Covexillary Schubert varieties and Kazhdan–Lusztig polynomials

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Abstract

We establish combinatorial and inductive formulas for Kazhdan–Lusztig polynomials associated to covexillary elements in classical types, extending results of Boe, Lascoux–Schützenberger, Sankaran–Vanchinathan, and Zelevinsky for Grassmannians of classical types. The proof uses intersection cohomology theory and the isomorphism of Kazhdan–Lusztig varieties from Anderson–Ikeda–Jeon–Kawago.

Keywords: Schubert varieties; Kazhdan–Lusztig polynomials; Hilbert–Poincaré polynomials

1. Introduction

The celebrated Kazhdan–Lusztig polynomials were introduced by Kazhdan and Lusztig in their study of representations of Hecke algebras [16]. Later the Kazhdan–Lusztig polynomials $P_{v,w}(q)$ were realized as Hilbert–Poincaré polynomials for the stalk intersection cohomology sheaf of the Schubert variety X_w at a torus-fixed point $p_v \in X_w$. Here w and v are elements in the Weyl group W of a semisimple, simply connected algebraic group G over an algebraically closed field [17]. This realization is related to the singularities of Schubert varieties, since the Kazhdan–Lusztig polynomial $P_{v,w}(q)$ for $w, v \in W$ equals 1 if and only if the torus-fixed point p_v is a rationally smooth point in the Schubert variety X_w. Besides Kazhdan–Lusztig polynomials, singularities of Schubert varieties have been actively studied for decades, see, e.g., [5]. Further it has been a long standing open problem to describe the coefficients of $P_{v,w}(q)$ combinatorially. The study of Kazhdan–Lusztig polynomials is also significant in representation

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theory of Lie algebra $\mathfrak{g}$ of G, see [5, pg. 76]. For more recent developments, and further references, see [27].

Our main results give combinatorial rules for the coefficients of the Kazhdan–Lusztig polynomials associated to *covexillary elements*. Specifically, in this paper, we consider classical algebraic groups G of rank n over $\mathbb{C}$: $G = SL(n)$ for type A, $G = SO(2n + 1)$ for type B, $G = Sp(2n)$ for type C and $G = SO(2n)$ for type D.

In type A, W may be identified with the symmetric group S_n of permutations of $\{1, 2, \ldots, n\}$. The *vexillary permutations* $w \in S_n$ are those that avoid the pattern 2143; in other words, there is no sequence of four numbers $a < b < c < d$ with $w(c) < w(d) < w(a) < w(b)$. Billey and Lam [6] extended the notion of vexillary elements $w \in W$ from type A to Weyl groups for the other classical types. Anderson and Fulton [2] reformulated the definition of these vexillary elements with a "triple" in the geometric context of degeneracy loci. In this paper, we use the definition and properties of a vexillary (signed) permutation from the point of view in [2].

Recall, each Weyl group W is generated by a set of simple reflections $S = \{s_i : i = 1, 2, \ldots, r\}$ where r is the rank of the associated Lie group G. The length $\ell(w)$ of $w \in W$ is the minimal length of any factorization of $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ into simple reflections. Geometrically, $\ell(w)$ is the dimension of X_w as an algebraic variety. Now, each W has a unique longest length element, denoted $w_\circ$.

For a vexillary element w, we call $w_\circ w$ a *covexillary* element. In general, the case of covexillary elements includes the case of *cograssmannian* elements, which are those with at most one right ascent, i.e., $\ell(ws) > \ell(w)$ for at most one simple reflection $s \in W$. The terminology "cograssmannian" is borrowed from [13].

Theorem 1.1 (*Theorem* 4.7). *Let w be a covexillary element in W, and v be any element that satisfies $w \geq v$. Then there are some cograssmannian elements w_λ, v_μ in some Weyl group W, with $w_\lambda \geq v_\mu$ such that*

$$P_{v,w}(q) = P_{v_\mu, w_\lambda}(q).$$

In Section 4.1 we will explicitly describe w_λ, w_μ along with the Weyl group W. Our proof of Theorem 1.1 uses the interpretation of the Kazhdan–Lusztig polynomials as Hilbert–Poincaré polynomials for intersection cohomology of Schubert varieties associated to covexillary Schubert variety X_w at a torus-fixed point $p_v \in X_w$. We use the local isomorphism between vexillary Schubert varieties and Grassmannian Schubert varieties, recently established in the author's joint work [3, Theorem 7.1]. This local isomorphism (4.2) induces a ring isomorphism of the respective intersection cohomologies (Theorem 4.6), allowing us to extend results [8,19,24,28] for cograssmannians Schubert varieties to covexillary cases.

More precisely, we deduce combinatorial (Corollary 4.8) and inductive (Corollary 4.9) formulas computing the Kazhdan–Lusztig polynomial $P_{v,w}$ for a covexillary element $w \in W$ and any v with $v \leq w$.

Our generalization of (co)grassmannian cases to (co)vexillary cases, outside the type A result by Lascoux, is new. The combinatorial formulas (Corollary 4.8) are generalizations of the result by Lascoux and Schützenberger for type A and Boe for the other classical types, providing a new proof of Lascoux [18] and an analogous result by Li and Yong [20] for type A (co)vexillary Schubert varieties.

Lascoux and Schützenberger [19] developed an algorithmic way of computing the Kazhdan–Lusztig polynomials for a cograssmannian element $w_\circ w_v$ in terms of weighted trees for type

A. Boe [8] has carried over Lascoux and Schützenberger's formulas for the classical Lie types. Lascoux [18] generalized the algorithm of [19] to the type A covexillary case. Using Lascoux's rule [18], Li and Yong [20] formulated another combinatorial rule for Kazhdan–Lusztig polynomials in type A, in terms of drift configurations. Our results are independent of [18].

Stimulated by Lascoux and Schützenberger's result [19], Zelevinsky [28] gave a geometric explanation and provided inductive formulas by constructing small resolutions for type A Schubert varieties X_{w_λ} associated to (co)grassmannian elements w_λ, w_μ [28]. Sankaran and Vanchinathan [24] obtained similar results for types B, C and D under certain conditions. Our inductive formula (Corollary 4.9) generalizes results of Zelevinsky for type A and of Sankaran and Vanchinathan for types B, C and D.

The paper is organized as follows. Section 2 defines the Kazhdan–Lusztig polynomials in terms of the stalk intersection cohomology sheaf of Schubert varieties. In Section 3, we discuss covexillary Schubert varieties and related notions that will be used for the formulas in Corollary 4.8 and Corollary 4.9. In Section 4 we provide the proof of Theorem 1.1 as well as Corollaries 4.8 and 4.9.

2. Schubert varieties and Kazhdan–Lusztig polynomials

We provide notions and definitions for the Kazhdan–Lusztig polynomials in terms of the sheaf-theoretical *intersection cohomology* theory of Deligne–Goresky–MacPherson [11,22], just as in [17].

Let $G := G(n)$ be a classical algebraic group of rank n over $\mathbb{C}$. Thus, $G = SL(n)$ for type A, $G = SO(2n+1)$ for type B, $G = Sp(2n)$ for type C and $G = SO(2n)$ for type D. We use the standard terminology that can be found in [9]. We fix a maximal torus T of G. Let B be a Borel subgroup of upper triangular matrices such that $T \subset B \subset G$, B^- an opposite Borel subgroup with $B \cap B^- = T$ and $W_n := N_T/T$ the Weyl group of G for the normalizer N_T of T. (We hope the standard use of B for Borel subgroup and our use of B for the type B will not cause confusion. The distinction should be clear from context.)

In fact, the Weyl group $W := W_n$ of type A is isomorphic to the symmetric group S_n. To be specific, S_n can be identified with permutations of $1, \ldots, n$ for type A. The Weyl group W of type B and C are regarded as a group of signed permutations $S_n \ltimes \{\pm 1\}^n$. Specifically, the Weyl group W of type B can be identified with permutations of

$$\bar{n}, \ldots, \bar{1}, 0, 1, \ldots, n.$$

Here the barred numbers indicate the negation, i.e., $\bar{a} = -a$. For type C, it is identified with permutations of $\bar{n}, \ldots, \bar{1}, 1, \ldots, n$. As for type D, we take W to be the subgroup of the signed permutations, a set of permutations of $\bar{n}, \ldots, \bar{1}, 1, \ldots, n$ with even number of barred numbers. In this paper, we will use one-line notation to represent an element w of the Weyl group W: we write $w = w(1)\ w(2)\ \cdots w(n) \in S_n$ in one-line notation for type A. Following [2,3], the signed permutation w of types B, C and D only lists the values on positive integers $w = w(1)\ w(2)\ \cdots w(n)$ in one-line notation, since $w(\bar{i}) = \overline{w(i)}$ for each i. Here we take the natural order

$$\bar{n} < \overline{n-1} < \cdots < n-1 < n.$$

Let $Fl = G/B$ denote a flag variety. The *Bruhat decomposition* is $Fl = \bigsqcup_{w \in W} X_w^\circ$, where the *Schubert cell* $X_w^\circ := BwB/B$ is locally closed subset isomorphic to the affine space of

dimension $\ell(w)$. The *Schubert variety* X_w is the closure of X_w° such that $X_w = \bigsqcup_{v \leq w} X_v^\circ$ where $\leq$ denotes Bruhat order on W. One characterization of Bruhat order is that $v \leq w$ in W_n if and only if a T-fixed point p_v is in X_w.

Let Y be a pure n-dimensional algebraic variety, admitting a *stratification* [14]. A stratification of Y is a filtration

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n = Y$$

of Y by closed subvarieties such that $Y_i - Y_{i-1}$ is smooth of pure dimension i along which Y is "equisingular". The notion of equisingularity is a technical notion we will not need to inspect in our proofs. By letting $U_i = Y - Y_{n-i}$, we have the filtration

$$\emptyset \subset U_1 \subset \cdots \subset U_n.$$

Let $j_i : U_i \hookrightarrow U_{i+1}$ be the inclusion. Then the *intersection cohomology (Deligne) complex* $\mathcal{IC}(Y)$ can be determined by the canonical isomorphism

$$\mathcal{IC}(Y) \cong \tau_{\leq n-1} R j_{n*} \cdots \tau_{\leq 0} R j_{1*} \underline{\mathbb{Q}}_{U_1} \tag{2.1}$$

where $\underline{\mathbb{Q}}_{U_1}$ is the constant sheaf of rational numbers in degree 0 on U_1, $\tau_{\leq k}$ is the truncation functor that replaces the sheaf cohomology by zero in degree above k, and $R j_{i*}$ is higher direct image for all i.

Given any stratified variety Y, we let $\mathcal{H}^*(\mathcal{IC}(Y))$ denote the sheaf cohomology of the intersection cohomology complex $\mathcal{IC}(Y)$. We write $\mathcal{H}_y(\mathcal{IC}(Y))$ for the stalk of $\mathcal{H}(\mathcal{IC}(Y))$ at a point $y \in Y$. We note that Schubert varieties have the stratifications as the Whitney stratification: X_w of dimension $\ell(w)$ is the union of Schubert cells X_v° for all $v \leq w$. Each Schubert cell is smooth of pure dimension and its closure is a union of strata with partial order on W.

For $v \in W$ with $v \leq w$ the Kazhdan–Lusztig polynomial $P_{v,w}(q)$ can be obtained by

$$P_{v,w}(q) = \sum_{i \geq 0} \dim \mathcal{H}^{2i}_{p_v}(\mathcal{IC}(X_w)) q^i, \tag{2.2}$$

as the Hilbert–Poincaré polynomial for the stalk intersection cohomology sheaf of X_w at the T-fixed point $p_v \in X_w$ [17, Theorem 4.3].

3. Covexillary Schubert varieties in classical types

Throughout the rest of this manuscript, we express covexillary elements in terms of vexillary element associated to a triple τ as in [2]. Let $w_\circ$ be the longest element in W, more precisely, it takes i to $n + 1 - i$ for type A and $w(i)$ to $\overline{w(i)}$ for types B, C and D. In this section, we define covexillary Schubert varieties $X_{w_\circ w}$ for a vexillary element w in each type.

3.1. Type A

We start with a Type A triple τ to generate a vexillary element $w(\tau)$ as the first step. Given a triple, the vexillary element arises uniquely as in [2].

A *triple* $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ of length s denotes three sequences $\mathbf{k} = (0 < k_1 < \cdots < k_s)$, $\mathbf{p} = (0 < p_1 \leq \cdots \leq p_s \leq n)$ and $\mathbf{q} = (0 < q_1 \leq \cdots \leq q_s < n)$ of integers with a condition

$$k_{i+1} - k_i < (q_{i+1} - q_i) - (p_{i+1} - p_i)$$

for $1 \leq i < s$.

Proposition 3.1. *Given a triple $\tau = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, there is a unique vexillary permutation $w(\tau) \in W$. The vexillary element $w(\tau)$ is minimal in Bruhat order in S_n such that $\#\{a \leq p_i \mid w(a) > n - q_i\} = k_i$ for all i and $1 \leq a \leq n$. The permutations arising in this way are vexillary and all vexillary permutations are from a unique triple.*

Proof. After replacing q_i by $n - q_i$, we can construct a permutation $w(\tau)$ from a triple as in [3, §3.1.1]. Then the statements follow by [1, p. 13], [2] and [3, §3.1]. $\quad\square$

Let us describe the construction of $w(\tau)$ from a triple as follows. We start by putting k_1 integers $n - q_1 + 1, \ldots, n - q_1 + k_1$ in increasing order to the positions from $p_1 - k_1 + 1$ to p_1. Then insert $k_2 - k_1$ different integers, specifically, $n - q_2 + 1, \ldots, n - q_2 + k_2 - k_1$ in increasing order, as far right as possible but to the left of position $p_2 + 1$; i.e., place $n - q_2 + k_2 - k_1$ at position p_2, and then place $n - q_2 + k_2 - k_1 - 1$ at position $p_2 - 1$ if it is available. If the position $p_2 - 1$ is already occupied, then place $n - q_2 + k_2 - k_1 - 1$ at position $p_2 - 2$, and then continue this process until all $k_2 - k_1$ different integers are placed. Continue in this way until k_s integers have been inserted. Then enter unused integers in the rest of positions in increasing order.

For instance, for the triple $\tau = (2\ 3\ 5\ 6, 2\ 4\ 5\ 6, 2\ 4\ 7\ 8)$ for $n = 9$, we recover the permutation as follows. First we put 8,9 in positions 1 and 2, and then put 6 in position 4. Then place 3,4 at the positions 3 and 5, and then put 2 at position 6. Then fill in the unused integers $1, 5, 7$ in positions $7, 8, 9$ to get $w(\tau) = 8\ 9\ 3\ 6\ 4\ 2\ 1\ 5\ 7$ in S_9.

For the next step, we define a covexillary Schubert variety associated to the covexillary element $w_\circ w(\tau)$. Let V be a n-dimensional vector space over $\mathbb{C}$. The flag variety $SL(n)/B$ is the variety $Fl^A(V)$ of complete flags

$$E_\bullet : \{\mathbf{0}\} \subset E_1 \subset E_2 \subset \cdots \subset E_n = V$$

where the dimension of E_i is i for $1 \leq i \leq n$.

Given a fixed partial flag

$$F_{q_1} \subseteq \cdots \subseteq F_{q_s} \subseteq V$$

of V with $\dim(F_{q_i}) = q_i$, the *covexillary Schubert variety* $X_{w_\circ w(\tau)}$ is given by

$$X_{w_\circ w(\tau)} = \{E_\bullet \mid \dim(E_{p_i} \cap F_{q_i}) \geq k_i \ \text{ for } \ 1 \leq i \leq s\} \subseteq Fl^A(V). \tag{3.1}$$

Here $X_{w_\circ w(\tau)}$ is the Zariski closure of the B-orbit of the T-fixed point $p_{w_\circ w(\tau)}$.

3.1.1. Partition and $\alpha\beta$-word of a covexillary permutation

For a vexillary permutation w with a triple τ, there is an associated partition $\nu := \nu(\tau) = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{k_s})$ such that

$$\nu_{k_i} = n - q_i - p_i + k_i \tag{3.2}$$

and $\nu_k = \nu_{k_i}$ for $k_{i-1} < k \leq k_i$ with $k_0 = 0$ for $1 \leq i \leq s$. The partition λ of a covexillary permutation $w_\circ w(\tau)$ is obtained by a partition inside the $n \times n$ rectangle such that $\lambda_i = n - \nu_i$ for $1 \leq i \leq n$.

We define a function

$$\mathfrak{H} : \lambda \mapsto \begin{pmatrix} a_1 & \cdots & a_d \\ b_0 & \cdots & b_{d-1} \end{pmatrix}_\lambda \tag{3.3}$$

where a_i indicates the number of repeated λ_i in $\lambda = \lambda_1^{a_1} \lambda_2^{a_2} \cdots \lambda_d^{a_d}$ and $b_0 = \lambda_1$, $b_i = \lambda_{i+1} - \lambda_i$ for $1 \leq i < d$. In this way, the partition λ determines a $2 \times d$ matrix.

For each covexillary element $w_\circ w(\tau) \in W_n$ with a triple τ, we associate $w_\circ w(\tau)$ with a word in two symbols α and β by

$$\lambda(w_\circ w(\tau)) = \beta^{b_0} \alpha^{a_1} \cdots \beta^{b_{d-1}} \alpha^{a_d}.$$

The following is an example of finding the partition of a covexillary permutation $w_\circ w(\tau)$.

Example 3.2. Let $n = 8$. Let $w(\tau) = 1\ 4\ 7\ 5\ 2\ 3\ 6\ 8$ in S_8 be a vexillary permutation with a triple $\tau = (1\ 3, 3\ 4, 2\ 5)$. The partition associated to $w(\tau)$ is $\nu = (4, 2, 2)$. And, the partition λ of the covexillary permutation $w_\circ w(\tau)$ is $(4, 6, 6, 8, 8, 8, 8, 8)$ from $\lambda_1 = 4, \lambda_2 = 6$, and $\lambda_3 = 8$.

Furthermore, we have $\mathfrak{H}(\lambda) = \begin{pmatrix} 1 & 2 & 5 \\ 4 & 2 & 2 \end{pmatrix}_\lambda$, and $\lambda(w_\circ w(\tau)) = \beta^4 \alpha^1 \beta^2 \alpha^2 \beta^2 \alpha^5$.

3.1.2. Partition and $\alpha\beta$-word of a pair

Let $w(\tau)$ be a vexillary permutation in S_n with a triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$. We may write w for $w(\tau)$ for simplicity. Let v be any element in S_n such that $w \le v$ in Bruhat order.

We define a weakly increasing sequence $\mathbf{k}' = (0 < k'_1 \le \cdots \le k'_s)$ defined by

$$k'_i = \#\{a \le p_i \mid v(a) > n - q_i\}.$$

It is worthwhile to note that $\mathbf{p} = (0 < p_1 \le \cdots \le p_s \le n)$ and $\mathbf{q} = (0 < q_1 \le \cdots \le q_s < n)$ depend on $w(\tau)$ while $\mathbf{k}'$ relies on v. Since $v \ge w$ in Bruhat order, we have $k'_i \ge k_i$. See [3, §3.1.3] for this property.

A *weak triple* $\tau'(w, v) = (\mathbf{k}', \mathbf{p}, \mathbf{q})$ consists of the three sequences $\mathbf{k}'$ from v and $\mathbf{p}, \mathbf{q}$ from $w(\tau)$, which satisfies

$$k'_{i+1} - k'_i \le (q_{i+1} - q_i) - (p_{i+1} - p_i)$$

for $1 \le i < s$. To construct a (strong) triple without the equality, we remove k'_i, p_i, q_i if the equality holds and $k'_{i+1}, p_{i+1}, q_{i+1}$ if $k'_i = k'_{i+1}$ from τ' to have another triple $\tau''(w, v) = (\mathbf{k}'', \mathbf{p}'', \mathbf{q}'')$. In other words,

$$(\mathbf{k}'', \mathbf{p}'', \mathbf{q}'') := \begin{cases} (\cdots \le k'_{i-1} \le k'_{i+1} \le \cdots, \ldots \le p_{i-1} \le p_{i+1} \le \cdots, \ldots \le q_{i-1} \le q_{i+1} \le \cdots) \\ \qquad \text{if } k'_{i+1} - k'_i = (q_{i+1} - q_i) - (p_{i+1} - p_i), \\ (\cdots \le k'_i < k'_{i+2} \le \cdots, \ldots \le p_i \le p_{i+2} \le \cdots, \ldots \le q_i \le q_{i+2} \le \cdots) \\ \qquad \text{if } k'_i = k'_{i+1}. \end{cases}$$

We replace $\tau'(w, v)$ by $\tau''(w, v)$ and will not distinguish $\tau'(w, v)$ and $\tau''(w, v)$.

The partition μ of a pair $(w_\circ w, w_\circ v)$ with a weak triple $\tau' := \tau'(w, v)$ is obtained as the partition λ of $w_\circ w$ with a triple $\tau(w)$: have a partition $\xi := \xi(\tau')$ by setting $\xi_{k'_i} = n - q_i - p_i + k'_i$ and $\xi_k = \xi_{k'_i}$ for $k'_{i-1} < k \le k'_i$. The partition μ of a pair $(w_\circ w, w_\circ v)$ is obtained by taking the partition inside $n \times n$ rectangle such that $\mu_i = n - \xi_i$ for $1 \le i \le n$.

We have the $2 \times d$ matrix corresponding to the partition $\mu := \mu(w_\circ w, w_\circ v)$ with τ' as the image of the partition μ under the function

$$\mathfrak{H} : \mu \mapsto \begin{pmatrix} e_1 & \cdots & e_d \\ f_0 & \cdots & f_{d-1} \end{pmatrix}_\mu$$

defined in the same way as (3.3) with a partition μ: a_i indicates the number of repeated μ_i in $\mu = \mu_1^{e_1} \mu_2^{e_2} \cdots \mu_d^{e_d}$ and $f_0 = \mu_1$, $f_i = \mu_{i+1} - \mu_i$ for $1 \le i < d$. From e_i and f_i, we have a $\alpha\beta$-word

$$\mu(w_\circ w, w_\circ v) = \beta^{f_0} \alpha^{e_1} \cdots \beta^{f_{d-1}} \alpha^{e_d}.$$

associated to the pair $(w_\circ w, w_\circ v)$.

The following is an example of getting the partition μ from a pair of a covexillary permutation $w_\circ w$ and $w_\circ v$ for any $v \geq w$ in S_n as well as its $\alpha\beta$-word.

Example 3.3. Let $n = 8$. Let $w(\tau) = 1\,4\,7\,5\,2\,3\,6\,8$ in S_8 be a vexillary permutation with a triple $\tau = (1\ 3, 3\ 4, 2\ 5)$. Let $v = 8\,7\,6\,5\,4\,3\,2\,1 \geq w(\tau)$. Then we get a weak triple $\tau' = (2\ 4, 3\ 4, 2\ 5)$ such that $\mu = (3, 3, 5, 5, 8, 8, 8, 8)$. Thus, we have

$$\mathfrak{H}(\mu) = \begin{pmatrix} 2 & 2 & 4 \\ 3 & 2 & 3 \end{pmatrix}_\mu \quad \text{and} \quad \mu(w_\circ w, w_\circ v) = \beta^3\alpha^2\beta^2\alpha^2\beta^3\alpha^4$$

3.2. *Type C*

Let us move on to the type C covexillary cases with similar steps as type A. We begin with a type C triple τ to get a (signed) vexillary element $w := w(\tau)$.

A *triple* $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ of type C is a collection of $\mathbf{k} = (0 < k_1 < \cdots < k_s)$, $\mathbf{p} = (0 < p_1 \leq \cdots \leq p_s \leq n)$ and $\mathbf{q} = (0 < q_1 \leq \cdots \leq q_s \leq n)$, requiring

$$k_{i+1} - k_i < (p_{i+1} - p_i) + (q_{i+1} - q_i) \tag{3.4}$$

for $1 \leq i < s$.

Given a triple τ, one can determine a signed permutation $w = w(\tau) \in W_n$, in one-line notation, such that w is minimal with $\#\{a \geq n + 1 - p_i \mid w(a) \leq \overline{n + 1 - q_i}\} = k_i$ for all i, see [1, p. 16] and [2]. Specifically, we begin with inserting a sequence of k_1 consecutive barred elements

$$\overline{n - q_1 + k_1}, \overline{n - q_1 + k_1 - 1}, \ldots, \overline{n - q_1 + 2}, \overline{n - q_1 + 1}$$

in positions $n - p_1 + 1$ to $n - p_1 + k_1$. Then insert $k_2 - k_1$ barred numbers, bars of the smallest remaining integers $a_{k_2-k_1} > \cdots > a_1 \geq n - q_2 + 1$, placed in order (bars of decreasing sequences, $\overline{a_{k_2-k_1}}, \ldots, \overline{a_1}$) at or to the right of position $n - p_2 + 1$ but as close as possible to that position. Repeat this until one places k_s barred integers. Then place the remaining positive integers in increasing order in the unfilled positions.

Example 3.4. Let $n = 8$ and $\tau = (2\ 4\ 5\ 6, 3\ 4\ 6\ 8, 3\ 6\ 6\ 7)$. Then we put $\overline{7}$ and $\overline{6}$ in positions 6,7. Then place $\overline{4}, \overline{3}$ in positions 5 and 8. Then put $\overline{5}$ at position 3. Then put $\overline{2}$ at position 1. Then we place remaining positive integers 1,8 in the unfilled positions 2 and 4. In this way, we get a permutation $w(\tau) = \overline{2}\,1\,\overline{5}\,8\,\overline{4}\,\overline{7}\,\overline{6}\,\overline{3}$ in W_8.

A signed permutation w in the Weyl group W is *vexillary* if $w = w(\tau)$ for some triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$, as in type A. In particular, the "empty" triple is a triple so that the identity element is vexillary [2, Definition 2.1].

From a vexillary signed permutation, one can reconstruct the triple as follows. $n + 1 - p_1$ is the position after the last descent. We start at that position and find a sequence of k_1 consecutive barred integers ending in $\overline{n + 1 - q_1}$. Then continue this process on the remaining, but considering elements as being in sequence if the only missing integers have already been taken cared of, see [1, p. 16] and [2] for more details.

We remark that there is a different definition of the vexillary elements from [25, Theorem 2]. But our definition of vexillary permutations is derived from the ones presented in [1–3], and we do not use properties deduced from [25].

Next, we define a covexillary Schubert variety associated to a covexillary element $w_\circ w$. Let V be a vector space of dimension $2n$ over $\mathbb{C}$ equipped with a nondegenerate skew-symmetric form. A vector space E is *isotropic* if $E \subset E^\perp$ with respect to the given form. An *isotropic (complete) flag* is a filtration

$$E_\bullet : \{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset V \cong \mathbb{C}^{2n},$$

of linear subspaces where E_i are isotropic with $\dim(E_i) = i$. The variety $Fl^C(V) = Sp(2n)/B_n$ is identified with the set of isotropic flags.

We fix an isotropic partial flag

$$F_{q_1} \subseteq \cdots \subseteq F_{q_s} \subseteq V$$

where the subscripts indicate their dimension. A *covexillary Schubert variety* $X_{w_\circ w}$ associated to a triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ is given by

$$X_{w_\circ w} = \{E_\bullet \mid \dim(E_{p_i} \cap F_{q_i}) \geq k_i \ \text{ for } \ 1 \leq i \leq s\} \subset Fl^C(V)$$

where $\dim(E_{p_i}) = p_i$, as described in type A.

3.2.1. Partition and $\alpha\beta$-word of a covexillary permutation

For each covexillary signed permutation $w_\circ w(\tau)$ with a triple τ, there is the largest strict partition $\lambda(\tau)$ such that

$$\lambda_{k_i} = p_i + q_i \ \text{ for } 1 \leq i \leq s.$$

That is, $\lambda(\tau) = (0 < \lambda_1 < \cdots < \lambda_{k_s})$ has parts $\lambda_k = p_i + q_i - k_i + k$ whenever $k_{i-1} \leq k < k_i$.

We recall the notation of [24]. Given a partition λ, a maximal subsequence of consecutive integers in this partition λ is referred to as a *block* of λ. The partition λ is the concatenation of its blocks. Let a_i be the length of the ith block. Then λ determines a $2 \times s$ matrix as the image of the following function

$$\mathfrak{H} : \lambda \mapsto \begin{pmatrix} a_1 & \cdots & a_s \\ b_0 & \cdots & b_{s-1} \end{pmatrix}_\lambda \tag{3.5}$$

where the image is obtained by

(1) $b_0 = \lambda_{a_1} - a_1$,
(2) and $b_i = \lambda_{k_{i+1}} - (a_1 + \cdots + a_{i+1}) - (b_0 + \cdots + b_{i-1})$ for $1 \leq i \leq s-1$, $k_i = a_1 + \cdots + a_i$.

To each covexillary element $w_\circ w(\tau) \in W_n$ with a triple τ, we associate $w_\circ w(\tau)$ with a word in two symbols α and β by

$$\lambda(w_\circ w(\tau)) = \beta^{b_0} \alpha^{a_1} \cdots \beta^{b_{s-1}} \alpha^{a_s} \quad \text{or} \quad \beta^{b_0} \alpha^{a_1} \cdots \beta^{b_{s-1}} \alpha^{a_s} \beta^{a_{s+1}}$$

for $\sum_i a_i + \sum_j b_j = 2n$.

Example 3.5. Let $w(\tau) = 5\,\bar{4}\,\bar{3}\,6\,\bar{1}\,\bar{2}\,7$ be a vexillary permutation in W_7 with a triple $\tau = (1\,2\,4, 2\,3\,6, 6\,7\,7)$.

The partition λ associated to the covexillary (signed) permutation $w_\circ w(\tau)$ is $(8, 10, 12, 13)$ via $\lambda_1 = 8$, $\lambda_2 = 10$, and $\lambda_4 = 13$. Moreover, we have

$$\mathfrak{H}(\lambda) = \begin{pmatrix} 1 & 1 & 2 \\ 7 & 1 & 1 \end{pmatrix}_\lambda, \quad \text{and} \quad \lambda(w_\circ w(\tau)) = \beta^7 \alpha^1 \beta^1 \alpha^1 \beta^1 \alpha^2 \beta.$$

3.2.2. Partition and αβ-word of a pair

Given a vexillary signed permutation $w := w(\tau)$ with a triple τ, we take any v such that $v \geq w$ in W_n. As in type A, there is a *weak triple* $\tau' = (\mathbf{k}', \mathbf{p}, \mathbf{q})$, obtained from $k_i' = \#\{a \geq n + 1 - p_i \mid v(a) \leq n + 1 - q_i\}$ such that it satisfies

$$k_{i+1}' - k_i' \leq (p_i - p_{i+1}) + (q_i - q_{i+1}) \tag{3.6}$$

for $i = 1, \ldots, s - 1$. We use the same convention as before. That is, we delete k_i', p_i, q_i if the equality holds and $k_{i+1}', p_{i+1}, q_{i+1}$ if $k_i' = k_{i+1}'$ from τ'.

The partition μ of a pair $(w_\circ w, w_\circ v)$ with a weak triple τ' is given by $\mu_{k_i'} = p_i + q_i$ and $\mu_k = p_i + q_i - k_i' + k$ whenever $k_{i-1}' \leq k < k_i'$.

For a partition μ, we have the $2 \times s$ matrix $\mathfrak{H}(\mu) = \begin{pmatrix} e_1 & \cdots & e_s \\ f_0 & \cdots & f_{s-1} \end{pmatrix}_\mu$ from a function $\mathfrak{H}$ defined as (3.5).

For each pair $(w_\circ w, w_\circ v)$ with a weak triple τ', the $\alpha\beta$-word associated to the pair is given by

$$\mu(w_\circ w, w_\circ v) = \beta^{f_0} \alpha^{e_1} \cdots \beta^{f_{s-1}} \alpha^{e_s} \quad \text{or} \quad \beta^{f_0} \alpha^{e_1} \cdots \beta^{f_{s-1}} \alpha^{e_s} \beta^{e_{s+1}}$$

for $\sum_i e_i + \sum_j f_j = 2n$.

Example 3.6. Let $w(\tau) = 5\ \bar{4}\ \bar{3}\ 6\ \bar{1}\ \bar{2}\ 7$ be a vexillary permutation in W_7 with a triple $\tau = (1\ 2\ 4, 2\ 3\ 6, 6\ 7\ 7)$. Let $v = 7\ \bar{6}\ \bar{5}\ \bar{4}\ \bar{3}\ \bar{2}\ \bar{1} \in W_7$. Then we have a weak triple $\tau' = (1\ 3\ 6, 2\ 3\ 6, 6\ 7\ 7)$ associated to the pair $(w_\circ w(\tau), w_\circ v)$.

The partition μ of the pair is given by $(8, 9, 10, 11, 12, 13)$ from $\mu_1 = 8, \mu_3 = 10, \mu_6 = 13$, and

$$\mathfrak{H}(\mu) = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 0 & 0 \end{pmatrix}_\mu \quad \text{and} \quad \mu(w_\circ w, w_\circ v) = \beta^7 \alpha^1 \beta^0 \alpha^2 \beta^0 \alpha^3 \beta = \beta^7 \alpha^6 \beta.$$

3.3. Type D

We follow similar procedures as before.

A type D triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ is $0 < k_1 < \cdots < k_s$, $0 \leq p_1 \leq \cdots \leq p_s < n$, and $0 \leq q_1 \leq \cdots \leq q_s < n$ with the same inequality condition (3.4) in type C. In particular τ is obtained by subtracting 1 from each p_i and q_i of the sequences in type C. If k_s is odd, the triple is replaced by one with $k_{s+1} = k_s + 1$ and $p_{s+1} = q_{s+1} = n$.

A (signed) vexillary permutation $w := w(\tau)$ in W_n is characterized as minimal in Bruhat order such that $\#\{a \geq n - p_i \mid w(a) \leq \overline{n - q_i}\} = k_i$ for all i. In particular, one can construct a vexillary signed permutation $w(\tau)$ for type D as follows. We get a triple τ^+ of type C uniquely by adding 1 to each p_i and each q_i. Then we construct $w(\tau) = w(\tau^+)$ with τ^+ from type C.

Example 3.7. For $\tau = (1\ 2\ 4, 0\ 2\ 5, 0\ 2\ 5)$ we have $w = \bar{2}\ \bar{1}\ 3\ \bar{4}\ 5\ \bar{6}$ in $W_6, n = 6$.

A vector space V has dimension $2n$ over $\mathbb{C}$ and is equipped with a symmetric bilinear form. An *isotropic flag* is a sequence of subspaces

$$E_\bullet : \{\mathbf{0}\} \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n \subset V \cong \mathbb{C}^{2n}$$

where all E_i are isotropic with respect to the form, i.e., $E_i \subset E_i^\perp$, and has dimension of i. We note that E_n is a maximal isotropic subspace of V. The variety $Fl^D(V) = SO(2n)/B$ consists of the set of all isotropic flags.

We take a partial flag of isotropic subspaces

$$F_{q_1} \subseteq \cdots \subseteq F_{q_s} \subseteq F_n \subseteq V$$

for $\dim(F_{q_i}) = q_i$. In fact F_n is the maximal isotropic subspace of V. The *covexillary Schubert variety* $X_{w_\circ w}$ is obtained by the closure of the Schubert cell

$$X^\circ_{w_\circ w} = \{E_\bullet \mid \dim(E_{p_i} \cap F_{q_i}) = k_i \text{ for } 1 \le i \le s\} \subseteq Fl^D(V)$$

with $\dim(E_{p_i}) = p_i$ for all i. The condition $p_s + q_s = 2n$ can be contained in the definition of $X_{w_\circ w}$ so that $\dim(E_n \cap F_n) \equiv n \pmod 2$ holds on $X_{w_\circ w}$.

3.3.1. Partition and $\alpha\beta$-word of a covexillary permutation

Given a signed vexillary permutation $w := w(\tau)$ with a triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$, the partition λ of the covexillary (signed) permutation $w_\circ w$ is given by the largest strict partition such that

$$\lambda_{k_i} = p_i + q_i + 1 \text{ for } 1 \le i \le s.$$

and $\lambda_k = p_i + q_i - k_i + k$ whenever $k_{i-1} \le k < k_i$. If $p_s = q_s = n$, then $\lambda_{k_s} = 2n$. Alternatively, we have a triple τ^+ by adding 1 to each p_i and each q_i. Then the partition $\lambda^+ = (\lambda_1 + 1, \ldots, \lambda_s + 1)$ is the strict partition from $(\tau')^+$ for type C.

The partition λ determines a $2 \times d$ matrix $\mathfrak{H}(\lambda)$, and have a $\alpha\beta$-word $\lambda(w_\circ w)$ associated to $w_\circ w$ as before in type C.

Example 3.8. Let $w(\tau) = \bar{2}\,\bar{1}\,3\,\bar{4}\,5\,\bar{6}$ be a (signed) vexillary permutation in $W_6, n = 6$ with $\tau = (1\,2\,4, 0\,2\,5, 0\,2\,5)$. Then $\lambda = (1, 5, 10, 11)$ from $\lambda_1 = 1, \lambda_2 = 5$, and $\lambda_4 = 11$.

Also, we have

$$\mathfrak{H}(\lambda) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}_\lambda \quad and \quad \lambda(w) = \beta^1 \alpha^1 \beta^2 \alpha^1 \beta^4 \alpha^2 \beta$$

3.3.2. Partition and $\alpha\beta$-word of a pair

Given a tripe τ with a vexillary (signed) permutation $w := w(\tau)$, and any $v \ge w$, the weak triple $\tau' = (\mathbf{k}', \mathbf{p}, \mathbf{q})$ associated to a pair $(w_\circ w, w_\circ v)$ is obtained by $k'_i = \#\{a \ge n - p_i \mid v(a) \le \overline{n - q_i}\}$.

The partition $\mu = (\mu_1, \ldots, \mu_s)$ associated to the pair can be determined as before with the weak triple τ'. Alternatively, we may get the partition $\mu^+((\tau')^+) = (\mu_1 + 1, \ldots, \mu_s + 1)$ for type C. As in type C, we have $2 \times s$ matrices $\mathfrak{H}(\mu)$ and $\alpha\beta$-word $\mu(w_\circ w, w_\circ v)$ with the weak triple $(\tau')^+$.

Example 3.9. Let $w(\tau) = \bar{2}\,\bar{1}\,3\,\bar{4}\,5\,\bar{6}$ be a (signed) vexillary permutation in W_6 with $\tau = (1\,2\,4, 0\,2\,5, 0\,2\,5)$. Let $v = \bar{3}\,\bar{2}\,\bar{1}\,\bar{5}\,\bar{4}\,\bar{6}$. The weak triple τ' associated to $(w_\circ w(\tau), w_\circ v)$ is given by $(1\,3\,6, 0\,2\,5, 0\,2\,5)$.

The partition $\mu = (1, 4, 5, 9, 10, 11)$ is obtained from $\mu_1 = 1, \mu_3 = 5$ and $\mu_6 = 11$. In addition, we have

$$\mathfrak{H}(\mu) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}_\mu \quad and \quad \mu(w_\circ w(\tau), w_\circ v) = \beta^1 \alpha^1 \beta^1 \alpha^2 \beta^3 \alpha^3 \beta.$$

3.4. Type B

This section closely resembles type C. Let V be a $2n+1$ dimensional vector space equipped with a symmetric bilinear form on it. The isotropic flag variety $Fl^B(V) = SO(2n + 1)/B$ is identified with the set of isotropic flags

$$E_\bullet : \{\mathbf{0}\} \subset E_1 \subset \cdots E_{n-1} \subset E_n \subset V \cong \mathbb{C}^{2n+1}$$

where E_i are isotropic with $\dim(E_i) = i$. Especially, E_n is a maximal isotropic subspace of V.

We use the same triple τ as in type C for a vexillary (signed) permutation $w = w(\tau) \in W_n$. The *covexillary Schubert variety* $X_{w_\circ w}$ in $Fl^B(V)$, the partition λ associated to a covexillary (signed) permutation $w_\circ w$, μ associated to the pair $(w_\circ w, w_\circ v)$ with any $v \geq w$ are obtained in the same way as type C.

4. Covexillary formulas for Kazhdan–Lusztig polynomials

4.1. Relations to Grassmanianns of classical types

In this section we review the relation of (co)vexillary elements with some inverse grassmannian ones in [3]. Here, an inverse grassmannian element is an element that is the inverse of a Grassmannian element in the Weyl group.

Let P be a maximal parabolic subgroup which contains the Borel subgroup B of G of rank n. Then G/P has a structure of a homogeneous projective space. Let W_P be the Weyl group of P in the Weyl group $W := W_n$ of G. The T-fixed points $(G/P)^T$ in G/P are indexed by the set of minimal representatives W^P of the coset space W/W_P. For the classical Lie groups G, each element w of W^P can be represented by certain partitions. Specifically it can be represented by a weakly decreasing sequence $\rho = (\rho_1 \geq \cdots \geq \rho_r > 0)$ for some r for type A, and strictly decreasing sequence $\rho = (\rho_1 > \cdots > \rho_r > 0)$ for types B, C, and D. We express such elements as $w_\rho \in W^P$. One may refer [7] for these standard notations.

Let $\overline{G} := G(2n)$ be a simple linear algebraic group of classical type of rank $2n$ over $\mathbb{C}$. For the remaining of this article we use the notation G for $G(n)$ of rank n and $\overline{G}$ for $G(2n)$ of rank $2n$ to avoid any confusion that might occur, since we will deal with two different flag varieties.

Let $w_\nu \in \overline{W}^P$ be a grassmannian element represented by the partition ν defined below for each classical type. Let λ be the partition associated to a covexillary (signed) permutation $w_\circ w(\tau)$ with a triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$.

(1) Type A: Let $\overline{G} = SL(2n)$. Let $\overline{P} = P_{k_s}$ be the maximal parabolic subgroup associated to the dth simple root where the simple roots are indexed as in [5, p. 207]. The grassmannian element $w_\nu \in \overline{W}^P$ is represented by the partition $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{k_s})$ defined by (3.2).

(2) Type C: Let $\overline{G} = Sp(4n)$. The maximal parabolic subgroup $\overline{P} = P_{2n}$ of $\overline{G}$ is the one corresponding to the omission of the right end root, $2n$-th simple root in [5, p. 207]. The partition ν is strictly decreasing defined by $\nu_{k_i} = 3n+1-\lambda_{k_i}$ for all i and $\nu_k = \nu_{k_i}+k_i-k$ for $k_{i-1} < k \leq k_i$.

(3) Type D: Here $\overline{G} = SO(4n)$ and $\overline{P} = P_{2n}$ is the maximal parabolic subgroup associated to the $2n$-th simple root as before. The partition ν is given by $\nu_{k_i} = 3n-\lambda_{k_i}$ for $1 \leq i \leq s$ and $\nu_k = \nu_{k_i} + k_i - k$ for $k_{i-1} < k \leq k_i$.

(4) Type B: The algebraic group is $\overline{G} = SO(4n+1)$. Let $\overline{P} = P_{2n}$ be the maximal parabolic subgroup that omits the $2n$-th simple root as in type C. Also, the partition ν is defined in the same way as in type C.

Let $w_\xi \in \overline{W}^{\overline{P}}$ be the element represented by a partition ξ where ξ is defined similarly as above with μ and the weak triple $\tau' = (\mathbf{k}', \mathbf{p}, \mathbf{q})$ in Section 3.

We define the opposite Schubert cell by $\Omega_w^\circ = B^- wB/B$ and the opposite Schubert variety to be the closure of Ω_w° in a flag variety G/B for w in the Weyl group. Theorem 4.1 is a heart of the proof for Theorem 4.7. The following theorem is [3, Theorem 7.1] by specializing the Weyl group elements to our situation.

Theorem 4.1. *Let* $w := w(\tau)$ *and* $v := w(\tau')$ *be vexillary (signed) permutations in* W, $w \leq v$. *Let* ν *and* ξ *be as above. For* w_ν^{-1} *and* v_ξ^{-1} *in* $\overline{W}$ *with* $w_\nu^{-1} \leq v_\xi^{-1}$, *we have a scheme-theoretic isomorphism*

$$G/B = Fl^X(V) \supset \Omega_w \cap X_v^\circ \cong \Omega_{w_\nu^{-1}} \cap X_{v_\xi^{-1}}^\circ \subset Fl^X(V \oplus V) = \overline{G}/\overline{B}. \qquad (4.1)$$

Here $X = A, B, C$ *or* D.

We note that the isomorphism (4.1) holds induced from the direct sum embedding

$$\sum : Fl^X(V) \to Fl^X(V \oplus V)$$

in [3, §6]. In particular, the image of $\Omega_w \cap X_v^\circ$ under the direct sum embedding $\sum$ is given by $\sum(\Omega_w \cap X_v^\circ) \cong \sum(X_v^\circ) \cap \Omega_{w_\nu^{-1}}$, and then certain matrix manipulation of $\sum(X_v^\circ)$ in [3, §7] establishes $\sum(X_v^\circ) \cong X_{w_\xi^{-1}}^\circ$.

A *Kazhdan–Lusztig variety*, so named in [26], is the intersection of a Schubert variety and the opposite Schubert cell in a flag variety. For any $w \in W$ the opposite Schubert variety Ω_w is isomorphic to the Schubert variety $X_{w_\circ w}$ [17, §1.2]. We use $w_\circ$ for the longest element in both W and $\overline{W}$ for abuse of notation, but it should be read in different Weyl groups. We will consider the Kazhdan–Lusztig varieties of the form $X_w \cap \Omega_v^\circ$ for $v \leq w$ for any $w, v \in W$. Then the isomorphism (4.1) can be interpreted as an isomorphism

$$X_{w_\circ w} \cap \Omega_{w_\circ v}^\circ \cong X_{w_\circ w_\nu^{-1}} \cap \Omega_{w_\circ v_\xi^{-1}}^\circ \qquad (4.2)$$

of Kazhdan–Lusztig varieties for vexillaries $w, v \in W$ with $v \leq w$ and inverse grassmannians $w_\nu^{-1}, v_\xi^{-1} \in \overline{W}$ with $w_\nu^{-1} \leq v_\xi^{-1}$. Accordingly the isomorphism (4.2) is an isomorphism of Kazhdan–Lusztig varieties in different flag varieties. It is noteworthy that the Kazhdan–Lusztig variety $X_{w_\circ w} \cap \Omega_{w_\circ v}^\circ$ associated to vexillary elements w, v is in $Fl^X(V) = G/B$ while $X_{w_\circ w_\nu^{-1}} \cap \Omega_{w_\circ v_\xi^{-1}}^\circ$ associated to inverse grassmannians w_ν^{-1}, v_ξ^{-1} lies inside $Fl^X(V \oplus V) = \overline{G}/\overline{B}$.

4.2. Isomorphisms and main theorem

In this section, we show Theorem 4.7. Let us denote by $\mathbb{H}^*(Y, \mathcal{IC}(Y))$ the *hypercohomology group* of Deligne complex $\mathcal{IC}(Y)$ in $D^b(Y)$. The following proposition provides a relation between the hypercohomology and the stalk cohomology sheaf for the Kazhdan–Lusztig variety. It will allow us to relate (4.1) with the coefficients from (2.2), and eventually get Theorem 4.7.

According to [17, Lemma 4.5], if Z is a closed subvariety of $\mathbb{C}^N$ for some $N \geq 2$ invariant under the $(\mathbb{A}_{\mathbb{C}}^1)^\times$-action by $t \cdot (z_1, \ldots, z_s) = (t^{\alpha_1} z_1, \ldots, t^{\alpha_s} z_s)$ for $\alpha_i > 0$, $i = 1, \ldots, s$, then

$\mathbb{H}^i(Z) = \mathcal{H}^i_{\mathbf{z}}(Z)$ for all i where $\mathbf{z}$ is the cone point of $\mathbb{C}^N$. Furthermore, the Kazhdan–Lusztig varieties $X_x \cap \Omega^\circ_y$ be for x and y in W_n can be considered as $(\mathbb{A}^1_{\mathbb{C}})^\times$-stable closed subvarieties of $\mathbb{C}^{\ell(w_\circ) - \ell(y)}$ [17, §1.5]. To be specific, $X_x \cap \Omega^\circ_y$ is a T-stable *attractive slice* to X°_y at y in X_x. See [10, §2.1] for the definition of the attractive slice. That is, y is an attractive fixed point for the T-action on $X_x \cap \Omega^\circ_y$. So, by [10, Proposition A2], there exists a one parameter subgroup $\alpha : (\mathbb{A}^1_{\mathbb{C}})^\times \to T$ such that $\lim_{t \to 0} \alpha(t) \cdot z = y$ for all z in a neighborhood of y. Moreover, the set of all $z \in X_x \cap \Omega^\circ_y$ satisfying $\lim_{t \to 0} \alpha(t) \cdot z = y$ forms an open affine T-stable neighborhood of y. Since $y\Omega_{id}$ is an open neighborhood of y in Fl and we restrict it on $X_x \cap \Omega^\circ_y$, all $z \in X_x \cap \Omega^\circ_y$ satisfies $\lim_{t \to 0} \alpha(t) \cdot z = y$. Note that $\chi_i \circ \alpha : (\mathbb{A}^1_{\mathbb{C}})^\times \to (\mathbb{A}^1_{\mathbb{C}})^\times$ is given by $t \to t^{\alpha_i}$ where χ_i are characters of T. In addition, by [10, §1.3] $Y := X_x \cap \Omega^\circ_y$ can be embedded into $T_y Y$ and all the weights of the $(\mathbb{A}^1_{\mathbb{C}})^\times$-action on $T_y Y$ are positive, i.e., $\alpha_i > 0$ for all i. Thus we have the following proposition.

Proposition 4.2. *Given $X_x \cap \Omega^\circ_y$ for x and y in W with $x \geq y$ in Bruhat order, we have*

$$\mathbb{H}^i(X_x \cap \Omega^\circ_y, \mathcal{IC}(X_x \cap \Omega^\circ_y)) \cong \mathcal{H}^i_y(\mathcal{IC}(X_x \cap \Omega^\circ_y))$$

for all i.

Now, we move toward Lemma 4.3. This lemma is an application of Theorem 4.1. Let us denote by $i : y\Omega^\circ_{id} \hookrightarrow Fl$ the open immersion, where $y\Omega^\circ_{id}$ is the open neighborhood of a point p_y. Let $X_x = X_\ell \supset X_{\ell-2} \supset \cdots \supset X_0 \supset \emptyset$ be the stratification of X_x by closed subschemes of the Schubert variety X_x of dimension $\ell := \ell(x)$. Then the closed embedding $j : X_x \hookrightarrow Fl$ induces a stratification on Fl. So, with this stratification and (2.1), we obtain the Deligne sheaves $\mathcal{IC}(X_x) := j_! \mathcal{IC}(X_x)$ on Fl. Moreover, with the similar argument and the map i, we have $\mathcal{IC}(X_x \cap y\Omega^\circ_{id})$ from $i^{-1}\mathcal{IC}(X_x) \cong \mathcal{IC}(i^{-1}X_x)$.

The following lemma shows that $f : X_x \cap \Omega^\circ_y \to X_{x'} \cap \Omega^\circ_{y'}$ induces an isomorphism of hypercohomologies of $\mathcal{IC}$-sheaves on Kazhdan–Lusztig varieties.

Lemma 4.3. *Suppose that $f : X_x \cap \Omega^\circ_y \to X_{x'} \cap \Omega^\circ_{y'}$ is a scheme-theoretic isomorphism for pairs $x, y \in W$ and $x', y' \in \overline{W}$ with $y \leq x$ and $y' \leq x'$. We have*

$$\mathbb{H}^*(X_x \cap \Omega^\circ_y, \mathcal{IC}(X_x \cap \Omega^\circ_y)) \cong \mathbb{H}^*(X_{x'} \cap \Omega^\circ_{y'}, \mathcal{IC}(X_{x'} \cap \Omega^\circ_{y'})).$$

Proof. Let $y\Omega^\circ_{id} = X^\circ_y \times \Omega^\circ_y$ be the open neighborhood of a point $p_y \in X_x$. As discussed earlier, we can get a $\mathcal{IC}$-sheaf on the open neighborhood $X_x \cap y\Omega^\circ_{id}$ of p_y in X_x. In addition, for $x, y \in W$ with $x \leq y$, we know that $X_x \cap y\Omega^\circ_{id} \cong X^\circ_y \times (X_x \cap \Omega^\circ_y)$ by [17, §1.4].

We define a T-equivariant isomorphism

$$\phi := id^{-1} \times f^{-1} : X^\circ_y \times (X_{x'} \cap \Omega^\circ_{y'}) \to X^\circ_y \times (X_x \cap \Omega^\circ_y). \qquad (4.3)$$

Then by topological invariance of $\mathcal{IC}$-sheaves [4, p. 97], we have

$$\phi^* \mathcal{IC}(X^\circ_y \times (X_x \cap \Omega^\circ_y)) \cong \mathcal{IC}(\phi^{-1}(X^\circ_y \times (X_x \cap \Omega^\circ_y))).$$

Thus, by [4, Theorem 4.3.1], we obtain $\mathcal{IC}(\phi^{-1}(X^\circ_y \times (X_x \cap \Omega^\circ_y))) \cong \mathcal{IC}(X^\circ_y \times (X_{x'} \cap \Omega^\circ_{y'}))$. As ϕ is an isomorphism, we get

$$\mathcal{IC}(X^\circ_y \times (X_x \cap \Omega^\circ_y)) \cong \mathcal{IC}(X^\circ_y \times (X_{x'} \cap \Omega^\circ_{y'})).$$

Then we have

$$\mathcal{IC}(X_y^\circ) \boxtimes \mathcal{IC}(X_x \cap \Omega_y^\circ) \cong \mathcal{IC}(X_y^\circ) \boxtimes \mathcal{IC}(X_{x'} \cap \Omega_{y'}^\circ)$$

from [21, §5] where $\boxtimes$ is the geometrical external tensor product of perverse sheaves.

As X_y° is smooth and $\mathcal{IC}$ sheaf is constructible, its intersection cohomology sheaf is constant on X_y°. Hence, we conclude $\mathcal{IC}(X_x \cap \Omega_y^\circ) \cong \mathcal{IC}(X_{x'} \cap \Omega_{y'}^\circ)$ of Deligne complexes of sheaves, which immediately gives the statement. $\square$

Remark 4.4. It is expected that if $f : A \to B$ is a scheme-theoretic isomorphism, then $\mathcal{IC}(A) \cong \mathcal{IC}(B)$. However, to the best of the author's knowledge, this is only true if there exists a $\mathcal{IC}$-sheaf on B, and A and B admit stratifications which are homeomorphic, according to [4, p. 97]. To resolve this, in Lemma 4.3, we pull back the stratification of $X_x \cap y\Omega_{id}^\circ$ under the isomorphism (4.3), and then get a $\mathcal{IC}$-sheaf on the Kazhdan–Lusztig varieties $X_{x'} \cap \Omega_{y'}^\circ$ and $X_x \cap \Omega_y^\circ$ as in [23, Lemma 3.2].

The following proposition explains that two relevant T-fixed points are included in a Schubert variety. We consider the locally closed set $\mathcal{Z}_{\tau'} = \{E_\bullet \mid \dim(E_{p_i} \cap F_{q_i}) = k_i'\} \subset Fl = G/B$ where $\tau' = (\mathbf{k}', \mathbf{p}, \mathbf{q})$ is a weak triple with a vexillary element $v' = w(\tau')$ in the Weyl group W. Suppose that the stabilizer of the action on the partial flag F_{q_i} is Q. Then the set $\mathcal{Z}_{\tau'}$ is a Q-orbit of $p_{w_\circ v'}$ as a disjoint union of Schubert cells containing both T-fixed points $p_{w_\circ v}$ and $p_{w_\circ v'}$. That is,

Proposition 4.5. *Let w be a vexillary element. A Schubert variety $X_{w_\circ w}$ contains both $p_{w_\circ v}$ and $p_{w_\circ v'}$ for any v, and a vexillary (signed) permutation $v'(\tau')$ induced by the weak triple in W.*

Proof. We fix $g \in G$ satisfying $gp_{w_\circ v} = p_{w_\circ v'}$. Essentially g stabilizes the partial flag $F_{q_1} \subseteq \cdots \subseteq F_{q_s}$, which implies that $\mathcal{Z}_{\tau'}$ is preserved by the multiplication by g, that is, $g\mathcal{Z}_{\tau'} = \mathcal{Z}_{\tau'}$. For a similar reason, $X_{w_\circ w}$ is invariant under the action of g so that $gX_{w_\circ w} = X_{w_\circ w}$. One can check by the invariance under g-action that $g(X_{w_\circ w} \cap \mathcal{Z}_{\tau'}) = gX_{w_\circ w} \cap g\mathcal{Z}_{\tau'} = X_{w_\circ w} \cap \mathcal{Z}_{\tau'}$. This completes the proposition. $\square$

We observe that $p_{w_\circ v}$ and $p_{w_\circ v'}$ can be in different strata of $X_{w_\circ w}$.

Let X and Y be varieties with points $x \in X$ and $y \in Y$. We say that the singularity of X at x and the singularity of Y at y are *smoothly equivalent* if there is a variety Z such that two maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ are smooth at z with $f_1(z) = x$ and $f_2(z) = y$, [15, Definition 3.7].

Finally, we are in the position to show Theorem 4.6.

Theorem 4.6. *For a covexillary Schubert variety $X_{w_\circ w}$ and any torus-fixed point $p_{w_\circ v} \in X_{w_\circ w}$ in $Fl = G/B$, there exist $X_{w_\circ w_v^{-1}}$ and a torus-fixed point $p_{w_\circ v_\xi^{-1}} \in X_{w_\circ w_v^{-1}}$ in $\overline{G}/B$ such that*

$$\mathcal{H}^{2q}_{p_{w_\circ v}}(\mathcal{IC}(X_{w_\circ w})) = \mathcal{H}^{2q}_{p_{w_\circ v_\xi^{-1}}}(\mathcal{IC}(X_{w_\circ w_v^{-1}}))$$

for all $q \geq 0$. In particular, $P_{w_\circ v, w_\circ w}(q) = P_{w_\circ v_\xi^{-1}, w_\circ w_v^{-1}}(q)$.

Proof. Let m_x be the multiplication map by $x \in G$ and e the identity of G. We take the commutative diagram

$$
\begin{array}{ccccccc}
X_{w_\circ w} & \xleftarrow{f_1} & X_{w_\circ w} \cap p_{w_\circ v'}\Omega^\circ_{id} & \xleftarrow{m_g} & X_{w_\circ w} \cap p_{w_\circ v}\Omega^\circ_{id} & \xrightarrow{f_2} & X_{w_\circ w} \\
{\scriptstyle u_1}\uparrow & & {\scriptstyle u_1}\uparrow & & {\scriptstyle u_2}\uparrow & & {\scriptstyle u_2}\uparrow \\
p_{w_\circ v'} & \xleftarrow{m_e} & p_{w_\circ v'} & \xleftarrow{m_g} & p_{w_\circ v} & \xrightarrow{m_e} & p_{w_\circ v}
\end{array}
$$

with arcs labelled h (top) and m_g (bottom),

where u_i is the inclusion, f_i is an open immersion, and $h = f_1 \circ m_g$ is the composition of f_1 and m_g. Since f_i is smooth at $p_{w_\circ v}$, the singularity of $X_{w_\circ w}$ at $p_{w_\circ v'}$ is smoothly equivalent to the singularity of $X_{w_\circ w}$ at $p_{w_\circ v}$. By virtue of [15, Proposition 3.8], we deduce $\mathcal{IC}(X_{w_\circ w})_{p_{w_\circ v}} \cong \mathcal{IC}(X_{w_\circ w})_{p_{w_\circ v'}}$ so that

$$\mathcal{H}^{2q}_{p_{w_\circ v}}(\mathcal{IC}(X_{w_\circ w})) \cong \mathcal{H}^{2q}_{p_{w_\circ v'}}(\mathcal{IC}(X_{w_\circ w})). \tag{4.4}$$

The definition of stalk for the intersection cohomology sheaf shows the isomorphism

$$\mathcal{H}^{2q}_{p_{w_\circ v'}}(\mathcal{IC}(X_{w_\circ w})) \cong \mathcal{H}^{2q}_{p_{w_\circ v'}}(\mathcal{IC}(X_{w_\circ w} \cap \Omega^\circ_{w_\circ v'})). \tag{4.5}$$

We know that $w = w(\tau)$ and v' are vexillary elements in W. Combined with (4.4), Proposition 4.2, (4.2) and Lemma 4.3, we obtain

$$\mathcal{H}^{2q}_{p_{w_\circ v}}(\mathcal{IC}(X_{w_\circ w})) \cong \mathbb{H}^{2q}(X_{w_\circ w_v^{-1}} \cap \Omega^\circ_{w_\circ v_\xi^{-1}}, \mathcal{IC}(X_{w_\circ w_v^{-1}} \cap \Omega^\circ_{w_\circ v_\xi^{-1}}))$$

$$\cong \mathcal{H}^{2q}_{p_{w_\circ v_\xi^{-1}}}(\mathcal{IC}(X_{w_\circ w_v^{-1}} \cap \Omega^\circ_{w_\circ v_\xi^{-1}})).$$

Since $\mathcal{H}^{2q}_{p_{w_\circ v_\xi^{-1}}}(\mathcal{IC}(X_{w_\circ w_v^{-1}})) \cong \mathcal{H}^{2q}_{p_{w_\circ v_\xi^{-1}}}(\mathcal{IC}(X_{w_\circ w_v^{-1}} \cap \Omega^\circ_{w_\circ v_\xi^{-1}}))$, we have thereby proved that their Hilbert–Poincaré polynomials coincide. $\square$

Finally, we finish a proof of our main results by the following theorem.

Theorem 4.7. *Let $w_\circ w$ be a covexillary element in W, and v be any element with $w \leq v$. Then there are some grassmannian elements $w_v, v_\xi \in \overline{W}^P$ with $w_v \geq v_\xi$ such that*

$$P_{w_\circ v, w_\circ w}(q) = P_{w_\circ v_\xi, w_\circ w_v}(q).$$

Proof. We know from [9, Cor. 4.4], [12, Prop. 7.6 and Sec. 7.13] that $P_{y,w}(q) = P_{y^{-1}, v^{-1}}(q) = P_{w_\circ y w_\circ, w_\circ w w_\circ}(q)$ for $u, v \in W$. Since

$$P_{w_\circ v_\xi^{-1}, w_\circ w_v^{-1}}(q) = P_{v_\xi w_\circ^{-1}, w_v w_\circ^{-1}}(q) = P_{w_\circ v_\xi, w_\circ w_v}(q),$$

we conclude the statement with Theorem 4.7. $\square$

4.3. Combinatorial formulas for covexillary cases

In this section, we discuss the combinatorial and inductive formulas for Kazhdan–Lusztig polynomials in covexillary case, generalizing the cograssmannian cases in classical types.

Let $w_\circ w(\tau)$ be a covexillary (signed) permutation with a triple τ, and $w_\circ v$ be any element with $w(\tau) \leq v$. Recall from Section 3 that $\lambda(w_\circ w(\tau))$ and $\mu(w_\circ w(\tau), w_\circ v)$ be the $\alpha\beta$-words.

Given the $\alpha\beta$-words $\lambda(w_\circ w(\tau))$ and $\mu(w_\circ w(\tau), w_\circ v)$, one can define the set

$$A^G(\lambda(w_\circ w(\tau))/\mu(w_\circ v)) \tag{4.6}$$

of all labelings of tree(s) $\mathcal{T} := \mathcal{T}(\lambda(w_\circ w(\tau)))$ for $G = SL(n)$ of type A by [19, 6.5] and for $G = Sp(2n)$ of type C, $G = SO(2n)$ of type D and $G = SO(2n+1)$ of type B by [8]. We note that the construction of tree(s) $\mathcal{T}$ and the set A^G of labeled tree(s) in [8,19] arising only from $\alpha\beta$-words. So, the set (4.6) and tree(s) $\mathcal{T}$ are well-defined with $\alpha\beta$-words.

The following gives the combinatorial formulas for Kazhdan–Lusztig polynomials associated to covexillary elements.

Corollary 4.8. *Let G be an algebraic group of classical types. Let $w := w(\tau)$ be a vexillary element with a triple τ in W. Let v be any element in W with $w \leq v$ in Bruhat order. Then the Kazhdan–Lusztig polynomial is given by*

$$P_{w_\circ v, w_\circ w}(q) = \sum q^{|\mathcal{T}|}$$

where the sums take over all edge labelings of $\mathcal{T}$ in $A^G(\lambda(w_\circ w), \mu(w_\circ w, w_\circ v))$. $|\mathcal{T}|$ denotes the sum of all the edge labels.

Proof. Let w_ν and w_ξ be the grassmannian elements defined in Section 4.1. Since the corresponding $\alpha\beta$ words of cograssmannian elements $w_\circ w_\nu$ and $w_\circ v_\xi$ are $\lambda(w_\circ w)$ and $\mu(w_\circ w, w_\circ v)$ respectively, by [19, Theorem 7.8] for type A and [8] for types C, D and B, we have

$$P_{w_\circ v_\xi, w_\circ w_\nu}(q) = \sum q^{|\mathcal{T}|}.$$

Combined with Theorem 4.7, we establish the result. □

We remark that the Kazhdan–Lusztig polynomials depend only on triples τ and τ' from vexillary element w and any v with $w \leq v$ in W. In particular if $v' \geq w$ is another element with $\tau'(v', w) = \tau(v, w)$, the Kazhdan–Lusztig polynomials are equal. In other words, the $\alpha\beta$-words for covexillary cases arise from the sequences of τ and τ', while the ones for cograssmannian cases are only from the partitions.

4.4. Covexillary version of inductive formulas for Kazhdan–Lusztig polynomials

Recall that λ is the partition associated to a covexillary (signed) permutation $w_\circ w(\tau)$ with a triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ and μ be the partition associated to the pair $(w_\circ w, w_\circ v)$ for any element v with $w \leq v$.

The algorithm of this second result can be realized by two steps with the matrix $\mathfrak{H}(\lambda) = \begin{pmatrix} a_1 & \cdots & a_d \\ b_0 & \cdots & b_{d-1} \end{pmatrix}_\lambda$ defined in Section 3. First, we find the smallest i such that $b_i \leq a_i$ and $a_{i+1} \leq b_{i+1}$. Next we replace $\mathfrak{H}(\lambda) = \begin{pmatrix} \cdots & a_i & a_{i+1} & a_{i+2} & \cdots \\ \cdots & b_{i-1} & b_i & b_{i+1} & \cdots \end{pmatrix}_\lambda$ by

$$\mathfrak{H}_1 := \begin{pmatrix} \cdots & a_i + a_{i+1} & a_{i+2} & \cdots \\ \cdots & b_{i-1} & b_i + b_{i+1} & \cdots \end{pmatrix}.$$

Note that $d = s$ or $d = s+1$, depending on their types, i.e., type D could have $d = s+1$ for a triple τ of length s.

In addition we impose two conditions on $\mathfrak{H}(\lambda)$ by

(1) $\lambda_d < N - a_d$

(2) $(a_d + a_{d-1} + \cdots + a_i) - (b_{d-1} + \cdots + b_i) < N - \lambda_d,$ for $1 \leq i,$

where $\lambda_d = p_d + q_d$ and $N = 2n + 1$ if $G = Sp(2n)$ or $SO(2n + 1)$, and $\lambda_d = p_d + q_d + 2$ and $N = 2n$ if $G = SO(2n)$. There are no such hypotheses on type A.

Given $\mu \leq \lambda$, $c(\mu, \lambda)$ is the *depth* sequence of μ in λ such that

$$
\mathfrak{H}(\mu) = \begin{pmatrix} e_1 & e_2 & \cdots & e_m & e_{m+1} \\ f_0 & f_1 & \cdots & f_{m-1} & f_m \end{pmatrix}_\mu
$$

$$
= \begin{pmatrix} a_1 + c_1 & a_2 + c_2 - c_1 & \cdots & a_m + c_m - c_{m-1} & 1 \\ f_0 & f_1 & \cdots & f_{m-1} & f_m \end{pmatrix}_\mu
$$

where the last column appears precisely when $G = SO(2n)$ and c_m is odd, and $2n = \sum_{i=1}^{m+1} e_i + \sum_{i=0}^{m} f_i$ [24, 3.2].

Let

$$
\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{(q^\alpha - 1) \cdots (q^{\alpha - \beta + 1} - 1)}{(q^\beta - 1)(q^{\beta - 1} - 1) \cdots (q - 1)}
$$

where $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$ if $\beta < 0$ or $\alpha < \beta$, and $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ as well as $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha - \beta \end{bmatrix}$.

Our inductive formula can be presented as follows.

Corollary 4.9. *Let $w = w(\tau)$ be a vexillary (signed) element in W. Let v be any element with $w \leq v$. The inductive formula for Kazhdan–Lusztig polynomial associated to $w_\circ w$ and $w_\circ v$ is given by*

$$
P_{w_\circ v, w_\circ w}(q) = \sum_t q^{(c_i - t)(c_{i+1} - t)} \begin{bmatrix} a_{i+1} - c_i + c_{i+1} \\ c_{i+1} - t \end{bmatrix} \begin{bmatrix} b_i + c_i - c_{i+1} \\ c_i - t \end{bmatrix} P_{\mathfrak{H}_1, c(t)}(q),
$$

where $c(t) = (c_0, \ldots, c_{i-1}, t, c_{i+2}, \ldots, c_d)$ for a capacity $c(\mu, \lambda) = (c_0, \ldots, c_d)$ and $c_0 = 0$.

Proof. Let w_v and w_ξ be the grassmannian elements defined in Section 4.1. Since the corresponding $2 \times m$ matrix of cograssmannian elements $w_\circ w_v$ and $w_\circ v_\xi$ are $\mathfrak{H}(\lambda(w_\circ w))$ and $\mathfrak{H}(\mu(w_\circ w, w_\circ v))$ respectively, by [28, Theorem 2] for type A and [24] for types C, D and B or [5, Chapter 9.1.20], we have

$$
P_{w_\circ v_\xi, w_\circ w_v}(q) = \sum_t q^{(c_i - t)(c_{i+1} - t)} \begin{bmatrix} a_{i+1} - c_i + c_{i+1} \\ c_{i+1} - t \end{bmatrix} \begin{bmatrix} b_i + c_i - c_{i+1} \\ c_i - t \end{bmatrix} P_{\mathfrak{H}_1, c(t)}(q),
$$

where $c(t) = (c_0, \ldots, c_{i-1}, t, c_{i+2}, \ldots, c_d)$ for a capacity $c = (c_0, \ldots, c_d)$ and $c_0 = 0$. Hence, the result follows by Theorem 4.7. $\square$

Acknowledgments

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