PHYS 512 - Problem Set 3 Solutions Explanation

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1 Question 1

With the initial conditions of $y(-20) = 1$ for $x \in [-20, 20]$, using RK4 integrator with a single step, we get 19.940303857712177 and if we use the two steps of length $h/2$ we get 19.940307381100723. The actual value we expect to get is 19.94030691362081. We observe that the second method is closer to the actual value than with the first method.

We also note that both methods use the same number of function calls. This was achieved by observing that rk4_stepd calls rk4_step 3 times. So in order to get the same number of total function calls, the initial number of steps used for rk4_stepd is a factor of 3 smaller than rk_step. The following plot graphs the difference between the value obtained using either method and the actual value, in order to take a look at how the accuracy of the values varies between the two methods.

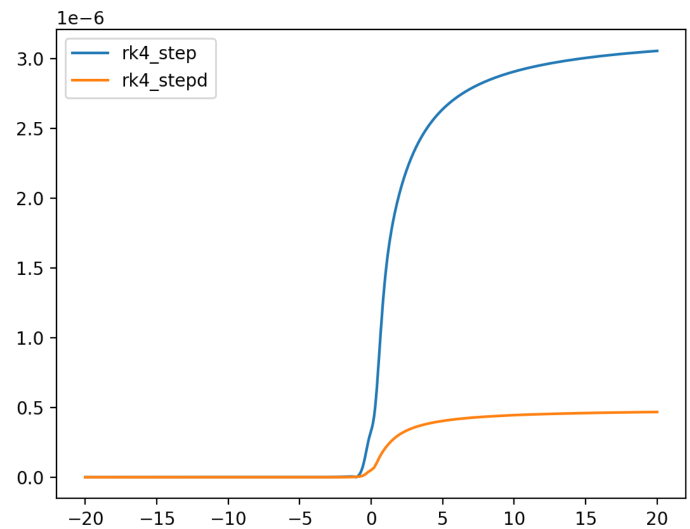

Figure 1: The difference between the actual values and values obtained using either method.

2 Question 2

2.1 Solving for decay products of U238

The Radau solver was used to solve this problem as it is much faster than using the rk4 method.

2.2 Ratio of Products

The following plots were obtained for the ratio between Pb206 and U238, and Th230 and U234:

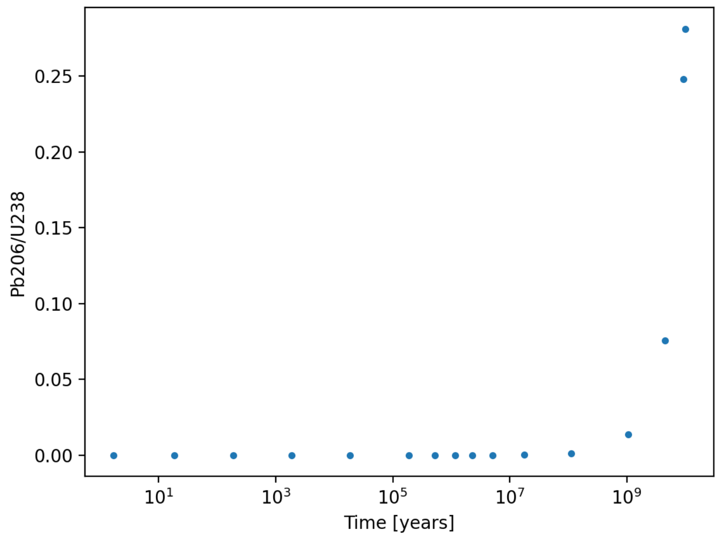

Figure 2: Modelling the ratio between Pb206 and U238 for t= 1e10 years.

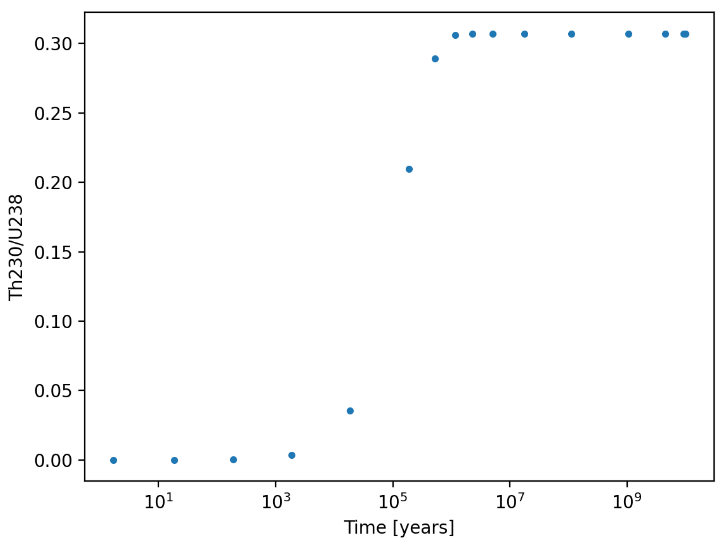

Figure 3: Modelling the ratio between Th230 and U234 for t= 1e10 years.

The first plot for the ratio between Pb206 and U238 makes sense as the amount of stable Pb206 is not produced until the very end. Thus, the graph remains at 0 until 10e9 years.

3 Question 3

3.1 New parameter set

If we expand the non-linear function as follows

$$z - z_0 = a\left((x - x_0)^2 + (y - y_0)^2\right) = a\left(x^2 + y^2 - 2xx_0 - 2yy_0 + x_0^2 + y_0^2\right), \tag{1}$$

we observe that we can make this into a linear problem by defining a function f as

$$f(x^2 + y^2, x, y) = z = a\left(x^2 + y^2\right) + bx + cy + d, \tag{2}$$

where

$$\begin{aligned} a &= a \\ b &= -2ax_0 \\ c &= -2ay_0 \\ d &= a(x_0^2 + y_0^2) + z_0 \end{aligned}$$

3.2 Best Fit Values

Now, we can fit this function as a linear problem! Assume the noise matrix to be I. The best fit parameters come out to

$$\begin{aligned} a &= 1.66704455e - 04 \\ b &= 4.53599028e - 04 \\ c &= -1.94115589e - 02 \\ d &= -1.51231182e + 03 \end{aligned}$$

and if using these values, we get the following residuals in z:

3.3 Noise Analysis

Let's take the noise of the data to be the standard deviation in the residuals. From this we can calculate the covariance matrix for our fit using the following:

$$cov = \frac{\left(A^T A\right)^{-1}}{noise}. \tag{3}$$

From this, the uncertainty in the value a is the square root of the first diagonal entry, which gives us a = 1.66704455e-04 $\pm$ 8.819880974884928e-09. Now to calculate the focal length, we observe that for a dish centered at (0,0), the function is a parabola along any vertical slice through the center. Now we note that these parabolas could be expressed as a function in cylindricals using z and r (for $r^2 = x^2 + y^2$). This means we can write the following:

$$z^2 = \frac{r^2}{4f}. \tag{4}$$

Now, we recall these points are still on the dish itself, and thus will satisfy the original equation of the paraboloid. Since we centered the dish $x_0 = y_0 = 0$. So,

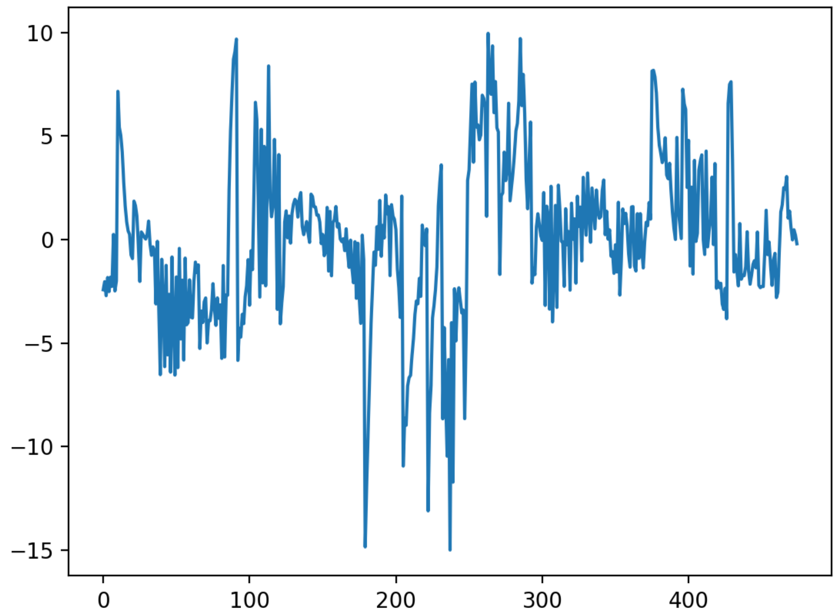

Figure 4: The difference between the z values measured and the ones obtained from fit.

$$z^2 = a(x^2 + y^2) = \frac{r^2}{4f}$$
$$a = \frac{1}{4f}$$

$$f = \frac{1}{4a}. \tag{5}$$

Great! Since we have our value for a, we can solve for f and use error propagation to calculate the uncertainty in f.

$$\delta f = \sqrt{\left(\frac{df}{da}\delta a\right)^2}$$
$$\frac{df}{da} = -\frac{1}{4a^2}$$
$$\delta f = \sqrt{\left(-\frac{1}{4a^2}\delta a\right)^2}$$

Subbing in the values of a and the uncertainty of a gives us $f = 1.4996599841252194 \pm$ 7.934294605811652e-05 m.