

## CHAPTER 4

# RATIONAL FUNCTIONS

### 4.1 INTRODUCTION TO RATIONAL FUNCTIONS

If we add, subtract or multiply polynomial functions according to the function arithmetic rules defined in Section 1.5, we will produce another polynomial function. If, on the other hand, we divide two polynomial functions, the result may not be a polynomial. In this chapter we study **rational functions** - functions which are ratios of polynomials.

**Definition 4.1.** A **rational function** is a function which is the ratio of polynomial functions. Said differently,  $r$  is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where  $p$  and  $q$  are polynomial functions.<sup>a</sup>

<sup>a</sup>According to this definition, all polynomial functions are also rational functions. (Take  $q(x) = 1$ ).

As we recall from Section 1.4, we have domain issues anytime the denominator of a fraction is zero. In the example below, we review this concept as well as some of the arithmetic of rational expressions.

**Example 4.1.1.** Find the domain of the following rational functions. Write them in the form  $\frac{p(x)}{q(x)}$  for polynomial functions  $p$  and  $q$  and simplify.

1.  $f(x) = \frac{2x - 1}{x + 1}$

2.  $g(x) = 2 - \frac{3}{x + 1}$

3.  $h(x) = \frac{2x^2 - 1}{x^2 - 1} - \frac{3x - 2}{x^2 - 1}$

4.  $r(x) = \frac{2x^2 - 1}{x^2 - 1} \div \frac{3x - 2}{x^2 - 1}$

**Solution.**

1. To find the domain of  $f$ , we proceed as we did in Section 1.4: we find the zeros of the denominator and exclude them from the domain. Setting  $x + 1 = 0$  results in  $x = -1$ . Hence,

our domain is  $(-\infty, -1) \cup (-1, \infty)$ . The expression  $f(x)$  is already in the form requested and when we check for common factors among the numerator and denominator we find none, so we are done.

2. Proceeding as before, we determine the domain of  $g$  by solving  $x + 1 = 0$ . As before, we find the domain of  $g$  is  $(-\infty, -1) \cup (-1, \infty)$ . To write  $g(x)$  in the form requested, we need to get a common denominator

$$\begin{aligned} g(x) &= 2 - \frac{3}{x+1} = \frac{2}{1} - \frac{3}{x+1} = \frac{(2)(x+1)}{(1)(x+1)} - \frac{3}{x+1} \\ &= \frac{(2x+2)-3}{x+1} = \frac{2x-1}{x+1} \end{aligned}$$

This formula is now completely simplified.

3. The denominators in the formula for  $h(x)$  are both  $x^2 - 1$  whose zeros are  $x = \pm 1$ . As a result, the domain of  $h$  is  $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ . We now proceed to simplify  $h(x)$ . Since we have the same denominator in both terms, we subtract the numerators. We then factor the resulting numerator and denominator, and cancel out the common factor.

$$\begin{aligned} h(x) &= \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1} = \frac{(2x^2-1)-(3x-2)}{x^2-1} \\ &= \frac{2x^2-1-3x+2}{x^2-1} = \frac{2x^2-3x+1}{x^2-1} \\ &= \frac{(2x-1)(x-1)}{(x+1)(x-1)} = \frac{(2x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

4. To find the domain of  $r$ , it may help to temporarily rewrite  $r(x)$  as

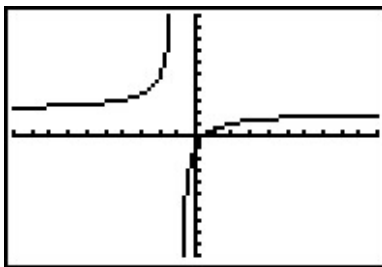
$$r(x) = \frac{\frac{2x^2-1}{x^2-1}}{\frac{3x-2}{x^2-1}}$$

We need to set all of the denominators equal to zero which means we need to solve not only  $x^2 - 1 = 0$ , but also  $\frac{3x-2}{x^2-1} = 0$ . We find  $x = \pm 1$  for the former and  $x = \frac{2}{3}$  for the latter. Our domain is  $(-\infty, -1) \cup (-1, \frac{2}{3}) \cup (\frac{2}{3}, 1) \cup (1, \infty)$ . We simplify  $r(x)$  by rewriting the division as multiplication by the reciprocal and then by canceling the common factor

$$\begin{aligned}
 r(x) &= \frac{2x^2 - 1}{x^2 - 1} \div \frac{3x - 2}{x^2 - 1} = \frac{2x^2 - 1}{x^2 - 1} \cdot \frac{x^2 - 1}{3x - 2} = \frac{(2x^2 - 1)(x^2 - 1)}{(x^2 - 1)(3x - 2)} \\
 &= \frac{(2x^2 - 1)\cancel{(x^2 - 1)}}{\cancel{(x^2 - 1)}(3x - 2)} = \frac{2x^2 - 1}{3x - 2} \quad \square
 \end{aligned}$$

A few remarks about Example 4.1.1 are in order. Note that the expressions for  $f(x)$ ,  $g(x)$  and  $h(x)$  work out to be the same. However, only two of these functions are actually equal. Recall that functions are ultimately sets of ordered pairs,<sup>1</sup> so for two functions to be equal, they need, among other things, to have the same domain. Since  $f(x) = g(x)$  and  $f$  and  $g$  have the same domain, they are equal functions. Even though the formula  $h(x)$  is the same as  $f(x)$ , the domain of  $h$  is different than the domain of  $f$ , and thus they are different functions.

We now turn our attention to the graphs of rational functions. Consider the function  $f(x) = \frac{2x-1}{x+1}$  from Example 4.1.1. Using a graphing calculator, we obtain



Two behaviors of the graph are worthy of further discussion. First, note that the graph appears to ‘break’ at  $x = -1$ . We know from our last example that  $x = -1$  is not in the domain of  $f$  which means  $f(-1)$  is undefined. When we make a table of values to study the behavior of  $f$  near  $x = -1$  we see that we can get ‘near’  $x = -1$  from two directions. We can choose values a little less than  $-1$ , for example  $x = -1.1$ ,  $x = -1.01$ ,  $x = -1.001$ , and so on. These values are said to ‘approach  $-1$  from the left.’ Similarly, the values  $x = -0.9$ ,  $x = -0.99$ ,  $x = -0.999$ , etc., are said to ‘approach  $-1$  from the right.’ If we make two tables, we find that the numerical results confirm what we see graphically.

$x$	$f(x)$	$(x, f(x))$
$-1.1$	32	$(-1.1, 32)$
$-1.01$	302	$(-1.01, 302)$
$-1.001$	3002	$(-1.001, 3002)$
$-1.0001$	30002	$(-1.001, 30002)$

$x$	$f(x)$	$(x, f(x))$
$-0.9$	-28	$(-0.9, -28)$
$-0.99$	-298	$(-0.99, -298)$
$-0.999$	-2998	$(-0.999, -2998)$
$-0.9999$	-29998	$(-0.9999, -29998)$

As the  $x$  values approach  $-1$  from the left, the function values become larger and larger positive numbers.<sup>2</sup> We express this symbolically by stating as  $x \rightarrow -1^-$ ,  $f(x) \rightarrow \infty$ . Similarly, using analogous notation, we conclude from the table that as  $x \rightarrow -1^+$ ,  $f(x) \rightarrow -\infty$ . For this type of

<sup>1</sup>You should review Sections 1.2 and 1.3 if this statement caught you off guard.

<sup>2</sup>We would need Calculus to confirm this analytically.

unbounded behavior, we say the graph of  $y = f(x)$  has a **vertical asymptote** of  $x = -1$ . Roughly speaking, this means that near  $x = -1$ , the graph looks very much like the vertical line  $x = -1$ .

The other feature worthy of note about the graph of  $y = f(x)$  is that it seems to ‘level off’ on the left and right hand sides of the screen. This is a statement about the end behavior of the function. As we discussed in Section 3.1, the end behavior of a function is its behavior as  $x$  attains larger<sup>3</sup> and larger negative values without bound,  $x \rightarrow -\infty$ , and as  $x$  becomes large without bound,  $x \rightarrow \infty$ . Making tables of values, we find

$x$	$f(x)$	$(x, f(x))$
-10	$\approx 2.3333$	$\approx (-10, 2.3333)$
-100	$\approx 2.0303$	$\approx (-100, 2.0303)$
-1000	$\approx 2.0030$	$\approx (-1000, 2.0030)$
-10000	$\approx 2.0003$	$\approx (-10000, 2.0003)$

$x$	$f(x)$	$(x, f(x))$
10	$\approx 1.7273$	$\approx (10, 1.7273)$
100	$\approx 1.9703$	$\approx (100, 1.9703)$
1000	$\approx 1.9970$	$\approx (1000, 1.9970)$
10000	$\approx 1.9997$	$\approx (10000, 1.9997)$

From the tables, we see that as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 2^+$  and as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 2^-$ . Here the ‘+’ means ‘from above’ and the ‘-’ means ‘from below’. In this case, we say the graph of  $y = f(x)$  has a **horizontal asymptote** of  $y = 2$ . This means that the end behavior of  $f$  resembles the horizontal line  $y = 2$ , which explains the ‘leveling off’ behavior we see in the calculator’s graph. We formalize the concepts of vertical and horizontal asymptotes in the following definitions.

**Definition 4.2.** The line  $x = c$  is called a **vertical asymptote** of the graph of a function  $y = f(x)$  if as  $x \rightarrow c^-$  or as  $x \rightarrow c^+$ , either  $f(x) \rightarrow \infty$  or  $f(x) \rightarrow -\infty$ .

**Definition 4.3.** The line  $y = c$  is called a **horizontal asymptote** of the graph of a function  $y = f(x)$  if as  $x \rightarrow -\infty$  or as  $x \rightarrow \infty$ ,  $f(x) \rightarrow c$ .

Note that in Definition 4.3, we write  $f(x) \rightarrow c$  (not  $f(x) \rightarrow c^+$  or  $f(x) \rightarrow c^-$ ) because we are unconcerned from which direction the values  $f(x)$  approach the value  $c$ , just as long as they do so.<sup>4</sup> In our discussion following Example 4.1.1, we determined that, despite the fact that the formula for  $h(x)$  reduced to the same formula as  $f(x)$ , the functions  $f$  and  $h$  are different, since  $x = 1$  is in the domain of  $f$ , but  $x = 1$  is not in the domain of  $h$ . If we graph  $h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1}$  using a graphing calculator, we are surprised to find that the graph looks identical to the graph of  $y = f(x)$ . There is a vertical asymptote at  $x = -1$ , but near  $x = 1$ , everything seem fine. Tables of values provide numerical evidence which supports the graphical observation.

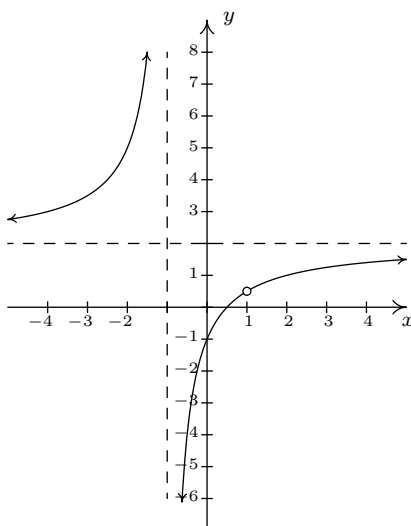
<sup>3</sup>Here, the word ‘larger’ means larger in absolute value.

<sup>4</sup>As we shall see in the next section, the graphs of rational functions may, in fact, *cross* their horizontal asymptotes. If this happens, however, it does so only a *finite* number of times, and so for each choice of  $x \rightarrow -\infty$  and  $x \rightarrow \infty$ ,  $f(x)$  will approach  $c$  from either below (in the case  $f(x) \rightarrow c^-$ ) or above (in the case  $f(x) \rightarrow c^+$ .) We leave  $f(x) \rightarrow c$  generic in our definition, however, to allow this concept to apply to less tame specimens in the Precalculus zoo, such as Exercise 50 in Section 10.5.

$x$	$h(x)$	$(x, h(x))$
0.9	$\approx 0.4210$	$\approx (0.9, 0.4210)$
0.99	$\approx 0.4925$	$\approx (0.99, 0.4925)$
0.999	$\approx 0.4992$	$\approx (0.999, 0.4992)$
0.9999	$\approx 0.4999$	$\approx (0.9999, 0.4999)$

$x$	$h(x)$	$(x, h(x))$
1.1	$\approx 0.5714$	$\approx (1.1, 0.5714)$
1.01	$\approx 0.5075$	$\approx (1.01, 0.5075)$
1.001	$\approx 0.5007$	$\approx (1.001, 0.5007)$
1.0001	$\approx 0.5001$	$\approx (1.0001, 0.5001)$

We see that as  $x \rightarrow 1^-$ ,  $h(x) \rightarrow 0.5^-$  and as  $x \rightarrow 1^+$ ,  $h(x) \rightarrow 0.5^+$ . In other words, the points on the graph of  $y = h(x)$  are approaching  $(1, 0.5)$ , but since  $x = 1$  is not in the domain of  $h$ , it would be inaccurate to fill in a point at  $(1, 0.5)$ . As we've done in past sections when something like this occurs,<sup>5</sup> we put an open circle (also called a **hole** in this case<sup>6</sup>) at  $(1, 0.5)$ . Below is a detailed graph of  $y = h(x)$ , with the vertical and horizontal asymptotes as dashed lines.



Neither  $x = -1$  nor  $x = 1$  are in the domain of  $h$ , yet the behavior of the graph of  $y = h(x)$  is drastically different near these  $x$ -values. The reason for this lies in the second to last step when we simplified the formula for  $h(x)$  in Example 4.1.1, where we had  $h(x) = \frac{(2x-1)(x-1)}{(x+1)(x-1)}$ . The reason  $x = -1$  is not in the domain of  $h$  is because the factor  $(x + 1)$  appears in the denominator of  $h(x)$ ; similarly,  $x = 1$  is not in the domain of  $h$  because of the factor  $(x - 1)$  in the denominator of  $h(x)$ . The major difference between these two factors is that  $(x - 1)$  cancels with a factor in the numerator whereas  $(x + 1)$  does not. Loosely speaking, the trouble caused by  $(x - 1)$  in the denominator is canceled away while the factor  $(x + 1)$  remains to cause mischief. This is why the graph of  $y = h(x)$  has a vertical asymptote at  $x = -1$  but only a hole at  $x = 1$ . These observations are generalized and summarized in the theorem below, whose proof is found in Calculus.

<sup>5</sup>For instance, graphing piecewise defined functions in Section 1.6.

<sup>6</sup>In Calculus, we will see how these ‘holes’ can be ‘plugged’ when embarking on a more advanced study of continuity.

**Theorem 4.1. Location of Vertical Asymptotes and Holes:**<sup>a</sup> Suppose  $r$  is a rational function which can be written as  $r(x) = \frac{p(x)}{q(x)}$  where  $p$  and  $q$  have no common zeros.<sup>b</sup> Let  $c$  be a real number which is not in the domain of  $r$ .

- If  $q(c) \neq 0$ , then the graph of  $y = r(x)$  has a hole at  $\left(c, \frac{p(c)}{q(c)}\right)$ .
- If  $q(c) = 0$ , then the line  $x = c$  is a vertical asymptote of the graph of  $y = r(x)$ .

<sup>a</sup>Or, ‘How to tell your asymptote from a hole in the graph.’

<sup>b</sup>In other words,  $r(x)$  is in lowest terms.

In English, Theorem 4.1 says that if  $x = c$  is not in the domain of  $r$  but, when we simplify  $r(x)$ , it no longer makes the denominator 0, then we have a hole at  $x = c$ . Otherwise, the line  $x = c$  is a vertical asymptote of the graph of  $y = r(x)$ .

**Example 4.1.2.** Find the vertical asymptotes of, and/or holes in, the graphs of the following rational functions. Verify your answers using a graphing calculator, and describe the behavior of the graph near them using proper notation.

$$1. f(x) = \frac{2x}{x^2 - 3}$$

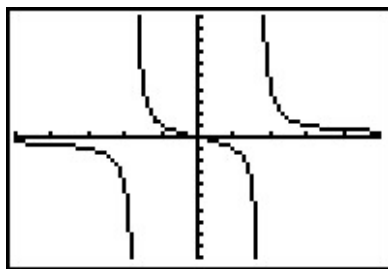
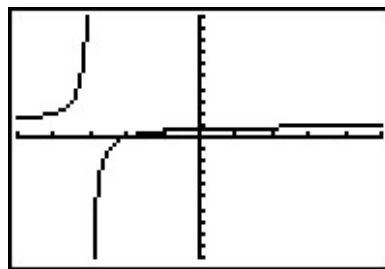
$$2. g(x) = \frac{x^2 - x - 6}{x^2 - 9}$$

$$3. h(x) = \frac{x^2 - x - 6}{x^2 + 9}$$

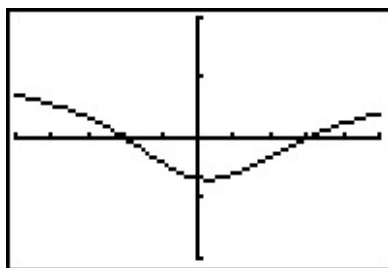
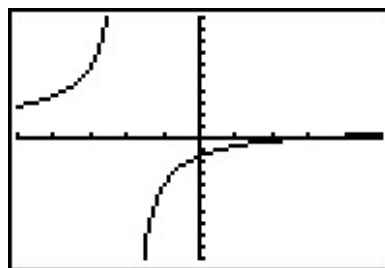
$$4. r(x) = \frac{x^2 - x - 6}{x^2 + 4x + 4}$$

**Solution.**

1. To use Theorem 4.1, we first find all of the real numbers which aren't in the domain of  $f$ . To do so, we solve  $x^2 - 3 = 0$  and get  $x = \pm\sqrt{3}$ . Since the expression  $f(x)$  is in lowest terms, there is no cancellation possible, and we conclude that the lines  $x = -\sqrt{3}$  and  $x = \sqrt{3}$  are vertical asymptotes to the graph of  $y = f(x)$ . The calculator verifies this claim, and from the graph, we see that as  $x \rightarrow -\sqrt{3}^-$ ,  $f(x) \rightarrow -\infty$ , as  $x \rightarrow -\sqrt{3}^+$ ,  $f(x) \rightarrow \infty$ , as  $x \rightarrow \sqrt{3}^-$ ,  $f(x) \rightarrow -\infty$ , and finally as  $x \rightarrow \sqrt{3}^+$ ,  $f(x) \rightarrow \infty$ .
2. Solving  $x^2 - 9 = 0$  gives  $x = \pm 3$ . In lowest terms  $g(x) = \frac{x^2 - x - 6}{x^2 - 9} = \frac{(x-3)(x+2)}{(x-3)(x+3)} = \frac{x+2}{x+3}$ . Since  $x = -3$  continues to make trouble in the denominator, we know the line  $x = -3$  is a vertical asymptote of the graph of  $y = g(x)$ . Since  $x = 3$  no longer produces a 0 in the denominator, we have a hole at  $x = 3$ . To find the  $y$ -coordinate of the hole, we substitute  $x = 3$  into  $\frac{x+2}{x+3}$  and find the hole is at  $\left(3, \frac{5}{6}\right)$ . When we graph  $y = g(x)$  using a calculator, we clearly see the vertical asymptote at  $x = -3$ , but everything seems calm near  $x = 3$ . Hence, as  $x \rightarrow -3^-$ ,  $g(x) \rightarrow \infty$ , as  $x \rightarrow -3^+$ ,  $g(x) \rightarrow -\infty$ , as  $x \rightarrow 3^-$ ,  $g(x) \rightarrow \frac{5}{6}^-$ , and as  $x \rightarrow 3^+$ ,  $g(x) \rightarrow \frac{5}{6}^+$ .

The graph of  $y = f(x)$ The graph of  $y = g(x)$ 

3. The domain of  $h$  is all real numbers, since  $x^2 + 9 = 0$  has no real solutions. Accordingly, the graph of  $y = h(x)$  is devoid of both vertical asymptotes and holes.
4. Setting  $x^2 + 4x + 4 = 0$  gives us  $x = -2$  as the only real number of concern. Simplifying, we see  $r(x) = \frac{x^2 - x - 6}{x^2 + 4x + 4} = \frac{(x-3)(x+2)}{(x+2)^2} = \frac{x-3}{x+2}$ . Since  $x = -2$  continues to produce a 0 in the denominator of the reduced function, we know  $x = -2$  is a vertical asymptote to the graph. The calculator bears this out, and, moreover, we see that as  $x \rightarrow -2^-$ ,  $r(x) \rightarrow \infty$  and as  $x \rightarrow -2^+$ ,  $r(x) \rightarrow -\infty$ .

The graph of  $y = h(x)$ The graph of  $y = r(x)$ 

□

Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named ‘doomsday’ equations.<sup>7</sup>

**Example 4.1.3.** A mathematical model for the population  $P$ , in thousands, of a certain species of bacteria,  $t$  days after it is introduced to an environment is given by  $P(t) = \frac{100}{(5-t)^2}$ ,  $0 \leq t < 5$ .

1. Find and interpret  $P(0)$ .
2. When will the population reach 100,000?
3. Determine the behavior of  $P$  as  $t \rightarrow 5^-$ . Interpret this result graphically and within the context of the problem.

<sup>7</sup>These functions arise in Differential Equations. The unfortunate name will make sense shortly.

**Solution.**

1. Substituting  $t = 0$  gives  $P(0) = \frac{100}{(5-0)^2} = 4$ , which means 4000 bacteria are initially introduced into the environment.
2. To find when the population reaches 100,000, we first need to remember that  $P(t)$  is measured in *thousands*. In other words, 100,000 bacteria corresponds to  $P(t) = 100$ . Substituting for  $P(t)$  gives the equation  $\frac{100}{(5-t)^2} = 100$ . Clearing denominators and dividing by 100 gives  $(5-t)^2 = 1$ , which, after extracting square roots, produces  $t = 4$  or  $t = 6$ . Of these two solutions, only  $t = 4$  is in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.
3. To determine the behavior of  $P$  as  $t \rightarrow 5^-$ , we can make a table

$t$	$P(t)$
4.9	10000
4.99	1000000
4.999	100000000
4.9999	10000000000

In other words, as  $t \rightarrow 5^-$ ,  $P(t) \rightarrow \infty$ . Graphically, the line  $t = 5$  is a vertical asymptote of the graph of  $y = P(t)$ . Physically, this means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason,  $t = 5$  is called the ‘doomsday’ for this population. There is no way any environment can support infinitely many bacteria, so shortly before  $t = 5$  the environment would collapse.  $\square$

Now that we have thoroughly investigated vertical asymptotes, we can turn our attention to horizontal asymptotes. The next theorem tells us when to expect horizontal asymptotes.

**Theorem 4.2. Location of Horizontal Asymptotes:** Suppose  $r$  is a rational function and  $r(x) = \frac{p(x)}{q(x)}$ , where  $p$  and  $q$  are polynomial functions with leading coefficients  $a$  and  $b$ , respectively.

- If the degree of  $p(x)$  is the same as the degree of  $q(x)$ , then  $y = \frac{a}{b}$  is the<sup>a</sup> horizontal asymptote of the graph of  $y = r(x)$ .
- If the degree of  $p(x)$  is less than the degree of  $q(x)$ , then  $y = 0$  is the horizontal asymptote of the graph of  $y = r(x)$ .
- If the degree of  $p(x)$  is greater than the degree of  $q(x)$ , then the graph of  $y = r(x)$  has no horizontal asymptotes.

<sup>a</sup>The use of the definite article will be justified momentarily.

Like Theorem 4.1, Theorem 4.2 is proved using Calculus. Nevertheless, we can understand the idea behind it using our example  $f(x) = \frac{2x-1}{x+1}$ . If we interpret  $f(x)$  as a division problem,  $(2x-1) \div (x+1)$ ,



we find that the quotient is 2 with a remainder of  $-3$ . Using what we know about polynomial division, specifically Theorem 3.4, we get  $2x - 1 = 2(x + 1) - 3$ . Dividing both sides by  $(x + 1)$  gives  $\frac{2x-1}{x+1} = 2 - \frac{3}{x+1}$ . (You may remember this as the formula for  $g(x)$  in Example 4.1.1.) As  $x$  becomes unbounded in either direction, the quantity  $\frac{3}{x+1}$  gets closer and closer to 0 so that the values of  $f(x)$  become closer and closer<sup>8</sup> to 2. In symbols, as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 2$ , and we have the result.<sup>9</sup> Notice that the graph gets close to the same  $y$  value as  $x \rightarrow -\infty$  or  $x \rightarrow \infty$ . This means that the graph can have only one horizontal asymptote if it is going to have one at all. Thus we were justified in using ‘the’ in the previous theorem.

Alternatively, we can use what we know about end behavior of polynomials to help us understand this theorem. From Theorem 3.2, we know the end behavior of a polynomial is determined by its leading term. Applying this to the numerator and denominator of  $f(x)$ , we get that as  $x \rightarrow \pm\infty$ ,  $f(x) = \frac{2x-1}{x+1} \approx \frac{2x}{x} = 2$ . This last approach is useful in Calculus, and, indeed, is made rigorous there. (Keep this in mind for the remainder of this paragraph.) Applying this reasoning to the general case, suppose  $r(x) = \frac{p(x)}{q(x)}$  where  $a$  is the leading coefficient of  $p(x)$  and  $b$  is the leading coefficient of  $q(x)$ . As  $x \rightarrow \pm\infty$ ,  $r(x) \approx \frac{ax^n}{bx^m}$ , where  $n$  and  $m$  are the degrees of  $p(x)$  and  $q(x)$ , respectively. If the degree of  $p(x)$  and the degree of  $q(x)$  are the same, then  $n = m$  so that  $r(x) \approx \frac{a}{b}$ , which means  $y = \frac{a}{b}$  is the horizontal asymptote in this case. If the degree of  $p(x)$  is less than the degree of  $q(x)$ , then  $n < m$ , so  $m - n$  is a positive number, and hence,  $r(x) \approx \frac{a}{bx^{m-n}} \rightarrow 0$  as  $x \rightarrow \pm\infty$ . If the degree of  $p(x)$  is greater than the degree of  $q(x)$ , then  $n > m$ , and hence  $n - m$  is a positive number and  $r(x) \approx \frac{ax^{n-m}}{b}$ , which becomes unbounded as  $x \rightarrow \pm\infty$ . As we said before, if a rational function has a horizontal asymptote, then it will have only one. (This is not true for other types of functions we shall see in later chapters.)

**Example 4.1.4.** List the horizontal asymptotes, if any, of the graphs of the following functions. Verify your answers using a graphing calculator, and describe the behavior of the graph near them using proper notation.

$$1. f(x) = \frac{5x}{x^2 + 1} \qquad 2. g(x) = \frac{x^2 - 4}{x + 1} \qquad 3. h(x) = \frac{6x^3 - 3x + 1}{5 - 2x^3}$$

**Solution.**

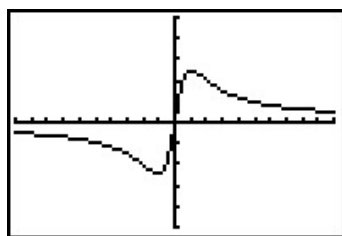
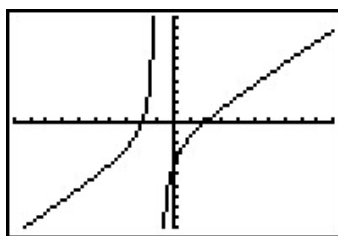
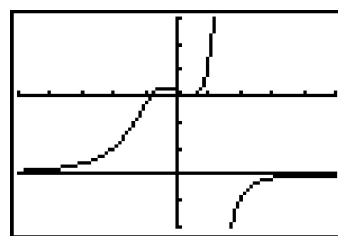
1. The numerator of  $f(x)$  is  $5x$ , which has degree 1. The denominator of  $f(x)$  is  $x^2 + 1$ , which has degree 2. Applying Theorem 4.2,  $y = 0$  is the horizontal asymptote. Sure enough, we see from the graph that as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$  and as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ .
2. The numerator of  $g(x)$ ,  $x^2 - 4$ , has degree 2, but the degree of the denominator,  $x + 1$ , has degree 1. By Theorem 4.2, there is no horizontal asymptote. From the graph, we see that the graph of  $y = g(x)$  doesn't appear to level off to a constant value, so there is no horizontal asymptote.<sup>10</sup>

<sup>8</sup>As seen in the tables immediately preceding Definition 4.2.

<sup>9</sup>More specifically, as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 2^+$ , and as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 2^-$ .

<sup>10</sup>Sit tight! We'll revisit this function and its end behavior shortly.

3. The degrees of the numerator and denominator of  $h(x)$  are both three, so Theorem 4.2 tells us  $y = \frac{6}{-2} = -3$  is the horizontal asymptote. We see from the calculator's graph that as  $x \rightarrow -\infty$ ,  $h(x) \rightarrow -3^+$ , and as  $x \rightarrow \infty$ ,  $h(x) \rightarrow -3^-$ .

The graph of  $y = f(x)$ The graph of  $y = g(x)$ The graph of  $y = h(x)$ 

□

Our next example of the section gives us a real-world application of a horizontal asymptote.<sup>11</sup>

**Example 4.1.5.** The number of students  $N$  at local college who have had the flu  $t$  months after the semester begins can be modeled by the formula  $N(t) = 500 - \frac{450}{1+3t}$  for  $t \geq 0$ .

1. Find and interpret  $N(0)$ .
2. How long will it take until 300 students will have had the flu?
3. Determine the behavior of  $N$  as  $t \rightarrow \infty$ . Interpret this result graphically and within the context of the problem.

**Solution.**

1.  $N(0) = 500 - \frac{450}{1+3(0)} = 50$ . This means that at the beginning of the semester, 50 students have had the flu.
2. We set  $N(t) = 300$  to get  $500 - \frac{450}{1+3t} = 300$  and solve. Isolating the fraction gives  $\frac{450}{1+3t} = 200$ . Clearing denominators gives  $450 = 200(1+3t)$ . Finally, we get  $t = \frac{5}{12}$ . This means it will take  $\frac{5}{12}$  months, or about 13 days, for 300 students to have had the flu.
3. To determine the behavior of  $N$  as  $t \rightarrow \infty$ , we can use a table.

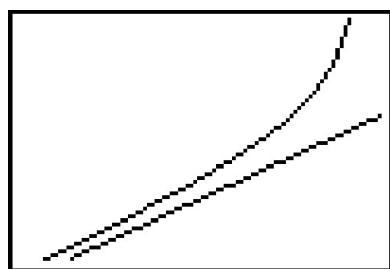
$t$	$N(t)$
10	$\approx 485.48$
100	$\approx 498.50$
1000	$\approx 499.85$
10000	$\approx 499.98$

The table suggests that as  $t \rightarrow \infty$ ,  $N(t) \rightarrow 500$ . (More specifically,  $500^-$ .) This means as time goes by, only a total of 500 students will have ever had the flu. □

<sup>11</sup>Though the population below is more accurately modeled with the functions in Chapter 6, we approximate it (using Calculus, of course!) using a rational function.

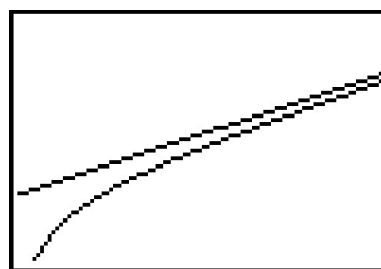
We close this section with a discussion of the *third* (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function  $g(x) = \frac{x^2-4}{x+1}$  in Example 4.1.4. Performing long division,<sup>12</sup> we get  $g(x) = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$ . Since the term  $\frac{3}{x+1} \rightarrow 0$  as  $x \rightarrow \pm\infty$ , it stands to reason that as  $x$  becomes unbounded, the function values  $g(x) = x - 1 - \frac{3}{x+1} \approx x - 1$ . Geometrically, this means that the graph of  $y = g(x)$  should resemble the line  $y = x - 1$  as  $x \rightarrow \pm\infty$ . We see this play out both numerically and graphically below.

$x$	$g(x)$	$x - 1$
-10	$\approx -10.6667$	-11
-100	$\approx -100.9697$	-101
-1000	$\approx -1000.9970$	-1001
-10000	$\approx -10000.9997$	-10001



$y = g(x)$  and  $y = x - 1$   
as  $x \rightarrow -\infty$

$x$	$g(x)$	$x - 1$
10	$\approx 8.7273$	9
100	$\approx 98.9703$	99
1000	$\approx 998.9970$	999
10000	$\approx 9998.9997$	9999



$y = g(x)$  and  $y = x - 1$   
as  $x \rightarrow \infty$

The way we symbolize the relationship between the end behavior of  $y = g(x)$  with that of the line  $y = x - 1$  is to write ‘as  $x \rightarrow \pm\infty$ ,  $g(x) \rightarrow x - 1$ .’ In this case, we say the line  $y = x - 1$  is a **slant asymptote**<sup>13</sup> to the graph of  $y = g(x)$ . Informally, the graph of a rational function has a slant asymptote if, as  $x \rightarrow \infty$  or as  $x \rightarrow -\infty$ , the graph resembles a non-horizontal, or ‘slanted’ line. Formally, we define a slant asymptote as follows.

**Definition 4.4.** The line  $y = mx + b$  where  $m \neq 0$  is called a **slant asymptote** of the graph of a function  $y = f(x)$  if as  $x \rightarrow -\infty$  or as  $x \rightarrow \infty$ ,  $f(x) \rightarrow mx + b$ .

A few remarks are in order. First, note that the stipulation  $m \neq 0$  in Definition 4.4 is what makes the ‘slant’ asymptote ‘slanted’ as opposed to the case when  $m = 0$  in which case we’d have a horizontal asymptote. Secondly, while we have motivated what we mean intuitively by the notation ‘ $f(x) \rightarrow mx + b$ ,’ like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase ‘ $f(x) \rightarrow mx + b$ ’ as ‘ $f(x) - (mx + b) \rightarrow 0$ .’ In other words, the graph of  $y = f(x)$  has the *slant* asymptote  $y = mx + b$  if and only if the graph of  $y = f(x) - (mx + b)$  has a *horizontal* asymptote  $y = 0$ .

<sup>12</sup>See the remarks following Theorem 4.2.

<sup>13</sup>Also called an ‘oblique’ asymptote in some, ostensibly higher class (and more expensive), texts.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of  $g(x) = \frac{x^2-4}{x+1}$ , the degree of the numerator  $x^2 - 4$  is 2, which is *exactly one more* than the degree of its denominator  $x + 1$  which is 1. This results in a *linear* quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us the following theorem.<sup>14</sup>

**Theorem 4.3. Determination of Slant Asymptotes:** Suppose  $r$  is a rational function and  $r(x) = \frac{p(x)}{q(x)}$ , where the degree of  $p$  is *exactly* one more than the degree of  $q$ . Then the graph of  $y = r(x)$  has the slant asymptote  $y = L(x)$  where  $L(x)$  is the quotient obtained by dividing  $p(x)$  by  $q(x)$ .

In the same way that Theorem 4.2 gives us an easy way to see if the graph of a rational function  $r(x) = \frac{p(x)}{q(x)}$  has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem 4.3 gives us an easy way to check for slant asymptotes. Unlike Theorem 4.2, which gives us a quick way to *find* the horizontal asymptotes (if any exist), Theorem 4.3 gives us no such ‘short-cut’. If a slant asymptote exists, we have no recourse but to use long division to find it.<sup>15</sup>

**Example 4.1.6.** Find the slant asymptotes of the graphs of the following functions if they exist. Verify your answers using a graphing calculator and describe the behavior of the graph near them using proper notation.

$$1. f(x) = \frac{x^2 - 4x + 2}{1 - x} \qquad 2. g(x) = \frac{x^2 - 4}{x - 2} \qquad 3. h(x) = \frac{x^3 + 1}{x^2 - 4}$$

**Solution.**

1. The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem 4.3 guarantees us a slant asymptote. To find it, we divide  $1 - x = -x + 1$  into  $x^2 - 4x + 2$  and get a quotient of  $-x + 3$ , so our slant asymptote is  $y = -x + 3$ . We confirm this graphically, and we see that as  $x \rightarrow -\infty$ , the graph of  $y = f(x)$  approaches the asymptote from below, and as  $x \rightarrow \infty$ , the graph of  $y = f(x)$  approaches the asymptote from above.<sup>16</sup>
2. As with the previous example, the degree of the numerator  $g(x) = \frac{x^2-4}{x-2}$  is 2 and the degree of the denominator is 1, so Theorem 4.3 applies. In this case,

$$g(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{(x - 2)} = \frac{(x + 2)\cancel{(x - 2)}}{\cancel{(x - 2)}^1} = x + 2, \quad x \neq 2$$

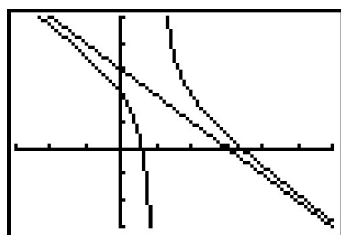
<sup>14</sup>Once again, this theorem is brought to you courtesy of Theorem 3.4 and Calculus.

<sup>15</sup>That’s OK, though. In the next section, we’ll use long division to analyze end behavior and it’s worth the effort!

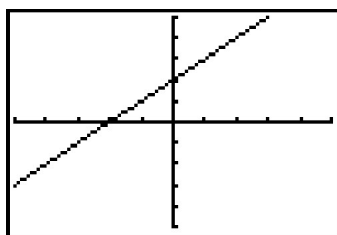
<sup>16</sup>Note that we are purposefully avoiding notation like ‘as  $x \rightarrow \infty$ ,  $f(x) \rightarrow (-x + 3)^+$ ’. While it is possible to define these notions formally with Calculus, it is not standard to do so. Besides, with the introduction of the symbol ‘?’ in the next section, the authors feel we are in enough trouble already.

so we have that the slant asymptote  $y = x + 2$  is identical to the graph of  $y = g(x)$  except at  $x = 2$  (where the latter has a ‘hole’ at  $(2, 4)$ .) The calculator supports this claim.<sup>17</sup>

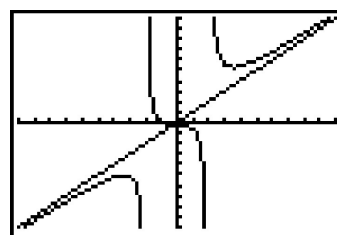
3. For  $h(x) = \frac{x^3+1}{x^2-4}$ , the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division  $(x^3 + 1) \div (x^2 - 4)$  gives a quotient of just  $x$ , so our slant asymptote is the line  $y = x$ . The calculator confirms this, and we find that as  $x \rightarrow -\infty$ , the graph of  $y = h(x)$  approaches the asymptote from below, and as  $x \rightarrow \infty$ , the graph of  $y = h(x)$  approaches the asymptote from above.



The graph of  $y = f(x)$



The graph of  $y = g(x)$



The graph of  $y = h(x)$

□

The reader may be a bit disappointed with the authors at this point owing to the fact that in Examples 4.1.2, 4.1.4, and 4.1.6, we used the *calculator* to determine function behavior near asymptotes. We rectify that in the next section where we, in excruciating detail, demonstrate the usefulness of ‘number sense’ to reveal this behavior analytically.

<sup>17</sup>While the word ‘asymptote’ has the connotation of ‘approaching but not equaling,’ Definitions 4.3 and 4.4 invite the same kind of pathologies we saw with Definitions 1.11 in Section 1.6.

## 4.1.1 EXERCISES

In Exercises 1 - 18, for the given rational function  $f$ :

- Find the domain of  $f$ .
- Identify any vertical asymptotes of the graph of  $y = f(x)$ .
- Identify any holes in the graph.
- Find the horizontal asymptote, if it exists.
- Find the slant asymptote, if it exists.
- Graph the function using a graphing utility and describe the behavior near the asymptotes.

1.  $f(x) = \frac{x}{3x - 6}$

2.  $f(x) = \frac{3 + 7x}{5 - 2x}$

3.  $f(x) = \frac{x}{x^2 + x - 12}$

4.  $f(x) = \frac{x}{x^2 + 1}$

5.  $f(x) = \frac{x + 7}{(x + 3)^2}$

6.  $f(x) = \frac{x^3 + 1}{x^2 - 1}$

7.  $f(x) = \frac{4x}{x^2 + 4}$

8.  $f(x) = \frac{4x}{x^2 - 4}$

9.  $f(x) = \frac{x^2 - x - 12}{x^2 + x - 6}$

10.  $f(x) = \frac{3x^2 - 5x - 2}{x^2 - 9}$

11.  $f(x) = \frac{x^3 + 2x^2 + x}{x^2 - x - 2}$

12.  $f(x) = \frac{x^3 - 3x + 1}{x^2 + 1}$

13.  $f(x) = \frac{2x^2 + 5x - 3}{3x + 2}$

14.  $f(x) = \frac{-x^3 + 4x}{x^2 - 9}$

15.  $f(x) = \frac{-5x^4 - 3x^3 + x^2 - 10}{x^3 - 3x^2 + 3x - 1}$

16.  $f(x) = \frac{x^3}{1 - x}$

17.  $f(x) = \frac{18 - 2x^2}{x^2 - 9}$

18.  $f(x) = \frac{x^3 - 4x^2 - 4x - 5}{x^2 + x + 1}$

19. The cost  $C$  in dollars to remove  $p\%$  of the invasive species of Ippizuti fish from Sasquatch Pond is given by

$$C(p) = \frac{1770p}{100 - p}, \quad 0 \leq p < 100$$

- Find and interpret  $C(25)$  and  $C(95)$ .
  - What does the vertical asymptote at  $x = 100$  mean within the context of the problem?
  - What percentage of the Ippizuti fish can you remove for \$40000?
20. In Exercise 71 in Section 1.4, the population of Sasquatch in Portage County was modeled by the function

$$P(t) = \frac{150t}{t + 15},$$

where  $t = 0$  represents the year 1803. Find the horizontal asymptote of the graph of  $y = P(t)$  and explain what it means.

21. Recall from Example 1.5.3 that the cost  $C$  (in dollars) to make  $x$  dOpi media players is  $C(x) = 100x + 2000$ ,  $x \geq 0$ .
- Find a formula for the average cost  $\overline{C}(x)$ . Recall:  $\overline{C}(x) = \frac{C(x)}{x}$ .
  - Find and interpret  $\overline{C}(1)$  and  $\overline{C}(100)$ .
  - How many dOpis need to be produced so that the average cost per dOpi is \$200?
  - Interpret the behavior of  $\overline{C}(x)$  as  $x \rightarrow 0^+$ . (HINT: You may want to find the fixed cost  $C(0)$  to help in your interpretation.)
  - Interpret the behavior of  $\overline{C}(x)$  as  $x \rightarrow \infty$ . (HINT: You may want to find the variable cost (defined in Example 2.1.5 in Section 2.1) to help in your interpretation.)
22. In Exercise 35 in Section 3.1, we fit a few polynomial models to the following electric circuit data. (The circuit was built with a variable resistor. For each of the following resistance values (measured in kilo-ohms,  $k\Omega$ ), the corresponding power to the load (measured in milliwatts,  $mW$ ) is given in the table below.)<sup>18</sup>

Resistance: ( $k\Omega$ )	1.012	2.199	3.275	4.676	6.805	9.975
Power: ( $mW$ )	1.063	1.496	1.610	1.613	1.505	1.314

Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power to resistance. For this circuit, it is  $P(x) = \frac{25x}{(x+3.9)^2}$ , where  $x$  is the resistance value,  $x \geq 0$ .

- Graph the data along with the function  $y = P(x)$  on your calculator.
  - Use your calculator to approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?
  - Find and interpret the end behavior of  $P(x)$  as  $x \rightarrow \infty$ .
23. In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed. (This paper, which is now in the public domain and can be found [here](#), is from a bygone era when students at business schools took typing classes on manual typewriters.) Using his original notation and original language, we have  $Y = \frac{L(X+P)}{(X+P)+R}$  where  $L$  is the predicted practice limit in terms of speed units,  $X$  is pages written,  $Y$  is writing speed in terms of words in four minutes,  $P$  is equivalent previous practice in terms of pages and  $R$  is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve  $Y = \frac{216(X+19)}{X+148}$ . Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.

<sup>18</sup>The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

## 4.1.2 ANSWERS

1.  $f(x) = \frac{x}{3x-6}$

Domain:  $(-\infty, 2) \cup (2, \infty)$ Vertical asymptote:  $x = 2$ As  $x \rightarrow 2^-$ ,  $f(x) \rightarrow -\infty$ As  $x \rightarrow 2^+$ ,  $f(x) \rightarrow \infty$ 

No holes in the graph

Horizontal asymptote:  $y = \frac{1}{3}$ As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \frac{1}{3}^-$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \frac{1}{3}^+$ 

2.  $f(x) = \frac{3+7x}{5-2x}$

Domain:  $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$ Vertical asymptote:  $x = \frac{5}{2}$ As  $x \rightarrow \frac{5}{2}^-$ ,  $f(x) \rightarrow \infty$ As  $x \rightarrow \frac{5}{2}^+$ ,  $f(x) \rightarrow -\infty$ 

No holes in the graph

Horizontal asymptote:  $y = -\frac{7}{2}$ As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\frac{7}{2}^+$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\frac{7}{2}^-$ 

3.  $f(x) = \frac{x}{x^2+x-12} = \frac{x}{(x+4)(x-3)}$

Domain:  $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$ Vertical asymptotes:  $x = -4, x = 3$ As  $x \rightarrow -4^-$ ,  $f(x) \rightarrow -\infty$ As  $x \rightarrow -4^+$ ,  $f(x) \rightarrow \infty$ As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow -\infty$ As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow \infty$ 

No holes in the graph

Horizontal asymptote:  $y = 0$ As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ 

4.  $f(x) = \frac{x}{x^2+1}$

Domain:  $(-\infty, \infty)$ 

No vertical asymptotes

No holes in the graph

Horizontal asymptote:  $y = 0$ As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ 

5.  $f(x) = \frac{x+7}{(x+3)^2}$

Domain:  $(-\infty, -3) \cup (-3, \infty)$ Vertical asymptote:  $x = -3$ As  $x \rightarrow -3^-$ ,  $f(x) \rightarrow \infty$ As  $x \rightarrow -3^+$ ,  $f(x) \rightarrow -\infty$ 

No holes in the graph

Horizontal asymptote:  $y = 0$ <sup>19</sup>As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ 

6.  $f(x) = \frac{x^3+1}{x^2-1} = \frac{x^2-x+1}{x-1}$

Domain:  $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ Vertical asymptote:  $x = 1$ As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow -\infty$ As  $x \rightarrow 1^+$ ,  $f(x) \rightarrow \infty$ Hole at  $(-1, -\frac{3}{2})$ Slant asymptote:  $y = x$ As  $x \rightarrow -\infty$ , the graph is below  $y = x$ As  $x \rightarrow \infty$ , the graph is above  $y = x$ <sup>19</sup>This is hard to see on the calculator, but trust me, the graph is below the  $x$ -axis to the left of  $x = -7$ .



$$7. f(x) = \frac{4x}{x^2 + 4}$$

Domain:  $(-\infty, \infty)$

No vertical asymptotes

No holes in the graph

Horizontal asymptote:  $y = 0$

As  $x \rightarrow -\infty, f(x) \rightarrow 0^-$

As  $x \rightarrow \infty, f(x) \rightarrow 0^+$

$$8. f(x) = \frac{4x}{x^2 - 4} = \frac{4x}{(x+2)(x-2)}$$

Domain:  $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Vertical asymptotes:  $x = -2, x = 2$

As  $x \rightarrow -2^-, f(x) \rightarrow -\infty$

As  $x \rightarrow -2^+, f(x) \rightarrow \infty$

As  $x \rightarrow 2^-, f(x) \rightarrow -\infty$

As  $x \rightarrow 2^+, f(x) \rightarrow \infty$

No holes in the graph

Horizontal asymptote:  $y = 0$

As  $x \rightarrow -\infty, f(x) \rightarrow 0^-$

As  $x \rightarrow \infty, f(x) \rightarrow 0^+$

$$9. f(x) = \frac{x^2 - x - 12}{x^2 + x - 6} = \frac{x - 4}{x - 2}$$

Domain:  $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$

Vertical asymptote:  $x = 2$

As  $x \rightarrow 2^-, f(x) \rightarrow \infty$

As  $x \rightarrow 2^+, f(x) \rightarrow -\infty$

Hole at  $(-3, \frac{7}{5})$

Horizontal asymptote:  $y = 1$

As  $x \rightarrow -\infty, f(x) \rightarrow 1^+$

As  $x \rightarrow \infty, f(x) \rightarrow 1^-$

$$10. f(x) = \frac{3x^2 - 5x - 2}{x^2 - 9} = \frac{(3x+1)(x-2)}{(x+3)(x-3)}$$

Domain:  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Vertical asymptotes:  $x = -3, x = 3$

As  $x \rightarrow -3^-, f(x) \rightarrow \infty$

As  $x \rightarrow -3^+, f(x) \rightarrow -\infty$

As  $x \rightarrow 3^-, f(x) \rightarrow -\infty$

As  $x \rightarrow 3^+, f(x) \rightarrow \infty$

No holes in the graph

Horizontal asymptote:  $y = 3$

As  $x \rightarrow -\infty, f(x) \rightarrow 3^+$

As  $x \rightarrow \infty, f(x) \rightarrow 3^-$

$$11. f(x) = \frac{x^3 + 2x^2 + x}{x^2 - x - 2} = \frac{x(x+1)}{x-2}$$

Domain:  $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

Vertical asymptote:  $x = 2$

As  $x \rightarrow 2^-, f(x) \rightarrow -\infty$

As  $x \rightarrow 2^+, f(x) \rightarrow \infty$

Hole at  $(-1, 0)$

Slant asymptote:  $y = x + 3$

As  $x \rightarrow -\infty$ , the graph is below  $y = x + 3$

As  $x \rightarrow \infty$ , the graph is above  $y = x + 3$

$$12. f(x) = \frac{x^3 - 3x + 1}{x^2 + 1}$$

Domain:  $(-\infty, \infty)$

No vertical asymptotes

No holes in the graph

Slant asymptote:  $y = x$

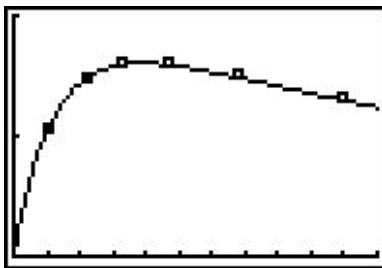
As  $x \rightarrow -\infty$ , the graph is above  $y = x$

As  $x \rightarrow \infty$ , the graph is below  $y = x$

13.  $f(x) = \frac{2x^2 + 5x - 3}{3x + 2}$   
 Domain:  $(-\infty, -\frac{2}{3}) \cup (-\frac{2}{3}, \infty)$   
 Vertical asymptote:  $x = -\frac{2}{3}$   
 As  $x \rightarrow -\frac{2}{3}^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -\frac{2}{3}^+$ ,  $f(x) \rightarrow -\infty$   
 No holes in the graph  
 Slant asymptote:  $y = \frac{2}{3}x + \frac{11}{9}$   
 As  $x \rightarrow -\infty$ , the graph is above  $y = \frac{2}{3}x + \frac{11}{9}$   
 As  $x \rightarrow \infty$ , the graph is below  $y = \frac{2}{3}x + \frac{11}{9}$
14.  $f(x) = \frac{-x^3 + 4x}{x^2 - 9} = \frac{-x^3 + 4x}{(x-3)(x+3)}$   
 Domain:  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$   
 Vertical asymptotes:  $x = -3, x = 3$   
 As  $x \rightarrow -3^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -3^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow -\infty$   
 No holes in the graph  
 Slant asymptote:  $y = -x$   
 As  $x \rightarrow -\infty$ , the graph is above  $y = -x$   
 As  $x \rightarrow \infty$ , the graph is below  $y = -x$
15.  $f(x) = \frac{-5x^4 - 3x^3 + x^2 - 10}{x^3 - 3x^2 + 3x - 1}$   
 $= \frac{-5x^4 - 3x^3 + x^2 - 10}{(x-1)^3}$   
 Domain:  $(-\infty, 1) \cup (1, \infty)$   
 Vertical asymptotes:  $x = 1$   
 As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 1^+$ ,  $f(x) \rightarrow -\infty$   
 No holes in the graph  
 Slant asymptote:  $y = -5x - 18$   
 As  $x \rightarrow -\infty$ , the graph is above  $y = -5x - 18$   
 As  $x \rightarrow \infty$ , the graph is below  $y = -5x - 18$
16.  $f(x) = \frac{x^3}{1-x}$   
 Domain:  $(-\infty, 1) \cup (1, \infty)$   
 Vertical asymptote:  $x = 1$   
 As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 1^+$ ,  $f(x) \rightarrow -\infty$   
 No holes in the graph  
 No horizontal or slant asymptote  
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$
17.  $f(x) = \frac{18 - 2x^2}{x^2 - 9} = -2$   
 Domain:  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$   
 No vertical asymptotes  
 Holes in the graph at  $(-3, -2)$  and  $(3, -2)$   
 Horizontal asymptote  $y = -2$   
 As  $x \rightarrow \pm\infty$ ,  $f(x) = -2$
18.  $f(x) = \frac{x^3 - 4x^2 - 4x - 5}{x^2 + x + 1} = x - 5$   
 Domain:  $(-\infty, \infty)$   
 No vertical asymptotes  
 No holes in the graph  
 Slant asymptote:  $y = x - 5$   
 $f(x) = x - 5$  everywhere.

19. (a)  $C(25) = 590$  means it costs \$590 to remove 25% of the fish and  $C(95) = 33630$  means it would cost \$33630 to remove 95% of the fish from the pond.
- (b) The vertical asymptote at  $x = 100$  means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.
- (c) For \$40000 you could remove about 95.76% of the fish.

20. The horizontal asymptote of the graph of  $P(t) = \frac{150t}{t+15}$  is  $y = 150$  and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.
21. (a)  $\overline{C}(x) = \frac{100x+2000}{x}$ ,  $x > 0$ .  
 (b)  $\overline{C}(1) = 2100$  and  $\overline{C}(100) = 120$ . When just 1 dOpi is produced, the cost per dOpi is \$2100, but when 100 dOpis are produced, the cost per dOpi is \$120.  
 (c)  $\overline{C}(x) = 200$  when  $x = 20$ . So to get the cost per dOpi to \$200, 20 dOpis need to be produced.  
 (d) As  $x \rightarrow 0^+$ ,  $\overline{C}(x) \rightarrow \infty$ . This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of \$2000 ( $C(0) = 2000$ ), we are trying to spread that \$2000 over fewer and fewer dOpis.  
 (e) As  $x \rightarrow \infty$ ,  $\overline{C}(x) \rightarrow 100^+$ . This means that as more and more dOpis are produced, the cost per dOpi approaches \$100, but is always a little more than \$100. Since \$100 is the variable cost per dOpi ( $C(x) = 100x + 2000$ ), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the \$2000 fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.
22. (a)

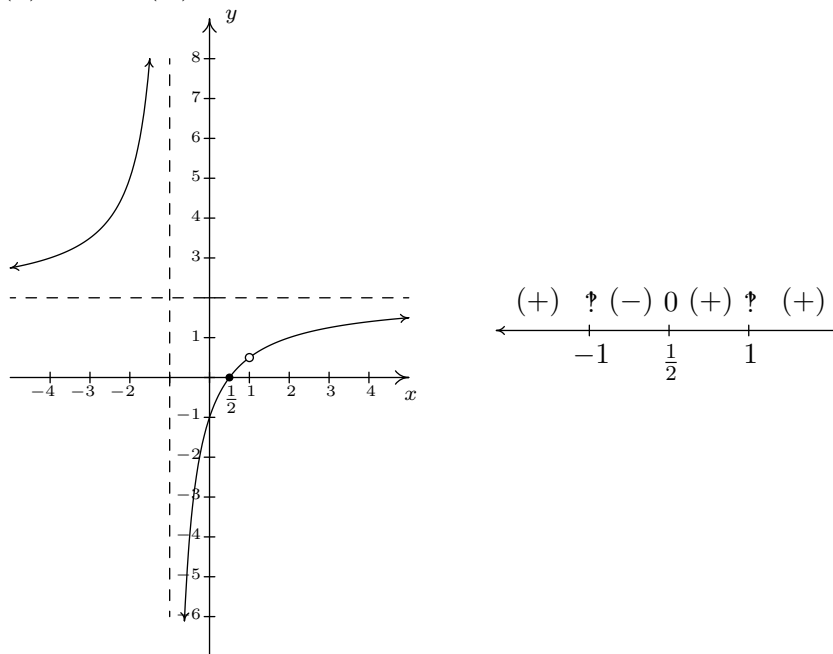


- (b) The maximum power is approximately  $1.603 \text{ mW}$  which corresponds to  $3.9 \text{ k}\Omega$ .  
 (c) As  $x \rightarrow \infty$ ,  $P(x) \rightarrow 0^+$  which means as the resistance increases without bound, the power diminishes to zero.

## 4.2 GRAPHS OF RATIONAL FUNCTIONS

In this section, we take a closer look at graphing rational functions. In Section 4.1, we learned that the graphs of rational functions may have holes in them and could have vertical, horizontal and slant asymptotes. Theorems 4.1, 4.2 and 4.3 tell us exactly when and where these behaviors will occur, and if we combine these results with what we already know about graphing functions, we will quickly be able to generate reasonable graphs of rational functions.

One of the standard tools we will use is the sign diagram which was first introduced in Section 2.4, and then revisited in Section 3.1. In those sections, we operated under the belief that a function couldn't change its sign without its graph crossing through the  $x$ -axis. The major theorem we used to justify this belief was the Intermediate Value Theorem, Theorem 3.1. It turns out the Intermediate Value Theorem applies to all *continuous* functions,<sup>1</sup> not just polynomials. Although rational functions are continuous on their domains,<sup>2</sup> Theorem 4.1 tells us that vertical asymptotes and holes occur at the values excluded from their domains. In other words, rational functions aren't continuous at these excluded values which leaves open the possibility that the function could change sign *without* crossing through the  $x$ -axis. Consider the graph of  $y = h(x)$  from Example 4.1.1, recorded below for convenience. We have added its  $x$ -intercept at  $(\frac{1}{2}, 0)$  for the discussion that follows. Suppose we wish to construct a sign diagram for  $h(x)$ . Recall that the intervals where  $h(x) > 0$ , or (+), correspond to the  $x$ -values where the graph of  $y = h(x)$  is *above* the  $x$ -axis; the intervals on which  $h(x) < 0$ , or (−) correspond to where the graph is *below* the  $x$ -axis.



As we examine the graph of  $y = h(x)$ , reading from left to right, we note that from  $(-\infty, -1)$ , the graph is above the  $x$ -axis, so  $h(x)$  is (+) there. At  $x = -1$ , we have a vertical asymptote, at which point the graph 'jumps' across the  $x$ -axis. On the interval  $(-1, \frac{1}{2})$ , the graph is below the

<sup>1</sup>Recall that, for our purposes, this means the graphs are devoid of any breaks, jumps or holes

<sup>2</sup>Another result from Calculus.

$x$ -axis, so  $h(x)$  is  $(-)$  there. The graph crosses through the  $x$ -axis at  $(\frac{1}{2}, 0)$  and remains above the  $x$ -axis until  $x = 1$ , where we have a ‘hole’ in the graph. Since  $h(1)$  is undefined, there is no sign here. So we have  $h(x)$  as  $(+)$  on the interval  $(\frac{1}{2}, 1)$ . Continuing, we see that on  $(1, \infty)$ , the graph of  $y = h(x)$  is above the  $x$ -axis, so we mark  $(+)$  there. To construct a sign diagram from this information, we not only need to denote the zero of  $h$ , but also the places not in the domain of  $h$ . As is our custom, we write ‘0’ above  $\frac{1}{2}$  on the sign diagram to remind us that it is a zero of  $h$ . We need a different notation for  $-1$  and  $1$ , and we have chosen to use ‘?’ - a nonstandard symbol called the [interrobang](#). We use this symbol to convey a sense of surprise, caution and wonderment - an appropriate attitude to take when approaching these points. The moral of the story is that when constructing sign diagrams for rational functions, we include the zeros as well as the values excluded from the domain.

### Steps for Constructing a Sign Diagram for a Rational Function

Suppose  $r$  is a rational function.

1. Place any values excluded from the domain of  $r$  on the number line with an ‘?’ above them.
2. Find the zeros of  $r$  and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of  $r(x)$  for each test value in step 3, and write that sign above the corresponding interval.

We now present our procedure for graphing rational functions and apply it to a few exhaustive examples. Please note that we decrease the amount of detail given in the explanations as we move through the examples. The reader should be able to fill in any details in those steps which we have abbreviated.

### Steps for Graphing Rational Functions

Suppose  $r$  is a rational function.

1. Find the domain of  $r$ .
2. Reduce  $r(x)$  to lowest terms, if applicable.
3. Find the  $x$ - and  $y$ -intercepts of the graph of  $y = r(x)$ , if they exist.
4. Determine the location of any vertical asymptotes or holes in the graph, if they exist. Analyze the behavior of  $r$  on either side of the vertical asymptotes, if applicable.
5. Analyze the end behavior of  $r$ . Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph of  $y = r(x)$ .

**Example 4.2.1.** Sketch a detailed graph of  $f(x) = \frac{3x}{x^2 - 4}$ .

**Solution.** We follow the six step procedure outlined above.

1. As usual, we set the denominator equal to zero to get  $x^2 - 4 = 0$ . We find  $x = \pm 2$ , so our domain is  $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ .
2. To reduce  $f(x)$  to lowest terms, we factor the numerator and denominator which yields  $f(x) = \frac{3x}{(x-2)(x+2)}$ . There are no common factors which means  $f(x)$  is already in lowest terms.
3. To find the  $x$ -intercepts of the graph of  $y = f(x)$ , we set  $y = f(x) = 0$ . Solving  $\frac{3x}{(x-2)(x+2)} = 0$  results in  $x = 0$ . Since  $x = 0$  is in our domain,  $(0, 0)$  is the  $x$ -intercept. To find the  $y$ -intercept, we set  $x = 0$  and find  $y = f(0) = 0$ , so that  $(0, 0)$  is our  $y$ -intercept as well.<sup>3</sup>
4. The two numbers excluded from the domain of  $f$  are  $x = -2$  and  $x = 2$ . Since  $f(x)$  didn't reduce at all, both of these values of  $x$  still cause trouble in the denominator. Thus by Theorem 4.1,  $x = -2$  and  $x = 2$  are vertical asymptotes of the graph. We can actually go a step further at this point and determine exactly how the graph approaches the asymptote near each of these values. Though not absolutely necessary,<sup>4</sup> it is good practice for those heading off to Calculus. For the discussion that follows, it is best to use the factored form of  $f(x) = \frac{3x}{(x-2)(x+2)}$ .

- *The behavior of  $y = f(x)$  as  $x \rightarrow -2$ :* Suppose  $x \rightarrow -2^-$ . If we were to build a table of values, we'd use  $x$ -values a little less than  $-2$ , say  $-2.1$ ,  $-2.01$  and  $-2.001$ . While there is no harm in actually building a table like we did in Section 4.1, we want to develop a 'number sense' here. Let's think about each factor in the formula of  $f(x)$  as we imagine substituting a number like  $x = -2.000001$  into  $f(x)$ . The quantity  $3x$  would be very close to  $-6$ , the quantity  $(x - 2)$  would be very close to  $-4$ , and the factor  $(x + 2)$  would be very close to  $0$ . More specifically,  $(x + 2)$  would be a little less than  $0$ , in this case,  $-0.000001$ . We will call such a number a 'very small  $(-)$ ', 'very small' meaning close to zero in absolute value. So, mentally, as  $x \rightarrow -2^-$ , we estimate

$$f(x) = \frac{3x}{(x-2)(x+2)} \approx \frac{-6}{(-4)(\text{very small } (-))} = \frac{3}{2(\text{very small } (-))}$$

Now, the closer  $x$  gets to  $-2$ , the smaller  $(x + 2)$  will become, so even though we are multiplying our 'very small  $(-)$ ' by  $2$ , the denominator will continue to get smaller and smaller, and remain negative. The result is a fraction whose numerator is positive, but whose denominator is very small and negative. Mentally,

$$f(x) \approx \frac{3}{2(\text{very small } (-))} \approx \frac{3}{\text{very small } (-)} \approx \text{very big } (-)$$

<sup>3</sup>As we mentioned at least once earlier, since functions can have at most one  $y$ -intercept, once we find that  $(0, 0)$  is on the graph, we know it is the  $y$ -intercept.

<sup>4</sup>The sign diagram in step 6 will also determine the behavior near the vertical asymptotes.

The term ‘very big  $(-)$ ’ means a number with a large absolute value which is negative.<sup>5</sup> What all of this means is that as  $x \rightarrow -2^-$ ,  $f(x) \rightarrow -\infty$ . Now suppose we wanted to determine the behavior of  $f(x)$  as  $x \rightarrow -2^+$ . If we imagine substituting something a little larger than  $-2$  in for  $x$ , say  $-1.999999$ , we mentally estimate

$$f(x) \approx \frac{-6}{(-4)(\text{very small } (+))} = \frac{3}{2(\text{very small } (+))} \approx \frac{3}{\text{very small } (+)} \approx \text{very big } (+)$$

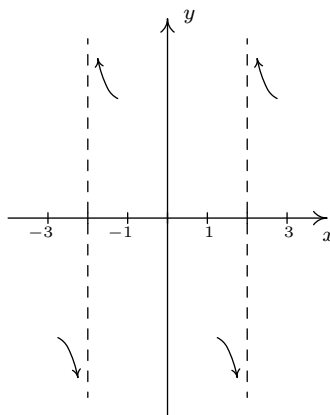
We conclude that as  $x \rightarrow -2^+$ ,  $f(x) \rightarrow \infty$ .

- *The behavior of  $y = f(x)$  as  $x \rightarrow 2$ :* Consider  $x \rightarrow 2^-$ . We imagine substituting  $x = 1.999999$ . Approximating  $f(x)$  as we did above, we get

$$f(x) \approx \frac{6}{(\text{very small } (-))(4)} = \frac{3}{2(\text{very small } (-))} \approx \frac{3}{\text{very small } (-)} \approx \text{very big } (-)$$

We conclude that as  $x \rightarrow 2^-$ ,  $f(x) \rightarrow -\infty$ . Similarly, as  $x \rightarrow 2^+$ , we imagine substituting  $x = 2.000001$  to get  $f(x) \approx \frac{3}{\text{very small } (+)} \approx \text{very big } (+)$ . So as  $x \rightarrow 2^+$ ,  $f(x) \rightarrow \infty$ .

Graphically, we have that near  $x = -2$  and  $x = 2$  the graph of  $y = f(x)$  looks like<sup>6</sup>



5. Next, we determine the end behavior of the graph of  $y = f(x)$ . Since the degree of the numerator is 1, and the degree of the denominator is 2, Theorem 4.2 tells us that  $y = 0$  is the horizontal asymptote. As with the vertical asymptotes, we can glean more detailed information using ‘number sense’. For the discussion below, we use the formula  $f(x) = \frac{3x}{x^2-4}$ .

- *The behavior of  $y = f(x)$  as  $x \rightarrow -\infty$ :* If we were to make a table of values to discuss the behavior of  $f$  as  $x \rightarrow -\infty$ , we would substitute very ‘large’ negative numbers in for  $x$ , say for example,  $x = -1$  billion. The numerator  $3x$  would then be  $-3$  billion, whereas

<sup>5</sup>The actual retail value of  $f(-2.000001)$  is approximately  $-1,500,000$ .

<sup>6</sup>We have deliberately left off the labels on the  $y$ -axis because we know only the behavior near  $x = \pm 2$ , not the actual function values.

the denominator  $x^2 - 4$  would be  $(-1 \text{ billion})^2 - 4$ , which is pretty much the same as  $1(\text{billion})^2$ . Hence,

$$f(-1 \text{ billion}) \approx \frac{-3 \text{ billion}}{1(\text{billion})^2} \approx -\frac{3}{\text{billion}} \approx \text{very small } (-)$$

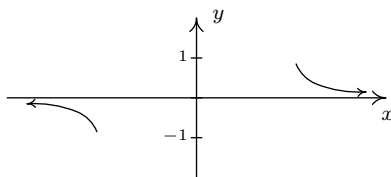
Notice that if we substituted in  $x = -1$  trillion, essentially the same kind of cancellation would happen, and we would be left with an even ‘smaller’ negative number. This not only confirms the fact that as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0$ , it tells us that  $f(x) \rightarrow 0^-$ . In other words, the graph of  $y = f(x)$  is a little bit *below* the  $x$ -axis as we move to the far left.

- *The behavior of  $y = f(x)$  as  $x \rightarrow \infty$ :* On the flip side, we can imagine substituting very large positive numbers in for  $x$  and looking at the behavior of  $f(x)$ . For example, let  $x = 1$  billion. Proceeding as before, we get

$$f(1 \text{ billion}) \approx \frac{3 \text{ billion}}{1(\text{billion})^2} \approx \frac{3}{\text{billion}} \approx \text{very small } (+)$$

The larger the number we put in, the smaller the positive number we would get out. In other words, as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ , so the graph of  $y = f(x)$  is a little bit *above* the  $x$ -axis as we look toward the far right.

Graphically, we have<sup>7</sup>

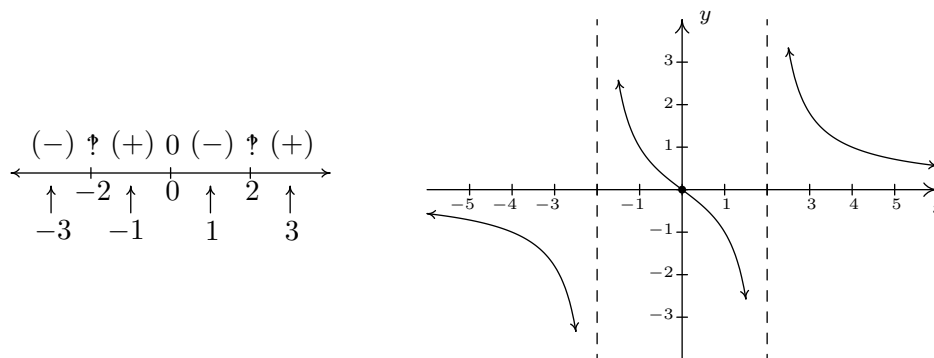


6. Lastly, we construct a sign diagram for  $f(x)$ . The  $x$ -values excluded from the domain of  $f$  are  $x = \pm 2$ , and the only zero of  $f$  is  $x = 0$ . Displaying these appropriately on the number line gives us four test intervals, and we choose the test values<sup>8</sup>  $x = -3$ ,  $x = -1$ ,  $x = 1$  and  $x = 3$ . We find  $f(-3)$  is  $(-)$ ,  $f(-1)$  is  $(+)$ ,  $f(1)$  is  $(-)$  and  $f(3)$  is  $(+)$ . Combining this with our previous work, we get the graph of  $y = f(x)$  below.

<sup>7</sup>As with the vertical asymptotes in the previous step, we know only the behavior of the graph as  $x \rightarrow \pm\infty$ . For that reason, we provide no  $x$ -axis labels.

<sup>8</sup>In this particular case, we can eschew test values, since our analysis of the behavior of  $f$  near the vertical asymptotes and our end behavior analysis have given us the signs on each of the test intervals. In general, however, this won't always be the case, so for demonstration purposes, we continue with our usual construction.





□

A couple of notes are in order. First, the graph of  $y = f(x)$  certainly seems to possess symmetry with respect to the origin. In fact, we can check  $f(-x) = -f(x)$  to see that  $f$  is an odd function. In some textbooks, checking for symmetry is part of the standard procedure for graphing rational functions; but since it happens comparatively rarely<sup>9</sup> we'll just point it out when we see it. Also note that while  $y = 0$  is the horizontal asymptote, the graph of  $f$  actually crosses the  $x$ -axis at  $(0, 0)$ . The myth that graphs of rational functions can't cross their horizontal asymptotes is completely false,<sup>10</sup> as we shall see again in our next example.

**Example 4.2.2.** Sketch a detailed graph of  $g(x) = \frac{2x^2 - 3x - 5}{x^2 - x - 6}$ .

**Solution.**

1. Setting  $x^2 - x - 6 = 0$  gives  $x = -2$  and  $x = 3$ . Our domain is  $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$ .
2. Factoring  $g(x)$  gives  $g(x) = \frac{(2x-5)(x+1)}{(x-3)(x+2)}$ . There is no cancellation, so  $g(x)$  is in lowest terms.
3. To find the  $x$ -intercept we set  $y = g(x) = 0$ . Using the factored form of  $g(x)$  above, we find the zeros to be the solutions of  $(2x - 5)(x + 1) = 0$ . We obtain  $x = \frac{5}{2}$  and  $x = -1$ . Since both of these numbers are in the domain of  $g$ , we have two  $x$ -intercepts,  $(\frac{5}{2}, 0)$  and  $(-1, 0)$ . To find the  $y$ -intercept, we set  $x = 0$  and find  $y = g(0) = \frac{5}{6}$ , so our  $y$ -intercept is  $(0, \frac{5}{6})$ .
4. Since  $g(x)$  was given to us in lowest terms, we have, once again by Theorem 4.1 vertical asymptotes  $x = -2$  and  $x = 3$ . Keeping in mind  $g(x) = \frac{(2x-5)(x+1)}{(x-3)(x+2)}$ , we proceed to our analysis near each of these values.

- *The behavior of  $y = g(x)$  as  $x \rightarrow -2$ :* As  $x \rightarrow -2^-$ , we imagine substituting a number a little bit less than  $-2$ . We have

$$g(x) \approx \frac{(-9)(-1)}{(-5)(\text{very small } (-))} \approx \frac{9}{\text{very small } (+)} \approx \text{very big } (+)$$

<sup>9</sup>And Jeff doesn't think much of it to begin with...

<sup>10</sup>That's why we called it a MYTH!

so as  $x \rightarrow -2^-$ ,  $g(x) \rightarrow \infty$ . On the flip side, as  $x \rightarrow -2^+$ , we get

$$g(x) \approx \frac{9}{\text{very small } (-)} \approx \text{very big } (-)$$

so  $g(x) \rightarrow -\infty$ .

- *The behavior of  $y = g(x)$  as  $x \rightarrow 3$ :* As  $x \rightarrow 3^-$ , we imagine plugging in a number just shy of 3. We have

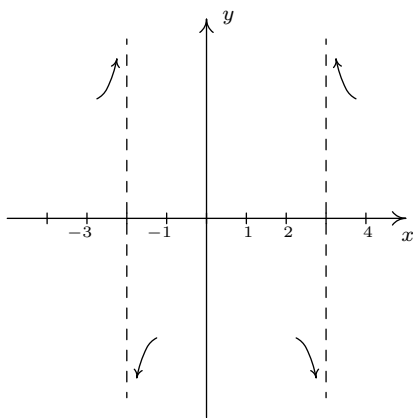
$$g(x) \approx \frac{(1)(4)}{(\text{very small } (-))(5)} \approx \frac{4}{\text{very small } (-)} \approx \text{very big } (-)$$

Hence, as  $x \rightarrow 3^-$ ,  $g(x) \rightarrow -\infty$ . As  $x \rightarrow 3^+$ , we get

$$g(x) \approx \frac{4}{\text{very small } (+)} \approx \text{very big } (+)$$

so  $g(x) \rightarrow \infty$ .

Graphically, we have (again, without labels on the  $y$ -axis)



5. Since the degrees of the numerator and denominator of  $g(x)$  are the same, we know from Theorem 4.2 that we can find the horizontal asymptote of the graph of  $g$  by taking the ratio of the leading terms coefficients,  $y = \frac{2}{1} = 2$ . However, if we take the time to do a more detailed analysis, we will be able to reveal some ‘hidden’ behavior which would be lost otherwise.<sup>11</sup> As in the discussion following Theorem 4.2, we use the result of the long division  $(2x^2 - 3x - 5) \div (x^2 - x - 6)$  to rewrite  $g(x) = \frac{2x^2 - 3x - 5}{x^2 - x - 6}$  as  $g(x) = 2 - \frac{x-7}{x^2 - x - 6}$ . We focus our attention on the term  $\frac{x-7}{x^2 - x - 6}$ .

<sup>11</sup>That is, if you use a calculator to graph. Once again, Calculus is the ultimate graphing power tool.

- *The behavior of  $y = g(x)$  as  $x \rightarrow -\infty$ :* If imagine substituting  $x = -1$  billion into  $\frac{x-7}{x^2-x-6}$ , we estimate  $\frac{x-7}{x^2-x-6} \approx \frac{-1 \text{ billion}}{1 \text{ billion}^2} \approx \text{very small } (-)$ .<sup>12</sup> Hence,

$$g(x) = 2 - \frac{x-7}{x^2-x-6} \approx 2 - \text{very small } (-) = 2 + \text{very small } (+)$$

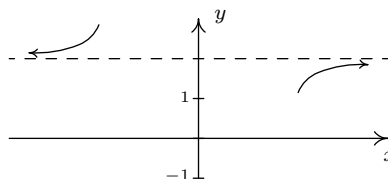
In other words, as  $x \rightarrow -\infty$ , the graph of  $y = g(x)$  is a little bit *above* the line  $y = 2$ .

- *The behavior of  $y = g(x)$  as  $x \rightarrow \infty$ .* To consider  $\frac{x-7}{x^2-x-6}$  as  $x \rightarrow \infty$ , we imagine substituting  $x = 1$  billion and, going through the usual mental routine, find

$$\frac{x-7}{x^2-x-6} \approx \text{very small } (+)$$

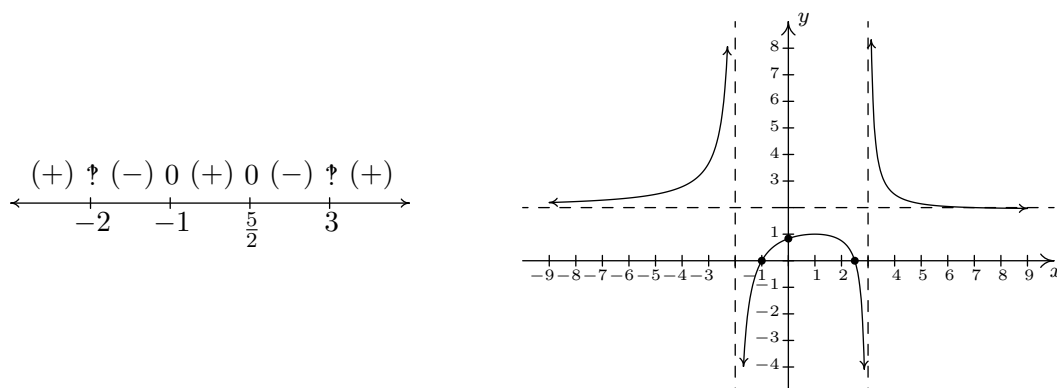
Hence,  $g(x) \approx 2 - \text{very small } (+)$ , in other words, the graph of  $y = g(x)$  is just *below* the line  $y = 2$  as  $x \rightarrow \infty$ .

On  $y = g(x)$ , we have (again, without labels on the  $x$ -axis)



- Finally we construct our sign diagram. We place an ‘?’ above  $x = -2$  and  $x = 3$ , and a ‘0’ above  $x = \frac{5}{2}$  and  $x = -1$ . Choosing test values in the test intervals gives us  $f(x)$  is (+) on the intervals  $(-\infty, -2)$ ,  $(-1, \frac{5}{2})$  and  $(3, \infty)$ , and (–) on the intervals  $(-2, -1)$  and  $(\frac{5}{2}, 3)$ . As we piece together all of the information, we note that the graph must cross the horizontal asymptote at some point after  $x = 3$  in order for it to approach  $y = 2$  from underneath. This is the subtlety that we would have missed had we skipped the long division and subsequent end behavior analysis. We can, in fact, find exactly when the graph crosses  $y = 2$ . As a result of the long division, we have  $g(x) = 2 - \frac{x-7}{x^2-x-6}$ . For  $g(x) = 2$ , we would need  $\frac{x-7}{x^2-x-6} = 0$ . This gives  $x - 7 = 0$ , or  $x = 7$ . Note that  $x - 7$  is the remainder when  $2x^2 - 3x - 5$  is divided by  $x^2 - x - 6$ , so it makes sense that for  $g(x)$  to equal the quotient 2, the remainder from the division must be 0. Sure enough, we find  $g(7) = 2$ . Moreover, it stands to reason that  $g$  must attain a relative minimum at some point past  $x = 7$ . Calculus verifies that at  $x = 13$ , we have such a minimum at exactly  $(13, 1.96)$ . The reader is challenged to find calculator windows which show the graph crossing its horizontal asymptote on one window, and the relative minimum in the other.

<sup>12</sup>In the denominator, we would have  $(1 \text{ billion})^2 - 1 \text{ billion} - 6$ . It’s easy to see why the 6 is insignificant, but to ignore the 1 billion seems criminal. However, compared to  $(1 \text{ billion})^2$ , it’s on the insignificant side; it’s  $10^{18}$  versus  $10^9$ . We are once again using the fact that for polynomials, end behavior is determined by the leading term, so in the denominator, the  $x^2$  term wins out over the  $x$  term.



□

Our next example gives us an opportunity to more thoroughly analyze a slant asymptote.

**Example 4.2.3.** Sketch a detailed graph of  $h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$ .

**Solution.**

1. For domain, you know the drill. Solving  $x^2 + 3x + 2 = 0$  gives  $x = -2$  and  $x = -1$ . Our answer is  $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$ .
2. To reduce  $h(x)$ , we need to factor the numerator and denominator. To factor the numerator, we use the techniques<sup>13</sup> set forth in Section 3.3 and we get

$$h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2} = \frac{(2x+1)(x+1)^2}{(x+2)(x+1)} = \frac{(2x+1)(x+1)^{\cancel{2}}}{(x+2)\cancel{(x+1)}} = \frac{(2x+1)(x+1)}{x+2}$$

We will use this reduced formula for  $h(x)$  as long as we're not substituting  $x = -1$ . To make this exclusion specific, we write  $h(x) = \frac{(2x+1)(x+1)}{x+2}$ ,  $x \neq -1$ .

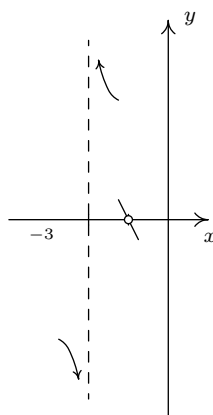
3. To find the  $x$ -intercepts, as usual, we set  $h(x) = 0$  and solve. Solving  $\frac{(2x+1)(x+1)}{x+2} = 0$  yields  $x = -\frac{1}{2}$  and  $x = -1$ . The latter isn't in the domain of  $h$ , so we exclude it. Our only  $x$ -intercept is  $(-\frac{1}{2}, 0)$ . To find the  $y$ -intercept, we set  $x = 0$ . Since  $0 \neq -1$ , we can use the reduced formula for  $h(x)$  and we get  $h(0) = \frac{1}{2}$  for a  $y$ -intercept of  $(0, \frac{1}{2})$ .
4. From Theorem 4.1, we know that since  $x = -2$  still poses a threat in the denominator of the reduced function, we have a vertical asymptote there. As for  $x = -1$ , the factor  $(x+1)$  was canceled from the denominator when we reduced  $h(x)$ , so it no longer causes trouble there. This means that we get a hole when  $x = -1$ . To find the  $y$ -coordinate of the hole, we substitute  $x = -1$  into  $\frac{(2x+1)(x+1)}{x+2}$ , per Theorem 4.1 and get 0. Hence, we have a hole on

<sup>13</sup>Bet you never thought you'd never see *that* stuff again before the Final Exam!

the  $x$ -axis at  $(-1, 0)$ . It should make you uncomfortable plugging  $x = -1$  into the reduced formula for  $h(x)$ , especially since we've made such a big deal concerning the stipulation about not letting  $x = -1$  for that formula. What we are really doing is taking a Calculus short-cut to the more detailed kind of analysis near  $x = -1$  which we will show below. Speaking of which, for the discussion that follows, we will use the formula  $h(x) = \frac{(2x+1)(x+1)}{x+2}$ ,  $x \neq -1$ .

- *The behavior of  $y = h(x)$  as  $x \rightarrow -2$ :* As  $x \rightarrow -2^-$ , we imagine substituting a number a little bit less than  $-2$ . We have  $h(x) \approx \frac{(-3)(-1)}{(\text{very small } (-))} \approx \frac{3}{(\text{very small } (-))} \approx \text{very big } (-)$  thus as  $x \rightarrow -2^-$ ,  $h(x) \rightarrow -\infty$ . On the other side of  $-2$ , as  $x \rightarrow -2^+$ , we find that  $h(x) \approx \frac{3}{\text{very small } (+)} \approx \text{very big } (+)$ , so  $h(x) \rightarrow \infty$ .
- *The behavior of  $y = h(x)$  as  $x \rightarrow -1$ .* As  $x \rightarrow -1^-$ , we imagine plugging in a number a bit less than  $x = -1$ . We have  $h(x) \approx \frac{(-1)(\text{very small } (-))}{1} = \text{very small } (+)$  Hence, as  $x \rightarrow -1^-$ ,  $h(x) \rightarrow 0^+$ . This means that as  $x \rightarrow -1^-$ , the graph is a bit above the point  $(-1, 0)$ . As  $x \rightarrow -1^+$ , we get  $h(x) \approx \frac{(-1)(\text{very small } (+))}{1} = \text{very small } (-)$ . This gives us that as  $x \rightarrow -1^+$ ,  $h(x) \rightarrow 0^-$ , so the graph is a little bit lower than  $(-1, 0)$  here.

Graphically, we have

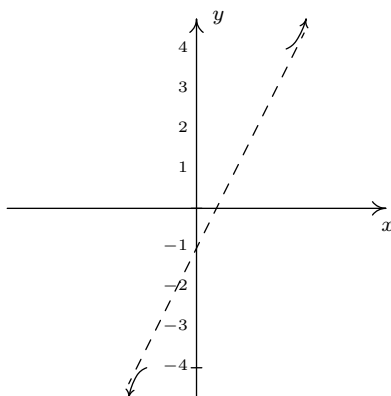


5. For end behavior, we note that the degree of the numerator of  $h(x)$ ,  $2x^3 + 5x^2 + 4x + 1$ , is 3 and the degree of the denominator,  $x^2 + 3x + 2$ , is 2 so by Theorem 4.3, the graph of  $y = h(x)$  has a slant asymptote. For  $x \rightarrow \pm\infty$ , we are far enough away from  $x = -1$  to use the reduced formula,  $h(x) = \frac{(2x+1)(x+1)}{x+2}$ ,  $x \neq -1$ . To perform long division, we multiply out the numerator and get  $h(x) = \frac{2x^2+3x+1}{x+2}$ ,  $x \neq -1$ , and rewrite  $h(x) = 2x - 1 + \frac{3}{x+2}$ ,  $x \neq -1$ . By Theorem 4.3, the slant asymptote is  $y = 2x - 1$ , and to better see *how* the graph approaches the asymptote, we focus our attention on the term generated from the remainder,  $\frac{3}{x+2}$ .

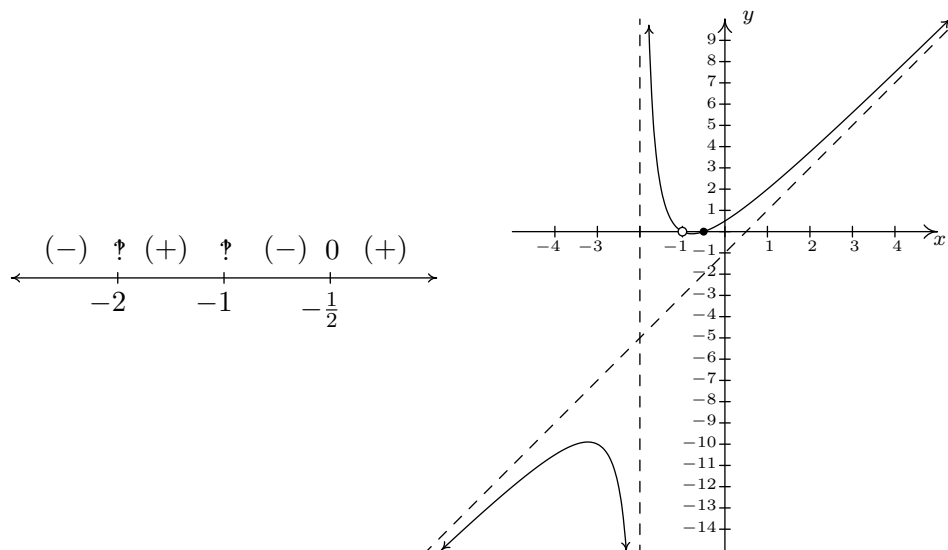
- *The behavior of  $y = h(x)$  as  $x \rightarrow -\infty$ :* Substituting  $x = -1$  billion into  $\frac{3}{x+2}$ , we get the estimate  $\frac{3}{-1 \text{ billion}} \approx \text{very small } (-)$ . Hence,  $h(x) = 2x - 1 + \frac{3}{x+2} \approx 2x - 1 + \text{very small } (-)$ . This means the graph of  $y = h(x)$  is a little bit *below* the line  $y = 2x - 1$  as  $x \rightarrow -\infty$ .

- *The behavior of  $y = h(x)$  as  $x \rightarrow \infty$ :* If  $x \rightarrow \infty$ , then  $\frac{3}{x+2} \approx$  very small (+). This means  $h(x) \approx 2x - 1 +$  very small (+), or that the graph of  $y = h(x)$  is a little bit *above* the line  $y = 2x - 1$  as  $x \rightarrow \infty$ .

Graphically we have



6. To make our sign diagram, we place an ‘?’ above  $x = -2$  and  $x = -1$  and a ‘0’ above  $x = -\frac{1}{2}$ . On our four test intervals, we find  $h(x)$  is (+) on  $(-2, -1)$  and  $(-\frac{1}{2}, \infty)$  and  $h(x)$  is (-) on  $(-\infty, -2)$  and  $(-1, -\frac{1}{2})$ . Putting all of our work together yields the graph below.



We could ask whether the graph of  $y = h(x)$  crosses its slant asymptote. From the formula  $h(x) = 2x - 1 + \frac{3}{x+2}$ ,  $x \neq -1$ , we see that if  $h(x) = 2x - 1$ , we would have  $\frac{3}{x+2} = 0$ . Since this will never happen, we conclude the graph never crosses its slant asymptote.<sup>14</sup>  $\square$

<sup>14</sup>But rest assured, some graphs do!

We end this section with an example that shows it's not all pathological weirdness when it comes to rational functions and technology still has a role to play in studying their graphs at this level.

**Example 4.2.4.** Sketch the graph of  $r(x) = \frac{x^4 + 1}{x^2 + 1}$ .

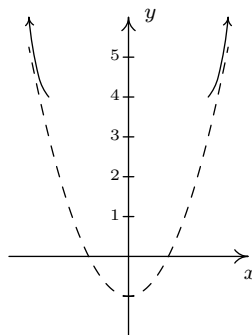
**Solution.**

1. The denominator  $x^2 + 1$  is never zero so the domain is  $(-\infty, \infty)$ .
2. With no real zeros in the denominator,  $x^2 + 1$  is an irreducible quadratic. Our only hope of reducing  $r(x)$  is if  $x^2 + 1$  is a factor of  $x^4 + 1$ . Performing long division gives us

$$\frac{x^4 + 1}{x^2 + 1} = x^2 - 1 + \frac{2}{x^2 + 1}$$

The remainder is not zero so  $r(x)$  is already reduced.

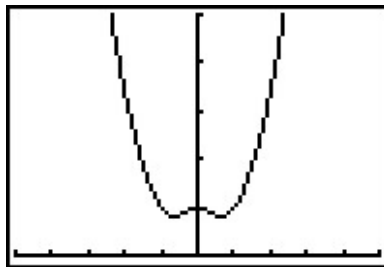
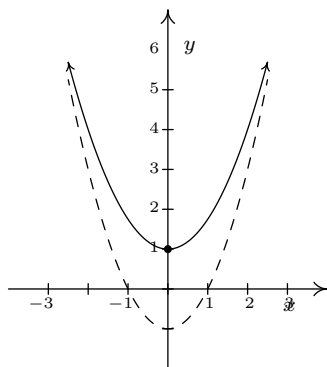
3. To find the  $x$ -intercept, we'd set  $r(x) = 0$ . Since there are no real solutions to  $\frac{x^4+1}{x^2+1} = 0$ , we have no  $x$ -intercepts. Since  $r(0) = 1$ , we get  $(0, 1)$  as the  $y$ -intercept.
4. This step doesn't apply to  $r$ , since its domain is all real numbers.
5. For end behavior, we note that since the degree of the numerator is exactly *two* more than the degree of the denominator, neither Theorems 4.2 nor 4.3 apply.<sup>15</sup> We know from our attempt to reduce  $r(x)$  that we can rewrite  $r(x) = x^2 - 1 + \frac{2}{x^2+1}$ , so we focus our attention on the term corresponding to the remainder,  $\frac{2}{x^2+1}$ . It should be clear that as  $x \rightarrow \pm\infty$ ,  $\frac{2}{x^2+1} \approx$  very small (+), which means  $r(x) \approx x^2 - 1 +$  very small (+). So the graph  $y = r(x)$  is a little bit *above* the graph of the parabola  $y = x^2 - 1$  as  $x \rightarrow \pm\infty$ . Graphically,



6. There isn't much work to do for a sign diagram for  $r(x)$ , since its domain is all real numbers and it has no zeros. Our sole test interval is  $(-\infty, \infty)$ , and since we know  $r(0) = 1$ , we conclude  $r(x)$  is (+) for all real numbers. At this point, we don't have much to go on for

<sup>15</sup>This won't stop us from giving it the old community college try, however!

a graph.<sup>16</sup> Below is a comparison of what we have determined analytically versus what the calculator shows us. We have no way to detect the relative extrema analytically<sup>17</sup> apart from brute force plotting of points, which is done more efficiently by the calculator.



□

As usual, the authors offer no apologies for what may be construed as ‘pedantry’ in this section. We feel that the detail presented in this section is necessary to obtain a firm grasp of the concepts presented here and it also serves as an introduction to the methods employed in Calculus. As we have said many times in the past, your instructor will decide how much, if any, of the kinds of details presented here are ‘mission critical’ to your understanding of Precalculus. Without further delay, we present you with this section’s Exercises.

<sup>16</sup>So even Jeff at this point may check for symmetry! We leave it to the reader to show  $r(-x) = r(x)$  so  $r$  is even, and, hence, its graph is symmetric about the  $y$ -axis.

<sup>17</sup>Without appealing to Calculus, of course.



## 4.2.1 EXERCISES

In Exercises 1 - 16, use the six-step procedure to graph the rational function. Be sure to draw any asymptotes as dashed lines.

1.  $f(x) = \frac{4}{x+2}$

2.  $f(x) = \frac{5x}{6-2x}$

3.  $f(x) = \frac{1}{x^2}$

4.  $f(x) = \frac{1}{x^2 + x - 12}$

5.  $f(x) = \frac{2x-1}{-2x^2-5x+3}$

6.  $f(x) = \frac{x}{x^2 + x - 12}$

7.  $f(x) = \frac{4x}{x^2 + 4}$

8.  $f(x) = \frac{4x}{x^2 - 4}$

9.  $f(x) = \frac{x^2 - x - 12}{x^2 + x - 6}$

10.  $f(x) = \frac{3x^2 - 5x - 2}{x^2 - 9}$

11.  $f(x) = \frac{x^2 - x - 6}{x + 1}$

12.  $f(x) = \frac{x^2 - x}{3 - x}$

13.  $f(x) = \frac{x^3 + 2x^2 + x}{x^2 - x - 2}$

14.  $f(x) = \frac{-x^3 + 4x}{x^2 - 9}$

15.  $f(x) = \frac{x^3 - 2x^2 + 3x}{2x^2 + 2}$

16.<sup>18</sup>  $f(x) = \frac{x^2 - 2x + 1}{x^3 + x^2 - 2x}$

In Exercises 17 - 20, graph the rational function by applying transformations to the graph of  $y = \frac{1}{x}$ .

17.  $f(x) = \frac{1}{x-2}$

18.  $g(x) = 1 - \frac{3}{x}$

19.  $h(x) = \frac{-2x+1}{x}$  (Hint: Divide)

20.  $j(x) = \frac{3x-7}{x-2}$  (Hint: Divide)

21. Discuss with your classmates how you would graph  $f(x) = \frac{ax+b}{cx+d}$ . What restrictions must be placed on  $a, b, c$  and  $d$  so that the graph is indeed a transformation of  $y = \frac{1}{x}$ ?

22. In Example 3.1.1 in Section 3.1 we showed that  $p(x) = \frac{4x+x^3}{x}$  is not a polynomial even though its formula reduced to  $4 + x^2$  for  $x \neq 0$ . However, it is a rational function similar to those studied in the section. With the help of your classmates, graph  $p(x)$ .

---

<sup>18</sup>Once you've done the six-step procedure, use your calculator to graph this function on the viewing window  $[0, 12] \times [0, 0.25]$ . What do you see?

23. Let  $g(x) = \frac{x^4 - 8x^3 + 24x^2 - 72x + 135}{x^3 - 9x^2 + 15x - 7}$ . With the help of your classmates, find the  $x$ - and  $y$ - intercepts of the graph of  $g$ . Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Find all of the asymptotes of the graph of  $g$  and any holes in the graph, if they exist. Be sure to show all of your work including any polynomial or synthetic division. Sketch the graph of  $g$ , using more than one picture if necessary to show all of the important features of the graph.

Example 4.2.4 showed us that the six-step procedure cannot tell us everything of importance about the graph of a rational function. Without Calculus, we need to use our graphing calculators to reveal the hidden mysteries of rational function behavior. Working with your classmates, use a graphing calculator to examine the graphs of the rational functions given in Exercises 24 - 27. Compare and contrast their features. Which features can the six-step process reveal and which features cannot be detected by it?

24.  $f(x) = \frac{1}{x^2 + 1}$

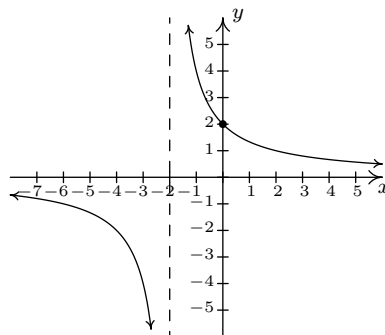
25.  $f(x) = \frac{x}{x^2 + 1}$

26.  $f(x) = \frac{x^2}{x^2 + 1}$

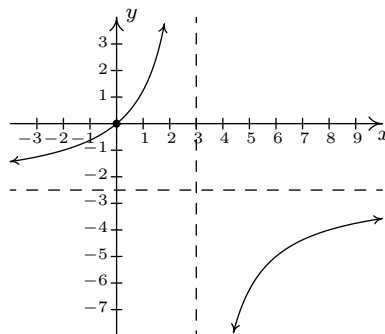
27.  $f(x) = \frac{x^3}{x^2 + 1}$

## 4.2.2 ANSWERS

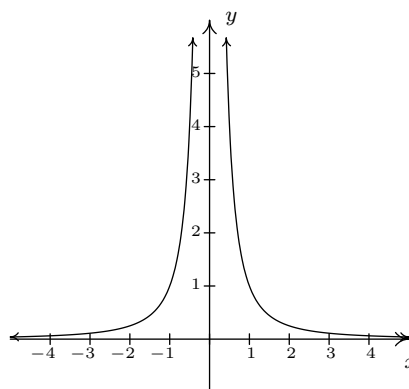
1.  $f(x) = \frac{4}{x+2}$

Domain:  $(-\infty, -2) \cup (-2, \infty)$ No  $x$ -intercepts $y$ -intercept:  $(0, 2)$ Vertical asymptote:  $x = -2$ As  $x \rightarrow -2^-$ ,  $f(x) \rightarrow -\infty$ As  $x \rightarrow -2^+$ ,  $f(x) \rightarrow \infty$ Horizontal asymptote:  $y = 0$ As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ 

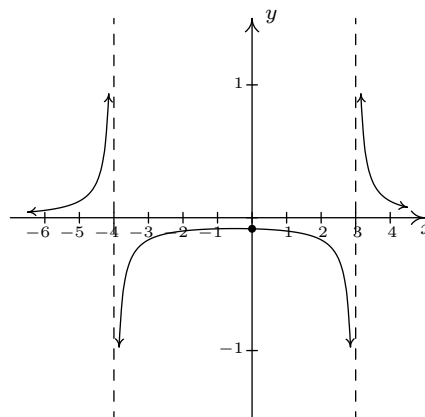
2.  $f(x) = \frac{5x}{6-2x}$

Domain:  $(-\infty, 3) \cup (3, \infty)$  $x$ -intercept:  $(0, 0)$  $y$ -intercept:  $(0, 0)$ Vertical asymptote:  $x = 3$ As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow \infty$ As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow -\infty$ Horizontal asymptote:  $y = -\frac{5}{2}$ As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\frac{5}{2}^+$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\frac{5}{2}^-$ 

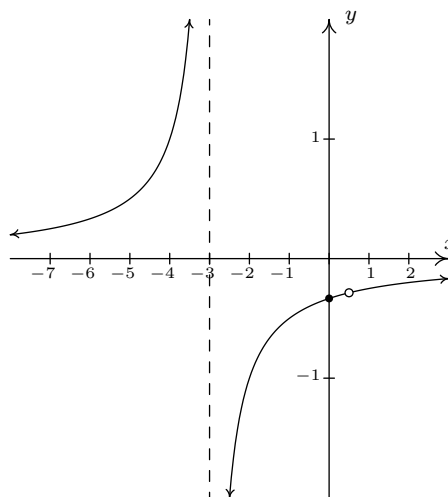
3.  $f(x) = \frac{1}{x^2}$

Domain:  $(-\infty, 0) \cup (0, \infty)$ No  $x$ -interceptsNo  $y$ -interceptsVertical asymptote:  $x = 0$ As  $x \rightarrow 0^-$ ,  $f(x) \rightarrow \infty$ As  $x \rightarrow 0^+$ ,  $f(x) \rightarrow \infty$ Horizontal asymptote:  $y = 0$ As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^+$ As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ 

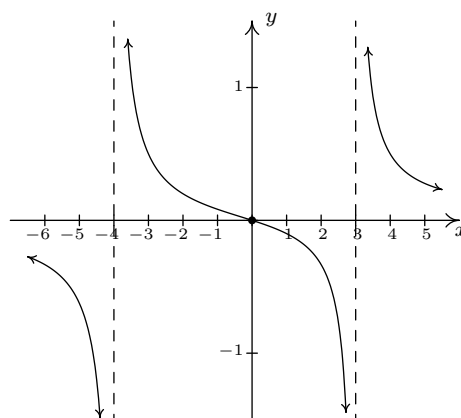
4.  $f(x) = \frac{1}{x^2 + x - 12} = \frac{1}{(x-3)(x+4)}$   
 Domain:  $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$   
 No  $x$ -intercepts  
 $y$ -intercept:  $(0, -\frac{1}{12})$   
 Vertical asymptotes:  $x = -4$  and  $x = 3$   
 As  $x \rightarrow -4^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -4^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow \infty$   
 Horizontal asymptote:  $y = 0$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^+$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$



5.  $f(x) = \frac{2x-1}{-2x^2-5x+3} = -\frac{2x-1}{(2x-1)(x+3)}$   
 Domain:  $(-\infty, -3) \cup (-3, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$   
 No  $x$ -intercepts  
 $y$ -intercept:  $(0, -\frac{1}{3})$   
 $f(x) = \frac{-1}{x+3}$ ,  $x \neq \frac{1}{2}$   
 Hole in the graph at  $(\frac{1}{2}, -\frac{2}{7})$   
 Vertical asymptote:  $x = -3$   
 As  $x \rightarrow -3^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -3^+$ ,  $f(x) \rightarrow -\infty$   
 Horizontal asymptote:  $y = 0$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^+$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^-$



6.  $f(x) = \frac{x}{x^2 + x - 12} = \frac{x}{(x-3)(x+4)}$   
 Domain:  $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$   
 $x$ -intercept:  $(0, 0)$   
 $y$ -intercept:  $(0, 0)$   
 Vertical asymptotes:  $x = -4$  and  $x = 3$   
 As  $x \rightarrow -4^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -4^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow \infty$   
 Horizontal asymptote:  $y = 0$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$



7.  $f(x) = \frac{4x}{x^2 + 4}$

Domain:  $(-\infty, \infty)$

$x$ -intercept:  $(0, 0)$

$y$ -intercept:  $(0, 0)$

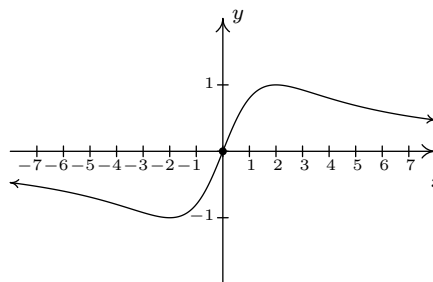
No vertical asymptotes

No holes in the graph

Horizontal asymptote:  $y = 0$

As  $x \rightarrow -\infty, f(x) \rightarrow 0^-$

As  $x \rightarrow \infty, f(x) \rightarrow 0^+$



8.  $f(x) = \frac{4x}{x^2 - 4} = \frac{4x}{(x+2)(x-2)}$

Domain:  $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$x$ -intercept:  $(0, 0)$

$y$ -intercept:  $(0, 0)$

Vertical asymptotes:  $x = -2, x = 2$

As  $x \rightarrow -2^-, f(x) \rightarrow -\infty$

As  $x \rightarrow -2^+, f(x) \rightarrow \infty$

As  $x \rightarrow 2^-, f(x) \rightarrow -\infty$

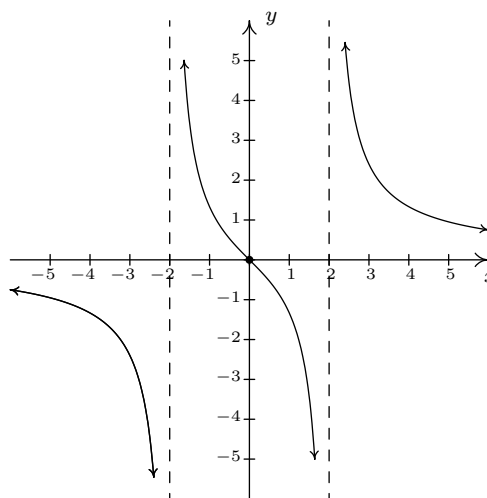
As  $x \rightarrow 2^+, f(x) \rightarrow \infty$

No holes in the graph

Horizontal asymptote:  $y = 0$

As  $x \rightarrow -\infty, f(x) \rightarrow 0^-$

As  $x \rightarrow \infty, f(x) \rightarrow 0^+$



9.  $f(x) = \frac{x^2 - x - 12}{x^2 + x - 6} = \frac{x-4}{x-2}, x \neq -3$

Domain:  $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$

$x$ -intercept:  $(4, 0)$

$y$ -intercept:  $(0, 2)$

Vertical asymptote:  $x = 2$

As  $x \rightarrow 2^-, f(x) \rightarrow \infty$

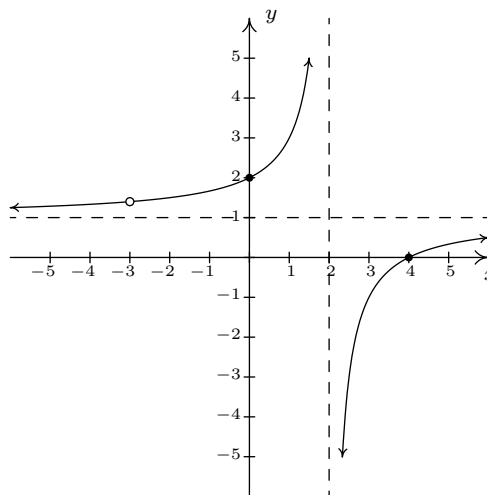
As  $x \rightarrow 2^+, f(x) \rightarrow -\infty$

Hole at  $(-3, \frac{7}{5})$

Horizontal asymptote:  $y = 1$

As  $x \rightarrow -\infty, f(x) \rightarrow 1^+$

As  $x \rightarrow \infty, f(x) \rightarrow 1^-$



$$10. f(x) = \frac{3x^2 - 5x - 2}{x^2 - 9} = \frac{(3x+1)(x-2)}{(x+3)(x-3)}$$

Domain:  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

$x$ -intercepts:  $(-\frac{1}{3}, 0), (2, 0)$

$y$ -intercept:  $(0, \frac{2}{9})$

Vertical asymptotes:  $x = -3, x = 3$

As  $x \rightarrow -3^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow -3^+$ ,  $f(x) \rightarrow -\infty$

As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow -\infty$

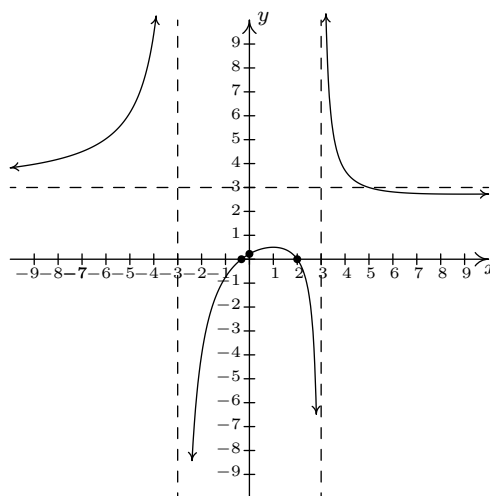
As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow \infty$

No holes in the graph

Horizontal asymptote:  $y = 3$

As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 3^+$

As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 3^-$



$$11. f(x) = \frac{x^2 - x - 6}{x + 1} = \frac{(x-3)(x+2)}{x+1}$$

Domain:  $(-\infty, -1) \cup (-1, \infty)$

$x$ -intercepts:  $(-2, 0), (3, 0)$

$y$ -intercept:  $(0, -6)$

Vertical asymptote:  $x = -1$

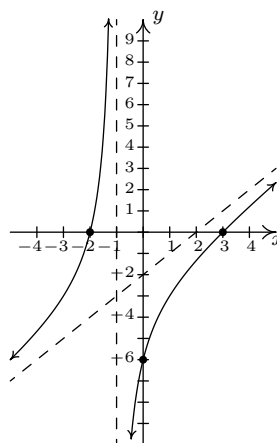
As  $x \rightarrow -1^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow -1^+$ ,  $f(x) \rightarrow -\infty$

Slant asymptote:  $y = x - 2$

As  $x \rightarrow -\infty$ , the graph is above  $y = x - 2$

As  $x \rightarrow \infty$ , the graph is below  $y = x - 2$



$$12. f(x) = \frac{x^2 - x}{3 - x} = \frac{x(x-1)}{3-x}$$

Domain:  $(-\infty, 3) \cup (3, \infty)$

$x$ -intercepts:  $(0, 0), (1, 0)$

$y$ -intercept:  $(0, 0)$

Vertical asymptote:  $x = 3$

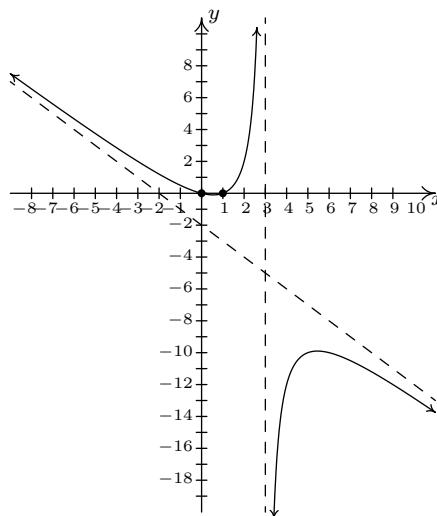
As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow -\infty$

Slant asymptote:  $y = -x - 2$

As  $x \rightarrow -\infty$ , the graph is above  $y = -x - 2$

As  $x \rightarrow \infty$ , the graph is below  $y = -x - 2$



13.  $f(x) = \frac{x^3 + 2x^2 + x}{x^2 - x - 2} = \frac{x(x+1)}{x-2} \quad x \neq -1$

Domain:  $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

$x$ -intercept:  $(0, 0)$

$y$ -intercept:  $(0, 0)$

Vertical asymptote:  $x = 2$

As  $x \rightarrow 2^-$ ,  $f(x) \rightarrow -\infty$

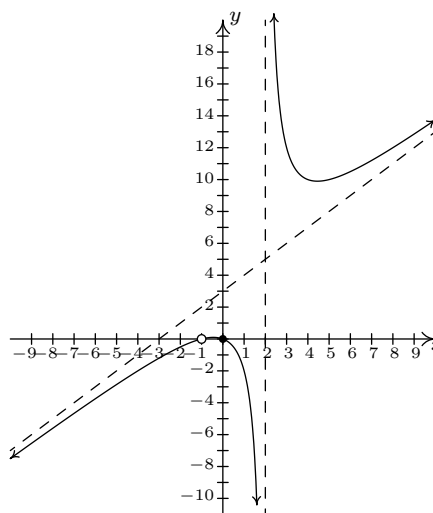
As  $x \rightarrow 2^+$ ,  $f(x) \rightarrow \infty$

Hole at  $(-1, 0)$

Slant asymptote:  $y = x + 3$

As  $x \rightarrow -\infty$ , the graph is below  $y = x + 3$

As  $x \rightarrow \infty$ , the graph is above  $y = x + 3$



14.  $f(x) = \frac{-x^3 + 4x}{x^2 - 9}$

Domain:  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

$x$ -intercepts:  $(-2, 0), (0, 0), (2, 0)$

$y$ -intercept:  $(0, 0)$

Vertical asymptotes:  $x = -3, x = 3$

As  $x \rightarrow -3^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow -3^+$ ,  $f(x) \rightarrow -\infty$

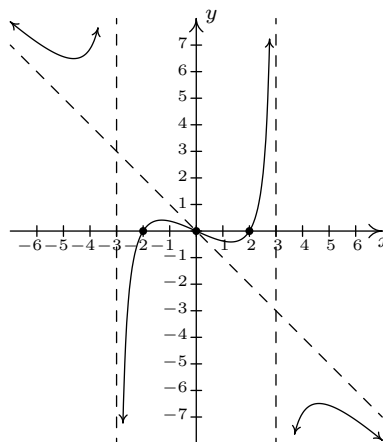
As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow -\infty$

Slant asymptote:  $y = -x$

As  $x \rightarrow -\infty$ , the graph is above  $y = -x$

As  $x \rightarrow \infty$ , the graph is below  $y = -x$



15.  $f(x) = \frac{x^3 - 2x^2 + 3x}{2x^2 + 2}$

Domain:  $(-\infty, \infty)$

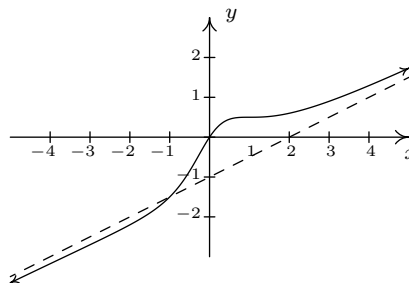
$x$ -intercept:  $(0, 0)$

$y$ -intercept:  $(0, 0)$

Slant asymptote:  $y = \frac{1}{2}x - 1$

As  $x \rightarrow -\infty$ , the graph is below  $y = \frac{1}{2}x - 1$

As  $x \rightarrow \infty$ , the graph is above  $y = \frac{1}{2}x - 1$



16.  $f(x) = \frac{x^2 - 2x + 1}{x^3 + x^2 - 2x}$

Domain:  $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, \infty)$

$$f(x) = \frac{x-1}{x(x+2)}, x \neq 1$$

No  $x$ -intercepts

No  $y$ -intercepts

Vertical asymptotes:  $x = -2$  and  $x = 0$

As  $x \rightarrow -2^-$ ,  $f(x) \rightarrow -\infty$

As  $x \rightarrow -2^+$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow 0^-$ ,  $f(x) \rightarrow \infty$

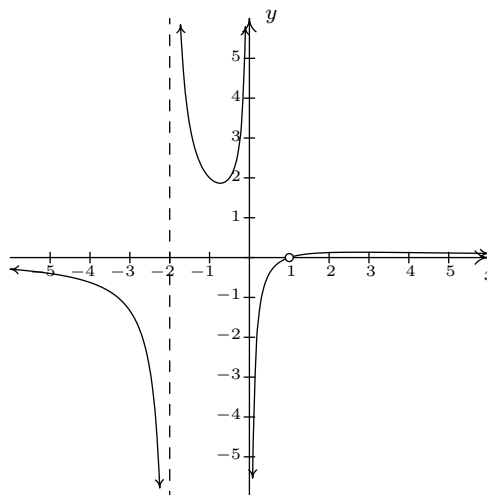
As  $x \rightarrow 0^+$ ,  $f(x) \rightarrow -\infty$

Hole in the graph at  $(1, 0)$

Horizontal asymptote:  $y = 0$

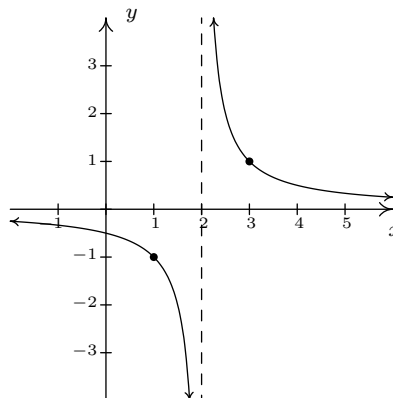
As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$

As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$



17.  $f(x) = \frac{1}{x-2}$

Shift the graph of  $y = \frac{1}{x}$   
to the right 2 units.

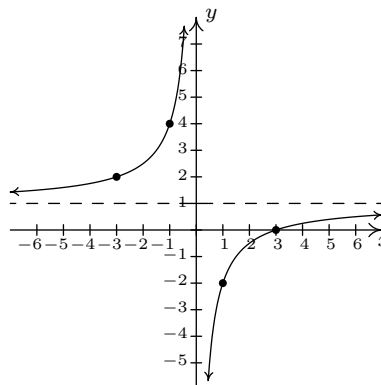


18.  $g(x) = 1 - \frac{3}{x}$

Vertically stretch the graph of  $y = \frac{1}{x}$   
by a factor of 3.

Reflect the graph of  $y = \frac{3}{x}$   
about the  $x$ -axis.

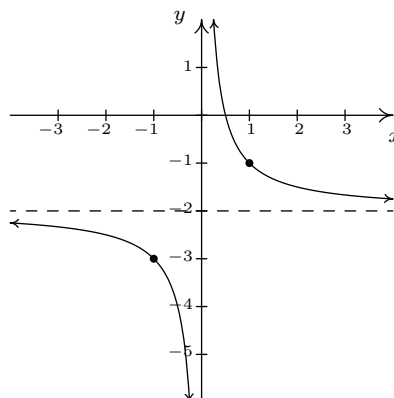
Shift the graph of  $y = -\frac{3}{x}$   
up 1 unit.





19.  $h(x) = \frac{-2x + 1}{x} = -2 + \frac{1}{x}$

Shift the graph of  $y = \frac{1}{x}$   
down 2 units.

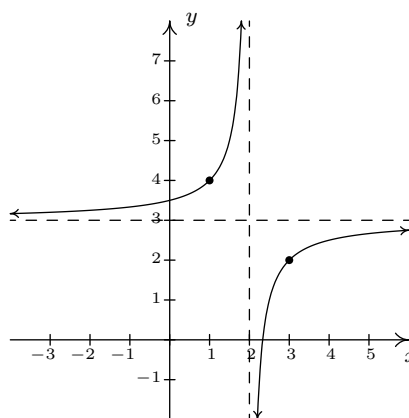


20.  $j(x) = \frac{3x - 7}{x - 2} = 3 - \frac{1}{x - 2}$

Shift the graph of  $y = \frac{1}{x}$   
to the right 2 units.

Reflect the graph of  $y = \frac{1}{x - 2}$   
about the  $x$ -axis.

Shift the graph of  $y = -\frac{1}{x - 2}$   
up 3 units.



### 4.3 RATIONAL INEQUALITIES AND APPLICATIONS

In this section, we solve equations and inequalities involving rational functions and explore associated application problems. Our first example showcases the critical difference in procedure between solving a rational equation and a rational inequality.

**Example 4.3.1.**

1. Solve  $\frac{x^3 - 2x + 1}{x - 1} = \frac{1}{2}x - 1$ .
2. Solve  $\frac{x^3 - 2x + 1}{x - 1} \geq \frac{1}{2}x - 1$ .
3. Use your calculator to graphically check your answers to 1 and 2.

**Solution.**

1. To solve the equation, we clear denominators

$$\begin{aligned}
 \frac{x^3 - 2x + 1}{x - 1} &= \frac{1}{2}x - 1 \\
 \left(\frac{x^3 - 2x + 1}{x - 1}\right) \cdot 2(x - 1) &= \left(\frac{1}{2}x - 1\right) \cdot 2(x - 1) \\
 2x^3 - 4x + 2 &= x^2 - 3x + 2 && \text{expand} \\
 2x^3 - x^2 - x &= 0 \\
 x(2x + 1)(x - 1) &= 0 && \text{factor} \\
 x &= -\frac{1}{2}, 0, 1
 \end{aligned}$$

Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that  $x = 1$  does not satisfy the original equation and must be discarded. Our solutions are  $x = -\frac{1}{2}$  and  $x = 0$ .

2. To solve the inequality, it may be tempting to begin as we did with the equation – namely by multiplying both sides by the quantity  $(x - 1)$ . The problem is that, depending on  $x$ ,  $(x - 1)$  may be positive (which doesn't affect the inequality) or  $(x - 1)$  could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page 321 in Section 4.2.

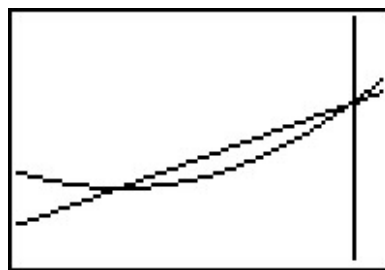
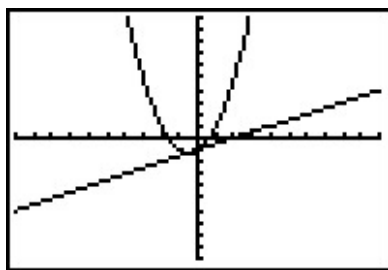
$$\begin{aligned}
 \frac{x^3 - 2x + 1}{x - 1} &\geq \frac{1}{2}x - 1 \\
 \frac{x^3 - 2x + 1}{x - 1} - \frac{1}{2}x + 1 &\geq 0 \\
 \frac{2(x^3 - 2x + 1) - x(x - 1) + 1(2(x - 1))}{2(x - 1)} &\geq 0 && \text{get a common denominator} \\
 \frac{2x^3 - x^2 - x}{2x - 2} &\geq 0 && \text{expand}
 \end{aligned}$$

Viewing the left hand side as a rational function  $r(x)$  we make a sign diagram. The only value excluded from the domain of  $r$  is  $x = 1$  which is the solution to  $2x - 2 = 0$ . The zeros of  $r$  are the solutions to  $2x^3 - x^2 - x = 0$ , which we have already found to be  $x = 0$ ,  $x = -\frac{1}{2}$  and  $x = 1$ , the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we construct the sign diagram below.

$$\begin{array}{ccccccc} (+) & 0 & (-) & 0 & (+) & ? & (+) \\ \leftarrow & -\frac{1}{2} & & 0 & & 1 & \rightarrow \end{array}$$

We are interested in where  $r(x) \geq 0$ . We find  $r(x) > 0$ , or  $(+)$ , on the intervals  $(-\infty, -\frac{1}{2})$ ,  $(0, 1)$  and  $(1, \infty)$ . We add to these intervals the zeros of  $r$ ,  $-\frac{1}{2}$  and  $0$ , to get our final solution:  $(-\infty, -\frac{1}{2}] \cup [0, 1) \cup (1, \infty)$ .

3. Geometrically, if we set  $f(x) = \frac{x^3 - 2x + 1}{x - 1}$  and  $g(x) = \frac{1}{2}x - 1$ , the solutions to  $f(x) = g(x)$  are the  $x$ -coordinates of the points where the graphs of  $y = f(x)$  and  $y = g(x)$  intersect. The solution to  $f(x) \geq g(x)$  represents not only where the graphs meet, but the intervals over which the graph of  $y = f(x)$  is above  $(>)$  the graph of  $g(x)$ . We obtain the graphs below.



The ‘Intersect’ command confirms that the graphs cross when  $x = -\frac{1}{2}$  and  $x = 0$ . It is clear from the calculator that the graph of  $y = f(x)$  is above the graph of  $y = g(x)$  on  $(-\infty, -\frac{1}{2})$  as well as on  $(0, \infty)$ . According to the calculator, our solution is then  $(-\infty, -\frac{1}{2}] \cup [0, \infty)$  which *almost* matches the answer we found analytically. We have to remember that  $f$  is not defined at  $x = 1$ , and, even though it isn’t shown on the calculator, there is a hole<sup>1</sup> in the graph of  $y = f(x)$  when  $x = 1$  which is why  $x = 1$  is not part of our final answer.  $\square$

Next, we explore how rational equations can be used to solve some classic problems involving rates.

**Example 4.3.2.** Carl decides to explore the Meander River, the location of several recent Sasquatch sightings. From camp, he canoes downstream five miles to check out a purported Sasquatch nest. Finding nothing, he immediately turns around, retraces his route (this time traveling upstream),

<sup>1</sup>There is no asymptote at  $x = 1$  since the graph is well behaved near  $x = 1$ . According to Theorem 4.1, there must be a hole there.

and returns to camp 3 hours after he left. If Carl canoes at a rate of 6 miles per hour in still water, how fast was the Meander River flowing on that day?

**Solution.** We are given information about distances, rates (speeds) and times. The basic principle relating these quantities is:

$$\text{distance} = \text{rate} \cdot \text{time}$$

The first observation to make, however, is that the distance, rate and time given to us aren't 'compatible': the distance given is the distance for only *part* of the trip, the rate given is the speed Carl can canoe in still water, not in a flowing river, and the time given is the duration of the *entire* trip. Ultimately, we are after the speed of the river, so let's call that  $R$  measured in miles per hour to be consistent with the other rate given to us. To get started, let's divide the trip into its two parts: the initial trip downstream and the return trip upstream. For the downstream trip, all we know is that the distance traveled is 5 miles.

$$\begin{aligned}\text{distance downstream} &= \text{rate traveling downstream} \cdot \text{time traveling downstream} \\ 5 \text{ miles} &= \text{rate traveling downstream} \cdot \text{time traveling downstream}\end{aligned}$$

Since the return trip upstream followed the same route as the trip downstream, we know that the distance traveled upstream is also 5 miles.

$$\begin{aligned}\text{distance upstream} &= \text{rate traveling upstream} \cdot \text{time traveling upstream} \\ 5 \text{ miles} &= \text{rate traveling upstream} \cdot \text{time traveling upstream}\end{aligned}$$

We are told Carl can canoe at a rate of 6 miles per hour in still water. How does this figure into the rates traveling upstream and downstream? The speed the canoe travels in the river is a combination of the speed at which Carl can propel the canoe in still water, 6 miles per hour, and the speed of the river, which we're calling  $R$ . When traveling downstream, the river is helping Carl along, so we *add* these two speeds:

$$\begin{aligned}\text{rate traveling downstream} &= \text{rate Carl propels the canoe} + \text{speed of the river} \\ &= 6 \frac{\text{miles}}{\text{hour}} + R \frac{\text{miles}}{\text{hour}}\end{aligned}$$

So our downstream speed is  $(6 + R) \frac{\text{miles}}{\text{hour}}$ . Substituting this into our 'distance-rate-time' equation for the downstream part of the trip, we get:

$$\begin{aligned}5 \text{ miles} &= \text{rate traveling downstream} \cdot \text{time traveling downstream} \\ 5 \text{ miles} &= (6 + R) \frac{\text{miles}}{\text{hour}} \cdot \text{time traveling downstream}\end{aligned}$$

When traveling upstream, Carl works against the current. Since the canoe manages to travel upstream, the speed Carl can canoe in still water is greater than the river's speed, so we *subtract* the river's speed *from* Carl's canoing speed to get:

$$\begin{aligned}\text{rate traveling upstream} &= \text{rate Carl propels the canoe} - \text{river speed} \\ &= 6 \frac{\text{miles}}{\text{hour}} - R \frac{\text{miles}}{\text{hour}}\end{aligned}$$

Proceeding as before, we get

$$\begin{aligned} 5 \text{ miles} &= \text{rate traveling upstream} \cdot \text{time traveling upstream} \\ 5 \text{ miles} &= (6 - R) \frac{\text{miles}}{\text{hour}} \cdot \text{time traveling upstream} \end{aligned}$$

The last piece of information given to us is that the total trip lasted 3 hours. If we let  $t_{\text{down}}$  denote the time of the downstream trip and  $t_{\text{up}}$  the time of the upstream trip, we have:  $t_{\text{down}} + t_{\text{up}} = 3$  hours. Substituting  $t_{\text{down}}$  and  $t_{\text{up}}$  into the ‘distance-rate-time’ equations, we get (suppressing the units) *three equations in three unknowns*:<sup>2</sup>

$$\begin{cases} E1 & (6 + R)t_{\text{down}} = 5 \\ E2 & (6 - R)t_{\text{up}} = 5 \\ E3 & t_{\text{down}} + t_{\text{up}} = 3 \end{cases}$$

Since we are ultimately after  $R$ , we need to use these three equations to get at least one equation involving only  $R$ . To that end, we solve  $E1$  for  $t_{\text{down}}$  by dividing both sides<sup>3</sup> by the quantity  $(6 + R)$  to get  $t_{\text{down}} = \frac{5}{6+R}$ . Similarly, we solve  $E2$  for  $t_{\text{up}}$  and get  $t_{\text{up}} = \frac{5}{6-R}$ . Substituting these into  $E3$ , we get:<sup>4</sup>

$$\frac{5}{6+R} + \frac{5}{6-R} = 3.$$

Clearing denominators, we get  $5(6 - R) + 5(6 + R) = 3(6 + R)(6 - R)$  which reduces to  $R^2 = 16$ . We find  $R = \pm 4$ , and since  $R$  represents the speed of the river, we choose  $R = 4$ . On the day in question, the Meander River is flowing at a rate of 4 miles per hour.  $\square$

One of the important lessons to learn from Example 4.3.2 is that speeds, and more generally, rates, are additive. As we see in our next example, the concept of rate and its associated principles can be applied to a wide variety of problems - not just ‘distance-rate-time’ scenarios.

**Example 4.3.3.** Working alone, Taylor can weed the garden in 4 hours. If Carl helps, they can weed the garden in 3 hours. How long would it take for Carl to weed the garden on his own?

**Solution.** The key relationship between work and time which we use in this problem is:

$$\text{amount of work done} = \text{rate of work} \cdot \text{time spent working}$$

We are told that, working alone, Taylor can weed the garden in 4 hours. In Taylor’s case then:

$$\begin{aligned} \text{amount of work Taylor does} &= \text{rate of Taylor working} \cdot \text{time Taylor spent working} \\ 1 \text{ garden} &= (\text{rate of Taylor working}) \cdot (4 \text{ hours}) \end{aligned}$$

So we have that the rate Taylor works is  $\frac{1 \text{ garden}}{4 \text{ hours}} = \frac{1}{4} \frac{\text{garden}}{\text{hour}}$ . We are also told that when working together, Taylor and Carl can weed the garden in just 3 hours. We have:

<sup>2</sup>This is called a *system* of equations. No doubt, you’ve had experience with these things before, and we will study systems in greater detail in Chapter 8.

<sup>3</sup>While we usually discourage dividing both sides of an equation by a variable expression, we know  $(6 + R) \neq 0$  since otherwise we couldn’t possibly multiply it by  $t_{\text{down}}$  and get 5.

<sup>4</sup>The reader is encouraged to verify that the units in this equation are the same on both sides. To get you started, the units on the ‘3’ is ‘hours.’

$$\begin{aligned}\text{amount of work done together} &= \text{rate of working together} \cdot \text{time spent working together} \\ 1 \text{ garden} &= (\text{rate of working together}) \cdot (3 \text{ hours})\end{aligned}$$

From this, we find that the rate of Taylor and Carl working together is  $\frac{1 \text{ garden}}{3 \text{ hours}} = \frac{1}{3} \frac{\text{garden}}{\text{hour}}$ . We are asked to find out how long it would take for Carl to weed the garden on his own. Let us call this unknown  $t$ , measured in hours to be consistent with the other times given to us in the problem. Then:

$$\begin{aligned}\text{amount of work Carl does} &= \text{rate of Carl working} \cdot \text{time Carl spent working} \\ 1 \text{ garden} &= (\text{rate of Carl working}) \cdot (t \text{ hours})\end{aligned}$$

In order to find  $t$ , we need to find the rate of Carl working, so let's call this quantity  $R$ , with units  $\frac{\text{garden}}{\text{hour}}$ . Using the fact that rates are additive, we have:

$$\begin{aligned}\text{rate working together} &= \text{rate of Taylor working} + \text{rate of Carl working} \\ \frac{1}{3} \frac{\text{garden}}{\text{hour}} &= \frac{1}{4} \frac{\text{garden}}{\text{hour}} + R \frac{\text{garden}}{\text{hour}}\end{aligned}$$

so that  $R = \frac{1}{12} \frac{\text{garden}}{\text{hour}}$ . Substituting this into our 'work-rate-time' equation for Carl, we get:

$$\begin{aligned}1 \text{ garden} &= (\text{rate of Carl working}) \cdot (t \text{ hours}) \\ 1 \text{ garden} &= \left( \frac{1}{12} \frac{\text{garden}}{\text{hour}} \right) \cdot (t \text{ hours})\end{aligned}$$

Solving  $1 = \frac{1}{12}t$ , we get  $t = 12$ , so it takes Carl 12 hours to weed the garden on his own.<sup>5</sup> □

As is common with 'word problems' like Examples 4.3.2 and 4.3.3, there is no short-cut to the answer. We encourage the reader to carefully think through and apply the basic principles of rate to each (potentially different!) situation. It is time well spent. We also encourage the tracking of units, especially in the early stages of the problem. Not only does this promote uniformity in the units, it also serves as a quick means to check if an equation makes sense.<sup>6</sup>

Our next example deals with the average cost function, first introduced on page 82, as applied to PortaBoy Game systems from Example 2.1.5 in Section 2.1.

**Example 4.3.4.** Given a cost function  $C(x)$ , which returns the total cost of producing  $x$  items, recall that the average cost function,  $\overline{C}(x) = \frac{C(x)}{x}$  computes the cost per item when  $x$  items are produced. Suppose the cost  $C$ , in dollars, to produce  $x$  PortaBoy game systems for a local retailer is  $C(x) = 80x + 150$ ,  $x \geq 0$ .

1. Find an expression for the average cost function  $\overline{C}(x)$ .
2. Solve  $\overline{C}(x) < 100$  and interpret.

<sup>5</sup>Carl would much rather spend his time writing open-source Mathematics texts than gardening anyway.

<sup>6</sup>In other words, make sure you don't try to add apples to oranges!

3. Determine the behavior of  $\overline{C}(x)$  as  $x \rightarrow \infty$  and interpret.

**Solution.**

1. From  $\overline{C}(x) = \frac{C(x)}{x}$ , we obtain  $\overline{C}(x) = \frac{80x+150}{x}$ . The domain of  $C$  is  $x \geq 0$ , but since  $x = 0$  causes problems for  $\overline{C}(x)$ , we get our domain to be  $x > 0$ , or  $(0, \infty)$ .
2. Solving  $\overline{C}(x) < 100$  means we solve  $\frac{80x+150}{x} < 100$ . We proceed as in the previous example.

$$\begin{aligned}\frac{80x+150}{x} &< 100 \\ \frac{80x+150}{x} - 100 &< 0 \\ \frac{80x+150-100x}{x} &< 0 \quad \text{common denominator} \\ \frac{150-20x}{x} &< 0\end{aligned}$$

If we take the left hand side to be a rational function  $r(x)$ , we need to keep in mind that the applied domain of the problem is  $x > 0$ . This means we consider only the positive half of the number line for our sign diagram. On  $(0, \infty)$ ,  $r$  is defined everywhere so we need only look for zeros of  $r$ . Setting  $r(x) = 0$  gives  $150 - 20x = 0$ , so that  $x = \frac{15}{2} = 7.5$ . The test intervals on our domain are  $(0, 7.5)$  and  $(7.5, \infty)$ . We find  $r(x) < 0$  on  $(7.5, \infty)$ .

$$\begin{array}{ccccccc} \uparrow & (+) & & 0 & & (-) & \\ | & & & | & & & \\ 0 & & & 7.5 & & & \end{array} \rightarrow$$

In the context of the problem,  $x$  represents the number of PortaBoy games systems produced and  $\overline{C}(x)$  is the average cost to produce each system. Solving  $\overline{C}(x) < 100$  means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution,  $(7.5, \infty)$  tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is  $[8, \infty)$ .

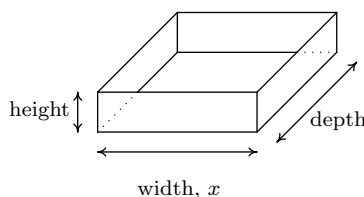
3. When we apply Theorem 4.2 to  $\overline{C}(x)$  we find that  $y = 80$  is a horizontal asymptote to the graph of  $y = \overline{C}(x)$ . To more precisely determine the behavior of  $\overline{C}(x)$  as  $x \rightarrow \infty$ , we first use long division<sup>7</sup> and rewrite  $\overline{C}(x) = 80 + \frac{150}{x}$ . As  $x \rightarrow \infty$ ,  $\frac{150}{x} \rightarrow 0^+$ , which means  $\overline{C}(x) \approx 80 + \text{very small } (+)$ . Thus the average cost per system is getting closer to \$80 per system. If we set  $\overline{C}(x) = 80$ , we get  $\frac{150}{x} = 0$ , which is impossible, so we conclude that  $\overline{C}(x) > 80$  for all  $x > 0$ . This means that the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example 2.1.5, we realize \$80 is the variable cost per system —

<sup>7</sup>In this case, long division amounts to term-by-term division.

the cost per system above and beyond the fixed initial cost of \$150. Another way to interpret our answer is that ‘infinitely’ many systems would need to be produced to effectively ‘zero out’ the fixed cost.  $\square$

Our next example is another classic ‘box with no top’ problem.

**Example 4.3.5.** A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimeters. Let  $x$  denote the width of the box, in centimeters as seen below.



1. Express the height  $h$  in centimeters as a function of the width  $x$  and state the applied domain.
2. Solve  $h(x) \geq x$  and interpret.
3. Find and interpret the behavior of  $h(x)$  as  $x \rightarrow 0^+$  and as  $x \rightarrow \infty$ .
4. Express the surface area  $S$  of the box as a function of  $x$  and state the applied domain.
5. Use a calculator to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

**Solution.**

1. We are told that the volume of the box is 1000 cubic centimeters and that  $x$  represents the width, in centimeters. From geometry, we know  $\text{Volume} = \text{width} \times \text{height} \times \text{depth}$ . Since the base of the box is a square, the width and the depth are both  $x$  centimeters. Using  $h$  for the height, we have  $1000 = x^2 h$ , so that  $h = \frac{1000}{x^2}$ . Using function notation,<sup>8</sup>  $h(x) = \frac{1000}{x^2}$ . As for the applied domain, in order for there to be a box at all,  $x > 0$ , and since every such choice of  $x$  will return a positive number for the height  $h$  we have no other restrictions and conclude our domain is  $(0, \infty)$ .
2. To solve  $h(x) \geq x$ , we proceed as before and collect all nonzero terms on one side of the inequality in order to use a sign diagram.

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<sup>8</sup>That is,  $h(x)$  means ‘ $h$  of  $x$ ’, not ‘ $h$  times  $x$ ’ here.



$$\begin{aligned}
 h(x) &\geq x \\
 \frac{1000}{x^2} &\geq x \\
 \frac{1000}{x^2} - x &\geq 0 \\
 \frac{1000 - x^3}{x^2} &\geq 0 \quad \text{common denominator}
 \end{aligned}$$

We consider the left hand side of the inequality as our rational function  $r(x)$ . We see  $r$  is undefined at  $x = 0$ , but, as in the previous example, the applied domain of the problem is  $x > 0$ , so we are considering only the behavior of  $r$  on  $(0, \infty)$ . The sole zero of  $r$  comes when  $1000 - x^3 = 0$ , which is  $x = 10$ . Choosing test values in the intervals  $(0, 10)$  and  $(10, \infty)$  gives the following diagram.

$$\begin{array}{ccccccc}
 & \text{?} & (+) & & 0 & & (-) \\
 & & & & | & & \\
 \hline
 & 0 & & & 10 & & \rightarrow
 \end{array}$$

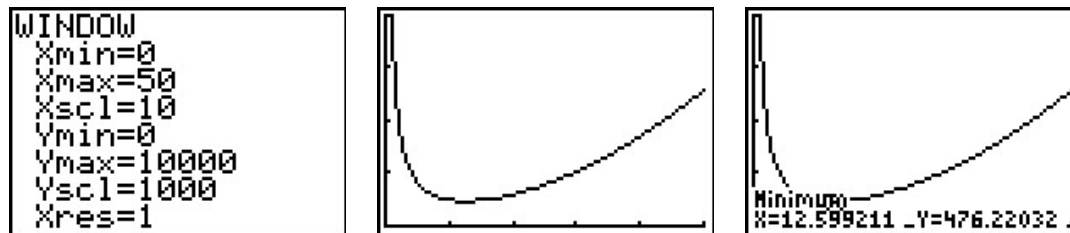
We see  $r(x) > 0$  on  $(0, 10)$ , and since  $r(x) = 0$  at  $x = 10$ , our solution is  $(0, 10]$ . In the context of the problem,  $h$  represents the height of the box while  $x$  represents the width (and depth) of the box. Solving  $h(x) \geq x$  is tantamount to finding the values of  $x$  which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimeters for this to happen.

- As  $x \rightarrow 0^+$ ,  $h(x) = \frac{1000}{x^2} \rightarrow \infty$ . This means that the smaller the width  $x$  (and, in this case, depth), the larger the height  $h$  has to be in order to maintain a volume of 1000 cubic centimeters. As  $x \rightarrow \infty$ , we find  $h(x) \rightarrow 0^+$ , which means that in order to maintain a volume of 1000 cubic centimeters, the width and depth must get bigger as the height becomes smaller.
- Since the box has no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions  $x$  by  $x$ , and each side has dimensions  $x$  by  $h$ . We get the surface area,  $S = x^2 + 4xh$ . To get  $S$  as a function of  $x$ , we substitute  $h = \frac{1000}{x^2}$  to obtain  $S = x^2 + 4x \left( \frac{1000}{x^2} \right)$ . Hence, as a function of  $x$ ,  $S(x) = x^2 + \frac{4000}{x}$ . The domain of  $S$  is the same as  $h$ , namely  $(0, \infty)$ , for the same reasons as above.
- A first attempt at the graph of  $y = S(x)$  on the calculator may lead to frustration. Chances are good that the first window chosen to view the graph will suggest  $y = S(x)$  has the  $x$ -axis as a horizontal asymptote. From the formula  $S(x) = x^2 + \frac{4000}{x}$ , however, we get  $S(x) \approx x^2$  as  $x \rightarrow \infty$ , so  $S(x) \rightarrow \infty$ . Readjusting the window, we find  $S$  does possess a relative minimum at  $x \approx 12.60$ . As far as we can tell,<sup>9</sup> this is the only relative extremum, so it is the absolute minimum as well. This means that the width and depth of the box should each measure

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<sup>9</sup>without Calculus, that is...

approximately 12.60 centimeters. To determine the height, we find  $h(12.60) \approx 6.30$ , so the height of the box should be approximately 6.30 centimeters.



□

#### 4.3.1 VARIATION

In many instances in the sciences, rational functions are encountered as a result of fundamental natural laws which are typically a result of assuming certain basic relationships between variables. These basic relationships are summarized in the definition below.

**Definition 4.5.** Suppose  $x$ ,  $y$  and  $z$  are variable quantities. We say

- $y$  **varies directly with** (or is **directly proportional to**)  $x$  if there is a constant  $k$  such that  $y = kx$ .
- $y$  **varies inversely with** (or is **inversely proportional to**)  $x$  if there is a constant  $k$  such that  $y = \frac{k}{x}$ .
- $z$  **varies jointly with** (or is **jointly proportional to**)  $x$  and  $y$  if there is a constant  $k$  such that  $z = kxy$ .

The constant  $k$  in the above definitions is called the **constant of proportionality**.

**Example 4.3.6.** Translate the following into mathematical equations using Definition 4.5.

1. Hooke's Law: The force  $F$  exerted on a spring is directly proportional the extension  $x$  of the spring.
2. Boyle's Law: At a constant temperature, the pressure  $P$  of an ideal gas is inversely proportional to its volume  $V$ .
3. The volume  $V$  of a right circular cone varies jointly with the height  $h$  of the cone and the square of the radius  $r$  of the base.
4. Ohm's Law: The current  $I$  through a conductor between two points is directly proportional to the voltage  $V$  between the two points and inversely proportional to the resistance  $R$  between the two points.

5. [Newton's Law of Universal Gravitation](#): Suppose two objects, one of mass  $m$  and one of mass  $M$ , are positioned so that the distance between their centers of mass is  $r$ . The gravitational force  $F$  exerted on the two objects varies directly with the product of the two masses and inversely with the square of the distance between their centers of mass.

**Solution.**

1. Applying the definition of direct variation, we get  $F = kx$  for some constant  $k$ .
2. Since  $P$  and  $V$  are inversely proportional, we write  $P = \frac{k}{V}$ .
3. There is a bit of ambiguity here. It's clear that the volume and the height of the cone are represented by the quantities  $V$  and  $h$ , respectively, but does  $r$  represent the radius of the base or the square of the radius of the base? It is the former. Usually, if an algebraic operation is specified (like squaring), it is meant to be expressed in the formula. We apply Definition 4.5 to get  $V = khr^2$ .
4. Even though the problem doesn't use the phrase 'varies jointly', it is implied by the fact that the current  $I$  is related to two different quantities. Since  $I$  varies directly with  $V$  but inversely with  $R$ , we write  $I = \frac{kV}{R}$ .
5. We write the product of the masses  $mM$  and the square of the distance as  $r^2$ . We have that  $F$  varies directly with  $mM$  and inversely with  $r^2$ , so  $F = \frac{kmM}{r^2}$ .  $\square$

In many of the formulas in the previous example, more than two varying quantities are related. In practice, however, usually all but two quantities are held constant in an experiment and the data collected is used to relate just two of the variables. Comparing just two varying quantities allows us to view the relationship between them as functional, as the next example illustrates.

**Example 4.3.7.** According to this [website](#) the actual data relating the volume  $V$  of a gas and its pressure  $P$  used by Boyle and his assistant in 1662 to verify the gas law that bears his name is given below.

$V$	48	46	44	42	40	38	36	34	32	30	28	26	24
$P$	29.13	30.56	31.94	33.5	35.31	37	39.31	41.63	44.19	47.06	50.31	54.31	58.81

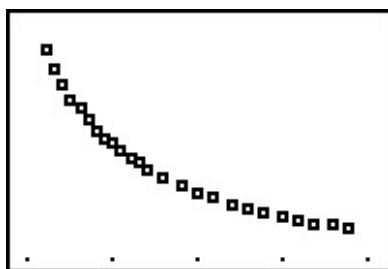
$V$	23	22	21	20	19	18	17	16	15	14	13	12
$P$	61.31	64.06	67.06	70.69	74.13	77.88	82.75	87.88	93.06	100.44	107.81	117.56

1. Use your calculator to generate a scatter diagram for these data using  $V$  as the independent variable and  $P$  as the dependent variable. Does it appear from the graph that  $P$  is inversely proportional to  $V$ ? Explain.
2. Assuming that  $P$  and  $V$  do vary inversely, use the data to approximate the constant of proportionality.

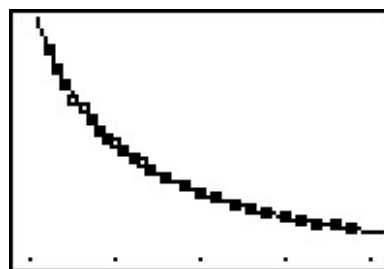
3. Use your calculator to determine a ‘Power Regression’ for this data<sup>10</sup> and use it verify your results in 1 and 2.

**Solution.**

1. If  $P$  really does vary inversely with  $V$ , then  $P = \frac{k}{V}$  for some constant  $k$ . From the data plot, the points do seem to lie along a curve like  $y = \frac{k}{x}$ .
2. To determine the constant of proportionality, we note that from  $P = \frac{k}{V}$ , we get  $k = PV$ . Multiplying each of the volume numbers times each of the pressure numbers,<sup>11</sup> we produce a number which is always approximately 1400. We suspect that  $P = \frac{1400}{V}$ . Graphing  $y = \frac{1400}{x}$  along with the data gives us good reason to believe our hypotheses that  $P$  and  $V$  are, in fact, inversely related.

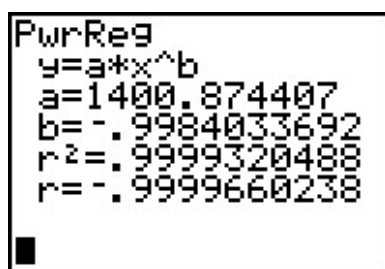


The graph of the data



The data with  $y = \frac{1400}{x}$

3. After performing a ‘Power Regression’, the calculator fits the data to the curve  $y = ax^b$  where  $a \approx 1400$  and  $b \approx -1$  with a correlation coefficient which is darned near perfect.<sup>12</sup> In other words,  $y = 1400x^{-1}$  or  $y = \frac{1400}{x}$ , as we guessed.



□

<sup>10</sup>We will talk more about this in the coming chapters.

<sup>11</sup>You can use tell the calculator to do this arithmetic on the lists and save yourself some time.

<sup>12</sup>We will revisit this example once we have developed logarithms in Chapter 6 to see how we can actually ‘linearize’ this data and do a linear regression to obtain the same result.

## 4.3.2 EXERCISES

In Exercises 1 - 6, solve the rational equation. Be sure to check for extraneous solutions.

1.  $\frac{x}{5x+4} = 3$

2.  $\frac{3x-1}{x^2+1} = 1$

3.  $\frac{1}{x+3} + \frac{1}{x-3} = \frac{x^2-3}{x^2-9}$

4.  $\frac{2x+17}{x+1} = x+5$

5.  $\frac{x^2-2x+1}{x^3+x^2-2x} = 1$

6.  $\frac{-x^3+4x}{x^2-9} = 4x$

In Exercises 7 - 20, solve the rational inequality. Express your answer using interval notation.

7.  $\frac{1}{x+2} \geq 0$

8.  $\frac{x-3}{x+2} \leq 0$

9.  $\frac{x}{x^2-1} > 0$

10.  $\frac{4x}{x^2+4} \geq 0$

11.  $\frac{x^2-x-12}{x^2+x-6} > 0$

12.  $\frac{3x^2-5x-2}{x^2-9} < 0$

13.  $\frac{x^3+2x^2+x}{x^2-x-2} \geq 0$

14.  $\frac{x^2+5x+6}{x^2-1} > 0$

15.  $\frac{3x-1}{x^2+1} \leq 1$

16.  $\frac{2x+17}{x+1} > x+5$

17.  $\frac{-x^3+4x}{x^2-9} \geq 4x$

18.  $\frac{1}{x^2+1} < 0$

19.  $\frac{x^4-4x^3+x^2-2x-15}{x^3-4x^2} \geq x$

20.  $\frac{5x^3-12x^2+9x+10}{x^2-1} \geq 3x-1$

21. Carl and Mike start a 3 mile race at the same time. If Mike ran the race at 6 miles per hour and finishes the race 10 minutes before Carl, how fast does Carl run?
22. One day, Donnie observes that the wind is blowing at 6 miles per hour. A unladen swallow nesting near Donnie's house flies three quarters of a mile down the road (in the direction of the wind), turns around, and returns exactly 4 minutes later. What is the airspeed of the unladen swallow? (Here, 'airspeed' is the speed that the swallow can fly in still air.)
23. In order to remove water from a flooded basement, two pumps, each rated at 40 gallons per minute, are used. After half an hour, the one pump burns out, and the second pump finishes removing the water half an hour later. How many gallons of water were removed from the basement?
24. A faucet can fill a sink in 5 minutes while a drain will empty the same sink in 8 minutes. If the faucet is turned on and the drain is left open, how long will it take to fill the sink?
25. Working together, Daniel and Donnie can clean the llama pen in 45 minutes. On his own, Daniel can clean the pen in an hour. How long does it take Donnie to clean the llama pen on his own?

26. In Exercise 32, the function  $C(x) = .03x^3 - 4.5x^2 + 225x + 250$ , for  $x \geq 0$  was used to model the cost (in dollars) to produce  $x$  PortaBoy game systems. Using this cost function, find the number of PortaBoys which should be produced to minimize the average cost  $\bar{C}$ . Round your answer to the nearest number of systems.
27. Suppose we are in the same situation as Example 4.3.5. If the volume of the box is to be 500 cubic centimeters, use your calculator to find the dimensions of the box which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.
28. The box for the new Sasquatch-themed cereal, ‘Crypt-Os’, is to have a volume of 140 cubic inches. For aesthetic reasons, the height of the box needs to be 1.62 times the width of the base of the box.<sup>13</sup> Find the dimensions of the box which will minimize the surface area of the box. What is the minimum surface area? Round your answers to two decimal places.
29. Sally is Skippy’s neighbor from Exercise 19 in Section 2.3. Sally also wants to plant a vegetable garden along the side of her home. She doesn’t have any fencing, but wants to keep the size of the garden to 100 square feet. What are the dimensions of the garden which will minimize the amount of fencing she needs to buy? What is the minimum amount of fencing she needs to buy? Round your answers to the nearest foot. (Note: Since one side of the garden will border the house, Sally doesn’t need fencing along that side.)
30. Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one pint. (For dry goods, one pint is equal to 33.6 cubic inches.)<sup>14</sup>
- Find an expression for the volume  $V$  of the can in terms of the height  $h$  and the base radius  $r$ .
  - Find an expression for the surface area  $S$  of the can in terms of the height  $h$  and the base radius  $r$ . (Hint: The top and bottom of the can are circles of radius  $r$  and the side of the can is really just a rectangle that has been bent into a cylinder.)
  - Using the fact that  $V = 33.6$ , write  $S$  as a function of  $r$  and state its applied domain.
  - Use your graphing calculator to find the dimensions of the can which has minimal surface area.
31. A right cylindrical drum is to hold 7.35 cubic feet of liquid. Find the dimensions (radius of the base and height) of the drum which would minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.
32. In Exercise 71 in Section 1.4, the population of Sasquatch in Portage County was modeled by the function  $P(t) = \frac{150t}{t+15}$ , where  $t = 0$  represents the year 1803. When were there fewer than 100 Sasquatch in Portage County?

<sup>13</sup>1.62 is a crude approximation of the so-called ‘Golden Ratio’  $\phi = \frac{1+\sqrt{5}}{2}$ .

<sup>14</sup>According to [www.dictionary.com](http://www.dictionary.com), there are different values given for this conversion. We will stick with 33.6in<sup>3</sup> for this problem.

In Exercises 33 - 38, translate the following into mathematical equations.

33. At a constant pressure, the temperature  $T$  of an ideal gas is directly proportional to its volume  $V$ . (This is [Charles's Law](#))
34. The frequency of a wave  $f$  is inversely proportional to the wavelength of the wave  $\lambda$ .
35. The density  $d$  of a material is directly proportional to the mass of the object  $m$  and inversely proportional to its volume  $V$ .
36. The square of the orbital period of a planet  $P$  is directly proportional to the cube of the semi-major axis of its orbit  $a$ . (This is [Kepler's Third Law of Planetary Motion](#) )
37. The drag of an object traveling through a fluid  $D$  varies jointly with the density of the fluid  $\rho$  and the square of the velocity of the object  $\nu$ .
38. Suppose two electric point charges, one with charge  $q$  and one with charge  $Q$ , are positioned  $r$  units apart. The electrostatic force  $F$  exerted on the charges varies directly with the product of the two charges and inversely with the square of the distance between the charges. (This is [Coulomb's Law](#))
39. According to [this webpage](#), the frequency  $f$  of a vibrating string is given by  $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$  where  $T$  is the tension,  $\mu$  is the linear mass<sup>15</sup> of the string and  $L$  is the length of the vibrating part of the string. Express this relationship using the language of variation.
40. According to the Centers for Disease Control and Prevention [www.cdc.gov](http://www.cdc.gov), a person's Body Mass Index  $B$  is directly proportional to his weight  $W$  in pounds and inversely proportional to the square of his height  $h$  in inches.
  - (a) Express this relationship as a mathematical equation.
  - (b) If a person who was 5 feet, 10 inches tall weighed 235 pounds had a Body Mass Index of 33.7, what is the value of the constant of proportionality?
  - (c) Rewrite the mathematical equation found in part 40a to include the value of the constant found in part 40b and then find your Body Mass Index.
41. We know that the circumference of a circle varies directly with its radius with  $2\pi$  as the constant of proportionality. (That is, we know  $C = 2\pi r$ .) With the help of your classmates, compile a list of other basic geometric relationships which can be seen as variations.

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<sup>15</sup>Also known as the linear density. It is simply a measure of mass per unit length.

## 4.3.3 ANSWERS

1.  $x = -\frac{6}{7}$
2.  $x = 1, x = 2$
3.  $x = -1$
4.  $x = -6, x = 2$
5. No solution
6.  $x = 0, x = \pm 2\sqrt{2}$
7.  $(-2, \infty)$
8.  $(-2, 3]$
9.  $(-1, 0) \cup (1, \infty)$
10.  $[0, \infty)$
11.  $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$
12.  $(-3, -\frac{1}{3}) \cup (2, 3)$
13.  $(-1, 0] \cup (2, \infty)$
14.  $(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$
15.  $(-\infty, 1] \cup [2, \infty)$
16.  $(-\infty, -6) \cup (-1, 2)$
17.  $(-\infty, -3) \cup [-2\sqrt{2}, 0] \cup [2\sqrt{2}, 3)$
18. No solution
19.  $[-3, 0) \cup (0, 4) \cup [5, \infty)$
20.  $(-1, -\frac{1}{2}] \cup (1, \infty)$
21. 4.5 miles per hour
22. 24 miles per hour
23. 3600 gallons
24.  $\frac{40}{3} \approx 13.33$  minutes
25. 3 hours
26. The absolute minimum of  $y = \overline{C}(x)$  occurs at  $\approx (75.73, 59.57)$ . Since  $x$  represents the number of game systems, we check  $\overline{C}(75) \approx 59.58$  and  $\overline{C}(76) \approx 59.57$ . Hence, to minimize the average cost, 76 systems should be produced at an average cost of \$59.57 per system.
27. The width (and depth) should be 10.00 centimeters, the height should be 5.00 centimeters. The minimum surface area is 300.00 square centimeters.
28. The width of the base of the box should be  $\approx 4.12$  inches, the height of the box should be  $\approx 6.67$  inches, and the depth of the base of the box should be  $\approx 5.09$  inches; minimum surface area  $\approx 164.91$  square inches.
29. The dimensions are  $\approx 7$  feet by  $\approx 14$  feet; minimum amount of fencing required  $\approx 28$  feet.
30. (a)  $V = \pi r^2 h$
- (b)  $S = 2\pi r^2 + 2\pi r h$
- (c)  $S(r) = 2\pi r^2 + \frac{67.2}{r}$ , Domain  $r > 0$
- (d)  $r \approx 1.749$  in. and  $h \approx 3.498$  in.
31. The radius of the drum should be  $\approx 1.05$  feet and the height of the drum should be  $\approx 2.12$  feet. The minimum surface area of the drum is  $\approx 20.93$  cubic feet.
32.  $P(t) < 100$  on  $(-15, 30)$ , and the portion of this which lies in the applied domain is  $[0, 30)$ . Since  $t = 0$  corresponds to the year 1803, from 1803 through the end of 1832, there were fewer than 100 Sasquatch in Portage County.



33.  $T = kV$

34. <sup>16</sup>  $f = \frac{k}{\lambda}$

35.  $d = \frac{km}{V}$

36.  $P^2 = ka^3$

37. <sup>17</sup>  $D = k\rho\nu^2$

38. <sup>18</sup>  $F = \frac{kqQ}{r^2}$

39. Rewriting  $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$  as  $f = \frac{\frac{1}{2}\sqrt{T}}{L\sqrt{\mu}}$  we see that the frequency  $f$  varies directly with the square root of the tension and varies inversely with the length and the square root of the linear mass.

40. (a)  $B = \frac{kW}{h^2}$

(b) <sup>19</sup>  $k = 702.68$

(c)  $B = \frac{702.68W}{h^2}$

---

<sup>16</sup>The character  $\lambda$  is the lower case Greek letter ‘lambda.’

<sup>17</sup>The characters  $\rho$  and  $\nu$  are the lower case Greek letters ‘rho’ and ‘nu,’ respectively.

<sup>18</sup>Note the similarity to this formula and Newton’s Law of Universal Gravitation as discussed in Example 5.

<sup>19</sup>The CDC uses 703.



## CHAPTER 5

# FURTHER TOPICS IN FUNCTIONS

### 5.1 FUNCTION COMPOSITION

Before we embark upon any further adventures with functions, we need to take some time to gather our thoughts and gain some perspective. Chapter 1 first introduced us to functions in Section 1.3. At that time, functions were specific kinds of relations - sets of points in the plane which passed the Vertical Line Test, Theorem 1.1. In Section 1.4, we developed the idea that functions are processes - rules which match inputs to outputs - and this gave rise to the concepts of domain and range. We spoke about how functions could be combined in Section 1.5 using the four basic arithmetic operations, took a more detailed look at their graphs in Section 1.6 and studied how their graphs behaved under certain classes of transformations in Section 1.7. In Chapter 2, we took a closer look at three families of functions: linear functions (Section 2.1), absolute value functions<sup>1</sup> (Section 2.2), and quadratic functions (Section 2.3). Linear and quadratic functions were special cases of polynomial functions, which we studied in generality in Chapter 3. Chapter 3 culminated with the Real Factorization Theorem, Theorem 3.16, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions. Our next step was to enlarge our field<sup>2</sup> of study to rational functions in Chapter 4. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Chapters 2, 3, and 4 can be thought of as an exhaustive study of linear and quadratic<sup>3</sup> functions and their arithmetic combinations as described in Section 1.5. We now wish to study other algebraic functions, such as  $f(x) = \sqrt{x}$  and  $g(x) = x^{2/3}$ , and the purpose of the first two sections of this chapter is to see how these kinds of functions arise from polynomial and rational functions. To that end, we first study a new way to combine functions as defined below.

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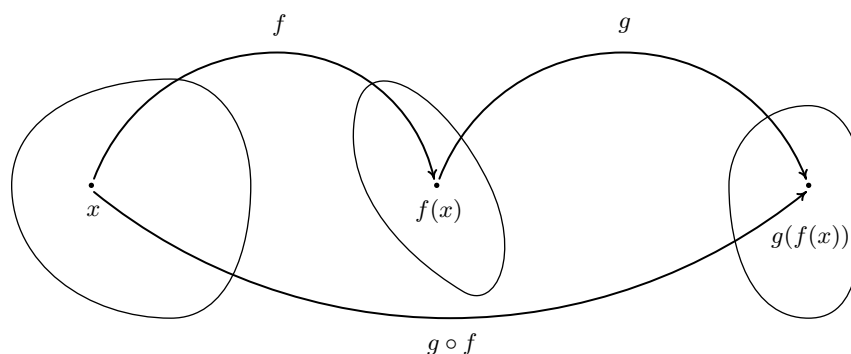
<sup>1</sup>These were introduced, as you may recall, as piecewise-defined linear functions.

<sup>2</sup>This is a really bad math pun.

<sup>3</sup>If we broaden our concept of functions to allow for complex valued coefficients, the Complex Factorization Theorem, Theorem 3.14, tells us every function we have studied thus far is a combination of linear functions.

**Definition 5.1.** Suppose  $f$  and  $g$  are two functions. The **composite** of  $g$  with  $f$ , denoted  $g \circ f$ , is defined by the formula  $(g \circ f)(x) = g(f(x))$ , provided  $x$  is an element of the domain of  $f$  and  $f(x)$  is an element of the domain of  $g$ .

The quantity  $g \circ f$  is also read ‘ $g$  composed with  $f$ ’ or, more simply ‘ $g$  of  $f$ .’ At its most basic level, Definition 5.1 tells us to obtain the formula for  $(g \circ f)(x)$ , we replace every occurrence of  $x$  in the formula for  $g(x)$  with the formula we have for  $f(x)$ . If we take a step back and look at this from a procedural, ‘inputs and outputs’ perspective, Definition 5.1 tells us the output from  $g \circ f$  is found by taking the output from  $f$ ,  $f(x)$ , and then making that the input to  $g$ . The result,  $g(f(x))$ , is the output from  $g \circ f$ . From this perspective, we see  $g \circ f$  as a two step process taking an input  $x$  and first applying the procedure  $f$  then applying the procedure  $g$ . Abstractly, we have



In the expression  $g(f(x))$ , the function  $f$  is often called the ‘inside’ function while  $g$  is often called the ‘outside’ function. There are two ways to go about evaluating composite functions - ‘inside out’ and ‘outside in’ - depending on which function we replace with its formula first. Both ways are demonstrated in the following example.

**Example 5.1.1.** Let  $f(x) = x^2 - 4x$ ,  $g(x) = 2 - \sqrt{x + 3}$ , and  $h(x) = \frac{2x}{x + 1}$ .

In numbers 1 - 3, find the indicated function value.

1.  $(g \circ f)(1)$
2.  $(f \circ g)(1)$
3.  $(g \circ g)(6)$

In numbers 4 - 10, find and simplify the indicated composite functions. State the domain of each.

4.  $(g \circ f)(x)$
5.  $(f \circ g)(x)$
6.  $(g \circ h)(x)$
7.  $(h \circ g)(x)$
8.  $(h \circ h)(x)$
9.  $(h \circ (g \circ f))(x)$
10.  $((h \circ g) \circ f)(x)$

**Solution.**

1. Using Definition 5.1,  $(g \circ f)(1) = g(f(1))$ . We find  $f(1) = -3$ , so

$$(g \circ f)(1) = g(f(1)) = g(-3) = 2$$

2. As before, we use Definition 5.1 to write  $(f \circ g)(1) = f(g(1))$ . We find  $g(1) = 0$ , so

$$(f \circ g)(1) = f(g(1)) = f(0) = 0$$

3. Once more, Definition 5.1 tells us  $(g \circ g)(6) = g(g(6))$ . That is, we evaluate  $g$  at 6, then plug that result back into  $g$ . Since  $g(6) = -1$ ,

$$(g \circ g)(6) = g(g(6)) = g(-1) = 2 - \sqrt{2}$$

4. By definition,  $(g \circ f)(x) = g(f(x))$ . We now illustrate *two* ways to approach this problem.

- *inside out*: We insert the expression  $f(x)$  into  $g$  first to get

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 4x) = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

Hence,  $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ .

- *outside in*: We use the formula for  $g$  first to get

$$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

We get the same answer as before,  $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ .

To find the domain of  $g \circ f$ , we need to find the elements in the domain of  $f$  whose outputs  $f(x)$  are in the domain of  $g$ . We accomplish this by following the rule set forth in Section 1.4, that is, we find the domain *before* we simplify. To that end, we examine  $(g \circ f)(x) = 2 - \sqrt{(x^2 - 4x) + 3}$ . To keep the square root happy, we solve the inequality  $x^2 - 4x + 3 \geq 0$  by creating a sign diagram. If we let  $r(x) = x^2 - 4x + 3$ , we find the zeros of  $r$  to be  $x = 1$  and  $x = 3$ . We obtain

$$\begin{array}{ccccccc} & (+) & 0 & & (-) & & 0 & (+) \\ & & | & & & & | & \\ \leftarrow & & 1 & & & & 3 & \rightarrow \end{array}$$

Our solution to  $x^2 - 4x + 3 \geq 0$ , and hence the domain of  $g \circ f$ , is  $(-\infty, 1] \cup [3, \infty)$ .

5. To find  $(f \circ g)(x)$ , we find  $f(g(x))$ .

- *inside out*: We insert the expression  $g(x)$  into  $f$  first to get

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = f(2 - \sqrt{x + 3}) \\ &= (2 - \sqrt{x + 3})^2 - 4(2 - \sqrt{x + 3}) \\ &= 4 - 4\sqrt{x + 3} + (\sqrt{x + 3})^2 - 8 + 4\sqrt{x + 3} \\ &= 4 + x + 3 - 8 \\ &= x - 1 \end{aligned}$$

- *outside in*: We use the formula for  $f(x)$  first to get

$$\begin{aligned}
 (f \circ g)(x) &= f(g(x)) = (g(x))^2 - 4(g(x)) \\
 &= (2 - \sqrt{x+3})^2 - 4(2 - \sqrt{x+3}) \\
 &= x - 1
 \end{aligned}$$

same algebra as before

Thus we get  $(f \circ g)(x) = x - 1$ . To find the domain of  $(f \circ g)$ , we look to the step before we did any simplification and find  $(f \circ g)(x) = (2 - \sqrt{x+3})^2 - 4(2 - \sqrt{x+3})$ . To keep the square root happy, we set  $x + 3 \geq 0$  and find our domain to be  $[-3, \infty)$ .

6. To find  $(g \circ h)(x)$ , we compute  $g(h(x))$ .

- *inside out*: We insert the expression  $h(x)$  into  $g$  first to get

$$\begin{aligned}
 (g \circ h)(x) &= g(h(x)) = g\left(\frac{2x}{x+1}\right) \\
 &= 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3} \\
 &= 2 - \sqrt{\frac{2x}{x+1} + \frac{3(x+1)}{x+1}} \quad \text{get common denominators} \\
 &= 2 - \sqrt{\frac{5x+3}{x+1}}
 \end{aligned}$$

- *outside in*: We use the formula for  $g(x)$  first to get

$$\begin{aligned}
 (g \circ h)(x) &= g(h(x)) = 2 - \sqrt{h(x) + 3} \\
 &= 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3} \\
 &= 2 - \sqrt{\frac{5x+3}{x+1}} \quad \text{get common denominators as before}
 \end{aligned}$$

To find the domain of  $(g \circ h)$ , we look to the step before we began to simplify:

$$(g \circ h)(x) = 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3}$$

To avoid division by zero, we need  $x \neq -1$ . To keep the radical happy, we need to solve

$$\frac{2x}{x+1} + 3 = \frac{5x+3}{x+1} \geq 0$$

Defining  $r(x) = \frac{5x+3}{x+1}$ , we see  $r$  is undefined at  $x = -1$  and  $r(x) = 0$  at  $x = -\frac{3}{5}$ . We get

$$\begin{array}{ccccccc} & (+) & ? & (-) & 0 & (+) & \\ & | & & | & | & & \\ \leftarrow & -1 & & -\frac{3}{5} & & & \rightarrow \end{array}$$

Our domain is  $(-\infty, -1) \cup [-\frac{3}{5}, \infty)$ .

7. We find  $(h \circ g)(x)$  by finding  $h(g(x))$ .

- *inside out*: We insert the expression  $g(x)$  into  $h$  first to get

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) = h(2 - \sqrt{x+3}) \\ &= \frac{2(2 - \sqrt{x+3})}{(2 - \sqrt{x+3}) + 1} \\ &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}} \end{aligned}$$

- *outside in*: We use the formula for  $h(x)$  first to get

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) = \frac{2(g(x))}{(g(x)) + 1} \\ &= \frac{2(2 - \sqrt{x+3})}{(2 - \sqrt{x+3}) + 1} \\ &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}} \end{aligned}$$

To find the domain of  $h \circ g$ , we look to the step before any simplification:

$$(h \circ g)(x) = \frac{2(2 - \sqrt{x+3})}{(2 - \sqrt{x+3}) + 1}$$

To keep the square root happy, we require  $x+3 \geq 0$  or  $x \geq -3$ . Setting the denominator equal to zero gives  $(2 - \sqrt{x+3}) + 1 = 0$  or  $\sqrt{x+3} = 3$ . Squaring both sides gives us  $x+3 = 9$ , or  $x = 6$ . Since  $x = 6$  checks in the original equation,  $(2 - \sqrt{x+3}) + 1 = 0$ , we know  $x = 6$  is the only zero of the denominator. Hence, the domain of  $h \circ g$  is  $[-3, 6) \cup (6, \infty)$ .

8. To find  $(h \circ h)(x)$ , we substitute the function  $h$  into itself,  $h(h(x))$ .

- *inside out*: We insert the expression  $h(x)$  into  $h$  to get

$$(h \circ h)(x) = h(h(x)) = h\left(\frac{2x}{x+1}\right)$$

$$\begin{aligned}
&= \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right) + 1} \\
&= \frac{\frac{4x}{x+1}}{\frac{2x}{x+1} + 1} \cdot \frac{(x+1)}{(x+1)} \\
&= \frac{\frac{4x}{x+1} \cdot (x+1)}{\left(\frac{2x}{x+1}\right) \cdot (x+1) + 1 \cdot (x+1)} \\
&= \frac{\cancel{(x+1)} \cdot 4x}{\cancel{(x+1)} \cdot 2x + \cancel{(x+1)} + x + 1} \\
&= \frac{4x}{3x+1}
\end{aligned}$$

- *outside in*: This approach yields

$$\begin{aligned}
(h \circ h)(x) &= h(h(x)) = \frac{2(h(x))}{h(x) + 1} \\
&= \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right) + 1} \\
&= \frac{4x}{3x+1} \qquad \text{same algebra as before}
\end{aligned}$$

To find the domain of  $h \circ h$ , we analyze

$$(h \circ h)(x) = \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right) + 1}$$

To keep the denominator  $x+1$  happy, we need  $x \neq -1$ . Setting the denominator

$$\frac{2x}{x+1} + 1 = 0$$

gives  $x = -\frac{1}{3}$ . Our domain is  $(-\infty, -1) \cup (-1, -\frac{1}{3}) \cup (-\frac{1}{3}, \infty)$ .



9. The expression  $(h \circ (g \circ f))(x)$  indicates that we first find the composite,  $g \circ f$  and compose the function  $h$  with the result. We know from number 1 that  $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ . We now proceed as usual.

- *inside out*: We insert the expression  $(g \circ f)(x)$  into  $h$  first to get

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

- *outside in*: We use the formula for  $h(x)$  first to get

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = \frac{2((g \circ f)(x))}{((g \circ f)(x)) + 1} \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

To find the domain of  $(h \circ (g \circ f))$ , we look at the step before we began to simplify,

$$(h \circ (g \circ f))(x) = \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1}$$

For the square root, we need  $x^2 - 4x + 3 \geq 0$ , which we determined in number 1 to be  $(-\infty, 1] \cup [3, \infty)$ . Next, we set the denominator to zero and solve:  $\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1 = 0$ . We get  $\sqrt{x^2 - 4x + 3} = 3$ , and, after squaring both sides, we have  $x^2 - 4x + 3 = 9$ . To solve  $x^2 - 4x - 6 = 0$ , we use the quadratic formula and get  $x = 2 \pm \sqrt{10}$ . The reader is encouraged to check that both of these numbers satisfy the original equation,  $\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1 = 0$ . Hence we must exclude these numbers from the domain of  $h \circ (g \circ f)$ . Our final domain for  $h \circ (f \circ g)$  is  $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup [3, 2 + \sqrt{10}) \cup (2 + \sqrt{10}, \infty)$ .

10. The expression  $((h \circ g) \circ f)(x)$  indicates that we first find the composite  $h \circ g$  and then compose that with  $f$ . From number 4, we have

$$(h \circ g)(x) = \frac{4 - 2\sqrt{x + 3}}{3 - \sqrt{x + 3}}$$

We now proceed as before.

- *inside out*: We insert the expression  $f(x)$  into  $h \circ g$  first to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = (h \circ g)(x^2 - 4x) \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

- *outside in*: We use the formula for  $(h \circ g)(x)$  first to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = \frac{4 - 2\sqrt{f(x) + 3}}{3 - \sqrt{f(x) + 3}} \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

We note that the formula for  $((h \circ g) \circ f)(x)$  before simplification is identical to that of  $(h \circ (g \circ f))(x)$  before we simplified it. Hence, the two functions have the same domain,  $h \circ (f \circ g)$  is  $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup [3, 2 + \sqrt{10}) \cup (2 + \sqrt{10}, \infty)$ .  $\square$

It should be clear from Example 5.1.1 that, in general, when you compose two functions, such as  $f$  and  $g$  above, the order matters.<sup>4</sup> We found that the functions  $f \circ g$  and  $g \circ f$  were different as were  $g \circ h$  and  $h \circ g$ . Thinking of functions as processes, this isn't all that surprising. If we think of one process as putting on our socks, and the other as putting on our shoes, the order in which we do these two tasks does matter.<sup>5</sup> Also note the importance of finding the domain of the composite function *before* simplifying. For instance, the domain of  $f \circ g$  is much different than its simplified formula would indicate. Composing a function with itself, as in the case of finding  $(g \circ g)(6)$  and  $(h \circ h)(x)$ , may seem odd. Looking at this from a procedural perspective, however, this merely indicates performing a task  $h$  and then doing it again - like setting the washing machine to do a 'double rinse'. Composing a function with itself is called 'iterating' the function, and we could easily spend an entire course on just that. The last two problems in Example 5.1.1 serve to demonstrate the **associative** property of functions. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. This property as well as another important property are listed in the theorem below.

<sup>4</sup>This shows us function composition isn't **commutative**. An example of an operation we perform on two functions which is commutative is function addition, which we defined in Section 1.5. In other words, the functions  $f + g$  and  $g + f$  are always equal. Which of the remaining operations on functions we have discussed are commutative?

<sup>5</sup>A more mathematical example in which the order of two processes matters can be found in Section 1.7. In fact, all of the transformations in that section can be viewed in terms of composing functions with linear functions.

**Theorem 5.1. Properties of Function Composition:** Suppose  $f$ ,  $g$ , and  $h$  are functions.

- $h \circ (g \circ f) = (h \circ g) \circ f$ , provided the composite functions are defined.
- If  $I$  is defined as  $I(x) = x$  for all real numbers  $x$ , then  $I \circ f = f \circ I = f$ .

By repeated applications of Definition 5.1, we find  $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$ . Similarly,  $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$ . This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality  $h \circ (g \circ f) = (h \circ g) \circ f$ . A consequence of the associativity of function composition is that there is no need for parentheses when we write  $h \circ g \circ f$ . The second property can also be verified using Definition 5.1. Recall that the function  $I(x) = x$  is called the *identity function* and was introduced in Exercise 73 in Section 2.1. If we compose the function  $I$  with a function  $f$ , then we have  $(I \circ f)(x) = I(f(x)) = f(x)$ , and a similar computation shows  $(f \circ I)(x) = f(x)$ . This establishes that we have an identity for function composition much in the same way the real number 1 is an identity for real number multiplication. That is, just as for any real number  $x$ ,  $1 \cdot x = x \cdot 1 = x$ , we have for any function  $f$ ,  $I \circ f = f \circ I = f$ . We shall see the concept of an identity take on great significance in the next section. Out in the wild, function composition is often used to relate two quantities which may not be directly related, but have a variable in common, as illustrated in our next example.

**Example 5.1.2.** The surface area  $S$  of a sphere is a function of its radius  $r$  and is given by the formula  $S(r) = 4\pi r^2$ . Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula  $r(t) = 3t^2$ , where  $t$  is measured in seconds,  $t \geq 0$ , and  $r$  is measured in inches. Find and interpret  $(S \circ r)(t)$ .

**Solution.** If we look at the functions  $S(r)$  and  $r(t)$  individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time,  $t$ , we could find the radius at that time,  $r(t)$  and feed that into  $S(r)$  to find the surface area at that time. From this we see that the surface area  $S$  is ultimately a function of time  $t$  and we find  $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi(3t^2)^2 = 36\pi t^4$ . This formula allows us to compute the surface area directly given the time without going through the ‘middle man’  $r$ .  $\square$

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates.

**Example 5.1.3.** Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

$$1. F(x) = |3x - 1| \qquad 2. G(x) = \frac{2}{x^2 + 1} \qquad 3. H(x) = \frac{\sqrt{x} + 1}{\sqrt{x} - 1}$$

**Solution.** There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function  $F$  as  $F = g \circ f$  for functions  $g$  and  $f$ . From Definition 5.1, we know  $F(x) = g(f(x))$ , and we can think of  $f(x)$  as being the ‘inside’ function and  $g$  as being the ‘outside’ function. Looking at  $F(x) = |3x - 1|$  from an ‘inside versus outside’ perspective, we can think of  $3x - 1$  being inside the absolute value symbols. Taking this cue, we define  $f(x) = 3x - 1$ . At this point, we have  $F(x) = |f(x)|$ . What is the outside function? The function which takes the absolute value of its input,  $g(x) = |x|$ . Sure enough,  $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$ , so we are done.
2. We attack deconstructing  $G$  from an operational approach. Given an input  $x$ , the first step is to square  $x$ , then add 1, then divide the result into 2. We will assign each of these steps a function so as to write  $G$  as a composite of three functions:  $f$ ,  $g$  and  $h$ . Our first function,  $f$ , is the function that squares its input,  $f(x) = x^2$ . The next function is the function that adds 1 to its input,  $g(x) = x + 1$ . Our last function takes its input and divides it into 2,  $h(x) = \frac{2}{x}$ . The claim is that  $G = h \circ g \circ f$ . We find

$$(h \circ g \circ f)(x) = h(g(f(x))) = h(g(x^2)) = h(x^2 + 1) = \frac{2}{x^2 + 1} = G(x),$$

so we are done.

3. If we look  $H(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$  with an eye towards building a complicated function from simpler functions, we see the expression  $\sqrt{x}$  is a simple piece of the larger function. If we define  $f(x) = \sqrt{x}$ , we have  $H(x) = \frac{f(x)+1}{f(x)-1}$ . If we want to decompose  $H = g \circ f$ , then we can glean the formula for  $g(x)$  by looking at what is being done to  $f(x)$ . We take  $g(x) = \frac{x+1}{x-1}$ , so

$$(g \circ f)(x) = g(f(x)) = \frac{f(x) + 1}{f(x) - 1} = \frac{\sqrt{x} + 1}{\sqrt{x} - 1} = H(x),$$

as required. □



In Exercises 25 - 30, use  $f(x) = -2x$ ,  $g(x) = \sqrt{x}$  and  $h(x) = |x|$  to find and simplify expressions for the following functions and state the domain of each using interval notation.

25.  $(h \circ g \circ f)(x)$

26.  $(h \circ f \circ g)(x)$

27.  $(g \circ f \circ h)(x)$

28.  $(g \circ h \circ f)(x)$

29.  $(f \circ h \circ g)(x)$

30.  $(f \circ g \circ h)(x)$

In Exercises 31 - 40, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

31.  $p(x) = (2x + 3)^3$

32.  $P(x) = (x^2 - x + 1)^5$

33.  $h(x) = \sqrt{2x - 1}$

34.  $H(x) = |7 - 3x|$

35.  $r(x) = \frac{2}{5x + 1}$

36.  $R(x) = \frac{7}{x^2 - 1}$

37.  $q(x) = \frac{|x| + 1}{|x| - 1}$

38.  $Q(x) = \frac{2x^3 + 1}{x^3 - 1}$

39.  $v(x) = \frac{2x + 1}{3 - 4x}$

40.  $w(x) = \frac{x^2}{x^4 + 1}$

41. Write the function  $F(x) = \sqrt{\frac{x^3 + 6}{x^3 - 9}}$  as a composition of three or more non-identity functions.

42. Let  $g(x) = -x$ ,  $h(x) = x + 2$ ,  $j(x) = 3x$  and  $k(x) = x - 4$ . In what order must these functions be composed with  $f(x) = \sqrt{x}$  to create  $F(x) = 3\sqrt{-x + 2} - 4$ ?

43. What linear functions could be used to transform  $f(x) = x^3$  into  $F(x) = -\frac{1}{2}(2x - 7)^3 + 1$ ? What is the proper order of composition?

In Exercises 44 - 55, let  $f$  be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let  $g$  be the function defined

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}$$

. Find the value if it exists.

44.  $(f \circ g)(3)$

45.  $f(g(-1))$

46.  $(f \circ f)(0)$

47.  $(f \circ g)(-3)$

48.  $(g \circ f)(3)$

49.  $g(f(-3))$

50.  $(g \circ g)(-2)$

51.  $(g \circ f)(-2)$

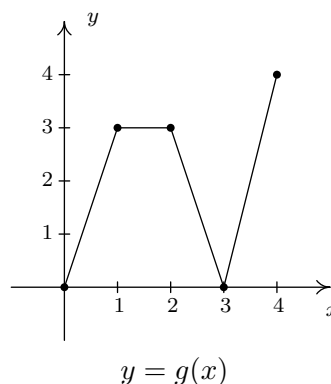
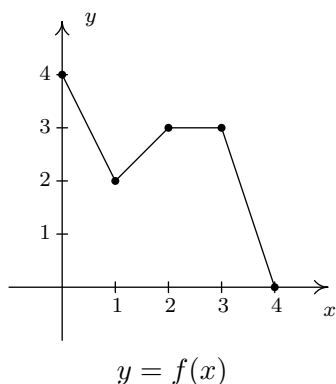
52.  $g(f(g(0)))$

53.  $f(f(f(-1)))$

54.  $f(f(f(f(f(1))))))$

55.  $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0)$

In Exercises 56 - 61, use the graphs of  $y = f(x)$  and  $y = g(x)$  below to find the function value.



56.  $(g \circ f)(1)$

57.  $(f \circ g)(3)$

58.  $(g \circ f)(2)$

59.  $(f \circ g)(0)$

60.  $(f \circ f)(1)$

61.  $(g \circ g)(1)$

62. The volume  $V$  of a cube is a function of its side length  $x$ . Let's assume that  $x = t + 1$  is also a function of time  $t$ , where  $x$  is measured in inches and  $t$  is measured in minutes. Find a formula for  $V$  as a function of  $t$ .
63. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour  $x$  is given by  $x(t) = -4t^2 + 20t + 92$ , where  $t$  is the number of hours since 10 AM,  $0 \leq t \leq 4$ .
- Find an expression for the revenue per hour  $R$  as a function of  $x$ .
  - Find and simplify  $(R \circ x)(t)$ . What does this represent?
  - What is the revenue per hour at noon?
64. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

## 5.1.2 ANSWERS

1. For  $f(x) = x^2$  and  $g(x) = 2x + 1$ ,

$$\begin{array}{lll} \bullet (g \circ f)(0) = 1 & \bullet (f \circ g)(-1) = 1 & \bullet (f \circ f)(2) = 16 \\ \bullet (g \circ f)(-3) = 19 & \bullet (f \circ g)\left(\frac{1}{2}\right) = 4 & \bullet (f \circ f)(-2) = 16 \end{array}$$

2. For  $f(x) = 4 - x$  and  $g(x) = 1 - x^2$ ,

$$\begin{array}{lll} \bullet (g \circ f)(0) = -15 & \bullet (f \circ g)(-1) = 4 & \bullet (f \circ f)(2) = 2 \\ \bullet (g \circ f)(-3) = -48 & \bullet (f \circ g)\left(\frac{1}{2}\right) = \frac{13}{4} & \bullet (f \circ f)(-2) = -2 \end{array}$$

3. For  $f(x) = 4 - 3x$  and  $g(x) = |x|$ ,

$$\begin{array}{lll} \bullet (g \circ f)(0) = 4 & \bullet (f \circ g)(-1) = 1 & \bullet (f \circ f)(2) = 10 \\ \bullet (g \circ f)(-3) = 13 & \bullet (f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2} & \bullet (f \circ f)(-2) = -26 \end{array}$$

4. For  $f(x) = |x - 1|$  and  $g(x) = x^2 - 5$ ,

$$\begin{array}{lll} \bullet (g \circ f)(0) = -4 & \bullet (f \circ g)(-1) = 5 & \bullet (f \circ f)(2) = 0 \\ \bullet (g \circ f)(-3) = 11 & \bullet (f \circ g)\left(\frac{1}{2}\right) = \frac{23}{4} & \bullet (f \circ f)(-2) = 2 \end{array}$$

5. For  $f(x) = 4x + 5$  and  $g(x) = \sqrt{x}$ ,

$$\begin{array}{lll} \bullet (g \circ f)(0) = \sqrt{5} & \bullet (f \circ g)(-1) \text{ is not real} & \bullet (f \circ f)(2) = 57 \\ \bullet (g \circ f)(-3) \text{ is not real} & \bullet (f \circ g)\left(\frac{1}{2}\right) = 5 + 2\sqrt{2} & \bullet (f \circ f)(-2) = -7 \end{array}$$

6. For  $f(x) = \sqrt{3 - x}$  and  $g(x) = x^2 + 1$ ,

$$\begin{array}{lll} \bullet (g \circ f)(0) = 4 & \bullet (f \circ g)(-1) = 1 & \bullet (f \circ f)(2) = \sqrt{2} \\ \bullet (g \circ f)(-3) = 7 & \bullet (f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt{7}}{2} & \bullet (f \circ f)(-2) = \sqrt{3 - \sqrt{5}} \end{array}$$



7. For  $f(x) = 6 - x - x^2$  and  $g(x) = x\sqrt{x+10}$ ,

- $(g \circ f)(0) = 24$                       •  $(f \circ g)(-1) = 0$                       •  $(f \circ f)(2) = 6$
- $(g \circ f)(-3) = 0$                       •  $(f \circ g)\left(\frac{1}{2}\right) = \frac{27-2\sqrt{42}}{8}$                       •  $(f \circ f)(-2) = -14$

8. For  $f(x) = \sqrt[3]{x+1}$  and  $g(x) = 4x^2 - x$ ,

- $(g \circ f)(0) = 3$                       •  $(f \circ g)(-1) = \sqrt[3]{6}$                       •  $(f \circ f)(2) = \sqrt[3]{\sqrt[3]{3}+1}$
- $(g \circ f)(-3) = 4\sqrt[3]{4} + \sqrt[3]{2}$                       •  $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt[3]{12}}{2}$                       •  $(f \circ f)(-2) = 0$

9. For  $f(x) = \frac{3}{1-x}$  and  $g(x) = \frac{4x}{x^2+1}$ ,

- $(g \circ f)(0) = \frac{6}{5}$                       •  $(f \circ g)(-1) = 1$                       •  $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$                       •  $(f \circ g)\left(\frac{1}{2}\right) = -5$                       •  $(f \circ f)(-2)$  is undefined

10. For  $f(x) = \frac{x}{x+5}$  and  $g(x) = \frac{2}{7-x^2}$ ,

- $(g \circ f)(0) = \frac{2}{7}$                       •  $(f \circ g)(-1) = \frac{1}{16}$                       •  $(f \circ f)(2) = \frac{2}{37}$
- $(g \circ f)(-3) = \frac{8}{19}$                       •  $(f \circ g)\left(\frac{1}{2}\right) = \frac{8}{143}$                       •  $(f \circ f)(-2) = -\frac{2}{13}$

11. For  $f(x) = \frac{2x}{5-x^2}$  and  $g(x) = \sqrt{4x+1}$ ,

- $(g \circ f)(0) = 1$                       •  $(f \circ g)(-1)$  is not real                      •  $(f \circ f)(2) = -\frac{8}{11}$
- $(g \circ f)(-3) = \sqrt{7}$                       •  $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$                       •  $(f \circ f)(-2) = \frac{8}{11}$

12. For  $f(x) = \sqrt{2x+5}$  and  $g(x) = \frac{10x}{x^2+1}$ ,

- $(g \circ f)(0) = \frac{5\sqrt{5}}{3}$                       •  $(f \circ g)(-1)$  is not real                      •  $(f \circ f)(2) = \sqrt{11}$
- $(g \circ f)(-3)$  is not real                      •  $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{13}$                       •  $(f \circ f)(-2) = \sqrt{7}$

13. For  $f(x) = 2x + 3$  and  $g(x) = x^2 - 9$

- $(g \circ f)(x) = 4x^2 + 12x$ , domain:  $(-\infty, \infty)$
- $(f \circ g)(x) = 2x^2 - 15$ , domain:  $(-\infty, \infty)$
- $(f \circ f)(x) = 4x + 9$ , domain:  $(-\infty, \infty)$

14. For  $f(x) = x^2 - x + 1$  and  $g(x) = 3x - 5$

- $(g \circ f)(x) = 3x^2 - 3x - 2$ , domain:  $(-\infty, \infty)$
- $(f \circ g)(x) = 9x^2 - 33x + 31$ , domain:  $(-\infty, \infty)$
- $(f \circ f)(x) = x^4 - 2x^3 + 2x^2 - x + 1$ , domain:  $(-\infty, \infty)$

15. For  $f(x) = x^2 - 4$  and  $g(x) = |x|$

- $(g \circ f)(x) = |x^2 - 4|$ , domain:  $(-\infty, \infty)$
- $(f \circ g)(x) = |x|^2 - 4 = x^2 - 4$ , domain:  $(-\infty, \infty)$
- $(f \circ f)(x) = x^4 - 8x^2 + 12$ , domain:  $(-\infty, \infty)$

16. For  $f(x) = 3x - 5$  and  $g(x) = \sqrt{x}$

- $(g \circ f)(x) = \sqrt{3x - 5}$ , domain:  $[\frac{5}{3}, \infty)$
- $(f \circ g)(x) = 3\sqrt{x} - 5$ , domain:  $[0, \infty)$
- $(f \circ f)(x) = 9x - 20$ , domain:  $(-\infty, \infty)$

17. For  $f(x) = |x + 1|$  and  $g(x) = \sqrt{x}$

- $(g \circ f)(x) = \sqrt{|x + 1|}$ , domain:  $(-\infty, \infty)$
- $(f \circ g)(x) = |\sqrt{x} + 1| = \sqrt{x} + 1$ , domain:  $[0, \infty)$
- $(f \circ f)(x) = ||x + 1| + 1| = |x + 1| + 1$ , domain:  $(-\infty, \infty)$

18. For  $f(x) = 3 - x^2$  and  $g(x) = \sqrt{x + 1}$

- $(g \circ f)(x) = \sqrt{4 - x^2}$ , domain:  $[-2, 2]$
- $(f \circ g)(x) = 2 - x$ , domain:  $[-1, \infty)$
- $(f \circ f)(x) = -x^4 + 6x^2 - 6$ , domain:  $(-\infty, \infty)$

19. For  $f(x) = |x|$  and  $g(x) = \sqrt{4 - x}$

- $(g \circ f)(x) = \sqrt{4 - |x|}$ , domain:  $[-4, 4]$
- $(f \circ g)(x) = |\sqrt{4 - x}| = \sqrt{4 - x}$ , domain:  $(-\infty, 4]$
- $(f \circ f)(x) = ||x|| = |x|$ , domain:  $(-\infty, \infty)$

20. For  $f(x) = x^2 - x - 1$  and  $g(x) = \sqrt{x - 5}$

- $(g \circ f)(x) = \sqrt{x^2 - x - 6}$ , domain:  $(-\infty, -2] \cup [3, \infty)$
- $(f \circ g)(x) = x - 6 - \sqrt{x - 5}$ , domain:  $[5, \infty)$
- $(f \circ f)(x) = x^4 - 2x^3 - 2x^2 + 3x + 1$ , domain:  $(-\infty, \infty)$

21. For  $f(x) = 3x - 1$  and  $g(x) = \frac{1}{x+3}$

- $(g \circ f)(x) = \frac{1}{3x+2}$ , domain:  $(-\infty, -\frac{2}{3}) \cup (-\frac{2}{3}, \infty)$
- $(f \circ g)(x) = -\frac{x}{x+3}$ , domain:  $(-\infty, -3) \cup (-3, \infty)$
- $(f \circ f)(x) = 9x - 4$ , domain:  $(-\infty, \infty)$

22. For  $f(x) = \frac{3x}{x-1}$  and  $g(x) = \frac{x}{x-3}$

- $(g \circ f)(x) = x$ , domain:  $(-\infty, 1) \cup (1, \infty)$
- $(f \circ g)(x) = x$ , domain:  $(-\infty, 3) \cup (3, \infty)$
- $(f \circ f)(x) = \frac{9x}{2x+1}$ , domain:  $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 1) \cup (1, \infty)$

23. For  $f(x) = \frac{x}{2x+1}$  and  $g(x) = \frac{2x+1}{x}$

- $(g \circ f)(x) = \frac{4x+1}{x}$ , domain:  $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 0) \cup (0, \infty)$
- $(f \circ g)(x) = \frac{2x+1}{5x+2}$ , domain:  $(-\infty, -\frac{2}{5}) \cup (-\frac{2}{5}, 0) \cup (0, \infty)$
- $(f \circ f)(x) = \frac{x}{4x+1}$ , domain:  $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, -\frac{1}{4}) \cup (-\frac{1}{4}, \infty)$

24. For  $f(x) = \frac{2x}{x^2-4}$  and  $g(x) = \sqrt{1-x}$

- $(g \circ f)(x) = \sqrt{\frac{x^2-2x-4}{x^2-4}}$ , domain:  $(-\infty, -2) \cup [1 - \sqrt{5}, 2) \cup [1 + \sqrt{5}, \infty)$
- $(f \circ g)(x) = -\frac{2\sqrt{1-x}}{x+3}$ , domain:  $(-\infty, -3) \cup (-3, 1]$
- $(f \circ f)(x) = \frac{4x-x^3}{x^4-9x^2+16}$ , domain:  $(-\infty, -\frac{1+\sqrt{17}}{2}) \cup (-\frac{1+\sqrt{17}}{2}, -2) \cup (-2, \frac{1-\sqrt{17}}{2}) \cup (\frac{1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}) \cup (\frac{-1+\sqrt{17}}{2}, 2) \cup (2, \frac{1+\sqrt{17}}{2}) \cup (\frac{1+\sqrt{17}}{2}, \infty)$

25.  $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$ , domain:  $(-\infty, 0]$

26.  $(h \circ f \circ g)(x) = |-2\sqrt{x}| = 2\sqrt{x}$ , domain:  $[0, \infty)$

27.  $(g \circ f \circ h)(x) = \sqrt{-2|x|}$ , domain:  $\{0\}$

28.  $(g \circ h \circ f)(x) = \sqrt{|-2x|} = \sqrt{2|x|}$ , domain:  $(-\infty, \infty)$

29.  $(f \circ h \circ g)(x) = -2|\sqrt{x}| = -2\sqrt{x}$ , domain:  $[0, \infty)$

30.  $(f \circ g \circ h)(x) = -2\sqrt{|x|}$ , domain:  $(-\infty, \infty)$
31. Let  $f(x) = 2x + 3$  and  $g(x) = x^3$ , then  $p(x) = (g \circ f)(x)$ .
32. Let  $f(x) = x^2 - x + 1$  and  $g(x) = x^5$ ,  $P(x) = (g \circ f)(x)$ .
33. Let  $f(x) = 2x - 1$  and  $g(x) = \sqrt{x}$ , then  $h(x) = (g \circ f)(x)$ .
34. Let  $f(x) = 7 - 3x$  and  $g(x) = |x|$ , then  $H(x) = (g \circ f)(x)$ .
35. Let  $f(x) = 5x + 1$  and  $g(x) = \frac{2}{x}$ , then  $r(x) = (g \circ f)(x)$ .
36. Let  $f(x) = x^2 - 1$  and  $g(x) = \frac{7}{x}$ , then  $R(x) = (g \circ f)(x)$ .
37. Let  $f(x) = |x|$  and  $g(x) = \frac{x+1}{x-1}$ , then  $q(x) = (g \circ f)(x)$ .
38. Let  $f(x) = x^3$  and  $g(x) = \frac{2x+1}{x-1}$ , then  $Q(x) = (g \circ f)(x)$ .
39. Let  $f(x) = 2x$  and  $g(x) = \frac{x+1}{3-2x}$ , then  $v(x) = (g \circ f)(x)$ .
40. Let  $f(x) = x^2$  and  $g(x) = \frac{x}{x^2+1}$ , then  $w(x) = (g \circ f)(x)$ .
41.  $F(x) = \sqrt{\frac{x^3+6}{x^3-9}} = (h(g(f(x))))$  where  $f(x) = x^3$ ,  $g(x) = \frac{x+6}{x-9}$  and  $h(x) = \sqrt{x}$ .
42.  $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$
43. One possible solution is  $F(x) = -\frac{1}{2}(2x-7)^3 + 1 = k(j(f(h(g(x)))))$  where  $g(x) = 2x$ ,  $h(x) = x - 7$ ,  $j(x) = -\frac{1}{2}x$  and  $k(x) = x + 1$ . You could also have  $F(x) = H(f(G(x)))$  where  $G(x) = 2x - 7$  and  $H(x) = -\frac{1}{2}x + 1$ .
44.  $(f \circ g)(3) = f(g(3)) = f(2) = 4$
45.  $f(g(-1)) = f(-4)$  which is undefined
46.  $(f \circ f)(0) = f(f(0)) = f(1) = 3$
47.  $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$
48.  $(g \circ f)(3) = g(f(3)) = g(-1) = -4$
49.  $g(f(-3)) = g(4)$  which is undefined
50.  $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$
51.  $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$
52.  $g(f(g(0))) = g(f(0)) = g(1) = -3$
53.  $f(f(f(-1))) = f(f(0)) = f(1) = 3$
54.  $f(f(f(f(f(1)))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$
55.  $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0) = 0$

56.  $(g \circ f)(1) = 3$

57.  $(f \circ g)(3) = 4$

58.  $(g \circ f)(2) = 0$

59.  $(f \circ g)(0) = 4$

60.  $(f \circ f)(1) = 3$

61.  $(g \circ g)(1) = 0$

62.  $V(x) = x^3$  so  $V(x(t)) = (t + 1)^3$

63. (a)  $R(x) = 2x$

(b)  $(R \circ x)(t) = -8t^2 + 40t + 184$ ,  $0 \leq t \leq 4$ . This gives the revenue per hour as a function of time.

(c) Noon corresponds to  $t = 2$ , so  $(R \circ x)(2) = 232$ . The hourly revenue at noon is \$232 per hour.

## 5.2 INVERSE FUNCTIONS

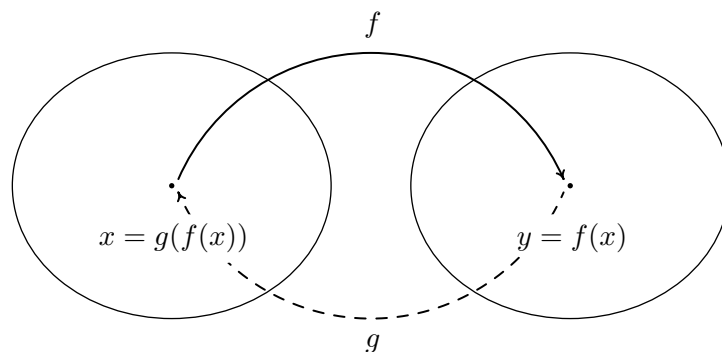
Thinking of a function as a process like we did in Section 1.4, in this section we seek another function which might reverse that process. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function which is reversible,  $f(x) = 3x + 4$ . Thinking of  $f$  as a process, we start with an input  $x$  and apply two steps, as we saw in Section 1.4

1. multiply by 3
2. add 4

To reverse this process, we seek a function  $g$  which will undo each of these steps and take the output from  $f$ ,  $3x + 4$ , and return the input  $x$ . If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function  $g$  should undo the second step of  $f$  first. That is, the function  $g$  should

1. *subtract* 4
2. *divide* by 3

Following this procedure, we get  $g(x) = \frac{x-4}{3}$ . Let's check to see if the function  $g$  does the job. If  $x = 5$ , then  $f(5) = 3(5) + 4 = 15 + 4 = 19$ . Taking the output 19 from  $f$ , we substitute it into  $g$  to get  $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$ , which is our original input to  $f$ . To check that  $g$  does the job for all  $x$  in the domain of  $f$ , we take the generic output from  $f$ ,  $f(x) = 3x + 4$ , and substitute that into  $g$ . That is,  $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$ , which is our original input to  $f$ . If we carefully examine the arithmetic as we simplify  $g(f(x))$ , we actually see  $g$  first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3. Not only does  $g$  undo  $f$ , but  $f$  also undoes  $g$ . That is, if we take the output from  $g$ ,  $g(x) = \frac{x-4}{3}$ , and put that into  $f$ , we get  $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$ . Using the language of function composition developed in Section 5.1, the statements  $g(f(x)) = x$  and  $f(g(x)) = x$  can be written as  $(g \circ f)(x) = x$  and  $(f \circ g)(x) = x$ , respectively. Abstractly, we can visualize the relationship between  $f$  and  $g$  in the diagram below.



The main idea to get from the diagram is that  $g$  takes the outputs from  $f$  and returns them to their respective inputs, and conversely,  $f$  takes outputs from  $g$  and returns them to their respective inputs. We now have enough background to state the central definition of the section.

**Definition 5.2.** Suppose  $f$  and  $g$  are two functions such that

1.  $(g \circ f)(x) = x$  for all  $x$  in the domain of  $f$  **and**
2.  $(f \circ g)(x) = x$  for all  $x$  in the domain of  $g$

then  $f$  and  $g$  are said to be **inverses** of each other. The functions  $f$  and  $g$  are said to be **invertible**.

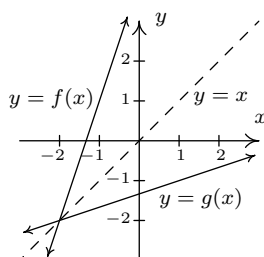
We now formalize the concept that inverse functions exchange inputs and outputs.

**Theorem 5.2. Properties of Inverse Functions:** Suppose  $f$  and  $g$  are inverse functions.

- The range<sup>a</sup> of  $f$  is the domain of  $g$  and the domain of  $f$  is the range of  $g$
- $f(a) = b$  if and only if  $g(b) = a$
- $(a, b)$  is on the graph of  $f$  if and only if  $(b, a)$  is on the graph of  $g$

<sup>a</sup>Recall this is the set of all outputs of a function.

Theorem 5.2 is a consequence of Definition 5.2 and the Fundamental Graphing Principle for Functions. We note the third property in Theorem 5.2 tells us that the graphs of inverse functions are reflections about the line  $y = x$ . For a proof of this, see Example 1.1.7 in Section 1.1 and Exercise 72 in Section 2.1. For example, we plot the inverse functions  $f(x) = 3x + 4$  and  $g(x) = \frac{x-4}{3}$  below.



If we abstract one step further, we can express the sentiment in Definition 5.2 by saying that  $f$  and  $g$  are inverses if and only if  $g \circ f = I_1$  and  $f \circ g = I_2$  where  $I_1$  is the identity function restricted<sup>1</sup> to the domain of  $f$  and  $I_2$  is the identity function restricted to the domain of  $g$ . In other words,  $I_1(x) = x$  for all  $x$  in the domain of  $f$  and  $I_2(x) = x$  for all  $x$  in the domain of  $g$ . Using this description of inverses along with the properties of function composition listed in Theorem 5.1, we can show that function inverses are unique.<sup>2</sup> Suppose  $g$  and  $h$  are both inverses of a function

<sup>1</sup>The identity function  $I$ , which was introduced in Section 2.1 and mentioned in Theorem 5.1, has a domain of all real numbers. Since the domains of  $f$  and  $g$  may not be all real numbers, we need the restrictions listed here.

<sup>2</sup>In other words, invertible functions have exactly one inverse.

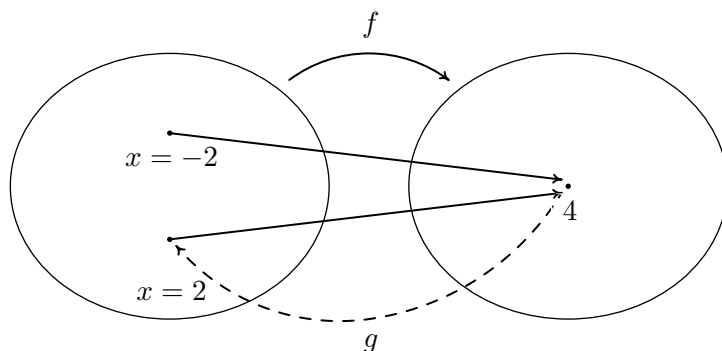
$f$ . By Theorem 5.2, the domain of  $g$  is equal to the domain of  $h$ , since both are the range of  $f$ . This means the identity function  $I_2$  applies both to the domain of  $h$  and the domain of  $g$ . Thus  $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$ , as required.<sup>3</sup> We summarize the discussion of the last two paragraphs in the following theorem.<sup>4</sup>

**Theorem 5.3. Uniqueness of Inverse Functions and Their Graphs :** Suppose  $f$  is an invertible function.

- There is exactly one inverse function for  $f$ , denoted  $f^{-1}$  (read  $f$ -inverse)
- The graph of  $y = f^{-1}(x)$  is the reflection of the graph of  $y = f(x)$  across the line  $y = x$ .

The notation  $f^{-1}$  is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as  $\frac{1}{f}$ . This is most definitely *not* the case since, for instance,  $f(x) = 3x + 4$  has as its inverse  $f^{-1}(x) = \frac{x-4}{3}$ , which is certainly different than  $\frac{1}{f(x)} = \frac{1}{3x+4}$ . Why does this confusing notation persist? As we mentioned in Section 5.1, the identity function  $I$  is to function composition what the real number 1 is to real number multiplication. The choice of notation  $f^{-1}$  alludes to the property that  $f^{-1} \circ f = I_1$  and  $f \circ f^{-1} = I_2$ , in much the same way as  $3^{-1} \cdot 3 = 1$  and  $3 \cdot 3^{-1} = 1$ .

Let's turn our attention to the function  $f(x) = x^2$ . Is  $f$  invertible? A likely candidate for the inverse is the function  $g(x) = \sqrt{x}$ . Checking the composition yields  $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$ , which is not equal to  $x$  for all  $x$  in the domain  $(-\infty, \infty)$ . For example, when  $x = -2$ ,  $f(-2) = (-2)^2 = 4$ , but  $g(4) = \sqrt{4} = 2$ , which means  $g$  failed to return the input  $-2$  from its output 4. What  $g$  did, however, is match the output 4 to a *different* input, namely 2, which satisfies  $f(2) = 4$ . This issue is presented schematically in the picture below.



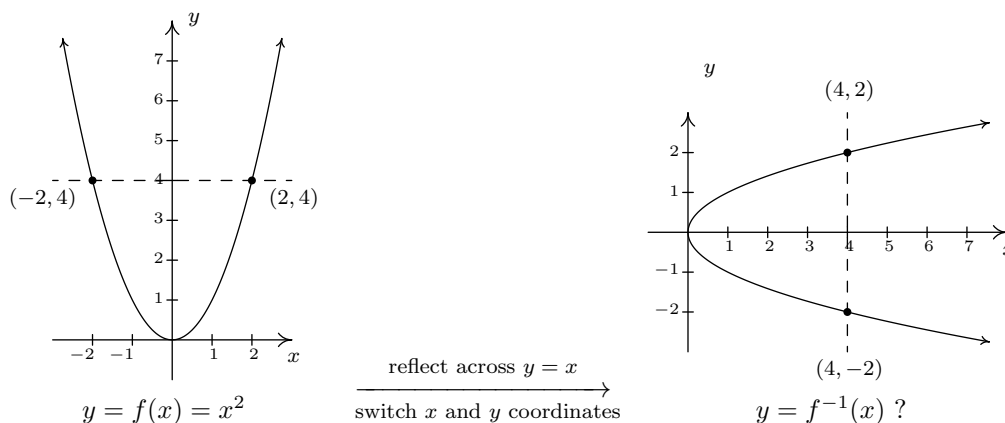
We see from the diagram that since both  $f(-2)$  and  $f(2)$  are 4, it is impossible to construct a *function* which takes 4 back to *both*  $x = 2$  and  $x = -2$ . (By definition, a function matches a real number with exactly one other real number.) From a graphical standpoint, we know that if

<sup>3</sup>It is an excellent exercise to explain each step in this string of equalities.

<sup>4</sup>In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter 8.



$y = f^{-1}(x)$  exists, its graph can be obtained by reflecting  $y = x^2$  about the line  $y = x$ , in accordance with Theorem 5.3. Doing so produces



We see that the line  $x = 4$  intersects the graph of the supposed inverse twice - meaning the graph fails the Vertical Line Test, Theorem 1.1, and as such, does not represent  $y$  as a function of  $x$ . The vertical line  $x = 4$  on the graph on the right corresponds to the *horizontal line*  $y = 4$  on the graph of  $y = f(x)$ . The fact that the horizontal line  $y = 4$  intersects the graph of  $f$  twice means two *different* inputs, namely  $x = -2$  and  $x = 2$ , are matched with the *same* output, 4, which is the cause of all of the trouble. In general, for a function to have an inverse, *different* inputs must go to *different* outputs, or else we will run into the same problem we did with  $f(x) = x^2$ . We give this property a name.

**Definition 5.3.** A function  $f$  is said to be **one-to-one** if  $f$  matches different inputs to different outputs. Equivalently,  $f$  is one-to-one if and only if whenever  $f(c) = f(d)$ , then  $c = d$ .

Graphically, we detect one-to-one functions using the test below.

**Theorem 5.4. The Horizontal Line Test:** A function  $f$  is one-to-one if and only if no horizontal line intersects the graph of  $f$  more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test. We have argued that if  $f$  is invertible, then  $f$  must be one-to-one, otherwise the graph given by reflecting the graph of  $y = f(x)$  about the line  $y = x$  will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility. To see this, we think of  $f$  as the set of ordered pairs which constitute its graph. If switching the  $x$ - and  $y$ -coordinates of the points results in a function, then  $f$  is invertible and we have found  $f^{-1}$ . This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes  $x$  as a function of  $y$ . We summarize these results below.

**Theorem 5.5. Equivalent Conditions for Invertibility:** Suppose  $f$  is a function. The following statements are equivalent.

- $f$  is invertible
- $f$  is one-to-one
- The graph of  $f$  passes the Horizontal Line Test

We put this result to work in the next example.

**Example 5.2.1.** Determine if the following functions are one-to-one in two ways: (a) analytically using Definition 5.3 and (b) graphically using the Horizontal Line Test.

$$1. f(x) = \frac{1-2x}{5}$$

$$2. g(x) = \frac{2x}{1-x}$$

$$3. h(x) = x^2 - 2x + 4$$

$$4. F = \{(-1, 1), (0, 2), (2, 1)\}$$

**Solution.**

1. (a) To determine if  $f$  is one-to-one analytically, we assume  $f(c) = f(d)$  and attempt to deduce that  $c = d$ .

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence,  $f$  is one-to-one.

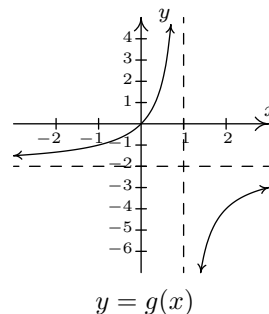
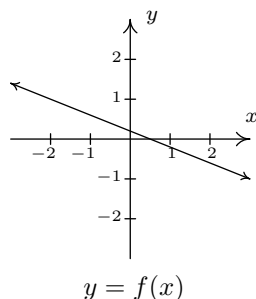
- (b) To check if  $f$  is one-to-one graphically, we look to see if the graph of  $y = f(x)$  passes the Horizontal Line Test. We have that  $f$  is a non-constant linear function, which means its graph is a non-horizontal line. Thus the graph of  $f$  passes the Horizontal Line Test.

2. (a) We begin with the assumption that  $g(c) = g(d)$  and try to show  $c = d$ .

$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c - 2cd &= 2d - 2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

We have shown that  $g$  is one-to-one.

- (b) We can graph  $g$  using the six step procedure outlined in Section 4.2. We get the sole intercept at  $(0, 0)$ , a vertical asymptote  $x = 1$  and a horizontal asymptote (which the graph never crosses)  $y = -2$ . We see from that the graph of  $g$  passes the Horizontal Line Test.

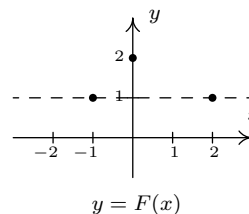
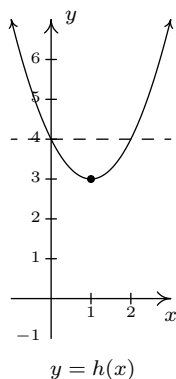


3. (a) We begin with  $h(c) = h(d)$ . As we work our way through the problem, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a 0 on the right and factor accordingly.

$$\begin{aligned}
 h(c) &= h(d) \\
 c^2 - 2c + 4 &= d^2 - 2d + 4 \\
 c^2 - 2c &= d^2 - 2d \\
 c^2 - d^2 - 2c + 2d &= 0 \\
 (c + d)(c - d) - 2(c - d) &= 0 \\
 (c - d)((c + d) - 2) &= 0 && \text{factor by grouping} \\
 c - d = 0 &\text{ or } c + d - 2 = 0 \\
 c = d &\text{ or } c = 2 - d
 \end{aligned}$$

We get  $c = d$  as one possibility, but we also get the possibility that  $c = 2 - d$ . This suggests that  $f$  may not be one-to-one. Taking  $d = 0$ , we get  $c = 0$  or  $c = 2$ . With  $f(0) = 4$  and  $f(2) = 4$ , we have produced two different inputs with the same output meaning  $f$  is not one-to-one.

- (b) We note that  $h$  is a quadratic function and we graph  $y = h(x)$  using the techniques presented in Section 2.3. The vertex is  $(1, 3)$  and the parabola opens upwards. We see immediately from the graph that  $h$  is not one-to-one, since there are several horizontal lines which cross the graph more than once.
4. (a) The function  $F$  is given to us as a set of ordered pairs. The condition  $F(c) = F(d)$  means the outputs from the function (the  $y$ -coordinates of the ordered pairs) are the same. We see that the points  $(-1, 1)$  and  $(2, 1)$  are both elements of  $F$  with  $F(-1) = 1$  and  $F(2) = 1$ . Since  $-1 \neq 2$ , we have established that  $F$  is *not* one-to-one.
- (b) Graphically, we see the horizontal line  $y = 1$  crosses the graph more than once. Hence, the graph of  $F$  fails the Horizontal Line Test.



□

We have shown that the functions  $f$  and  $g$  in Example 5.2.1 are one-to-one. This means they are invertible, so it is natural to wonder what  $f^{-1}(x)$  and  $g^{-1}(x)$  would be. For  $f(x) = \frac{1-2x}{5}$ , we can think our way through the inverse since there is only one occurrence of  $x$ . We can track step-by-step what is done to  $x$  and reverse those steps as we did at the beginning of the chapter. The function  $g(x) = \frac{2x}{1-x}$  is a bit trickier since  $x$  occurs in two places. When one evaluates  $g(x)$  for a specific value of  $x$ , which is first, the  $2x$  or the  $1-x$ ? We can imagine functions more complicated than these so we need to develop a general methodology to attack this problem. Theorem 5.2 tells us equation  $y = f^{-1}(x)$  is equivalent to  $f(y) = x$  and this is the basis of our algorithm.

#### Steps for finding the Inverse of a One-to-one Function

1. Write  $y = f(x)$
2. Interchange  $x$  and  $y$
3. Solve  $x = f(y)$  for  $y$  to obtain  $y = f^{-1}(x)$

Note that we could have simply written ‘Solve  $x = f(y)$  for  $y$ ’ and be done with it. The act of interchanging the  $x$  and  $y$  is there to remind us that we are finding the inverse function by switching the inputs and outputs.

**Example 5.2.2.** Find the inverse of the following one-to-one functions. Check your answers analytically using function composition and graphically.

1.  $f(x) = \frac{1-2x}{5}$

2.  $g(x) = \frac{2x}{1-x}$

**Solution.**

1. As we mentioned earlier, it is possible to think our way through the inverse of  $f$  by recording the steps we apply to  $x$  and the order in which we apply them and then reversing those steps in the reverse order. We encourage the reader to do this. We, on the other hand, will practice the algorithm. We write  $y = f(x)$  and proceed to switch  $x$  and  $y$

$$\begin{aligned}
y &= f(x) \\
y &= \frac{1-2x}{5} \\
x &= \frac{1-2y}{5} && \text{switch } x \text{ and } y \\
5x &= 1-2y \\
5x-1 &= -2y \\
\frac{5x-1}{-2} &= y \\
y &= -\frac{5}{2}x + \frac{1}{2}
\end{aligned}$$

We have  $f^{-1}(x) = -\frac{5}{2}x + \frac{1}{2}$ . To check this answer analytically, we first check that  $(f^{-1} \circ f)(x) = x$  for all  $x$  in the domain of  $f$ , which is all real numbers.

$$\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\
&= -\frac{5}{2}f(x) + \frac{1}{2} \\
&= -\frac{5}{2}\left(\frac{1-2x}{5}\right) + \frac{1}{2} \\
&= -\frac{1}{2}(1-2x) + \frac{1}{2} \\
&= -\frac{1}{2} + x + \frac{1}{2} \\
&= x \quad \checkmark
\end{aligned}$$

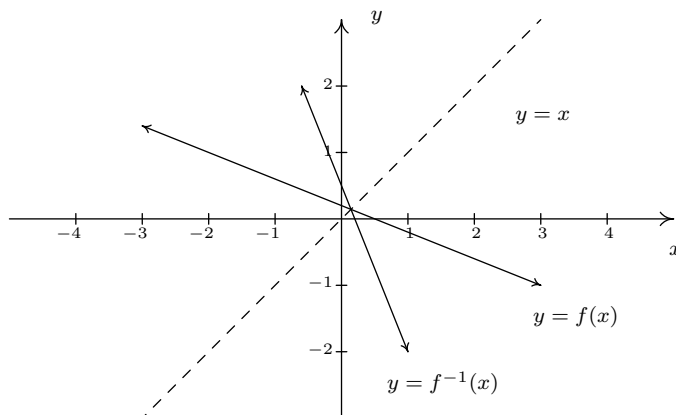
We now check that  $(f \circ f^{-1})(x) = x$  for all  $x$  in the range of  $f$  which is also all real numbers. (Recall that the domain of  $f^{-1}$  is the range of  $f$ .)

$$\begin{aligned}
(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\
&= \frac{1-2f^{-1}(x)}{5} \\
&= \frac{1-2\left(-\frac{5}{2}x + \frac{1}{2}\right)}{5} \\
&= \frac{1+5x-1}{5} \\
&= \frac{5x}{5} \\
&= x \quad \checkmark
\end{aligned}$$

To check our answer graphically, we graph  $y = f(x)$  and  $y = f^{-1}(x)$  on the same set of axes.<sup>5</sup> They appear to be reflections across the line  $y = x$ .

---

<sup>5</sup>Note that if you perform your check on a calculator for more sophisticated functions, you'll need to take advantage of the 'ZoomSquare' feature to get the correct geometric perspective.



2. To find  $g^{-1}(x)$ , we start with  $y = g(x)$ . We note that the domain of  $g$  is  $(-\infty, 1) \cup (1, \infty)$ .

$$\begin{aligned}
 y &= g(x) \\
 y &= \frac{2x}{1-x} \\
 x &= \frac{2y}{1-y} && \text{switch } x \text{ and } y \\
 x(1-y) &= 2y \\
 x - xy &= 2y \\
 x &= xy + 2y \\
 x &= y(x+2) && \text{factor} \\
 y &= \frac{x}{x+2}
 \end{aligned}$$

We obtain  $g^{-1}(x) = \frac{x}{x+2}$ . To check this analytically, we first check  $(g^{-1} \circ g)(x) = x$  for all  $x$  in the domain of  $g$ , that is, for all  $x \neq 1$ .

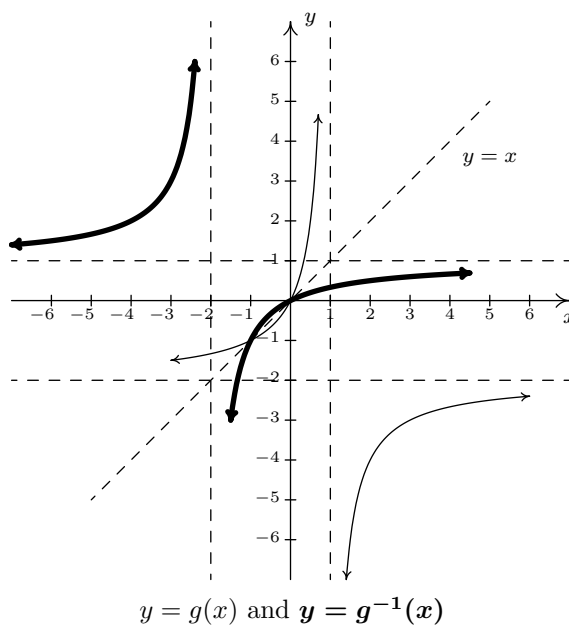
$$\begin{aligned}
 (g^{-1} \circ g)(x) &= g^{-1}(g(x)) \\
 &= g^{-1}\left(\frac{2x}{1-x}\right) \\
 &= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right) + 2} \\
 &= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right) + 2} \cdot \frac{(1-x)}{(1-x)} && \text{clear denominators}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2x}{2x + 2(1 - x)} \\
&= \frac{2x}{2x + 2 - 2x} \\
&= \frac{2x}{2} \\
&= x \quad \checkmark
\end{aligned}$$

Next, we check  $g(g^{-1}(x)) = x$  for all  $x$  in the range of  $g$ . From the graph of  $g$  in Example 5.2.1, we have that the range of  $g$  is  $(-\infty, -2) \cup (-2, \infty)$ . This matches the domain we get from the formula  $g^{-1}(x) = \frac{x}{x+2}$ , as it should.

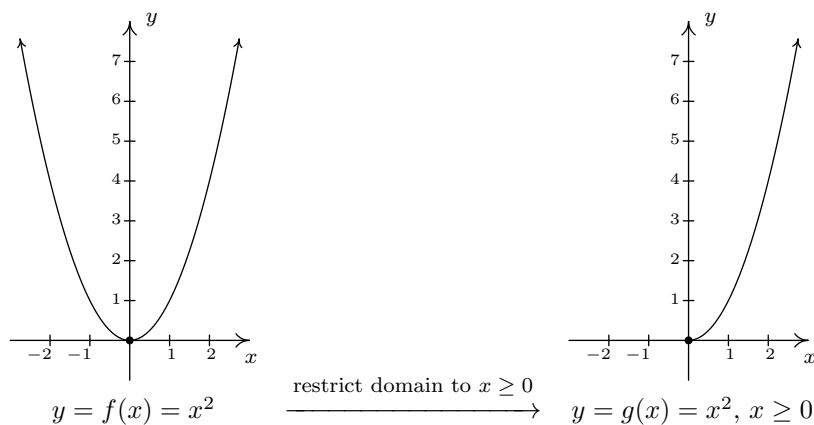
$$\begin{aligned}
(g \circ g^{-1})(x) &= g(g^{-1}(x)) \\
&= g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1 - \left(\frac{x}{x+2}\right)} \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1 - \left(\frac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} \quad \text{clear denominators} \\
&= \frac{2x}{(x+2) - x} \\
&= \frac{2x}{2} \\
&= x \quad \checkmark
\end{aligned}$$

Graphing  $y = g(x)$  and  $y = g^{-1}(x)$  on the same set of axes is busy, but we can see the symmetric relationship if we thicken the curve for  $y = g^{-1}(x)$ . Note that the vertical asymptote  $x = 1$  of the graph of  $g$  corresponds to the horizontal asymptote  $y = 1$  of the graph of  $g^{-1}$ , as it should since  $x$  and  $y$  are switched. Similarly, the horizontal asymptote  $y = -2$  of the graph of  $g$  corresponds to the vertical asymptote  $x = -2$  of the graph of  $g^{-1}$ .



□

We now return to  $f(x) = x^2$ . We know that  $f$  is not one-to-one, and thus, is not invertible. However, if we restrict the domain of  $f$ , we can produce a new function  $g$  which is one-to-one. If we define  $g(x) = x^2, x \geq 0$ , then we have

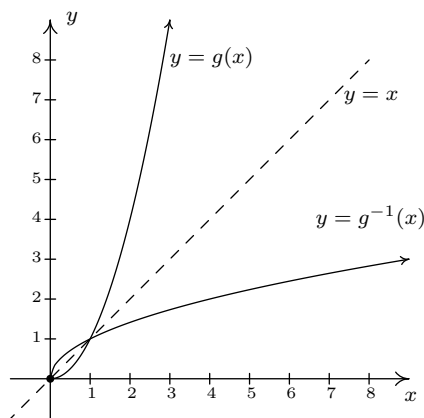


The graph of  $g$  passes the Horizontal Line Test. To find an inverse of  $g$ , we proceed as usual

$$\begin{aligned}
 y &= g(x) \\
 y &= x^2, \quad x \geq 0 \\
 x &= y^2, \quad y \geq 0 && \text{switch } x \text{ and } y \\
 y &= \pm\sqrt{x} \\
 y &= \sqrt{x} && \text{since } y \geq 0
 \end{aligned}$$



We get  $g^{-1}(x) = \sqrt{x}$ . At first it looks like we'll run into the same trouble as before, but when we check the composition, the domain restriction on  $g$  saves the day. We get  $(g^{-1} \circ g)(x) = g^{-1}(g(x)) = g^{-1}(x^2) = \sqrt{x^2} = |x| = x$ , since  $x \geq 0$ . Checking  $(g \circ g^{-1})(x) = g(g^{-1}(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x$ . Graphing<sup>6</sup>  $g$  and  $g^{-1}$  on the same set of axes shows that they are reflections about the line  $y = x$ .



Our next example continues the theme of domain restriction.

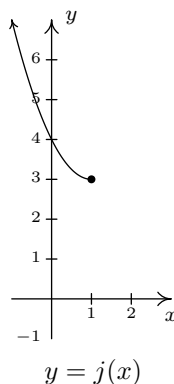
**Example 5.2.3.** Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1.  $j(x) = x^2 - 2x + 4, x \leq 1$ .

2.  $k(x) = \sqrt{x+2} - 1$

**Solution.**

1. The function  $j$  is a restriction of the function  $h$  from Example 5.2.1. Since the domain of  $j$  is restricted to  $x \leq 1$ , we are selecting only the ‘left half’ of the parabola. We see that the graph of  $j$  passes the Horizontal Line Test and thus  $j$  is invertible.



<sup>6</sup>We graphed  $y = \sqrt{x}$  in Section 1.7.

We now use our algorithm<sup>7</sup> to find  $j^{-1}(x)$ .

$$\begin{aligned}
 y &= j(x) \\
 y &= x^2 - 2x + 4, \quad x \leq 1 \\
 x &= y^2 - 2y + 4, \quad y \leq 1 && \text{switch } x \text{ and } y \\
 0 &= y^2 - 2y + 4 - x \\
 y &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} && \text{quadratic formula, } c = 4 - x \\
 y &= \frac{2 \pm \sqrt{4x - 12}}{2} \\
 y &= \frac{2 \pm \sqrt{4(x-3)}}{2} \\
 y &= \frac{2 \pm 2\sqrt{x-3}}{2} \\
 y &= \frac{2(1 \pm \sqrt{x-3})}{2} \\
 y &= 1 \pm \sqrt{x-3} \\
 y &= 1 - \sqrt{x-3} && \text{since } y \leq 1.
 \end{aligned}$$

We have  $j^{-1}(x) = 1 - \sqrt{x-3}$ . When we simplify  $(j^{-1} \circ j)(x)$ , we need to remember that the domain of  $j$  is  $x \leq 1$ .

$$\begin{aligned}
 (j^{-1} \circ j)(x) &= j^{-1}(j(x)) \\
 &= j^{-1}(x^2 - 2x + 4), \quad x \leq 1 \\
 &= 1 - \sqrt{(x^2 - 2x + 4) - 3} \\
 &= 1 - \sqrt{x^2 - 2x + 1} \\
 &= 1 - \sqrt{(x-1)^2} \\
 &= 1 - |x-1| \\
 &= 1 - (-(x-1)) && \text{since } x \leq 1 \\
 &= x \quad \checkmark
 \end{aligned}$$

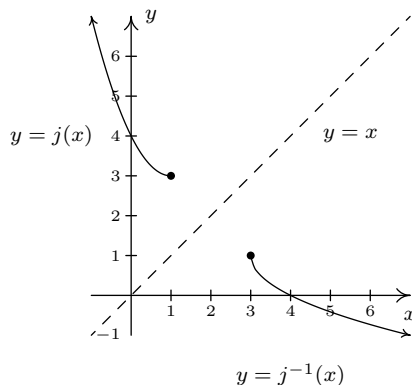
Checking  $j \circ j^{-1}$ , we get

$$\begin{aligned}
 (j \circ j^{-1})(x) &= j(j^{-1}(x)) \\
 &= j(1 - \sqrt{x-3}) \\
 &= (1 - \sqrt{x-3})^2 - 2(1 - \sqrt{x-3}) + 4 \\
 &= 1 - 2\sqrt{x-3} + (\sqrt{x-3})^2 - 2 + 2\sqrt{x-3} + 4 \\
 &= 3 + x - 3 \\
 &= x \quad \checkmark
 \end{aligned}$$

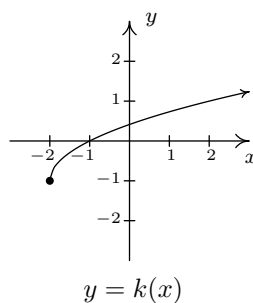
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<sup>7</sup>Here, we use the Quadratic Formula to solve for  $y$ . For ‘completeness,’ we note you can (and should!) also consider solving for  $y$  by ‘completing’ the square.

Using what we know from Section 1.7, we graph  $y = j^{-1}(x)$  and  $y = j(x)$  below.



2. We graph  $y = k(x) = \sqrt{x+2} - 1$  using what we learned in Section 1.7 and see  $k$  is one-to-one.



We now try to find  $k^{-1}$ .

$$\begin{aligned}
 y &= k(x) \\
 y &= \sqrt{x+2} - 1 \\
 x &= \sqrt{y+2} - 1 \quad \text{switch } x \text{ and } y \\
 x+1 &= \sqrt{y+2} \\
 (x+1)^2 &= (\sqrt{y+2})^2 \\
 x^2 + 2x + 1 &= y + 2 \\
 y &= x^2 + 2x - 1
 \end{aligned}$$

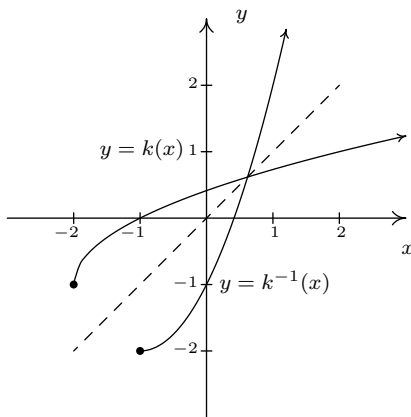
We have  $k^{-1}(x) = x^2 + 2x - 1$ . Based on our experience, we know something isn't quite right. We determined  $k^{-1}$  is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted. Theorem 5.2 tells us that the domain of  $k^{-1}$  is the range of  $k$ . From the graph of  $k$ , we see that the range is  $[-1, \infty)$ , which means we restrict the domain of  $k^{-1}$  to  $x \geq -1$ . We now check that this works in our compositions.

$$\begin{aligned}
(k^{-1} \circ k)(x) &= k^{-1}(k(x)) \\
&= k^{-1}(\sqrt{x+2}-1), \quad x \geq -2 \\
&= (\sqrt{x+2}-1)^2 + 2(\sqrt{x+2}-1) - 1 \\
&= (\sqrt{x+2})^2 - 2\sqrt{x+2} + 1 + 2\sqrt{x+2} - 2 - 1 \\
&= x + 2 - 2 \\
&= x \quad \checkmark
\end{aligned}$$

and

$$\begin{aligned}
(k \circ k^{-1})(x) &= k(x^2 + 2x - 1) \quad x \geq -1 \\
&= \sqrt{(x^2 + 2x - 1) + 2} - 1 \\
&= \sqrt{x^2 + 2x + 1} - 1 \\
&= \sqrt{(x+1)^2} - 1 \\
&= |x+1| - 1 \\
&= x + 1 - 1 \quad \text{since } x \geq -1 \\
&= x \quad \checkmark
\end{aligned}$$

Graphically, everything checks out as well, provided that we remember the domain restriction on  $k^{-1}$  means we take the right half of the parabola.



□

Our last example of the section gives an application of inverse functions.

**Example 5.2.4.** Recall from Section 2.1 that the price-demand equation for the PortaBoy game system is  $p(x) = -1.5x + 250$  for  $0 \leq x \leq 166$ , where  $x$  represents the number of systems sold weekly and  $p$  is the price per system in dollars.

1. Explain why  $p$  is one-to-one and find a formula for  $p^{-1}(x)$ . State the restricted domain.
2. Find and interpret  $p^{-1}(220)$ .
3. Recall from Section 2.3 that the weekly profit  $P$ , in dollars, as a result of selling  $x$  systems is given by  $P(x) = -1.5x^2 + 170x - 150$ . Find and interpret  $(P \circ p^{-1})(x)$ .
4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 2.3.3.

**Solution.**

1. We leave to the reader to show the graph of  $p(x) = -1.5x + 250$ ,  $0 \leq x \leq 166$ , is a line segment from  $(0, 250)$  to  $(166, 1)$ , and as such passes the Horizontal Line Test. Hence,  $p$  is one-to-one. We find the expression for  $p^{-1}(x)$  as usual and get  $p^{-1}(x) = \frac{500-2x}{3}$ . The domain of  $p^{-1}$  should match the range of  $p$ , which is  $[1, 250]$ , and as such, we restrict the domain of  $p^{-1}$  to  $1 \leq x \leq 250$ .
2. We find  $p^{-1}(220) = \frac{500-2(220)}{3} = 20$ . Since the function  $p$  took as inputs the weekly sales and furnished the price per system as the output,  $p^{-1}$  takes the price per system and returns the weekly sales as its output. Hence,  $p^{-1}(220) = 20$  means 20 systems will be sold in a week if the price is set at \$220 per system.
3. We compute  $(P \circ p^{-1})(x) = P(p^{-1}(x)) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$ . After a hefty amount of Elementary Algebra,<sup>8</sup> we obtain  $(P \circ p^{-1})(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$ . To understand what this means, recall that the original profit function  $P$  gave us the weekly profit as a function of the weekly sales. The function  $p^{-1}$  gives us the weekly sales as a function of the price. Hence,  $P \circ p^{-1}$  takes as its input a price. The function  $p^{-1}$  returns the weekly sales, which in turn is fed into  $P$  to return the weekly profit. Hence,  $(P \circ p^{-1})(x)$  gives us the weekly profit (in dollars) as a function of the price per system,  $x$ , using the weekly sales  $p^{-1}(x)$  as the ‘middle man’.
4. We know from Section 2.3 that the graph of  $y = (P \circ p^{-1})(x)$  is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the  $x$ -coordinate of the vertex. Identifying  $a = -\frac{2}{3}$  and  $b = 220$ , we get, by the Vertex Formula, Equation 2.4,  $x = -\frac{b}{2a} = 165$ . Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example 2.3.3, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately.  $\square$

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<sup>8</sup>It is good review to actually do this!

## 5.2.1 EXERCISES

In Exercises 1 - 20, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify that the range of  $f$  is the domain of  $f^{-1}$  and vice-versa.

1.  $f(x) = 6x - 2$

2.  $f(x) = 42 - x$

3.  $f(x) = \frac{x-2}{3} + 4$

4.  $f(x) = 1 - \frac{4+3x}{5}$

5.  $f(x) = \sqrt{3x-1} + 5$

6.  $f(x) = 2 - \sqrt{x-5}$

7.  $f(x) = 3\sqrt{x-1} - 4$

8.  $f(x) = 1 - 2\sqrt{2x+5}$

9.  $f(x) = \sqrt[5]{3x-1}$

10.  $f(x) = 3 - \sqrt[3]{x-2}$

11.  $f(x) = x^2 - 10x, x \geq 5$

12.  $f(x) = 3(x+4)^2 - 5, x \leq -4$

13.  $f(x) = x^2 - 6x + 5, x \leq 3$

14.  $f(x) = 4x^2 + 4x + 1, x < -1$

15.  $f(x) = \frac{3}{4-x}$

16.  $f(x) = \frac{x}{1-3x}$

17.  $f(x) = \frac{2x-1}{3x+4}$

18.  $f(x) = \frac{4x+2}{3x-6}$

19.  $f(x) = \frac{-3x-2}{x+3}$

20.  $f(x) = \frac{x-2}{2x-1}$

With help from your classmates, find the inverses of the functions in Exercises 21 - 24.

21.  $f(x) = ax + b, a \neq 0$

22.  $f(x) = a\sqrt{x-h} + k, a \neq 0, x \geq h$

23.  $f(x) = ax^2 + bx + c$  where  $a \neq 0, x \geq -\frac{b}{2a}$ .

24.  $f(x) = \frac{ax+b}{cx+d},$  (See Exercise 33 below.)

25. In Example 1.5.3, the price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales  $x$  according to the formula  $p(x) = 450 - 15x$  for  $0 \leq x \leq 30$ .

(a) Find  $p^{-1}(x)$  and state its domain.

(b) Find and interpret  $p^{-1}(105)$ .

(c) In Example 1.5.3, we determined that the profit (in dollars) made from producing and selling  $x$  dOpis per week is  $P(x) = -15x^2 + 350x - 2000$ , for  $0 \leq x \leq 30$ . Find  $(P \circ p^{-1})(x)$  and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?

26. Show that the Fahrenheit to Celsius conversion function found in Exercise 35 in Section 2.1 is invertible and that its inverse is the Celsius to Fahrenheit conversion function.
27. Analytically show that the function  $f(x) = x^3 + 3x + 1$  is one-to-one. Since finding a formula for its inverse is beyond the scope of this textbook, use Theorem 5.2 to help you compute  $f^{-1}(1)$ ,  $f^{-1}(5)$ , and  $f^{-1}(-3)$ .
28. Let  $f(x) = \frac{2x}{x^2-1}$ . Using the techniques in Section 4.2, graph  $y = f(x)$ . Verify that  $f$  is one-to-one on the interval  $(-1, 1)$ . Use the procedure outlined on Page 384 and your graphing calculator to find the formula for  $f^{-1}(x)$ . Note that since  $f(0) = 0$ , it should be the case that  $f^{-1}(0) = 0$ . What goes wrong when you attempt to substitute  $x = 0$  into  $f^{-1}(x)$ ? Discuss with your classmates how this problem arose and possible remedies.
29. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.
30. If  $f$  is odd and invertible, prove that  $f^{-1}$  is also odd.
31. Let  $f$  and  $g$  be invertible functions. With the help of your classmates show that  $(f \circ g)$  is one-to-one, hence invertible, and that  $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$ .
32. What graphical feature must a function  $f$  possess for it to be its own inverse?
33. What conditions must you place on the values of  $a, b, c$  and  $d$  in Exercise 24 in order to guarantee that the function is invertible?

## 5.2.2 ANSWERS

1.  $f^{-1}(x) = \frac{x+2}{6}$
2.  $f^{-1}(x) = 42 - x$
3.  $f^{-1}(x) = 3x - 10$
4.  $f^{-1}(x) = -\frac{5}{3}x + \frac{1}{3}$
5.  $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}, x \geq 5$
6.  $f^{-1}(x) = (x-2)^2 + 5, x \leq 2$
7.  $f^{-1}(x) = \frac{1}{9}(x+4)^2 + 1, x \geq -4$
8.  $f^{-1}(x) = \frac{1}{8}(x-1)^2 - \frac{5}{2}, x \leq 1$
9.  $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
10.  $f^{-1}(x) = -(x-3)^3 + 2$
11.  $f^{-1}(x) = 5 + \sqrt{x+25}$
12.  $f^{-1}(x) = -\sqrt{\frac{x+5}{3}} - 4$
13.  $f^{-1}(x) = 3 - \sqrt{x+4}$
14.  $f^{-1}(x) = -\frac{\sqrt{x+1}}{2}, x > 1$
15.  $f^{-1}(x) = \frac{4x-3}{x}$
16.  $f^{-1}(x) = \frac{x}{3x+1}$
17.  $f^{-1}(x) = \frac{4x+1}{2-3x}$
18.  $f^{-1}(x) = \frac{6x+2}{3x-4}$
19.  $f^{-1}(x) = \frac{-3x-2}{x+3}$
20.  $f^{-1}(x) = \frac{x-2}{2x-1}$
25. (a)  $p^{-1}(x) = \frac{450-x}{15}$ . The domain of  $p^{-1}$  is the range of  $p$  which is  $[0, 450]$   
 (b)  $p^{-1}(105) = 23$ . This means that if the price is set to \$105 then 23 dOpis will be sold.  
 (c)  $(P \circ p^{-1})(x) = -\frac{1}{15}x^2 + \frac{110}{3}x - 5000, 0 \leq x \leq 450$ . The graph of  $y = (P \circ p^{-1})(x)$  is a parabola opening downwards with vertex  $(275, \frac{125}{3}) \approx (275, 41.67)$ . This means that the maximum profit is a whopping \$41.67 when the price per dOpi is set to \$275. At this price, we can produce and sell  $p^{-1}(275) = 11.\bar{6}$  dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find  $p(11) = 285$  and  $p(12) = 270$ , which means we set the price per dOpi at either \$285 or \$270, respectively. The profits at these prices are  $(P \circ p^{-1})(285) = 35$  and  $(P \circ p^{-1})(270) = 40$ , so it looks as if the maximum profit is \$40 and it is made by producing and selling 12 dOpis a week at a price of \$270 per dOpi.
27. Given that  $f(0) = 1$ , we have  $f^{-1}(1) = 0$ . Similarly  $f^{-1}(5) = 1$  and  $f^{-1}(-3) = -1$



## 5.3 OTHER ALGEBRAIC FUNCTIONS

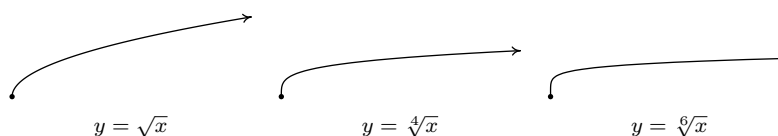
This section serves as a watershed for functions which are combinations of polynomial, and more generally, rational functions, with the operations of radicals. It is business of Calculus to discuss these functions in all the detail they demand so our aim in this section is to help shore up the requisite skills needed so that the reader can answer Calculus's call when the time comes. We briefly recall the definition and some of the basic properties of radicals from Intermediate Algebra.<sup>1</sup>

**Definition 5.4.** Let  $x$  be a real number and  $n$  a natural number.<sup>a</sup> If  $n$  is odd, the **principal  $n^{\text{th}}$  root** of  $x$ , denoted  $\sqrt[n]{x}$  is the unique real number satisfying  $(\sqrt[n]{x})^n = x$ . If  $n$  is even,  $\sqrt[n]{x}$  is defined similarly<sup>b</sup> provided  $x \geq 0$  and  $\sqrt[n]{x} \geq 0$ . The **index** is the number  $n$  and the **radicand** is the number  $x$ . For  $n = 2$ , we write  $\sqrt{x}$  instead of  $\sqrt[2]{x}$ .

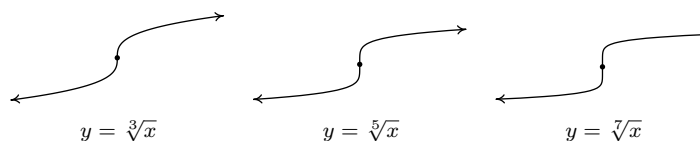
<sup>a</sup>Recall this means  $n = 1, 2, 3, \dots$

<sup>b</sup>Recall both  $x = -2$  and  $x = 2$  satisfy  $x^4 = 16$ , but  $\sqrt[4]{16} = 2$ , not  $-2$ .

It is worth remarking that, in light of Section 5.2, we could define  $f(x) = \sqrt[n]{x}$  functionally as the inverse of  $g(x) = x^n$  with the stipulation that when  $n$  is even, the domain of  $g$  is restricted to  $[0, \infty)$ . From what we know about  $g(x) = x^n$  from Section 3.1 along with Theorem 5.3, we can produce the graphs of  $f(x) = \sqrt[n]{x}$  by reflecting the graphs of  $g(x) = x^n$  across the line  $y = x$ . Below are the graphs of  $y = \sqrt{x}$ ,  $y = \sqrt[4]{x}$  and  $y = \sqrt[6]{x}$ . The point  $(0, 0)$  is indicated as a reference. The axes are hidden so we can see the vertical steepening near  $x = 0$  and the horizontal flattening as  $x \rightarrow \infty$ .



The odd-indexed radical functions also follow a predictable trend - steepening near  $x = 0$  and flattening as  $x \rightarrow \pm\infty$ . In the exercises, you'll have a chance to graph some basic radical functions using the techniques presented in Section 1.7.



We have used all of the following properties at some point in the textbook for the case  $n = 2$  (the square root), but we list them here in generality for completeness.

<sup>1</sup>Although we discussed imaginary numbers in Section 3.4, we restrict our attention to real numbers in this section. See the epilogue on page 293 for more details.

**Theorem 5.6. Properties of Radicals:** Let  $x$  and  $y$  be real numbers and  $m$  and  $n$  be natural numbers. If  $\sqrt[n]{x}$ ,  $\sqrt[n]{y}$  are real numbers, then

- **Product Rule:**  $\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$
- **Powers of Radicals:**  $\sqrt[n]{x^m} = (\sqrt[n]{x})^m$
- **Quotient Rule:**  $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$ , provided  $y \neq 0$ .
- If  $n$  is odd,  $\sqrt[n]{x^n} = x$ ; if  $n$  is even,  $\sqrt[n]{x^n} = |x|$ .

The proof of Theorem 5.6 is based on the definition of the principal roots and properties of exponents. To establish the product rule, consider the following. If  $n$  is odd, then by definition  $\sqrt[n]{xy}$  is the unique real number such that  $(\sqrt[n]{xy})^n = xy$ . Given that  $(\sqrt[n]{x} \sqrt[n]{y})^n = (\sqrt[n]{x})^n (\sqrt[n]{y})^n = xy$ , it must be the case that  $\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$ . If  $n$  is even, then  $\sqrt[n]{xy}$  is the unique non-negative real number such that  $(\sqrt[n]{xy})^n = xy$ . Also note that since  $n$  is even,  $\sqrt[n]{x}$  and  $\sqrt[n]{y}$  are also non-negative and hence so is  $\sqrt[n]{x} \sqrt[n]{y}$ . Proceeding as above, we find that  $\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$ . The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as  $\sqrt[n]{x}$  is a real number to start with.<sup>2</sup> The last property is an application of the power rule when  $n$  is odd, and the occurrence of the absolute value when  $n$  is even is due to the requirement that  $\sqrt[n]{x} \geq 0$  in Definition 5.4. For instance,  $\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2 = |-2|$ , not  $-2$ . It's this last property which makes compositions of roots and powers delicate. This is especially true when we use exponential notation for radicals. Recall the following definition.

**Definition 5.5.** Let  $x$  be a real number,  $m$  an integer<sup>a</sup> and  $n$  a natural number.

- $x^{\frac{1}{n}} = \sqrt[n]{x}$  and is defined whenever  $\sqrt[n]{x}$  is defined.
- $x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$ , whenever  $(\sqrt[n]{x})^m$  is defined.

<sup>a</sup>Recall this means  $m = 0, \pm 1, \pm 2, \dots$

The rational exponents defined in Definition 5.5 behave very similarly to the usual integer exponents from Elementary Algebra with one critical exception. Consider the expression  $(x^{2/3})^{3/2}$ . Applying the usual laws of exponents, we'd be tempted to simplify this as  $(x^{2/3})^{3/2} = x^{\frac{2}{3} \cdot \frac{3}{2}} = x^1 = x$ . However, if we substitute  $x = -1$  and apply Definition 5.5, we find  $(-1)^{2/3} = (\sqrt[3]{-1})^2 = (-1)^2 = 1$  so that  $((-1)^{2/3})^{3/2} = 1^{3/2} = (\sqrt{1})^3 = 1^3 = 1$ . We see in this case that  $(x^{2/3})^{3/2} \neq x$ . If we take the time to rewrite  $(x^{2/3})^{3/2}$  with radicals, we see

$$(x^{2/3})^{3/2} = ((\sqrt[3]{x})^2)^{3/2} = \left(\sqrt{(\sqrt[3]{x})^2}\right)^3 = (|\sqrt[3]{x}|)^3 = |(\sqrt[3]{x})^3| = |x|$$

<sup>2</sup>Otherwise we'd run into the same paradox we did in Section 3.4.

In the play-by-play analysis, we see that when we canceled the 2's in multiplying  $\frac{2}{3} \cdot \frac{3}{2}$ , we were, in fact, attempting to cancel a square with a square root. The fact that  $\sqrt{x^2} = |x|$  and not simply  $x$  is the root<sup>3</sup> of the trouble. It may amuse the reader to know that  $(x^{3/2})^{2/3} = x$ , and this verification is left as an exercise. The moral of the story is that when simplifying fractional exponents, it's usually best to rewrite them as radicals.<sup>4</sup> The last major property we will state, and leave to Calculus to prove, is that radical functions are continuous on their domains, so the Intermediate Value Theorem, Theorem 3.1, applies. This means that if we take combinations of radical functions with polynomial and rational functions to form what the authors consider the **algebraic functions**,<sup>5</sup> we can make sign diagrams using the procedure set forth in Section 4.2.

### Steps for Constructing a Sign Diagram for an Algebraic Function

Suppose  $f$  is an algebraic function.

1. Place any values excluded from the domain of  $f$  on the number line with an ‘?’ above them.
2. Find the zeros of  $f$  and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of  $f(x)$  for each test value in step 3, and write that sign above the corresponding interval.

Our next example reviews quite a bit of Intermediate Algebra and demonstrates some of the new features of these graphs.

**Example 5.3.1.** For the following functions, state their domains and create sign diagrams. Check your answer graphically using your calculator.

$$1. f(x) = 3x\sqrt[3]{2-x}$$

$$2. g(x) = \sqrt{2 - \sqrt[4]{x+3}}$$

$$3. h(x) = \sqrt[3]{\frac{8x}{x+1}}$$

$$4. k(x) = \frac{2x}{\sqrt{x^2-1}}$$

**Solution.**

1. As far as domain is concerned,  $f(x)$  has no denominators and no even roots, which means its domain is  $(-\infty, \infty)$ . To create the sign diagram, we find the zeros of  $f$ .

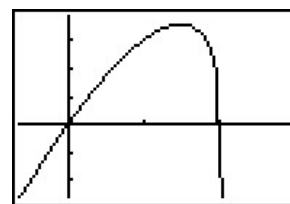
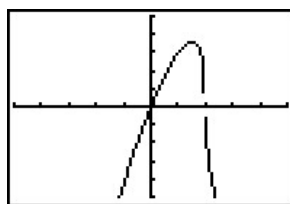
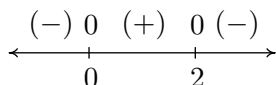
<sup>3</sup>Did you like that pun?

<sup>4</sup>In most other cases, though, rational exponents are preferred.

<sup>5</sup>As mentioned in Section 2.2,  $f(x) = \sqrt{x^2} = |x|$  so that absolute value is also considered an algebraic functions.

$$\begin{aligned}
 f(x) &= 0 \\
 3x\sqrt[3]{2-x} &= 0 \\
 3x = 0 &\text{ or } \sqrt[3]{2-x} = 0 \\
 x = 0 &\text{ or } (\sqrt[3]{2-x})^3 = 0^3 \\
 x = 0 &\text{ or } 2-x = 0 \\
 x = 0 &\text{ or } x = 2
 \end{aligned}$$

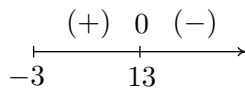
The zeros 0 and 2 divide the real number line into three test intervals. The sign diagram and accompanying graph are below. Note that the intervals on which  $f$  is (+) correspond to where the graph of  $f$  is above the  $x$ -axis, and where the graph of  $f$  is below the  $x$ -axis we have that  $f$  is (-). The calculator suggests something mysterious happens near  $x = 2$ . Zooming in shows the graph becomes nearly vertical there. You'll have to wait until Calculus to fully understand this phenomenon.



$$y = f(x)$$

$$y = f(x) \text{ near } x = 2.$$

2. In  $g(x) = \sqrt{2 - \sqrt[4]{x+3}}$ , we have two radicals both of which are even indexed. To satisfy  $\sqrt[4]{x+3}$ , we require  $x+3 \geq 0$  or  $x \geq -3$ . To satisfy  $\sqrt{2 - \sqrt[4]{x+3}}$ , we need  $2 - \sqrt[4]{x+3} \geq 0$ . While it may be tempting to write this as  $2 \geq \sqrt[4]{x+3}$  and take both sides to the fourth power, there are times when this technique will produce erroneous results.<sup>6</sup> Instead, we solve  $2 - \sqrt[4]{x+3} \geq 0$  using a sign diagram. If we let  $r(x) = 2 - \sqrt[4]{x+3}$ , we know  $x \geq -3$ , so we concern ourselves with only this portion of the number line. To find the zeros of  $r$  we set  $r(x) = 0$  and solve  $2 - \sqrt[4]{x+3} = 0$ . We get  $\sqrt[4]{x+3} = 2$  so that  $(\sqrt[4]{x+3})^4 = 2^4$  from which we obtain  $x+3 = 16$  or  $x = 13$ . Since we raised both sides of an equation to an even power, we need to check to see if  $x = 13$  is an extraneous solution.<sup>7</sup> We find  $x = 13$  does check since  $2 - \sqrt[4]{x+3} = 2 - \sqrt[4]{13+3} = 2 - \sqrt[4]{16} = 2 - 2 = 0$ . Below is our sign diagram for  $r$ .

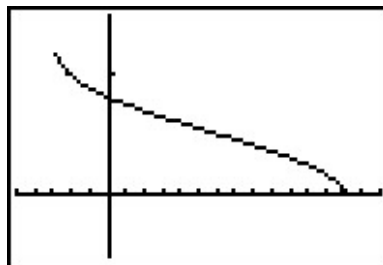
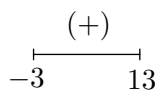


We find  $2 - \sqrt[4]{x+3} \geq 0$  on  $[-3, 13]$  so this is the domain of  $g$ . To find a sign diagram for  $g$ , we look for the zeros of  $g$ . Setting  $g(x) = 0$  is equivalent to  $\sqrt{2 - \sqrt[4]{x+3}} = 0$ . After squaring

<sup>6</sup>For instance,  $-2 \geq \sqrt[4]{x+3}$ , which has no solution or  $-2 \leq \sqrt[4]{x+3}$  whose solution is  $[-3, \infty)$ .

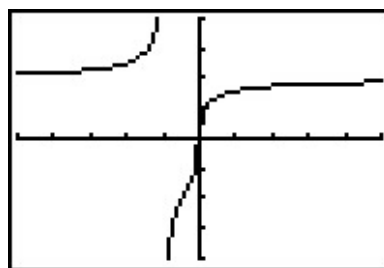
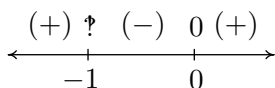
<sup>7</sup>Recall, this means we have produced a candidate which doesn't satisfy the original equation. Do you remember how raising both sides of an equation to an even power could cause this?

both sides, we get  $2 - \sqrt[4]{x+3} = 0$ , whose solution we have found to be  $x = 13$ . Since we squared both sides, we double check and find  $g(13)$  is, in fact, 0. Our sign diagram and graph of  $g$  are below. Since the domain of  $g$  is  $[-3, 13]$ , what we have below is not just a *portion* of the graph of  $g$ , but the *complete* graph. It is always above or on the  $x$ -axis, which verifies our sign diagram.



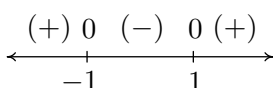
The complete graph of  $y = g(x)$ .

3. The radical in  $h(x)$  is odd, so our only concern is the denominator. Setting  $x + 1 = 0$  gives  $x = -1$ , so our domain is  $(-\infty, -1) \cup (-1, \infty)$ . To find the zeros of  $h$ , we set  $h(x) = 0$ . To solve  $\sqrt[3]{\frac{8x}{x+1}} = 0$ , we cube both sides to get  $\frac{8x}{x+1} = 0$ . We get  $8x = 0$ , or  $x = 0$ . Below is the resulting sign diagram and corresponding graph. From the graph, it appears as though  $x = -1$  is a vertical asymptote. Carrying out an analysis as  $x \rightarrow -1$  as in Section 4.2 confirms this. (We leave the details to the reader.) Near  $x = 0$ , we have a situation similar to  $x = 2$  in the graph of  $f$  in number 1 above. Finally, it appears as if the graph of  $h$  has a horizontal asymptote  $y = 2$ . Using techniques from Section 4.2, we find as  $x \rightarrow \pm\infty$ ,  $\frac{8x}{x+1} \rightarrow 8$ . From this, it is hardly surprising that as  $x \rightarrow \pm\infty$ ,  $h(x) = \sqrt[3]{\frac{8x}{x+1}} \approx \sqrt[3]{8} = 2$ .

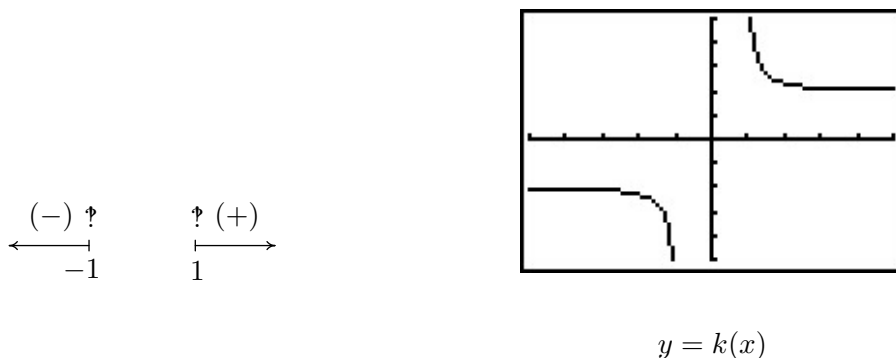


$y = h(x)$

4. To find the domain of  $k$ , we have both an even root and a denominator to concern ourselves with. To satisfy the square root,  $x^2 - 1 \geq 0$ . Setting  $r(x) = x^2 - 1$ , we find the zeros of  $r$  to be  $x = \pm 1$ , and we find the sign diagram of  $r$  to be



We find  $x^2 - 1 \geq 0$  for  $(-\infty, -1] \cup [1, \infty)$ . To keep the denominator of  $k(x)$  away from zero, we set  $\sqrt{x^2 - 1} = 0$ . We leave it to the reader to verify the solutions are  $x = \pm 1$ , both of which must be excluded from the domain. Hence, the domain of  $k$  is  $(-\infty, -1) \cup (1, \infty)$ . To build the sign diagram for  $k$ , we need the zeros of  $k$ . Setting  $k(x) = 0$  results in  $\frac{2x}{\sqrt{x^2 - 1}} = 0$ . We get  $2x = 0$  or  $x = 0$ . However,  $x = 0$  isn't in the domain of  $k$ , which means  $k$  has no zeros. We construct our sign diagram on the domain of  $k$  below alongside the graph of  $k$ . It appears that the graph of  $k$  has two vertical asymptotes, one at  $x = -1$  and one at  $x = 1$ . The gap in the graph between the asymptotes is because of the gap in the domain of  $k$ . Concerning end behavior, there appear to be two horizontal asymptotes,  $y = 2$  and  $y = -2$ . To see why this is the case, we think of  $x \rightarrow \pm\infty$ . The radicand of the denominator  $x^2 - 1 \approx x^2$ , and as such,  $k(x) = \frac{2x}{\sqrt{x^2 - 1}} \approx \frac{2x}{\sqrt{x^2}} = \frac{2x}{|x|}$ . As  $x \rightarrow \infty$ , we have  $|x| = x$  so  $k(x) \approx \frac{2x}{x} = 2$ . On the other hand, as  $x \rightarrow -\infty$ ,  $|x| = -x$ , and as such  $k(x) \approx \frac{2x}{-x} = -2$ . Finally, it appears as though the graph of  $k$  passes the Horizontal Line Test which means  $k$  is one to one and  $k^{-1}$  exists. Computing  $k^{-1}$  is left as an exercise.



□

As the previous example illustrates, the graphs of general algebraic functions can have features we've seen before, like vertical and horizontal asymptotes, but they can occur in new and exciting ways. For example,  $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$  had two distinct horizontal asymptotes. You'll recall that rational functions could have at most one horizontal asymptote. Also some new characteristics like 'unusual steepness'<sup>8</sup> and cusps<sup>9</sup> can appear in the graphs of arbitrary algebraic functions. Our next example first demonstrates how we can use sign diagrams to solve nonlinear inequalities. (Don't panic. The technique is very similar to the ones used in Chapters 2, 3 and 4.) We then check our answers graphically with a calculator and see some of the new graphical features of the functions in this extended family.

**Example 5.3.2.** Solve the following inequalities. Check your answers graphically with a calculator.

<sup>8</sup>The proper Calculus term for this is 'vertical tangent', but for now we'll be okay calling it 'unusual steepness'.

<sup>9</sup>See page 241 for the first reference to this feature.

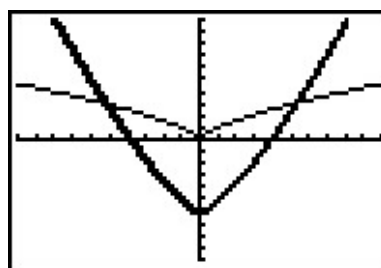
1.  $x^{2/3} < x^{4/3} - 6$

2.  $3(2-x)^{1/3} \leq x(2-x)^{-2/3}$

**Solution.**

1. To solve  $x^{2/3} < x^{4/3} - 6$ , we get 0 on one side and attempt to solve  $x^{4/3} - x^{2/3} - 6 > 0$ . We set  $r(x) = x^{4/3} - x^{2/3} - 6$  and note that since the denominators in the exponents are 3, they correspond to cube roots, which means the domain of  $r$  is  $(-\infty, \infty)$ . To find the zeros for the sign diagram, we set  $r(x) = 0$  and attempt to solve  $x^{4/3} - x^{2/3} - 6 = 0$ . At this point, it may be unclear how to proceed. We could always try as a last resort converting back to radical notation, but in this case we can take a cue from Example 3.3.4. Since there are three terms, and the exponent on one of the variable terms,  $x^{4/3}$ , is exactly twice that of the other,  $x^{2/3}$ , we have ourselves a ‘quadratic in disguise’ and we can rewrite  $x^{4/3} - x^{2/3} - 6 = 0$  as  $(x^{2/3})^2 - x^{2/3} - 6 = 0$ . If we let  $u = x^{2/3}$ , then in terms of  $u$ , we get  $u^2 - u - 6 = 0$ . Solving for  $u$ , we obtain  $u = -2$  or  $u = 3$ . Replacing  $x^{2/3}$  back in for  $u$ , we get  $x^{2/3} = -2$  or  $x^{2/3} = 3$ . To avoid the trouble we encountered in the discussion following Definition 5.5, we now convert back to radical notation. By interpreting  $x^{2/3}$  as  $\sqrt[3]{x^2}$  we have  $\sqrt[3]{x^2} = -2$  or  $\sqrt[3]{x^2} = 3$ . Cubing both sides of these equations results in  $x^2 = -8$ , which admits no real solution, or  $x^2 = 27$ , which gives  $x = \pm 3\sqrt{3}$ . We construct a sign diagram and find  $x^{4/3} - x^{2/3} - 6 > 0$  on  $(-\infty, -3\sqrt{3}) \cup (3\sqrt{3}, \infty)$ . To check our answer graphically, we set  $f(x) = x^{2/3}$  and  $g(x) = x^{4/3} - 6$ . The solution to  $x^{2/3} < x^{4/3} - 6$  corresponds to the inequality  $f(x) < g(x)$ , which means we are looking for the  $x$  values for which the graph of  $f$  is below the graph of  $g$ . Using the ‘Intersect’ command we confirm<sup>10</sup> that the graphs cross at  $x = \pm 3\sqrt{3}$ . We see that the graph of  $f$  is below the graph of  $g$  (the thicker curve) on  $(-\infty, -3\sqrt{3}) \cup (3\sqrt{3}, \infty)$ .

$$\begin{array}{ccccccc} & (+) & 0 & (-) & 0 & (+) & \\ \leftarrow & & & & & & \rightarrow \\ & -3\sqrt{3} & & & & 3\sqrt{3} & \end{array}$$

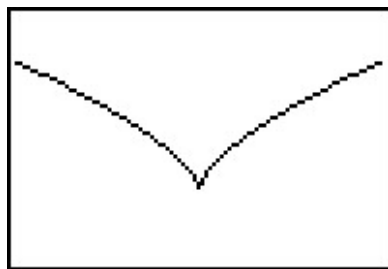
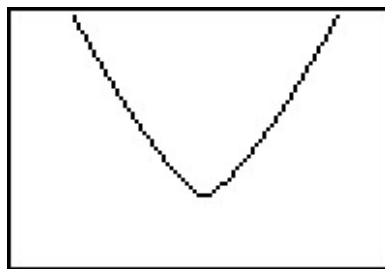


$$y = f(x) \text{ and } y = g(x)$$

As a point of interest, if we take a closer look at the graphs of  $f$  and  $g$  near  $x = 0$  with the axes off, we see that despite the fact they both involve cube roots, they exhibit different behavior near  $x = 0$ . The graph of  $f$  has a sharp turn, or cusp, while  $g$  does not.<sup>11</sup>

<sup>10</sup>Or at least confirm to several decimal places

<sup>11</sup>Again, we introduced this feature on page 241 as a feature which makes the graph of a function ‘not smooth’.

 $y = f(x)$  near  $x = 0$  $y = g(x)$  near  $x = 0$ 

2. To solve  $3(2-x)^{1/3} \leq x(2-x)^{-2/3}$ , we gather all the nonzero terms on one side and obtain  $3(2-x)^{1/3} - x(2-x)^{-2/3} \leq 0$ . We set  $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$ . As in number 1, the denominators of the rational exponents are odd, which means there are no domain concerns there. However, the negative exponent on the second term indicates a denominator. Rewriting  $r(x)$  with positive exponents, we obtain

$$r(x) = 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}}$$

Setting the denominator equal to zero we get  $(2-x)^{2/3} = 0$ , or  $\sqrt[3]{(2-x)^2} = 0$ . After cubing both sides, and subsequently taking square roots, we get  $2-x = 0$ , or  $x = 2$ . Hence, the domain of  $r$  is  $(-\infty, 2) \cup (2, \infty)$ . To find the zeros of  $r$ , we set  $r(x) = 0$ . There are two school of thought on how to proceed and we demonstrate both.

- *Factoring Approach.* From  $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$ , we note that the quantity  $(2-x)$  is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, since  $-\frac{2}{3} < \frac{1}{3}$ , we factor  $(2-x)^{-2/3}$  from both quantities. While it may seem odd to do so, we need to factor  $(2-x)^{-2/3}$  from  $(2-x)^{1/3}$ , which results in subtracting the exponent  $-\frac{2}{3}$  from  $\frac{1}{3}$ . We proceed using the usual properties of exponents.<sup>12</sup>

$$\begin{aligned} r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\ &= (2-x)^{-2/3} \left[ 3(2-x)^{\frac{1}{3}-(-\frac{2}{3})} - x \right] \\ &= (2-x)^{-2/3} [3(2-x)^{3/3} - x] \\ &= (2-x)^{-2/3} [3(2-x)^1 - x] && \text{since } \sqrt[3]{u^3} = (\sqrt[3]{u})^3 = u \\ &= (2-x)^{-2/3} (6-4x) \\ &= (2-x)^{-2/3} (6-4x) \end{aligned}$$

To solve  $r(x) = 0$ , we set  $(2-x)^{-2/3} (6-4x) = 0$ , or  $\frac{6-4x}{(2-x)^{2/3}} = 0$ . We have  $6-4x = 0$  or  $x = \frac{3}{2}$ .

<sup>12</sup>And we exercise special care when reducing the  $\frac{3}{3}$  power to 1.

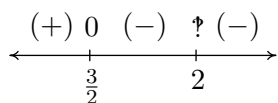


- *Common Denominator Approach.* We rewrite

$$\begin{aligned}
 r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\
 &= 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}} \\
 &= \frac{3(2-x)^{1/3}(2-x)^{2/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{common denominator} \\
 &= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
 &= \frac{3(2-x)^{3/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
 &= \frac{3(2-x)^1}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{since } \sqrt[3]{u^3} = (\sqrt[3]{u})^3 = u \\
 &= \frac{3(2-x) - x}{(2-x)^{2/3}} \\
 &= \frac{6-4x}{(2-x)^{2/3}}
 \end{aligned}$$

As before, when we set  $r(x) = 0$  we obtain  $x = \frac{3}{2}$ .

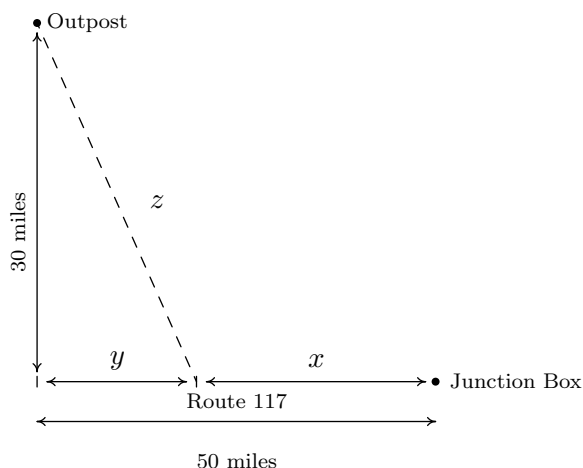
We now create our sign diagram and find  $3(2-x)^{1/3} - x(2-x)^{-2/3} \leq 0$  on  $[\frac{3}{2}, 2) \cup (2, \infty)$ . To check this graphically, we set  $f(x) = 3(2-x)^{1/3}$  and  $g(x) = x(2-x)^{-2/3}$  (the thicker curve). We confirm that the graphs intersect at  $x = \frac{3}{2}$  and the graph of  $f$  is below the graph of  $g$  for  $x \geq \frac{3}{2}$ , with the exception of  $x = 2$  where it appears the graph of  $g$  has a vertical asymptote.



$y = f(x)$  and  $y = g(x)$  □

One application of algebraic functions was given in Example 1.6.6 in Section 1.1. Our last example is a more sophisticated application of distance.

**Example 5.3.3.** Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles downroad from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.



1. Express the total cost  $C$  of connecting the Junction Box to the Outpost as a function of  $x$ , the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.
2. Use your calculator to graph  $y = C(x)$  on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?

**Solution.**

1. The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. Since  $x$  represents the miles of cable run along Route 117, the cost for that portion is  $15x$ . From the diagram, we see that the number of miles the cable is run off road is  $z$ , so the cost of that portion is  $20z$ . Hence, the total cost is  $C = 15x + 20z$ . Our next goal is to determine  $z$  as a function of  $x$ . The diagram suggests we can use the Pythagorean Theorem to get  $y^2 + 30^2 = z^2$ . But we also see  $x + y = 50$  so that  $y = 50 - x$ . Hence,  $z^2 = (50 - x)^2 + 900$ . Solving for  $z$ , we obtain  $z = \pm\sqrt{(50 - x)^2 + 900}$ . Since  $z$  represents a distance, we choose  $z = \sqrt{(50 - x)^2 + 900}$  so that our cost as a function of  $x$  only is given by

$$C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$$

From the context of the problem, we have  $0 \leq x \leq 50$ .

2. Graphing  $y = C(x)$  on a calculator and using the 'Minimum' feature, we find the relative minimum (which is also the absolute minimum in this case) to two decimal places to be (15.98, 1146.86). Here the  $x$ -coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The  $y$ -coordinate tells us that the minimum cost, in dollars, to do so is \$1146.86. The ability to stream live SasquatchCasts? Priceless.  $\square$

## 5.3.1 EXERCISES

For each function in Exercises 1 - 10 below

- Find its domain.
- Create a sign diagram.
- Use your calculator to help you sketch its graph and identify any vertical or horizontal asymptotes, ‘unusual steepness’ or cusps.

1.  $f(x) = \sqrt{1 - x^2}$

2.  $f(x) = \sqrt{x^2 - 1}$

3.  $f(x) = x\sqrt{1 - x^2}$

4.  $f(x) = x\sqrt{x^2 - 1}$

5.  $f(x) = \sqrt[4]{\frac{16x}{x^2 - 9}}$

6.  $f(x) = \frac{5x}{\sqrt[3]{x^3 + 8}}$

7.  $f(x) = x^{\frac{2}{3}}(x - 7)^{\frac{1}{3}}$

8.  $f(x) = x^{\frac{3}{2}}(x - 7)^{\frac{1}{3}}$

9.  $f(x) = \sqrt{x(x + 5)(x - 4)}$

10.  $f(x) = \sqrt[3]{x^3 + 3x^2 - 6x - 8}$

In Exercises 11 - 16, sketch the graph of  $y = g(x)$  by starting with the graph of  $y = f(x)$  and using the transformations presented in Section 1.7.

11.  $f(x) = \sqrt[3]{x}, g(x) = \sqrt[3]{x - 1} - 2$

12.  $f(x) = \sqrt[3]{x}, g(x) = -2\sqrt[3]{x + 1} + 4$

13.  $f(x) = \sqrt[4]{x}, g(x) = \sqrt[4]{x - 1} - 2$

14.  $f(x) = \sqrt[4]{x}, g(x) = 3\sqrt[4]{x - 7} - 1$

15.  $f(x) = \sqrt[5]{x}, g(x) = \sqrt[5]{x + 2} + 3$

16.  $f(x) = \sqrt[8]{x}, g(x) = \sqrt[8]{-x} - 2$

In Exercises 17 - 35, solve the equation or inequality.

17.  $x + 1 = \sqrt{3x + 7}$

18.  $2x + 1 = \sqrt{3 - 3x}$

19.  $x + \sqrt{3x + 10} = -2$

20.  $3x + \sqrt{6 - 9x} = 2$

21.  $2x - 1 = \sqrt{x + 3}$

22.  $x^{\frac{3}{2}} = 8$

23.  $x^{\frac{2}{3}} = 4$

24.  $\sqrt{x - 2} + \sqrt{x - 5} = 3$

25.  $\sqrt{2x + 1} = 3 + \sqrt{4 - x}$

26.  $5 - (4 - 2x)^{\frac{2}{3}} = 1$

27.  $10 - \sqrt{x - 2} \leq 11$

28.  $\sqrt[3]{x} \leq x$

29.  $2(x-2)^{-\frac{1}{3}} - \frac{2}{3}x(x-2)^{-\frac{4}{3}} \leq 0$
30.  $-\frac{4}{3}(x-2)^{-\frac{4}{3}} + \frac{8}{9}x(x-2)^{-\frac{7}{3}} \geq 0$
31.  $2x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}} + x^{\frac{2}{3}}(x-3)^{-\frac{2}{3}} \geq 0$
32.  $\sqrt[3]{x^3 + 3x^2 - 6x - 8} > x + 1$
33.  $\frac{1}{3}x^{\frac{3}{4}}(x-3)^{-\frac{2}{3}} + \frac{3}{4}x^{-\frac{1}{4}}(x-3)^{\frac{1}{3}} < 0$
34.  $x^{-\frac{1}{3}}(x-3)^{-\frac{2}{3}} - x^{-\frac{4}{3}}(x-3)^{-\frac{5}{3}}(x^2 - 3x + 2) \geq 0$
35.  $\frac{2}{3}(x+4)^{\frac{3}{5}}(x-2)^{-\frac{1}{3}} + \frac{3}{5}(x+4)^{-\frac{2}{5}}(x-2)^{\frac{2}{3}} \geq 0$
36. Rework Example 5.3.3 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.
37. The volume  $V$  of a right cylindrical cone depends on the radius of its base  $r$  and its height  $h$  and is given by the formula  $V = \frac{1}{3}\pi r^2 h$ . The surface area  $S$  of a right cylindrical cone also depends on  $r$  and  $h$  according to the formula  $S = \pi r \sqrt{r^2 + h^2}$ . Suppose a cone is to have a volume of 100 cubic centimeters.
- Use the formula for volume to find the height  $h$  as a function of  $r$ .
  - Use the formula for surface area and your answer to 37a to find the surface area  $S$  as a function of  $r$ .
  - Use your calculator to find the values of  $r$  and  $h$  which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.
38. The [National Weather Service](#) uses the following formula to calculate the wind chill:

$$W = 35.74 + 0.6215 T_a - 35.75 V^{0.16} + 0.4275 T_a V^{0.16}$$

where  $W$  is the wind chill temperature in  $^{\circ}\text{F}$ ,  $T_a$  is the air temperature in  $^{\circ}\text{F}$ , and  $V$  is the wind speed in miles per hour. Note that  $W$  is defined only for air temperatures at or lower than  $50^{\circ}\text{F}$  and wind speeds above 3 miles per hour.

- Suppose the air temperature is  $42^{\circ}$  and the wind speed is 7 miles per hour. Find the wind chill temperature. Round your answer to two decimal places.
  - Suppose the air temperature is  $37^{\circ}\text{F}$  and the wind chill temperature is  $30^{\circ}\text{F}$ . Find the wind speed. Round your answer to two decimal places.
39. As a follow-up to Exercise 38, suppose the air temperature is  $28^{\circ}\text{F}$ .
- Use the formula from Exercise 38 to find an expression for the wind chill temperature as a function of the wind speed,  $W(V)$ .
  - Solve  $W(V) = 0$ , round your answer to two decimal places, and interpret.
  - Graph the function  $W$  using your calculator and check your answer to part 39b.

40. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where  $L$  is the length of the pendulum in meters and  $g = 9.8$  meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs  $T = \frac{1}{2}$  second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

41. The Cobb-Douglas production model states that the yearly total dollar value of the production output  $P$  in an economy is a function of labor  $x$  (the total number of hours worked in a year) and capital  $y$  (the total dollar value of all of the stuff purchased in order to make things). Specifically,  $P = ax^by^{1-b}$ . By fixing  $P$ , we create what's known as an 'isoquant' and we can then solve for  $y$  as a function of  $x$ . Let's assume that the Cobb-Douglas production model for the country of Sasquatchia is  $P = 1.23x^{0.4}y^{0.6}$ .

- (a) Let  $P = 300$  and solve for  $y$  in terms of  $x$ . If  $x = 100$ , what is  $y$ ?
- (b) Graph the isoquant  $300 = 1.23x^{0.4}y^{0.6}$ . What information does an ordered pair  $(x, y)$  which makes  $P = 300$  give you? With the help of your classmates, find several different combinations of labor and capital all of which yield  $P = 300$ . Discuss any patterns you may see.

42. According to Einstein's Theory of Special Relativity, the observed mass  $m$  of an object is a function of how fast the object is traveling. Specifically,

$$m(x) = \frac{m_r}{\sqrt{1 - \frac{x^2}{c^2}}}$$

where  $m(0) = m_r$  is the mass of the object at rest,  $x$  is the speed of the object and  $c$  is the speed of light.

- (a) Find the applied domain of the function.
- (b) Compute  $m(.1c)$ ,  $m(.5c)$ ,  $m(.9c)$  and  $m(.999c)$ .
- (c) As  $x \rightarrow c^-$ , what happens to  $m(x)$ ?
- (d) How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest?

43. Find the inverse of  $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$ .

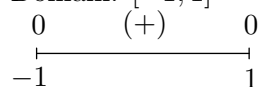
44. Suppose Fritzzy the Fox, positioned at a point  $(x, y)$  in the first quadrant, spots Chewbacca the Bunny at  $(0, 0)$ . Chewbacca begins to run along a fence (the positive  $y$ -axis) towards his warren. Fritzzy, of course, takes chase and constantly adjusts his direction so that he is always running directly at Chewbacca. If Chewbacca's speed is  $v_1$  and Fritzzy's speed is  $v_2$ , the path Fritzzy will take to intercept Chewbacca, provided  $v_2$  is directly proportional to, but not equal to,  $v_1$  is modeled by

$$y = \frac{1}{2} \left( \frac{x^{1+v_1/v_2}}{1+v_1/v_2} - \frac{x^{1-v_1/v_2}}{1-v_1/v_2} \right) + \frac{v_1 v_2}{v_2^2 - v_1^2}$$

- (a) Determine the path that Fritzzy will take if he runs exactly twice as fast as Chewbacca; that is,  $v_2 = 2v_1$ . Use your calculator to graph this path for  $x \geq 0$ . What is the significance of the  $y$ -intercept of the graph?
  - (b) Determine the path Fritzzy will take if Chewbacca runs exactly twice as fast as he does; that is,  $v_1 = 2v_2$ . Use your calculator to graph this path for  $x > 0$ . Describe the behavior of  $y$  as  $x \rightarrow 0^+$  and interpret this physically.
  - (c) With the help of your classmates, generalize parts (a) and (b) to two cases:  $v_2 > v_1$  and  $v_2 < v_1$ . We will discuss the case of  $v_1 = v_2$  in Exercise 32 in Section 6.5.
45. Verify the Quotient Rule for Radicals in Theorem 5.6.
46. Show that  $\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = x$  for all  $x \geq 0$ .
47. Show that  $\sqrt[3]{2}$  is an irrational number by first showing that it is a zero of  $p(x) = x^3 - 2$  and then showing  $p$  has no rational zeros. (You'll need the Rational Zeros Theorem, Theorem 3.9, in order to show this last part.)
48. With the help of your classmates, generalize Exercise 47 to show that  $\sqrt[n]{c}$  is an irrational number for any natural numbers  $c \geq 2$  and  $n \geq 2$  provided that  $c \neq p^n$  for some natural number  $p$ .

## 5.3.2 ANSWERS

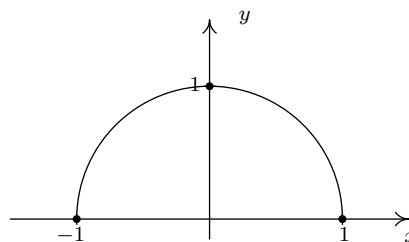
1.  $f(x) = \sqrt{1 - x^2}$

Domain:  $[-1, 1]$ 

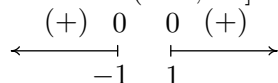
No asymptotes

Unusual steepness at  $x = -1$  and  $x = 1$ 

No cusps



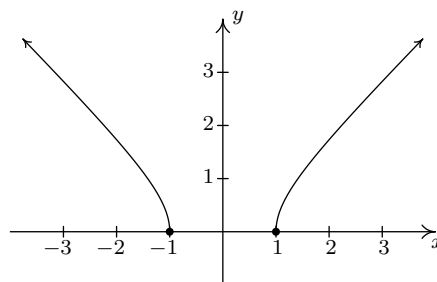
2.  $f(x) = \sqrt{x^2 - 1}$

Domain:  $(-\infty, -1] \cup [1, \infty)$ 

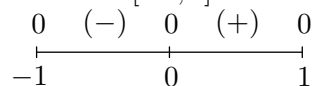
No asymptotes

Unusual steepness at  $x = -1$  and  $x = 1$ 

No cusps



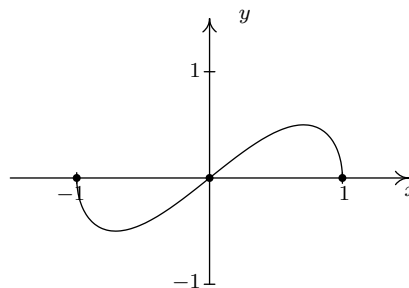
3.  $f(x) = x\sqrt{1 - x^2}$

Domain:  $[-1, 1]$ 

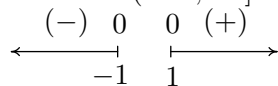
No asymptotes

Unusual steepness at  $x = -1$  and  $x = 1$ 

No cusps



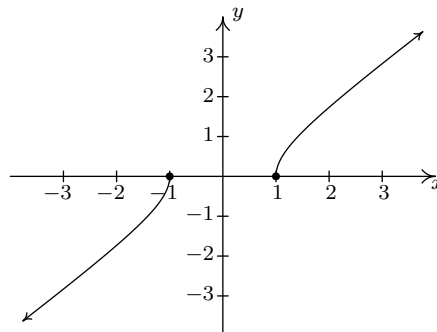
4.  $f(x) = x\sqrt{x^2 - 1}$

Domain:  $(-\infty, -1] \cup [1, \infty)$ 

No asymptotes

Unusual steepness at  $x = -1$  and  $x = 1$ 

No cusps



5.  $f(x) = \sqrt[4]{\frac{16x}{x^2 - 9}}$

Domain:  $(-3, 0] \cup (3, \infty)$

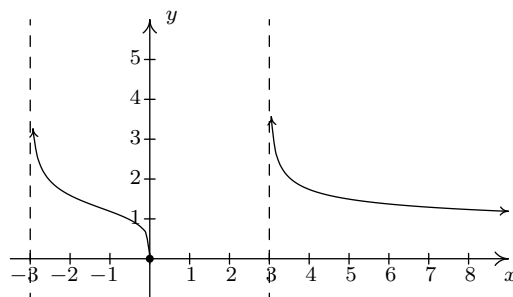
$\begin{array}{ccccccc} \uparrow & (+) & 0 & & \uparrow & (+) & \\ \hline -3 & & 0 & & 3 & & \end{array}$

Vertical asymptotes:  $x = -3$  and  $x = 3$

Horizontal asymptote:  $y = 0$

Unusual steepness at  $x = 0$

No cusps



6.  $f(x) = x^{\frac{2}{3}}(x - 7)^{\frac{1}{3}}$

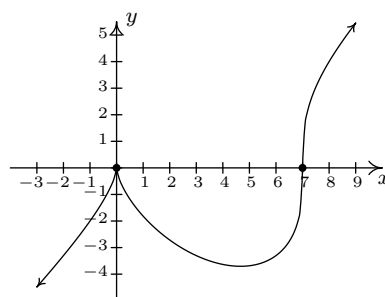
Domain:  $(-\infty, \infty)$

$\begin{array}{ccccccc} (-) & 0 & & (-) & & 0 & (+) \\ \hline & 0 & & 7 & & & \end{array}$

No vertical or horizontal asymptotes<sup>13</sup>

Unusual steepness at  $x = 7$

Cusp at  $x = 0$



7.  $f(x) = \frac{5x}{\sqrt[3]{x^3 + 8}}$

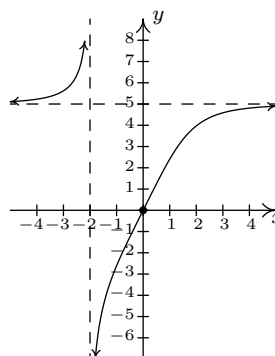
Domain:  $(-\infty, -2) \cup (-2, \infty)$

$\begin{array}{ccccccc} (+) & \uparrow & (-) & 0 & (+) & & \\ \hline & -2 & & 0 & & & \end{array}$

Vertical asymptote  $x = -2$

Horizontal asymptote  $y = 5$

No unusual steepness or cusps



8.  $f(x) = x^{\frac{3}{2}}(x - 7)^{\frac{1}{3}}$

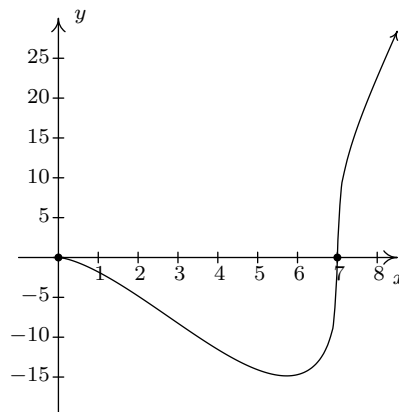
Domain:  $[0, \infty)$

$\begin{array}{ccccccc} 0 & & (-) & & 0 & (+) & \\ \hline 0 & & & & 7 & & \end{array}$

No asymptotes

Unusual steepness at  $x = 7$

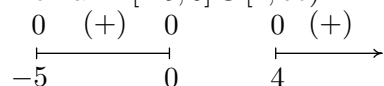
No cusps



<sup>13</sup>Using Calculus it can be shown that  $y = x - \frac{7}{3}$  is a slant asymptote of this graph.



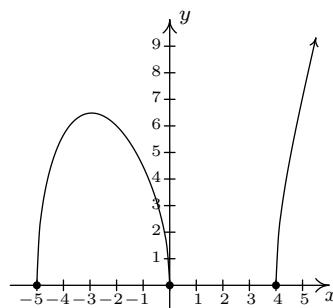
9.  $f(x) = \sqrt{x(x+5)(x-4)}$

Domain:  $[-5, 0] \cup [4, \infty)$ 

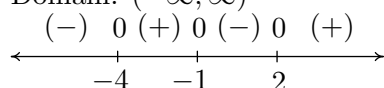
No asymptotes

Unusual steepness at  $x = -5, x = 0$  and  $x = 4$ 

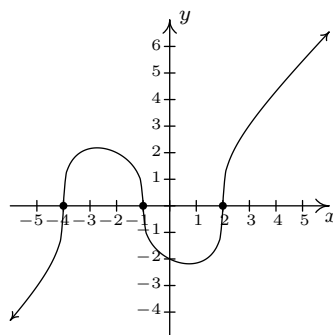
No cusps



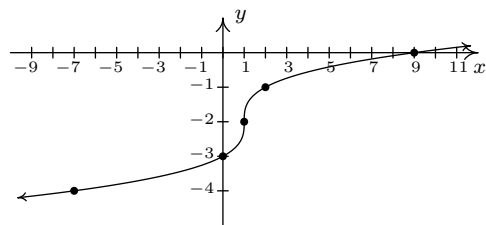
10.  $f(x) = \sqrt[3]{x^3 + 3x^2 - 6x - 8}$

Domain:  $(-\infty, \infty)$ No vertical or horizontal asymptotes<sup>14</sup>Unusual steepness at  $x = -4, x = -1$  and  $x = 2$ 

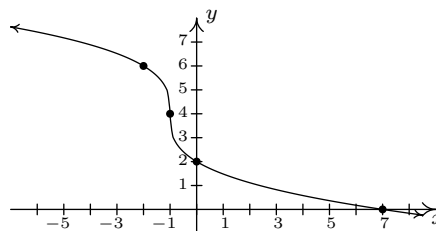
No cusps



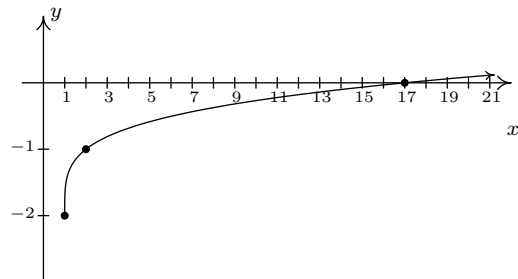
11.  $g(x) = \sqrt[3]{x-1} - 2$



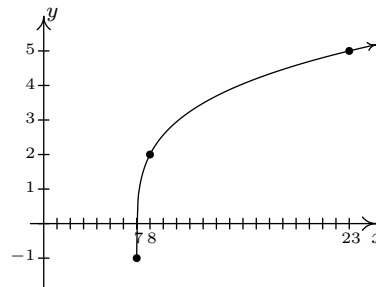
12.  $g(x) = -2\sqrt[3]{x+1} + 4$



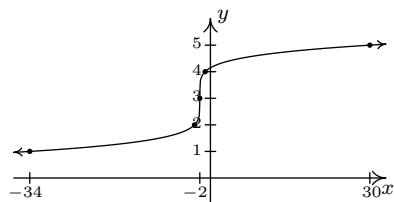
13.  $g(x) = \sqrt[4]{x-1} - 2$



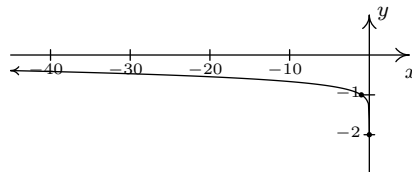
14.  $g(x) = 3\sqrt[4]{x-7} - 1$

<sup>14</sup>Using Calculus it can be shown that  $y = x + 1$  is a slant asymptote of this graph.

15.  $g(x) = \sqrt[5]{x+2} + 3$



16.  $g(x) = \sqrt[8]{-x} - 2$



17.  $x = 3$

18.  $x = \frac{1}{4}$

19.  $x = -3$

20.  $x = -\frac{1}{3}, \frac{2}{3}$

21.  $x = \frac{5+\sqrt{57}}{8}$

22.  $x = 4$

23.  $x = \pm 8$

24.  $x = 6$

25.  $x = 4$

26.  $x = -2, 6$

27.  $[2, \infty)$

28.  $[-1, 0] \cup [1, \infty)$

29.  $(-\infty, 2) \cup (2, 3]$

30.  $(2, 6]$

31.  $(-\infty, 0) \cup [2, 3) \cup (3, \infty)$

32.  $(-\infty, -1)$

33.  $(0, \frac{27}{13})$

34.  $(-\infty, 0) \cup (0, 3)$

35.  $(-\infty, -4) \cup (-4, -\frac{22}{19}] \cup (2, \infty)$

36.  $C(x) = 15x + 20\sqrt{100 + (30 - x)^2}$ ,  $0 \leq x \leq 30$ . The calculator gives the absolute minimum at  $\approx (18.66, 582.29)$ . This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off the road and heading towards the outpost. The minimum cost to run the cable is approximately \$582.29.

37. (a)  $h(r) = \frac{300}{\pi r^2}$ ,  $r > 0$ .

(b)  $S(r) = \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$ ,  $r > 0$

(c) The calculator gives the absolute minimum at the point  $\approx (4.07, 90.23)$ . This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.

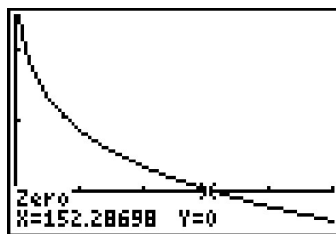
38. (a)  $W \approx 37.55^\circ\text{F}$ .

(b)  $V \approx 9.84$  miles per hour.

39. (a)  $W(V) = 53.142 - 23.78V^{0.16}$ . Since we are told in Exercise 38 that wind chill is only effect for wind speeds of more than 3 miles per hour, we restrict the domain to  $V > 3$ .

(b)  $W(V) = 0$  when  $V \approx 152.29$ . This means, according to the model, for the wind chill temperature to be  $0^\circ\text{F}$ , the wind speed needs to be 152.29 miles per hour.

(c) The graph is below.



40.  $9.8 \left( \frac{1}{4\pi} \right)^2 \approx 0.062$  meters or 6.2 centimeters

41. (a) First rewrite the model as  $P = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$ . Then  $300 = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$  yields  $y = \left( \frac{300}{1.23x^{\frac{2}{5}}} \right)^{\frac{5}{3}}$ .  
If  $x = 100$  then  $y \approx 441.93687$ .

42. (a)  $[0, c)$

(b)

$$\begin{aligned} m(.1c) &= \frac{m_r}{\sqrt{.99}} \approx 1.005m_r & m(.5c) &= \frac{m_r}{\sqrt{.75}} \approx 1.155m_r \\ m(.9c) &= \frac{m_r}{\sqrt{.19}} \approx 2.294m_r & m(.999c) &= \frac{m_r}{\sqrt{.001999}} \approx 22.366m_r \end{aligned}$$

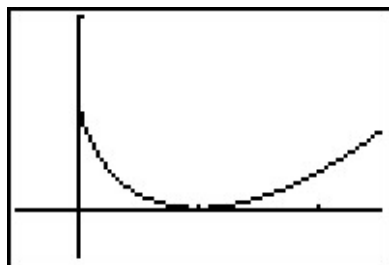
(c) As  $x \rightarrow c^-$ ,  $m(x) \rightarrow \infty$

(d) If the object is traveling no faster than approximately 0.99995 times the speed of light, then its observed mass will be no greater than  $100m_r$ .

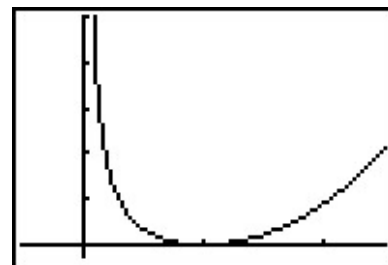
43.  $k^{-1}(x) = \frac{x}{\sqrt{x^2 - 4}}$

44. (a)  $y = \frac{1}{3}x^{3/2} - \sqrt{x} + \frac{2}{3}$ . The point  $(0, \frac{2}{3})$  is when Fritzy's path crosses Chewbacca's path - in other words, where Fritzy catches Chewbacca.

(b)  $y = \frac{1}{6}x^3 + \frac{1}{2x} - \frac{2}{3}$ . Using the techniques from Chapter 4, we find as  $x \rightarrow 0^+$ ,  $y \rightarrow \infty$  which means, in this case, Fritzy's pursuit never ends; he never catches Chewbacca. This makes sense since Chewbacca has a head start and is running faster than Fritzy.



$$y = \frac{1}{3}x^{3/2} - \sqrt{x} + \frac{2}{3}$$



$$y = \frac{1}{6}x^3 + \frac{1}{2x} - \frac{2}{3}$$

