s,\ t]$ 纹理坐标为 $uvA=[u, v]$

$$\frac{d[s,t]}{duvA} = \boxed{\begin{bmatrix} uvdA.x = a & uvdA.y = b \\ uvdA.z = c & uvdA.w = d \end{bmatrix}} \rightarrow uvdA$$

uvdA的特征值是像素位置为 (s,t) 处footprint的大小

为了求 λ_1^2, λ_2^2 的平方 我们可以求 $uvdA^T uvdA$ 的特征值.

$$uvdA^T uvdA = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+ca & b^2+d^2 \end{bmatrix}$$

令 $A = a^2+b^2 \quad B = b^2+d^2 \quad C = ab+cd$

$$\begin{vmatrix} A-\lambda & C \\ C & B-\lambda \end{vmatrix} = (A-\lambda)(B-\lambda)-C^2$$

$$= (\lambda-A)(\lambda-B)-C^2$$

$$= \lambda^2-(A+B)\lambda+AB-C^2$$

$$\lambda = \frac{(A+B) \pm \sqrt{(A+B)^2-4(AB-C^2)}}{2}$$

$$= \frac{A+B \pm \sqrt{(A-B)^2+4C^2}}{2}$$

$$= \frac{A+B}{2} \pm \sqrt{\frac{(A-B)^2}{4}+C^2}$$

令 $l2b = \frac{A+B}{2} \quad l2n = \frac{(A-B)^2}{4}+C^2, \quad l2a = \sqrt{l2n}$

linMinSqr

$\therefore \lambda_{min}^2 = l2b - l2a, \quad \lambda_{max}^2 = l2b + l2a \rightarrow$ lenMajorSqr

$$\therefore \quad level = \log_2 \lambda_{max} = \frac{1}{2} \cdot 2 \log_2 \lambda_{max}$$

$$= \frac{1}{2} \log_2 \lambda_{max}^2$$

$$= \frac{1}{2} \log_2 (lenMajorSqr)$$

$$\frac{\partial\, level}{\partial\, lenMajorSqr} = \frac{1}{2} \frac{1}{\ln 2} \frac{1}{lenMajorSqr} = \frac{1}{2\ln 2} \frac{1}{l2b + l2a}$$

$$\frac{\partial\, l2a}{\partial\, l2n} = \frac{1}{2\sqrt{l2n}} = \frac{1}{2\, l2a}$$

$$\frac{\partial\, l2n}{\partial A} = \frac{1}{2}(A-B) \qquad \frac{\partial\, l2n}{\partial B} = -\frac{1}{2}(A-B) \qquad \frac{\partial\, l2n}{\partial C} = 2C$$

$$\frac{\partial\, l2b}{\partial A} = \frac{1}{2} \qquad \frac{\partial\, l2b}{\partial B} = \frac{1}{2}$$

$$\frac{\partial\, level}{\partial [A,B,C]} = \frac{1}{2\ln 2(l2b + l2a)} \left[\frac{1}{2l2a}\left(l2a + \frac{1}{2}(A-B)\right), \frac{1}{2l2a}\left(l2a + \frac{-1}{2}(A-B)\right), \frac{1}{2l2a} 2C \right]$$

$$= \frac{1}{4\ln 2(l2a\, l2b + l2n)} \left[l2a + \frac{1}{2}(A-B),\ l2a - \frac{1}{2}(A-B),\ 2C \right]$$

$$\frac{\partial [A,B,C]}{\partial [a,b,c,d]} = \begin{bmatrix} 2a & 0 & 2c & 0 \\ & 2b & & 2d \\ b & a & d & c \end{bmatrix}$$

$$\frac{\partial\, level}{\partial [a,b,c,d]} = \frac{\partial\, level}{\partial [A,B,C]} \frac{\partial [A,B,C]}{\partial [a,b,c,d]}$$

$$= \frac{1}{4\ln 2(l2a\, l2b + l2n)} \begin{bmatrix} [l2a + \frac{1}{2}(A-B)]2a + 2bc \\ [l2a - \frac{1}{2}(A-B)]2b + 2ac \\ [l2a + \frac{1}{2}(A-B)]2c + 2dc \\ [l2a - \frac{1}{2}(A-B)]2d + 2 \cdot cc \end{bmatrix}$$

$$= \frac{1}{\underbrace{2\ln 2(l2a\,l2b + l2n)}_{dw}} \begin{bmatrix} [l2a + \frac{1}{2}(A-B)]a + bc \\ [l2a - \frac{1}{2}(A-B)]b + ac \\ [l2a + \frac{1}{2}(A-B)]c + dc \\ [l2a - \frac{1}{2}(A-B)]d + cc \end{bmatrix}$$

$$= \begin{bmatrix} [\underbrace{l2a \cdot dw}_{l2aw} + \underbrace{\frac{1}{2}(A-B) \cdot dw}_{AB}]a + \underbrace{dw \cdot c}_{Cw} \cdot b \\ [l2a \cdot dw - \frac{1}{2}(A-B)dw]b + dw \cdot c \cdot a \\ [l2a \cdot dw + \frac{1}{2}(A-B)dw]c + dw \cdot c \cdot d \\ [l2a \cdot dw - \frac{1}{2}(A-B)dw]d + dw \cdot c \cdot c \end{bmatrix}$$

$$= \begin{bmatrix} (l2aw + AB)a + b \cdot Cw \\ (l2aw - AB) \cdot b + a \cdot Cw \\ (l2aw + AB) \cdot c + d \cdot Cw \\ (l2aw - AB)d + c \cdot Cw \end{bmatrix}$$

Texture对纹理和纹理坐标的导数

$$a = \underbrace{(1-u)(1-v)}_{uvo00} a_{00} + \underbrace{u(1-v)}_{uvo10} a_{10} + \underbrace{(1-u)v}_{uvo01} a_{01} + \underbrace{uv}_{uvo11} a_{11}$$

纹理滤波 a 对纹理 a_{00}, a_{10}, a_{01}, a_{11} 的导数 | 单层线性插值

$$\frac{\partial a}{\partial a_{00}} = wv_{000} \qquad \frac{\partial a}{\partial a_{10}} = wv_{010} \qquad \frac{\partial a}{\partial a_{01}} = wv_{001} \qquad \frac{\partial a}{\partial a_{11}} = wv_{011}$$

渲染纹理值 a 对图像坐标的导数

$$\frac{\partial a}{\partial u} = -(1-v)a_{00} + (1-v)a_{10} - v a_{01} + v a_{11}$$

$$= v\underbrace{(a_{00} - a_{10} - a_{01} + a_{11})}_{ad} + a_{10} - a_{00}$$

$$\frac{\partial a}{\partial v} = -(1-u)a_{00} - u a_{10} + (1-u) a_{01} + u a_{11}$$

$$= u(a_{00} - a_{10} - a_{01} + a_{11}) - a_{00} + a_{01}$$

$$= u\underbrace{(a_{00} + a_{11} - a_{10} - a_{01})}_{ad} + a_{01} - a_{00}$$

总的 Loss 对纹理值，渲染图像的纹理坐标的导数

$$\frac{\partial L}{\partial a_{ij}} = \frac{\partial L}{\partial a} \frac{\partial a}{\partial a_{ij}}$$

$$\frac{\partial L}{\partial [u,v]} = \frac{\partial L}{\partial a} \frac{\partial a}{\partial [u,v]}$$

渲染图像对纹理图像texel和渲染图像坐标的导数 | 3线性插值

level0的双线性插值得到的像素值 a

$$a = {}^0w_{00}a_{00} + {}^0w_{10}a_{10} + {}^0w_{01}a_{01} + {}^0w_{11}a_{11}$$

level1的双线性插值得到的像素值 b

$$b = {}^1w_{00}b_{00} + {}^1w_{10}b_{10} + {}^1w_{01}b_{01} + {}^1w_{11}b_{11}$$

level0与level1双线性插值得到的像素值为 c

$$c = (1-flevel)\cdot a + flevel\cdot b$$

$$\frac{\partial c}{\partial flevel} = b-a$$

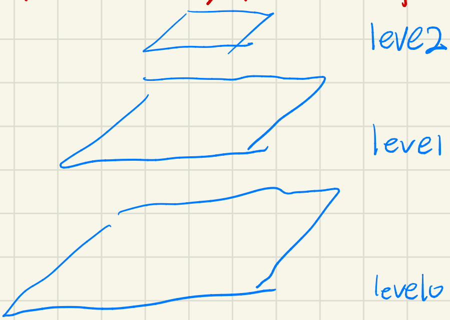
$$flevel = level + level\_bias$$

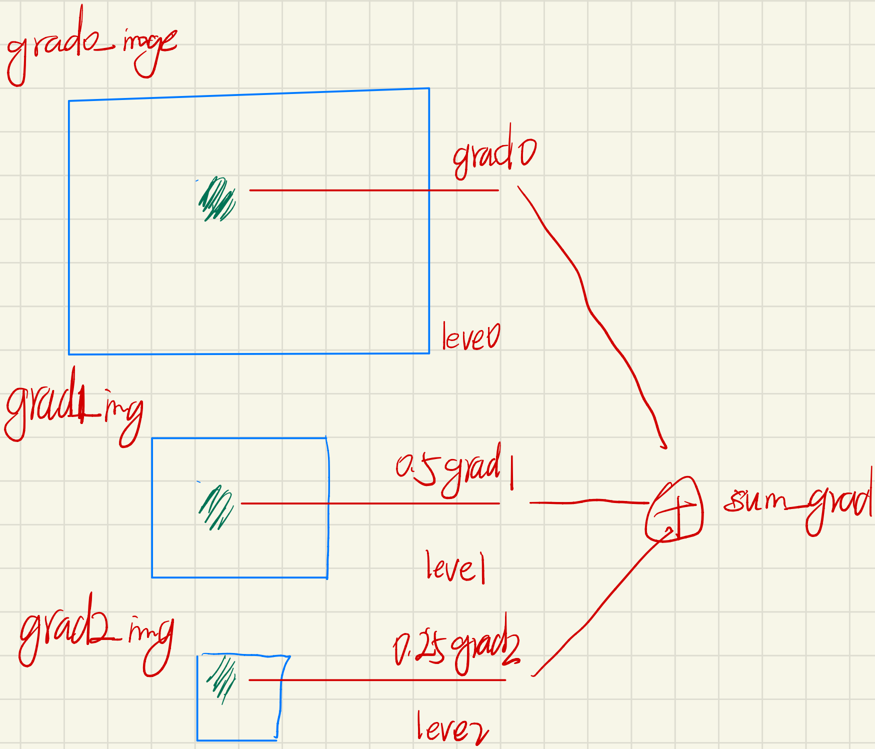
$$\frac{\partial flevel}{\partial level} = 1 \qquad \frac{\partial flevel}{\partial level\_bias} = 1$$

flevel的纹理坐标屏幕空间的导数 uvdA=[a,b,c,d] 的导数为

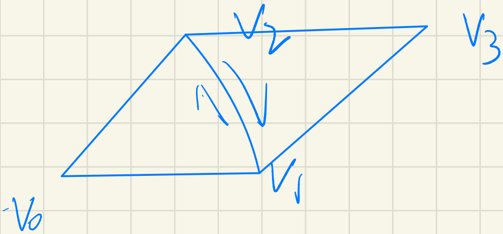
$$\frac{\partial flevel}{\partial [a,b,c,d]} = \frac{\partial flevel}{\partial level}\underbrace{\frac{\partial level}{[a,b,c,d]}}_{dw}$$

对纹理图像的texel的梯度 | 多层Mip-map纹理梯度的汇聚





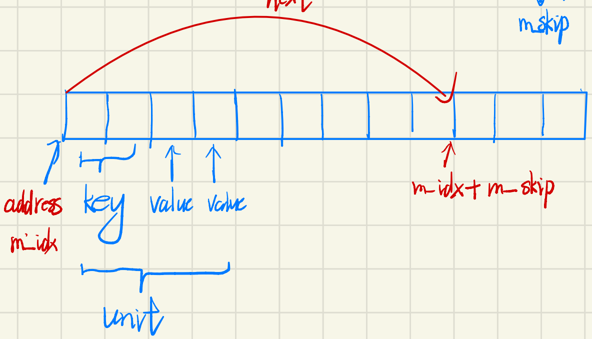
Antialise



Edge Hash Function

Edge ID	Value		key	Value
V_1V_2	V_0	⟶	$V_2 \ll 32 \mid V_1$	V_0
V_2V_0	V_1		$V_2 \ll 32 \mid V_0$	V_1
V_0V_1	V_2		$V_1 \ll 32 \mid V_0$	V_2
V_2V_1	V_3		$V_2 \ll 32 \mid V_1$	V_3

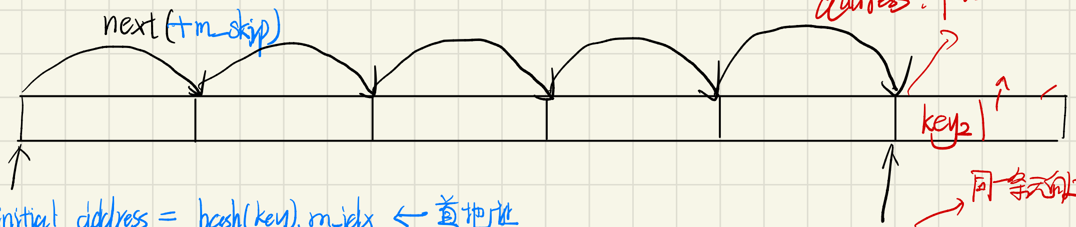
$\underbrace{V_2 \ll 32}_{\text{hash高} \to \text{m\_skip}} \mid \underbrace{V_1}_{\text{hash低} \to \text{m\_idx}}$



如果一条边是 border edge,
那么只有一个 value, 如这条边
是内部边, 那么存储两个 value

i

hash insert (key, value)



initial address = hash(key).m_idx ← 首地址

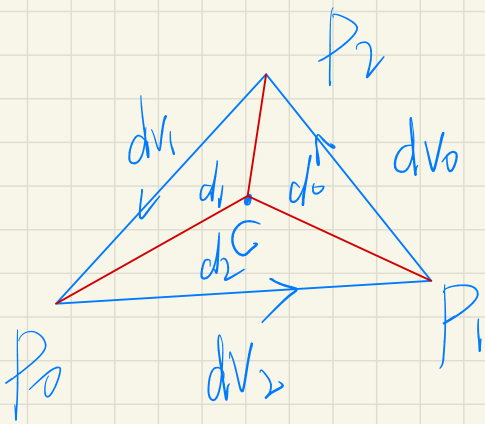
m_skip = hash(key).m_skip ← 步长

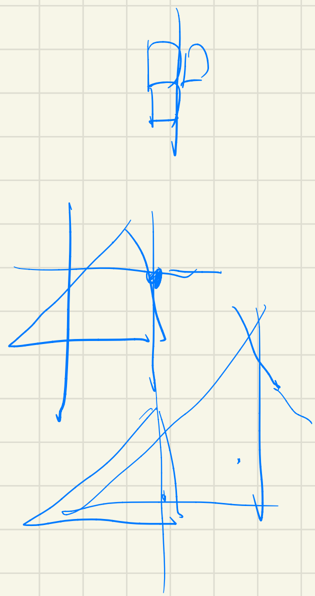
同一条元向址

End cond: key = key_2 or

key_2 == 0 未被使用过

process: if prev[2] == 0:
prev[2] = value
else
prev[3] = value

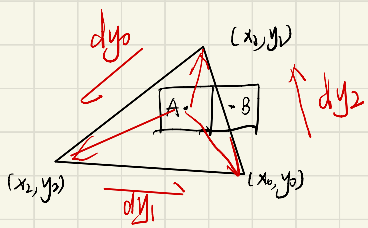


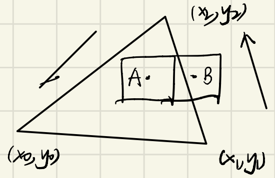


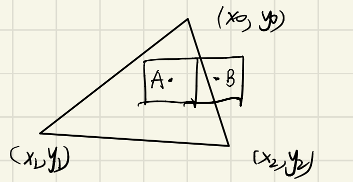
$$d_0 = |\overrightarrow{P_1C} \times \overrightarrow{dV_0}|$$

$$d_1 = |\overrightarrow{P_2C} \times \overrightarrow{dV_1}|$$

$$d_2 = |\overrightarrow{P_0C} \times \overrightarrow{dV_2}|$$







$$x = x_0 + \frac{x_1 - x_0}{y_1 - y_0}(y_{center} - y_0)$$
$$= x_0 + \frac{dx_2}{dy_2}(y_{center} - y_0)$$

$$x = x_1 + \frac{x_2 - x_1}{y_2 - y_1}(y_{center} - y_1)$$
$$= x_1 + \frac{dx_0}{dy_0}(y_{center} - y_1)$$

$$x = x_2 + \frac{x_0 - x_2}{y_0 - y_2}(y_{center} - y_2)$$
$$= x_2 + \frac{dx_1}{dy_1}(y_{center} - y_2)$$

我们将坐标系建立在pixel A的中心点，上述的结果可以简化为

$$x = x_0 - \frac{dx_2}{dy_2} y_0$$
$$x = \frac{x_0 dy_2 - dx_2 y_0}{dy_2}$$
$$= \frac{d_2}{dy_2}$$

$$x = x_1 - \frac{dx_0}{dy_0} y_1$$
$$x = \frac{x_1 dy_0 - dx_0 y_1}{dy_0}$$
$$= \frac{d_0}{dy_0}$$

$$x = x_2 - \frac{dx_1}{dy_1} y_2$$
$$x = \frac{x_2 dy_1 - dx_1 y_2}{dy_1}$$
$$= \frac{d_1}{dy_1}$$

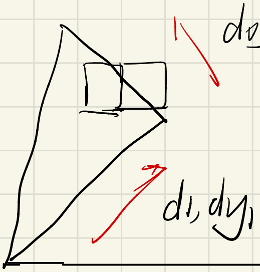
如何判断像素AB靠近哪个边界呢？

如果 dy_0 与 dy_1 异号，则靠近 dy_0 这条边

Examp: $d_0 = 2.0$ $dy_0 = 3.0$ $d_1 = 2.0$ $dy_1 = -3.0$

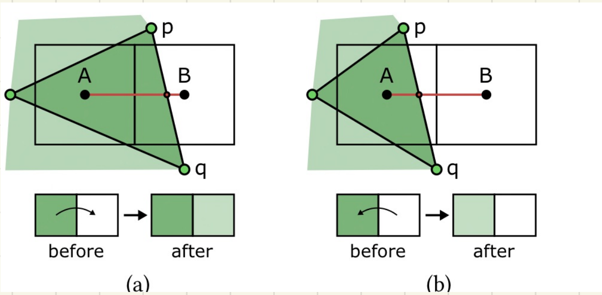
$d_0 dy_1 < d_1 dy_0$

所以靠近 dy_0 这条边



如果 dy_0 dy_1 同号，则 $\frac{d_0}{dy_0}$ 越大，AB的~~交点越靠近~~

由于 d_1 的两个顶点在 pixel A B的同侧，我们将 d_1 设置为 -INF，则 $d_0 dy_1 > d_1 dy_0$，表示离 d_0 近



(a) (b)

pq与AB的交点的 x轴坐标的取值范围为 $\overset{dc}{[0, 1)}$ 坐标轴原点在 pixel A的中心点，blend的系数 alpha

$\alpha = 0.5 - dc \quad [-0.5, 0.5]$

如果 $\alpha > 0$，如图(b)所示，则

increase pixel_B的比例

$$pixel\_A\_new = pixel\_A\_old + \alpha(pixel\_B\_old - pixel\_A\_old)$$

如果 $\alpha < 0$，如果(a)所示，则

$$pixel\_B\_new = pixel\_B\_old + \alpha(pixel\_B\_old - pixel\_A\_old)$$

increase pixel_A的比例

Backward:

$$\frac{\partial pixel\_A\_new}{\partial pixel\_A\_old} = 1-\alpha$$

$$\frac{\partial pixel\_A\_new}{\partial pixel\_B\_old} = \alpha$$

$$\frac{\partial pixel\_B\_new}{\partial pixel\_B\_old} = 1+\alpha$$

$$\frac{\partial pixel\_B\_old}{\partial pixel\_A\_old} = -\alpha$$

Antialis图像对顶点位置的导数

$$\begin{cases} \alpha = 0.5 - dc \\ x = x_1 + \frac{x_2 - x_1}{y_2 - y_1}(-y_1) \\ \quad = x_1 + \frac{dx}{dy}(-y_1) = \frac{x_1 dy - y_1 dx}{dy} \end{cases}$$

$$\frac{\partial x}{\partial x_1} = 1 + \frac{-y_1}{y_2 - y_1}(-1) = 1 + \frac{y_1}{y_2 - y_1} = \frac{y_2}{y_2 - y_1} = \frac{y_2}{dy} = y_2 \cdot iy$$

$$\frac{\partial x}{\partial x_2} = \frac{-y_1}{y_2 - y_1} = -y_1 \cdot iy$$

$$\frac{\partial x}{\partial y_1} = -\frac{dx}{dy} + \frac{dx \cdot y_1}{dy^2}(-1) = \frac{-(dx\,dy + dx\,y_1)}{dy^2}$$

$$= \frac{-y_1 dx + x_1 dy - x_1 dy - dx\,dy}{dy^2}$$

$$= \frac{1}{dy}\left[\frac{db}{dy} - x_1 - (x_2 - x_1)\right]$$

$$= iy\left[db \cdot iy - x_2\right]$$

$$\frac{\partial x}{\partial y_2} = \frac{dx\,y_1}{dy^2} = \frac{-(x_1 dy - y_1 dx) + x_1 dy}{dy^2}$$

$$= \frac{1}{dy}\left[-\frac{db}{dy} + x_1\right]$$

$$= -iy\left[db \cdot iy - x_1\right]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{width \cdot X}{W} \\ \frac{height \cdot Y}{W} \end{bmatrix} \qquad \frac{\partial [x, y]}{\partial [X, Y, W]} = \begin{bmatrix} \frac{width}{W} & 0 & -\frac{width}{W^2} \\ & \frac{height}{W} & -\frac{height}{W^2} \end{bmatrix}$$

$$\frac{\partial x}{\partial [x,y]} \frac{\partial [x,y]}{\partial [X,Y,W]} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \frac{width}{W} & 0 & \frac{-width X}{W^2} \\ 0 & \frac{height}{W} & \frac{-height Y}{W^2} \end{bmatrix}$$

$$= \begin{bmatrix} a\frac{width}{W} & \frac{b\,height}{W} & -\frac{a\,width X}{W^2} - \frac{b\,height Y}{W^2} \end{bmatrix}$$

$$\Uparrow \qquad \Uparrow \qquad \Uparrow$$

$$\begin{bmatrix} g_x & g_y & -(x \cdot g_x + y g_y) \end{bmatrix}$$