PHYS 512 - Problem Set 5 Solutions

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1 Question 1

A function was written to shift an array by an arbitrary amount using a convolution. This function (shift_conv) is written in the functions.py file. We get the following:

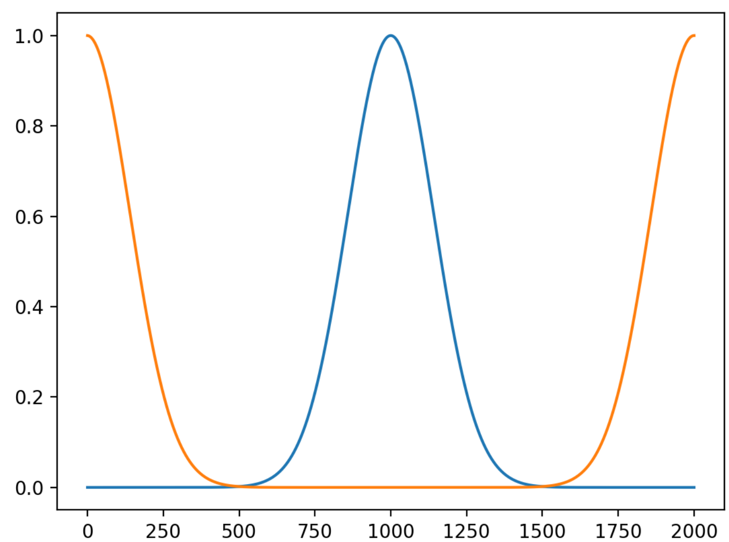

Figure 1: Shift the a Gaussian by half the array length.

2 Question 2

It can be shown that the correlation function $f \star g = ift(dft(f) \times conj(dft(g)))$. This function (corr) is also written in the functions.py file and calling the corr of a gaussian and itself, we get the following:

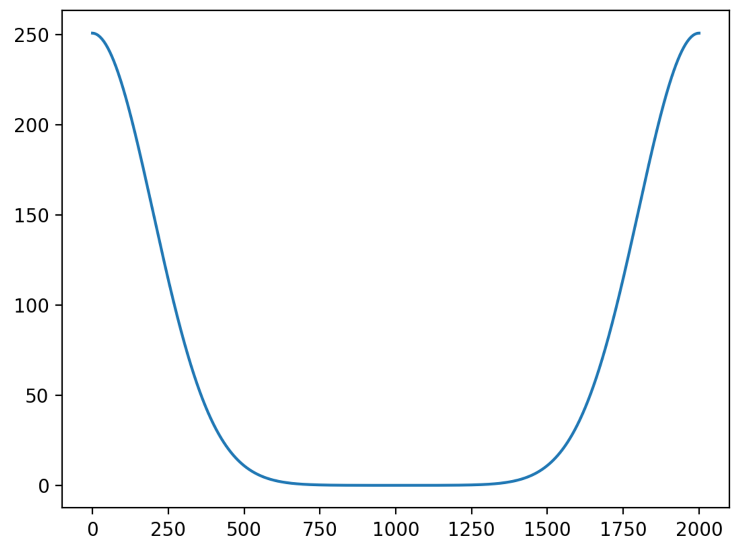

Figure 2: Calculate the correlation of a gaussian with itself.

3 Question 3

Using the functions from 1 and 2, we can plot the correlation of a gaussian with a shifted gaussian:

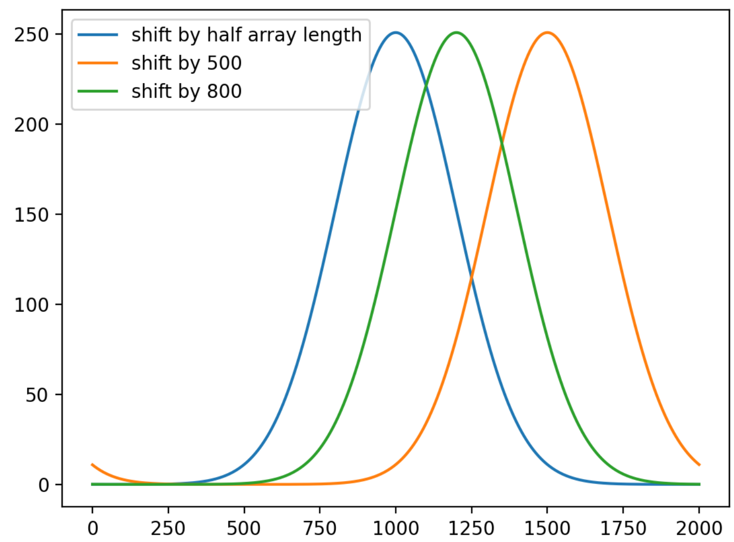

Figure 3: Correlation between gaussian and a shifted gaussian.

We observe that the correlation function shifts by the amount the other gaussian is shifted from the tail end of the gaussian.

4 Question 4

In order to avoid any wrap around issues with the fourier transform (notice in shift by 500 in Fig 3), we can pad either end of the function to avoid this problem. To be safe, I padded the array such that the final array would be sum of the lengths of the initial arrays. Doing so, and applying a fourier transform, we get the following:

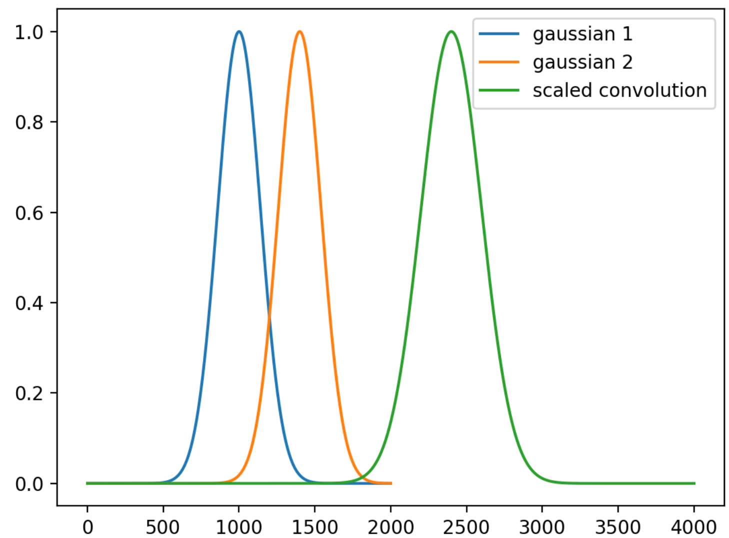

Figure 4: Comparison of the gaussians and the convolution.

5 Question 5

5.1 Show that:

$$\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = \frac{1 - exp(-2\pi ik)}{1 - exp(-2\pi ik/N)}$$

First, let's observe the sum of the first few terms of the summation:

$$\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = 1 + exp(-2\pi ik/N) + exp(-4\pi ik/N) + exp(-6\pi ik/N) + \cdots .$$

We see that the series starts at one and increases by a factor of $exp(-2\pi ik/N)$ between consecutive terms. Thus we can see this as a geometric series when trying to evaluate the sum. We note for a geometric series starting with a and has a factor of r between terms, had the sum

$$\sum_{x} ar^x = a\left(\frac{1 - r^{n+1}}{1 - r}\right).$$

For our sum, we have $a = 1$ and $r = exp(-2\pi ik/N)$, so

$$\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = (1)\left(\frac{1-(exp(-2\pi ik/N))^{N-1+1}}{1-exp(-2\pi ik/N)}\right)$$
$$\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = \frac{1-exp(-2\pi ik)}{1-exp(-2\pi ik/N)}$$

as required to show.

5.2 Show that this approaches N as k approaches zero, and is zero for any integer k that is not a multiple of N.

Recall from previous question:

$$\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = \frac{1-exp(-2\pi ik)}{1-exp(-2\pi ik/N)}$$

Note, if we were to simply set k to 0, the numerator and denominator both goes to zero, which is undefined. Instead, we can use l'Hopital's rule when dealing with the limit:

$$\lim_{k\to 0}\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = \lim_{k\to 0}\frac{1-exp(-2\pi ik)}{1-exp(-2\pi ik/N)}$$
$$\lim_{k\to 0}\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = \lim_{k\to 0}\frac{(2\pi ik)exp(-2\pi ik)}{(2\pi ik/N)exp(-2\pi ik/N)}$$
$$\lim_{k\to 0}\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = \lim_{k\to 0} N\frac{exp(-2\pi ik)}{exp(-2\pi ik/N)}$$
$$\lim_{k\to 0}\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = N$$

as required to show.

For any integer k, we note that the numerator $exp(-2\pi ik)$ is always equal to 0. The denominator will be non-zero for any k that is not a multiple of N, as required to show.

5.3 Analytic DFT vs FFT

Using similar methods as section a, we can show that

$$\sum_{x=0}^{N-1} sin(2\pi kx/N)exp(-2\pi ikx/N) = \frac{1}{2i}\left[\frac{1-exp(-2\pi i(k'-k))}{1-exp(-2\pi i(k'-k)/N)} - \frac{1-exp(-2\pi i(k'+k))}{1-exp(-2\pi i(k'+k)/N)}\right].$$

Plotting the two methods we get:

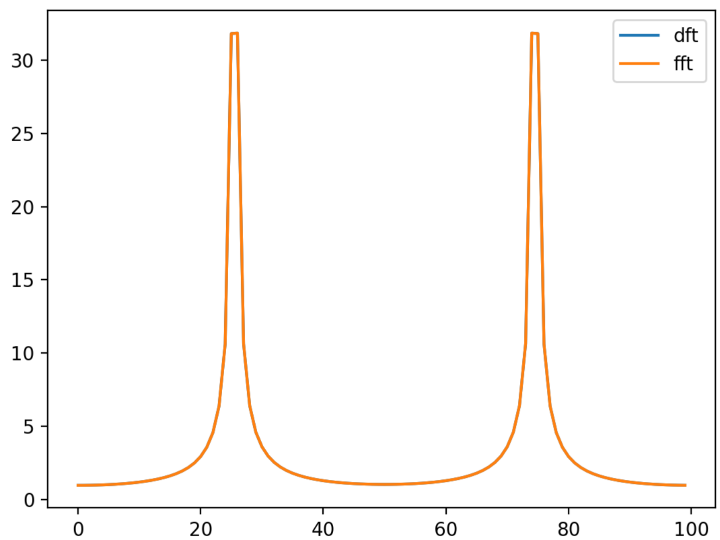

Figure 5: Analytic DFT vs FFT.

We observe that the functions are consists of two peaks at approximately $k' - k$ and $k' + k$. These functions are not quite delta functions as they have a slight slope, but they are similar to it.

Plotting the difference between the two methods, we get the following:

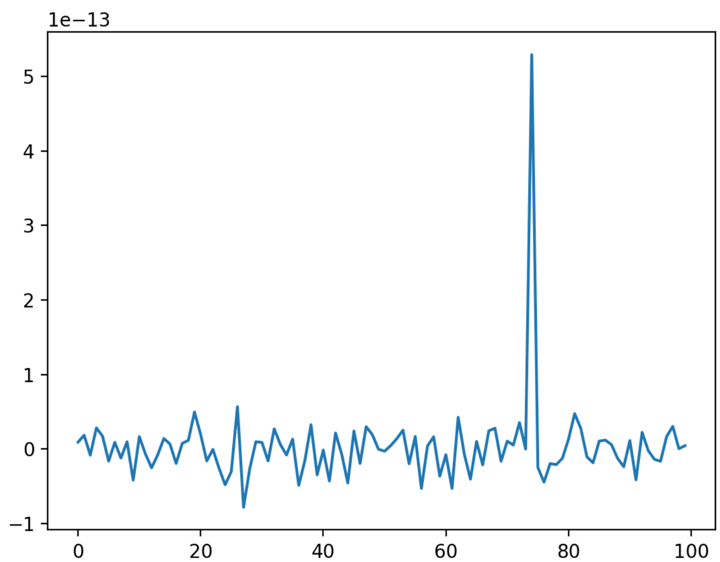

Figure 6: Difference between the two methods

From the residuals, the average difference is of order 10^{-15} which is close to machine precision.

5.4 Window Functions

Applying the window function, we observe the following:

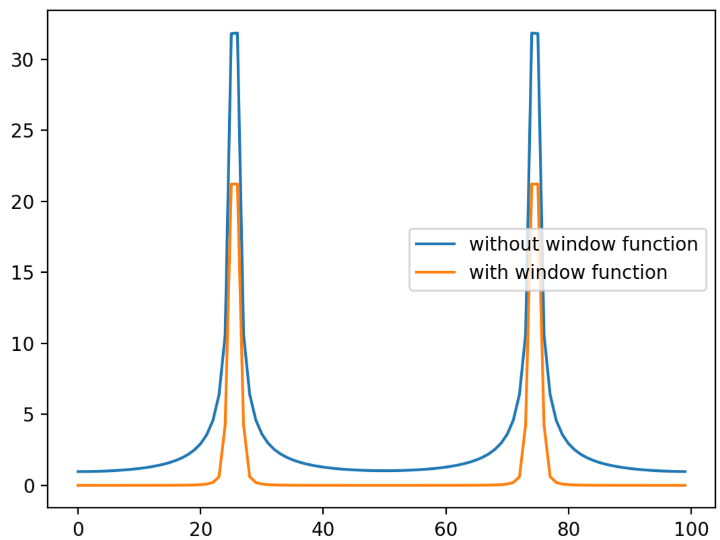

Figure 7: FFT with window function.

We see that the width of the function decreases drastically, and regions out of the peak go to 0. This is a better approximation of the expected delta function.

5.5 Window Functions Pt 2

We observe with the following plot that the fourier transform of the window functions gives values of N/2, -N/4, a series of zeros, then -N/4 again:

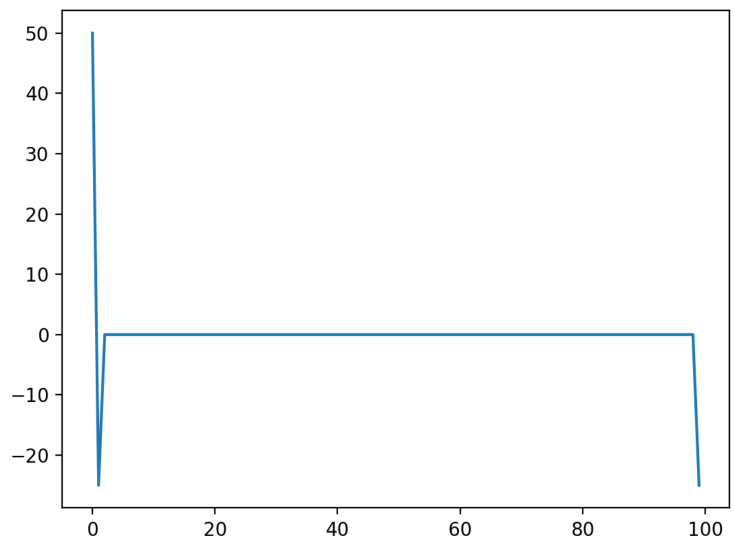

Figure 8: FFT of the window function.

Using this, we note that we can express our function

$$F(N) = -\frac{F(N-1)}{4} + \frac{F(N)}{2} - \frac{F(N+1)}{4}$$

using values from the fft of the window function. Plotting the equation above, we see that we get the same result as the previous section:

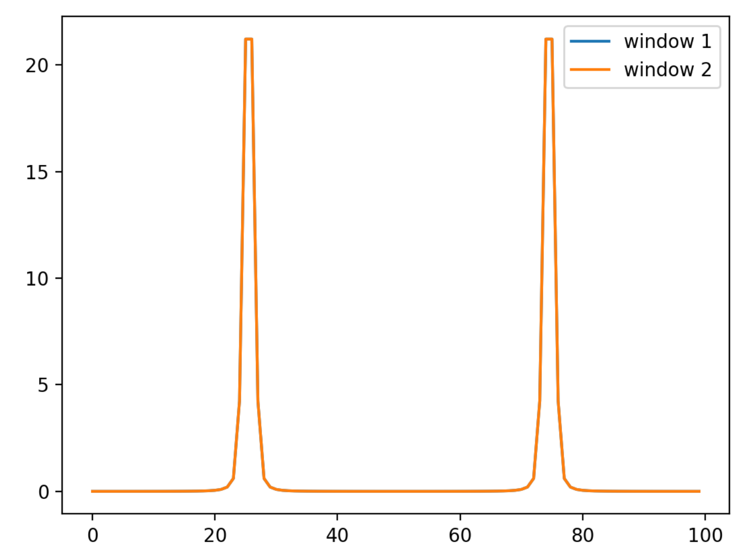

Figure 9: Comparison of the window functions using two different methods.

6 Question 6

6.1 Show power spectrum scales as factor of k^{-2}

$$< (f(x) - f(x+dx))^2 >= g(dx)$$
$$< f(x)^2 + f(x+dx)^2 - 2f(x)f(x+dx) >= ndx$$

For large timescales, $f(x+dx)^2 \approx f(x)^2$ and $< f(x)^2 >\approx N$.

$$< 2f(x)^2 - 2f(x)f(x+dx) >= ndx$$
$$2N - 2 < f(x)f(x+dx) >= ndx$$
$$< f(x)f(x+dx) >= N - ndx/2$$

Now if we take the fourier transform of this, we will see that

$$< F(k)F(k') >= c/k^2. \tag{1}$$

6.2 Random Walk

The code was written for a random walk, and using the correlation function written in question 2, the power spectrum was calculated and plotted with the k^{-2} function as shown below:

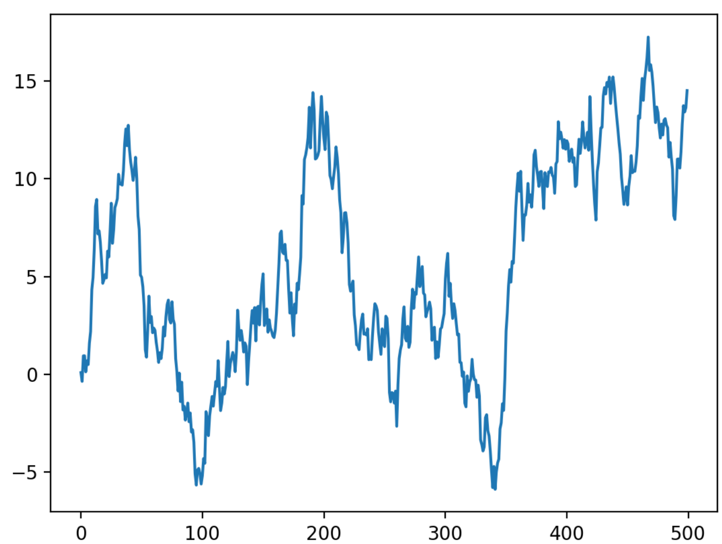

Figure 10: Random walk with 500 steps

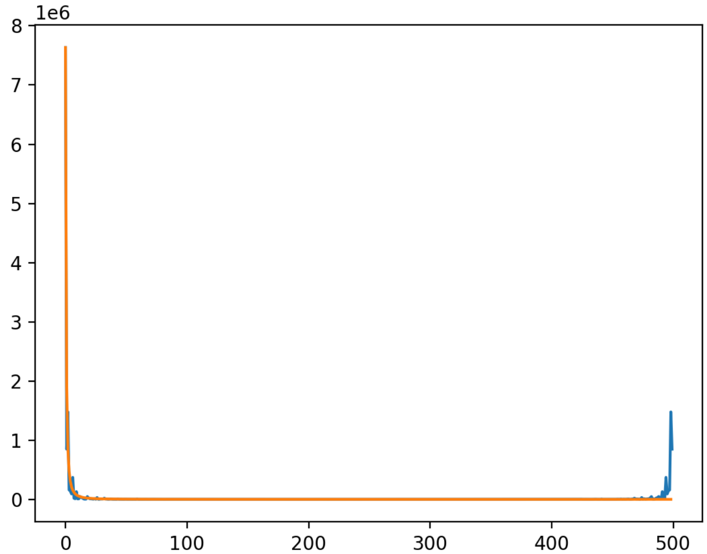

Figure 11: The power spectrum compared to the k^{-2} functions, We can see that the fit is pretty good.