Problem Set 1

Due: April 26, 2021 (in class; subject to change if COVID restrictions apply)

Problem 1

Suppose you have n i.i.d. observations from the nonlinear regression

$$y_i = (x_i + \beta)^2 + \epsilon_i \quad \epsilon_i | x_i \sim \mathcal{N}(0, 1)$$

1. Find the asymptotic distribution of the nonlinear least squares estimator of β.
2. Find the asymptotic distribution of the maximum likelihood estimator of β. Compare the two asymptotic distributions.

Hint: For NLS $m(x_i; \beta) = -[y_i - (x_i + \beta)^2]^2$. For MLE
$m(x_i; \beta) = \log f(y|x; \beta) = \log(1) - \log(\sqrt{2\pi}) - \frac{1}{2}[y - (x + \beta)^2]^2$

Problem 2

Consider the linear regression model with normal errors, whose conditional density for observation i is

$$\log f(y_i | \mathbf{x_i}; \beta, \sigma^2) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{(\mathbf{y_i} - \mathbf{x_i'}\beta)^2}{2\sigma^2}$$

Let $\left(\hat{\beta}, \hat{\sigma}^2\right)$ be the unrestricted ML estimate of $\theta = \left(\beta, \sigma^2\right)$ and let $\left(\tilde{\beta}, \tilde{\sigma}^2\right)$ be the restricted ML estimate subject to the constraint $\mathbf{R}\beta = \mathbf{c}$ where $\mathbf{R}$ is an $r \times X$ matrix of known constraints. Assume that $\Theta = \mathbb{R}^K \times \mathbb{R}_+ +$ and that $\mathbb{E}[x_i x_i']$ is nonsingular. Also, let

$$\hat{\Sigma} = \begin{bmatrix} \frac{1}{\hat{\sigma^2}} \frac{1}{n} \sum_1^n x_i x_i' & 0 \\ 0 & \frac{1}{2(\hat{\sigma}^2)^2} \end{bmatrix}$$

$$\tilde{\Sigma} = \begin{bmatrix} \frac{1}{\tilde{\sigma^2}} \frac{1}{n} \sum_1^n x_i x_i' & 0 \\ 0 & \frac{1}{2(\tilde{\sigma}^2)^2} \end{bmatrix}$$

- (a) Verify that $\hat{\beta}$ minimizes the sum of squared residuals. So it is the OLS estimator. Verify that $\tilde{\beta}$ minimizes the sum of squared residuals subject to the constraint $\mathbf{R}\beta = \mathbf{c}$. so it is the restricted least squares estimator. **Hint**: to set up a problem, please refer to Restrictions section from EE.pdf notes. You can replace the term $\sum(y_i - \mathbf{x_i'}\beta)^2$ with the sum of squaredresidual: i.e., $SSR(\beta) \equiv \sum(y_i - \mathbf{x_i'}\beta)^2$.
- (b) Let $Q_n(\theta) = \log f(y_i | \mathbf{x_i}; \beta, \sigma^2)$. Show that

$$Q_n(\hat{\theta}) = -\frac{1}{2}\log(2\pi) - \frac{1}{2} - \frac{1}{2}\log\big(\frac{SSR_U}{n}\big)$$

$$Q_n(\hat{\theta}) = -\frac{1}{2}\log(2\pi) - \frac{1}{2} - \frac{1}{2}\log\big(\frac{SSR_R}{n}\big)$$

where $SSR_U \equiv \sum(y_i - \mathbf{x_i'}\hat{\beta})^{\mathbf{2}}$ is the unrestricted sum of squared residuals and $SSR_R \equiv \sum(y_i - \mathbf{x_i'}\tilde{\beta})^{\mathbf{2}}$ is the restricted sum of squared residuals.

- (c) Verify that the $\hat{\Sigma}$ given here, although not the same as $-\frac{1}{n}\sum_1^n \mathbf{H(w_i;}\hat{\theta}\mathbf{)}$, is consistent for $\mathbb{E}\big[\mathbf{w_i;}\,\theta_\mathbf{0}\big]$. Verify that the $\tilde{\Sigma}$ given here, although not the same as $-\frac{1}{n}\sum_1^n \mathbf{H(w_i;}\tilde{\theta}\mathbf{)}$, is consistent for $\mathbb{E}\big[\mathbf{w_i;}\,\theta_\mathbf{0}\big]$.

- (d) Show that the Wald, LM, and LR statistics using $\hat{\Sigma}$ and $\tilde{\Sigma}$ given here can be written as

$$W = n \cdot \frac{(\mathbf{R}\hat{\beta} - \mathbf{c})'[\mathbf{R(X'X)^{-1}R'}]^{-\mathbf{1}}(\mathbf{R}\hat{\beta} - \mathbf{c})}{SSR_U} \tag{1}$$

$$LM = n \cdot \frac{(y - \mathbf{X}\tilde{\beta})'\mathbf{P}(\mathbf{y} - \mathbf{X}\tilde{\beta})}{SSR_R} \tag{2}$$

$$LR = n \cdot \left[\log\left(\frac{SSR_R}{n}\right) - \log\left(\frac{SSR_U}{n}\right)\right] \tag{3}$$

where y is a vector with dimension $(n \times 1)$ and $\mathbf{X}$ is a matrix with dimension $(n \times K)$.

- (e) Show that the three statistics can also be written as

$$W = n \cdot \frac{SSR_R - SSR_U}{SSR_U} \tag{4}$$

$$LM = n \cdot \frac{SSR_R - SSR_U}{SSR_R} \tag{5}$$

$$LR = n \cdot \log\left(\frac{SSR_R}{SSR_U}\right) \tag{6}$$

- (f) Show that $W \geq LR \geq LM$. These inequality do not always hold in nonlinear regression models.