# CHAPTER 8

# SYSTEMS OF EQUATIONS AND MATRICES

## 8.1 SYSTEMS OF LINEAR EQUATIONS: GAUSSIAN ELIMINATION

Up until now, when we concerned ourselves with solving different types of equations there was only one equation to solve at a time. Given an equation $f(x) = g(x)$, we could check our solutions geometrically by finding where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The $x$-coordinates of these intersection points correspond to the solutions to the equation $f(x) = g(x)$, and the $y$-coordinates were largely ignored. If we modify the problem and ask for the intersection points of the graphs of $y = f(x)$ and $y = g(x)$, where both the solution to $x$ and $y$ are of interest, we have what is known as a **system of equations**, usually written as

$$\begin{cases} y & = & f(x) \\ y & = & g(x) \end{cases}$$

The 'curly bracket' notation means we are to find all **pairs** of points $(x, y)$ which satisfy **both** equations. We begin our study of systems of equations by reviewing some basic notions from Intermediate Algebra.

> **Definition 8.1.** A **linear equation in two variables** is an equation of the form $a_1x + a_2y = c$ where $a_1$, $a_2$ and $c$ are real numbers and at least one of $a_1$ and $a_2$ is nonzero.

For reasons which will become clear later in the section, we are using subscripts in Definition 8.1 to indicate different, but fixed, real numbers and those subscripts have no mathematical meaning beyond that. For example, $3x - \frac{y}{2} = 0.1$ is a linear equation in two variables with $a_1 = 3$, $a_2 = -\frac{1}{2}$ and $c = 0.1$. We can also consider $x = 5$ to be a linear equation in two variables[1] by identifying $a_1 = 1$, $a_2 = 0$, and $c = 5$. If $a_1$ and $a_2$ are both 0, then depending on $c$, we get either an equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c = 0$, then we get $0 = 0$, which is always true. If $c \neq 0$, then we'd have $0 \neq 0$, which is never true.) Even though identities and contradictions have a large role to play

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[1]Critics may argue that $x = 5$ is clearly an equation in one variable. It can also be considered an equation in 117 variables with the coefficients of 116 variables set to 0. As with many conventions in Mathematics, the context will clarify the situation.

in the upcoming sections, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. We leave it to the reader to explain why these do not satisfy Definition 8.1. From what we know from Sections 1.2 and 2.1, the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example reviews some of the basic techniques first learned in Intermediate Algebra.

**Example 8.1.1.** Solve the following systems of equations. Check your answer algebraically and graphically.

1. $\begin{cases} 2x - y &= 1 \\ y &= 3 \end{cases}$

3. $\begin{cases} \frac{x}{3} - \frac{4y}{5} &= \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} &= \frac{1}{2} \end{cases}$

5. $\begin{cases} 6x + 3y &= 9 \\ 4x + 2y &= 12 \end{cases}$

2. $\begin{cases} 3x + 4y &= -2 \\ -3x - y &= 5 \end{cases}$

4. $\begin{cases} 2x - 4y &= 6 \\ 3x - 6y &= 9 \end{cases}$

6. $\begin{cases} x - y &= 0 \\ x + y &= 2 \\ -2x + y &= -2 \end{cases}$

**Solution.**

1. Our first system is nearly solved for us. The second equation tells us that $y = 3$. To find the corresponding value of $x$, we **substitute** this value for $y$ into the the first equation to obtain $2x - 3 = 1$, so that $x = 2$. Our solution to the system is $(2, 3)$. To check this algebraically, we substitute $x = 2$ and $y = 3$ into each equation and see that they are satisfied. We see $2(2) - 3 = 1$, and $3 = 3$, as required. To check our answer graphically, we graph the lines $2x - y = 1$ and $y = 3$ and verify that they intersect at $(2, 3)$.

2. To solve the second system, we use the **addition** method to **eliminate** the variable $x$. We take the two equations as given and 'add equals to equals' to obtain

$$\begin{array}{rcl} 3x + 4y &=& -2 \\ +\quad (-3x - y &=& 5) \\ \hline 3y &=& 3 \end{array}$$

This gives us $y = 1$. We now substitute $y = 1$ into either of the two equations, say $-3x - y = 5$, to get $-3x - 1 = 5$ so that $x = -2$. Our solution is $(-2, 1)$. Substituting $x = -2$ and $y = 1$ into the first equation gives $3(-2) + 4(1) = -2$, which is true, and, likewise, when we check $(-2, 1)$ in the second equation, we get $-3(-2) - 1 = 5$, which is also true. Geometrically, the lines $3x + 4y = -2$ and $-3x - y = 5$ intersect at $(-2, 1)$.

$2x - y = 1$
$\mathbf{y = 3}$

$3x + 4y = -2$
$\mathbf{-3x - y = 5}$
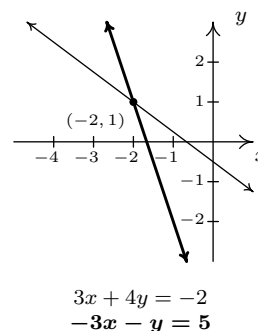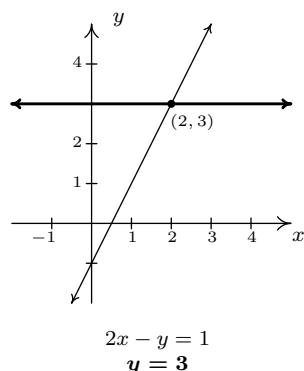
3. The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18 to obtain the kinder, gentler system

$$\begin{cases} 5x - 12y &=& 21 \\ 4x + 6y &=& 9 \end{cases}$$

Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by $-5$, we will be in a position to eliminate the $x$ term

$$\begin{array}{rrcr} & 20x - 48y &=& 84 \\ + & (-20x - 30y &=& -45) \\ \hline & -78y &=& 39 \end{array}$$

From this we get $y = -\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y = -\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into $5x - 12y = 21$. We get $5x + 6 = 21$ which gives $x = 3$. Our answer is $\left(3, -\frac{1}{2}\right)$. At this point, we have no choice — in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x = 3$ and $y = -\frac{1}{2}$ into both $\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3, -\frac{1}{2}\right)$.

4. An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and the both sides of the second equation by $-2$, we are ready to eliminate the $x$

$$
\begin{array}{rcl}
6x - 12y &=& 18 \\
+ \quad (-6x + 12y &=& -18) \\
\hline
0 &=& 0
\end{array}
$$

We eliminated not only the $x$, but the $y$ as well and we are left with the identity $0 = 0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair $(x, y)$ satisfies the equation $2x - 4y = 6$, it *automatically* satisfies the equation $3x - 6y = 9$. One way to describe the solution set to this system is to use the roster method[2] and write $\{(x, y) \,|\, 2x - 4y = 6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution to a system**. Our first step is to solve $2x - 4y = 6$ for one of the variables, say $y = \frac{1}{2}x - \frac{3}{2}$. For each value of $x$, the formula $y = \frac{1}{2}x - \frac{3}{2}$ determines the corresponding $y$-value of a solution. Since we have no restriction on $x$, it is called a **free variable**. We let $x = t$, a so-called 'parameter', and get $y = \frac{1}{2}t - \frac{3}{2}$. Our set of solutions can then be described as $\{(t, \frac{1}{2}t - \frac{3}{2}) \,|\, -\infty < t < \infty\}$.[3] For specific values of $t$, we can generate solutions. For example, $t = 0$ gives us the solution $(0, -\frac{3}{2})$; $t = 117$ gives us $(117, 57)$, and while we can readily check each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as always. We claim that for any real number $t$, the pair $(t, \frac{1}{2}t - \frac{3}{2})$ satisfies both equations. Substituting $x = t$ and $y = \frac{1}{2}t - \frac{3}{2}$ into $2x - 4y = 6$ gives $2t - 4\left(\frac{1}{2}t - \frac{3}{2}\right) = 6$. Simplifying, we get $2t - 2t + 6 = 6$, which is always true. Similarly, when we make these substitutions in the equation $3x - 6y = 9$, we get $3t - 6\left(\frac{1}{2}t - \frac{3}{2}\right) = 9$ which reduces to $3t - 3t + 9 = 9$, so it checks out, too. Geometrically, $2x - 4y = 6$ and $3x - 6y = 9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.
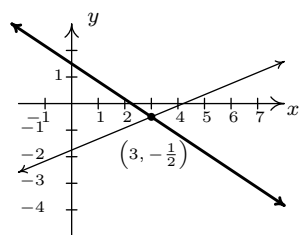


$$\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$$
$$\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$$
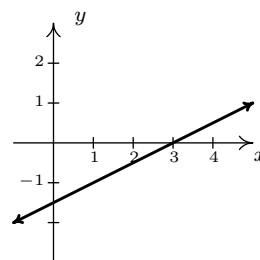
$$2x - 4y = 6$$
$$3x - 6y = 9$$
(Same line.)

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[2]See Section 1.2 for a review of this.

[3]Note that we could have just as easily chosen to solve $2x - 4y = 6$ for $x$ to obtain $x = 2y + 3$. Letting $y$ be the parameter $t$, we have that for any value of $t$, $x = 2t + 3$, which gives $\{(2t + 3, t) \,|\, -\infty < t < \infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.

5. Multiplying both sides of the first equation by 2 and the both sides of the second equation by $-3$, we set the stage to eliminate $x$

$$
\begin{array}{rcr}
12x + 6y & = & 18 \\
+ \quad (-12x - 6y & = & -36) \\
\hline
0 & = & -18
\end{array}
$$

As in the previous example, both $x$ and $y$ dropped out of the equation, but we are left with an irrevocable contradiction, $0 = -18$. This tells us that it is impossible to find a pair $(x, y)$ which satisfies both equations; in other words, the system has no solution. Graphically, the lines $6x + 3y = 9$ and $4x + 2y = 12$ are distinct and parallel, so they do not intersect.

6. We can begin to solve our last system by adding the first two equations

$$
\begin{array}{rcr}
x - y & = & 0 \\
+ \quad (x + y & = & 2) \\
\hline
2x & = & 2
\end{array}
$$

which gives $x = 1$. Substituting this into the first equation gives $1 - y = 0$ so that $y = 1$. We seem to have determined a solution to our system, $(1, 1)$. While this checks in the first two equations, when we substitute $x = 1$ and $y = 1$ into the third equation, we get $-2(1) + (1) = -2$ which simplifies to the contradiction $-1 = -2$. Graphing the lines $x - y = 0$, $x + y = 2$, and $-2x + y = -2$, we see that the first two lines do, in fact, intersect at $(1, 1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found.



$6x + 3y = 9$
**$4x + 2y = 12$**

$y - x = 0$
$y + x = 2$
$-2x + y = -2$

□
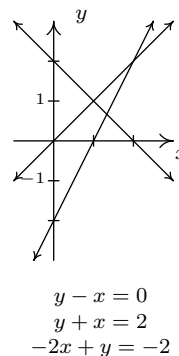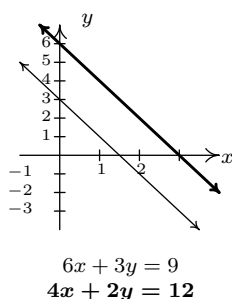
A few remarks about Example 8.1.1 are in order. It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish the two different types of behavior among

consistent systems. Those which admit free variables are called **dependent**; those with no free variables are called **independent**.[4] Using this new vocabulary, we classify numbers 1, 2 and 3 in Example 8.1.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[5] The system in 6 above is called **overdetermined**, since we have more equations than variables.[6] Not surprisingly, a system with more variables than equations is called **underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases. Likewise, there are both consistent and inconsistent underdetermined systems,[7] but a consistent underdetermined system of linear equations is necessarily dependent.[8]

In order to move this section beyond a review of Intermediate Algebra, we now define what is meant by a linear equation in $n$ variables.

> **Definition 8.2.** A **linear equation in $n$ variables**, $x_1$, $x_2$, ..., $x_n$ is an equation of the form $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = c$ where $a_1$, $a_2$, ...$a_n$ and $c$ are real numbers and at least one of $a_1$, $a_2$, ..., $a_n$ is nonzero.

Instead of using more familiar variables like $x$, $y$, and even $z$ and/or $w$ in Definition 8.2, we use subscripts to distinguish the different variables. We have no idea how many variables may be involved, so we use numbers to distinguish them instead of letters. (There is an endless supply of distinct numbers.) As an example, the linear equation $3x_1 - x_2 = 4$ represents the same relationship between the variables $x_1$ and $x_2$ as the equation $3x - y = 4$ does between the variables $x$ and $y$. In addition, just as we cannot combine the terms in the expression $3x - y$, we cannot combine the terms in the expression $3x_1 - x_2$. Coupling more than one linear equation in $n$ variables results in a **system of linear equations in $n$ variables**. When solving these systems, it becomes increasingly important to keep track of what operations are performed to which equations and to develop a strategy based on the kind of manipulations we've already employed. To this end, we first remind ourselves of the maneuvers which can be applied to a system of linear equations that result in an equivalent system.[9]

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[4]In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[5]The adjectives 'dependent' and 'independent' apply only to *consistent* systems – they describe the *type* of solutions. Is there a free variable (dependent) or not (independent)?

[6]If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

[7]We need more than two variables to give an example of the latter.

[8]Again, experience with systems with more variables helps to see this here, as does a solid course in Linear Algebra.

[9]That is, a system with the same solution set.

> **Theorem 8.1.** Given a system of equations, the following moves will result in an equivalent system of equations.
>
> - Interchange the position of any two equations.
>
> - Replace an equation with a nonzero multiple of itself.[a]
>
> - Replace an equation with itself plus a nonzero multiple of another equation.
>
> _____
>
> [a]That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

We have seen plenty of instances of the second and third moves in Theorem 8.1 when we solved the systems Example 8.1.1. The first move, while it obviously admits an equivalent system, seems silly. Our perception will change as we consider more equations and more variables in this, and later sections.

Consider the system of equations

$$\left\{ \begin{array}{rcl} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

Clearly $z = -1$, and we substitute this into the second equation $y - \frac{1}{2}(-1) = 4$ to obtain $y = \frac{7}{2}$. Finally, we substitute $y = \frac{7}{2}$ and $z = -1$ into the first equation to get $x - \frac{1}{3}\left(\frac{7}{2}\right) + \frac{1}{2}(-1) = 1$, so that $x = \frac{8}{3}$. The reader can verify that these values of $x$, $y$ and $z$ satisfy all three original equations. It is tempting for us to write the solution to this system by extending the usual $(x, y)$ notation to $(x, y, z)$ and list our solution as $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$. The question quickly becomes what does an 'ordered triple' like $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space.[10] Moreover, just as equations involving the variables $x$ and $y$ describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving variables $x$, $y$, and $z$ describe objects called **surfaces** in three-dimensional space. Each of the equations in the above system can be visualized as a plane situated in three-space. Geometrically, the system is trying to find the intersection, or common point, of all three planes. If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect. Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line,[11] so our intersection point is where all three of these lines meet.

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[10]You were asked to think about this in Exercise 40 in Section 1.1.

[11]In fact, these lines are described by the parametric solutions to the systems formed by taking any two of these equations by themselves.
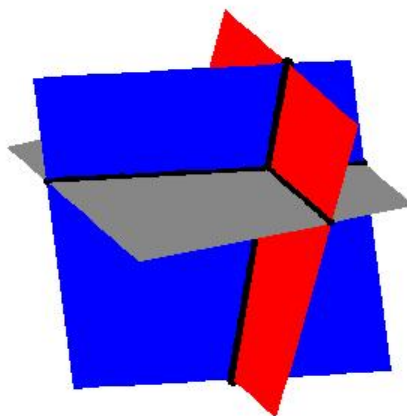
Since the geometry for equations involving more than two variables is complicated, we will focus our efforts on the algebra. Returning to the system

$$\begin{cases} x - \frac{1}{3}y + \frac{1}{2}z &=& 1 \\ y - \frac{1}{2}z &=& 4 \\ z &=& -1 \end{cases}$$

we note the reason it was so easy to solve is that the third equation is solved for $z$, the second equation involves only $y$ and $z$, and since the coefficient of $y$ is 1, it makes it easy to solve for $y$ using our known value for $z$. Lastly, the coefficient of $x$ in the first equation is 1 making it easy to substitute the known values of $y$ and $z$ and then solve for $x$. We formalize this pattern below for the most general systems of linear equations. Again, we use subscripted variables to describe the general case. The variable with the smallest subscript in a given equation is typically called the **leading variable** of that equation.

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**Definition 8.3.** A system of linear equations with variables $x_1$, $x_2$, $\ldots x_n$ is said to be in **triangular form** provided all of the following conditions hold:

1. The subscripts of the variables in each equation are always increasing from left to right.

2. The leading variable in each equation has coefficient 1.

3. The subscript on the leading variable in a given equation is greater than the subscript on the leading variable in the equation above it.

4. Any equation without variables[a] cannot be placed above an equation with variables.

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[a]necessarily an identity or contradiction

In our previous system, if we make the obvious choices $x = x_1$, $y = x_2$, and $z = x_3$, we see that the system is in triangular form.[12] An example of a more complicated system in triangular form is

$$\begin{cases} x_1 - 4x_3 + x_4 - x_6 &=& 6 \\ x_2 + 2x_3 &=& 1 \\ x_4 + 3x_5 - x_6 &=& 8 \\ x_5 + 9x_6 &=& 10 \end{cases}$$

Our goal henceforth will be to transform a given system of linear equations into triangular form using the moves in Theorem 8.1.

**Example 8.1.2.** Use Theorem 8.1 to put the following systems into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

1. $\begin{cases} 3x - y + z &=& 3 \\ 2x - 4y + 3z &=& 16 \\ x - y + z &=& 5 \end{cases}$
2. $\begin{cases} 2x + 3y - z &=& 1 \\ 10x - z &=& 2 \\ 4x - 9y + 2z &=& 5 \end{cases}$
3. $\begin{cases} 3x_1 + x_2 + x_4 &=& 6 \\ 2x_1 + x_2 - x_3 &=& 4 \\ x_2 - 3x_3 - 2x_4 &=& 0 \end{cases}$

**Solution.**

1. For definitiveness, we label the topmost equation in the system $E1$, the equation beneath that $E2$, and so forth. We now attempt to put the system in triangular form using an algorithm known as **Gaussian Elimination**. What this means is that, starting with $x$, we transform the system so that conditions 2 and 3 in Definition 8.3 are satisfied. Then we move on to the next variable, in this case $y$, and repeat. Since the variables in all of the equations have a consistent ordering from left to right, our first move is to get an $x$ in $E1$'s spot with a coefficient of 1. While there are many ways to do this, the easiest is to apply the first move listed in Theorem 8.1 and interchange $E1$ and $E3$.

$$\begin{cases} (E1) & 3x - y + z &=& 3 \\ (E2) & 2x - 4y + 3z &=& 16 \\ (E3) & x - y + z &=& 5 \end{cases} \xrightarrow{\text{Switch } E1 \text{ and } E3} \begin{cases} (E1) & x - y + z &=& 5 \\ (E2) & 2x - 4y + 3z &=& 16 \\ (E3) & 3x - y + z &=& 3 \end{cases}$$

To satisfy Definition 8.3, we need to eliminate the $x$'s from $E2$ and $E3$. We accomplish this by replacing each of them with a sum of themselves and a multiple of $E1$. To eliminate the $x$ from $E2$, we need to multiply $E1$ by $-2$ then add; to eliminate the $x$ from $E3$, we need to multiply $E1$ by $-3$ then add. Applying the third move listed in Theorem 8.1 twice, we get

$$\begin{cases} (E1) & x - y + z &=& 5 \\ (E2) & 2x - 4y + 3z &=& 16 \\ (E3) & 3x - y + z &=& 3 \end{cases} \xrightarrow[\text{Replace } E3 \text{ with } -3E1 + E3]{\text{Replace } E2 \text{ with } -2E1 + E2} \begin{cases} (E1) & x - y + z &=& 5 \\ (E2) & -2y + z &=& 6 \\ (E3) & 2y - 2z &=& -12 \end{cases}$$

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[12]If letters are used instead of subscripted variables, Definition 8.3 can be suitably modified using alphabetical order of the variables instead of numerical order on the subscripts of the variables.

Now we enforce the conditions stated in Definition 8.3 for the variable $y$. To that end we need to get the coefficient of $y$ in $E2$ equal to 1. We apply the second move listed in Theorem 8.1 and replace $E2$ with itself times $-\frac{1}{2}$.

$$\begin{cases} (E1) & x - y + z & = & 5 \\ (E2) & -2y + z & = & 6 \\ (E3) & 2y - 2z & = & -12 \end{cases} \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{2}E2} \begin{cases} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & 2y - 2z & = & -12 \end{cases}$$

To eliminate the $y$ in $E3$, we add $-2E2$ to it.

$$\begin{cases} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & 2y - 2z & = & -12 \end{cases} \xrightarrow{\text{Replace } E3 \text{ with } -2E2 + E3} \begin{cases} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{cases}$$

Finally, we apply the second move from Theorem 8.1 one last time and multiply $E3$ by $-1$ to satisfy the conditions of Definition 8.3 for the variable $z$.

$$\begin{cases} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{cases} \xrightarrow{\text{Replace } E3 \text{ with } -1E3} \begin{cases} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & z & = & 6 \end{cases}$$

Now we proceed to substitute. Plugging in $z = 6$ into $E2$ gives $y - 3 = -3$ so that $y = 0$. With $y = 0$ and $z = 6$, $E1$ becomes $x - 0 + 6 = 5$, or $x = -1$. Our solution is $(-1, 0, 6)$. We leave it to the reader to check that substituting the respective values for $x$, $y$, and $z$ into the original system results in three identities. Since we have found a solution, the system is consistent; since there are no free variables, it is independent.

2. Proceeding as we did in 1, our first step is to get an equation with $x$ in the $E1$ position with 1 as its coefficient. Since there is no easy fix, we multiply $E1$ by $\frac{1}{2}$.

$$\begin{cases} (E1) & 2x + 3y - z & = & 1 \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{cases} \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{2}E1} \begin{cases} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{cases}$$

Now it's time to take care of the $x$'s in $E2$ and $E3$.

$$\begin{cases} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{cases} \xrightarrow[\text{Replace } E3 \text{ with } -4E1 + E3]{\text{Replace } E2 \text{ with } -10E1 + E2} \begin{cases} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{cases}$$

Our next step is to get the coefficient of $y$ in $E2$ equal to 1. To that end, we have

$$\left\{\begin{array}{rl} (E1) & x + \frac{3}{2}y - \frac{1}{2}z = \frac{1}{2} \\ (E2) & -15y + 4z = -3 \\ (E3) & -15y + 4z = 3 \end{array}\right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{15}E2} \left\{\begin{array}{rl} (E1) & x + \frac{3}{2}y - \frac{1}{2}z = \frac{1}{2} \\ (E2) & y - \frac{4}{15}z = \frac{1}{5} \\ (E3) & -15y + 4z = 3 \end{array}\right.$$

Finally, we rid $E3$ of $y$.

$$\left\{\begin{array}{rl} (E1) & x + \frac{3}{2}y - \frac{1}{2}z = \frac{1}{2} \\ (E2) & y - \frac{4}{15}z = \frac{1}{5} \\ (E3) & -15y + 4z = 3 \end{array}\right. \xrightarrow{\text{Replace } E3 \text{ with } 15E2 + E3} \left\{\begin{array}{rl} (E1) & x - y + z = 5 \\ (E2) & y - \frac{1}{2}z = -3 \\ (E3) & 0 = 6 \end{array}\right.$$

The last equation, $0 = 6$, is a contradiction so the system has no solution. According to Theorem 8.1, since this system has no solutions, neither does the original, thus we have an inconsistent system.

3. For our last system, we begin by multiplying $E1$ by $\frac{1}{3}$ to get a coefficient of 1 on $x_1$.

$$\left\{\begin{array}{rl} (E1) & 3x_1 + x_2 + x_4 = 6 \\ (E2) & 2x_1 + x_2 - x_3 = 4 \\ (E3) & x_2 - 3x_3 - 2x_4 = 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{rl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 = 2 \\ (E2) & 2x_1 + x_2 - x_3 = 4 \\ (E3) & x_2 - 3x_3 - 2x_4 = 0 \end{array}\right.$$

Next we eliminate $x_1$ from $E2$

$$\left\{\begin{array}{rl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 = 2 \\ (E2) & 2x_1 + x_2 - x_3 = 4 \\ (E3) & x_2 - 3x_3 - 2x_4 = 0 \end{array}\right. \xrightarrow[\text{with } -2E1 + E2]{\text{Replace } E2} \left\{\begin{array}{rl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 = 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 = 0 \\ (E3) & x_2 - 3x_3 - 2x_4 = 0 \end{array}\right.$$

We switch $E2$ and $E3$ to get a coefficient of 1 for $x_2$.

$$\left\{\begin{array}{rl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 = 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 = 0 \\ (E3) & x_2 - 3x_3 - 2x_4 = 0 \end{array}\right. \xrightarrow{\text{Switch } E2 \text{ and } E3} \left\{\begin{array}{rl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 = 2 \\ (E2) & x_2 - 3x_3 - 2x_4 = 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 = 0 \end{array}\right.$$

Finally, we eliminate $x_2$ in $E3$.

$$\left\{\begin{array}{llr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -\frac{1}{3}E2 + E3]{\text{Replace } E3} \left\{\begin{array}{llr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & 0 & = & 0 \end{array}\right.$$

Equation $E3$ reduces to $0 = 0$, which is always true. Since we have no equations with $x_3$ or $x_4$ as leading variables, they are both free, which means we have a consistent dependent system. We parametrize the solution set by letting $x_3 = s$ and $x_4 = t$ and obtain from $E2$ that $x_2 = 3s + 2t$. Substituting this and $x_4 = t$ into $E1$, we have $x_1 + \frac{1}{3}(3s + 2t) + \frac{1}{3}t = 2$ which gives $x_1 = 2 - s - t$. Our solution is the set $\{(2 - s - t, 2s + 3t, s, t) \,|\, -\infty < s, t < \infty\}$.[13] We leave it to the reader to verify that the substitutions $x_1 = 2 - s - t$, $x_2 = 3s + 2t$, $x_3 = s$ and $x_4 = t$ satisfy the equations in the original system.                                                        □

Like all algorithms, Gaussian Elimination has the advantage of always producing what we need, but it can also be inefficient at times. For example, when solving 2 above, it is clear after we eliminated the $x$'s in the second step to get the system

$$\left\{\begin{array}{llr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array}\right.$$

that equations $E2$ and $E3$ when taken together form a contradiction since we have identical left hand sides and different right hand sides. The algorithm takes two more steps to reach this contradiction. We also note that substitution in Gaussian Elimination is delayed until all the elimination is done, thus it gets called **back-substitution**. This may also be inefficient in many cases. Rest assured, the technique of substitution as you may have learned it in Intermediate Algebra will once again take center stage in Section 8.7. Lastly, we note that the system in 3 above is underdetermined, and as it is consistent, we have free variables in our answer. We close this section with a standard 'mixture' type application of systems of linear equations.

**Example 8.1.3.** Lucas needs to create a 500 milliliters (mL) of a 40% acid solution. He has stock solutions of 30% and 90% acid as well as all of the distilled water he wants. Set-up and solve a system of linear equations which determines all of the possible combinations of the stock solutions and water which would produce the required solution.

**Solution.** We are after three unknowns, the amount (in mL) of the 30% stock solution (which we'll call $x$), the amount (in mL) of the 90% stock solution (which we'll call $y$) and the amount (in mL) of water (which we'll call $w$). We now need to determine some relationships between these variables. Our goal is to produce 500 milliliters of a 40% acid solution. This product has two defining characteristics. First, it must be 500 mL; second, it must be 40% acid. We take each

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[13]Here, any choice of $s$ and $t$ will determine a solution which is a point in 4-dimensional space. Yeah, we have trouble visualizing that, too.

of these qualities in turn. First, the total volume of 500 mL must be the sum of the contributed volumes of the two stock solutions and the water. That is

amount of 30% stock solution + amount of 90% stock solution + amount of water = 500 mL

Using our defined variables, this reduces to $x + y + w = 500$. Next, we need to make sure the final solution is 40% acid. Since water contains no acid, the acid will come from the stock solutions only. We find 40% of 500 mL to be 200 mL which means the final solution must contain 200 mL of acid. We have

amount of acid in 30% stock solution + amount of acid 90% stock solution = 200 mL

The amount of acid in $x$ mL of 30% stock is $0.30x$ and the amount of acid in $y$ mL of 90% solution is $0.90y$. We have $0.30x + 0.90y = 200$. Converting to fractions,[14] our system of equations becomes

$$\begin{cases} x + y + w & = & 500 \\ \frac{3}{10}x + \frac{9}{10}y & = & 200 \end{cases}$$

We first eliminate the $x$ from the second equation

$$\begin{cases} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{10}x + \frac{9}{10}y & = & 200 \end{cases} \xrightarrow{\text{Replace } E2 \text{ with } -\frac{3}{10}E1 + E2} \begin{cases} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w & = & 50 \end{cases}$$

Next, we get a coefficient of 1 on the leading variable in $E2$

$$\begin{cases} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w & = & 50 \end{cases} \xrightarrow{\text{Replace } E2 \text{ with } \frac{5}{3}E2} \begin{cases} (E1) & x + y + w & = & 500 \\ (E2) & y - \frac{1}{2}w & = & \frac{250}{3} \end{cases}$$

Notice that we have no equation to determine $w$, and as such, $w$ is free. We set $w = t$ and from $E2$ get $y = \frac{1}{2}t + \frac{250}{3}$. Substituting into $E1$ gives $x + \left(\frac{1}{2}t + \frac{250}{3}\right) + t = 500$ so that $x = -\frac{3}{2}t + \frac{1250}{3}$. This system is consistent, dependent and its solution set is $\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid -\infty < t < \infty\}$. While this answer checks algebraically, we have neglected to take into account that $x$, $y$ and $w$, being amounts of acid and water, need to be nonnegative. That is, $x \geq 0$, $y \geq 0$ and $w \geq 0$. The constraint $x \geq 0$ gives us $-\frac{3}{2}t + \frac{1250}{3} \geq 0$, or $t \leq \frac{2500}{9}$. From $y \geq 0$, we get $\frac{1}{2}t + \frac{250}{3} \geq 0$ or $t \geq -\frac{500}{3}$. The condition $z \geq 0$ yields $t \geq 0$, and we see that when we take the set theoretic intersection of these intervals, we get $0 \leq t \leq \frac{2500}{9}$. Our final answer is $\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid 0 \leq t \leq \frac{2500}{9}\}$. Of what practical use is our answer? Suppose there is only 100 mL of the 90% solution remaining and it is due to expire. Can we use all of it to make our required solution? We would have $y = 100$ so that $\frac{1}{2}t + \frac{250}{3} = 100$, and we get $t = \frac{100}{3}$. This means the amount of 30% solution required is $x = -\frac{3}{2}t + \frac{1250}{3} = -\frac{3}{2}\left(\frac{100}{3}\right) + \frac{1250}{3} = \frac{1100}{3}$ mL, and for the water, $w = t = \frac{100}{3}$ mL. The reader is invited to check that mixing these three amounts of our constituent solutions produces the required 40% acid mix. $\square$

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[14] We do this only because we believe students can use all of the practice with fractions they can get!

### 8.1.1   EXERCISES

(Review Exercises) In Exercises 1 - 8, take a trip down memory lane and solve the given system using substitution and/or elimination. Classify each system as consistent independent, consistent dependent, or inconsistent. Check your answers both algebraically and graphically.

1. $\begin{cases} x + 2y & = & 5 \\ x & = & 6 \end{cases}$

2. $\begin{cases} 2y - 3x & = & 1 \\ y & = & -3 \end{cases}$

3. $\begin{cases} \frac{x+2y}{4} & = & -5 \\ \frac{3x-y}{2} & = & 1 \end{cases}$

4. $\begin{cases} \frac{2}{3}x - \frac{1}{5}y & = & 3 \\ \frac{1}{2}x + \frac{3}{4}y & = & 1 \end{cases}$

5. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y & = & -1 \\ 2y - 3x & = & 6 \end{cases}$

6. $\begin{cases} x + 4y & = & 6 \\ \frac{1}{12}x + \frac{1}{3}y & = & \frac{1}{2} \end{cases}$

7. $\begin{cases} 3y - \frac{3}{2}x & = & -\frac{15}{2} \\ \frac{1}{2}x - y & = & \frac{3}{2} \end{cases}$

8. $\begin{cases} \frac{5}{6}x + \frac{5}{3}y & = & -\frac{7}{3} \\ -\frac{10}{3}x - \frac{20}{3}y & = & 10 \end{cases}$

In Exercises 9 - 26, put each system of linear equations into triangular form and solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

9. $\begin{cases} -5x + y & = & 17 \\ x + y & = & 5 \end{cases}$

10. $\begin{cases} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{cases}$

11. $\begin{cases} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 5 \end{cases}$

12. $\begin{cases} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 6 \end{cases}$

13. $\begin{cases} x + y + z & = & -17 \\ y - 3z & = & 0 \end{cases}$

14. $\begin{cases} x - 2y + 3z & = & 7 \\ -3x + y + 2z & = & -5 \\ 2x + 2y + z & = & 3 \end{cases}$

15. $\begin{cases} 3x - 2y + z & = & -5 \\ x + 3y - z & = & 12 \\ x + y + 2z & = & 0 \end{cases}$

16. $\begin{cases} 2x - y + z & = & -1 \\ 4x + 3y + 5z & = & 1 \\ 5y + 3z & = & 4 \end{cases}$

17. $\begin{cases} x - y + z & = & -4 \\ -3x + 2y + 4z & = & -5 \\ x - 5y + 2z & = & -18 \end{cases}$

18. $\begin{cases} 2x - 4y + z & = & -7 \\ x - 2y + 2z & = & -2 \\ -x + 4y - 2z & = & 3 \end{cases}$

19. $\begin{cases} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{cases}$

20. $\begin{cases} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{cases}$

21. $\begin{cases} x + y + z & = & 4 \\ 2x - 4y - z & = & -1 \\ x - y & = & 2 \end{cases}$

22. $\begin{cases} x - y + z & = & 8 \\ 3x + 3y - 9z & = & -6 \\ 7x - 2y + 5z & = & 39 \end{cases}$

23. $\begin{cases} 2x - 3y + z & = & -1 \\ 4x - 4y + 4z & = & -13 \\ 6x - 5y + 7z & = & -25 \end{cases}$

24. $\begin{cases} 2x_1 + x_2 - 12x_3 - x_4 & = & 16 \\ -x_1 + x_2 + 12x_3 - 4x_4 & = & -5 \\ 3x_1 + 2x_2 - 16x_3 - 3x_4 & = & 25 \\ x_1 + 2x_2 - 5x_4 & = & 11 \end{cases}$

25. $\begin{cases} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{cases}$

26. $\begin{cases} x_1 - x_2 - 5x_3 + 3x_4 & = & -1 \\ x_1 + x_2 + 5x_3 - 3x_4 & = & 0 \\ x_2 + 5x_3 - 3x_4 & = & 1 \\ x_1 - 2x_2 - 10x_3 + 6x_4 & = & -1 \end{cases}$

27. Find two other forms of the parametric solution to Exercise 11 above by reorganizing the equations so that $x$ or $y$ can be the free variable.

28. A local buffet charges \$7.50 per person for the basic buffet and \$9.25 for the deluxe buffet (which includes crab legs.) If 27 diners went out to eat and the total bill was \$227.00 before taxes, how many chose the basic buffet and how many chose the deluxe buffet?

29. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

30. Skippy has a total of \$10,000 to split between two investments. One account offers 3% simple interest, and the other account offers 8% simple interest. For tax reasons, he can only earn \$500 in interest the entire year. How much money should Skippy invest in each account to earn \$500 in interest for the year?

31. A 10% salt solution is to be mixed with pure water to produce 75 gallons of a 3% salt solution. How much of each are needed?

32. At The Crispy Critter's Head Shop and Patchouli Emporium along with their dried up weeds, sunflower seeds and astrological postcards they sell an herbal tea blend. By weight, Type I herbal tea is 30% peppermint, 40% rose hips and 30% chamomile, Type II has percents 40%, 20% and 40%, respectively, and Type III has percents 35%, 30% and 35%, respectively. How much of each Type of tea is needed to make 2 pounds of a new blend of tea that is equal parts peppermint, rose hips and chamomile?

33. Discuss with your classmates how you would approach Exercise 32 above if they needed to use up a pound of Type I tea to make room on the shelf for a new canister.

34. If you were to try to make 100 mL of a 60% acid solution using stock solutions at 20% and 40%, respectively, what would the triangular form of the resulting system look like? Explain.

## 8.1.2   ANSWERS

1. Consistent independent
   Solution $\left(6, -\frac{1}{2}\right)$

2. Consistent independent
   Solution $\left(-\frac{7}{3}, -3\right)$

3. Consistent independent
   Solution $\left(-\frac{16}{7}, -\frac{62}{7}\right)$

4. Consistent independent
   Solution $\left(\frac{49}{12}, -\frac{25}{18}\right)$

5. Consistent dependent
   Solution $\left(t, \frac{3}{2}t + 3\right)$
   for all real numbers $t$

6. Consistent dependent
   Solution $(6 - 4t, t)$
   for all real numbers $t$

7. Inconsistent
   No solution

8. Inconsistent
   No solution

Because triangular form is not unique, we give only one possible answer to that part of the question. Yours may be different and still be correct.

9. $\begin{cases} x + y &= 5 \\ y &= 7 \end{cases}$

   Consistent independent
   Solution $(-2, 7)$

10. $\begin{cases} x - \frac{5}{3}y - \frac{7}{3}z &= -\frac{7}{3} \\ y + \frac{5}{4}z &= 2 \\ z &= 0 \end{cases}$

   Consistent independent
   Solution $(1, 2, 0)$

11. $\begin{cases} x - \frac{1}{4}y + \frac{1}{4}z &= \frac{5}{4} \\ y + 3z &= 15 \\ 0 &= 0 \end{cases}$

   Consistent dependent
   Solution $(-t + 5, -3t + 15, t)$
   for all real numbers $t$

12. $\begin{cases} x - \frac{1}{4}y + \frac{1}{4}z &= \frac{5}{4} \\ y + 3z &= 15 \\ 0 &= 1 \end{cases}$

   Inconsistent
   No solution

13. $\begin{cases} x + y + z &= -17 \\ y - 3z &= 0 \end{cases}$

   Consistent dependent
   Solution $(-4t - 17, 3t, t)$
   for all real numbers $t$

14. $\begin{cases} x - 2y + 3z &= 7 \\ y - \frac{11}{5}z &= -\frac{16}{5} \\ z &= 1 \end{cases}$

   Consistent independent
   Solution $(2, -1, 1)$

15. $\begin{cases} x + y + 2z &= 0 \\ y - \frac{3}{2}z &= 6 \\ z &= -2 \end{cases}$

Consistent independent
Solution $(1, 3, -2)$

16. $\begin{cases} x - \frac{1}{2}y + \frac{1}{2}z &= -\frac{1}{2} \\ y + \frac{3}{5}z &= \frac{3}{5} \\ 0 &= 1 \end{cases}$

Inconsistent
no solution

17. $\begin{cases} x - y + z &= -4 \\ y - 7z &= 17 \\ z &= -2 \end{cases}$

Consistent independent
Solution $(1, 3, -2)$

18. $\begin{cases} x - 2y + 2z &= -2 \\ y &= \frac{1}{2} \\ z &= 1 \end{cases}$

Consistent independent
Solution $\left(-3, \frac{1}{2}, 1\right)$

19. $\begin{cases} x - \frac{1}{2}y + \frac{1}{2}z &= \frac{1}{2} \\ y - \frac{2}{3}z &= 0 \\ z &= 1 \end{cases}$

Consistent independent
Solution $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$

20. $\begin{cases} x - 3y - 4z &= 3 \\ y + \frac{11}{13}z &= \frac{4}{13} \\ 0 &= 0 \end{cases}$

Consistent dependent
Solution $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$
for all real numbers $t$

21. $\begin{cases} x + y + z &= 4 \\ y + \frac{1}{2}z &= \frac{3}{2} \\ 0 &= 1 \end{cases}$

Inconsistent
no solution

22. $\begin{cases} x - y + z &= 8 \\ y - 2z &= -5 \\ z &= 1 \end{cases}$

Consistent independent
Solution $(4, -3, 1)$

23. $\begin{cases} x - \frac{3}{2}y + \frac{1}{2}z &= -\frac{1}{2} \\ y + z &= -\frac{11}{2} \\ 0 &= 0 \end{cases}$

Consistent dependent
Solution $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$
for all real numbers $t$

24. $\begin{cases} x_1 + \frac{2}{3}x_2 - \frac{16}{3}x_3 - x_4 &= \frac{25}{3} \\ x_2 + 4x_3 - 3x_4 &= 2 \\ 0 &= 0 \\ 0 &= 0 \end{cases}$

Consistent dependent
Solution $(8s - t + 7, -4s + 3t + 2, s, t)$
for all real numbers $s$ and $t$

25. $\begin{cases} x_1 - x_3 &= -2 \\ x_2 - \frac{1}{2}x_4 &= 0 \\ x_3 - \frac{1}{2}x_4 &= 1 \\ x_4 &= 4 \end{cases}$

Consistent independent
Solution $(1, 2, 3, 4)$

26. $\begin{cases} x_1 - x_2 - 5x_3 + 3x_4 &=& -1 \\ x_2 + 5x_3 - 3x_4 &=& \frac{1}{2} \\ 0 &=& 1 \\ 0 &=& 0 \end{cases}$      Inconsistent
                                                   No solution

27. If $x$ is the free variable then the solution is $(t, 3t, -t+5)$ and if $y$ is the free variable then the solution is $\left(\frac{1}{3}t, t, -\frac{1}{3}t + 5\right)$.

28. 13 chose the basic buffet and 14 chose the deluxe buffet.

29. Mavis needs 20 pounds of $3 per pound coffee and 30 pounds of $8 per pound coffee.

30. Skippy needs to invest $6000 in the 3% account and $4000 in the 8% account.

31. 22.5 gallons of the 10% solution and 52.5 gallons of pure water.

32. $\frac{4}{3} - \frac{1}{2}t$ pounds of Type I, $\frac{2}{3} - \frac{1}{2}t$ pounds of Type II and $t$ pounds of Type III where $0 \le t \le \frac{4}{3}$.

## 8.2    Systems of Linear Equations: Augmented Matrices

In Section 8.1 we introduced Gaussian Elimination as a means of transforming a system of linear equations into triangular form with the ultimate goal of producing an equivalent system of linear equations which is easier to solve. If we take a step back and study the process, we see that all of our moves are determined entirely by the *coefficients* of the variables involved, and not the variables themselves. Much the same thing happened when we studied long division in Section 3.2. Just as we developed synthetic division to streamline that process, in this section, we introduce a similar bookkeeping device to help us solve systems of linear equations. To that end, we define a **matrix** as a rectangular array of real numbers. We typically enclose matrices with square brackets, '[' and ']', and we size matrices by the number of rows and columns they have. For example, the **size** (sometimes called the **dimension**) of

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$$

is $2 \times 3$ because it has 2 rows and 3 columns. The individual numbers in a matrix are called its **entries** and are usually labeled with double subscripts: the first tells which row the element is in and the second tells which column it is in. The rows are numbered from top to bottom and the columns are numbered from left to right. Matrices themselves are usually denoted by uppercase letters ($A$, $B$, $C$, etc.) while their entries are usually denoted by the corresponding letter. So, for instance, if we have

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$$

then $a_{11} = 3$, $a_{12} = 0$, $a_{13} = -1$, $a_{21} = 2$, $a_{22} = -5$, and $a_{23} = 10$. We shall explore matrices as mathematical objects with their own algebra in Section 8.3 and introduce them here solely as a bookkeeping device. Consider the system of linear equations from number 2 in Example 8.1.2

$$\begin{cases} (E1) & 2x + 3y - z & = & 1 \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{cases}$$

We encode this system into a matrix by assigning each equation to a corresponding row. Within that row, each variable and the constant gets its own column, and to separate the variables on the left hand side of the equation from the constants on the right hand side, we use a vertical bar, |. Note that in $E2$, since $y$ is not present, we record its coefficient as 0. The matrix associated with this system is

$$\begin{array}{c} \\ (E1) \rightarrow \\ (E2) \rightarrow \\ (E3) \rightarrow \end{array} \begin{array}{cccc} x & y & z & c \\ \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right] \end{array}$$

This matrix is called an **augmented matrix** because the column containing the constants is appended to the matrix containing the coefficients.[1] To solve this system, we can use the same kind operations on the *rows* of the matrix that we performed on the *equations* of the system. More specifically, we have the following analog of Theorem 8.1 below.

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**Theorem 8.2. Row Operations:** Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix which corresponds to an equivalent system of linear equations.

- Interchange any two rows.

- Replace a row with a nonzero multiple of itself.[a]

- Replace a row with itself plus a nonzero multiple of another row.[b]

---

[a]That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[b]Where we add entries in corresponding columns.

---

As a demonstration of the moves in Theorem 8.2, we revisit some of the steps that were used in solving the systems of linear equations in Example 8.1.2 of Section 8.1. The reader is encouraged to perform the indicated operations on the rows of the augmented matrix to see that the machinations are identical to what is done to the coefficients of the variables in the equations. We first see a demonstration of switching two rows using the first step of part 1 in Example 8.1.2.

$$\left\{ \begin{array}{rrcr} (E1) & 3x - y + z & = & 3 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & x - y + z & = & 5 \end{array} \right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{ \begin{array}{rrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array} \right.$$

$$\left[ \begin{array}{rrr|r} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array} \right] \xrightarrow{\text{Switch } R1 \text{ and } R3} \left[ \begin{array}{rrr|r} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array} \right]$$

Next, we have a demonstration of replacing a row with a nonzero multiple of itself using the first step of part 3 in Example 8.1.2.

$$\left\{ \begin{array}{rrcr} (E1) & 3x_1 + x_2 + x_4 & = & 6 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array} \right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{ \begin{array}{rrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array} \right.$$

$$\left[ \begin{array}{rrrr|r} 3 & 1 & 0 & 1 & 6 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array} \right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R1} \left[ \begin{array}{rrrr|r} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 2 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array} \right]$$

Finally, we have an example of replacing a row with itself plus a multiple of another row using the second step from part 2 in Example 8.1.2.

---

[1]We shall study the coefficient and constant matrices separately in Section 8.3.

$$\begin{cases} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{cases} \xrightarrow[\text{Replace } E3 \text{ with } -4E1 + E3]{\text{Replace } E2 \text{ with } -10E1 + E2} \begin{cases} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -4R1 + R3]{\text{Replace } R2 \text{ with } -10R1 + R2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -15 & 4 & -3 \\ 0 & -15 & 4 & 3 \end{array}\right]$$

The matrix equivalent of 'triangular form' is **row echelon form**. The reader is encouraged to refer to Definition 8.3 for comparison. Note that the analog of 'leading variable' of an equation is 'leading entry' of a row. Specifically, the first nonzero entry (if it exists) in a row is called the **leading entry** of that row.

> **Definition 8.4.** A matrix is said to be in **row echelon form** provided all of the following conditions hold:
>
> 1. The first nonzero entry in each row is 1.
>
> 2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
>
> 3. Any row of all zeros cannot be placed above a row with nonzero entries.

To solve a system of a linear equations using an augmented matrix, we encode the system into an augmented matrix and apply Gaussian Elimination to the rows to get the matrix into row-echelon form. We then decode the matrix and back substitute. The next example illustrates this nicely.

**Example 8.2.1.** Use an augmented matrix to transform the following system of linear equations into triangular form. Solve the system.

$$\begin{cases} 3x - y + z & = & 8 \\ x + 2y - z & = & 4 \\ 2x + 3y - 4z & = & 10 \end{cases}$$

**Solution.** We first encode the system into an augmented matrix.

$$\begin{cases} 3x - y + z & = & 8 \\ x + 2y - z & = & 4 \\ 2x + 3y - 4z & = & 10 \end{cases} \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

Thinking back to Gaussian Elimination at an equations level, our first order of business is to get $x$ in $E1$ with a coefficient of 1. At the matrix level, this means getting a leading 1 in $R1$. This is in accordance with the first criteria in Definition 8.4. To that end, we interchange $R1$ and $R2$.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

Our next step is to eliminate the $x$'s from $E2$ and $E3$. From a matrix standpoint, this means we need 0's below the leading 1 in $R1$. This guarantees the leading 1 in $R2$ will be to the right of the leading 1 in $R1$ in accordance with the second requirement of Definition 8.4.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -2R1 + R3]{\text{Replace } R2 \text{ with } -3R1 + R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right]$$

Now we repeat the above process for the variable $y$ which means we need to get the leading entry in $R2$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -\frac{1}{7}R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right]$$

To guarantee the leading 1 in $R3$ is to the right of the leading 1 in $R2$, we get a 0 in the second column of $R3$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } R2 + R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right]$$

Finally, we get the leading entry in $R3$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{7}{18}R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Decoding from the matrix gives a system in triangular form

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x + 2y - z & = & 4 \\ y - \frac{4}{7}z & = & \frac{4}{7} \\ z & = & -1 \end{array}\right.$$

We get $z = -1$, $y = \frac{4}{7}z + \frac{4}{7} = \frac{4}{7}(-1) + \frac{4}{7} = 0$ and $x = -2y + z + 4 = -2(0) + (-1) + 4 = 3$ for a final answer of $(3, 0, -1)$. We leave it to the reader to check.                                    □

As part of Gaussian Elimination, we used row operations to obtain 0's beneath each leading 1 to put the matrix into row echelon form. If we also require that 0's are the only numbers above a leading 1, we have what is known as the **reduced row echelon form** of the matrix.

---

**Definition 8.5.** A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.

2. The leading 1s are the only nonzero entry in their respective columns.

Of what significance is the reduced row echelon form of a matrix? To illustrate, let's take the row echelon form from Example 8.2.1 and perform the necessary steps to put into reduced row echelon form. We start by using the leading 1 in $R3$ to zero out the numbers in the rows above it.

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\text{Replace } R2 \text{ with } \frac{4}{7}R3 + R2]{\text{Replace } R1 \text{ with } R3 + R1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Finally, we take care of the 2 in $R1$ above the leading 1 in $R2$.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{Replace } R1 \text{ with } -2R2 + R1} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

To our surprise and delight, when we decode this matrix, we obtain the solution instantly without having to deal with any back-substitution at all.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{Decode from the matrix}} \begin{cases} x & = & 3 \\ y & = & 0 \\ z & = & -1 \end{cases}$$

Note that in the previous discussion, we could have started with $R2$ and used it to get a zero above its leading 1 and then done the same for the leading 1 in $R3$. By starting with $R3$, however, we get more zeros first, and the more zeros there are, the faster the remaining calculations will be.[2] It is also worth noting that while a matrix has several[3] row echelon forms, it has only one reduced row echelon form. The process by which we have put a matrix into reduced row echelon form is called **Gauss-Jordan Elimination**.

**Example 8.2.2.** Solve the following system using an augmented matrix. Use Gauss-Jordan Elimination to put the augmented matrix into reduced row echelon form.

$$\begin{cases} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{cases}$$

**Solution.** We first encode the system into a matrix. (Pay attention to the subscripts!)

$$\begin{cases} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{cases} \xrightarrow{\text{Encode into the matrix}} \begin{bmatrix} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{bmatrix}$$

Next, we get a leading 1 in the first column of $R1$.

$$\begin{bmatrix} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{bmatrix} \xrightarrow{\text{Replace } R1 \text{ with } -\frac{1}{3}R1} \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{bmatrix}$$

---

[2] Carl also finds starting with $R3$ to be more symmetric, in a purely poetic way.

[3] infinite, in fact

Now we eliminate the nonzero entry below our leading 1.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & \Big| & 5 \\ 0 & 4 & 0 & -1 & \Big| & 3 \end{bmatrix} \xrightarrow{\text{Replace } R2 \text{ with } -2R1 + R2} \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \Big| & \frac{19}{3} \\ 0 & 4 & 0 & -1 & \Big| & 3 \end{bmatrix}$$

We proceed to get a leading 1 in $R2$.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \Big| & \frac{19}{3} \\ 0 & 4 & 0 & -1 & \Big| & 3 \end{bmatrix} \xrightarrow{\text{Replace } R2 \text{ with } \frac{3}{2}R2} \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \Big| & \frac{19}{2} \\ 0 & 4 & 0 & -1 & \Big| & 3 \end{bmatrix}$$

We now zero out the entry below the leading 1 in $R2$.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \Big| & \frac{19}{2} \\ 0 & 4 & 0 & -1 & \Big| & 3 \end{bmatrix} \xrightarrow{\text{Replace } R3 \text{ with } -4R2 + R3} \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \Big| & \frac{19}{2} \\ 0 & 0 & -24 & -5 & \Big| & -35 \end{bmatrix}$$

Next, it's time for a leading 1 in $R3$.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \Big| & \frac{19}{2} \\ 0 & 0 & -24 & -5 & \Big| & -35 \end{bmatrix} \xrightarrow{\text{Replace } R3 \text{ with } -\frac{1}{24}R3} \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \Big| & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \Big| & \frac{35}{24} \end{bmatrix}$$

The matrix is now in row echelon form. To get the reduced row echelon form, we start with the last leading 1 we produced and work to get 0's above it.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \Big| & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \Big| & \frac{35}{24} \end{bmatrix} \xrightarrow{\text{Replace } R2 \text{ with } -6R3 + R2} \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \Big| & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \Big| & \frac{35}{24} \end{bmatrix}$$

Lastly, we get a 0 above the leading 1 of $R2$.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & \Big| & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \Big| & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \Big| & \frac{35}{24} \end{bmatrix} \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R2 + R1} \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{12} & \Big| & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \Big| & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \Big| & \frac{35}{24} \end{bmatrix}$$

At last, we decode to get

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{12} & \Big| & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \Big| & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \Big| & \frac{35}{24} \end{bmatrix} \xrightarrow{\text{Decode from the matrix}} \begin{cases} x_1 - \frac{5}{12}x_4 & = & -\frac{5}{12} \\ x_2 - \frac{1}{4}x_4 & = & \frac{3}{4} \\ x_3 + \frac{5}{24}x_4 & = & \frac{35}{24} \end{cases}$$

We have that $x_4$ is free and we assign it the parameter $t$. We obtain $x_3 = -\frac{5}{24}t + \frac{35}{24}$, $x_2 = \frac{1}{4}t + \frac{3}{4}$, and $x_1 = \frac{5}{12}t - \frac{5}{12}$. Our solution is $\left\{ \left( \frac{5}{12}t - \frac{5}{12}, \frac{1}{4}t + \frac{3}{4}, -\frac{5}{24}t + \frac{35}{24}, t \right) : -\infty < t < \infty \right\}$ and leave it to the reader to check. $\qquad\square$
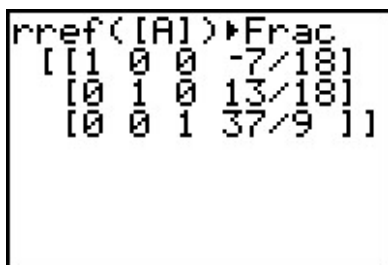
Like all good algorithms, putting a matrix in row echelon or reduced row echelon form can easily be programmed into a calculator, and, doubtless, your graphing calculator has such a feature. We use this in our next example.

**Example 8.2.3.** Find the quadratic function passing through the points $(-1, 3)$, $(2, 4)$, $(5, -2)$.

**Solution.** According to Definition 2.5, a quadratic function has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. Our goal is to find $a$, $b$ and $c$ so that the three given points are on the graph of $f$. If $(-1, 3)$ is on the graph of $f$, then $f(-1) = 3$, or $a(-1)^2 + b(-1) + c = 3$ which reduces to $a - b + c = 3$, an honest-to-goodness linear equation with the variables $a$, $b$ and $c$. Since the point $(2, 4)$ is also on the graph of $f$, then $f(2) = 4$ which gives us the equation $4a + 2b + c = 4$. Lastly, the point $(5, -2)$ is on the graph of $f$ gives us $25a + 5b + c = -2$. Putting these together, we obtain a system of three linear equations. Encoding this into an augmented matrix produces

$$\left\{ \begin{array}{rcr} a - b + c & = & 3 \\ 4a + 2b + c & = & 4 \\ 25a + 5b + c & = & -2 \end{array} \right. \quad \xrightarrow{\text{Encode into the matrix}} \quad \left[ \begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 4 & 2 & 1 & 4 \\ 25 & 5 & 1 & -2 \end{array} \right]$$

Using a calculator,[4] we find $a = -\frac{7}{18}$, $b = \frac{13}{18}$ and $c = \frac{37}{9}$. Hence, the one and only quadratic which fits the bill is $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$. To verify this analytically, we see that $f(-1) = 3$, $f(2) = 4$, and $f(5) = -2$. We can use the calculator to check our solution as well by plotting the three data points and the function $f$.
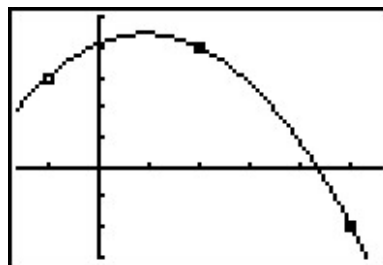




The graph of $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$
with the points $(-1, 3)$, $(2, 4)$ and $(5, -2)$

$\square$

---

[4] We've tortured you enough already with fractions in this exposition!

## 8.2.1  Exercises

In Exercises 1 - 6, state whether the given matrix is in reduced row echelon form, row echelon form only or in neither of those forms.

1. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

5. $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right]$

In Exercises 7 - 12, the following matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system is inconsistent.

7. $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

9. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$

10. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

11. $\left[\begin{array}{cccc|c} 1 & 0 & -8 & 1 & 7 \\ 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

12. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 13 - 26, solve the following systems of linear equations using the techniques discussed in this section. Compare and contrast these techniques with those you used to solve the systems in the Exercises in Section 8.1.

13. $\begin{cases} -5x + y &= 17 \\ x + y &= 5 \end{cases}$

14. $\begin{cases} x + y + z &= 3 \\ 2x - y + z &= 0 \\ -3x + 5y + 7z &= 7 \end{cases}$

15. $\begin{cases} 4x - y + z &= 5 \\ 2y + 6z &= 30 \\ x + z &= 5 \end{cases}$

16. $\begin{cases} x - 2y + 3z &= 7 \\ -3x + y + 2z &= -5 \\ 2x + 2y + z &= 3 \end{cases}$

17. $\begin{cases} 3x - 2y + z &= -5 \\ x + 3y - z &= 12 \\ x + y + 2z &= 0 \end{cases}$

18. $\begin{cases} 2x - y + z &= -1 \\ 4x + 3y + 5z &= 1 \\ 5y + 3z &= 4 \end{cases}$

19. $\begin{cases} x - y + z &= -4 \\ -3x + 2y + 4z &= -5 \\ x - 5y + 2z &= -18 \end{cases}$

20. $\begin{cases} 2x - 4y + z &= -7 \\ x - 2y + 2z &= -2 \\ -x + 4y - 2z &= 3 \end{cases}$

21. $\begin{cases} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{cases}$

22. $\begin{cases} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{cases}$

23. $\begin{cases} x + y + z & = & 4 \\ 2x - 4y - z & = & -1 \\ x - y & = & 2 \end{cases}$

24. $\begin{cases} x - y + z & = & 8 \\ 3x + 3y - 9z & = & -6 \\ 7x - 2y + 5z & = & 39 \end{cases}$

25. $\begin{cases} 2x - 3y + z & = & -1 \\ 4x - 4y + 4z & = & -13 \\ 6x - 5y + 7z & = & -25 \end{cases}$

26. $\begin{cases} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{cases}$

27. It's time for another meal at our local buffet. This time, 22 diners (5 of whom were children) feasted for $162.25, before taxes. If the kids buffet is $4.50, the basic buffet is $7.50, and the deluxe buffet (with crab legs) is $9.25, find out how many diners chose the deluxe buffet.

28. Carl wants to make a party mix consisting of almonds (which cost $7 per pound), cashews (which cost $5 per pound), and peanuts (which cost $2 per pound.) If he wants to make a 10 pound mix with a budget of $35, what are the possible combinations almonds, cashews, and peanuts? (You may find it helpful to review Example 8.1.3 in Section 8.1.)

29. Find the quadratic function passing through the points $(-2, 1)$, $(1, 4)$, $(3, -2)$

30. At 9 PM, the temperature was 60°F; at midnight, the temperature was 50°F; and at 6 AM, the temperature was 70°F . Use the technique in Example 8.2.3 to fit a quadratic function to these data with the temperature, $T$, measured in degrees Fahrenheit, as the dependent variable, and the number of hours after 9 PM, $t$, measured in hours, as the independent variable. What was the coldest temperature of the night? When did it occur?

31. The price for admission into the Stitz-Zeager Sasquatch Museum and Research Station is $15 for adults and $8 for kids 13 years old and younger. When the Zahlenreich family visits the museum their bill is $38 and when the Nullsatz family visits their bill is $39. One day both families went together and took an adult babysitter along to watch the kids and the total admission charge was $92. Later that summer, the adults from both families went without the kids and the bill was $45. Is that enough information to determine how many adults and children are in each family? If not, state whether the resulting system is inconsistent or consistent dependent. In the latter case, give at least two plausible solutions.

32. Use the technique in Example 8.2.3 to find the line between the points $(-3, 4)$ and $(6, 1)$. How does your answer compare to the slope-intercept form of the line in Equation 2.3?

33. With the help of your classmates, find at least two different row echelon forms for the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 12 & 8 \end{bmatrix}$$

## 8.2.2   Answers

1. Reduced row echelon form

2. Neither

3. Row echelon form only

4. Reduced row echelon form

5. Reduced row echelon form

6. Row echelon form only

7. $(-2, 7)$

8. $(-3, 20, 19)$

9. $(-3t + 4, -6t - 6, 2, t)$
   for all real numbers $t$

10. Inconsistent

11. $(8s - t + 7, -4s + 3t + 2, s, t)$
    for all real numbers $s$ and $t$

12. $(-9t - 3, 4t + 20, t)$
    for all real numbers $t$

13. $(-2, 7)$

14. $(1, 2, 0)$

15. $(-t + 5, -3t + 15, t)$
    for all real numbers $t$

16. $(2, -1, 1)$

17. $(1, 3, -2)$

18. Inconsistent

19. $(1, 3, -2)$

20. $\left(-3, \frac{1}{2}, 1\right)$

21. $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$

22. $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$
    for all real numbers $t$

23. Inconsistent

24. $(4, -3, 1)$

25. $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$
    for all real numbers $t$

26. $(1, 2, 3, 4)$

27. This time, 7 diners chose the deluxe buffet.

28. If $t$ represents the amount (in pounds) of peanuts, then we need $1.5t - 7.5$ pounds of almonds and $17.5 - 2.5t$ pounds of cashews. Since we can't have a negative amount of nuts, $5 \leq t \leq 7$.

29. $f(x) = -\frac{4}{5}x^2 + \frac{1}{5}x + \frac{23}{5}$

30. $T(t) = \frac{20}{27}t^2 - \frac{50}{9}t + 60$. Lowest temperature of the evening $\frac{595}{12} \approx 49.58°$F at 12:45 AM.

31. Let $x_1$ and $x_2$ be the numbers of adults and children, respectively, in the Zahlenreich family and let $x_3$ and $x_4$ be the numbers of adults and children, respectively, in the Nullsatz family. The system of equations determined by the given information is

$$\begin{cases} 15x_1 + 8x_2 & = & 38 \\ 15x_3 + 8x_4 & = & 39 \\ 15x_1 + 8x_2 + 15x_3 + 8x_4 & = & 77 \\ 15x_1 + 15x_3 & = & 45 \end{cases}$$

We subtracted the cost of the babysitter in E3 so the constant is 77, not 92. This system is consistent dependent and its solution is $\left(\frac{8}{15}t + \frac{2}{5}, -t + 4, -\frac{8}{15}t + \frac{13}{5}, t\right)$. Our variables represent numbers of adults and children so they must be whole numbers. Running through the values $t = 0, 1, 2, 3, 4$ yields only one solution where all four variables are whole numbers; $t = 3$ gives us $(2, 1, 1, 3)$. Thus there are 2 adults and 1 child in the Zahlenreichs and 1 adult and 3 kids in the Nullsatzs.

## 8.3    MATRIX ARITHMETIC

In Section 8.2, we used a special class of matrices, the augmented matrices, to assist us in solving systems of linear equations. In this section, we study matrices as mathematical objects of their own accord, temporarily divorced from systems of linear equations. To do so conveniently requires some more notation. When we write $A = [a_{ij}]_{m \times n}$, we mean $A$ is an $m$ by $n$ matrix[1] and $a_{ij}$ is the entry found in the $i$th row and $j$th column. Schematically, we have

$$
A \;=\; \overset{\substack{j \text{ counts columns} \\ \text{from left to right} \\ \longrightarrow}}{\left[\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{array}\right]} \Big\downarrow \quad \substack{i \text{ counts rows} \\ \text{from top to bottom}}
$$

With this new notation we can define what it means for two matrices to be equal.

**Definition 8.6. Matrix Equality:** Two matrices are said to be **equal** if they are the same size and their corresponding entries are equal. More specifically, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times r}$, we write $A = B$ provided

1. $m = p$ and $n = r$

2. $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and all $1 \leq j \leq n$.

Essentially, two matrices are equal if they are the same size and they have the same numbers in the same spots.[2] For example, the two $2 \times 3$ matrices below are, despite appearances, equal.

$$
\left[\begin{array}{ccc}
0 & -2 & 9 \\
25 & 117 & -3
\end{array}\right] = \left[\begin{array}{ccc}
\ln(1) & \sqrt[3]{-8} & e^{2\ln(3)} \\
125^{2/3} & 3^2 \cdot 13 & \log(0.001)
\end{array}\right]
$$

Now that we have an agreed upon understanding of what it means for two matrices to equal each other, we may begin defining arithmetic operations on matrices. Our first operation is addition.

**Definition 8.7. Matrix Addition:** Given two matrices of the same size, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices. More specifically, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, we define

$$
A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}
$$

As an example, consider the sum below.

---

[1]Recall that means $A$ has $m$ rows and $n$ columns.

[2]Critics may well ask: Why not leave it at that? Why the need for all the notation in Definition 8.6? It is the authors' attempt to expose you to the wonderful world of mathematical precision.

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ -5 & -3 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 2+(-1) & 3+4 \\ 4+(-5) & (-1)+(-3) \\ 0+8 & (-7)+1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -1 & -4 \\ 8 & -6 \end{bmatrix}$$

It is worth the reader's time to think what would have happened had we reversed the order of the summands above. As we would expect, we arrive at the same answer. In general, $A + B = B + A$ for matrices $A$ and $B$, provided they are the same size so that the sum is defined in the first place. This is the **commutative property** of matrix addition. To see why this is true in general, we appeal to the definition of matrix addition. Given $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$,

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n} = [b_{ij} + a_{ij}]_{m \times n} = [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} = B + A$$

where the second equality is the definition of $A + B$, the third equality holds by the commutative law of real number addition, and the fourth equality is the definition of $B + A$. In other words, matrix addition is commutative because real number addition is. A similar argument shows the **associative property** of matrix addition also holds, inherited in turn from the associative law of real number addition. Specifically, for matrices $A$, $B$, and $C$ of the same size, $(A + B) + C = A + (B + C)$. In other words, when adding more than two matrices, it doesn't matter how they are grouped. This means that we can write $A + B + C$ without parentheses and there is no ambiguity as to what this means.[3] These properties and more are summarized in the following theorem.

---

**Theorem 8.3. Properties of Matrix Addition**

- **Commutative Property:** For all $m \times n$ matrices, $A + B = B + A$

- **Associative Property:** For all $m \times n$ matrices, $(A + B) + C = A + (B + C)$

- **Identity Property:** If $0_{m \times n}$ is the $m \times n$ matrix whose entries are all 0, then $0_{m \times n}$ is called the **$m \times n$ additive identity** and for all $m \times n$ matrices $A$

$$A + 0_{m \times n} = 0_{m \times n} + A = A$$

- **Inverse Property:** For every given $m \times n$ matrix $A$, there is a unique matrix denoted $-A$ called the **additive inverse of $A$** such that

$$A + (-A) = (-A) + A = 0_{m \times n}$$

---

The identity property is easily verified by resorting to the definition of matrix addition; just as the number 0 is the additive identity for real numbers, the matrix comprised of all 0's does the same job for matrices. To establish the inverse property, given a matrix $A = [a_{ij}]_{m \times n}$, we are looking for a matrix $B = [b_{ij}]_{m \times n}$ so that $A + B = 0_{m \times n}$. By the definition of matrix addition, we must

---

[3]A technical detail which is sadly lost on most readers.

have that $a_{ij} + b_{ij} = 0$ for all $i$ and $j$. Solving, we get $b_{ij} = -a_{ij}$. Hence, given a matrix $A$, its additive inverse, which we call $-A$, does exist and is unique and, moreover, is given by the formula: $-A = [-a_{ij}]_{m \times n}$. The long and short of this is: to get the additive inverse of a matrix, take additive inverses of each of its entries. With the concept of additive inverse well in hand, we may now discuss what is meant by subtracting matrices. You may remember from arithmetic that $a - b = a + (-b)$; that is, subtraction is defined as 'adding the opposite (inverse).' We extend this concept to matrices. For two matrices $A$ and $B$ of the same size, we define $A - B = A + (-B)$. At the level of entries, this amounts to

$$A - B = A + (-B) = [a_{ij}]_{m \times n} + [-b_{ij}]_{m \times n} = [a_{ij} + (-b_{ij})]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$$

Thus to subtract two matrices of equal size, we subtract their corresponding entries. Surprised?

Our next task is to define what it means to multiply a matrix by a real number. Thinking back to arithmetic, you may recall that multiplication, at least by a natural number, can be thought of as 'rapid addition.' For example, $2 + 2 + 2 = 3 \cdot 2$. We know from algebra[4] that $3x = x + x + x$, so it seems natural that given a matrix $A$, we define $3A = A + A + A$. If $A = [a_{ij}]_{m \times n}$, we have

$$3A = A + A + A = [a_{ij}]_{m \times n} + [a_{ij}]_{m \times n} + [a_{ij}]_{m \times n} = [a_{ij} + a_{ij} + a_{ij}]_{m \times n} = [3a_{ij}]_{m \times n}$$

In other words, multiplying the *matrix* in this fashion by 3 is the same as multiplying *each entry* by 3. This leads us to the following definition.

---

**Definition 8.8. Scalar[a] Multiplication:** We define the product of a real number and a matrix to be the matrix obtained by multiplying each of its entries by said real number. More specifically, if $k$ is a real number and $A = [a_{ij}]_{m \times n}$, we define

$$kA = k[a_{ij}]_{m \times n} = [ka_{ij}]_{m \times n}$$

---

[a]The word 'scalar' here refers to real numbers. 'Scalar multiplication' in this context means we are multiplying a matrix by a real number (a scalar).

---

One may well wonder why the word 'scalar' is used for 'real number.' It has everything to do with 'scaling' factors.[5] A point $P(x, y)$ in the plane can be represented by its position matrix, $P$:

$$(x, y) \leftrightarrow P = \left[ \begin{array}{c} x \\ y \end{array} \right]$$

Suppose we take the point $(-2, 1)$ and multiply its position matrix by 3. We have

$$3P = 3 \left[ \begin{array}{c} -2 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 3(-2) \\ 3(1) \end{array} \right] = \left[ \begin{array}{c} -6 \\ 3 \end{array} \right]$$

which corresponds to the point $(-6, 3)$. We can imagine taking $(-2, 1)$ to $(-6, 3)$ in this fashion as a dilation by a factor of 3 in both the horizontal and vertical directions. Doing this to all points $(x, y)$ in the plane, therefore, has the effect of magnifying (scaling) the plane by a factor of 3.

---

[4]The Distributive Property, in particular.

[5]See Section 1.7.

As did matrix addition, scalar multiplication inherits many properties from real number arithmetic. Below we summarize these properties.

---

**Theorem 8.4. Properties of Scalar Multiplication**

- **Associative Property:** For every $m \times n$ matrix $A$ and scalars $k$ and $r$, $(kr)A = k(rA)$.

- **Identity Property:** For all $m \times n$ matrices $A$, $1A = A$.

- **Additive Inverse Property:** For all $m \times n$ matrices $A$, $-A = (-1)A$.

- **Distributive Property of Scalar Multiplication over Scalar Addition:** For every $m \times n$ matrix $A$ and scalars $k$ and $r$,

$$(k + r)A = kA + rA$$

- **Distributive Property of Scalar Multiplication over Matrix Addition:** For all $m \times n$ matrices $A$ and $B$ scalars $k$,

$$k(A + B) = kA + kB$$

- **Zero Product Property:** If $A$ is an $m \times n$ matrix and $k$ is a scalar, then

$$kA = 0_{m \times n} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad A = 0_{m \times n}$$

---

As with the other results in this section, Theorem 8.4 can be proved using the definitions of scalar multiplication and matrix addition. For example, to prove that $k(A + B) = kA + kB$ for a scalar $k$ and $m \times n$ matrices $A$ and $B$, we start by adding $A$ and $B$, then multiplying by $k$ and seeing how that compares with the sum of $kA$ and $kB$.

$$k(A + B) = k\left([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}\right) = k\left[a_{ij} + b_{ij}\right]_{m \times n} = \left[k\left(a_{ij} + b_{ij}\right)\right]_{m \times n} = \left[ka_{ij} + kb_{ij}\right]_{m \times n}$$

As for $kA + kB$, we have

$$kA + kB = k\left[a_{ij}\right]_{m \times n} + k\left[b_{ij}\right]_{m \times n} = \left[ka_{ij}\right]_{m \times n} + \left[kb_{ij}\right]_{m \times n} = \left[ka_{ij} + kb_{ij}\right]_{m \times n} \checkmark$$

which establishes the property. The remaining properties are left to the reader. The properties in Theorems 8.3 and 8.4 establish an algebraic system that lets us treat matrices and scalars more or less as we would real numbers and variables, as the next example illustrates.

**Example 8.3.1.** Solve for the matrix $A$: $3A - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A\right) = \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \dfrac{1}{3}\begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix}$ using the definitions and properties of matrix arithmetic.

**Solution.**

$$3A - \left( \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) = \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix}$$

$$3A + \left\{ -\left( \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) \right\} = \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{3}\right)(9) & \left(\frac{1}{3}\right)(12) \\ \left(\frac{1}{3}\right)(-3) & \left(\frac{1}{3}\right)(39) \end{bmatrix}$$

$$3A + (-1) \left( \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) = \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & 13 \end{bmatrix}$$

$$3A + \left\{ (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) \right\} = \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix}$$

$$3A + (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) = \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix}$$

$$3A + \begin{bmatrix} (-1)(2) & (-1)(-1) \\ (-1)(3) & (-1)(5) \end{bmatrix} + ((-1)(5))A = \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix}$$

$$3A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + (-5)A = \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix}$$

$$3A + (-5)A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix}$$

$$(3 + (-5))A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + \left( - \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right) = \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} + \left( - \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right)$$

$$(-2)A + 0_{2\times 2} = \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix}$$

$$(-2)A = \begin{bmatrix} -1 - (-2) & 6 - 1 \\ 5 - (-3) & 11 - (-5) \end{bmatrix}$$

$$(-2)A = \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix}$$

$$\left( -\frac{1}{2} \right) ((-2)A) = -\frac{1}{2} \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix}$$

$$\left( \left( -\frac{1}{2} \right) (-2) \right) A = \begin{bmatrix} \left(-\frac{1}{2}\right)(1) & \left(-\frac{1}{2}\right)(5) \\ \left(-\frac{1}{2}\right)(8) & \left(-\frac{1}{2}\right)(16) \end{bmatrix}$$

$$1A = \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -\frac{16}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -8 \end{bmatrix}$$

The reader is encouraged to check our answer in the original equation.                    □

While the solution to the previous example is written in excruciating detail, in practice many of the steps above are omitted. We have spelled out each step in this example to encourage the reader to justify each step using the definitions and properties we have established thus far for matrix arithmetic. The reader is encouraged to solve the equation in Example 8.3.1 as they would any other linear equation, for example: $3a - (2 + 5a) = -4 + \frac{1}{3}(9)$.

We now turn our attention to **matrix multiplication** - that is, multiplying a matrix by another matrix. Based on the 'no surprises' trend so far in the section, you may expect that in order to multiply two matrices, they must be of the same size and you find the product by multiplying the corresponding entries. While this kind of product is used in other areas of mathematics,[6] we define matrix multiplication to serve us in solving systems of linear equations. To that end, we begin by defining the product of a row and a column. We motivate the general definition with an example. Consider the two matrices $A$ and $B$ below.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

Let $R1$ denote the first row of $A$ and $C1$ denote the first column of $B$. To find the 'product' of $R1$ with $C1$, denoted $R1 \cdot C1$, we first find the product of the first entry in $R1$ and the first entry in $C1$. Next, we add to that the product of the second entry in $R1$ and the second entry in $C1$. Finally, we take that sum and we add to that the product of the last entry in $R1$ and the last entry in $C1$. Using entry notation, $R1 \cdot C1 = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (2)(3) + (0)(4) + (-1)(5) = 6 + 0 + (-5) = 1$. We can visualize this schematically as follows



To find $R2 \cdot C3$ where $R2$ denotes the second row of $A$ and $C3$ denotes the third column of $B$, we proceed similarly. We start with finding the product of the first entry of $R2$ with the first entry in $C3$ then add to it the product of the second entry in $R2$ with the second entry in $C3$, and so forth. Using entry notation, we have $R2 \cdot C3 = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = (-10)(2) + (3)(-5) + (5)(-2) = -45$. Schematically,



---

[6]See this article on the Hadamard Product.

$$\underbrace{\boxed{-10}\quad 3\quad 5 \quad\quad \begin{array}{c}\boxed{2}\\-5\\-2\end{array}}_{a_{21}b_{13}=(-10)(2)=-20} \quad+\quad \underbrace{-10\quad \boxed{3}\quad 5 \quad\quad \begin{array}{c}2\\\boxed{-5}\\-2\end{array}}_{a_{22}b_{23}=(3)(-5)=-15} \quad+\quad \underbrace{-10\quad 3\quad \boxed{5} \quad\quad \begin{array}{c}2\\-5\\\boxed{-2}\end{array}}_{a_{23}b_{33}=(5)(-2)=-10}$$

Generalizing this process, we have the following definition.

**Definition 8.9. Product of a Row and a Column:** Suppose $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times r}$. Let $Ri$ denote the $i$th row of $A$ and let $Cj$ denote the $j$th column of $B$. The **product of $R_i$ and $C_j$, denoted $R_i \cdot C_j$** is the real number defined by

$$Ri \cdot Cj = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots a_{in}b_{nj}$$

Note that in order to multiply a row by a column, the number of entries in the row must match the number of entries in the column. We are now in the position to define matrix multiplication.

**Definition 8.10. Matrix Multiplication:** Suppose $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times r}$. Let $Ri$ denote the $i$th row of $A$ and let $Cj$ denote the $j$th column of $B$. The **product of $A$ and $B$**, denoted $AB$, is the matrix defined by

$$AB = [Ri \cdot Cj]_{m\times r}$$

that is

$$AB = \begin{bmatrix} R1\cdot C1 & R1\cdot C2 & \ldots & R1\cdot Cr \\ R2\cdot C1 & R2\cdot C2 & \ldots & R2\cdot Cr \\ \vdots & \vdots & & \vdots \\ Rm\cdot C1 & Rm\cdot C2 & \ldots & Rm\cdot Cr \end{bmatrix}$$

There are a number of subtleties in Definition 8.10 which warrant closer inspection. First and foremost, Definition 8.10 tells us that the $ij$-entry of a matrix product $AB$ is the $i$th row of $A$ times the $j$th column of $B$. In order for this to be defined, the number of entries in the rows of $A$ must match the number of entries in the columns of $B$. This means that the number of columns of $A$ must match[7] the number of rows of $B$. In other words, to multiply $A$ times $B$, the second dimension of $A$ must match the first dimension of $B$, which is why in Definition 8.10, $A_{m\times \underline{n}}$ is being multiplied by a matrix $B_{\underline{n}\times r}$. Furthermore, the product matrix $AB$ has as many rows as $A$ and as many columns of $B$. As a result, when multiplying a matrix $A_{\underline{m}\times n}$ by a matrix $B_{n\times \underline{r}}$, the result is the matrix $AB_{\underline{m}\times\underline{r}}$. Returning to our example matrices below, we see that $A$ is a $2 \times \underline{3}$ matrix and $B$ is a $\underline{3} \times 4$ matrix. This means that the product matrix $AB$ is defined and will be a $2 \times 4$ matrix.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

---

[7]The reader is encouraged to think this through carefully.

Using $Ri$ to denote the $i$th row of $A$ and $Cj$ to denote the $j$th column of $B$, we form $AB$ according to Definition 8.10.

$$AB \;=\; \left[ \begin{array}{cccc} R1 \cdot C1 & R1 \cdot C2 & R1 \cdot C3 & R1 \cdot C4 \\ R2 \cdot C1 & R2 \cdot C2 & R2 \cdot C3 & R2 \cdot C4 \end{array} \right] \;=\; \left[ \begin{array}{cccc} 1 & 2 & 6 & -4 \\ 7 & 14 & -45 & 47 \end{array} \right]$$

Note that the product $BA$ is not defined, since $B$ is a $3 \times \underline{4}$ matrix while $A$ is a $\underline{2} \times 3$ matrix; $B$ has more columns than $A$ has rows, and so it is not possible to multiply a row of $B$ by a column of $A$. Even when the dimensions of $A$ and $B$ are compatible such that $AB$ and $BA$ are both defined, the product $AB$ and $BA$ aren't necessarily equal.[8] In other words, $AB$ may not equal $BA$. Although there is no commutative property of matrix multiplication in general, several other real number properties are inherited by matrix multiplication, as illustrated in our next theorem.

---

**Theorem 8.5. Properties of Matrix Multiplication** Let $A$, $B$ and $C$ be matrices such that all of the matrix products below are defined and let $k$ be a real number.

- **Associative Property of Matrix Multiplication:** $(AB)C = A(BC)$

- **Associative Property with Scalar Multiplication:** $k(AB) = (kA)B = A(kB)$

- **Identity Property:** For a natural number $k$, the $\boldsymbol{k \times k}$ **identity matrix**, denoted $I_k$, is defined by $I_k = [d_{ij}]_{k \times k}$ where

$$d_{ij} = \left\{ \begin{array}{ll} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{array} \right.$$

  For all $m \times n$ matrices, $I_m A = A I_n = A$.

- **Distributive Property of Matrix Multiplication over Matrix Addition:**

$$A(B \pm C) = AB \pm AC \text{ and } (A \pm B)C = AC \pm BC$$

---

The one property in Theorem 8.5 which begs further investigation is, without doubt, the multiplicative identity. The entries in a matrix where $i = j$ comprise what is called the **main diagonal** of the matrix. The identity matrix has 1's along its main diagonal and 0's everywhere else. A few examples of the matrix $I_k$ mentioned in Theorem 8.5 are given below. The reader is encouraged to see how they match the definition of the identity matrix presented there.

$$[1] \qquad \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \qquad \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \qquad \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$I_1 \qquad\qquad I_2 \qquad\qquad\qquad I_3 \qquad\qquad\qquad\qquad I_4$$

---

[8]And may not even have the same dimensions. For example, if $A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 2$ matrix, then $AB$ is defined and is a $2 \times 2$ matrix while $BA$ is also defined... but is a $3 \times 3$ matrix!

The identity matrix is an example of what is called a **square matrix** as it has the same number of rows as columns. Note that to in order to verify that the identity matrix acts as a multiplicative identity, some care must be taken depending on the order of the multiplication. For example, take the matrix $2 \times 3$ matrix $A$ from earlier

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

In order for the product $I_k A$ to be defined, $k = 2$; similarly, for $AI_k$ to be defined, $k = 3$. We leave it to the reader to show $I_2 A = A$ and $AI_3 = A$. In other words,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

While the proofs of the properties in Theorem 8.5 are computational in nature, the notation becomes quite involved very quickly, so they are left to a course in Linear Algebra. The following example provides some practice with matrix multiplication and its properties. As usual, some valuable lessons are to be learned.

**Example 8.3.2.**

1. Find $AB$ for $A = \begin{bmatrix} -23 & -1 & 17 \\ 46 & 2 & -34 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$

2. Find $C^2 - 5C + 10I_2$ for $C = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

3. Suppose $M$ is a $4 \times 4$ matrix. Use Theorem 8.5 to expand $(M - 2I_4)(M + 3I_4)$.

**Solution.**

1. We have $AB = \begin{bmatrix} -23 & -1 & 17 \\ 46 & 2 & -34 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. Just as $x^2$ means $x$ times itself, $C^2$ denotes the matrix $C$ times itself. We get

$$C^2 - 5C + 10I_2 = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}^2 - 5 \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -10 \\ 15 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 10 \\ -15 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. We expand $(M - 2I_4)(M + 3I_4)$ with the same pedantic zeal we showed in Example 8.3.1. The reader is encouraged to determine which property of matrix arithmetic is used as we proceed from one step to the next.

$$\begin{aligned} (M - 2I_4)(M + 3I_4) &= (M - 2I_4)M + (M - 2I_4)(3I_4) \\ &= MM - (2I_4)M + M(3I_4) - (2I_4)(3I_4) \\ &= M^2 - 2(I_4 M) + 3(M I_4) - 2(I_4(3I_4)) \\ &= M^2 - 2M + 3M - 2(3(I_4 I_4)) \\ &= M^2 + M - 6I_4 \end{aligned}$$

$\square$

Example 8.3.2 illustrates some interesting features of matrix multiplication. First note that in part 1, neither $A$ nor $B$ is the zero matrix, yet the product $AB$ is the zero matrix. Hence, the the zero product property enjoyed by real numbers and scalar multiplication does not hold for matrix multiplication. Parts 2 and 3 introduce us to polynomials involving matrices. The reader is encouraged to step back and compare our expansion of the matrix product $(M - 2I_4)(M + 3I_4)$ in part 3 with the product $(x - 2)(x + 3)$ from real number algebra. The exercises explore this kind of parallel further.

As we mentioned earlier, a point $P(x, y)$ in the $xy$-plane can be represented as a $2 \times 1$ position matrix. We now show that matrix multiplication can be used to rotate these points, and hence graphs of equations.

**Example 8.3.3.** Let $R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

1. Plot $P(2, -2)$, $Q(4, 0)$, $S(0, 3)$, and $T(-3, -3)$ in the plane as well as the points $RP$, $RQ$, $RS$, and $RT$. Plot the lines $y = x$ and $y = -x$ as guides. What does $R$ appear to be doing to these points?

2. If a point $P$ is on the hyperbola $x^2 - y^2 = 4$, show that the point $RP$ is on the curve $y = \frac{2}{x}$.

**Solution.** For $P(2, -2)$, the position matrix is $P = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, and

$$
\begin{aligned}
RP &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}
\end{aligned}
$$

We have that $R$ takes $(2, -2)$ to $(2\sqrt{2}, 0)$. Similarly, we find $(4, 0)$ is moved to $(2\sqrt{2}, 2\sqrt{2})$, $(0, 3)$ is moved to $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$, and $(-3, -3)$ is moved to $(0, -3\sqrt{2})$. Plotting these in the coordinate plane along with the lines $y = x$ and $y = -x$, we see that the matrix $R$ is rotating these points counterclockwise by $45°$.
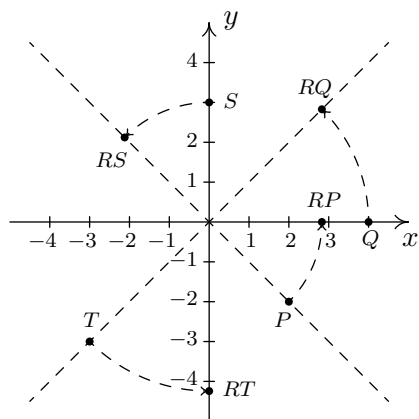


For a generic point $P(x, y)$ on the hyperbola $x^2 - y^2 = 4$, we have

$$
\begin{aligned}
RP &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{bmatrix}
\end{aligned}
$$

which means $R$ takes $(x, y)$ to $\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)$. To show that this point is on the curve $y = \frac{2}{x}$, we replace $x$ with $\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y$ with $\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ and simplify.

$$\begin{aligned}
y &= \frac{2}{x} \\
\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y &\stackrel{?}{=} \frac{2}{\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y} \\
\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right) &\stackrel{?}{=} \left(\frac{2}{\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y}\right)\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right) \\
\left(\frac{\sqrt{2}}{2}x\right)^2 - \left(\frac{\sqrt{2}}{2}y\right)^2 &\stackrel{?}{=} 2 \\
\frac{x^2}{2} - \frac{y^2}{2} &\stackrel{?}{=} 2 \\
x^2 - y^2 &\stackrel{\checkmark}{=} 4
\end{aligned}$$

Since $(x, y)$ is on the hyperbola $x^2 - y^2 = 4$, we know that this last equation is true. Since all of our steps are reversible, this last equation is equivalent to our original equation, which establishes the point is, indeed, on the graph of $y = \frac{2}{x}$. This means the graph of $y = \frac{2}{x}$ is a hyperbola, and it is none other than the hyperbola $x^2 - y^2 = 4$ rotated counterclockwise by $45°$.[9] Below we have the graph of $x^2 - y^2 = 4$ (solid line) and $y = \frac{2}{x}$ (dashed line) for comparison.
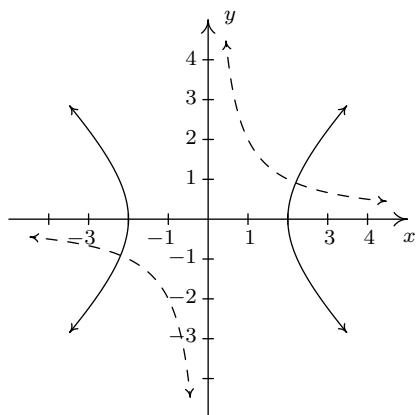


When we started this section, we mentioned that we would temporarily consider matrices as their own entities, but that the algebra developed here would ultimately allow us to solve systems of linear equations. To that end, consider the system

$$\begin{cases} 3x - y + z &= 8 \\ x + 2y - z &= 4 \\ 2x + 3y - 4z &= 10 \end{cases}$$

In Section 8.2, we encoded this system into the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

---

[9]See Section 7.5 for more details.

Recall that the entries to the left of the vertical line come from the coefficients of the variables in the system, while those on the right comprise the associated constants. For that reason, we may form the **coefficient matrix** $A$, the **unknowns matrix** $X$ and the **constant matrix** $B$ as below

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}$$

We now consider the matrix equation $AX = B$.

$$AX = B$$

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 3x - y + z \\ x + 2y - z \\ 2x + 3y - 4z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}$$

We see that finding a solution $(x, y, z)$ to the original system corresponds to finding a solution $X$ for the matrix equation $AX = B$. If we think about solving the real number equation $ax = b$, we would simply 'divide' both sides by $a$. Is it possible to 'divide' both sides of the matrix equation $AX = B$ by the matrix $A$? This is the central topic of Section 8.4.

### 8.3.1 Exercises

For each pair of matrices $A$ and $B$ in Exercises 1 - 7, find the following, if defined

- $3A$
- $-B$
- $A^2$

- $A - 2B$
- $AB$
- $BA$

1. $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -2 \\ 4 & 8 \end{bmatrix}$    2. $A = \begin{bmatrix} -1 & 5 \\ -3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 10 \\ -7 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 0 & 8 \\ -3 & 1 & 4 \end{bmatrix}$    4. $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & -5 \\ 7 & -9 & 11 \end{bmatrix}$

5. $A = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$    6. $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

In Exercises 8 - 21, use the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 10 & -\frac{11}{2} & 0 \\ \frac{3}{5} & 5 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & -13 \\ -\frac{4}{3} & 0 \\ 6 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

to compute the following or state that the indicated operation is undefined.

8. $7B - 4A$          9. $AB$          10. $BA$

11. $E + D$          12. $ED$          13. $CD + 2I_2 A$

14. $A - 4I_2$          15. $A^2 - B^2$          16. $(A + B)(A - B)$

17. $A^2 - 5A - 2I_2$          18. $E^2 + 5E - 36I_3$          19. $EDC$

20. $CDE$          21. $ABCEDI_2$

22. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Compute $E_1 A$, $E_2 A$ and $E_3 A$. What effect did each of the $E_i$ matrices have on the rows of $A$? Create $E_4$ so that its effect on $A$ is to multiply the bottom row by $-6$. How would you extend this idea to matrices with more than two rows?

In Exercises 23 - 29, consider the following scenario. In the small village of Pedimaxus in the country of Sasquatchia, all 150 residents get one of the two local newspapers. Market research has shown that in any given week, 90% of those who subscribe to the Pedimaxus Tribune want to keep getting it, but 10% want to switch to the Sasquatchia Picayune. Of those who receive the Picayune, 80% want to continue with it and 20% want switch to the Tribune. We can express this situation using matrices. Specifically, let $X$ be the 'state matrix' given by

$$X = \begin{bmatrix} T \\ P \end{bmatrix}$$

where $T$ is the number of people who get the Tribune and $P$ is the number of people who get the Picayune in a given week. Let $Q$ be the 'transition matrix' given by

$$Q = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix}$$

such that $QX$ will be the state matrix for the next week.

23. Let's assume that when Pedimaxus was founded, all 150 residents got the Tribune. (Let's call this Week 0.) This would mean

$$X = \begin{bmatrix} 150 \\ 0 \end{bmatrix}$$

Since 10% of that 150 want to switch to the Picayune, we should have that for Week 1, 135 people get the Tribune and 15 people get the Picayune. Show that $QX$ in this situation is indeed

$$QX = \begin{bmatrix} 135 \\ 15 \end{bmatrix}$$

24. Assuming that the percentages stay the same, we can get to the subscription numbers for Week 2 by computing $Q^2X$. How many people get each paper in Week 2?

25. Explain why the transition matrix does what we want it to do.

26. If the conditions do not change from week to week, then $Q$ remains the same and we have what's known as a **Stochastic Process**[10] because Week $n$'s numbers are found by computing $Q^nX$. Choose a few values of $n$ and, with the help of your classmates and calculator, find out how many people get each paper for that week. You should start to see a pattern as $n \to \infty$.

27. If you didn't see the pattern, we'll help you out. Let

$$X_s = \begin{bmatrix} 100 \\ 50 \end{bmatrix}.$$

Show that $QX_s = X_s$ This is called the **steady state** because the number of people who get each paper didn't change for the next week. Show that $Q^nX \to X_s$ as $n \to \infty$.

---

[10]More specifically, we have a Markov Chain, which is a special type of stochastic process.

28. Now let
$$S = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Show that $Q^n \to S$ as $n \to \infty$.

29. Show that $SY = X_s$ for any matrix $Y$ of the form
$$Y = \begin{bmatrix} y \\ 150 - y \end{bmatrix}$$

This means that no matter how the distribution starts in Pedimaxus, if $Q$ is applied often enough, we always end up with 100 people getting the Tribune and 50 people getting the Picayune.

30. Let $z = a + bi$ and $w = c + di$ be arbitrary complex numbers. Associate $z$ and $w$ with the matrices
$$Z = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ and } W = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

Show that complex number addition, subtraction and multiplication are mirrored by the associated *matrix* arithmetic. That is, show that $Z + W$, $Z - W$ and $ZW$ produce matrices which can be associated with the complex numbers $z + w$, $z - w$ and $zw$, respectively.

31. Let
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix}$$

Compare $(A + B)^2$ to $A^2 + 2AB + B^2$. Discuss with your classmates what constraints must be placed on two arbitrary matrices $A$ and $B$ so that both $(A+B)^2$ and $A^2 + 2AB + B^2$ exist. When will $(A + B)^2 = A^2 + 2AB + B^2$? In general, what is the correct formula for $(A + B)^2$?

In Exercises 32 - 36, consider the following definitions. A square matrix is said to be an **upper triangular matrix** if all of its entries below the main diagonal are zero and it is said to be a **lower triangular matrix** if all of its entries above the main diagonal are zero. For example,
$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

from Exercises 8 - 21 above is an upper triangular matrix whereas
$$F = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

is a lower triangular matrix. (Zeros are allowed on the main diagonal.) Discuss the following questions with your classmates.

32. Give an example of a matrix which is neither upper triangular nor lower triangular.

33. Is the product of two $n \times n$ upper triangular matrices always upper triangular?

34. Is the product of two $n \times n$ lower triangular matrices always lower triangular?

35. Given the matrix
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

    write $A$ as $LU$ where $L$ is a lower triangular matrix and $U$ is an upper triangular matrix?

36. Are there any matrices which are simultaneously upper and lower triangular?

### 8.3.2  Answers

1. For $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ 4 & 8 \end{bmatrix}$

- $3A = \begin{bmatrix} 6 & -9 \\ 3 & 12 \end{bmatrix}$
- $-B = \begin{bmatrix} -5 & 2 \\ -4 & -8 \end{bmatrix}$
- $A^2 = \begin{bmatrix} 1 & -18 \\ 6 & 13 \end{bmatrix}$

- $A - 2B = \begin{bmatrix} -8 & 1 \\ -7 & -12 \end{bmatrix}$
- $AB = \begin{bmatrix} -2 & -28 \\ 21 & 30 \end{bmatrix}$
- $BA = \begin{bmatrix} 8 & -23 \\ 16 & 20 \end{bmatrix}$

2. For $A = \begin{bmatrix} -1 & 5 \\ -3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 10 \\ -7 & 1 \end{bmatrix}$

- $3A = \begin{bmatrix} -3 & 15 \\ -9 & 18 \end{bmatrix}$
- $-B = \begin{bmatrix} -2 & -10 \\ 7 & -1 \end{bmatrix}$
- $A^2 = \begin{bmatrix} -14 & 25 \\ -15 & 21 \end{bmatrix}$

- $A - 2B = \begin{bmatrix} -5 & -15 \\ 11 & 4 \end{bmatrix}$
- $AB = \begin{bmatrix} -37 & -5 \\ -48 & -24 \end{bmatrix}$
- $BA = \begin{bmatrix} -32 & 70 \\ 4 & -29 \end{bmatrix}$

3. For $A = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 & 8 \\ -3 & 1 & 4 \end{bmatrix}$

- $3A = \begin{bmatrix} -3 & 9 \\ 15 & 6 \end{bmatrix}$
- $-B = \begin{bmatrix} -7 & 0 & -8 \\ 3 & -1 & -4 \end{bmatrix}$
- $A^2 = \begin{bmatrix} 16 & 3 \\ 5 & 19 \end{bmatrix}$

- $A - 2B$ is not defined
- $AB = \begin{bmatrix} -16 & 3 & 4 \\ 29 & 2 & 48 \end{bmatrix}$
- $BA$ is not defined

4. For $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & -5 \\ 7 & -9 & 11 \end{bmatrix}$

- $3A = \begin{bmatrix} 6 & 12 \\ 18 & 24 \end{bmatrix}$
- $-B = \begin{bmatrix} 1 & -3 & 5 \\ -7 & 9 & -11 \end{bmatrix}$
- $A^2 = \begin{bmatrix} 28 & 40 \\ 60 & 88 \end{bmatrix}$

- $A - 2B$ is not defined
- $AB = \begin{bmatrix} 26 & -30 & 34 \\ 50 & -54 & 58 \end{bmatrix}$
- $BA$ is not defined

5. For $A = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

- $3A = \begin{bmatrix} 21 \\ 24 \\ 27 \end{bmatrix}$

- $-B = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$

- $A^2$ is not defined

- $A - 2B$ is not defined

- $AB = \begin{bmatrix} 7 & 14 & 21 \\ 8 & 16 & 24 \\ 9 & 18 & 27 \end{bmatrix}$

- $BA = [50]$

6. For $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

- $3A = \begin{bmatrix} 3 & -6 \\ -9 & 12 \\ 15 & -18 \end{bmatrix}$

- $-B = \begin{bmatrix} 5 & -1 & -8 \end{bmatrix}$

- $A^2$ is not defined

- $A - 2B$ is not defined

- $AB$ is not defined

- $BA = \begin{bmatrix} 32 & -34 \end{bmatrix}$

7. For $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

- $3A = \begin{bmatrix} 6 & -9 & 15 \\ 9 & 3 & -6 \\ -21 & 3 & -3 \end{bmatrix}$

- $-B = \begin{bmatrix} -1 & -2 & -1 \\ -17 & -33 & -19 \\ -10 & -19 & -11 \end{bmatrix}$

- $A^2 = \begin{bmatrix} -40 & -4 & 11 \\ 23 & -10 & 15 \\ -4 & 21 & -36 \end{bmatrix}$

- $A - 2B = \begin{bmatrix} 0 & -7 & 3 \\ -31 & -65 & -40 \\ -27 & -37 & -23 \end{bmatrix}$

- $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $7B - 4A = \begin{bmatrix} -4 & -29 \\ -47 & -2 \end{bmatrix}$

9. $AB = \begin{bmatrix} -10 & 1 \\ -20 & -1 \end{bmatrix}$

10. $BA = \begin{bmatrix} -9 & -12 \\ 1 & -2 \end{bmatrix}$

11. $E + D$ is undefined

12. $ED = \begin{bmatrix} \frac{67}{3} & 11 \\ -\frac{178}{3} & -72 \\ -30 & -40 \end{bmatrix}$

13. $CD + 2I_2 A = \begin{bmatrix} \frac{238}{3} & -126 \\ \frac{863}{15} & \frac{361}{5} \end{bmatrix}$

14. $A - 4I_2 = \begin{bmatrix} -3 & 2 \\ 3 & 0 \end{bmatrix}$

15. $A^2 - B^2 = \begin{bmatrix} -8 & 16 \\ 25 & 3 \end{bmatrix}$

16. $(A + B)(A - B) = \begin{bmatrix} -7 & 3 \\ 46 & 2 \end{bmatrix}$

17. $A^2 - 5A - 2I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

18. $E^2 + 5E - 36I_3 = \begin{bmatrix} -30 & 20 & -15 \\ 0 & 0 & -36 \\ 0 & 0 & -36 \end{bmatrix}$

19. $EDC = \begin{bmatrix} \frac{3449}{15} & -\frac{407}{6} & 99 \\ -\frac{9548}{15} & -\frac{101}{3} & -648 \\ -324 & -35 & -360 \end{bmatrix}$

20. $CDE$ is undefined

21. $ABCEDI_2 = \begin{bmatrix} -\frac{90749}{15} & -\frac{28867}{5} \\ -\frac{156601}{15} & -\frac{47033}{5} \end{bmatrix}$

22. $E_1 A = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}$    $E_1$ interchanged $R1$ and $R2$ of $A$.

$E_2 A = \begin{bmatrix} 5a & 5b & 5c \\ d & e & f \end{bmatrix}$    $E_2$ multiplied $R1$ of $A$ by 5.

$E_3 A = \begin{bmatrix} a - 2d & b - 2e & c - 2f \\ d & e & f \end{bmatrix}$    $E_3$ replaced $R1$ in $A$ with $R1 - 2R2$.

$E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix}$

## 8.4    Systems of Linear Equations: Matrix Inverses

We concluded Section 8.3 by showing how we can rewrite a system of linear equations as the matrix equation $AX = B$ where $A$ and $B$ are known matrices and the solution matrix $X$ of the equation corresponds to the solution of the system. In this section, we develop the method for solving such an equation. To that end, consider the system

$$\begin{cases} 2x - 3y & = & 16 \\ 3x + 4y & = & 7 \end{cases}$$

To write this as a matrix equation, we follow the procedure outlined on page 590. We find the coefficient matrix $A$, the unknowns matrix $X$ and constant matrix $B$ to be

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 16 \\ 7 \end{bmatrix}$$

In order to motivate how we solve a matrix equation like $AX = B$, we revisit solving a similar equation involving real numbers. Consider the equation $3x = 5$. To solve, we simply divide both sides by 3 and obtain $x = \frac{5}{3}$. How can we go about defining an analogous process for matrices? To answer this question, we solve $3x = 5$ again, but this time, we pay attention to the properties of real numbers being used at each step. Recall that dividing by 3 is the same as multiplying by $\frac{1}{3} = 3^{-1}$, the so-called *multiplicative inverse*[1] of 3.

$$\begin{array}{rcll} 3x & = & 5 & \\ 3^{-1}(3x) & = & 3^{-1}(5) & \text{Multiply by the (multiplicative) inverse of 3} \\ \left(3^{-1} \cdot 3\right)x & = & 3^{-1}(5) & \text{Associative property of multiplication} \\ 1 \cdot x & = & 3^{-1}(5) & \text{Inverse property} \\ x & = & 3^{-1}(5) & \text{Multiplicative Identity} \end{array}$$

If we wish to check our answer, we substitute $x = 3^{-1}(5)$ into the original equation

$$\begin{array}{rcll} 3x & \overset{?}{=} & 5 & \\ 3\left(3^{-1}(5)\right) & \overset{?}{=} & 5 & \\ \left(3 \cdot 3^{-1}\right)(5) & \overset{?}{=} & 5 & \text{Associative property of multiplication} \\ 1 \cdot 5 & \overset{?}{=} & 5 & \text{Inverse property} \\ 5 & \overset{\checkmark}{=} & 5 & \text{Multiplicative Identity} \end{array}$$

Thinking back to Theorem 8.5, we know that matrix multiplication enjoys both an associative property and a multiplicative identity. What's missing from the mix is a multiplicative inverse for the coefficient matrix $A$. Assuming we can find such a beast, we can mimic our solution (and check) to $3x = 5$ as follows

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[1]Every nonzero real number $a$ has a multiplicative inverse, denoted $a^{-1}$, such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$.

$$\text{Solving } AX = B \qquad \text{Checking our answer}$$

$$
\begin{array}{rcl}
AX & = & B \\
A^{-1}(AX) & = & A^{-1}B \\
\left(A^{-1}A\right)X & = & A^{-1}B \\
I_2 X & = & A^{-1}B \\
X & = & A^{-1}B
\end{array}
\qquad
\begin{array}{rcl}
AX & \stackrel{?}{=} & B \\
A\left(A^{-1}B\right) & \stackrel{?}{=} & B \\
\left(AA^{-1}\right)B & \stackrel{?}{=} & B \\
I_2 B & \stackrel{?}{=} & B \\
B & \stackrel{\checkmark}{=} & B
\end{array}
$$

The matrix $A^{-1}$ is read '$A$-inverse' and we will define it formally later in the section. At this stage, we have no idea if such a matrix $A^{-1}$ exists, but that won't deter us from trying to find it.[2] We want $A^{-1}$ to satisfy two equations, $A^{-1}A = I_2$ and $AA^{-1} = I_2$, making $A^{-1}$ necessarily a $2 \times 2$ matrix.[3] Hence, we assume $A^{-1}$ has the form

$$A^{-1} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

for real numbers $x_1$, $x_2$, $x_3$ and $x_4$. For reasons which will become clear later, we focus our attention on the equation $AA^{-1} = I_2$. We have

$$AA^{-1} = I_2$$

$$\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - 3x_3 & 2x_2 - 3x_4 \\ 3x_1 + 4x_3 & 3x_2 + 4x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives rise to *two* more systems of equations

$$\left\{ \begin{array}{rcl} 2x_1 - 3x_3 & = & 1 \\ 3x_1 + 4x_3 & = & 0 \end{array} \right. \qquad \left\{ \begin{array}{rcl} 2x_2 - 3x_4 & = & 0 \\ 3x_2 + 4x_4 & = & 1 \end{array} \right.$$

At this point, it may seem absurd to continue with this venture. After all, the intent was to solve *one* system of equations, and in doing so, we have produced *two* more to solve. Remember, the objective of this discussion is to develop a general *method* which, when used in the correct scenarios, allows us to do far more than just solve a system of equations. If we set about to solve these systems using augmented matrices using the techniques in Section 8.2, we see that not only do both systems have the same coefficient matrix, this coefficient matrix is none other than the matrix $A$ itself. (We will come back to this observation in a moment.)

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[2] Much like Carl's quest to find Sasquatch.

[3] Since matrix multiplication isn't necessarily commutative, at this stage, these are two different equations.

$$\begin{cases} 2x_1 - 3x_3 &= 1 \\ 3x_1 + 4x_3 &= 0 \end{cases} \quad \xrightarrow{\text{Encode into a matrix}} \quad \left[ \begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array} \right]$$

$$\begin{cases} 2x_2 - 3x_4 &= 0 \\ 3x_2 + 4x_4 &= 1 \end{cases} \quad \xrightarrow{\text{Encode into a matrix}} \quad \left[ \begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array} \right]$$

To solve these two systems, we use Gauss-Jordan Elimination to put the augmented matrices into reduced row echelon form. (We leave the details to the reader.) For the first system, we get

$$\left[ \begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array} \right] \quad \xrightarrow{\text{Gauss Jordan Elimination}} \quad \left[ \begin{array}{rr|r} 1 & 0 & \frac{4}{17} \\ 0 & 1 & -\frac{3}{17} \end{array} \right]$$

which gives $x_1 = \frac{4}{17}$ and $x_3 = -\frac{3}{17}$. To solve the second system, we use the exact same row operations, in the same order, to put its augmented matrix into reduced row echelon form (Think about why that works.) and we obtain

$$\left[ \begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array} \right] \quad \xrightarrow{\text{Gauss Jordan Elimination}} \quad \left[ \begin{array}{rr|r} 1 & 0 & \frac{3}{17} \\ 0 & 1 & \frac{2}{17} \end{array} \right]$$

which means $x_2 = \frac{3}{17}$ and $x_4 = \frac{2}{17}$. Hence,

$$A^{-1} = \left[ \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right] = \left[ \begin{array}{cc} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right]$$

We can check to see that $A^{-1}$ behaves as it should by computing $AA^{-1}$

$$AA^{-1} = \left[ \begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] \left[ \begin{array}{cc} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I_2 \checkmark$$

As an added bonus,

$$A^{-1}A = \left[ \begin{array}{cc} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] \left[ \begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I_2 \checkmark$$

We can now return to the problem at hand. From our discussion at the beginning of the section on page , we know

$$X = A^{-1}B = \left[ \begin{array}{cc} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] \left[ \begin{array}{c} 16 \\ 7 \end{array} \right] = \left[ \begin{array}{r} 5 \\ -2 \end{array} \right]$$

so that our final solution to the system is $(x, y) = (5, -2)$.

As we mentioned, the point of this exercise was not just to solve the system of linear equations, but to develop a general method for finding $A^{-1}$. We now take a step back and analyze the foregoing discussion in a more general context. In solving for $A^{-1}$, we used two augmented matrices, both of which contained the same entries as $A$

$$\begin{bmatrix} 2 & -3 & \big| & 1 \\ 3 & 4 & \big| & 0 \end{bmatrix} = \begin{bmatrix} A & \big| & 1 \\ & \big| & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & \big| & 0 \\ 3 & 4 & \big| & 1 \end{bmatrix} = \begin{bmatrix} A & \big| & 0 \\ & \big| & 1 \end{bmatrix}$$

We also note that the reduced row echelon forms of these augmented matrices can be written as

$$\begin{bmatrix} 1 & 0 & \big| & \frac{4}{17} \\ 0 & 1 & \big| & -\frac{3}{17} \end{bmatrix} = \begin{bmatrix} I_2 & \big| & x_1 \\ & \big| & x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \big| & \frac{3}{17} \\ 0 & 1 & \big| & \frac{2}{17} \end{bmatrix} = \begin{bmatrix} I_2 & \big| & x_2 \\ & \big| & x_4 \end{bmatrix}$$

where we have identified the entries to the left of the vertical bar as the identity $I_2$ and the entries to the right of the vertical bar as the solutions to our systems. The long and short of the solution process can be summarized as

$$\begin{bmatrix} A & \big| & 1 \\ & \big| & 0 \end{bmatrix} \xrightarrow{\text{Gauss Jordan Elimination}} \begin{bmatrix} I_2 & \big| & x_1 \\ & \big| & x_3 \end{bmatrix}$$

$$\begin{bmatrix} A & \big| & 0 \\ & \big| & 1 \end{bmatrix} \xrightarrow{\text{Gauss Jordan Elimination}} \begin{bmatrix} I_2 & \big| & x_2 \\ & \big| & x_4 \end{bmatrix}$$

Since the row operations for both processes are the same, all of the arithmetic on the left hand side of the vertical bar is identical in both problems. The only difference between the two processes is what happens to the constants to the right of the vertical bar. As long as we keep these separated into columns, we can combine our efforts into one 'super-sized' augmented matrix and describe the above process as

$$\begin{bmatrix} A & \big| & 1 & 0 \\ & \big| & 0 & 1 \end{bmatrix} \xrightarrow{\text{Gauss Jordan Elimination}} \begin{bmatrix} I_2 & \big| & x_1 & x_2 \\ & \big| & x_3 & x_4 \end{bmatrix}$$

We have the identity matrix $I_2$ appearing as the right hand side of the first super-sized augmented matrix and the left hand side of the second super-sized augmented matrix. To our surprise and delight, the elements on the right hand side of the second super-sized augmented matrix are none other than those which comprise $A^{-1}$. Hence, we have

$$\begin{bmatrix} A & \big| & I_2 \end{bmatrix} \xrightarrow{\text{Gauss Jordan Elimination}} \begin{bmatrix} I_2 & \big| & A^{-1} \end{bmatrix}$$

In other words, the process of finding $A^{-1}$ for a matrix $A$ can be viewed as performing a series of row operations which transform $A$ into the identity matrix of the same dimension. We can view this process as follows. In trying to find $A^{-1}$, we are trying to 'undo' multiplication by the matrix $A$. The identity matrix in the super-sized augmented matrix $[A|I]$ keeps a running memory of all of the moves required to 'undo' $A$. This results in exactly what we want, $A^{-1}$. We are now ready

to formalize and generalize the foregoing discussion. We begin with the formal definition of an invertible matrix.

**Definition 8.11.** An $n \times n$ matrix $A$ is said to be **invertible** if there exists a matrix $A^{-1}$, read '$A$ inverse', such that $A^{-1}A = AA^{-1} = I_n$.

Note that, as a consequence of our definition, invertible matrices are square, and as such, the conditions in Definition 8.11 force the matrix $A^{-1}$ to be same dimensions as $A$, that is, $n \times n$. Since not all matrices are square, not all matrices are invertible. However, just because a matrix is square doesn't guarantee it is invertible. (See the exercises.) Our first result summarizes some of the important characteristics of invertible matrices and their inverses.

**Theorem 8.6.** Suppose $A$ is an $n \times n$ matrix.

1. If $A$ is invertible then $A^{-1}$ is unique.
2. $A$ is invertible if and only if $AX = B$ has a unique solution for every $n \times r$ matrix $B$.

The proofs of the properties in Theorem 8.6 rely on a healthy mix of definition and matrix arithmetic. To establish the first property, we assume that $A$ is invertible and suppose the matrices $B$ and $C$ act as inverses for $A$. That is, $BA = AB = I_n$ and $CA = AC = I_n$. We need to show that $B$ and $C$ are, in fact, the same matrix. To see this, we note that $B = I_nB = (CA)B = C(AB) = CI_n = C$. Hence, any two matrices that act like $A^{-1}$ are, in fact, the same matrix.[4] To prove the second property of Theorem 8.6, we note that if $A$ is invertible then the discussion on page 599 shows the solution to $AX = B$ to be $X = A^{-1}B$, and since $A^{-1}$ is unique, so is $A^{-1}B$. Conversely, if $AX = B$ has a unique solution for every $n \times r$ matrix $B$, then, in particular, there is a unique solution $X_0$ to the equation $AX = I_n$. The solution matrix $X_0$ is our candidate for $A^{-1}$. We have $AX_0 = I_n$ by definition, but we need to also show $X_0A = I_n$. To that end, we note that $A(X_0A) = (AX_0)A = I_nA = A$. In other words, the matrix $X_0A$ is a solution to the equation $AX = A$. Clearly, $X = I_n$ is also a solution to the equation $AX = A$, and since we are assuming every such equation as a *unique* solution, we must have $X_0A = I_n$. Hence, we have $X_0A = AX_0 = I_n$, so that $X_0 = A^{-1}$ and $A$ is invertible. The foregoing discussion justifies our quest to find $A^{-1}$ using our super-sized augmented matrix approach

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

We are, in essence, trying to find the unique solution to the equation $AX = I_n$ using row operations.

What does all of this mean for a system of linear equations? Theorem 8.6 tells us that if we write the system in the form $AX = B$, then if the coefficient matrix $A$ is invertible, there is only one solution to the system − that is, if $A$ is invertible, the system is consistent and independent.[5] We also know that the process by which we find $A^{-1}$ is determined completely by $A$, and not by the

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[4]If this proof sounds familiar, it should. See the discussion following Theorem 5.2 on page 380.

[5]It can be shown that a matrix is invertible if and only if when it serves as a coefficient matrix for a system of equations, the system is always consistent independent. It amounts to the second property in Theorem 8.6 where the matrices $B$ are restricted to being $n \times 1$ matrices. We note that, owing to how matrix multiplication is defined, being able to find unique solutions to $AX = B$ for $n \times 1$ matrices $B$ gives you the same statement about solving such equations for $n \times r$ matrices − since we can find a unique solution to them one column at a time.

constants in $B$. This answers the question as to why we would bother doing row operations on a super-sized augmented matrix to find $A^{-1}$ instead of an ordinary augmented matrix to solve a system; by finding $A^{-1}$ we have done all of the row operations we ever need to do, once and for all, since we can quickly solve *any* equation $AX = B$ using *one* multiplication, $A^{-1}B$.

**Example 8.4.1.** Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$

1. Use row operations to find $A^{-1}$. Check your answer by finding $A^{-1}A$ and $AA^{-1}$.

2. Use $A^{-1}$ to solve the following systems of equations

(a) $\begin{cases} 3x + y + 2z &= 26 \\ -y + 5z &= 39 \\ 2x + y + 4z &= 117 \end{cases}$ (b) $\begin{cases} 3x + y + 2z &= 4 \\ -y + 5z &= 2 \\ 2x + y + 4z &= 5 \end{cases}$ (c) $\begin{cases} 3x + y + 2z &= 1 \\ -y + 5z &= 0 \\ 2x + y + 4z &= 0 \end{cases}$

**Solution.**

1. We begin with a super-sized augmented matrix and proceed with Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{with } \frac{1}{3}R1]{\text{Replace } R1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R1 + R3]{\text{Replace } R3 \text{ with}} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right] \xrightarrow[\text{with } (-1)R2]{\text{Replace } R2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right] \xrightarrow[-\frac{1}{3}R2 + R3]{\text{Replace } R3 \text{ with}} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & \frac{13}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & \frac{13}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array}\right] \xrightarrow[\text{with } \frac{3}{13}R3]{\text{Replace } R3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \xrightarrow[\substack{\text{Replace } R1 \text{ with} \\ -\frac{2}{3}R3 + R1 \\ \\ \text{Replace } R2 \text{ with} \\ 5R3 + R2}]{} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 0 & \frac{17}{39} & -\frac{2}{39} & -\frac{2}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 0 & \frac{17}{39} & -\frac{2}{39} & -\frac{2}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \xrightarrow[\;-\frac{1}{3}R2 + R1\;]{\text{Replace } R1 \text{ with}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$$

We find $A^{-1} = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix}$. To check our answer, we compute

$$A^{-1}A = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

and

$$AA^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

2. Each of the systems in this part has $A$ as its coefficient matrix. The only difference between the systems is the constants which is the matrix $B$ in the associated matrix equation $AX = B$. We solve each of them using the formula $X = A^{-1}B$.

(a) $X = A^{-1}B = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 26 \\ 39 \\ 117 \end{bmatrix} = \begin{bmatrix} -39 \\ 91 \\ 26 \end{bmatrix}$. Our solution is $(-39, 91, 26)$.

(b) $X = A^{-1}B = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} \\ \frac{19}{13} \\ \frac{9}{13} \end{bmatrix}$. We get $\left(\frac{5}{13}, \frac{19}{13}, \frac{9}{13}\right)$.

(c) $X = A^{-1}B = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{13} \\ -\frac{10}{13} \\ -\frac{2}{13} \end{bmatrix}$. We find $\left(\frac{9}{13}, -\frac{10}{13}, -\frac{2}{13}\right)$.[6]

□

In Example 8.4.1, we see that finding one inverse matrix can enable us to solve an entire family of systems of linear equations. There are many examples of where this comes in handy 'in the wild', and we chose our example for this section from the field of electronics. We also take this opportunity to introduce the student to how we can compute inverse matrices using the calculator.

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[6]Note that the solution is the first column of the $A^{-1}$. The reader is encouraged to meditate on this 'coincidence'.

**Example 8.4.2.** Consider the circuit diagram below.[7] We have two batteries with source voltages $VB_1$ and $VB_2$, measured in volts $V$, along with six resistors with resistances $R_1$ through $R_6$, measured in kiloohms, $k\Omega$. Using Ohm's Law and Kirchhoff's Voltage Law, we can relate the voltage supplied to the circuit by the two batteries to the voltage drops across the six resistors in order to find the four 'mesh' currents: $i_1$, $i_2$, $i_3$ and $i_4$, measured in milliamps, $mA$. If we think of electrons flowing through the circuit, we can think of the voltage sources as providing the 'push' which makes the electrons move, the resistors as obstacles for the electrons to overcome, and the mesh current as a net rate of flow of electrons around the indicated loops.
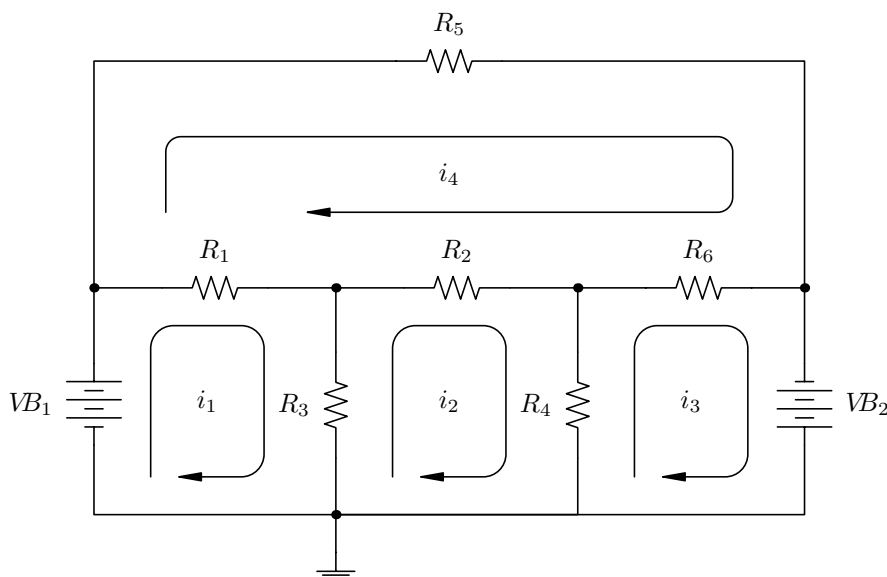


The system of linear equations associated with this circuit is

$$\begin{cases} (R_1 + R_3)\, i_1 - R_3 i_2 - R_1 i_4 &=& VB_1 \\ -R_3 i_1 + (R_2 + R_3 + R_4)\, i_2 - R_4 i_3 - R_2 i_4 &=& 0 \\ -R_4 i_2 + (R_4 + R_6)\, i_3 - R_6 i_4 &=& -VB_2 \\ -R_1 i_1 - R_2 i_2 - R_6 i_3 + (R_1 + R_2 + R_5 + R_6)\, i_4 &=& 0 \end{cases}$$

1. Assuming the resistances are all $1k\Omega$, find the mesh currents if the battery voltages are

   (a) $VB_1 = 10V$ and $VB_2 = 5V$

   (b) $VB_1 = 10V$ and $VB_2 = 0V$

   (c) $VB_1 = 0V$ and $VB_2 = 10V$

   (d) $VB_1 = 10V$ and $VB_2 = 10V$

2. Assuming $VB_1 = 10V$ and $VB_2 = 5V$, find the possible combinations of resistances which would yield the mesh currents you found in 1(a).

---

[7]The authors wish to thank Don Anthan of Lakeland Community College for the design of this example.

**Solution.**
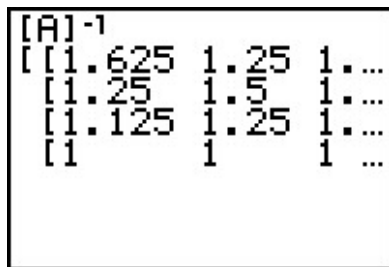
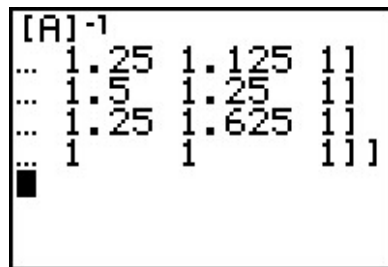1. Substituting the resistance values into our system of equations, we get

$$\begin{cases} 2i_1 - i_2 - i_4 = VB_1 \\ -i_1 + 3i_2 - i_3 - i_4 = 0 \\ -i_2 + 2i_3 - i_4 = -VB_2 \\ -i_1 - i_2 - i_3 + 4i_4 = 0 \end{cases}$$

This corresponds to the matrix equation $AX = B$ where

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \quad X = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} \quad B = \begin{bmatrix} VB_1 \\ 0 \\ -VB_2 \\ 0 \end{bmatrix}$$

When we input the matrix $A$ into the calculator, we find



from which we have $A^{-1} = \begin{bmatrix} 1.625 & 1.25 & 1.125 & 1 \\ 1.25 & 1.5 & 1.25 & 1 \\ 1.125 & 1.25 & 1.625 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

To solve the four systems given to us, we find $X = A^{-1}B$ where the value of $B$ is determined by the given values of $VB_1$ and $VB_2$

$$1 \text{ (a)} \quad B = \begin{bmatrix} 10 \\ 0 \\ -5 \\ 0 \end{bmatrix}, \quad 1 \text{ (b)} \quad B = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad 1 \text{ (c)} \quad B = \begin{bmatrix} 0 \\ 0 \\ -10 \\ 0 \end{bmatrix}, \quad 1 \text{ (d)} \quad B = \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

(a) For $VB_1 = 10V$ and $VB_2 = 5V$, the calculator gives $i_1 = 10.625 \ mA$, $i_2 = 6.25 \ mA$, $i_3 = 3.125 \ mA$, and $i_4 = 5 \ mA$. We include a calculator screenshot below for this part (and this part only!) for reference.

```
[A]⁻¹[B]
            [[10.625]
             [6.25  ]
             [3.125 ]
             [5     ]]
```

(b) By keeping $VB_1 = 10V$ and setting $VB_2 = 0V$, we are removing the effect of the second battery. We get $i_1 = 16.25\ mA$, $i_2 = 12.5\ mA$, $i_3 = 11.25\ mA$, and $i_4 = 10\ mA$.

(c) Part (c) is a symmetric situation to part (b) in so much as we are zeroing out $VB_1$ and making $VB_2 = 10$. We find $i_1 = -11.25\ mA$, $i_2 = -12.5\ mA$, $i_3 = -16.25\ mA$, and $i_4 = -10\ mA$, where the negatives indicate that the current is flowing in the opposite direction as is indicated on the diagram. The reader is encouraged to study the symmetry here, and if need be, hold up a mirror to the diagram to literally 'see' what is happening.

(d) For $VB_1 = 10V$ and $VB_2 = 10V$, we get $i_1 = 5\ mA$, $i_2 = 0\ mA$, $i_3 = -5\ mA$, and $i_4 = 0\ mA$. The mesh currents $i_2$ and $i_4$ being zero is a consequence of both batteries 'pushing' in equal but opposite directions, causing the net flow of electrons in these two regions to cancel out.

2. We now turn the tables and are given $VB_1 = 10V$, $VB_2 = 5V$, $i_1 = 10.625\ mA$, $i_2 = 6.25\ mA$, $i_3 = 3.125\ mA$ and $i_4 = 5\ mA$ and our unknowns are the resistance values. Rewriting our system of equations, we get

$$\begin{cases} 5.625R_1 + 4.375R_3 & = & 10 \\ 1.25R_2 - 4.375R_3 + 3.125R_4 & = & 0 \\ -3.125R_4 - 1.875R_6 & = & -5 \\ -5.625R_1 - 1.25R_2 + 5R_5 + 1.875R_6 & = & 0 \end{cases}$$

The coefficient matrix for this system is $4 \times 6$ (4 equations with 6 unknowns) and is therefore not invertible. We do know, however, this system is consistent, since setting all the resistance values equal to 1 corresponds to our situation in problem 1a. This means we have an underdetermined consistent system which is necessarily dependent. To solve this system, we encode it into an augmented matrix

$$\left[ \begin{array}{cccccc|c} 5.25 & 0 & 4.375 & 0 & 0 & 0 & 10 \\ 0 & 1.25 & -4.375 & 3.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.125 & 0 & -1.875 & -5 \\ -5.625 & -1.25 & 0 & 0 & 5 & 1.875 & 0 \end{array} \right]$$

and use the calculator to write in reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0.\overline{7} & 0 & 0 & 0 & | & 1.\overline{7} \\ 0 & 1 & -3.5 & 0 & 0 & -1.5 & | & -4 \\ 0 & 0 & 0 & 1 & 0 & 0.6 & | & 1.6 \\ 0 & 0 & 0 & 0 & 1 & 0 & | & 1 \end{bmatrix}$$
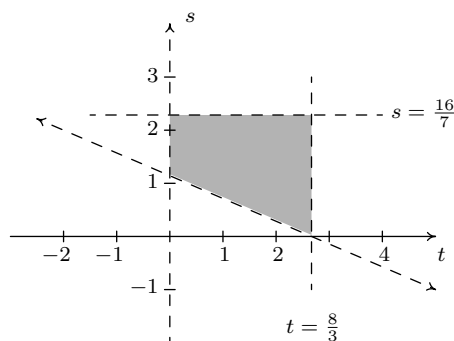
Decoding this system from the matrix, we get

$$\begin{cases} R_1 + 0.\overline{7}R_3 & = & 1.\overline{7} \\ R_2 - 3.5R_3 - 1.5R_6 & = & -4 \\ R_4 + 0.6R_6 & = & 1.6 \\ R_5 & = & 1 \end{cases}$$

We have can solve for $R_1$, $R_2$, $R_4$ and $R_5$ leaving $R_3$ and $R_6$ as free variables. Labeling $R_3 = s$ and $R_6 = t$, we have $R_1 = -0.\overline{7}s + 1.\overline{7}$, $R_2 = 3.5s + 1.5t - 4$, $R_4 = -0.6t + 1.6$ and $R_5 = 1$. Since resistance values are always positive, we need to restrict our values of $s$ and $t$. We know $R_3 = s > 0$ and when we combine that with $R_1 = -0.\overline{7}s + 1.\overline{7} > 0$, we get $0 < s < \frac{16}{7}$. Similarly, $R_6 = t > 0$ and with $R_4 = -0.6t + 1.6 > 0$, we find $0 < t < \frac{8}{3}$. In order visualize the inequality $R_2 = 3.5s + 1.5t - 4 > 0$, we graph the line $3.5s + 1.5t - 4 = 0$ on the $st$-plane and shade accordingly.[8] Imposing the additional conditions $0 < s < \frac{16}{7}$ and $0 < t < \frac{8}{3}$, we find our values of $s$ and $t$ restricted to the region depicted on the right. Using the roster method, the values of $s$ and $t$ are pulled from the region $\left\{ (s,t) : 0 < s < \frac{16}{7},\ 0 < t < \frac{8}{3},\ 3.5s + 1.5t - 4 > 0 \right\}$. The reader is encouraged to check that the solution presented in 1(a), namely all resistance values equal to 1, corresponds to a pair $(s,t)$ in the region.
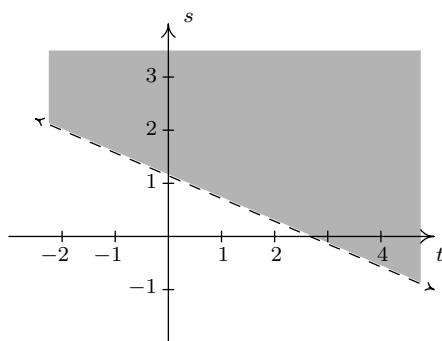


The region where $3.5s + 1.5t - 4 > 0$          The region for our parameters $s$ and $t$.

□

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[8]See Section 2.4 for a review of this procedure.

### 8.4.1   Exercises

In Exercises 1 - 8, find the inverse of the matrix or state that the matrix is not invertible.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

3. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

4. $D = \begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$

5. $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$

6. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

7. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 11, use one matrix inverse to solve the following systems of linear equations.

9. $\begin{cases} 3x + 7y & = & 26 \\ 5x + 12y & = & 39 \end{cases}$

10. $\begin{cases} 3x + 7y & = & 0 \\ 5x + 12y & = & -1 \end{cases}$

11. $\begin{cases} 3x + 7y & = & -7 \\ 5x + 12y & = & 5 \end{cases}$

In Exercises 12 - 14, use the inverse of $E$ from Exercise 5 above to solve the following systems of linear equations.

12. $\begin{cases} 3x + 4z & = & 1 \\ 2x - y + 3z & = & 0 \\ -3x + 2y - 5z & = & 0 \end{cases}$

13. $\begin{cases} 3x + 4z & = & 0 \\ 2x - y + 3z & = & 1 \\ -3x + 2y - 5z & = & 0 \end{cases}$

14. $\begin{cases} 3x + 4z & = & 0 \\ 2x - y + 3z & = & 0 \\ -3x + 2y - 5z & = & 1 \end{cases}$

15. This exercise is a continuation of Example 8.3.3 in Section 8.3 and gives another application of matrix inverses. Recall that given the position matrix $P$ for a point in the plane, the matrix $RP$ corresponds to a point rotated $45°$ counterclockwise from $P$ where

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) Find $R^{-1}$.

(b) If $RP$ rotates a point counterclockwise $45°$, what should $R^{-1}P$ do? Check your answer by finding $R^{-1}P$ for various points on the coordinate axes and the lines $y = \pm x$.

(c) Find $R^{-1}P$ where $P$ corresponds to a generic point $P(x, y)$. Verify that this takes points on the curve $y = \frac{2}{x}$ to points on the curve $x^2 - y^2 = 4$.

16. A Sasquatch's diet consists of three primary foods: Ippizuti Fish, Misty Mushrooms, and Sun Berries. Each serving of Ippizuti Fish is 500 calories, contains 40 grams of protein, and has no Vitamin X. Each serving of Misty Mushrooms is 50 calories, contains 1 gram of protein, and 5 milligrams of Vitamin X. Finally, each serving of Sun Berries is 80 calories, contains no protein, but has 15 milligrams of Vitamin X.[9]

    (a) If an adult male Sasquatch requires 3200 calories, 130 grams of protein, and 275 milligrams of Vitamin X daily, use a matrix inverse to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries he needs to eat each day.

    (b) An adult female Sasquatch requires 3100 calories, 120 grams of protein, and 300 milligrams of Vitamin X daily. Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

    (c) An adolescent Sasquatch requires 5000 calories, 400 grams of protein daily, but no Vitamin X daily.[10] Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

17. Matrices can be used in cryptography. Suppose we wish to encode the message 'BIGFOOT LIVES'. We start by assigning a number to each letter of the alphabet, say $A = 1$, $B = 2$ and so on. We reserve 0 to act as a space. Hence, our message 'BIGFOOT LIVES' corresponds to the string of numbers '2, 9, 7, 6, 15, 15, 20, 0, 12, 9, 22, 5, 19.' To encode this message, we use an invertible matrix. Any invertible matrix will do, but for this exercise, we choose

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$$

Since $A$ is $3 \times 3$ matrix, we encode our message string into a matrix $M$ with 3 rows. To do this, we take the first three numbers, 2 9 7, and make them our first column, the next three numbers, 6 15 15, and make them our second column, and so on. We put 0's to round out the matrix.

$$M = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$$

To encode the message, we find the product $AM$

$$AM = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix}$$

[9]Misty Mushrooms and Sun Berries are the only known fictional sources of Vitamin X.
[10]Vitamin X is needed to sustain Sasquatch longevity only.

So our coded message is '12, 1, $-12$, 42, 3, $-42$, 100, 36, $-152$, $-23$, 39, $-46$, 38, 57, $-133$.'
To decode this message, we start with this string of numbers, construct a message matrix as
we did earlier (we should get the matrix $AM$ again) and then multiply by $A^{-1}$.

(a) Find $A^{-1}$.

(b) Use $A^{-1}$ to decode the message and check this method actually works.

(c) Decode the message '14, 37, $-76$, 128, 21, $-151$, 31, 65, $-140$'

(d) Choose another invertible matrix and encode and decode your own messages.

18. Using the matrices $A$ from Exercise 1, $B$ from Exercise 2 and $D$ from Exercise 4, show
$AB = D$ and $D^{-1} = B^{-1}A^{-1}$. That is, show that $(AB)^{-1} = B^{-1}A^{-1}$.

19. Let $M$ and $N$ be invertible $n \times n$ matrices. Show that $(MN)^{-1} = N^{-1}M^{-1}$ and compare
your work to Exercise 31 in Section 5.2.

## 8.4.2   ANSWERS

1. $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

2. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

3. $C$ is not invertible

4. $D^{-1} = \begin{bmatrix} \frac{9}{2} & -\frac{1}{2} \\ 8 & -1 \end{bmatrix}$

5. $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

6. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

7. $G$ is not invertible

8. $H^{-1} = \begin{bmatrix} 16 & 0 & 3 & 0 \\ -90 & -\frac{1}{2} & -\frac{35}{2} & \frac{7}{2} \\ 5 & 0 & 1 & 0 \\ -36 & 0 & -7 & 1 \end{bmatrix}$

The coefficient matrix is $B^{-1}$ from Exercise 2 above so the inverse we need is $(B^{-1})^{-1} = B$.

9. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 26 \\ 39 \end{bmatrix} = \begin{bmatrix} 39 \\ -13 \end{bmatrix}$   So $x = 39$ and $y = -13$.

10. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$   So $x = 7$ and $y = -3$.

11. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} = \begin{bmatrix} -119 \\ 50 \end{bmatrix}$   So $x = -119$ and $y = 50$.

The coefficient matrix is $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$ from Exercise 5, so $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$   So $x = -1$, $y = 1$ and $z = 1$.

13. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -6 \end{bmatrix}$   So $x = 8$, $y = -3$ and $z = -6$.

14. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$   So $x = 4$, $y = -1$ and $z = -3$.

16.  (a) The adult male Sasquatch needs: 3 servings of Ippizuti Fish, 10 servings of Misty Mushrooms, and 15 servings of Sun Berries daily.

 (b) The adult female Sasquatch needs: 3 servings of Ippizuti Fish and 20 servings of Sun Berries daily. (No Misty Mushrooms are needed!)

 (c) The adolescent Sasquatch requires 10 servings of Ippizuti Fish daily. (No Misty Mushrooms or Sun Berries are needed!)

17.  (a) $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

 (b) $\begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix} \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$ ✓

 (c) 'LOGS RULE'

## 8.5   Determinants and Cramer's Rule

### 8.5.1   Definition and Properties of the Determinant

In this section we assign to each square matrix $A$ a real number, called the **determinant** of $A$, which will eventually lead us to yet another technique for solving consistent independent systems of linear equations. The determinant is defined recursively, that is, we define it for $1 \times 1$ matrices and give a rule by which we can reduce determinants of $n \times n$ matrices to a sum of determinants of $(n-1) \times (n-1)$ matrices.[1] This means we will be able to evaluate the determinant of a $2 \times 2$ matrix as a sum of the determinants of $1 \times 1$ matrices; the determinant of a $3 \times 3$ matrix as a sum of the determinants of $2 \times 2$ matrices, and so forth. To explain how we will take an $n \times n$ matrix and distill from it an $(n-1) \times (n-1)$, we use the following notation.

---

**Definition 8.12.** Given an $n \times n$ matrix $A$ where $n > 1$, the matrix $A_{ij}$ is the $(n-1) \times (n-1)$ matrix formed by deleting the $i$th row of $A$ and the $j$th column of $A$.

---

For example, using the matrix $A$ below, we find the matrix $A_{23}$ by deleting the second row and third column of $A$.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{\text{Delete } R2 \text{ and } C3} A_{23} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

We are now in the position to define the determinant of a matrix.

---

**Definition 8.13.** Given an $n \times n$ matrix $A$ the **determinant of $A$**, denoted $\det(A)$, is defined as follows

- If $n = 1$, then $A = [a_{11}]$ and $\det(A) = \det([a_{11}]) = a_{11}$.

- If $n > 1$, then $A = [a_{ij}]_{n \times n}$ and

$$\det(A) = \det\left([a_{ij}]_{n \times n}\right) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

---

There are two commonly used notations for the determinant of a matrix $A$: '$\det(A)$' and '$|A|$' We have chosen to use the notation $\det(A)$ as opposed to $|A|$ because we find that the latter is often confused with absolute value, especially in the context of a $1 \times 1$ matrix. In the expansion $a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n} \det(A_{1n})$, the notation '$+ - \ldots + (-1)^{1+n} a_{1n}$' means that the signs alternate and the final sign is dictated by the sign of the quantity $(-1)^{1+n}$. Since the entries $a_{11}$, $a_{12}$ and so forth up through $a_{1n}$ comprise the first row of $A$, we say we are finding the determinant of $A$ by 'expanding along the first row'. Later in the section, we will develop a formula for $\det(A)$ which allows us to find it by expanding along any row.

Applying Definition 8.13 to the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ we get

---

[1] We will talk more about the term 'recursively' in Section 9.1.

$$\det(A) \;=\; \det\left(\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}\right)$$

$$\begin{aligned} &= & 4\det\left(A_{11}\right) - (-3)\det\left(A_{12}\right) \\ &= & 4\det([1]) + 3\det([2]) \\ &= & 4(1) + 3(2) \\ &= & 10 \end{aligned}$$

For a generic $2\times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we get

$$\det(A) \;=\; \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$\begin{aligned} &= & a\det\left(A_{11}\right) - b\det\left(A_{12}\right) \\ &= & a\det\left([d]\right) - b\det\left([c]\right) \\ &= & ad - bc \end{aligned}$$

This formula is worth remembering

**Equation 8.1.** For a $2 \times 2$ matrix,
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Applying Definition 8.13 to the $3 \times 3$ matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ we obtain

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}\right) \\ &= 3\det\left(A_{11}\right) - 1\det\left(A_{12}\right) + 2\det\left(A_{13}\right) \\[2mm] &= 3\det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\right) \\[2mm] &= 3((-1)(4) - (5)(1)) - ((0)(4) - (5)(2)) + 2((0)(1) - (-1)(2)) \\ &= 3(-9) - (-10) + 2(2) \\ &= -13 \end{aligned}$$

To evaluate the determinant of a $4 \times 4$ matrix, we would have to evaluate the determinants of *four* $3 \times 3$ matrices, each of which involves the finding the determinants of *three* $2 \times 2$ matrices. As you can see, our method of evaluating determinants quickly gets out of hand and many of you may be reaching for the calculator. There is some mathematical machinery which can assist us in calculating determinants and we present that here. Before we state the theorem, we need some more terminology.

**Definition 8.14.** Let $A$ be an $n \times n$ matrix and $A_{ij}$ be defined as in Definition 8.12. The **$ij$ minor** of $A$, denoted $M_{ij}$ is defined by $M_{ij} = \det(A_{ij})$. The **$ij$ cofactor** of $A$, denoted $C_{ij}$ is defined by $C_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \det(A_{ij})$.

We note that in Definition 8.13, the sum

$$a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

can be rewritten as

$$a_{11}(-1)^{1+1} \det(A_{11}) + a_{12}(-1)^{1+2} \det(A_{12}) + \ldots + a_{1n}(-1)^{1+n} \det(A_{1n})$$

which, in the language of cofactors is

$$a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

We are now ready to state our main theorem concerning determinants.

**Theorem 8.7. Properties of the Determinant:** Let $A = [a_{ij}]_{n \times n}$.

- We may find the determinant by expanding along any row. That is, for any $1 \leq k \leq n$,

$$\det(A) = a_{k1}C_{k1} + a_{k2}C_{k2} + \ldots + a_{kn}C_{kn}$$

- If $A'$ is the matrix obtained from $A$ by:

    - interchanging any two rows, then $\det(A') = -\det(A)$.
    - replacing a row with a nonzero multiple (say $c$) of itself, then $\det(A') = c\det(A)$
    - replacing a row with itself plus a multiple of another row, then $\det(A') = \det(A)$

- If $A$ has two identical rows, or a row consisting of all 0's, then $\det(A) = 0$.

- If $A$ is upper or lower triangular,[a] then $\det(A)$ is the product of the entries on the main diagonal.[b]

- If $B$ is an $n \times n$ matrix, then $\det(AB) = \det(A)\det(B)$.

- $\det(A^n) = \det(A)^n$ for all natural numbers $n$.

- $A$ is invertible if and only if $\det(A) \neq 0$. In this case, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

---

[a]See Exercise 8.3.1 in 8.3.
[b]See page 585 in Section 8.3.

Unfortunately, while we can easily *demonstrate* the results in Theorem 8.7, the proofs of most of these properties are beyond the scope of this text. We could prove these properties for generic $2 \times 2$

or even $3 \times 3$ matrices by brute force computation, but this manner of proof belies the elegance and symmetry of the determinant. We will prove what few properties we can after we have developed some more tools such as the Principle of Mathematical Induction in Section 9.3.[2] For the moment, let us demonstrate some of the properties listed in Theorem 8.7 on the matrix $A$ below. (Others will be discussed in the Exercises.)

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

We found $\det(A) = -13$ by expanding along the first row. To take advantage of the 0 in the second row, we use Theorem 8.7 to find $\det(A) = -13$ by expanding along that row.

$$
\begin{aligned}
\det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}\right) &= 0C_{21} + (-1)C_{22} + 5C_{23} \\
&= (-1)(-1)^{2+2}\det(A_{22}) + 5(-1)^{2+3}\det(A_{23}) \\
&= -\det\left(\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\right) - 5\det\left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\right) \\
&= -((3)(4) - (2)(2)) - 5((3)(1) - (2)(1)) \\
&= -8 - 5 \\
&= -13 \checkmark
\end{aligned}
$$

In general, the sign of $(-1)^{i+j}$ in front of the minor in the expansion of the determinant follows an alternating pattern. Below is the pattern for $2 \times 2$, $3 \times 3$ and $4 \times 4$ matrices, and it extends naturally to higher dimensions.

$$
\begin{bmatrix} + & - \\ - & + \end{bmatrix}
\qquad
\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}
\qquad
\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}
$$

The reader is cautioned, however, against reading too much into these sign patterns. In the example above, we expanded the $3 \times 3$ matrix $A$ by its second row and the term which corresponds to the second entry ended up being negative even though the sign attached to the minor is $(+)$. These signs represent only the signs of the $(-1)^{i+j}$ in the formula; the sign of the corresponding entry as well as the minor itself determine the ultimate sign of the term in the expansion of the determinant.

To illustrate some of the other properties in Theorem 8.7, we use row operations to transform our $3 \times 3$ matrix $A$ into an upper triangular matrix, keeping track of the row operations, and labeling

---

[2]For a very elegant treatment, take a course in Linear Algebra. There, you will most likely see the treatment of determinants logically reversed than what is presented here. Specifically, the determinant is defined as a function which takes a square matrix to a real number and satisfies some of the properties in Theorem 8.7. From that function, a formula for the determinant is developed.

each successive matrix.[3]

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow[\text{with } -\frac{2}{3}R1 + R3]{\text{Replace } R3} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & \frac{1}{3} & \frac{8}{3} \end{bmatrix} \xrightarrow[\frac{1}{3}R2 + R3]{\text{Replace } R3 \text{ with}} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & \frac{13}{3} \end{bmatrix}$$
$$\qquad A \qquad\qquad\qquad\qquad\qquad\qquad B \qquad\qquad\qquad\qquad\qquad\qquad C$$

Theorem 8.7 guarantees us that $\det(A) = \det(B) = \det(C)$ since we are replacing a row with itself plus a multiple of another row moving from one matrix to the next. Furthermore, since $C$ is upper triangular, $\det(C)$ is the product of the entries on the main diagonal, in this case $\det(C) = (3)(-1)\left(\frac{13}{3}\right) = -13$. This demonstrates the utility of using row operations to assist in calculating determinants. This also sheds some light on the connection between a determinant and invertibility. Recall from Section 8.4 that in order to find $A^{-1}$, we attempt to transform $A$ to $I_n$ using row operations

$$\left[ \begin{array}{c|c} A & I_n \end{array} \right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[ \begin{array}{c|c} I_n & A^{-1} \end{array} \right]$$

As we apply our allowable row operations on $A$ to put it into reduced row echelon form, the determinant of the intermediate matrices can vary from the determinant of $A$ by at most a *nonzero* multiple. This means that if $\det(A) \neq 0$, then the determinant of $A$'s reduced row echelon form must also be nonzero, which, according to Definition 8.4 means that all the main diagonal entries on $A$'s reduced row echelon form must be 1. That is, $A$'s reduced row echelon form is $I_n$, and $A$ is invertible. Conversely, if $A$ is invertible, then $A$ can be transformed into $I_n$ using row operations. Since $\det(I_n) = 1 \neq 0$, our same logic implies $\det(A) \neq 0$. Basically, we have established that the determinant *determines* whether or not the matrix $A$ is invertible.[4]

It is worth noting that when we first introduced the notion of a matrix inverse, it was in the context of solving a linear matrix equation. In effect, we were trying to 'divide' both sides of the matrix equation $AX = B$ by the matrix $A$. Just like we cannot divide a real number by 0, Theorem 8.7 tells us we cannot 'divide' by a matrix whose *determinant* is 0. We also know that if the coefficient matrix of a system of linear equations is invertible, then system is consistent and independent. It follows, then, that if the determinant of said coefficient is not zero, the system is consistent and independent.

### 8.5.2 Cramer's Rule and Matrix Adjoints

In this section, we introduce a theorem which enables us to solve a system of linear equations by means of determinants only. As usual, the theorem is stated in full generality, using numbered unknowns $x_1$, $x_2$, etc., instead of the more familiar letters $x$, $y$, $z$, etc. The proof of the general case is best left to a course in Linear Algebra.

---

[3]Essentially, we follow the Gauss Jordan algorithm but we don't care about getting leading 1's.

[4]In Section 8.5.2, we learn determinants (specifically cofactors) are deeply connected with the inverse of a matrix.

> **Theorem 8.8. Cramer's Rule:** Suppose $AX = B$ is the matrix form of a system of $n$ linear equations in $n$ unknowns where $A$ is the coefficient matrix, $X$ is the unknowns matrix, and $B$ is the constant matrix. If $\det(A) \neq 0$, then the corresponding system is consistent and independent and the solution for unknowns $x_1$, $x_2$, $\ldots x_n$ is given by:
>
> $$x_j = \frac{\det(A_j)}{\det(A)},$$
>
> where $A_j$ is the matrix $A$ whose $j$th column has been replaced by the constants in $B$.

In words, Cramer's Rule tells us we can solve for each unknown, one at a time, by finding the ratio of the determinant of $A_j$ to that of the determinant of the coefficient matrix. The matrix $A_j$ is found by replacing the column in the coefficient matrix which holds the coefficients of $x_j$ with the constants of the system. The following example fleshes out this method.

**Example 8.5.1.** Use Cramer's Rule to solve for the indicated unknowns.

1. Solve $\begin{cases} 2x_1 - 3x_2 &= 4 \\ 5x_1 + x_2 &= -2 \end{cases}$ for $x_1$ and $x_2$

2. Solve $\begin{cases} 2x - 3y + z &= -1 \\ x - y + z &= 1 \\ 3x - 4z &= 0 \end{cases}$ for $z$.

**Solution.**

1. Writing this system in matrix form, we find

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

To find the matrix $A_1$, we remove the column of the coefficient matrix $A$ which holds the coefficients of $x_1$ and replace it with the corresponding entries in $B$. Likewise, we replace the column of $A$ which corresponds to the coefficients of $x_2$ with the constants to form the matrix $A_2$. This yields

$$A_1 = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 4 \\ 5 & -2 \end{bmatrix}$$

Computing determinants, we get $\det(A) = 17$, $\det(A_1) = -2$ and $\det(A_2) = -24$, so that

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{2}{17} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = -\frac{24}{17}$$

The reader can check that the solution to the system is $\left(-\frac{2}{17}, -\frac{24}{17}\right)$.

2. To use Cramer's Rule to find $z$, we identify $x_3$ as $z$. We have

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad A_3 = A_z = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Expanding both $\det(A)$ and $\det(A_z)$ along the third rows (to take advantage of the 0's) gives

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-12}{-10} = \frac{6}{5}$$

The reader is encouraged to solve this system for $x$ and $y$ similarly and check the answer.  □

Our last application of determinants is to develop an alternative method for finding the inverse of a matrix.[5] Let us consider the $3 \times 3$ matrix $A$ which we so extensively studied in Section 8.5.1

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

We found through a variety of methods that $\det(A) = -13$. To our surprise and delight, its inverse below has a remarkable number of 13's in the denominators of its entries. This is no coincidence.

$$A^{-1} = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix}$$

Recall that to find $A^{-1}$, we are essentially solving the matrix equation $AX = I_3$, where $X = [x_{ij}]_{3\times 3}$ is a $3 \times 3$ matrix. Because of how matrix multiplication is defined, the first column of $I_3$ is the product of $A$ with the first column of $X$, the second column of $I_3$ is the product of $A$ with the second column of $X$ and the third column of $I_3$ is the product of $A$ with the third column of $X$. In other words, we are solving three equations[6]

$$A \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can solve each of these systems using Cramer's Rule. Focusing on the first system, we have

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 5 \\ 2 & 0 & 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

---

[5]We are developing a *method* in the forthcoming discussion. As with the discussion in Section 8.4 when we developed the first algorithm to find matrix inverses, we ask that you indulge us.

[6]The reader is encouraged to stop and think this through.

If we expand $\det(A_1)$ along the first row, we get

$$\det(A_1) = \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 0 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right)$$

Amazingly, this is none other than the $C_{11}$ cofactor of $A$. The reader is invited to check this, as well as the claims that $\det(A_2) = C_{12}$ and $\det(A_3) = C_{13}$.[7] (To see this, though it seems unnatural to do so, expand along the first row.) Cramer's Rule tells us

$$x_{11} = \frac{\det(A_1)}{\det(A)} = \frac{C_{11}}{\det(A)}, \quad x_{21} = \frac{\det(A_2)}{\det(A)} = \frac{C_{12}}{\det(A)}, \quad x_{31} = \frac{\det(A_3)}{\det(A)} = \frac{C_{13}}{\det(A)}$$

So the first column of the inverse matrix $X$ is:

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} \dfrac{C_{11}}{\det(A)} \\ \dfrac{C_{12}}{\det(A)} \\ \dfrac{C_{13}}{\det(A)} \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

Notice the reversal of the subscripts going from the unknown to the corresponding cofactor of $A$. This trend continues and we get

$$\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} \qquad \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix}$$

Putting all of these together, we have obtained a new and surprising formula for $A^{-1}$, namely

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

To see that this does indeed yield $A^{-1}$, we find all of the cofactors of $A$

$$\begin{array}{rclrclrcl} C_{11} & = & -9, & C_{21} & = & -2, & C_{31} & = & 7 \\ C_{12} & = & 10, & C_{22} & = & 8, & C_{32} & = & -15 \\ C_{13} & = & 2, & C_{23} & = & -1, & C_{33} & = & -3 \end{array}$$

And, as promised,

---

[7] In a solid Linear Algebra course you will learn that the properties in Theorem 8.7 hold equally well if the word 'row' is replaced by the word 'column'. We're not going to get into column operations in this text, but they do make some of what we're trying to say easier to follow.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} -9 & -2 & 7 \\ 10 & 8 & -15 \\ 2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix}$$

To generalize this to invertible $n \times n$ matrices, we need another definition and a theorem. Our definition gives a special name to the cofactor matrix, and the theorem tells us how to use it along with $\det(A)$ to find the inverse of a matrix.

**Definition 8.15.** Let $A$ be an $n \times n$ matrix, and $C_{ij}$ denote the $ij$ cofactor of $A$. The **adjoint** of $A$, denoted $\mathrm{adj}(A)$ is the matrix whose $ij$-entry is the $ji$ cofactor of $A$, $C_{ji}$. That is

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix}$$

This new notation greatly shortens the statement of the formula for the inverse of a matrix.

**Theorem 8.9.** Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

For $2 \times 2$ matrices, Theorem 8.9 reduces to a fairly simple formula.

**Equation 8.2.** For an invertible $2 \times 2$ matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The proof of Theorem 8.9 is, like so many of the results in this section, best left to a course in Linear Algebra. In such a course, not only do you gain some more sophisticated proof techniques, you also gain a larger perspective. The authors assure you that persistence pays off. If you stick around a few semesters and take a course in Linear Algebra, you'll see just how pretty all things matrix really are - in spite of the tedious notation and sea of subscripts. Within the scope of this text, we will prove a few results involving determinants in Section 9.3 once we have the Principle of Mathematical Induction well in hand. Until then, make sure you have a handle on the *mechanics* of matrices and the theory will come eventually.

### 8.5.3  EXERCISES

In Exercises 1 - 8, compute the determinant of the given matrix. (Some of these matrices appeared in Exercises 1 - 8 in Section 8.4.)

1. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

2. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

3. $Q = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

4. $L = \begin{bmatrix} \dfrac{1}{x^3} & \dfrac{\ln(x)}{x^3} \\ -\dfrac{3}{x^4} & \dfrac{1 - 3\ln(x)}{x^4} \end{bmatrix}$

5. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

6. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

7. $V = \begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 14, use Cramer's Rule to solve the system of linear equations.

9. $\begin{cases} 3x + 7y &= 26 \\ 5x + 12y &= 39 \end{cases}$

10. $\begin{cases} 2x - 4y &= 5 \\ 10x + 13y &= -6 \end{cases}$

11. $\begin{cases} x + y &= 8000 \\ 0.03x + 0.05y &= 250 \end{cases}$

12. $\begin{cases} \frac{1}{2}x - \frac{1}{5}y &= 1 \\ 6x + 7y &= 3 \end{cases}$

13. $\begin{cases} x + y + z &= 3 \\ 2x - y + z &= 0 \\ -3x + 5y + 7z &= 7 \end{cases}$

14. $\begin{cases} 3x + y - 2z &= 10 \\ 4x - y + z &= 5 \\ x - 3y - 4z &= -1 \end{cases}$

In Exercises 15 - 16, use Cramer's Rule to solve for $x_4$.

15. $\begin{cases} x_1 - x_3 &= -2 \\ 2x_2 - x_4 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ -x_3 + x_4 &= 1 \end{cases}$

16. $\begin{cases} 4x_1 + x_2 &= 4 \\ x_2 - 3x_3 &= 1 \\ 10x_1 + x_3 + x_4 &= 0 \\ -x_2 + x_3 &= -3 \end{cases}$

In Exercises 17 - 18, find the inverse of the given matrix using their determinants and adjoints.

17. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

18. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

19. Carl's Sasquatch Attack! Game Card Collection is a mixture of common and rare cards. Each common card is worth $0.25 while each rare card is worth $0.75. If his entire 117 card collection is worth $48.75, how many of each kind of card does he own?

20. How much of a 5 gallon 40% salt solution should be replaced with pure water to obtain 5 gallons of a 15% solution?

21. How much of a 10 liter 30% acid solution must be replaced with pure acid to obtain 10 liters of a 50% solution?

22. Daniel's Exotic Animal Rescue houses snakes, tarantulas and scorpions. When asked how many animals of each kind he boards, Daniel answered: 'We board 49 total animals, and I am responsible for each of their 272 legs and 28 tails.' How many of each animal does the Rescue board? (Recall: tarantulas have 8 legs and no tails, scorpions have 8 legs and one tail, and snakes have no legs and one tail.)

23. This exercise is a continuation of Exercise 16 in Section 8.4. Just because a system is consistent independent doesn't mean it will admit a solution that makes sense in an applied setting. Using the nutrient values given for Ippizuti Fish, Misty Mushrooms, and Sun Berries, use Cramer's Rule to determine the number of servings of Ippizuti Fish needed to meet the needs of a daily diet which requires 2500 calories, 1000 grams of protein, and 400 milligrams of Vitamin X. Now use Cramer's Rule to find the number of servings of Misty Mushrooms required. Does a solution to this diet problem exist?

24. Let $R = \begin{bmatrix} -7 & 3 \\ 11 & 2 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -5 \\ 6 & 9 \end{bmatrix}$ $T = \begin{bmatrix} 11 & 2 \\ -7 & 3 \end{bmatrix}$, and $U = \begin{bmatrix} -3 & 15 \\ 6 & 9 \end{bmatrix}$

    (a) Show that $\det(RS) = \det(R)\det(S)$
    (b) Show that $\det(T) = -\det(R)$
    (c) Show that $\det(U) = -3\det(S)$

25. For $M$, $N$, and $P$ below, show that $\det(M) = 0$, $\det(N) = 0$ and $\det(P) = 0$.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

26. Let $A$ be an arbitrary invertible $3 \times 3$ matrix.

    (a) Show that $\det(I_3) = 1$.[8]

    (b) Using the facts that $AA^{-1} = I_3$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

The purpose of Exercises 27 - 30 is to introduce you to the eigenvalues and eigenvectors of a matrix.[9] We begin with an example using a $2 \times 2$ matrix and then guide you through some exercises using a $3 \times 3$ matrix. Consider the matrix

$$C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$$

from Exercise 2. We know that $\det(C) = 0$ which means that $CX = 0_{2\times 2}$ does not have a unique solution. So there is a nonzero matrix $Y$ with $CY = 0_{2\times 2}$. In fact, every matrix of the form

$$Y = \begin{bmatrix} -\frac{5}{2}t \\ t \end{bmatrix}$$

is a solution to $CX = 0_{2\times 2}$, so there are infinitely many matrices such that $CX = 0_{2\times 2}$. But consider the matrix

$$X_{41} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

It is NOT a solution to $CX = 0_{2\times 2}$, but rather,

$$CX_{41} = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 123 \\ 287 \end{bmatrix} = 41 \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

In fact, if $Z$ is of the form

$$Z = \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix}$$

then

$$CZ = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{123}{7}t \\ 41t \end{bmatrix} = 41 \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = 41Z$$

for all $t$. The big question is "How did we know to use 41?"

We need a number $\lambda$ such that $CX = \lambda X$ has nonzero solutions. We have demonstrated that $\lambda = 0$ and $\lambda = 41$ both worked. Are there others? If we look at the matrix equation more closely, what

---

[8] If you think about it for just a moment, you'll see that $\det(I_n) = 1$ for any natural number $n$. The formal proof of this fact requires the Principle of Mathematical Induction (Section 9.3) so we'll stick with $n = 3$ for the time being.

[9] This material is usually given its own chapter in a Linear Algebra book so clearly we're not able to tell you everything you need to know about eigenvalues and eigenvectors. They are a nice application of determinants, though, so we're going to give you enough background so that you can start playing around with them.

we *really* wanted was a nonzero solution to $(C - \lambda I_2)X = 0_{2\times 2}$ which we know exists if and only if the determinant of $C - \lambda I_2$ is zero.[10] So we computed

$$\det(C - \lambda I_2) = \det\left(\begin{bmatrix} 6 - \lambda & 15 \\ 14 & 35 - \lambda \end{bmatrix}\right) = (6 - \lambda)(35 - \lambda) - 14 \cdot 15 = \lambda^2 - 41\lambda$$

This is called the **characteristic polynomial** of the matrix $C$ and it has two zeros: $\lambda = 0$ and $\lambda = 41$. That's how we knew to use 41 in our work above. The fact that $\lambda = 0$ showed up as one of the zeros of the characteristic polynomial just means that $C$ itself had determinant zero which we already knew. Those two numbers are called the **eigenvalues** of $C$. The corresponding matrix solutions to $CX = \lambda X$ are called the **eigenvectors** of $C$ and the 'vector' portion of the name will make more sense after you've studied vectors.

Now it's your turn. In the following exercises, you'll be using the matrix $G$ from Exercise 6.

$$G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$$

27. Show that the characteristic polynomial of $G$ is $p(\lambda) = -\lambda(\lambda - 1)(\lambda - 22)$. That is, compute $\det(G - \lambda I_3)$.

28. Let $G_0 = G$. Find the parametric description of the solution to the system of linear equations given by $GX = 0_{3\times 3}$.

29. Let $G_1 = G - I_3$. Find the parametric description of the solution to the system of linear equations given by $G_1 X = 0_{3\times 3}$. Show that any solution to $G_1 X = 0_{3\times 3}$ also has the property that $GX = 1X$.

30. Let $G_{22} = G - 22I_3$. Find the parametric description of the solution to the system of linear equations given by $G_{22}X = 0_{3\times 3}$. Show that any solution to $G_{22}X = 0_{3\times 3}$ also has the property that $GX = 22X$.

---

[10]Think about this.

### 8.5.4   ANSWERS

1. $\det(B) = 1$

2. $\det(C) = 0$

3. $\det(Q) = x^2$

4. $\det(L) = \dfrac{1}{x^7}$

5. $\det(F) = -12$

6. $\det(G) = 0$

7. $\det(V) = 20i + 43j + 4k$

8. $\det(H) = -2$

9. $x = 39$, $y = -13$

10. $x = \frac{41}{66}$, $y = -\frac{31}{33}$

11. $x = 7500$, $y = 500$

12. $x = \frac{76}{47}$, $y = -\frac{45}{47}$

13. $x = 1$, $y = 2$, $z = 0$

14. $x = \frac{121}{60}$, $y = \frac{131}{60}$, $z = -\frac{53}{60}$

15. $x_4 = 4$

16. $x_4 = -1$

17. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

18. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

19. Carl owns 78 common cards and 39 rare cards.

20. 3.125 gallons.

21. $\frac{20}{7} \approx 2.85$ liters.

22. The rescue houses 15 snakes, 21 tarantulas and 13 scorpions.

23. Using Cramer's Rule, we find we need 53 servings of Ippizuti Fish to satisfy the dietary requirements. The number of servings of Misty Mushrooms required, however, is $-1120$. Since it's impossible to have a negative number of servings, there is no solution to the applied problem, despite there being a solution to the mathematical problem. A cautionary tale about using Cramer's Rule: just because you are guaranteed a mathematical answer for each variable doesn't mean the solution will make sense in the 'real' world.

## 8.6  PARTIAL FRACTION DECOMPOSITION

This section uses systems of linear equations to rewrite rational functions in a form more palatable to Calculus students. In College Algebra, the function

$$f(x) = \frac{x^2 - x - 6}{x^4 + x^2} \tag{1}$$

is written in the best form possible to construct a sign diagram and to find zeros and asymptotes, but certain applications in Calculus require us to rewrite $f(x)$ as

$$f(x) = \frac{x + 7}{x^2 + 1} - \frac{1}{x} - \frac{6}{x^2} \tag{2}$$

If we are given the form of $f(x)$ in (2), it is a matter of Intermediate Algebra to determine a common denominator to obtain the form of $f(x)$ given in (1). The focus of this section is to develop a method by which we start with $f(x)$ in the form of (1) and 'resolve it into **partial fractions**' to obtain the form in (2). Essentially, we need to reverse the least common denominator process. Starting with the form of $f(x)$ in (1), we begin by factoring the denominator

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2 \left(x^2 + 1\right)}$$

We now think about which individual denominators could contribute to obtain $x^2 \left(x^2 + 1\right)$ as the least common denominator. Certainly $x^2$ and $x^2 + 1$, but are there any other factors? Since $x^2 + 1$ is an irreducible quadratic[1] there are no factors of it that have real coefficients which can contribute to the denominator. The factor $x^2$, however, is not irreducible, since we can think of it as $x^2 = xx = (x - 0)(x - 0)$, a so-called 'repeated' linear factor.[2] This means it's possible that a term with a denominator of just $x$ contributed to the expression as well. What about something like $x \left(x^2 + 1\right)$? This, too, could contribute, but we would then wish to break down that denominator into $x$ and $\left(x^2 + 1\right)$, so we leave out a term of that form. At this stage, we have guessed

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2 \left(x^2 + 1\right)} = \frac{?}{x} + \frac{?}{x^2} + \frac{?}{x^2 + 1}$$

Our next task is to determine what form the unknown numerators take. It stands to reason that since the expression $\frac{x^2 - x - 6}{x^4 + x^2}$ is 'proper' in the sense that the degree of the numerator is less than the degree of the denominator, we are safe to make the ansatz that all of the partial fraction resolvents are also. This means that the numerator of the fraction with $x$ as its denominator is just a constant and the numerators on the terms involving the denominators $x^2$ and $x^2 + 1$ are at most linear polynomials. That is, we guess that there are real numbers $A$, $B$, $C$, $D$ and $E$ so that

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2 \left(x^2 + 1\right)} = \frac{A}{x} + \frac{Bx + C}{x^2} + \frac{Dx + E}{x^2 + 1}$$

---

[1] Recall this means it has no real zeros; see Section 3.4.
[2] Recall this means $x = 0$ is a zero of multiplicity 2.

However, if we look more closely at the term $\frac{Bx+C}{x^2}$, we see that $\frac{Bx+C}{x^2} = \frac{Bx}{x^2} + \frac{C}{x^2} = \frac{B}{x} + \frac{C}{x^2}$. The term $\frac{B}{x}$ has the same form as the term $\frac{A}{x}$ which means it contributes nothing new to our expansion. Hence, we drop it and, after re-labeling, we find ourselves with our new guess:

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2 \left(x^2 + 1\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

Our next task is to determine the values of our unknowns. Clearing denominators gives

$$x^2 - x - 6 = Ax\left(x^2 + 1\right) + B\left(x^2 + 1\right) + (Cx + D)x^2$$

Gathering the like powers of $x$ we have

$$x^2 - x - 6 = (A + C)x^3 + (B + D)x^2 + Ax + B$$

In order for this to hold for all values of $x$ in the domain of $f$, we equate the coefficients of corresponding powers of $x$ on each side of the equation[3] and obtain the system of linear equations

$$\begin{cases} (E1) & A + C & = & 0 & \text{From equating coefficients of } x^3 \\ (E2) & B + D & = & 1 & \text{From equating coefficients of } x^2 \\ (E3) & A & = & -1 & \text{From equating coefficients of } x \\ (E4) & B & = & -6 & \text{From equating the constant terms} \end{cases}$$

To solve this system of equations, we could use any of the methods presented in Sections 8.1 through 8.5, but none of these methods are as efficient as the good old-fashioned substitution you learned in Intermediate Algebra. From $E3$, we have $A = -1$ and we substitute this into $E1$ to get $C = 1$. Similarly, since $E4$ gives us $B = -6$, we have from $E2$ that $D = 7$. We get

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2 \left(x^2 + 1\right)} = -\frac{1}{x} - \frac{6}{x^2} + \frac{x + 7}{x^2 + 1}$$

which matches the formula given in (2). As we have seen in this opening example, resolving a rational function into partial fractions takes two steps: first, we need to determine the *form* of the decomposition, and then we need to determine the unknown coefficients which appear in said form. Theorem 3.16 guarantees that any polynomial with real coefficients can be factored over the real numbers as a product of linear factors and irreducible quadratic factors. Once we have this factorization of the denominator of a rational function, the next theorem tells us the form the decomposition takes. The reader is encouraged to review the Factor Theorem (Theorem 3.6) and its connection to the role of multiplicity to fully appreciate the statement of the following theorem.

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[3]We will justify this shortly.

**Theorem 8.10.** Suppose $R(x) = \dfrac{N(x)}{D(x)}$ is a rational function where the degree of $N(x)$ less than the degree of $D(x)$ [a] and $N(x)$ and $D(x)$ have no common factors.

- If $c$ is a real zero of $D$ of multiplicity $m$ which corresponds to the linear factor $ax + b$, the partial fraction decomposition includes

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \ldots + \frac{A_m}{(ax + b)^m}$$

  for real numbers $A_1$, $A_2$, $\ldots A_m$.

- If $c$ is a non-real zero of $D$ of multiplicity $m$ which corresponds to the irreducible quadratic $ax^2 + bx + c$, the partial fraction decomposition includes

$$\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \ldots + \frac{B_m x + C_m}{(ax^2 + bx + c)^m}$$

  for real numbers $B_1$, $B_2$, $\ldots B_m$ and $C_1$, $C_2$, $\ldots C_m$.

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[a] In other words, $R(x)$ is a proper rational function.

The proof of Theorem 8.10 is best left to a course in Abstract Algebra. Notice that the theorem provides for the general case, so we need to use subscripts, $A_1$, $A_2$, etc., to denote different unknown coefficients as opposed to the usual convention of $A$, $B$, etc.. The stress on multiplicities is to help us correctly group factors in the denominator. For example, consider the rational function

$$\frac{3x - 1}{(x^2 - 1)(2 - x - x^2)}$$

Factoring the denominator to find the zeros, we get $(x + 1)(x - 1)(1 - x)(2 + x)$. We find $x = -1$ and $x = -2$ are zeros of multiplicity one but that $x = 1$ is a zero of multiplicity two due to the two different factors $(x - 1)$ and $(1 - x)$. One way to handle this is to note that $(1 - x) = -(x - 1)$ so

$$\frac{3x - 1}{(x + 1)(x - 1)(1 - x)(2 + x)} = \frac{3x - 1}{-(x - 1)^2(x + 1)(x + 2)} = \frac{1 - 3x}{(x - 1)^2(x + 1)(x + 2)}$$

from which we proceed with the partial fraction decomposition

$$\frac{1 - 3x}{(x - 1)^2(x + 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1} + \frac{D}{x + 2}$$

Turning our attention to non-real zeros, we note that the tool of choice to determine the irreducibility of a quadratic $ax^2 + bx + c$ is the discriminant, $b^2 - 4ac$. If $b^2 - 4ac < 0$, the quadratic admits a *pair* of non-real complex conjugate zeros. Even though *one* irreducible quadratic gives *two* distinct non-real zeros, we list the terms with denominators involving a given irreducible quadratic only once to avoid duplication in the form of the decomposition. The trick, of course, is factoring the

denominator or otherwise finding the zeros and their multiplicities in order to apply Theorem 8.10. We recommend that the reader review the techniques set forth in Sections 3.3 and 3.4. Next, we state a theorem that if two polynomials are equal, the corresponding coefficients of the like powers of $x$ are equal. This is the principal by which we shall determine the unknown coefficients in our partial fraction decomposition.

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**Theorem 8.11.** Suppose

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = b_m x^m + m_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

for all $x$ in an open interval $I$. Then $n = m$ and $a_i = b_i$ for all $i = 1 \ldots n$.

---

Believe it or not, the proof of Theorem 8.11 is a consequence of Theorem 3.14. Define $p(x)$ to be the difference of the left hand side of the equation in Theorem 8.11 and the right hand side. Then $p(x) = 0$ for all $x$ in the open interval $I$. If $p(x)$ were a nonzero polynomial of degree $k$, then, by Theorem 3.14, $p$ could have at most $k$ zeros in $I$, and $k$ is a finite number. Since $p(x) = 0$ for all the $x$ in $I$, $p$ has infinitely many zeros, and hence, $p$ is the zero polynomial. This means there can be no nonzero terms in $p(x)$ and the theorem follows. Arguably, the best way to make sense of either of the two preceding theorems is to work some examples.

**Example 8.6.1.** Resolve the following rational functions into partial fractions.

1. $R(x) = \dfrac{x + 5}{2x^2 - x - 1}$
2. $R(x) = \dfrac{3}{x^3 - 2x^2 + x}$
3. $R(x) = \dfrac{3}{x^3 - x^2 + x}$

4. $R(x) = \dfrac{4x^3}{x^2 - 2}$
5. $R(x) = \dfrac{x^3 + 5x - 1}{x^4 + 6x^2 + 9}$
6. $R(x) = \dfrac{8x^2}{x^4 + 16}$

**Solution.**

1. We begin by factoring the denominator to find $2x^2 - x - 1 = (2x + 1)(x - 1)$. We get $x = -\frac{1}{2}$ and $x = 1$ are both zeros of multiplicity one and thus we know

$$\frac{x + 5}{2x^2 - x - 1} = \frac{x + 5}{(2x + 1)(x - 1)} = \frac{A}{2x + 1} + \frac{B}{x - 1}$$

   Clearing denominators, we get $x + 5 = A(x - 1) + B(2x + 1)$ so that $x + 5 = (A + 2B)x + B - A$. Equating coefficients, we get the system

$$\left\{ \begin{array}{rcl} A + 2B & = & 1 \\ -A + B & = & 5 \end{array} \right.$$

   This system is readily handled using the Addition Method from Section 8.1, and after adding both equations, we get $3B = 6$ so $B = 2$. Using back substitution, we find $A = -3$. Our answer is easily checked by getting a common denominator and adding the fractions.

$$\frac{x + 5}{2x^2 - x - 1} = \frac{2}{x - 1} - \frac{3}{2x + 1}$$

2. Factoring the denominator gives $x^3 - 2x^2 + x = x\left(x^2 - 2x + 1\right) = x(x-1)^2$ which gives $x = 0$ as a zero of multiplicity one and $x = 1$ as a zero of multiplicity two. We have

$$\frac{3}{x^3 - 2x^2 + x} = \frac{3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Clearing denominators, we get $3 = A(x-1)^2 + Bx(x-1) + Cx$, which, after gathering up the like terms becomes $3 = (A+B)x^2 + (-2A - B + C)x + A$. Our system is

$$\begin{cases} A + B & = & 0 \\ -2A - B + C & = & 0 \\ A & = & 3 \end{cases}$$

Substituting $A = 3$ into $A + B = 0$ gives $B = -3$, and substituting both for $A$ and $B$ in $-2A - B + C = 0$ gives $C = 3$. Our final answer is

$$\frac{3}{x^3 - 2x^2 + x} = \frac{3}{x} - \frac{3}{x-1} + \frac{3}{(x-1)^2}$$

3. The denominator factors as $x\left(x^2 - x + 1\right)$. We see immediately that $x = 0$ is a zero of multiplicity one, but the zeros of $x^2 - x + 1$ aren't as easy to discern. The quadratic doesn't factor easily, so we check the discriminant and find it to be $(-1)^2 - 4(1)(1) = -3 < 0$. We find its zeros are not real so it is an irreducible quadratic. The form of the partial fraction decomposition is then

$$\frac{3}{x^3 - x^2 + x} = \frac{3}{x\left(x^2 - x + 1\right)} = \frac{A}{x} + \frac{Bx + C}{x^2 - x + 1}$$

Proceeding as usual, we clear denominators and get $3 = A\left(x^2 - x + 1\right) + (Bx + C)x$ or $3 = (A+B)x^2 + (-A + C)x + A$. We get

$$\begin{cases} A + B & = & 0 \\ -A + C & = & 0 \\ A & = & 3 \end{cases}$$

From $A = 3$ and $A + B = 0$, we get $B = -3$. From $-A + C = 0$, we get $C = A = 3$. We get

$$\frac{3}{x^3 - x^2 + x} = \frac{3}{x} + \frac{3 - 3x}{x^2 - x + 1}$$

4. Since $\frac{4x^3}{x^2 - 2}$ isn't proper, we use long division and we get a quotient of $4x$ with a remainder of $8x$. That is, $\frac{4x^3}{x^2 - 2} = 4x + \frac{8x}{x^2 - 2}$ so we now work on resolving $\frac{8x}{x^2 - 2}$ into partial fractions. The quadratic $x^2 - 2$, though it doesn't factor nicely, is, nevertheless, reducible. Solving $x^2 - 2 = 0$

gives us $x = \pm\sqrt{2}$, and each of these zeros must be of multiplicity one since Theorem 3.14 enables us to now factor $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$. Hence,

$$\frac{8x}{x^2 - 2} = \frac{8x}{(x - \sqrt{2})(x + \sqrt{2})} = \frac{A}{x - \sqrt{2}} + \frac{B}{x + \sqrt{2}}$$

Clearing fractions, we get $8x = A(x + \sqrt{2}) + B(x - \sqrt{2})$ or $8x = (A + B)x + (A - B)\sqrt{2}$. We get the system

$$\begin{cases} A + B = 8 \\ (A - B)\sqrt{2} = 0 \end{cases}$$

From $(A - B)\sqrt{2} = 0$, we get $A = B$, which, when substituted into $A + B = 8$ gives $B = 4$. Hence, $A = B = 4$ and we get

$$\frac{4x^3}{x^2 - 2} = 4x + \frac{8x}{x^2 - 2} = 4x + \frac{4}{x + \sqrt{2}} + \frac{4}{x - \sqrt{2}}$$

5. At first glance, the denominator $D(x) = x^4 + 6x^2 + 9$ appears irreducible. However, $D(x)$ has three terms, and the exponent on the first term is exactly twice that of the second. Rewriting $D(x) = (x^2)^2 + 6x^2 + 9$, we see it is a quadratic in disguise and factor $D(x) = (x^2 + 3)^2$. Since $x^2 + 3$ clearly has no real zeros, it is irreducible and the form of the decomposition is

$$\frac{x^3 + 5x - 1}{x^4 + 6x^2 + 9} = \frac{x^3 + 5x - 1}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$

When we clear denominators, we find $x^3 + 5x - 1 = (Ax + B)(x^2 + 3) + Cx + D$ which yields $x^3 + 5x - 1 = Ax^3 + Bx^2 + (3A + C)x + 3B + D$. Our system is

$$\begin{cases} A = 1 \\ B = 0 \\ 3A + C = 5 \\ 3B + D = -1 \end{cases}$$

We have $A = 1$ and $B = 0$ from which we get $C = 2$ and $D = -1$. Our final answer is

$$\frac{x^3 + 5x - 1}{x^4 + 6x^2 + 9} = \frac{x}{x^2 + 3} + \frac{2x - 1}{(x^2 + 3)^2}$$

6. Once again, the difficulty in our last example is factoring the denominator. In an attempt to get a quadratic in disguise, we write

$$x^4 + 16 = (x^2)^2 + 4^2 = (x^2)^2 + 8x^2 + 4^2 - 8x^2 = (x^2 + 4)^2 - 8x^2$$

and obtain a difference of two squares: $\left(x^2 + 4\right)^2$ and $8x^2 = \left(2x\sqrt{2}\right)^2$. Hence,

$$x^4 + 16 = \left(x^2 + 4 - 2x\sqrt{2}\right)\left(x^2 + 4 + 2x\sqrt{2}\right) = \left(x^2 - 2x\sqrt{2} + 4\right)\left(x^2 + 2x\sqrt{2} + 4\right)$$

The discrimant of both of these quadratics works out to be $-8 < 0$, which means they are irreducible. We leave it to the reader to verify that, despite having the same discriminant, these quadratics have different zeros. The partial fraction decomposition takes the form

$$\frac{8x^2}{x^4 + 16} = \frac{8x^2}{\left(x^2 - 2x\sqrt{2} + 4\right)\left(x^2 + 2x\sqrt{2} + 4\right)} = \frac{Ax + B}{x^2 - 2x\sqrt{2} + 4} + \frac{Cx + D}{x^2 + 2x\sqrt{2} + 4}$$

We get $8x^2 = (Ax + B)\left(x^2 + 2x\sqrt{2} + 4\right) + (Cx + D)\left(x^2 - 2x\sqrt{2} + 4\right)$ or

$$8x^2 = (A + C)x^3 + (2A\sqrt{2} + B - 2C\sqrt{2} + D)x^2 + (4A + 2B\sqrt{2} + 4C - 2D\sqrt{2})x + 4B + 4D$$

which gives the system

$$\begin{cases} A + C & = & 0 \\ 2A\sqrt{2} + B - 2C\sqrt{2} + D & = & 8 \\ 4A + 2B\sqrt{2} + 4C - 2D\sqrt{2} & = & 0 \\ 4B + 4D & = & 0 \end{cases}$$

We choose substitution as the weapon of choice to solve this system. From $A + C = 0$, we get $A = -C$; from $4B + 4D = 0$, we get $B = -D$. Substituting these into the remaining two equations, we get

$$\begin{cases} -2C\sqrt{2} - D - 2C\sqrt{2} + D & = & 8 \\ -4C - 2D\sqrt{2} + 4C - 2D\sqrt{2} & = & 0 \end{cases}$$

or

$$\begin{cases} -4C\sqrt{2} & = & 8 \\ -4D\sqrt{2} & = & 0 \end{cases}$$

We get $C = -\sqrt{2}$ so that $A = -C = \sqrt{2}$ and $D = 0$ which means $B = -D = 0$. We get

$$\frac{8x^2}{x^4 + 16} = \frac{x\sqrt{2}}{x^2 - 2x\sqrt{2} + 4} - \frac{x\sqrt{2}}{x^2 + 2x\sqrt{2} + 4}$$

$\square$

## 8.6.1   Exercises

In Exercises 1 - 6, find only the *form* needed to begin the process of partial fraction decomposition. Do not create the system of linear equations or attempt to find the actual decomposition.

1. $\dfrac{7}{(x-3)(x+5)}$

2. $\dfrac{5x+4}{x(x-2)(2-x)}$

3. $\dfrac{m}{(7x-6)(x^2+9)}$

4. $\dfrac{ax^2+bx+c}{x^3(5x+9)(3x^2+7x+9)}$

5. $\dfrac{\text{A polynomial of degree } < 9}{(x+4)^5(x^2+1)^2}$

6. $\dfrac{\text{A polynomial of degree } < 7}{x(4x-1)^2(x^2+5)(9x^2+16)}$

In Exercises 7 - 18, find the partial fraction decomposition of the following rational expressions.

7. $\dfrac{2x}{x^2-1}$

8. $\dfrac{-7x+43}{3x^2+19x-14}$

9. $\dfrac{11x^2-5x-10}{5x^3-5x^2}$

10. $\dfrac{-2x^2+20x-68}{x^3+4x^2+4x+16}$

11. $\dfrac{-x^2+15}{4x^4+40x^2+36}$

12. $\dfrac{-21x^2+x-16}{3x^3+4x^2-3x+2}$

13. $\dfrac{5x^4-34x^3+70x^2-33x-19}{(x-3)^2}$

14. $\dfrac{x^6+5x^5+16x^4+80x^3-2x^2+6x-43}{x^3+5x^2+16x+80}$

15. $\dfrac{-7x^2-76x-208}{x^3+18x^2+108x+216}$

16. $\dfrac{-10x^4+x^3-19x^2+x-10}{x^5+2x^3+x}$

17. $\dfrac{4x^3-9x^2+12x+12}{x^4-4x^3+8x^2-16x+16}$

18. $\dfrac{2x^2+3x+14}{(x^2+2x+9)(x^2+x+5)}$

19. As we stated at the beginning of this section, the technique of resolving a rational function into partial fractions is a skill needed for Calculus. However, we hope to have shown you that it is worth doing if, for no other reason, it reinforces a hefty amount of algebra. One of the common algebraic errors the authors find students make is something along the lines of

$$\frac{8}{x^2-9} \neq \frac{8}{x^2} - \frac{8}{9}$$

Think about why if the above were true, this section would have no need to exist.

## 8.6.2 ANSWERS

1. $\dfrac{A}{x-3} + \dfrac{B}{x+5}$

2. $\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

3. $\dfrac{A}{7x-6} + \dfrac{Bx+C}{x^2+9}$

4. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{5x+9} + \dfrac{Ex+F}{3x^2+7x+9}$

5. $\dfrac{A}{x+4} + \dfrac{B}{(x+4)^2} + \dfrac{C}{(x+4)^3} + \dfrac{D}{(x+4)^4} + \dfrac{E}{(x+4)^5} + \dfrac{Fx+G}{x^2+1} + \dfrac{Hx+I}{(x^2+1)^2}$

6. $\dfrac{A}{x} + \dfrac{B}{4x-1} + \dfrac{C}{(4x-1)^2} + \dfrac{Dx+E}{x^2+5} + \dfrac{Fx+G}{9x^2+16}$

7. $\dfrac{2x}{x^2-1} = \dfrac{1}{x+1} + \dfrac{1}{x-1}$

8. $\dfrac{-7x+43}{3x^2+19x-14} = \dfrac{5}{3x-2} - \dfrac{4}{x+7}$

9. $\dfrac{11x^2-5x-10}{5x^3-5x^2} = \dfrac{3}{x} + \dfrac{2}{x^2} - \dfrac{4}{5(x-1)}$

10. $\dfrac{-2x^2+20x-68}{x^3+4x^2+4x+16} = -\dfrac{9}{x+4} + \dfrac{7x-8}{x^2+4}$

11. $\dfrac{-x^2+15}{4x^4+40x^2+36} = \dfrac{1}{2(x^2+1)} - \dfrac{3}{4(x^2+9)}$

12. $\dfrac{-21x^2+x-16}{3x^3+4x^2-3x+2} = -\dfrac{6}{x+2} - \dfrac{3x+5}{3x^2-2x+1}$

13. $\dfrac{5x^4-34x^3+70x^2-33x-19}{(x-3)^2} = 5x^2 - 4x + 1 + \dfrac{9}{x-3} - \dfrac{1}{(x-3)^2}$

14. $\dfrac{x^6+5x^5+16x^4+80x^3-2x^2+6x-43}{x^3+5x^2+16x+80} = x^3 + \dfrac{x+1}{x^2+16} - \dfrac{3}{x+5}$

15. $\dfrac{-7x^2-76x-208}{x^3+18x^2+108x+216} = -\dfrac{7}{x+6} + \dfrac{8}{(x+6)^2} - \dfrac{4}{(x+6)^3}$

16. $\dfrac{-10x^4+x^3-19x^2+x-10}{x^5+2x^3+x} = -\dfrac{10}{x} + \dfrac{1}{x^2+1} + \dfrac{x}{(x^2+1)^2}$

17. $\dfrac{4x^3-9x^2+12x+12}{x^4-4x^3+8x^2-16x+16} = \dfrac{1}{x-2} + \dfrac{4}{(x-2)^2} + \dfrac{3x+1}{x^2+4}$

18. $\dfrac{2x^2+3x+14}{(x^2+2x+9)(x^2+x+5)} = \dfrac{1}{x^2+2x+9} + \dfrac{1}{x^2+x+5}$

## 8.7    Systems of Non-Linear Equations and Inequalities

In this section, we study systems of non-linear equations and inequalities. Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and unusual function domains are once again present. Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far. To get the section rolling we begin with a fairly routine example.

**Example 8.7.1.** Solve the following systems of equations. Verify your answers algebraically and graphically.

1. $\begin{cases} x^2 + y^2 & = & 4 \\ 4x^2 + 9y^2 & = & 36 \end{cases}$

2. $\begin{cases} x^2 + y^2 & = & 4 \\ 4x^2 - 9y^2 & = & 36 \end{cases}$

3. $\begin{cases} x^2 + y^2 & = & 4 \\ y - 2x & = & 0 \end{cases}$

4. $\begin{cases} x^2 + y^2 & = & 4 \\ y - x^2 & = & 0 \end{cases}$

Solution:

1. Since both equations contain $x^2$ and $y^2$ only, we can eliminate one of the variables as we did in Section 8.1.

$$\begin{cases} (E1) & x^2 + y^2 & = & 4 \\ (E2) & 4x^2 + 9y^2 & = & 36 \end{cases} \xrightarrow[\text{$-4E1 + E2$}]{\text{Replace $E2$ with}} \begin{cases} (E1) & x^2 + y^2 & = & 4 \\ (E2) & 5y^2 & = & 20 \end{cases}$$

   From $5y^2 = 20$, we get $y^2 = 4$ or $y = \pm 2$. To find the associated $x$ values, we substitute each value of $y$ into one of the equations to find the resulting value of $x$. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution is thus $\{(0, 2), (0, -2)\}$. To check this algebraically, we need to show that both points satisfy both of the original equations. We leave it to the reader to verify this. To check our answer graphically, we sketch both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, whereas the graph of $4x^2 + 9y^2 = 36$, when written in the standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is easily recognized as an ellipse centered at $(0, 0)$ with a major axis along the $x$-axis of length 6 and a minor axis along the $y$-axis of length 4. We see from the graph that the two curves intersect at their $y$-intercepts only, $(0, \pm 2)$.

2. We proceed as before to eliminate one of the variables

$$\begin{cases} (E1) & x^2 + y^2 & = & 4 \\ (E2) & 4x^2 - 9y^2 & = & 36 \end{cases} \xrightarrow[\text{$-4E1 + E2$}]{\text{Replace $E2$ with}} \begin{cases} (E1) & x^2 + y^2 & = & 4 \\ (E2) & -13y^2 & = & 20 \end{cases}$$

Since the equation $-13y^2 = 20$ admits no real solution, the system is inconsistent. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we find a hyperbola centered at $(0,0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4. We see that the circle and the hyperbola have no points in common.



$$\text{Graphs for} \quad \left\{ \begin{array}{rcl} x^2 + y^2 & = & 4 \\ 4x^2 + 9y^2 & = & 36 \end{array} \right. \qquad\qquad \text{Graphs for} \quad \left\{ \begin{array}{rcl} x^2 + y^2 & = & 4 \\ 4x^2 - 9y^2 & = & 36 \end{array} \right.$$
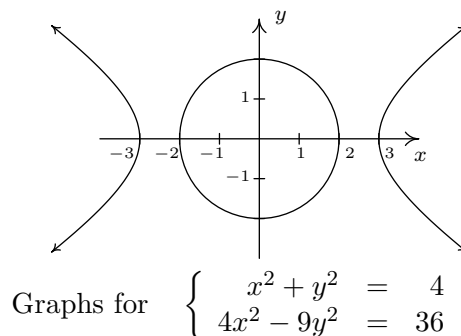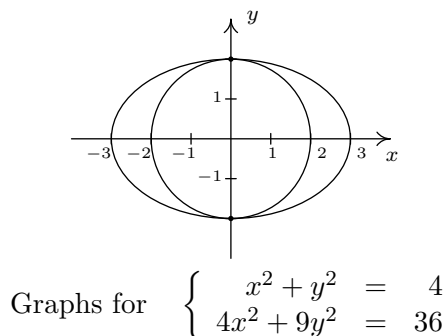
3. Since there are no like terms among the two equations, elimination won't do us any good. We turn to substitution and from the equation $y - 2x = 0$, we get $y = 2x$. Substituting this into $x^2 + y^2 = 4$ gives $x^2 + (2x)^2 = 4$. Solving, we find $5x^2 = 4$ or $x = \pm\frac{2\sqrt{5}}{5}$. Returning to the equation we used for the substitution, $y = 2x$, we find $y = \frac{4\sqrt{5}}{5}$ when $x = \frac{2\sqrt{5}}{5}$, so one solution is $\left( \frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5} \right)$. Similarly, we find the other solution to be $\left( -\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5} \right)$. We leave it to the reader that both points satisfy both equations, so that our final answer is $\left\{ \left( \frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5} \right), \left( -\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5} \right) \right\}$. The graph of $x^2 + y^2 = 4$ is our circle from before and the graph of $y - 2x = 0$ is a line through the origin with slope 2. Though we cannot verify the numerical values of the points of intersection from our sketch, we do see that we have two solutions: one in Quadrant I and one in Quadrant III as required.

4. While it may be tempting to solve $y - x^2 = 0$ as $y = x^2$ and substitute, we note that this system is set up for elimination.[1]

$$\left\{ \begin{array}{rrcl} (E1) & x^2 + y^2 & = & 4 \\ (E2) & y - x^2 & = & 0 \end{array} \right. \quad \xrightarrow[\;\; E1 + E2 \;\;]{\text{Replace } E2 \text{ with}} \quad \left\{ \begin{array}{rrcl} (E1) & x^2 + y^2 & = & 4 \\ (E2) & y^2 + y & = & 4 \end{array} \right.$$

From $y^2 + y = 4$ we get $y^2 + y - 4 = 0$ which gives $y = \frac{-1 \pm \sqrt{17}}{2}$. Due to the complicated nature of these answers, it is worth our time to make a quick sketch of both equations to head off any extraneous solutions we may encounter. We see that the circle $x^2 + y^2 = 4$ intersects the parabola $y = x^2$ exactly twice, and both of these points have a positive $y$ value. Of the two solutions for $y$, only $y = \frac{-1 + \sqrt{17}}{2}$ is positive, so to get our solution, we substitute this

---

[1] We encourage the reader to solve the system using substitution to see that you get the same solution.

into $y - x^2 = 0$ and solve for $x$. We get $x = \pm\sqrt{\frac{-1+\sqrt{17}}{2}} = \pm\frac{\sqrt{-2+2\sqrt{17}}}{2}$. Our solution is $\left\{\left(\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right), \left(-\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right)\right\}$, which we leave to the reader to verify.
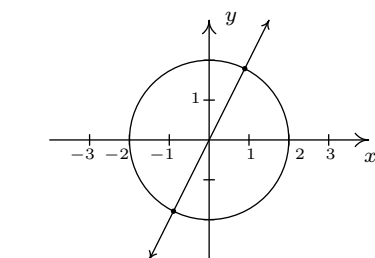
Graphs for $\begin{cases} x^2 + y^2 &= 4 \\ y - 2x &= 0 \end{cases}$
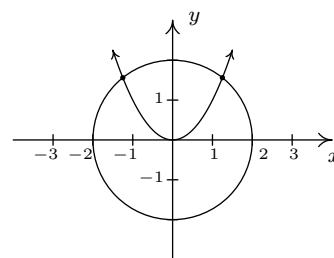
Graphs for $\begin{cases} x^2 + y^2 &= 4 \\ y - x^2 &= 36 \end{cases}$

□

A couple of remarks about Example 8.7.1 are in order. First note that, unlike systems of linear equations, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. In fact, while we characterize systems of nonlinear equations as being 'consistent' or 'inconsistent,' we generally don't use the labels 'dependent' or 'independent'. Secondly, as we saw with number 4, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the curves in a system of non-linear equations may not be easily visualized, it sometimes pays to take advantage when they are. Our next example provides some considerable review of many of the topics introduced in this text.

**Example 8.7.2.** Solve the following systems of equations. Verify your answers algebraically and graphically, as appropriate.

1. $\begin{cases} x^2 + 2xy - 16 &= 0 \\ y^2 + 2xy - 16 &= 0 \end{cases}$  2. $\begin{cases} y + 4e^{2x} &= 1 \\ y^2 + 2e^x &= 1 \end{cases}$  3. $\begin{cases} z(x - 2) &= x \\ yz &= y \\ (x - 2)^2 + y^2 &= 1 \end{cases}$

**Solution.**

1. At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, solving one of the equations for one variable and substituting it into the other, is full of unpleasantness. Returning to elimination, we note that it is possible to eliminate the troublesome $xy$ term, and the constant term as well, by elimination and doing so we get a more tractable relationship between $x$ and $y$

$$\begin{cases} (E1) & x^2 + 2xy - 16 &= 0 \\ (E2) & y^2 + 2xy - 16 &= 0 \end{cases} \xrightarrow{\substack{\text{Replace } E2 \text{ with} \\ -E1 + E2}} \begin{cases} (E1) & x^2 + 2xy - 16 &= 0 \\ (E2) & y^2 - x^2 &= 0 \end{cases}$$

We get $y^2 - x^2 = 0$ or $y = \pm x$. Substituting $y = x$ into $E1$ we get $x^2 + 2x^2 - 16 = 0$ so that $x^2 = \frac{16}{3}$ or $x = \pm\frac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into $E1$, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$ which gives no real solutions. Substituting each of $x = \pm\frac{4\sqrt{3}}{3}$ into the substitution equation $y = x$ yields the solution $\left\{ \left( \frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3} \right), \left( -\frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3} \right) \right\}$. We leave it to the reader to show that both points satisfy both equations and now turn to verifying our solution graphically. We begin by solving $x^2 + 2xy - 16 = 0$ for $y$ to obtain $y = \frac{16 - x^2}{2x}$. This function is easily graphed using the techniques of Section 4.2. Solving the second equation, $y^2 + 2xy - 16 = 0$, for $y$, however, is more complicated. We use the quadratic formula to obtain $y = -x \pm \sqrt{x^2 + 16}$ which would require the use of Calculus or a calculator to graph. Believe it or not, we don't need either because the equation $y^2 + 2xy - 16 = 0$ can be obtained from the equation $x^2 + 2xy - 16 = 0$ by interchanging $y$ and $x$. Thinking back to Section 5.2, this means we can obtain the graph of $y^2 + 2xy - 16 = 0$ by reflecting the graph of $x^2 + 2xy - 16 = 0$ across the line $y = x$. Doing so confirms that the two graphs intersect twice: once in Quadrant I, and once in Quadrant III as required.
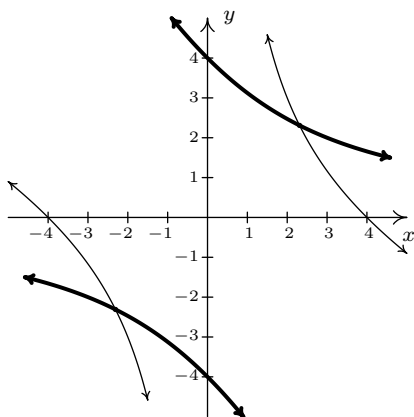


The graphs of $x^2 + 2xy - 16 = 0$ and $\mathbf{y^2 + 2xy - 16 = 0}$

2. Unlike the previous problem, there seems to be no avoiding substitution and a bit of algebraic unpleasantness. Solving $y + 4e^{2x} = 1$ for $y$, we get $y = 1 - 4e^{2x}$ which, when substituted into the second equation, yields $\left( 1 - 4e^{2x} \right)^2 + 2e^x = 1$. After expanding and gathering like terms, we get $16e^{4x} - 8e^{2x} + 2e^x = 0$. Factoring gives us $2e^x \left( 8e^{3x} - 4e^x + 1 \right) = 0$, and since $2e^x \neq 0$ for any real $x$, we are left with solving $8e^{3x} - 4e^x + 1 = 0$. We have three terms, and even though this is not a 'quadratic in disguise', we can benefit from the substitution $u = e^x$. The equation becomes $8u^3 - 4u + 1 = 0$. Using the techniques set forth in Section 3.3, we find $u = \frac{1}{2}$ is a zero and use synthetic division to factor the left hand side as $\left( u - \frac{1}{2} \right) \left( 8u^2 + 4u - 2 \right)$. We use the quadratic formula to solve $8u^2 + 4u - 2 = 0$ and find $u = \frac{-1 \pm \sqrt{5}}{4}$. Since $u = e^x$, we now must solve $e^x = \frac{1}{2}$ and $e^x = \frac{-1 \pm \sqrt{5}}{4}$. From $e^x = \frac{1}{2}$, we get $x = \ln\left( \frac{1}{2} \right) = -\ln(2)$. As for $e^x = \frac{-1 \pm \sqrt{5}}{4}$, we first note that $\frac{-1 - \sqrt{5}}{4} < 0$, so $e^x = \frac{-1 - \sqrt{5}}{4}$ has no real solutions. We are

left with $e^x = \frac{-1+\sqrt{5}}{4}$, so that $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$. We now return to $y = 1 - 4e^{2x}$ to find the accompanying $y$ values for each of our solutions for $x$. For $x = -\ln(2)$, we get

$$
\begin{aligned}
y &= 1 - 4e^{2x} \\
&= 1 - 4e^{-2\ln(2)} \\
&= 1 - 4e^{\ln\left(\frac{1}{4}\right)} \\
&= 1 - 4\left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

For $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$, we have

$$
\begin{aligned}
y &= 1 - 4e^{2x} \\
&= 1 - 4e^{2\ln\left(\frac{-1+\sqrt{5}}{4}\right)} \\
&= 1 - 4e^{\ln\left(\frac{-1+\sqrt{5}}{4}\right)^2} \\
&= 1 - 4\left(\frac{-1+\sqrt{5}}{4}\right)^2 \\
&= 1 - 4\left(\frac{3-\sqrt{5}}{8}\right) \\
&= \frac{-1+\sqrt{5}}{2}
\end{aligned}
$$

We get two solutions, $\left\{(0, -\ln(2)), \left(\ln\left(\frac{-1+\sqrt{5}}{4}\right), \frac{-1+\sqrt{5}}{2}\right)\right\}$. It is a good review of the properties of logarithms to verify both solutions, so we leave that to the reader. We are able to sketch $y = 1 - 4e^{2x}$ using transformations, but the second equation is more difficult and we resort to the calculator. We note that to graph $y^2 + 2e^x = 1$, we need to graph both the positive and negative roots, $y = \pm\sqrt{1 - 2e^x}$. After some careful zooming,[2] we get
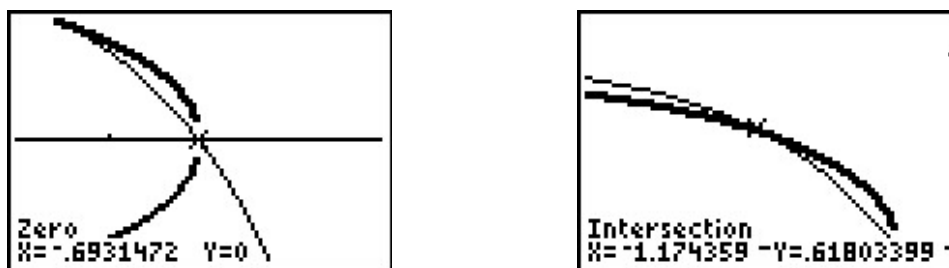


The graphs of $y = 1 - 4e^{2x}$ and $\boldsymbol{y = \pm\sqrt{1 - 2e^x}}$.

3. Our last system involves three variables and gives some insight on how to keep such systems organized. Labeling the equations as before, we have

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[2]The calculator has trouble confirming the solution $(-\ln(2), 0)$ due to its issues in graphing square root functions. If we mentally connect the two branches of the thicker curve, we see the intersection.

$$\begin{cases} E1 & z(x-2) & = & x \\ E2 & yz & = & y \\ E3 & (x-2)^2 + y^2 & = & 1 \end{cases}$$

The easiest equation to start with appears to be $E2$. While it may be tempting to divide both sides of $E2$ by $y$, we caution against this practice because it presupposes $y \neq 0$. Instead, we take $E2$ and rewrite it as $yz - y = 0$ so $y(z-1) = 0$. From this, we get two cases: $y = 0$ or $z = 1$. We take each case in turn.

CASE 1: $y = 0$. Substituting $y = 0$ into $E1$ and $E3$, we get

$$\begin{cases} E1 & z(x-2) & = & x \\ E3 & (x-2)^2 & = & 1 \end{cases}$$

Solving $E3$ for $x$ gives $x = 1$ or $x = 3$. Substituting these values into $E1$ gives $z = -1$ when $x = 1$ and $z = 3$ when $x = 3$. We obtain two solutions, $(1, 0, -1)$ and $(3, 0, 3)$.

CASE 2: $z = 1$. Substituting $z = 1$ into $E1$ and $E3$ gives us

$$\begin{cases} E1 & (1)(x-2) & = & x \\ E3 & (1-2)^2 + y^2 & = & 1 \end{cases}$$

Equation $E1$ gives us $x - 2 = x$ or $-2 = 0$, which is a contradiction. This means we have no solution to the system in this case, even though $E3$ is solvable and gives $y = 0$. Hence, our final answer is $\{(1, 0, -1), (3, 0, 3)\}$. These points are easy enough to check algebraically in our three original equations, so that is left to the reader. As for verifying these solutions graphically, they require plotting surfaces in three dimensions and looking for intersection points. While this is beyond the scope of this book, we provide a snapshot of the graphs of our three equations near one of the solution points, $(1, 0, -1)$.



$\square$
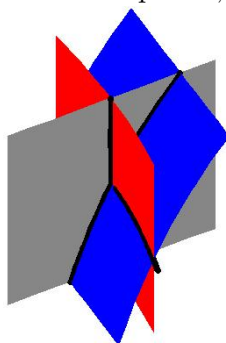
Example 8.7.2 showcases some of the ingenuity and tenacity mentioned at the beginning of the section. Sometimes you just have to look at a system the right way to find the most efficient method to solve it. Sometimes you just have to try something.

We close this section discussing how non-linear inequalities can be used to describe regions in the plane which we first introduced in Section 2.4. Before we embark on some examples, a little motivation is in order. Suppose we wish to solve $x^2 < 4 - y^2$. If we mimic the algorithms for solving nonlinear inequalities in one variable, we would gather all of the terms on one side and leave a 0 on the other to obtain $x^2 + y^2 - 4 < 0$. Then we would find the zeros of the left hand side, that is, where is $x^2 + y^2 - 4 = 0$, or $x^2 + y^2 = 4$. Instead of obtaining a few *numbers* which divide the real number *line* into *intervals*, we get an equation of a *curve*, in this case, a circle, which divides the *plane* into two *regions* - the 'inside' and 'outside' of the circle - with the circle itself as the boundary between the two. Just like we used test *values* to determine whether or not an interval belongs to the solution of the inequality, we use test *points* in the each of the regions to see which of these belong to our solution set.[3] We choose $(0,0)$ to represent the region inside the circle and $(0,3)$ to represent the points outside of the circle. When we substitute $(0,0)$ into $x^2 + y^2 - 4 < 0$, we get $-4 < 4$ which is true. This means $(0,0)$ and all the other points inside the circle are part of the solution. On the other hand, when we substitute $(0,3)$ into the same inequality, we get $5 < 0$ which is false. This means $(0,3)$ along with all other points outside the circle are not part of the solution. What about points on the circle itself? Choosing a point on the circle, say $(0,2)$, we get $0 < 0$, which means the circle itself does not satisfy the inequality.[4] As a result, we leave the circle dashed in the final diagram.
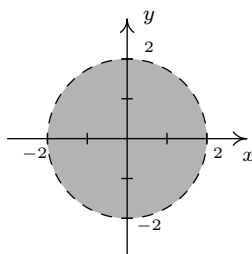


The solution to $x^2 < 4 - y^2$

We put this technique to good use in the following example.

**Example 8.7.3.** Sketch the solution to the following nonlinear inequalities in the plane.

1. $y^2 - 4 \leq x < y + 2$

2. $\begin{cases} x^2 + y^2 & \geq & 4 \\ x^2 - 2x + y^2 - 2y & \leq & 0 \end{cases}$

**Solution.**

1. The inequality $y^2 - 4 \leq x < y + 2$ is a compound inequality. It translates as $y^2 - 4 \leq x$ and $x < y + 2$. As usual, we solve each inequality and take the set theoretic intersection to determine the region which satisfies both inequalities. To solve $y^2 - 4 \leq x$, we write

---

[3]The theory behind why all this works is, surprisingly, the same theory which guarantees that sign diagrams work the way they do - continuity and the Intermediate Value Theorem - but in this case, applied to functions of more than one variable.

[4]Another way to see this is that points on the circle satisfy $x^2 + y^2 - 4 = 0$, so they do not satisfy $x^2 + y^2 - 4 < 0$.

$y^2 - x - 4 \le 0$. The curve $y^2 - x - 4 = 0$ describes a parabola since exactly one of the variables is squared. Rewriting this in standard form, we get $y^2 = x + 4$ and we see that the vertex is $(-4, 0)$ and the parabola opens to the right. Using the test points $(-5, 0)$ and $(0, 0)$, we find that the solution to the inequality includes the region to the right of, or 'inside', the parabola. The points on the parabola itself are also part of the solution, since the vertex $(-4, 0)$ satisfies the inequality. We now turn our attention to $x < y + 2$. Proceeding as before, we write $x - y - 2 < 0$ and focus our attention on $x - y - 2 = 0$, which is the line $y = x - 2$. Using the test points $(0, 0)$ and $(0, -4)$, we find points in the region above the line $y = x - 2$ satisfy the inequality. The points on the line $y = x - 2$ do not satisfy the inequality, since the $y$-intercept $(0, -2)$ does not. We see that these two regions do overlap, and to make the graph more precise, we seek the intersection of these two curves. That is, we need to solve the system of nonlinear equations

$$\begin{cases} (E1) & y^2 & = & x + 4 \\ (E2) & y & = & x - 2 \end{cases}$$

Solving $E1$ for $x$, we get $x = y^2 - 4$. Substituting this into $E2$ gives $y = y^2 - 4 - 2$, or $y^2 - y - 6 = 0$. We find $y = -2$ and $y = 3$ and since $x = y^2 - 4$, we get that the graphs intersect at $(0, -2)$ and $(5, 3)$. Putting all of this together, we get our final answer below.
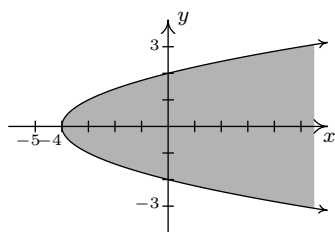


$$y^2 - 4 \le x \qquad\qquad x < y + 2 \qquad\qquad y^2 - 4 \le x < y + 2$$
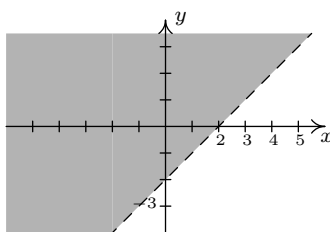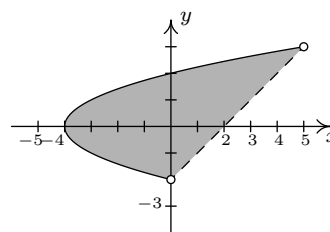
2. To solve this system of inequalities, we need to find all of the points $(x, y)$ which satisfy both inequalities. To do this, we solve each inequality separately and take the set theoretic intersection of the solution sets. We begin with the inequality $x^2 + y^2 \ge 4$ which we rewrite as $x^2 + y^2 - 4 \ge 0$. The points which satisfy $x^2 + y^2 - 4 = 0$ form our friendly circle $x^2 + y^2 = 4$. Using test points $(0, 0)$ and $(0, 3)$ we find that our solution comprises the region outside the circle. As far as the circle itself, the point $(0, 2)$ satisfies the inequality, so the circle itself is part of the solution set. Moving to the inequality $x^2 - 2x + y^2 - 2y \le 0$, we start with $x^2 - 2x + y^2 - 2y = 0$. Completing the squares, we obtain $(x - 1)^2 + (y - 1)^2 = 2$, which is a circle centered at $(1, 1)$ with a radius of $\sqrt{2}$. Choosing $(1, 1)$ to represent the inside of the circle, $(1, 3)$ as a point outside of the circle and $(0, 0)$ as a point on the circle, we find that the solution to the inequality is the inside of the circle, including the circle itself. Our final answer, then, consists of the points on or outside of the circle $x^2 + y^2 = 4$ which lie on or

inside the circle $(x-1)^2 + (y-1)^2 = 2$. To produce the most accurate graph, we need to find where these circles intersect. To that end, we solve the system

$$\begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & x^2 - 2x + y^2 - 2y = 0 \end{cases}$$

We can eliminate both the $x^2$ and $y^2$ by replacing $E2$ with $-E1 + E2$. Doing so produces $-2x - 2y = -4$. Solving this for $y$, we get $y = 2 - x$. Substituting this into $E1$ gives $x^2 + (2-x)^2 = 4$ which simplifies to $x^2 + 4 - 4x + x^2 = 4$ or $2x^2 - 4x = 0$. Factoring yields $2x(x-2)$ which gives $x = 0$ or $x = 2$. Substituting these values into $y = 2 - x$ gives the points $(0, 2)$ and $(2, 0)$. The intermediate graphs and final solution are below.
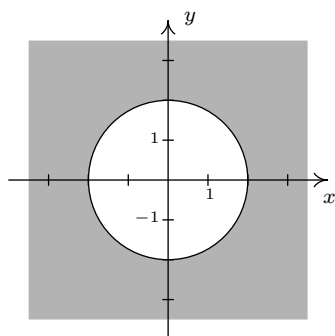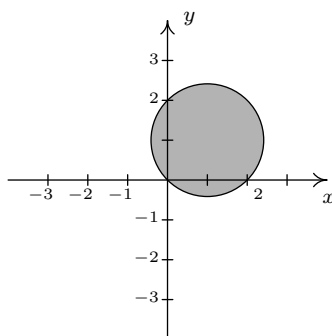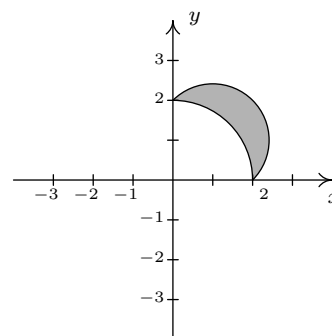


$$x^2 + y^2 \geq 4 \qquad\qquad x^2 - 2x + y^2 - 2y \leq 0 \qquad\qquad \text{Solution to the system.}$$

□

### 8.7.1    EXERCISES

In Exercises 1 - 6, solve the given system of nonlinear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

1. $\begin{cases} x^2 - y = 4 \\ x^2 + y^2 = 4 \end{cases}$  2. $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y = 5 \end{cases}$  3. $\begin{cases} x^2 + y^2 = 16 \\ 16x^2 + 4y^2 = 64 \end{cases}$

4. $\begin{cases} x^2 + y^2 = 16 \\ 9x^2 - 16y^2 = 144 \end{cases}$  5. $\begin{cases} x^2 + y^2 = 16 \\ \frac{1}{9}y^2 - \frac{1}{16}x^2 = 1 \end{cases}$  6. $\begin{cases} x^2 + y^2 = 16 \\ x - y = 2 \end{cases}$

In Exercises 9 - 15, solve the given system of nonlinear equations. Use a graph to help you avoid any potential extraneous solutions.

7. $\begin{cases} x^2 - y^2 = 1 \\ x^2 + 4y^2 = 4 \end{cases}$  8. $\begin{cases} \sqrt{x+1} - y = 0 \\ x^2 + 4y^2 = 4 \end{cases}$  9. $\begin{cases} x + 2y^2 = 2 \\ x^2 + 4y^2 = 4 \end{cases}$

10. $\begin{cases} (x - 2)^2 + y^2 = 1 \\ x^2 + 4y^2 = 4 \end{cases}$  11. $\begin{cases} x^2 + y^2 = 25 \\ y - x = 1 \end{cases}$  12. $\begin{cases} x^2 + y^2 = 25 \\ x^2 + (y - 3)^2 = 10 \end{cases}$

13. $\begin{cases} y = x^3 + 8 \\ y = 10x - x^2 \end{cases}$  14. $\begin{cases} x^2 - xy = 8 \\ y^2 - xy = 8 \end{cases}$  15. $\begin{cases} x^2 + y^2 = 25 \\ 4x^2 - 9y = 0 \\ 3y^2 - 16x = 0 \end{cases}$

16. A certain bacteria culture follows the Law of Uninbited Growth, Equation 6.4. After 10 minutes, there are 10,000 bacteria. Five minutes later, there are 14,000 bacteria. How many bacteria were present initially? How long before there are 50,000 bacteria?

Consider the system of nonlinear equations below

$$\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} = 1 \\ \dfrac{3}{x} + \dfrac{2}{y} = -1 \end{cases}$$

If we let $u = \frac{1}{x}$ and $v = \frac{1}{y}$ then the system becomes

$$\begin{cases} 4u + 3v = 1 \\ 3u + 2v = -1 \end{cases}$$

This associated system of linear equations can then be solved using any of the techniques presented earlier in the chapter to find that $u = -5$ and $v = 7$. Thus $x = \frac{1}{u} = -\frac{1}{5}$ and $y = \frac{1}{v} = \frac{1}{7}$.

We say that the original system is **linear in form** because its equations are not linear but a few substitutions reveal a structure that we can treat like a system of linear equations. Each system in Exercises 17 - 19 is linear in form. Make the appropriate substitutions and solve for $x$ and $y$.

17. $\begin{cases} 4x^3 + 3\sqrt{y} & = & 1 \\ 3x^3 + 2\sqrt{y} & = & -1 \end{cases}$  18. $\begin{cases} 4e^x + 3e^{-y} & = & 1 \\ 3e^x + 2e^{-y} & = & -1 \end{cases}$  19. $\begin{cases} 4\ln(x) + 3y^2 & = & 1 \\ 3\ln(x) + 2y^2 & = & -1 \end{cases}$

20. Solve the following system

$$\begin{cases} x^2 + \sqrt{y} + \log_2(z) & = & 6 \\ 3x^2 - 2\sqrt{y} + 2\log_2(z) & = & 5 \\ -5x^2 + 3\sqrt{y} + 4\log_2(z) & = & 13 \end{cases}$$

In Exercises 21 - 26, sketch the solution to each system of nonlinear inequalities in the plane.

21. $\begin{cases} x^2 - y^2 & \leq & 1 \\ x^2 + 4y^2 & \geq & 4 \end{cases}$

22. $\begin{cases} x^2 + y^2 & < & 25 \\ x^2 + (y-3)^2 & \geq & 10 \end{cases}$

23. $\begin{cases} (x-2)^2 + y^2 & < & 1 \\ x^2 + 4y^2 & < & 4 \end{cases}$

24. $\begin{cases} y & > & 10x - x^2 \\ y & < & x^3 + 8 \end{cases}$

25. $\begin{cases} x + 2y^2 & > & 2 \\ x^2 + 4y^2 & \leq & 4 \end{cases}$

26. $\begin{cases} x^2 + y^2 & \geq & 25 \\ y - x & \leq & 1 \end{cases}$

27. Systems of nonlinear equations show up in third semester Calculus in the midst of some really cool problems. The system below came from a problem in which we were asked to find the dimensions of a rectangular box with a volume of 1000 cubic inches that has minimal surface area. The variables $x$, $y$ and $z$ are the dimensions of the box and $\lambda$ is called a Lagrange multiplier. With the help of your classmates, solve the system.[5]

$$\begin{cases} 2y + 2z & = & \lambda yz \\ 2x + 2z & = & \lambda xz \\ 2y + 2x & = & \lambda xy \\ xyz & = & 1000 \end{cases}$$

28. According to Theorem 3.16 in Section 3.4, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In this exercise, we present a method for obtaining that factorization.

    (a) Show that $p$ has no real zeros.

    (b) Because $p$ has no real zeros, its factorization must be of the form $(x^2 + ax + b)(x^2 + cx + d)$ where each factor is an irreducible quadratic. Expand this quantity and gather like terms together.

    (c) Create and solve the system of nonlinear equations which results from equating the coefficients of the expansion found above with those of $x^4 + 4$. You should get four equations in the four unknowns $a$, $b$, $c$ and $d$. Write $p(x)$ in factored form.
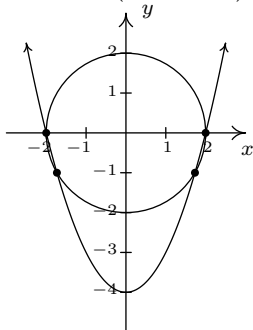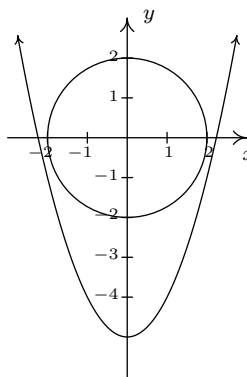
29. Factor $q(x) = x^4 + 6x^2 - 5x + 6$.

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[5]If using $\lambda$ bothers you, change it to $w$ when you solve the system.
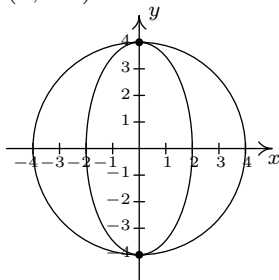
### 8.7.2   ANSWERS
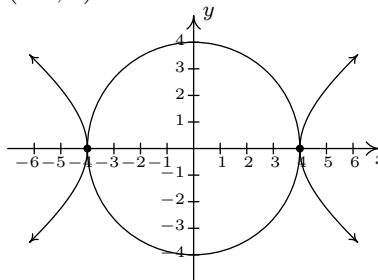
1. $(\pm 2, 0)$, $\left(\pm\sqrt{3}, -1\right)$
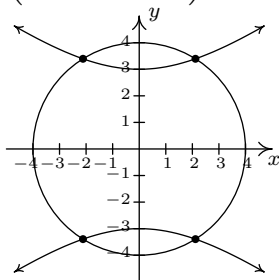


2. No solution



3. $(0, \pm 4)$



4. $(\pm 4, 0)$



5. $\left(\pm\frac{4\sqrt{7}}{5}, \pm\frac{12\sqrt{2}}{5}\right)$



6. $\left(1 + \sqrt{7}, -1 + \sqrt{7}\right)$, $\left(1 - \sqrt{7}, -1 - \sqrt{7}\right)$
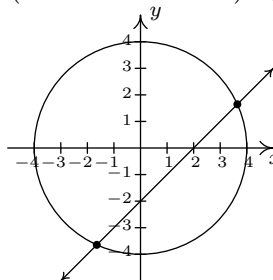


7. $\left(\pm\frac{2\sqrt{10}}{5}, \pm\frac{\sqrt{15}}{5}\right)$

8. $(0, 1)$

9. $(0, \pm 1)$, $(2, 0)$

10. $\left(\frac{4}{3}, \pm\frac{\sqrt{5}}{3}\right)$

11. $(3, 4)$, $(-4, -3)$

12. $(\pm 3, 4)$

13. $(-4, -56)$, $(1, 9)$, $(2, 16)$

14. $(-2, 2)$, $(2, -2)$

15. $(3, 4)$

16. Initially, there are $\frac{250000}{49} \approx 5102$ bacteria. It will take $\frac{5\ln(49/5)}{\ln(7/5)} \approx 33.92$ minutes for the colony to grow to 50,000 bacteria.
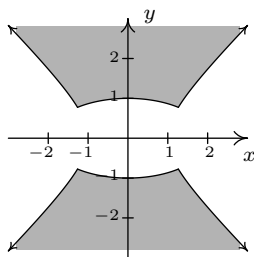
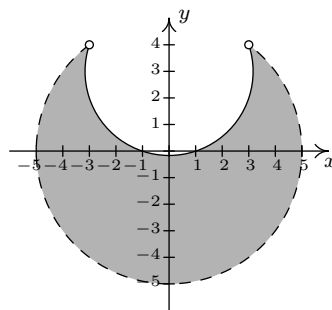17. $\left(-\sqrt[3]{5}, 49\right)$              18. No solution                    19. $\left(e^{-5}, \pm\sqrt{7}\right)$
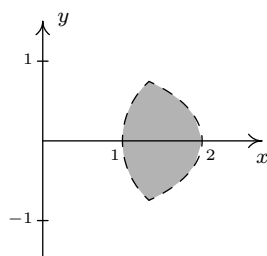
20. $(1, 4, 8)$, $(-1, 4, 8)$

21. $\begin{cases} x^2 - y^2 & \leq & 1 \\ x^2 + 4y^2 & \geq & 4 \end{cases}$                    22. $\begin{cases} x^2 + y^2 & < & 25 \\ x^2 + (y-3)^2 & \geq & 10 \end{cases}$

                    

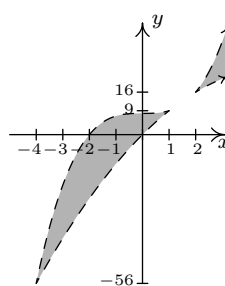23. $\begin{cases} (x-2)^2 + y^2 & < & 1 \\ x^2 + 4y^2 & < & 4 \end{cases}$                    24. $\begin{cases} y & > & 10x - x^2 \\ y & < & x^3 + 8 \end{cases}$
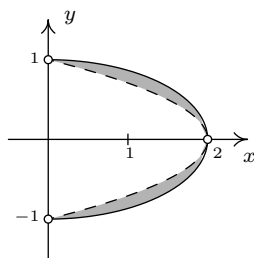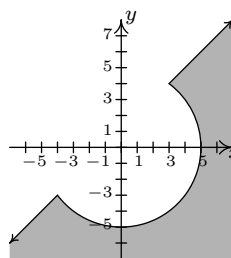
                    

25. $\begin{cases} x + 2y^2 & > & 2 \\ x^2 + 4y^2 & \leq & 4 \end{cases}$                    26. $\begin{cases} x^2 + y^2 & \geq & 25 \\ y - x & \leq & 1 \end{cases}$

                    

27. $x = 10$, $y = 10$, $z = 10$, $\lambda = \frac{2}{5}$

28. (c) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

29. $x^4 + 6x^2 - 5x + 6 = (x^2 - x + 1)(x^2 + x + 6)$

# CHAPTER 9

# SEQUENCES AND THE BINOMIAL THEOREM

## 9.1 SEQUENCES

When we first introduced a function as a special type of relation in Section 1.3, we did not put any restrictions on the domain of the function. All we said was that the set of $x$-coordinates of the points in the function $F$ is called the domain, and it turns out that any subset of the real numbers, regardless of how weird that subset may be, can be the domain of a function. As our exploration of functions continued beyond Section 1.3, we saw fewer and fewer functions with 'weird' domains. It is worth your time to go back through the text to see that the domains of the polynomial, rational, exponential, logarithmic and algebraic functions discussed thus far have fairly predictable domains which almost always consist of just a collection of intervals on the real line. This may lead some readers to believe that the only important functions in a College Algebra text have domains which consist of intervals and everything else was just introductory nonsense. In this section, we introduce **sequences** which are an important class of functions whose domains are the set of natural numbers.[1] Before we get to far ahead of ourselves, let's look at what the term 'sequence' means mathematically. Informally, we can think of a sequence as an infinite list of numbers. For example, consider the sequence

$$\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \ldots \tag{1}$$

As usual, the periods of ellipsis, ..., indicate that the proposed pattern continues forever. Each of the numbers in the list is called a **term**, and we call $\frac{1}{2}$ the 'first term', $-\frac{3}{4}$ the 'second term', $\frac{9}{8}$ the 'third term' and so forth. In numbering them this way, we are setting up a function, which we'll call $a$ per tradition, between the natural numbers and the terms in the sequence.

---

[1] Recall that this is the set $\{1, 2, 3, \ldots\}$.

| $n$ | $a(n)$ |
|-----|--------|
| 1 | $\frac{1}{2}$ |
| 2 | $-\frac{3}{4}$ |
| 3 | $\frac{9}{8}$ |
| 4 | $-\frac{27}{16}$ |
| $\vdots$ | $\vdots$ |

In other words, $a(n)$ is the $n^{\text{th}}$ term in the sequence. We formalize these ideas in our definition of a sequence and introduce some accompanying notation.

> **Definition 9.1.** A **sequence** is a function $a$ whose domain is the natural numbers. The value $a(n)$ is often written as $a_n$ and is called the $\mathbf{n^{th}}$ **term** of the sequence. The sequence itself is usually denoted using the notation: $a_n$, $n \geq 1$ or the notation: $\{a_n\}_{n=1}^{\infty}$.

Applying the notation provided in Definition 9.1 to the sequence given (1), we have $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{4}$, $a_3 = \frac{9}{8}$ and so forth. Now suppose we wanted to know $a_{117}$, that is, the $117^{\text{th}}$ term in the sequence. While the pattern of the sequence is apparent, it would benefit us greatly to have an explicit formula for $a_n$. Unfortunately, there is no general algorithm that will produce a formula for every sequence, so any formulas we do develop will come from that greatest of teachers, experience. In other words, it is time for an example.

**Example 9.1.1.** Write the first four terms of the following sequences.

1.  $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$

2.  $b_k = \dfrac{(-1)^k}{2k + 1}$, $k \geq 0$

3.  $\{2n - 1\}_{n=1}^{\infty}$

4.  $\left\{ \dfrac{1 + (-1)^i}{i} \right\}_{i=2}^{\infty}$

5.  $a_1 = 7$, $a_{n+1} = 2 - a_n$, $n \geq 1$

6.  $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$

**Solution.**

1.  Since we are given $n \geq 1$, the first four terms of the sequence are $a_1$, $a_2$, $a_3$ and $a_4$. Since the notation $a_1$ means the same thing as $a(1)$, we obtain our first term by replacing every occurrence of $n$ in the formula for $a_n$ with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.

2.  For this sequence we have $k \geq 0$, so the first four terms are $b_0$, $b_1$, $b_2$ and $b_3$. Proceeding as before, replacing in this case the variable $k$ with the appropriate whole number, beginning with 0, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. (This sequence is called an **alternating** sequence since the signs alternate between $+$ and $-$. The reader is encouraged to think what component of the formula is producing this effect.)

3. From $\{2n - 1\}_{n=1}^{\infty}$, we have that $a_n = 2n - 1$, $n \geq 1$. We get $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$. (The first four terms are the first four odd natural numbers. The reader is encouraged to examine whether or not this pattern continues indefinitely.)

4. Here, we are using the letter $i$ as a counter, not as the imaginary unit we saw in Section 3.4. Proceeding as before, we set $a_i = \frac{1+(-1)^i}{i}$, $i \geq 2$. We find $a_2 = 1$, $a_3 = 0$, $a_4 = \frac{1}{2}$ and $a_5 = 0$.

5. To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.

6. As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$. $\qquad\square$

Some remarks about Example 9.1.1 are in order. We first note that since sequences are functions, we can graph them in the same way we graph functions. For example, if we wish to graph the sequence $\{b_k\}_{k=0}^{\infty}$ from Example 9.1.1, we graph the equation $y = b(k)$ for the values $k \geq 0$. That is, we plot the points $(k, b(k))$ for the values of $k$ in the domain, $k = 0, 1, 2, \ldots$. The resulting collection of points is the graph of the sequence. Note that we do not connect the dots in a pleasing fashion as we are used to doing, because the domain is just the whole numbers in this case, not a collection of intervals of real numbers. If you feel a sense of nostalgia, you should see Section 1.2.
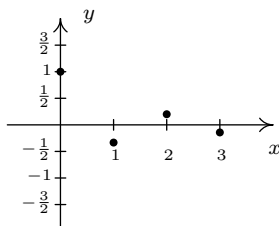


$$\text{Graphing } y = b_k = \frac{(-1)^k}{2k + 1}, \ k \geq 0$$

Speaking of $\{b_k\}_{k=0}^{\infty}$, the astute and mathematically minded reader will correctly note that this technically isn't a sequence, since according to Definition 9.1, sequences are functions whose domains are the *natural* numbers, not the *whole* numbers, as is the case with $\{b_k\}_{k=0}^{\infty}$. In other words, to satisfy Definition 9.1, we need to shift the variable $k$ so it starts at $k = 1$ instead of $k = 0$. To see how we can do this, it helps to think of the problem graphically. What we want is to shift the graph of $y = b(k)$ to the right one unit, and thinking back to Section 1.7, we can accomplish this by replacing $k$ with $k - 1$ in the definition of $\{b_k\}_{k=0}^{\infty}$. Specifically, let $c_k = b_{k-1}$ where $k - 1 \geq 0$. We get $c_k = \frac{(-1)^{k-1}}{2(k-1)+1} = \frac{(-1)^{k-1}}{2k-1}$, where now $k \geq 1$. We leave to the reader to verify that $\{c_k\}_{k=1}^{\infty}$ generates the same list of numbers as does $\{b_k\}_{k=0}^{\infty}$, but the former satisfies Definition

9.1, while the latter does not. Like so many things in this text, we acknowledge that this point is pedantic and join the vast majority of authors who adopt a more relaxed view of Definition 9.1 to include any function which generates a list of numbers which can then be matched up with the natural numbers.[2] Finally, we wish to note the sequences in parts 5 and 6 are examples of sequences described **recursively**. In each instance, an initial value of the sequence is given which is then followed by a **recursion equation** − a formula which enables us to use known terms of the sequence to determine other terms. The terms of the sequence in part 6 are given a special name: $f_n = n!$ is called **$n$-factorial**. Using the '!' notation, we can describe the factorial sequence as: $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. After $0! = 1$ the next four terms, written out in detail, are $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. From this, we see a more informal way of computing $n!$, which is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as a special case. (We will study factorials in greater detail in Section 9.4.) The world famous Fibonacci Numbers are defined recursively and are explored in the exercises. While none of the sequences worked out to be the sequence in (1), they do give us some insight into what kinds of patterns to look for. Two patterns in particular are given in the next definition.

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**Definition 9.2. Arithmetic and Geometric Sequences:** Suppose $\{a_n\}_{n=k}^{\infty}$ is a sequence[a]

- If there is a number $d$ so that $a_{n+1} = a_n + d$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **arithmetic sequence**. The number $d$ is called the **common difference**.

- If there is a number $r$ so that $a_{n+1} = ra_n$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **geometric sequence**. The number $r$ is called the **common ratio**.

---

[a]Note that we have adjusted for the fact that not all 'sequences' begin at $n = 1$.

---

Both arithmetic and geometric sequences are defined in terms of recursion equations. In English, an arithmetic sequence is one in which we proceed from one term to the next by always *adding* the fixed number $d$. The name 'common difference' comes from a slight rewrite of the recursion equation from $a_{n+1} = a_n + d$ to $a_{n+1} - a_n = d$. Analogously, a geometric sequence is one in which we proceed from one term to the next by always *multiplying* by the same fixed number $r$. If $r \neq 0$, we can rearrange the recursion equation to get $\frac{a_{n+1}}{a_n} = r$, hence the name 'common ratio.' Some sequences are arithmetic, some are geometric and some are neither as the next example illustrates.[3]

**Example 9.1.2.** Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference $d$; if geometric, find the common ratio $r$.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$

2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$

3. $\{2n - 1\}_{n=1}^{\infty}$

4. $\dfrac{1}{2}, -\dfrac{3}{4}, \dfrac{9}{8}, -\dfrac{27}{16}, \ldots$

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[2]We're basically talking about the 'countably infinite' subsets of the real number line when we do this.

[3]Sequences which are both arithmetic and geometric are discussed in the Exercises.

**Solution.** A good rule of thumb to keep in mind when working with sequences is "When in doubt, write it out!" Writing out the first several terms can help you identify the pattern of the sequence should one exist.

1. From Example 9.1.1, we know that the first four terms of this sequence are $\frac{1}{3}$, $\frac{5}{9}$, $\frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no 'common difference' and we have established that the sequence is *not* arithmetic. To investigate whether or not it is geometric, we compute the ratios of successive terms. The first three ratios

$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$

suggest that the sequence is geometric. To prove it, we must show that $\frac{a_{n+1}}{a_n} = r$ for all $n$.

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{5^{(n+1)-1}}{3^{n+1}}}{\dfrac{5^{n-1}}{3^n}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$

This sequence is geometric with common ratio $r = \frac{5}{3}$.

2. Again, we have Example 9.1.1 to thank for providing the first four terms of this sequence: $1$, $-\frac{1}{3}$, $\frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since there is no 'common ratio,' we conclude the sequence is not geometric, either.

3. As we saw in Example 9.1.1, the sequence $\{2n - 1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \ldots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To verify this, we find

$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$

This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

4. We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we assume that the pattern of ratios continue in this fashion and conclude that the sequence is geometric.    □

We are now one step away from determining an explicit formula for the sequence given in (1). We know that it is a geometric sequence and our next result gives us the explicit formula we require.

> **Equation 9.1. Formulas for Arithmetic and Geometric Sequences:**
>
> - An arithmetic sequence with first term $a$ and common difference $d$ is given by
>
> $$a_n = a + (n-1)d, \quad n \geq 1$$
>
> - A geometric sequence with first term $a$ and common ratio $r \neq 0$ is given by
>
> $$a_n = ar^{n-1}, \quad n \geq 1$$

While the formal proofs of the formulas in Equation 9.1 require the techniques set forth in Section 9.3, we attempt to motivate them here. According to Definition 9.2, given an arithmetic sequence with first term $a$ and common difference $d$, the way we get from one term to the next is by adding $d$. Hence, the terms of the sequence are: $a$, $a + d$, $a + 2d$, $a + 3d$, .... We see that to reach the $n$th term, we add $d$ to $a$ exactly $(n-1)$ times, which is what the formula says. The derivation of the formula for geometric series follows similarly. Here, we start with $a$ and go from one term to the next by multiplying by $r$. We get $a, ar, ar^2, ar^3$ and so forth. The $n$th term results from multiplying $a$ by $r$ exactly $(n-1)$ times. We note here that the reason $r = 0$ is excluded from Equation 9.1 is to avoid an instance of $0^0$ which is an indeterminant form.[4] With Equation 9.1 in place, we finally have the tools required to find an explicit formula for the $n$th term of the sequence given in (1). We know from Example 9.1.2 that it is geometric with common ratio $r = -\frac{3}{2}$. The first term is $a = \frac{1}{2}$ so by Equation 9.1 we get $a_n = ar^{n-1} = \frac{1}{2}\left(-\frac{3}{2}\right)^{n-1}$ for $n \geq 1$. After a touch of simplifying, we get $a_n = \frac{(-3)^{n-1}}{2^n}$ for $n \geq 1$. Note that we can easily check our answer by substituting in values of $n$ and seeing that the formula generates the sequence given in (1). We leave this to the reader. Our next example gives us more practice finding patterns.

**Example 9.1.3.** Find an explicit formula for the $n^{\text{th}}$ term of the following sequences.

1. $0.9, 0.09, 0.009, 0.0009, \ldots$    2. $\dfrac{2}{5}, 2, -\dfrac{2}{3}, -\dfrac{2}{7}, \ldots$    3. $1, -\dfrac{2}{7}, \dfrac{4}{13}, -\dfrac{8}{19}, \ldots$

**Solution.**

1. Although this sequence may seem strange, the reader can verify it is actually a geometric sequence with common ratio $r = 0.1 = \frac{1}{10}$. With $a = 0.9 = \frac{9}{10}$, we get $a_n = \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$ for $n \geq 0$. Simplifying, we get $a_n = \frac{9}{10^n}$, $n \geq 1$. There is more to this sequence than meets the eye and we shall return to this example in the next section.

2. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions. We have $\frac{2}{5}, \frac{2}{1}, -\frac{2}{3}, -\frac{2}{7}, \ldots$. If we associate the negative '$-$' of the last two terms with the denominators we get $\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, \frac{2}{-7}, \ldots$. This tells us that we can tentatively sketch out the formula for the sequence as $a_n = \frac{2}{d_n}$ where $d_n$ is the sequence of denominators.

---

[4]See the footnotes on page 237 in Section 3.1 and page 418 of Section 6.1.

Looking at the denominators $5, 1, -3, -7, \ldots$, we find that they go from one term to the next by subtracting 4 which is the same as adding $-4$. This means we have an arithmetic sequence on our hands. Using Equation 9.1 with $a = 5$ and $d = -4$, we get the $n$th denominator by the formula $d_n = 5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.

3.  The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting $c_n$ and $d_n$ denote the sequence of numerators and denominators, respectively, we have $a_n = \frac{c_n}{d_n}$. After some experimentation,[5] we choose to write the first term as a fraction and associate the negatives '$-$' with the numerators. This yields $\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \ldots$. The numerators form the sequence $1, -2, 4, -8, \ldots$ which is geometric with $a = 1$ and $r = -2$, so we get $c_n = (-2)^{n-1}$, for $n \geq 1$. The denominators $1, 7, 13, 19, \ldots$ form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $d_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$. We obtain our formula for $a_n = \frac{c_n}{d_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out. $\square$

While the last problem in Example 9.1.3 was neither geometric nor arithmetic, it did resolve into a combination of these two kinds of sequences. If handed the sequence $2, 5, 10, 17, \ldots$, we would be hard-pressed to find a formula for $a_n$ if we restrict our attention to these two archetypes. We said before that there is no general algorithm for finding the explicit formula for the $n$th term of a given sequence, and it is only through experience gained from evaluating sequences from explicit formulas that we learn to begin to recognize number patterns. The pattern $1, 4, 9, 16, \ldots$ is rather recognizable as the squares, so the formula $a_n = n^2$, $n \geq 1$ may not be too hard to determine. With this in mind, it's possible to see $2, 5, 10, 17, \ldots$ as the sequence $1 + 1, 4 + 1, 9 + 1, 16 + 1, \ldots$, so that $a_n = n^2 + 1$, $n \geq 1$. Of course, since we are given only a small *sample* of the sequence, we shouldn't be too disappointed to find out this isn't the *only* formula which generates this sequence. For example, consider the sequence defined by $b_n = -\frac{1}{4}n^4 + \frac{5}{2}n^3 - \frac{31}{4}n^2 + \frac{25}{2}n - 5$, $n \geq 1$. The reader is encouraged to verify that it also produces the terms $2, 5, 10, 17$. In fact, it can be shown that given any finite sample of a sequence, there are infinitely many explicit formulas all of which generate those same finite points. This means that there will be infinitely many correct answers to some of the exercises in this section.[6] Just because your answer doesn't match ours doesn't mean it's wrong. As always, when in doubt, write your answer out. As long as it produces the same terms in the same order as what the problem wants, your answer is correct.

Sequences play a major role in the Mathematics of Finance, as we have already seen with Equation 6.2 in Section 6.5. Recall that if we invest $P$ dollars at an annual percentage rate $r$ and compound the interest $n$ times per year, the formula for $A_k$, the amount in the account after $k$ compounding periods, is $A_k = P\left(1 + \frac{r}{n}\right)^k = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right)^{k-1}$, $k \geq 1$. We now spot this as a geometric sequence with first term $P\left(1 + \frac{r}{n}\right)$ and common ratio $\left(1 + \frac{r}{n}\right)$. In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an **annuity** and will be discussed in the next section once we have developed more mathematical machinery.

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[5]Here we take 'experimentation' to mean a frustrating guess-and-check session.
[6]For more on this, see When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test.

### 9.1.1   Exercises

In Exercises 1 - 13, write out the first four terms of the given sequence.

1. $a_n = 2^n - 1$, $n \geq 0$

2. $d_j = (-1)^{\frac{j(j+1)}{2}}$, $j \geq 1$

3. $\{5k - 2\}_{k=1}^{\infty}$

4. $\left\{\dfrac{n^2 + 1}{n + 1}\right\}_{n=0}^{\infty}$

5. $\left\{\dfrac{x^n}{n^2}\right\}_{n=1}^{\infty}$

6. $\left\{\dfrac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

7. $a_1 = 3$, $a_{n+1} = a_n - 1$, $n \geq 1$

8. $d_0 = 12$, $d_m = \dfrac{d_{m\text{-}1}}{100}$, $m \geq 1$

9. $b_1 = 2$, $b_{k+1} = 3b_k + 1$, $k \geq 1$

10. $c_0 = -2$, $c_j = \dfrac{c_{j\text{-}1}}{(j + 1)(j + 2)}$, $m \geq 1$

11. $a_1 = 117$, $a_{n+1} = \dfrac{1}{a_n}$, $n \geq 1$

12. $s_0 = 1$, $s_{n+1} = x^{n+1} + s_n$, $n \geq 0$

13. $F_0 = 1$, $F_1 = 1$, $F_n = F_{n\text{-}1} + F_{n\text{-}2}$, $n \geq 2$ (This is the famous Fibonacci Sequence )

In Exercises 14 - 21 determine if the given sequence is arithmetic, geometric or neither. If it is arithmetic, find the common difference $d$; if it is geometric, find the common ratio $r$.

14. $\{3n - 5\}_{n=1}^{\infty}$

15. $a_n = n^2 + 3n + 2$, $n \geq 1$

16. $\dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{12}, \dfrac{1}{24}, \ldots$

17. $\left\{3\left(\dfrac{1}{5}\right)^{n-1}\right\}_{n=1}^{\infty}$

18. $17, 5, -7, -19, \ldots$

19. $2, 22, 222, 2222, \ldots$

20. $0.9, 9, 90, 900, \ldots$

21. $a_n = \dfrac{n!}{2}$, $n \geq 0$.

In Exercises 22 - 30, find an explicit formula for the $n^{\text{th}}$ term of the given sequence. Use the formulas in Equation 9.1 as needed.

22. $3, 5, 7, 9, \ldots$

23. $1, -\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \ldots$

24. $1, \dfrac{2}{3}, \dfrac{4}{5}, \dfrac{8}{7}, \ldots$

25. $1, \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{4}{27}, \ldots$

26. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots$

27. $x, -\dfrac{x^3}{3}, \dfrac{x^5}{5}, -\dfrac{x^7}{7}, \ldots$

28. $0.9, 0.99, 0.999, 0.9999, \ldots$     29. $27, 64, 125, 216, \ldots$         30. $1, 0, 1, 0, \ldots$

31. Find a sequence which is both arithmetic and geometric. (Hint: Start with $a_n = c$ for all $n$.)

32. Show that a geometric sequence can be transformed into an arithmetic sequence by taking the natural logarithm of the terms.

33. Thomas Robert Malthus is credited with saying, "The power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second." (See this webpage for more information.) Discuss this quote with your classmates from a sequences point of view.

34. This classic problem involving sequences shows the power of geometric sequences. Suppose that a wealthy benefactor agrees to give you one penny today and then double the amount she gives you each day for 30 days. So, for example, you get two pennies on the second day and four pennies on the third day. How many pennies do you get on the $30^{\text{th}}$ day? What is the total dollar value of the gift you have received?

35. Research the terms 'arithmetic mean' and 'geometric mean.' With the help of your classmates, show that a given term of a arithmetic sequence $a_k$, $k \geq 2$ is the arithmetic mean of the term immediately preceding, $a_{k-1}$ it and immediately following it, $a_{k+1}$. State and prove an analogous result for geometric sequences.

36. Discuss with your classmates how the results of this section might change if we were to examine sequences of other mathematical things like complex numbers or matrices. Find an explicit formula for the $n^{\text{th}}$ term of the sequence $i, -1, -i, 1, i, \ldots$. List out the first four terms of the matrix sequences we discussed in Exercise 8.3.1 in Section 8.3.

## 9.1.2    Answers

1. $0, 1, 3, 7$

2. $-1, -1, 1, 1$

3. $3, 8, 13, 18$

4. $1, 1, \frac{5}{3}, \frac{5}{2}$

5. $x, \frac{x^2}{4}, \frac{x^3}{9}, \frac{x^4}{16}$

6. $0, \frac{\ln(2)}{2}, \frac{\ln(3)}{3}, \frac{\ln(4)}{4}$

7. $3, 2, 1, 0$

8. $12, 0.12, 0.0012, 0.000012$

9. $2, 7, 22, 67$

10. $-2, -\frac{1}{3}, -\frac{1}{36}, -\frac{1}{720}$

11. $117, \frac{1}{117}, 117, \frac{1}{117}$

12. $1, x+1, x^2 + x + 1, x^3 + x^2 + x + 1$

13. $1, 1, 2, 3$

14. arithmetic, $d = 3$

15. neither

16. geometric, $r = \frac{1}{2}$

17. geometric, $r = \frac{1}{5}$

18. arithmetic, $d = -12$

19. neither

20. geometric, $r = 10$

21. neither

22. $a_n = 1 + 2n,\ n \geq 1$

23. $a_n = \left(-\frac{1}{2}\right)^{n-1},\ n \geq 1$

24. $a_n = \frac{2^{n-1}}{2n-1},\ n \geq 1$

25. $a_n = \frac{n}{3^{n-1}},\ n \geq 1$

26. $a_n = \frac{1}{n^2},\ n \geq 1$

27. $\frac{(-1)^{n-1} x^{2n-1}}{2n-1},\ n \geq 1$

28. $a_n = \frac{10^n - 1}{10^n},\ n \geq 1$

29. $a_n = (n+2)^3,\ n \geq 1$

30. $a_n = \frac{1 + (-1)^{n-1}}{2},\ n \geq 1$

## 9.2 Summation Notation

In the previous section, we introduced sequences and now we shall present notation and theorems concerning the sum of terms of a sequence. We begin with a definition, which, while intimidating, is meant to make our lives easier.

---

**Definition 9.3. Summation Notation:** Given a sequence $\{a_n\}_{n=k}^{\infty}$ and numbers $m$ and $p$ satisfying $k \le m \le p$, the summation from $m$ to $p$ of the sequence $\{a_n\}$ is written

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p$$

The variable $n$ is called the **index of summation**. The number $m$ is called the **lower limit of summation** while the number $p$ is called the **upper limit of summation**.

---

In English, Definition 9.3 is simply defining a short-hand notation for adding up the terms of the sequence $\{a_n\}_{n=k}^{\infty}$ from $a_m$ through $a_p$. The symbol $\Sigma$ is the capital Greek letter sigma and is shorthand for 'sum'. The lower and upper limits of the summation tells us which term to start with and which term to end with, respectively. For example, using the sequence $a_n = 2n - 1$ for $n \ge 1$, we can write the sum $a_3 + a_4 + a_5 + a_6$ as

$$
\begin{aligned}
\sum_{n=3}^{6} (2n - 1) &= (2(3) - 1) + (2(4) - 1) + (2(5) - 1) + (2(6) - 1) \\
&= 5 + 7 + 9 + 11 \\
&= 32
\end{aligned}
$$

The index variable is considered a 'dummy variable' in the sense that it may be changed to any letter without affecting the value of the summation. For instance,

$$\sum_{n=3}^{6} (2n - 1) = \sum_{k=3}^{6} (2k - 1) = \sum_{j=3}^{6} (2j - 1)$$

One place you may encounter summation notation is in mathematical definitions. For example, summation notation allows us to define polynomials as functions of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

for real numbers $a_k$, $k = 0, 1, \ldots n$. The reader is invited to compare this with what is given in Definition 3.1. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the $i$th row $R_i$ of a matrix $A = [a_{ij}]_{m \times n}$ and the $j^{\text{th}}$ column $C_j$ of a matrix $B = [b_{ij}]_{n \times r}$ as

$$Ri \cdot Cj = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition 8.9. Our next example gives us practice with this new notation.

**Example 9.2.1.**

1. Find the following sums.

   (a) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$
   
   (b) $\displaystyle\sum_{n=0}^{4} \frac{n!}{2}$
   
   (c) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.

   (a) $1 + 3 + 5 + \ldots + 117$

   (b) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + - \ldots + \dfrac{1}{117}$

   (c) $0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0 \cdots 0}_{n-1 \text{ zeros}} 9$

**Solution.**

1. (a) We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$
\begin{aligned}
\sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\
&= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\
&= 0.13131313
\end{aligned}
$$

   (b) Proceeding as in (a), we replace every occurrence of $n$ with the values 0 through 4. We recall the factorials, $n!$ as defined in number Example 9.1.1, number 6 and get:

$$
\begin{aligned}
\sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{4!}{2} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{2 \cdot 1}{2} + \frac{3 \cdot 2 \cdot 1}{2} + \frac{4 \cdot 3 \cdot 2 \cdot 1}{2} \\
&= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12 \\
&= 17
\end{aligned}
$$

   (c) We proceed as before, replacing the index $n$, but *not* the variable $x$, with the values 1 through 5 and adding the resulting terms.

$$\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n = \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3$$

$$+ \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section 9.1.

(a) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. We get a formula for the $n$th term of the sequence using Equation 9.1 to get $a_n = 1 + (n-1)2 = 2n-1$, $n \geq 1$. At this stage, we have the formula for the terms, namely $2n - 1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of $n$ produces the term 117. Setting $a_n = 117$, we get $2n - 1 = 117$ or $n = 59$. Our final answer is

$$1 + 3 + 5 + \ldots + 117 = \sum_{n=1}^{59}(2n - 1)$$

(b) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '−' with the numerators to get

$$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \ldots + \frac{1}{117}$$

The numerators, 1, −1, etc. can be described by the geometric sequence[1] $c_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence[2] $d_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117} = \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

(c) Thanks to Example 9.1.3, we know that one formula for the $n^{\text{th}}$ term is $a_n = \frac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation. To determine the upper limit of summation, we note that to produce the $n - 1$ zeros to the right of the decimal point before the 9, we need a denominator of $10^n$. Hence, $n$ is

---

[1] This is indeed a geometric sequence with first term $a = 1$ and common ratio $r = -1$.
[2] It is an arithmetic sequence with first term $a = 1$ and common difference $d = 1$.

the upper limit of summation. Since $n$ is used in the limits of the summation, we need to choose a different letter for the index of summation.[3] We choose $k$ and get

$$0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots0}_{n-1\ \text{zeros}}9 \;=\; \sum_{k=1}^{n} \frac{9}{10^k}$$

□

The following theorem presents some general properties of summation notation. While we shall not have much need of these properties in Algebra, they do play a great role in Calculus. Moreover, there is much to be learned by thinking about why the properties hold. We invite the reader to prove these results. To get started, remember, "When in doubt, write it out!"

**Theorem 9.1. Properties of Summation Notation:** Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- $\displaystyle\sum_{n=m}^{p} (a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$

- $\displaystyle\sum_{n=m}^{p} c\,a_n = c \sum_{n=m}^{p} a_n$, for any real number $c$.

- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n$, for any natural number $m \le j < j+1 \le p$.

- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any whole number $r$.

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \ge 1$, we let $S$ denote the sum of the first $n$ terms. To derive a formula for $S$, we write it out in two different ways

$$
\begin{aligned}
S &= a &+& \quad (a+d) &+& \ldots &+& (a+(n-2)d) &+& (a+(n-1)d) \\
S &= (a+(n-1)d) &+& (a+(n-2)d) &+& \ldots &+& (a+d) &+& a
\end{aligned}
$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a+(n-1)d) + (2a+(n-1)d) + \ldots + (2a+(n-1)d) + (2a+(n-1)d)$$

The right hand side of this equation contains $n$ terms, all of which are equal to $(2a+(n-1)d)$ so we get $2S = n(2a+(n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

---

[3]To see why, try writing the summation using '$n$' as the index.

$$S = \frac{n}{2}(2a + (n-1)d)$$

If we rewrite the quantity $2a + (n-1)d$ as $a + (a + (n-1)d) = a_1 + a_n$, we get the formula

$$S = n\left(\frac{a_1 + a_n}{2}\right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms $n$ and the *average* of the first and $n^{\text{th}}$ terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \geq 1$, and let $S$ once again denote the sum of the first $n$ terms. Comparing $S$ and $rS$, we get

$$
\begin{array}{rcccccccccc}
S & = & a & + & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} \\
rS & = & & & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n
\end{array}
$$

Subtracting the second equation from the first forces all of the terms except $a$ and $ar^n$ to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a(1-r^n)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

If we distribute $a$ through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1-r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \ldots + a}_{n \text{ times}} = na$$

Our results are summarized below.

**Equation 9.2. Sums of Arithmetic and Geometric Sequences:**

- The sum $S$ of the first $n$ terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum $S$ of the first $n$ terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

1. $S = \sum_{k=1}^{n} a_k = \dfrac{a_1 - a_{n+1}}{1 - r} = a\left(\dfrac{1 - r^n}{1 - r}\right)$, if $r \neq 1$.

2. $S = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 9.2, formal proofs require the machinery in Section 9.3. An application of the arithmetic sum formula which proves useful in Calculus results in formula for the sum of the first $n$ natural numbers. The natural numbers themselves are a sequence[4] 1, 2, 3, ... which is arithmetic with $a = d = 1$. Applying Equation 9.2,

$$1 + 2 + 3 + \ldots + n \quad = \quad \frac{n(n+1)}{2}$$

So, for example, the sum of the first 100 natural numbers[5] is $\frac{100(101)}{2} = 5050$.

An important application of the geometric sum formula is the investment plan called an **annuity**. Annuities differ from the kind of investments we studied in Section 6.5 in that payments are deposited into the account on an on-going basis, and this complicates the mathematics a little.[6] Suppose you have an account with annual interest rate $r$ which is compounded $n$ times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. Suppose we wish to make ongoing deposits of $P$ dollars at the *end* of each compounding period. Let $A_k$ denote the amount in the account after $k$ compounding periods. Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on $A_1$ so that our initial investment has grown to $A_1(1+i) = P(1+i)$ in accordance with Equation 6.1. When we add our second payment at the end of the second period, we get

$$A_2 = A_1(1+i) + P = P(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)$$

The reason for factoring out the $P(1+i)$ will become apparent in short order. During the third compounding period, we earn interest on $A_2$ which then grows to $A_2(1+i)$. We add our third

---

[4]This is the identity function on the natural numbers!

[5]There is an interesting anecdote which says that the famous mathematician Carl Friedrich Gauss was given this problem in primary school and devised a very clever solution.

[6]The reader may wish to re-read the discussion on compound interest in Section 6.5 before proceeding.

payment at the end of the third compounding period to obtain

$$A_3 = A_2(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)(1+i) + P = P(1+i)^2\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2}\right)$$

During the fourth compounding period, $A_3$ grows to $A_3(1+i)$, and when we add the fourth payment, we factor out $P(1+i)^3$ to get

$$A_4 = P(1+i)^3\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3}\right)$$

This pattern continues so that at the end of the $k$th compounding, we get

$$A_k = P(1+i)^{k-1}\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first $k$ terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$. Using Equation 9.2, we get

$$1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}} = 1\left(\frac{1 - \dfrac{1}{(1+i)^k}}{1 - \dfrac{1}{1+i}}\right) = \frac{(1+i)\left(1 - (1+i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1}\left(\frac{(1+i)\left(1 - (1+i)^{-k}\right)}{i}\right) = \frac{P\left((1+i)^k - 1\right)}{i}$$

If we let $t$ be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the future value of an **ordinary annuity**.

---

**Equation 9.3. Future Value of an Ordinary Annuity:** Suppose an annuity offers an annual interest rate $r$ compounded $n$ times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit $P$ is made at the end of each compounding period, the amount $A$ in the account after $t$ years is given by

$$A = \frac{P\left((1+i)^{nt} - 1\right)}{i}$$

---

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 9.3 and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit $P$ is made a the *beginning* of the compounding period instead of at the end, the annuity is called an **annuity-due**. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

**Example 9.2.2.** An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of $50 are made, find the value of the annuity in 30 years.

2. How many years will it take for the annuity to grow to $100,000?

**Solution.**

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50\left((1 + 0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

Our final answer is $50,225.75.

2. To find how long it will take for the annuity to grow to $100,000, we set $A = 100000$ and solve for $t$. We isolate the exponential and take natural logs of both sides of the equation.

$$
\begin{aligned}
100000 &= \frac{50\left((1 + 0.005)^{12t} - 1\right)}{0.005} \\
10 &= (1.005)^{12t} - 1 \\
(1.005)^{12t} &= 11 \\
\ln\left((1.005)^{12t}\right) &= \ln(11) \\
12t\ln(1.005) &= \ln(11) \\
t &= \tfrac{\ln(11)}{12\ln(1.005)} \approx 40.06
\end{aligned}
$$

This means that it takes just over 40 years for the investment to grow to $100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering. $\qquad\square$

We close this section with a peek into Calculus by considering *infinite* sums, called **series**. Consider the number $0.\overline{9}$. We can write this number as

$$0.\overline{9} = 0.9999... = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$$

From Example 9.2.1, we know we can write the sum of the first $n$ of these terms as

$$0.\underbrace{9\cdots9}_{n \text{ nines}} = .9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^{n} \frac{9}{10^k}$$

Using Equation 9.2, we have

$$\sum_{k=1}^{n} \frac{9}{10^k} = \frac{9}{10} \left( \frac{1 - \dfrac{1}{10^{n+1}}}{1 - \dfrac{1}{10}} \right) = 1 - \frac{1}{10^{n+1}}$$

It stands to reason that $0.\overline{9}$ is the same value of $1 - \frac{1}{10^{n+1}}$ as $n \to \infty$. Our knowledge of exponential expressions from Section 6.1 tells us that $\frac{1}{10^{n+1}} \to 0$ as $n \to \infty$, so $1 - \frac{1}{10^{n+1}} \to 1$. We have just argued that $0.\overline{9} = 1$, which may cause some distress for some readers.[7] Any non-terminating decimal can be thought of as an infinite sum whose denominators are the powers of 10, so the phenomenon of adding up infinitely many terms and arriving at a finite number is not as foreign of a concept as it may appear. We end this section with a theorem concerning geometric series.

---

**Theorem 9.2. Geometric Series:** Given the sequence $a_k = ar^{k-1}$ for $k \geq 1$, where $|r| < 1$,

$$a + ar + ar^2 + \ldots = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1 - r}$$

If $|r| \geq 1$, the sum $a + ar + ar^2 + \ldots$ is not defined.

---

The justification of the result in Theorem 9.2 comes from taking the formula in Equation 9.2 for the sum of the first $n$ terms of a geometric sequence and examining the formula as $n \to \infty$. Assuming $|r| < 1$ means $-1 < r < 1$, so $r^n \to 0$ as $n \to \infty$. Hence as $n \to \infty$,

$$\sum_{k=1}^{n} ar^{k-1} = a \left( \frac{1 - r^n}{1 - r} \right) \to \frac{a}{1 - r}$$

As to what goes wrong when $|r| \geq 1$, we leave that to Calculus as well, but will explore some cases in the exercises.

---

[7]To make this more palatable, it is usually accepted that $0.\overline{3} = \frac{1}{3}$ so that $0.\overline{9} = 3\left(0.\overline{3}\right) = 3\left(\frac{1}{3}\right) = 1$. Feel better?

## 9.2.1    Exercises

In Exercises 1 - 8, find the value of each sum using Definition 9.3.

1. $\displaystyle\sum_{g=4}^{9}(5g+3)$

2. $\displaystyle\sum_{k=3}^{8}\frac{1}{k}$

3. $\displaystyle\sum_{j=0}^{5}2^j$

4. $\displaystyle\sum_{k=0}^{2}(3k-5)x^k$

5. $\displaystyle\sum_{i=1}^{4}\frac{1}{4}(i^2+1)$

6. $\displaystyle\sum_{n=1}^{100}(-1)^n$

7. $\displaystyle\sum_{n=1}^{5}\frac{(n+1)!}{n!}$

8. $\displaystyle\sum_{j=1}^{3}\frac{5!}{j!\,(5-j)!}$

In Exercises 9 - 16, rewrite the sum using summation notation.

9. $8+11+14+17+20$

10. $1-2+3-4+5-6+7-8$

11. $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}$

12. $1+2+4+\cdots+2^{29}$

13. $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}$

14. $-\ln(3)+\ln(4)-\ln(5)+\cdots+\ln(20)$

15. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}$

16. $\frac{1}{2}(x-5)+\frac{1}{4}(x-5)^2+\frac{1}{6}(x-5)^3+\frac{1}{8}(x-5)^4$

In Exercises 17 - 28, use the formulas in Equation 9.2 to find the sum.

17. $\displaystyle\sum_{n=1}^{10}5n+3$

18. $\displaystyle\sum_{n=1}^{20}2n-1$

19. $\displaystyle\sum_{k=0}^{15}3-k$

20. $\displaystyle\sum_{n=1}^{10}\left(\frac{1}{2}\right)^n$

21. $\displaystyle\sum_{n=1}^{5}\left(\frac{3}{2}\right)^n$

22. $\displaystyle\sum_{k=0}^{5}2\left(\frac{1}{4}\right)^k$

23. $1+4+7+\ldots+295$

24. $4+2+0-2-\ldots-146$

25. $1+3+9+\ldots+2187$

26. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{256}$

27. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+-\cdots+\frac{3}{256}$

28. $\displaystyle\sum_{n=1}^{10}-2n+\left(\frac{5}{3}\right)^n$

In Exercises 29 - 32, use Theorem 9.2 to express each repeating decimal as a fraction of integers.

29. $0.\overline{7}$

30. $0.\overline{13}$

31. $10.\overline{159}$

32. $-5.8\overline{67}$

In Exercises 33 - 38, use Equation 9.3 to compute the future value of the annuity with the given terms. In all cases, assume the payment is made monthly, the interest rate given is the annual rate, and interest is compounded monthly.

33. payments are $300, interest rate is 2.5%, term is 17 years.

34. payments are $50, interest rate is 1.0%, term is 30 years.

35. payments are $100, interest rate is 2.0%, term is 20 years

36. payments are $100, interest rate is 2.0%, term is 25 years

37. payments are $100, interest rate is 2.0%, term is 30 years

38. payments are $100, interest rate is 2.0%, term is 35 years

39. Suppose an ordinary annuity offers an annual interest rate of 2%, compounded monthly, for 30 years. What should the monthly payment be to have $100,000 at the end of the term?

40. Prove the properties listed in Theorem 9.1.

41. Show that the formula for the future value of an annuity due is

$$A = P(1 + i)\left[\frac{(1 + i)^{nt} - 1}{i}\right]$$

42. Discuss with your classmates what goes wrong when trying to find the following sums.[8]

(a) $\displaystyle\sum_{k=1}^{\infty} 2^{k-1}$

(b) $\displaystyle\sum_{k=1}^{\infty} (1.0001)^{k-1}$

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}$

---

[8]When in doubt, write them out!

## 9.2.2  Answers

1. 213

2. $\frac{341}{280}$

3. 63

4. $-5 - 2x + x^2$

5. $\frac{17}{2}$

6. 0

7. 20

8. 25

9. $\displaystyle\sum_{k=1}^{5}(3k+5)$

10. $\displaystyle\sum_{k=1}^{8}(-1)^{k-1}k$

11. $\displaystyle\sum_{k=1}^{4}(-1)^{k-1}\frac{x}{2k-1}$

12. $\displaystyle\sum_{k=1}^{30}2^{k-1}$

13. $\displaystyle\sum_{k=1}^{5}\frac{k+1}{k}$

14. $\displaystyle\sum_{k=3}^{20}(-1)^{k}\ln(k)$

15. $\displaystyle\sum_{k=1}^{6}\frac{(-1)^{k-1}}{k^2}$

16. $\displaystyle\sum_{k=1}^{4}\frac{1}{2k}(x-5)^k$

17. 305

18. 400

19. $-72$

20. $\dfrac{1023}{1024}$

21. $\dfrac{633}{32}$

22. $\dfrac{1365}{512}$

23. 14652

24. $-5396$

25. 3280

26. $\dfrac{255}{256}$

27. $\dfrac{513}{256}$

28. $\dfrac{17771050}{59049}$

29. $\dfrac{7}{9}$

30. $\dfrac{13}{99}$

31. $\dfrac{3383}{333}$

32. $-\dfrac{5809}{990}$

33. $76,163.67

34. $20,981.40

35. $29,479.69

36. $38,882.12

37. 49,272.55

38. 60,754.80

39. For $100,000, the monthly payment is $\approx$ $202.95.

## 9.3 Mathematical Induction

The Chinese philosopher Confucius is credited with the saying, "A journey of a thousand miles begins with a single step." In many ways, this is the central theme of this section. Here we introduce a method of proof, Mathematical Induction, which allows us to *prove* many of the formulas we have merely *motivated* in Sections 9.1 and 9.2 by starting with just a single step. A good example is the formula for arithmetic sequences we touted in Equation 9.1. Arithmetic sequences are defined recursively, starting with $a_1 = a$ and then $a_{n+1} = a_n + d$ for $n \geq 1$. This tells us that we start the sequence with $a$ and we go from one term to the next by successively adding $d$. In symbols,

$$a, a + d, a + 2d, a + 3d, a + 4d + \ldots$$

The pattern *suggested* here is that to reach the $n$th term, we start with $a$ and add $d$ to it exactly $n - 1$ times, which lead us to our formula $a_n = a + (n - 1)d$ for $n \geq 1$. But how do we *prove* this to be the case? We have the following.

---

**The Principle of Mathematical Induction (PMI):** Suppose $P(n)$ is a sentence involving the natural number $n$.

**IF**

1. $P(1)$ is true **and**

2. whenever $P(k)$ is true, it follows that $P(k + 1)$ is also true

**THEN** the sentence $P(n)$ is true for all natural numbers $n$.

---

The Principle of Mathematical Induction, or PMI for short, is exactly that - a principle.[1] It is a property of the natural numbers we either choose to accept or reject. In English, it says that if we want to prove that a formula works for all natural numbers $n$, we start by showing it is true for $n = 1$ (the '**base step**') and then show that if it is true for a generic natural number $k$, it must be true for the next natural number, $k + 1$ (the '**inductive step**'). The notation $P(n)$ acts just like function notation. For example, if $P(n)$ is the sentence (formula) '$n^2 + 1 = 3$', then $P(1)$ would be '$1^2 + 1 = 3$', which is false. The construction $P(k + 1)$ would be '$(k + 1)^2 + 1 = 3$'. As usual, this new concept is best illustrated with an example. Returning to our quest to prove the formula for an arithmetic sequence, we first identify $P(n)$ as the formula $a_n = a + (n - 1)d$. To prove this formula is valid for all natural numbers $n$, we need to do two things. First, we need to establish that $P(1)$ is true. In other words, is it true that $a_1 = a + (1 - 1)d$? The answer is yes, since this simplifies to $a_1 = a$, which is part of the definition of the arithmetic sequence. The second thing we need to show is that whenever $P(k)$ is true, it follows that $P(k + 1)$ is true. In other words, we *assume* $P(k)$ is true (this is called the '**induction hypothesis**') and *deduce* that $P(k + 1)$ is also true. Assuming $P(k)$ to be true seems to invite disaster - after all, isn't this essentially what we're trying to prove in the first place? To help explain this step a little better, we show how this works for specific values of $n$. We've already established $P(1)$ is true, and we now want to show that $P(2)$

---

[1] Another word for this you may have seen is 'axiom.'

is true. Thus we need to show that $a_2 = a + (2-1)d$. Since $P(1)$ is true, we have $a_1 = a$, and by the definition of an arithmetic sequence, $a_2 = a_1 + d = a + d = a + (2-1)d$. So $P(2)$ is true. We now use the fact that $P(2)$ is true to show that $P(3)$ is true. Using the fact that $a_2 = a + (2-1)d$, we show $a_3 = a + (3-1)d$. Since $a_3 = a_2 + d$, we get $a_3 = (a + (2-1)d) + d = a + 2d = a + (3-1)d$, so we have shown $P(3)$ is true. Similarly, we can use the fact that $P(3)$ is true to show that $P(4)$ is true, and so forth. In general, if $P(k)$ is true (i.e., $a_k = a + (k-1)d$) we set out to show that $P(k+1)$ is true (i.e., $a_{k+1} = a + ((k+1) - 1)d$). Assuming $a_k = a + (k-1)d$, we have by the definition of an arithmetic sequence that $a_{k+1} = a_k + d$ so we get $a_{k+1} = (a + (k-1)d) + d = a + kd = a + ((k+1) - 1)d$. Hence, $P(k+1)$ is true.

In essence, by showing that $P(k+1)$ must always be true when $P(k)$ is true, we are showing that the formula $P(1)$ can be used to get the formula $P(2)$, which in turn can be used to derive the formula $P(3)$, which in turn can be used to establish the formula $P(4)$, and so on. Thus as long as $P(k)$ is true for some natural number $k$, $P(n)$ is true for all of the natural numbers $n$ which follow $k$. Coupling this with the fact $P(1)$ is true, we have established $P(k)$ is true for all natural numbers which follow $n = 1$, in other words, all natural numbers $n$. One might liken Mathematical Induction to a repetitive process like climbing stairs.[2] If you are sure that (1) you can get on the stairs (the base case) and (2) you can climb from any one step to the next step (the inductive step), then presumably you can climb the entire staircase.[3] We get some more practice with induction in the following example.

**Example 9.3.1.** Prove the following assertions using the Principle of Mathematical Induction.

1. The sum formula for arithmetic sequences: $\displaystyle\sum_{j=1}^{n}(a + (j-1)d) = \frac{n}{2}(2a + (n-1)d)$.

2. For a complex number $z$, $(\overline{z})^n = \overline{z^n}$ for $n \geq 1$.

3. $3^n > 100n$ for $n > 5$.

4. Let $A$ be an $n \times n$ matrix and let $A'$ be the matrix obtained by replacing a row $R$ of $A$ with $cR$ for some real number $c$. Use the definition of determinant to show $\det(A') = c\det(A)$.

**Solution.**

1. We set $P(n)$ to be the equation we are asked to prove. For $n = 1$, we compare both sides of the equation given in $P(n)$

$$\sum_{j=1}^{1}(a + (j-1)d) \;\overset{?}{=}\; \frac{1}{2}(2a + (1-1)d)$$

$$a + (1-1)d \;\overset{?}{=}\; \frac{1}{2}(2a)$$

$$a \;=\; a \checkmark$$

---

[2]Falling dominoes is the most widely used metaphor in the mainstream College Algebra books.
[3]This is how Carl climbed the stairs in the Cologne Cathedral. Well, that, and encouragement from Kai.

This shows the base case $P(1)$ is true. Next we assume $P(k)$ is true, that is, we assume

$$\sum_{j=1}^{k}(a + (j-1)d) = \frac{k}{2}(2a + (k-1)d)$$

and attempt to use this to show $P(k+1)$ is true. Namely, we must show

$$\sum_{j=1}^{k+1}(a + (j-1)d) = \frac{k+1}{2}(2a + (k+1-1)d)$$

To see how we can use $P(k)$ in this case to prove $P(k+1)$, we note that the sum in $P(k+1)$ is the sum of the first $k+1$ terms of the sequence $a_k = a + (k-1)d$ for $k \geq 1$ while the sum in $P(k)$ is the sum of the first $k$ terms. We compare both side of the equation in $P(k+1)$.

$$\underbrace{\sum_{j=1}^{k+1}(a + (j-1)d)}_{\text{summing the first } k+1 \text{ terms}} \overset{?}{=} \frac{k+1}{2}(2a + (k+1-1)d)$$

$$\underbrace{\sum_{j=1}^{k}(a + (j-1)d)}_{\text{summing the first } k \text{ terms}} + \underbrace{(a + (k+1-1)d)}_{\text{adding the } (k+1)\text{st term}} \overset{?}{=} \frac{k+1}{2}(2a + kd)$$

$$\underbrace{\frac{k}{2}(2a + (k-1)d)}_{\text{Using } P(k)} + (a + kd) \overset{?}{=} \frac{(k+1)(2a + kd)}{2}$$

$$\frac{k(2a + (k-1)d) + 2(a + kd)}{2} \overset{?}{=} \frac{2ka + k^2d + 2a + kd}{2}$$

$$\frac{2ka + 2a + k^2d + kd}{2} = \frac{2ka + 2a + k^2d + kd}{2} \checkmark$$

Since all of our steps on both sides of the string of equations are reversible, we conclude that the two sides of the equation are equivalent and hence, $P(k+1)$ is true. By the Principle of Mathematical Induction, we have that $P(n)$ is true for all natural numbers $n$.

2. We let $P(n)$ be the formula $(\overline{z})^n = \overline{z^n}$. The base case $P(1)$ is $(\overline{z})^1 = \overline{z^1}$, which reduces to $\overline{z} = \overline{z}$ which is true. We now assume $P(k)$ is true, that is, we assume $(\overline{z})^k = \overline{z^k}$ and attempt to show that $P(k+1)$ is true. Since $(\overline{z})^{k+1} = (\overline{z})^k\,\overline{z}$, we can use the induction hypothesis and

write $(\overline{z})^k = \overline{z^k}$. Hence, $(\overline{z})^{k+1} = (\overline{z})^k \, \overline{z} = \overline{z^k} \, \overline{z}$. We now use the product rule for conjugates[4] to write $\overline{z^k} \, \overline{z} = \overline{z^k z} = \overline{z^{k+1}}$. This establishes $(\overline{z})^{k+1} = \overline{z^{k+1}}$, so that $P(k+1)$ is true. Hence, by the Principle of Mathematical Induction, $(\overline{z})^n = \overline{z^n}$ for all $n \geq 1$.

3. The first wrinkle we encounter in this problem is that we are asked to prove this formula for $n > 5$ instead of $n \geq 1$. Since $n$ is a natural number, this means our base step occurs at $n = 6$. We can still use the PMI in this case, but our conclusion will be that the formula is valid for all $n \geq 6$. We let $P(n)$ be the inequality $3^n > 100n$, and check that $P(6)$ is true. Comparing $3^6 = 729$ and $100(6) = 600$, we see $3^6 > 100(6)$ as required. Next, we assume that $P(k)$ is true, that is we assume $3^k > 100k$. We need to show that $P(k+1)$ is true, that is, we need to show $3^{k+1} > 100(k+1)$. Since $3^{k+1} = 3 \cdot 3^k$, the induction hypothesis gives $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k$. We are done if we can show $300k > 100(k+1)$ for $k \geq 6$. Solving $300k > 100(k+1)$ we get $k > \frac{1}{2}$. Since $k \geq 6$, we know this is true. Putting all of this together, we have $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k > 100(k+1)$, and hence $P(k+1)$ is true. By induction, $3^n > 100n$ for all $n \geq 6$.

4. To prove this determinant property, we use induction on $n$, where we take $P(n)$ to be that the property we wish to prove is true for all $n \times n$ matrices. For the base case, we note that if $A$ is a $1 \times 1$ matrix, then $A = [a]$ so $A' = [ca]$. By definition, $\det(A) = a$ and $\det(A') = ca$ so we have $\det(A') = c \det(A)$ as required. Now suppose that the property we wish to prove is true for all $k \times k$ matrices. Let $A$ be a $(k+1) \times (k+1)$ matrix. We have two cases, depending on whether or not the row $R$ being replaced is the first row of $A$.

CASE 1: The row $R$ being replaced is the first row of $A$. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p}$$

where the $1p$ cofactor of $A'$ is $C'_{1p} = (-1)^{(1+p)} \det\left(A'_{1p}\right)$ and $A'_{1p}$ is the $k \times k$ matrix obtained by deleting the 1st row and $p$th column of $A'$.[5] Since the first row of $A'$ is $c$ times the first row of $A$, we have $a'_{1p} = c\,a_{1p}$. In addition, since the remaining rows of $A'$ are identical to those of $A$, $A'_{1p} = A_{1p}$. (To obtain these matrices, the first row of $A'$ is removed.) Hence $\det\left(A'_{1p}\right) = \det\left(A_{1p}\right)$, so that $C'_{1p} = C_{1p}$. As a result, we get

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} c\,a_{1p} C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c \det(A),$$

as required. Hence, $P(k+1)$ is true in this case, which means the result is true in this case for all natural numbers $n \geq 1$. (You'll note that we did not use the induction hypothesis at all in this case. It is possible to restructure the proof so that induction is only used where

---

[4] See Exercise 54 in Section 3.4.
[5] See Section 8.5 for a review of this notation.

it is needed. While mathematically more elegant, it is less intuitive, and we stand by our approach because of its pedagogical value.)

CASE 2: The row $R$ being replaced is the not the first row of $A$. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p},$$

where in this case, $a'_{1p} = a_{1p}$, since the first rows of $A$ and $A'$ are the same. The matrices $A'_{1p}$ and $A_{1p}$, on the other hand, are different but in a very predictable way − the row in $A'_{1p}$ which corresponds to the row $cR$ in $A'$ is exactly $c$ times the row in $A_{1p}$ which corresponds to the row $R$ in $A$. In other words, $A'_{1p}$ and $A_{1p}$ are $k \times k$ matrices which satisfy the induction hypothesis. Hence, we know $\det\left(A'_{1p}\right) = c \det\left(A_{1p}\right)$ and $C'_{1p} = c\, C_{1p}$. We get

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} a_{1p} c\, C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c \det(A),$$

which establishes $P(k+1)$ to be true. Hence by induction, we have shown that the result holds in this case for $n \geq 1$ and we are done. $\qquad\square$

While we have used the Principle of Mathematical Induction to prove some of the formulas we have merely motivated in the text, our main use of this result comes in Section 9.4 to prove the celebrated Binomial Theorem. The ardent Mathematics student will no doubt see the PMI in many courses yet to come. Sometimes it is explicitly stated and sometimes it remains hidden in the background. If ever you see a property stated as being true 'for all natural numbers $n$', it's a solid bet that the formal proof requires the Principle of Mathematical Induction.

### 9.3.1   EXERCISES

In Exercises 1 - 7, prove each assertion using the Principle of Mathematical Induction.

1. $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

3. $2^n > 500n$ for $n > 12$

4. $3^n \geq n^3$ for $n \geq 4$

5. Use the Product Rule for Absolute Value to show $|x^n| = |x|^n$ for all real numbers $x$ and all natural numbers $n \geq 1$

6. Use the Product Rule for Logarithms to show $\log(x^n) = n\log(x)$ for all real numbers $x > 0$ and all natural numbers $n \geq 1$.

7. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ for $n \geq 1$.

8. Prove Equations 9.1 and 9.2 for the case of geometric sequences. That is:

    (a) For the sequence $a_1 = a$, $a_{n+1} = ra_n$, $n \geq 1$, prove $a_n = ar^{n-1}$, $n \geq 1$.

    (b) $\displaystyle\sum_{j=1}^{n} ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$, $\displaystyle\sum_{j=1}^{n} ar^{n-1} = na$, if $r = 1$.

9. Prove that the determinant of a lower triangular matrix is the product of the entries on the main diagonal. (See Exercise 8.3.1 in Section 8.3.) Use this result to then show $\det(I_n) = 1$ where $I_n$ is the $n \times n$ identity matrix.

10. Discuss the classic 'paradox' All Horses are the Same Color problem with your classmates.

## 9.3.2   SELECTED ANSWERS

1. Let $P(n)$ be the sentence $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. For the base case, $n = 1$, we get

$$\sum_{j=1}^{1} j^2 \overset{?}{=} \frac{(1)(1+1)(2(1)+1)}{6}$$

$$1^2 = 1 \checkmark$$

We now assume $P(k)$ is true and use it to show $P(k+1)$ is true. We have

$$\sum_{j=1}^{k+1} j^2 \overset{?}{=} \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\sum_{j=1}^{k} j^2 + (k+1)^2 \overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\underbrace{\frac{k(k+1)(2k+1)}{6}}_{\text{Using } P(k)} + (k+1)^2 \overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{(k+1)(k(2k+1) + 6(k+1))}{6} \overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{(k+1)\left(2k^2 + 7k + 6\right)}{6} \overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \checkmark$$

By induction, $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all natural numbers $n \geq 1$.

4. Let $P(n)$ be the sentence $3^n > n^3$. Our base case is $n = 4$ and we check $3^4 = 81$ and $4^3 = 64$ so that $3^4 > 4^3$ as required. We now assume $P(k)$ is true, that is $3^k > k^3$, and try to show $P(k+1)$ is true. We note that $3^{k+1} = 3 \cdot 3^k > 3k^3$ and so we are done if we can show $3k^3 > (k+1)^3$ for $k \geq 4$. We can solve the inequality $3x^3 > (x+1)^3$ using the techniques of Section 5.3, and doing so gives us $x > \frac{1}{\sqrt[3]{3}-1} \approx 2.26$. Hence, for $k \geq 4$, $3^{k+1} = 3 \cdot 3^k > 3k^3 > (k+1)^3$ so that $3^{k+1} > (k+1)^3$. By induction, $3^n > n^3$ is true for all natural numbers $n \geq 4$.

6. Let $P(n)$ be the sentence $\log(x^n) = n\log(x)$. For the duration of this argument, we assume $x > 0$. The base case $P(1)$ amounts checking that $\log(x^1) = 1\log(x)$ which is clearly true. Next we assume $P(k)$ is true, that is $\log(x^k) = k\log(x)$ and try to show $P(k+1)$ is true. Using the Product Rule for Logarithms along with the induction hypothesis, we get

$$\log\left(x^{k+1}\right) = \log\left(x^k \cdot x\right) = \log\left(x^k\right) + \log(x) = k\log(x) + \log(x) = (k+1)\log(x)$$

Hence, $\log\left(x^{k+1}\right) = (k+1)\log(x)$. By induction $\log(x^n) = n\log(x)$ is true for all $x > 0$ and all natural numbers $n \geq 1$.

9. Let $A$ be an $n \times n$ lower triangular matrix. We proceed to prove the $\det(A)$ is the product of the entries along the main diagonal by inducting on $n$. For $n = 1$, $A = [a]$ and $\det(A) = a$, so the result is (trivially) true. Next suppose the result is true for $k \times k$ lower triangular matrices. Let $A$ be a $(k+1) \times (k+1)$ lower triangular matrix. Expanding $\det(A)$ along the first row, we have

$$\det(A) = \sum_{p=1}^{n} a_{1p} C_{1p}$$

Since $a_{1p} = 0$ for $2 \leq p \leq k+1$, this simplifies $\det(A) = a_{11} C_{11}$. By definition, we know that $C_{11} = (-1)^{1+1}\det(A_{11}) = \det(A_{11})$ where $A_{11}$ is $k \times k$ matrix obtained by deleting the first row and first column of $A$. Since $A$ is lower triangular, so is $A_{11}$ and, as such, the induction hypothesis applies to $A_{11}$. In other words, $\det(A_{11})$ is the product of the entries along $A_{11}$'s main diagonal. Now, the entries on the main diagonal of $A_{11}$ are the entries $a_{22}$, $a_{33}$, ..., $a_{(k+1)(k+1)}$ from the main diagonal of $A$. Hence,

$$\det(A) = a_{11}\det(A_{11}) = a_{11}\left(a_{22}a_{33}\cdots a_{(k+1)(k+1)}\right) = a_{11}a_{22}a_{33}\cdots a_{(k+1)(k+1)}$$

We have $\det(A)$ is the product of the entries along its main diagonal. This shows $P(k+1)$ is true, and, hence, by induction, the result holds for all $n \times n$ upper triangular matrices. The $n \times n$ identity matrix $I_n$ is a lower triangular matrix whose main diagonal consists of all 1's. Hence, $\det(I_n) = 1$, as required.

## 9.4 The Binomial Theorem

In this section, we aim to prove the celebrated **Binomial Theorem**. Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers $n$. In Elementary and Intermediate Algebra, you should have seen specific instances of the formula, namely

$$
\begin{array}{rcl}
(a+b)^1 & = & a+b \\
(a+b)^2 & = & a^2 + 2ab + b^2 \\
(a+b)^3 & = & a^3 + 3a^2b + 3ab^2 + b^3
\end{array}
$$

If we wanted the expansion for $(a+b)^4$ we would write $(a+b)^4 = (a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get $(a+b)^4 = (a+b)\left(a^3 + 3a^2b + 3ab^2 + b^3\right) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Generalizing this a bit, we see that if we have a formula for $(a+b)^k$, we can obtain a formula for $(a+b)^{k+1}$ by rewriting the latter as $(a+b)^{k+1} = (a+b)(a+b)^k$. Clearly this means Mathematical Induction plays a major role in the proof of the Binomial Theorem.[1] Before we can state the theorem we need to revisit the sequence of factorials which were introduced in Example 9.1.1 number 6 in Section 9.1.

---

**Definition 9.4. Factorials:** For a whole number $n$, **$n$ factorial**, denoted $n!$, is the term $f_n$ of the sequence $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$.

---

Recall this means $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. Using the recursive definition, we get: $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Informally, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as our 'base case.' Our first example familiarizes us with some of the basic computations involving factorials.

**Example 9.4.1.**

1. Simplify the following expressions.

   (a) $\dfrac{3! \, 2!}{0!}$ 
   (b) $\dfrac{7!}{5!}$ 
   (c) $\dfrac{1000!}{998! \, 2!}$ 
   (d) $\dfrac{(k+2)!}{(k-1)!}$, $k \geq 1$

2. Prove $n! > 3^n$ for all $n \geq 7$.

**Solution.**

1. We keep in mind the mantra, "When in doubt, write it out!" as we simplify the following.

   (a) We have been programmed to react with alarm to the presence of a 0 in the denominator, but in this case $0! = 1$, so the fraction is defined after all. As for the numerator, $3! = 3 \cdot 2 \cdot 1 = 6$ and $2! = 2 \cdot 1 = 2$, so we have $\frac{3! \, 2!}{0!} = \frac{(6)(2)}{1} = 12$.

---

[1]It's pretty much the reason Section 9.3 is in the book.

(b) We have $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ while $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Dividing, we get $\frac{7!}{5!} = \frac{5040}{120} = 42$. While this is correct, we note that we could have saved ourselves some of time had we proceeded as follows

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42$$

In fact, should we want to fully exploit the recursive nature of the factorial, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$$

(c) Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998! \, 2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

(d) This problem continues the theme which we have seen in the previous two problems. We first note that since $k + 2$ is larger than $k - 1$, $(k + 2)!$ contains all of the factors of $(k - 1)!$ and as a result we can get the $(k - 1)!$ to cancel from the denominator. To see this, we begin by writing out $(k + 2)!$ starting with $(k + 2)$ and multiplying it by the numbers which precede it until we reach $(k - 1)$: $(k + 2)! = (k + 2)(k + 1)(k)(k - 1)!$. As a result, we have

$$\frac{(k + 2)!}{(k - 1)!} = \frac{(k + 2)(k + 1)(k)(k - 1)!}{(k - 1)!} = \frac{(k + 2)(k + 1)(k)\cancel{(k - 1)!}}{\cancel{(k - 1)!}} = k(k + 1)(k + 2)$$

The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

2. We proceed by induction and let $P(n)$ be the inequality $n! > 3^n$. The base case here is $n = 7$ and we see that $7! = 5040$ is larger than $3^7 = 2187$, so $P(7)$ is true. Next, we assume that $P(k)$ is true, that is, we assume $k! > 3^k$ and attempt to show $P(k+1)$ follows. Using the properties of the factorial, we have $(k + 1)! = (k + 1)k!$ and since $k! > 3^k$, we have $(k + 1)! > (k + 1)3^k$. Since $k \geq 7$, $k + 1 \geq 8$, so $(k + 1)3^k \geq 8 \cdot 3^k > 3 \cdot 3^k = 3^{k+1}$. Putting all of this together, we have $(k + 1)! = (k + 1)k! > (k + 1)3^k > 3^{k+1}$ which shows $P(k + 1)$ is true. By the Principle of Mathematical Induction, we have $n! > 3^n$ for all $n \geq 7$.                                                 □

Of all of the mathematical animals we have discussed in the text, factorials grow most quickly. In problem 2 of Example 9.4.1, we proved that $n!$ overtakes $3^n$ at $n = 7$. 'Overtakes' may be too polite a word, since $n!$ thoroughly trounces $3^n$ for $n \geq 7$, as any reasonable set of data will show. It can be shown that for any real number $x > 0$, not only does $n!$ eventually overtake $x^n$, but the ratio $\frac{x^n}{n!} \to 0$ as $n \to \infty$.[2]

Applications of factorials in the wild often involve counting arrangements. For example, if you have fifty songs on your mp3 player and wish arrange these songs in a playlist in which the order of the

---

[2]This fact is far more important than you could ever possibly imagine.

songs matters, it turns out that there are 50! different possible playlists. If you wish to select only ten of the songs to create a playlist, then there are $\frac{50!}{40!}$ such playlists. If, on the other hand, you just want to select ten song files out of the fifty to put on a flash memory card so that now the order no longer matters, there are $\frac{50!}{40!10!}$ ways to achieve this.[3] While some of these ideas are explored in the Exercises, the authors encourage you to take courses such as Finite Mathematics, Discrete Mathematics and Statistics. We introduce these concepts here because this is how the factorials make their way into the Binomial Theorem, as our next definition indicates.

---

**Definition 9.5. Binomial Coefficients:** Given two whole numbers $n$ and $j$ with $n \geq j$, the binomial coefficient $\dbinom{\boldsymbol{n}}{\boldsymbol{j}}$ (read, $n$ choose $j$) is the whole number given by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

---

The name 'binomial coefficient' will be justified shortly. For now, we can physically interpret $\binom{n}{j}$ as the number of ways to select $j$ items from $n$ items where the order of the items selected is unimportant. For example, suppose you won two free tickets to a special screening of the latest Hollywood blockbuster and have five good friends each of whom would love to accompany you to the movies. There are $\binom{5}{2}$ ways to choose who goes with you. Applying Definition 9.5, we get

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$$

So there are 10 different ways to distribute those two tickets among five friends. (Some will see it as 10 ways to decide which three friends have to stay home.) The reader is encouraged to verify this by actually taking the time to list all of the possibilities.

We now state anf prove a theorem which is crucial to the proof of the Binomial Theorem.

---

**Theorem 9.3.** For natural numbers $n$ and $j$ with $n \geq j$,

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$$

---

The proof of Theorem 9.3 is purely computational and uses the definition of binomial coefficients, the recursive property of factorials and common denominators.

---

[3]For reference,

$$
\begin{aligned}
50! &= 30414093201713378043612608166064768844377641568960512000000000000, \\
\frac{50!}{40!} &= 37276043023296000, \quad \text{and} \\
\frac{50!}{40!10!} &= 10272278170
\end{aligned}
$$

$$
\begin{aligned}
\binom{n}{j-1} + \binom{n}{j} &= \frac{n!}{(j-1)!(n-(j-1))!} + \frac{n!}{j!(n-j)!} \\[2mm]
&= \frac{n!}{(j-1)!(n-j+1)!} + \frac{n!}{j!(n-j)!} \\[2mm]
&= \frac{n!}{(j-1)!(n-j+1)(n-j)!} + \frac{n!}{j(j-1)!(n-j)!} \\[2mm]
&= \frac{n!\,j}{j(j-1)!(n-j+1)(n-j)!} + \frac{n!(n-j+1)}{j(j-1)!(n-j+1)(n-j)!} \\[2mm]
&= \frac{n!\,j}{j!(n-j+1)!} + \frac{n!(n-j+1)}{j!(n-j+1)!} \\[2mm]
&= \frac{n!\,j + n!(n-j+1)}{j!(n-j+1)!} \\[2mm]
&= \frac{n!\,(j+(n-j+1))}{j!(n-j+1)!} \\[2mm]
&= \frac{(n+1)n!}{j!(n+1-j))!} \\[2mm]
&= \frac{(n+1)!}{j!((n+1)-j))!} \\[2mm]
&= \binom{n+1}{j} \ \checkmark
\end{aligned}
$$

We are now in position to state and prove the Binomial Theorem where we see that binomial coefficients are just that - coefficients in the binomial expansion.

---

**Theorem 9.4. Binomial Theorem:** For nonzero real numbers $a$ and $b$,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

for all natural numbers $n$.

---

To get a feel of what this theorem is saying and how it really isn't as hard to remember as it may first appear, let's consider the specific case of $n = 4$. According to the theorem, we have

$$
\begin{aligned}
(a+b)^4 &= \sum_{j=0}^{4} \binom{4}{j} a^{4-j} b^{j} \\
&= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^{4-4} b^4 \\
&= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4
\end{aligned}
$$

We forgo the simplification of the coefficients in order to note the pattern in the expansion. First note that in each term, the total of the exponents is 4 which matched the exponent of the binomial $(a+b)^4$. The exponent on $a$ begins at 4 and decreases by one as we move from one term to the next while the exponent on $b$ starts at 0 and increases by one each time. Also note that the binomial coefficients themselves have a pattern. The upper number, 4, matches the exponent on the binomial $(a+b)^4$ whereas the lower number changes from term to term and matches the exponent of $b$ in that term. This is no coincidence and corresponds to the kind of counting we discussed earlier. If we think of obtaining $(a+b)^4$ by multiplying $(a+b)(a+b)(a+b)(a+b)$, our answer is the sum of all possible products with exactly four factors - some $a$, some $b$. If we wish to count, for instance, the number of ways we obtain 1 factor of $b$ out of a total of 4 possible factors, thereby forcing the remaining 3 factors to be $a$, the answer is $\binom{4}{1}$. Hence, the term $\binom{4}{1} a^3 b$ is in the expansion. The other terms which appear cover the remaining cases. While this discussion gives an indication as to *why* the theorem is true, a formal proof requires Mathematical Induction.[4]

To prove the Binomial Theorem, we let $P(n)$ be the expansion formula given in the statement of the theorem and we note that $P(1)$ is true since

$$
\begin{aligned}
(a+b)^1 &\stackrel{?}{=} \sum_{j=0}^{1} \binom{1}{j} a^{1-j} b^{j} \\
a+b &\stackrel{?}{=} \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1 \\
a+b &= a+b \checkmark
\end{aligned}
$$

Now we assume that $P(k)$ is true. That is, we assume that we can expand $(a+b)^k$ using the formula given in Theorem 9.4 and attempt to show that $P(k+1)$ is true.

---

[4]and a fair amount of tenacity and attention to detail.

$$
\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\[2em]
&= (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\[2em]
&= a\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j + b\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\[2em]
&= \sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1}
\end{aligned}
$$

Our goal is to combine as many of the terms as possible within the two summations. As the counter $j$ in the first summation runs from 0 through $k$, we get terms involving $a^{k+1}$, $a^k b$, $a^{k-1}b^2$, ..., $ab^k$. In the second summation, we get terms involving $a^k b$, $a^{k-1}b^2$, ..., $ab^k$, $b^{k+1}$. In other words, apart from the first term in the first summation and the last term in the second summation, we have terms common to both summations. Our next move is to 'kick out' the terms which we cannot combine and rewrite the summations so that we can combine them. To that end, we note

$$
\sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j
$$

and

$$
\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1} = \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}
$$

so that

$$
(a+b)^{k+1} = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}
$$

We now wish to write

$$
\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}
$$

as a single summation. The wrinkle is that the first summation starts with $j = 1$, while the second starts with $j = 0$. Even though the sums produce terms with the same powers of $a$ and $b$, they do so for different values of $j$. To resolve this, we need to shift the index on the second summation so that the index $j$ starts at $j = 1$ instead of $j = 0$ and we make use of Theorem 9.1 in the process.

$$\sum_{j=0}^{k-1} \binom{k}{j} a^{k-j} b^{j+1} \;=\; \sum_{j=0+1}^{k-1+1} \binom{k}{j-1} a^{k-(j-1)} b^{(j-1)+1}$$

$$= \sum_{j=1}^{k} \binom{k}{j-1} a^{k+1-j} b^{j}$$

We can now combine our two sums using Theorem 9.1 and simplify using Theorem 9.3

$$\sum_{j=1}^{k} \binom{k}{j} a^{k+1-j} b^{j} + \sum_{j=0}^{k-1} \binom{k}{j} a^{k-j} b^{j+1} \;=\; \sum_{j=1}^{k} \binom{k}{j} a^{k+1-j} b^{j} + \sum_{j=1}^{k} \binom{k}{j-1} a^{k+1-j} b^{j}$$

$$= \sum_{j=1}^{k} \left[ \binom{k}{j} + \binom{k}{j-1} \right] a^{k+1-j} b^{j}$$

$$= \sum_{j=1}^{k} \binom{k+1}{j} a^{k+1-j} b^{j}$$

Using this and the fact that $\binom{k+1}{0} = 1$ and $\binom{k+1}{k+1} = 1$, we get

$$(a+b)^{k+1} \;=\; a^{k+1} + \sum_{j=1}^{k} \binom{k+1}{j} a^{k+1-j} b^{j} + b^{k+1}$$

$$= \binom{k+1}{0} a^{k+1} b^0 + \sum_{j=1}^{k} \binom{k+1}{j} a^{k+1-j} b^{j} + \binom{k+1}{k+1} a^0 b^{k+1}$$

$$= \sum_{j=0}^{k+1} \binom{k+1}{j} a^{(k+1)-j} b^{j}$$

which shows that $P(k+1)$ is true. Hence, by induction, we have established that the Binomial Theorem holds for all natural numbers $n$.

**Example 9.4.2.** Use the Binomial Theorem to find the following.

1. $(x - 2)^4$                                              2. $2.1^3$

3. The term containing $x^3$ in the expansion $(2x + y)^5$

**Solution.**

1. Since $(x - 2)^4 = (x + (-2))^4$, we identify $a = x$, $b = -2$ and $n = 4$ and obtain

$$(x-2)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j}(-2)^j$$

$$= \binom{4}{0} x^{4-0}(-2)^0 + \binom{4}{1} x^{4-1}(-2)^1 + \binom{4}{2} x^{4-2}(-2)^2 + \binom{4}{3} x^{4-3}(-2)^3 + \binom{4}{4} x^{4-4}(-2)^4$$

$$= x^4 - 8x^3 + 24x^2 - 32x + 16$$

2. At first this problem seem misplaced, but we can write $2.1^3 = (2 + 0.1)^3$. Identifying $a = 2$, $b = 0.1 = \frac{1}{10}$ and $n = 3$, we get

$$\left(2 + \frac{1}{10}\right)^3 = \sum_{j=0}^{3} \binom{3}{j} 2^{3-j} \left(\frac{1}{10}\right)^j$$

$$= \binom{3}{0} 2^{3-0} \left(\frac{1}{10}\right)^0 + \binom{3}{1} 2^{3-1} \left(\frac{1}{10}\right)^1 + \binom{3}{2} 2^{3-2} \left(\frac{1}{10}\right)^2 + \binom{3}{3} 2^{3-3} \left(\frac{1}{10}\right)^3$$

$$= 8 + \frac{12}{10} + \frac{6}{100} + \frac{1}{1000}$$

$$= 8 + 1.2 + 0.06 + 0.001$$

$$= 9.261$$

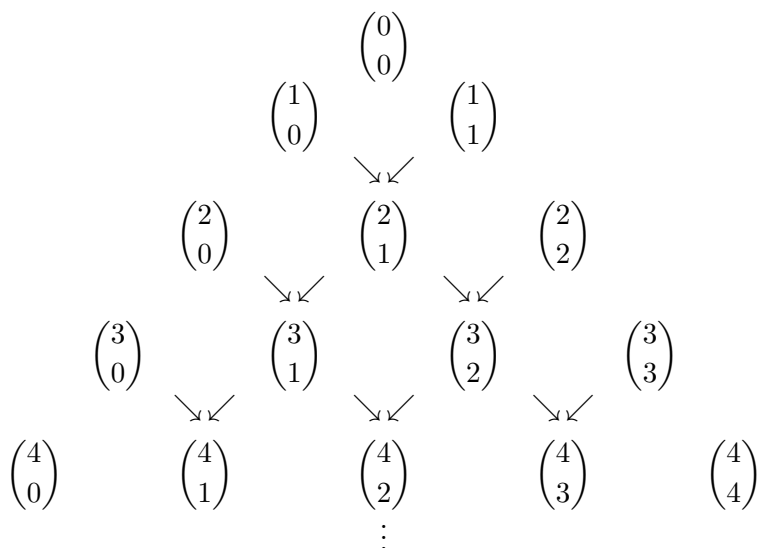3. Identifying $a = 2x$, $b = y$ and $n = 5$, the Binomial Theorem gives

$$(2x + y)^5 = \sum_{j=0}^{5} \binom{5}{j} (2x)^{5-j} y^j$$

Since we are concerned with only the term containing $x^3$, there is no need to expand the entire sum. The exponents on each term must add to 5 and if the exponent on $x$ is 3, the exponent on $y$ must be 2. Plucking out the term $j = 2$, we get

$$\binom{5}{2} (2x)^{5-2} y^2 = 10(2x)^3 y^2 = 80x^3 y^2$$

$\square$

We close this section with Pascal's Triangle, named in honor of the mathematician Blaise Pascal. Pascal's Triangle is obtained by arranging the binomial coefficients in the triangular fashion below.

$$\binom{0}{0}$$

$$\binom{1}{0} \qquad \binom{1}{1}$$

$$\binom{2}{0} \qquad \binom{2}{1} \qquad \binom{2}{2}$$

$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$

$$\binom{4}{0} \qquad \binom{4}{1} \qquad \binom{4}{2} \qquad \binom{4}{3} \qquad \binom{4}{4}$$

$$\vdots$$

Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all whole numbers $n$, we get that each row of Pascal's Triangle begins and ends with 1. To generate the numbers in the middle of the rows (from the third row onwards), we take advantage of the additive relationship expressed in Theorem 9.3. For instance, $\binom{1}{0} + \binom{1}{1} = \binom{2}{1}$, $\binom{2}{0} + \binom{2}{1} = \binom{3}{1}$ and so forth. This relationship is indicated by the arrows in the array above. With these two facts in hand, we can quickly generate Pascal's Triangle. We start with the first two rows, 1 and 1   1. From that point on, each successive row begins and ends with 1 and the middle numbers are generated using Theorem 9.3. Below we attempt to demonstrate this building process to generate the first five rows of Pascal's Triangle.
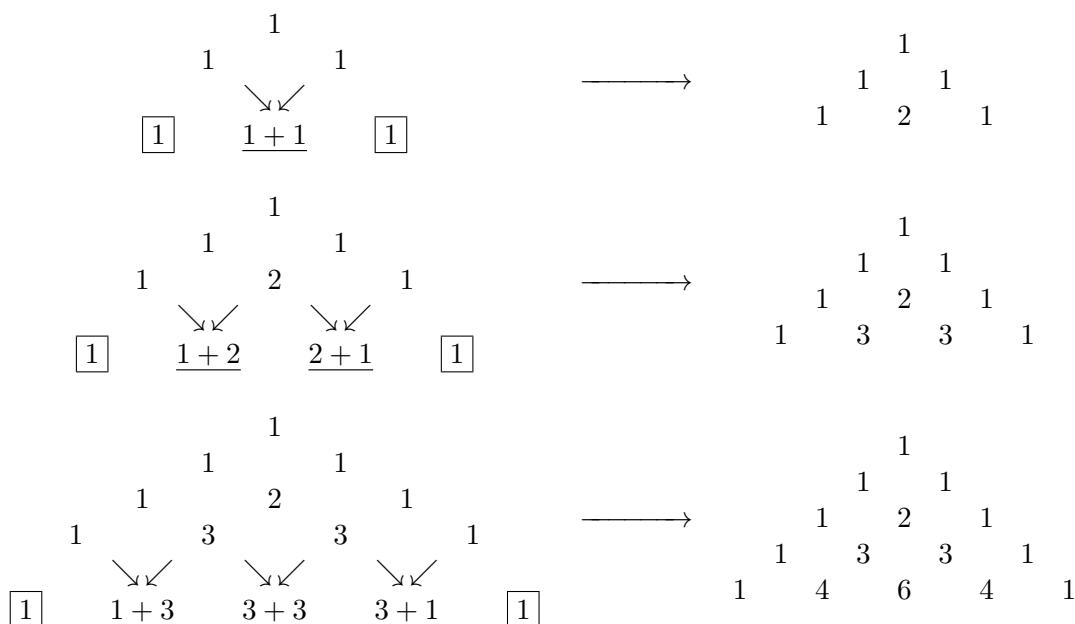
To see how we can use Pascal's Triangle to expedite the Binomial Theorem, suppose we wish to expand $(3x - y)^4$. The coefficients we need are $\binom{4}{j}$ for $j = 0, 1, 2, 3, 4$ and are the numbers which form the fifth row of Pascal's Triangle. Since we know that the exponent of $3x$ in the first term is 4 and then decreases by one as we go from left to right while the exponent of $-y$ starts at 0 in the first term and then increases by one as we move from left to right, we quickly obtain

$$
\begin{aligned}
(3x - y)^4 &= (1)(3x)^4 + (4)(3x)^3(-y) + (6)(3x)^2(-y)^2 + 4(3x)(-y)^3 + 1(-y)^4 \\
&= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4
\end{aligned}
$$

We would like to stress that Pascal's Triangle is a very quick method to expand an *entire* binomial. If only a term (or two or three) is required, then the Binomial Theorem is definitely the way to go.

### 9.4.1   Exercises

In Exercises 1 - 9, simplify the given expression.

1. $(3!)^2$

2. $\dfrac{10!}{7!}$

3. $\dfrac{7!}{2^3 3!}$

4. $\dfrac{9!}{4!3!2!}$

5. $\dfrac{(n+1)!}{n!}$, $n \geq 0$.

6. $\dfrac{(k-1)!}{(k+2)!}$, $k \geq 1$.

7. $\dbinom{8}{3}$

8. $\dbinom{117}{0}$

9. $\dbinom{n}{n-2}$, $n \geq 2$

In Exercises 10 - 13, use Pascal's Triangle to expand the given binomial.

10. $(x+2)^5$

11. $(2x-1)^4$

12. $\left(\frac{1}{3}x + y^2\right)^3$

13. $\left(x - x^{-1}\right)^4$

In Exercises 14 - 17, use Pascal's Triangle to simplify the given power of a complex number.

14. $(1+2i)^4$

15. $\left(-1 + i\sqrt{3}\right)^3$

16. $\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)^3$

17. $\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)^4$

In Exercises 18 - 22, use the Binomial Theorem to find the indicated term.

18. The term containing $x^3$ in the expansion $(2x - y)^5$

19. The term containing $x^{117}$ in the expansion $(x + 2)^{118}$

20. The term containing $x^{\frac{7}{2}}$ in the expansion $(\sqrt{x} - 3)^8$

21. The term containing $x^{-7}$ in the expansion $\left(2x - x^{-3}\right)^5$

22. The constant term in the expansion $\left(x + x^{-1}\right)^8$

23. Use the Prinicple of Mathematical Induction to prove $n! > 2^n$ for $n \geq 4$.

24. Prove $\displaystyle\sum_{j=0}^{n} \binom{n}{j} = 2^n$ for all natural numbers $n$. (HINT: Use the Binomial Theorem!)

25. With the help of your classmates, research Patterns and Properties of Pascal's Triangle.

26. You've just won three tickets to see the new film, '8.$\overline{9}$.' Five of your friends, Albert, Beth, Chuck, Dan, and Eugene, are interested in seeing it with you. With the help of your classmates, list all the possible ways to distribute your two extra tickets among your five friends. Now suppose you've come down with the flu. List all the different ways you can distribute the three tickets among these five friends. How does this compare with the first list you made? What does this have to do with the fact that $\binom{5}{2} = \binom{5}{3}$?

## 9.4.2 ANSWERS

1. 36

2. 720

3. 105

4. 1260

5. $n + 1$

6. $\frac{1}{k(k+1)(k+2)}$

7. 56

8. 1

9. $\frac{n(n-1)}{2}$

10. $(x + 2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

11. $(2x - 1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$

12. $\left(\frac{1}{3}x + y^2\right)^3 = \frac{1}{27}x^3 + \frac{1}{3}x^2y^2 + xy^4 + y^6$

13. $\left(x - x^{-1}\right)^4 = x^4 - 4x^2 + 6 - 4x^{-2} + x^{-4}$

14. $-7 - 24i$

15. 8

16. $i$

17. $-1$

18. $80x^3y^2$

19. $236x^{117}$

20. $-24x^{\frac{7}{2}}$

21. $-40x^{-7}$

22. 70